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Instanton counting on compact manifolds

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A Francesca

Abstract

In this thesis we analyze supersymmetric gauge theories on compact manifolds and their relation with representation theory of infinite Lie algebras associated to conformal field theories, and with the computation of geometric invariants and superconformal indices.

The thesis contains the work done by the candidate during the doctorate programme at SISSA under the supervision of A. Tanzini and G. Bonelli. This consists in the following publications

- in [17], reproduced in Chapter 2, we consider $\mathcal{N} = 2$ supersymmetric gauge theories on four manifolds admitting an isometry. Generalized Killing spinor equations are derived from the consistency of supersymmetry algebrae and solved in the case of four manifolds admitting a $U(1)$ isometry. This is used to explicitly compute the supersymmetric path integral on $S^2 \times S^2$ via equivariant localization. The building blocks of the resulting partition function are shown to contain the three point functions and the conformal blocks of Liouville Gravity.

- in [30], reproduced in Chapter 3, we provide a contour integral formula for the exact partition function of $\mathcal{N} = 2$ supersymmetric $U(N)$ gauge theories on compact toric four-manifolds by means of supersymmetric localisation. We perform the explicit evaluation of the contour integral for $U(2)$ $\mathcal{N} = 2^*$ theory on $\mathbb{P}^2$ for all instanton numbers. In the zero mass case, corresponding to the $\mathcal{N} = 4$ supersymmetric gauge theory, we obtain the generating function of the Euler characteristics of instanton moduli spaces in terms of mock-modular forms. In the decoupling limit of infinite mass we find that the generating function of local and surface observables computes equivariant Donaldson invariants, thus proving in this case a long-standing conjecture by N. Nekrasov. In the case of vanishing first Chern class the resulting equivariant Donaldson polynomials are new.

- in [23], reproduced in Chapter 4, we explore $\mathcal{N} = (1, 0)$ superconformal six-dimensional theories arising from M5 branes probing a transverse A_k singularity. Upon circle compactification to five dimensions, we describe this system with a dual pq-web of five-branes and propose the spectrum of basic five-dimensional instanton operators driving global symmetry enhancement. For a single M5 brane, we find that the exact partition function of the 5d quiver gauge theory matches the 6d $(1, 0)$ index, which we compute by letter counting. We finally show which relations among vertex correlators of $q\mathcal{W}$ algebrae are implied by the S-duality of the pq-web.

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Chapter 1

Introduction

The power of supersymmetry is the possibility to obtain *exact* results in quantum field theory. Exact means that all non-perturbative effects can be taken into account. The outcome is that in several cases the Feynman's path integral can be completely solved. It takes contributions only from those field configurations protected by the supersymmetry, which are called supersymmetric minima or vacua. This procedure is called supersymmetric localization.

The purpose of this work is the study and the complete accounting of non-perturbative effects for several examples of supersymmetric quantum field theories defined on compact manifolds. These non-perturbative effects are given by *instantons.*

The study of supersymmetric theory on compact manifolds in various dimensions has displayed several interesting links with other fields of study both in mathematics and in physics. Let us mention some of them. In mathematics supersymmetric theories on compact manifolds in various dimensions compute several types of topological invariats. For example $\mathcal{N} = 2$ theories on four dimensional compact manifolds describe moduli space of unframed instantons, and therefore can be used to compute Donaldson invariants. Moreover they are related with representation theory of infinite Lie algebras and with classical and quantum integrable systems. In physics, in all dimensions, they can compute invariants describing the spectrum of the theory, like Witten and superconformal indices. They also describe boundary theories in the context of holographic duality.

Even though many new exact results for partition functions of supersymmetric theories on compact manifolds have been recently derived, for the class of examples we consider in this thesis there are not so many general complete answers. This is because the complete evaluation of the partition function over these manifolds requires the integration of a complicated combinatorical function called Nekrasov partition function. at the moment, for general values of the parameters of the theory, this partition function has only a power series expression avaiable. Some complete results are known for certain specifications of these parameters, but the answer is still incomplete in the majority of cases. This thesis reports one of the few complete evaluation of this kind of partition function in a concrete example.

1.1 Historical background

The problem of defining supersymmetry on a curved background has been investigated thoroughly in the past thirty years.

In his seminal work [144] Witten shows that in the case of extended $\mathcal{N} = 2$ supersymmetry it is always possible to preserve some amount of supersymmetry on a generic Riemannian four manifold. This is done by the so called *topological twist* which consists of the following operation. The parameters generating the supersymmetry are sections of the spinor bundles (with chiralities $+$ or $-$) and of the R-symmetry bundle, which can be choosen arbitrarily. Identifying the R-symmetry bundle with the spinor bundle (of one chirality) changes the spins of the supersymmetry generators: starting from eight spinorial components, one of them becomes a scalar, three of them organize in a self-dual two-form and the last four components organize in a one-form. Therefore at least the scalar supersymmetry is preserved by every background metric.

The localization locus, that is the space of supersymmetry minima, for a $\mathcal{N} = 2$ gauge theory related to this scalar supercharge consists in anti self-dual connections that are *instantons*. So the path integral reduces to an integration over the moduli space of instantons. Although this can be quite hard, the evaluation of observables in such theories is known to produce interesting topological invariants of the manifold, such as Donaldson invariants [147] and knot invariants [104, 76].

The topological twist has also been defined in other dimensions. Topological twisted theory in two dimensions are a tool for curve-counting problems, computing Gromov-Witten invariants [145].

If the manifolds has some isometry it is possible to preserve also the one-form generators of the twisted supersymmetry, this allows to further localize on the moduli space of vacua. In either cases the result for the partition function does not depends on the localization scheme, and so, on the choice of the supersymmetry generators. With this choice the supersymmetry action squares to a torus action along the isometry, this torus action can be used to compute the integral over the moduli space by equivariant localization and to obtain the result as a sum over the fixed point locus. This program was completed by Nekrasov in [119] for the $\mathcal{N} = 2$ topological twisted theory on the so-called Omega-background, that is $\mathbb{R}^4$ with a preferred choice of isometries. The fixed point locus is given by *point-like* (zero size) instantons at the origin of $\mathbb{R}^4$. The aim of the work was the evaluation of the Seiberg-Witten prepotential [132] from first principles, indeed the result of localization procedure, called *Nekrasov partition function*, reproduces the SW prepotential when the Omega-background parameters vanish.

On the other hand Pestun in [126] derives supersymmetry generators preserved by the S^4 metric without the use of the topological twist. This was done by a conformal transformation of the constant generators of flat space. The generators obtained in this way are called *conformal Killing spinors*. The supersymmetric vacua of this theory are point-like instantons and anti-instantons located at the north and south poles of the sphere respectively. The square of the supersymmetry action generates an isometry action that realizes two copies of the Omega-background in the tangent space to the poles. The result for the partition function is then written in terms of two copies of the Nekrasov partition function, one complex conjugate to the other.

More recently a systematic approach in the classification of manifolds admitting supersymmetry was achieve by Dumitrescu, Festuccia and Seiberg [65, 64] for the $\mathcal{N} = 1$ theories, generalized to $\mathcal{N} = 2$ theories on [82, 99]. This approach is based on freezing some minimal supergravity theory on the background metric describing the manifold. The freezing procedure of the local supersymmetry to a global one succeeds if the variations of the gravitini can be put to zero, this defines equations that have to be satisfied by the generators of the supersymmetry on the manifold, they are called *generalized Killing spinors equations.*

Using this approach other results have been obtained via equivariant localization on four-manifolds as $\mathbb{C}^2/\Gamma$ [70, 44, 35, 18, 137, 43, 45], squashed S^4 [82], Hopf surfaces and elliptic fibrations [134, 48, 7, 22], $\mathbb{P}^2$ [130].

A difference between calculations on non-compact and compact spaces that we will encounter, is that in the second case one needs to integrate over the Coulomb branch parameters (which are the vacuum expectation values of the scalar field in the vector multiplet). We will see that in our setting this integration is prescribed by the treatment of extra fermionic zero modes appearing in the compact case. On the mathematical side, the Coulomb branch parameters are related with the framing of instantons, which corresponds to a specification of a fiber of the bundle at a point (for example, on $\mathbb{R}^4$, instantons can be considered framed at the point at infinity [119]). This implies that the moduli space includes global gauge transformations acting on the framing. The twisted theory on non-compact spaces localizes on *framed* instantons while on compact spaces the theory localizes on *unframed* instantons. Framed instanton moduli spaces are easier to deal with. For example the moduli space of instantons on $\mathbb{R}^4$ is hyperkähler and have deep links to representation theory of infinite dimensional Lie algebrae and Geometric Invariant Theory [115]. They are much more amenable to equivariant localization than unframed moduli spaces. On the other hand Donaldson invariants are formulated via intersection theory on the latter. The physical integration over the Coulomb branches consequently has a deep geometrical meaning, because it corresponds to the removal of the framing. Anyway this integral is usually very difficult, at least for generic values of the parameters of the theory.

The existence of these exact results obtained by supersymmetric localization has brought many developments in several fields of mathematics and physics. As the recent discovery of new types of dualities, whereof one of the most powerful is AGT duality discovered by Alday, Gaiotto and Tachikawa in [6], that will be considered repeatedly during this thesis. It relates a certain class ($\mathcal{S}$) of four-dimensional theories analyzed by Gaiotto in [72] with two dimensional conformal field theory on a punctured Riemann surface. Namely the Nekrasov partition functions for class $\mathcal{S}$ theories of type A_1 (gauge groups $SU(2)$) are recognized to be correlation functions of primary fields in Liouville theory. This was soon generalized by Wyllard [149] as a conjectural duality between $SU(N)$ theories and correlators in Toda field theory.

Another development is in the study of systems of multiple M5 branes, which is still an elusive problem of M-theory. The study of compactifications on various space-time backgrounds [148, 72] have given many insights on the BPS protected sector. The exact results obtained in supersymmetric gauge theories via localization and BPS state counting has been revealed to be a very powerful tool [121] in the understanding of these systems. On one hand this has produced new correspondences among quantum field theories, topological theories and two-dimensional conformal field theories, as for instance [6]. On the other it has stimulated the study of higher dimensional supersymmetric gauge theories as deformations of strongly coupled super-conformal field theories in six dimensions. We will focus on this second aspect, studying families of five-dimensional theories on compact spaces that have a ultraviolet completion as superconformal field theories in six dimensions. In this context BPS state counting has been used to capture information about the circle compactification of M5 branes in terms of superconformal indices [106, 93].

1.2 Project

The first part of this thesis addresses two questions. The first is to complete the localization program for equivariant $\mathcal{N} = 2$ twisted theories defined on a whole class of compact four-manifolds, namely compact toric surfaces. This will be done constructing equivariantly twisted generators for the supersymmetry, as in [119], with respect to the torus action. We will show that the theory localizes on point-like instantons sitting at the fixed points of the toric action, in a somehow similar fashion as for S^4. On top of these singular solutions there will be also global ones: they consist in quantized magnetic fluxes wrapping internal two-cycles of the manifolds. Eventually the partition function of the $\mathcal{N} = 2$ theory over a toric compact variety will be written as a contour integral expression, where the integrand is written as a product of copies of the Nekrasov partition function related to the fixed points under the toric action, summed over the values of the quantized magnetic fluxes. The precise prescription for the contour of integration will be given by an appropriate treatment of fermionic zero modes.

An new feature appearing in this calculation is the need to put constraints on the sum over the magnetic fluxes. This is related to the requirement of stability of the bundle given by the instanton equation [54]. This is a peculiar situation related with unframed instantons since framed instantons correspond directly to framed holomorphic bundles without any further condition [53] (see also [41]). We will conjecture that the stability of the bundle, demanded by the BPS equations, is achieved by a specific constraint on the sum over the fluxes linked with Klyachko's analysis of stability of equivariant vector bundles [100].

The second question is to interpret these results in the perspective of AGT duality, looking for possible dual conformal field theories as already done, for example, for theories defined on toric A_n singularities in [35, 43, 45]. We will succeed in doing so considering the specific example of $\mathbb{P}^1 \times \mathbb{P}^1$, and comparing the expression of the partition function with Liouville gravity correlators.

The complete evaluation of the partition function however will require the integration over the Coulomb branch parameters. We will be able to perform this operation in the specific example of $\mathbb{P}^2$. Doing so we will also test a conjecture by Nekrasov. In [120] is claimed that the partition function of the $\mathcal{N} = 2$ equivariant twisted theory, integrated over the Coulomb branch parameters produces equivariant version of the Donaldson invariants of the toric variety. This refinement of Donaldson invariants was studied by Göttsche, Nakajima and Yoshioka in [75], where formulae for the evaluation on toric manifolds are given. We will indeed identify the integrated partition function on $\mathbb{P}^2$ with the generating function of equivariant Donaldson invariants computed therein, specifying the proper contour of integration on the Coulomb parameters. In some sense this is natural since Nekrasov partition function is the equivariant deformation of the Seiberg-Witten prepotential, and the latter is known [114] to produce ordinary Donaldson invariants when integrated over the complex plane parametrizing the vacua of theory.

In the final part of the thesis we will consider $(1,0)$ $6d$ superconformal theories. These theories describe systems of multiple M5 branes. Even if they have been studied extensively recently, they remain rather misterious. For example they do not have a Lagrangian description. Our work is based in the identification of dual $5d$ theories with a Lagrangian description. This allow us to compute quantities of the M-theory system via localization calculation. We will identify these dual theories as some circular quiver gauge theories. The evaluation of the partition function of one such theory will be performed by the study of instantons operators in five dimensions. Following the result of [106, 93] we will compute the indices of the $6d$ $(1,0)$ superconformal theories from the evaluation of the partition function on the five-sphere.

1.3 Background material

We collect here some material that could be useful to the reader. This section does not aim to be a comprehensive collection of all the background material. We want mainly to introduce some basic notions which will be used in the thesis and to give the appropriate references.

1.3.1 Localization

The purpose of this work is to use supersymmetry to perform the exact evaluation of the partition function. This is possible via supersymmetric localization which can be seen as a infinite dimensional version of Duistermaat-Heckman theorem [63] and we are going to quickly review. A thorough exposition of the subject is in [133] (see also [86]).

The partition function of a theory is computed by the Feynman's path integral

$$\mathcal{Z} = \int [d\{\phi\}] e^{-S[\{\phi\}]} \tag{1.3.1}$$

where the action $S[\{\phi\}]$ is a local functional of a collection of fields $\{\phi\}$ describing the classical theory and $\int [d\{\phi\}]$ denotes an infinite dimensional integral over all the possible configurations of the fields, that is over all the sections of the bundles of the fields. The integration in (1.3.1) achieves the quantization of the theory. Although this integral is in general mathematically ill-defined, the use of supersymmetric localization reduces it to a well-defined finite dimensional integral.

The fields $\{\phi\}$ can be separated into two groups, depending on their statistics. Those with even statistics are called *bosons*, those with odd statistics are the *fermions*. The presence of supersymmetry means the existence of a nilpotent[1] operator $\mathcal{Q}$ acting on the fields $\{\phi\}$ which transforms bosons into fermions and viceversa. The theory is said to be supersymmetric if the action is invariant under the supersymmetry, that is

$$\mathcal{Q}S[\{\phi\}] = 0. \tag{1.3.2}$$

Supersymmetric localization consists of deforming the action by a $\mathcal{Q}$-exact term, weighted by a complex parameter $t \in \mathbb{C}$. The path integral of the deformed action should, in principle, depend on t

$$\mathcal{Z}[t] = \int [d\{\phi\}] e^{-S[\{\phi\}] - t\mathcal{Q}V[\{\phi\}]} \tag{1.3.3}$$

but turns out that is actually independent on t: $\frac{d}{dt}\mathcal{Z}[t] = 0$. This means that the partition function depends only on the $\mathcal{Q}$-cohomology class of the action and not on the action itself.

It is convenient to deform the action by a positive definite term and to take the limit $t \to \infty$. In this way the path integral is governed by those configurations $\{\phi_0\}$ of the fields for which the deformation vanishes $\mathcal{Q}V[\{\phi_0\}] = 0$ and which correspond also to its minima. These configurations are called supersymmetric (or BPS) minima or vacua. They define the moduli space of supersymmetric minima $\mathcal{M}$, which is a finite dimensional space.

In this limit we can expand the fields as $\phi = \phi_0 + t^{-\frac{1}{2}}\delta\phi$ where $\delta\phi$ are the fluctuations of the fields around the BPS minima ϕ_0, and write (1.3.3) in the limit of big t getting

$$\lim_{t\to\infty} \mathcal{Z}[t] = \int_{\mathcal{M}} d\{\phi_0\}\, e^{-S[\{\phi_0\}]} \int [d\{\delta\phi\}]\, e^{-\frac{1}{2}\langle \delta\phi, \mathcal{Q}V^{(2)}[\{\phi_0\}]\delta\phi\rangle} \tag{1.3.4}$$

where the first integral is over the finite dimensional moduli space of vacua and even though the second one is infinite dimensional, it is a Gaussian integral containing the Hessian matrix $\mathcal{Q}V^{(2)}$ of the deformation and can be factorized into a bosonic and a fermionic Gaussian integrals

$$\langle \delta\phi, \mathcal{Q}V^{(2)}[\{\phi_0\}]\delta\phi\rangle = \langle \delta\phi_f, K_f \delta\phi_f\rangle + \langle \delta\phi_b, K_b \delta\phi_b\rangle \tag{1.3.5}$$

where the label f and b are for fermions and bosons respectively. Therefore it can be computed using Berezin integration rules, and the result is the square root of the ratio

[1] Nilpotency can be relaxed to the condition that the supersymmetry action squares to a bosonic symmetry, such as parallel transport along an isometry or gauge transformations.

of the fermionic and bosonic part of the determinant of the Hessian calculated at the BPS minima

$$\left(\frac{\det_{\text{bosons}} \mathcal{Q}V^{(2)}[\{\phi_0\}]}{\det_{\text{fermions}} \mathcal{Q}V^{(2)}[\{\phi_0\}]}\right)^{1/2} = \left(\frac{\det\, K_b}{\det\, K_f}\right)^{1/2}. \tag{1.3.6}$$

Inserting (1.3.6) back into (1.3.4), the partition function can be calculated in terms of a finite dimensional integral with a measure given by (1.3.6). Although this integral can be difficult to perform, due to the possibly complicated structure of $\mathcal{M}$, there are several examples in which it has been evaluated, we are going to see one of these in the next section.

Instanton counting

In [119] Nekrasov conjectured the possibility of evaluating the Seiberg-Witten prepotential [132] through the exact evaluation of (1.3.3) for the $\mathcal{N} = 2$ supersymmetric version of Yang-Mills theory with gauge group $SU(N)$ on $\mathbb{R}^4$. The action is the supersymmetric completion of the Yang-Mills action

$$S_{\text{SYM}}[A, \dots] = \int_{\mathbb{R}^4} \frac{1}{8\pi}\text{Im}(\tau)\,\text{Tr}[F \wedge \star F + \cdots] + \text{Re}(\tau)\,\text{Tr}\, F \wedge F. \tag{1.3.7}$$

It is a functional of the gauge connection A (whose curvature is $F = dA + A \wedge A$) and of the other fields in the vector multiplet. These fields appear in the terms omitted, which are necessary to have $\mathcal{Q}S_{\text{SYM}} = 0$. τ is the (complexified) coupling of the theory.

The evaluation of (1.3.3) is independent of the choice of the generators of the supersymmetry that selects the localization scheme. The choice used in [119] is to consider the equivariant version of Witten's topologically twist. Anyway in flat space the theory is independent of the choice of the R-symmetry bundle, and so the result holds also for the untwisted theory.

The supersymmetric minima selected by this choice are given by the following equations for the curvature

$$F = - \star F, \qquad -\frac{1}{8\pi^2}\int \text{Tr}\, F \wedge F = k. \tag{1.3.8}$$

These are flat connections when $k = 0$ and instantons (anti self-dual connections) for $k > 0$. The space $\mathcal{M}(N, k)$ defined by $U(N)$ connections satisfying (1.3.8) modulo gauge transformations is called the moduli space of $SU(N)$ instantons on $\mathbb{R}^4$ with topological charge k, its dimension is $4kN$.

This leads to the integration over the moduli space of instantons

$$\mathcal{Z}^{\mathbb{R}^4}_{inst} = \sum_{k=0}^{\infty} q^k \int_{\mathcal{M}(N,k)} [dA] \qquad \text{where } q = \exp 2\pi i \tau. \tag{1.3.9}$$

The idea is to evaluate (1.3.9) via equivariant localization, using Atiyah-Bott localization formula [9], with respect to combined action of the maximal torus of the group of global gauge transformation (generated by the parameters $\vec{a}$) and the lift to the moduli space of the maximal torus of the group $SO(4)$ of rotations on the manifold (generated by the parameters ϵ_1, ϵ_2), to reduce the integral to the fixed locus on $\mathcal{M}(N,k)$.

The space $\mathcal{M}(N,k)$ is isomorphic to the space of holomorphic bundles on $\mathbb{P}^2$ with second Chern class k, trivialized on the line at infinity, modulo isomorphisms [53]. In this perspective the action generated by the parameters $\vec{a}$ is a change of the framing of the bundle.

This moduli space is in general singular. Moreover it does not contain any of the fixed points of the torus action. Therefore to evaluate (1.3.9) one consider a natural partial compactification of this space $\widetilde{\mathcal{M}}(N,k)$ introduced by Gieseker [73]. This is given including torsion-free coherent sheaves, this space is non-singular and contains all the fixed points of the torus action which are ideal sheaves and are classified by Young diagrams. The generalization of instantons corresponding to torsion-free sheaves is given by the gauge field on a non-commutative geometry [122], and the fixed points in $\widetilde{\mathcal{M}}(N,k)$ correspond to sets of zero size non-commutative $U(1)$ instantons sitting at the origin of $\mathbb{R}^4$.

The integral (1.3.9) can be then written

$$\mathcal{Z}^{\mathbb{R}^4}_{inst}(q;\vec{a},\epsilon_1,\epsilon_2) = \sum_{k=0}^{\infty} q^k \int_{\widetilde{\mathcal{M}}(N,k)} [\widetilde{\mathcal{M}}(N,k)]_T \tag{1.3.10}$$

where the integrand is the fundamental class in the T-equivariant cohomology of the moduli space $H_T(\widetilde{\mathcal{M}}(N,k))$, where T is the combined toric action mentioned previously. Since $\widetilde{\mathcal{M}}(N,k)$ is a non-compact space, the integral (1.3.10) is actually computed by a formal application of the Atiyah-Bott formula. Since the fixed points are isolated it reduces to the sum of the zero-form part of $[\widetilde{\mathcal{M}}(N,k)]_T$ evaluated at the fixed points and normalized by the equivariant Euler form of the tangent bundle. The partition function $\mathcal{Z}$ is a power series in the coupling q and depends on the toric action parameters $\vec{a},\vec{\epsilon}$.

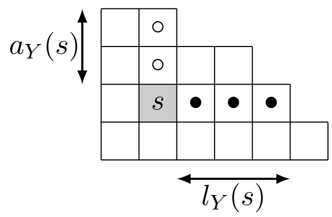


Figure 1.1: Young diagram.

Eventually the result for (1.3.10) is given in a combinatorical fashion as a sum over Young diagrams [119, 42, 69]

$$\mathcal{Z}^{\mathbb{R}^4}_{inst}(q;\vec{a},\epsilon_1,\epsilon_2) = \sum_{k=0}^{\infty} q^k \sum_{\substack{\{Y_\alpha\} \\ \sum_\alpha |Y_\alpha|=k}} \prod_{\alpha,\beta=1}^{N} \prod_{s\in Y_\alpha} \frac{1}{E(s)(E(s)-\epsilon_1-\epsilon_2)}$$
$$E(s) = a_\alpha - a_\beta - \epsilon_1\, l_{Y_\beta}(s) + \epsilon_2(a_{Y_\alpha}(s)+1) \tag{1.3.11}$$

where the arm and leg functions $a_Y(s)$, $l_Y(s)$ are depicted in figure 1.1

The complete evaluation of the path integral gives

$$\mathcal{Z}^{\mathbb{R}^4}_{full}(q;\vec{a},\epsilon_1,\epsilon_2) = e^{-\pi i\tau\frac{|\vec{a}|^2}{\epsilon_1\epsilon_2}}\mathcal{Z}^{\mathbb{R}^4}_{one-loop}(\vec{a},\epsilon_1,\epsilon_2)\mathcal{Z}^{\mathbb{R}^4}_{inst}(q;\vec{a},\epsilon_1,\epsilon_2) \tag{1.3.12}$$

where the exponential originates from the classical action $S[\{\phi_0\}]$ in (1.3.4) and $\mathcal{Z}^{\mathbb{R}^4}_{one-loop}$ is the contribution of the one-loop determinant (1.3.6) around the flat connections minima, where the integral over the moduli space is trivial. The latter can be calculated via Atiyah-Singer index theorem [8] for transversally elliptic differential operators [126]. In the specific case the complex to consider is the anti-self-dual complex

$$(d, d^-) : \Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d^-} \Omega^2. \tag{1.3.13}$$

From the evaluation of this index one obtains an expression for (1.3.6) as a infinite product formula that needs to be regularized. Via zeta-function regularization one gets

$$\mathcal{Z}^{\mathbb{R}^4}_{one-loop}(\vec{a},\epsilon_1,\epsilon_2) = \prod_{i,j}^{N} e^{-\gamma_{\epsilon_1,\epsilon_2}(a_i-a_j)} \tag{1.3.14}$$

where $\gamma_{\epsilon_1,\epsilon_2}$ is the logarithm of Barnes' double Gamma function [15] and it is defined as

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\frac{e^{-tx}}{(1-e^{\epsilon_1 t})(1-e^{\epsilon_2 t})}. \tag{1.3.15}$$

The conjecture by Nekrasov states that the non-equivariant limit of the partition function reproduces the Seiberg-Witten prepotential $\mathcal{F}(q;\vec{a})$, which was evaluated studying the infrared behaviour of the $\mathcal{N}=2$ theory

$$\lim_{\epsilon_1,\epsilon_2\to 0}\epsilon_1\epsilon_2\log\mathcal{Z}^{\mathbb{R}^4}_{full}(q;\vec{a},\epsilon_1,\epsilon_2) = \mathcal{F}(q;\vec{a}) \tag{1.3.16}$$

it was independently proved by three groups in [117, 118, 38].

1.3.2 AGT duality

Here we briefly review AGT-relation.

In [6] Alday, Gaiotto and Tachikawa discovered a duality between $2d$ conformal field theory and a certain class ($\mathcal{S}$) of supersymmetric field theories in $4d$ derived by Gaiotto in [72] obtained by the compactification of a $6d$ $(2,0)$ theory on a Riemann surface.

The duality identifies the partition function of $\mathcal{N}=2$ $SU(2)$ theory on S^4 with the n-point correlator function of primary operators in the Liouville conformal field theory on a Riemann surface.

Class $\mathcal{S}$ theories are classified by the punctured Riemann surface used to compactify the $(2,0)$ theory. Their weakly coupled limits are described by the possible ways to pant-decompose the Riemann surface. Eventually to each puncture corresponds a

$SU(2)_i$ flavor group with mass m_i and to each couple of punctures sewed together corresponds a gauge group $SU(2)_j$ with Coulomb parameter a_j. Moreover the moduli of the Riemann surface can be parametrized by the sewing parameters q_i, which correspond to the ultraviolet couplings of the $4d$ theory.

The punctured Riemann surface also states the correlator in Liouville theory with central charge $c = 1+6Q^2$ between primary fields $V_{\alpha_i} = e^{2\alpha_i\varphi(x_i)}$ with conformal dimensions $\Delta_{\alpha_i} = \alpha_i(Q-\alpha_i)$, inserted at the positions of the punctures x_i. Where $Q = b+b^{-1}$ and b is the coupling of the Liouville theory.

Two basic example are given in figure 1.2 and 1.3. On the right sides are displayed the $4d$ theories. In fig. 1.2 the theory has $SU(2)$ gauge group coupled to four hypermultiplets in (anti-)fundamental representation. The masses of the hypermultiplets are related to the mass parameters of the $SU(2)$ flavor groups, which in turn parametrize the conformal dimensions of the primaries in the four-point correlator function of Liouville theory on the sphere. In fig. 1.3 the $SU(2)$ gauge group is coupled to one hypermultiplets in the adjoint representation with mass m which parametrizes the conformal dimension of the primary in the one-point correlator function of Liouville theory on the torus.

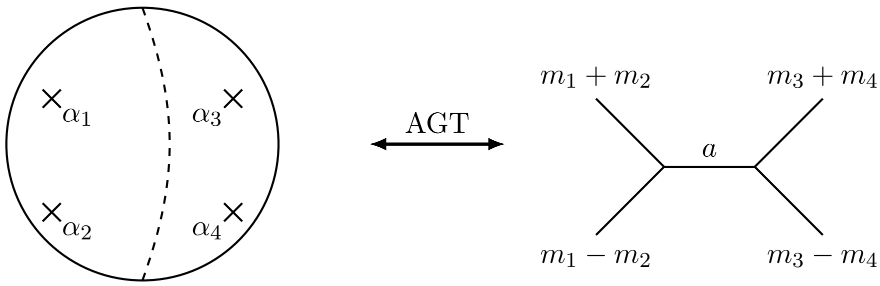


Figure 1.2: Four-punctured sphere: each puncture corresponds to the insertion of a primary in the $2d$ correlator. The dashed line displays the sewing of a pants decomposition. It represents a low energy limit of the $4d$ theory on the right, where the horizontal line represents the $SU(2)$ gauge group with Coulomb parameter a and the diagonal lines are hypermultiplets in the (anti-)fundamental representation with masses $m_1 \pm m_2$, $m_3 \pm m_4$.

Let us see the duality in the explicit example of figure 1.2. The dictionary between the two side of the AGT-duality is

$$\alpha = \frac{Q}{2} + a, \quad \alpha_i = \frac{Q}{2} + m_i, \quad Q = \epsilon_1 + \epsilon_2. \tag{1.3.17}$$

The duality can be summarize by

$$\langle V_{\alpha_1}(0) V_{\alpha_2}(1) V_{\alpha_3}(q) V_{\alpha_4}(\infty) \rangle_L \simeq \mathcal{Z}^{S^4}_{full}(q; \vec{a}, m_1, m_2, m_3, m_4, \epsilon_1, \epsilon_2) \tag{1.3.18}$$

where $\mathcal{Z}^{S^4}_{full}$ is the partition function for the theory with four hypermultiplets in the (anti-)fundamental representation on S^4, and it is expressed in terms of Nekrasov partition function (1.3.12) [126].

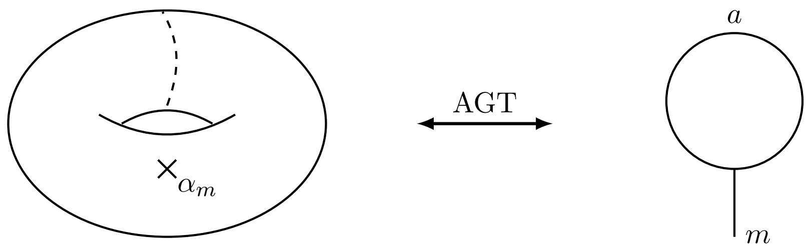


Figure 1.3: One-punctured torus: the puncture corresponds to the insertion of a primary in the $2d$ one-point correlator. The dashed line displays the sewing of the only pant. It represents the theory on the right, a $SU(2)$ gauge group (circular line) with a hypermultiplet in the adjoint representation (vertical line) of mass m.

This identification has two parts, the n-point correlator is constructed as the product of three-point functions and conformal blocks

$$\langle V_{\alpha_2}(\infty)V_{m_1}(1)V_{m_3}(q)V_{\alpha_4}(0)\rangle_L = \int d\alpha\, C_3(\alpha_2^*, \alpha_1, \alpha)C_3(\alpha^*, \alpha_3, \alpha_4)|\mathcal{B}_{\alpha_2}{}^{\alpha_1}{}_{\alpha}{}^{\alpha_3}{}_{\alpha_4}(q)|^2. \tag{1.3.19}$$

The instanton part of the Nekrasov partition function[2] (1.3.11) coincide with the conformal block $\mathcal{B}$ of the Virasoro algebra of the Liouville field theory. The perturbative part of the Nekrasov partition function coincide with the product of DOZZ three-point functions C_3 [58, 154, 138] in the correlator.

This result was soon generalized in [149] for a $SU(N)$ theory whose AGT dual was conjectured to be Toda conformal field theory.

Zamolodchikov's recursion relation

This duality provides an interesting insight into the structure of the Nekrasov partition function. This is one of the most important tools we use in this thesis.

It consists of the existence of a recursion relation for the conformal blocks of Virasoro algebra, this was discovered by Zamolodchikov in [155]. This formula enlightens the analytic structure of the Virasoro conformal block.

Indeed the poles of the Virasoro conformal block are known to be in one-to-one correspondence with degenerate fields in the conformal field theory and these are classified by the following theorem

Theorem 1 *(Kac, Feigin-Fuchs [87, Corollary 5.2]) Non-zero null vectors exists in the Verma module of degenerates fields* $\phi_{\alpha_{mn}}$

$$\Delta_{mn} = \alpha_{nm}(Q - \alpha_{nm}), \quad \alpha_{nm} = \frac{1-n}{2}b + \frac{1-m}{2}b^{-1}, \quad c = 1 + 6Q^2, \quad Q = b + b^{-1}, \tag{1.3.20}$$

these are at level $N = nm$ *and all non-zero null vectors are obtained in this way.*

[2] Actually equation (1.3.11) is defined for gauge group $U(2)$, so in the identification of [6] it is necessary to identify and factorize out the $U(1)$ part of the partition function.

Using AGT relation it is possible to translate this recursion relation for the instanton partition function, this was concretely done by Poghossian in [127]. The recursion relation is fundamental to perform the calculation of the $\mathcal{N} = 2$ partition function on toric compact manifolds, since the latter can be written as a contour integral of copies of the Nekrasov partition function and the recursion relation make the computation of the residues an easy task.

1.3.3 Donaldson invariants

In [56] Donaldson defined new kind of invariants for smooth four-manifolds using gauge theory. These invariants are polynomials in the cohomology classes of the manifolds and are defined by intersection theory on the moduli space of $SO(3)$ (or $SU(2)$) anti-self-dual connections, i.e. instantons, as we are going to briefly recall [57].

Let X be a smooth, simply connected, compact oriented Riemmanian four-manifold with metric g. Consider also $E \to X$ an $SO(3)$-bundle (or $SU(2)$), and the associate connection A. An anti-self-dual connection (or instanton) is defined by the following equation for the curvature

$$F^+ = \frac{1}{2}(F + \star F) = 0 \tag{1.3.21}$$

where $\star$ denotes the Hodge dual with respect to the metric g.

One important feature of instantons is that their action energy is proportional to the topology of the bundle, for a $SO(3)$ bundle

$$-\int_X F \wedge \star F = \int_X F \wedge F = -8\pi^2 p_1(E) \tag{1.3.22}$$

where p_1 is the first Pontryagin number (this is substitute by minus the second Chern number c_2 for $SU(2)$-bundle).

The moduli space of instantons $\mathcal{M}_E$ is the space of all anti-self-dual connections modulo gauge transformations, which are automorphisms of the bundle E. For a *generic* metric in the space of the Riemannian metrics, this moduli space has dimension

$$d = 2k(E) - 3(1 + b_2^+(X)), \qquad k(E) = \begin{cases} -p_1(E) & \text{for } SO(3) \\ 4c_2(E) & \text{for } SU(2) \end{cases} \tag{1.3.23}$$

Note that the parity of d does not actually depends on the bundle, but only on b_2^+, which is the dimension of the positive definite part of the intersection form in $H^2(X)$. *Non-generic* metrics are those metrics which admit reducible connections as solution to equation (1.3.21).

Moreover the cohomology of the moduli space is non-trivial only in even dimensions, and it is generated by the image of a map μ from the homology of the manifold

$$\mu : \begin{cases} H_2(X) \to H^2(\mathcal{M}_E) \\ H_0(X) \to H^4(\mathcal{M}_E) \end{cases} \tag{1.3.24}$$

Therefore if the dimension of the moduli space (1.3.23) is even, that is if b_2^+ is odd, one can define polynomials D on the homology of X, by intersection on the moduli space[3] of instantons associate to the metric g. For $\alpha \in H_2(X)$ and $p \in H_0(X)$

$$D_E^g(\alpha^n p^m) = \int_{\mathcal{M}_E} \mu(a)^n \mu(p)^m. \tag{1.3.25}$$

Donaldson proved that, even though the equation (1.3.21) depends on the metric g (and so does $\mathcal{M}_E$), the polynomials defined by (1.3.25) are actually independent of the metric. They are called *Donaldson invariants.* More precisely: if two metrics, in the subspace of generic Riemannian metrics, can be connected by a path then D_E^g is independent of g. Non-generic metrics form a subset of real co-dimension b_2^+ in the space of Riemannian metrics. This means that for $b_2^+ > 1$ every two generic metrics can be connected by a path, and so the polynomials D_E^g are invariants. On the other hand, if $b_2^+ = 1$, the subset of non-generic metric form a collection of *walls* and the polynomials D_E^g are only piece-wise invariants, displaying a wall-crossing phenomenon. This behavior is described by a chamber structure in $H^2(X, \mathbb{R})$ where the walls are located at $H^2(X, \mathbb{Z}) \cap H^{2,-}(X, \mathbb{R})$. A wall-crossing formula evaluates the discontinuity of Donaldson invariants when crossing a wall. In this case, one can compute Donaldson invariants in a specific chamber starting from an chamber in which the invarians are zero and sum all the wall-crossing contributions related to the walls crossed in the path between the two chambers.

From their original definition (1.3.25), Donaldson invariants are difficult to compute. But already in [144] Witten showed that $\mathcal{N} = 2$ supersymmetric theories give a powerful tool for their evaluation. This was further developed in [146] and in [114] where was shown that Donaldson invariants can be evaluated by the integration of Seiberg-Witten prepotential [132] in presence of local and surface observables, over the complex plane (u-plane) parametrizing the vacua of the infrared theory. The Donaldson invariants are constructed by two contributions, the smooth u-plane integral, which contributes only in the case $b_2^+ = 1$, and two special points in the u-plane where the measure is a δ-function. The latter contribution gives Seiberg-Witten invariants, which count solutions to the monopole equation of the infrared description of the theory [146].

On the mathematical side wall-crossing formulas and generating functions of Donaldson invariants were derived by Göttsche in [74], who gives a formula for the generating function of Donaldson invariants in terms of modular forms. This structure naturally arise in the perspective of [114], and was further studied in [77].

More recently Nekrasov gave a rigorous derivation of the SW prepotential [119]. This was used by Göttsche, Nakajima and Yoshioka in [75] to compute Donaldson invariants from instanton counting, using equivariant localization techniques. This, form the phisycs point of view, means studying the ultraviolet theory. They considered smooth toric manifolds (which have $b_2^+ = 1$ as all rational surfaces) and a refinement of Donaldson invariants *equivariant* with respect to the toric action, and they gave an explicit

[3] The moduli space $\mathcal{M}_E$ is actually non compact and one needs to use Uhlenbeck's compactification [141] of this space in (1.3.25).

expression for wall-crossing formula in terms of Nekrasov partition function. They considered also the special case on $\mathbb{P}^2$, where the chamber structure is not present, even though $b_2^+ = 1$. In this case they explicitly computed the generating function of equivariant Donaldson polynomials. Their work contributes to a deeper understanding of the structure of these invariants.

1.3.4 Superconformal index

We quickly review here the notion of superconformal index, following [31, 128].

The superconformal index is an invariant defined in superconformal field theory. It is a function of the spectrum of the theory forced by the superconformal algebra to be constant under continuous transformation of the spectrum itself.

We start introducing the Witten index $\mathcal{I}$ of a supersymmetric theory [143]

$$\mathcal{I} = \mathrm{Tr}_{\mathcal{H}}(-1)^F\, e^{-\beta H}, \qquad H = \{\mathcal{Q}, \mathcal{Q}^\dagger\}. \tag{1.3.26}$$

It is defined as the trace of the exponentiated Hamiltonian H, defined as the anticommutator of the chosen Poincaré supercharge $\mathcal{Q}$ and its conformal conjugate $\mathcal{Q}^\dagger$, weighted by $(-1)^F$, where F is the operator counting the fermion number of a state, which has eigenvalue $+1$ with bosons and -1 with fermions. The trace is taken over the Hilbert space $\mathcal{H}$ of the quantum theory on S^{d-1} relatated to the radial quantization of the superconformal theory in d dimensions

The unitarity of theory requires the Hamiltonian to be positive definite, that is $H\psi \geq 0$ for all $\psi \in \mathcal{H}$. Therefore a zero energy state $H\psi = 0$ needs to be annihilated by both $\mathcal{Q}$ and $\mathcal{Q}^\dagger$.

The index (1.3.26) counts only the zero energy states. Indeed can be shown that states which are not zero energy can be paired in couples as $(\psi, \mathcal{Q}\psi)$ with $\mathcal{Q}^\dagger\psi = 0$. Since the supercharges flip the statistic of states (that is they anticommute with the operator $(-1)^F$), the two states in the couple have opposite statistics and their total contribution in (1.3.26) is zero. This means that the index $\mathcal{I}$ actually does not depend on the temperature β.

Mathematically $\mathcal{Q}$ defines a nilpotent operator, and the index is counting $\mathcal{Q}$-cohomology classes with sign. In this perspective, the states which contribute, since are annihilated by both $\mathcal{Q}$ and $\mathcal{Q}^\dagger$, correspond to the harmonic representatives of the cohomology classes.

The Hilbert space of any unitary superconformal field theory can be decomposed into unitary irreducible, lowest energy representations of the superconfromal algebra. States annihilated by some of the supercharges are called BPS states and transform in *short* representations, which contain fewer states than generic representations (called *long*).

The superconformal index is a refinement of the Witten index (1.3.26) obtained adding a set of fugacities $\{q_i\}$ associated to a maximal set of commuting conserved charges $\{C_i\}$ which also commutes with the supercharges $\mathcal{Q}, \mathcal{Q}^\dagger$

$$\mathcal{I}(q_i) = \mathrm{Tr}_{\mathcal{H}}(-1)^F\, e^{-\beta H} \prod_i q_i^{C_i}. \tag{1.3.27}$$

Since the states in the couple $(\psi, \mathcal{Q}\psi)$ have the same quantum numbers, the precedent argument still holds and the states contributing to the index are only those annihilated by the supercharges.

If a theory flows, parametrized by some continuous coupling constant, short and long representations can rearrange the ones into the others. Anyway the index $\mathcal{I}$ is invariant. This means that two short representations joining together to form a long representation, must have opposite contribution to the index because the latter does not contribute.

In the case of a superconformal theory the index is completely invariant under all possible marginal deformations of the theory that preserve the superconformal algebra. This is because the spectrum of states is discrete and the eigenspaces associated to the quantum numbers $\{C_i\}$ are finite dimensional.

The index (1.3.27) can be also defined in a Lagrangian description. It is given by the supersymmetric partition function on $S^{d-1} \times S^1$. The various fugacities $\{q_i\}$ are encoded in twisted boundary conditions on the S^1. Therefore it can be computed via the localization techniques described in the previous subsections, as done in [106, 93]. Moreover this description makes sense also for non-conformal supersymmetric theories. Therefore one can compute the superconformal index starting from a non-conformal supersymmetric theory which flow under the renormalization group to the superconformal field theory. This can be used, for example, to compute the index of a superconformal field theory without a Lagrangian description, by computing the partition function of a Lagrangian supersymmetric theory.

Letter counting

In the simplest case of a free theory the superconformal index (1.3.27) can be evaluated simply by a combinatorial procedure. States are in one-to-one correspondence with operators inserted at the origin. These operators (*words*) are constructed as normal ordering of elementary fields and their derivatives (*letters*).

Since only zero energy states contribute to the index, and in the free theory the energy of a composite operator is given by the sum of the energy of its composing letters, one only need to consider zero energy letters, whose contribution to the index is called single-particle index

$$\mathcal{I}_{s.p.}(q_i) = f(q_1, q_2, \dots). \tag{1.3.28}$$

Then the multi-particle index is computed considering all the possible words obtainable with these letters. The result is given considering the plethystic exponential of the single-particle index

$$\mathcal{I}(q_i) = PE[\mathcal{I}_{s.p.}(q_i)] := \exp\left(\sum_{n=1}^{\infty} \frac{f(q_1^n, q_2^n, \dots)}{n}\right). \tag{1.3.29}$$

1.4 Organization of the material

The thesis is organized as follows.

In Chapter 2 we start the study of supersymmetric gauge theories on compact manifolds, focusing on $S^2 \times S^2$, and comparing the partition function of the $\mathcal{N} = 2$ theory with the three-point functions and the conformal blocks of a holomorphic version of Liouville Gravity. In sec. 2.1 we discuss supersymmetry on curved four manifolds and derive the generalized Killing spinor equations from the superalgebra. In sec. 2.2 we obtain some relevant solutions of these equations on $S^2 \times S^2$. In sec. 2.3 we use the results of the previous sections to compute the partition function of the supersymmetric gauge theory on $S^2 \times S^2$. In sec. 2.4 we compare the gauge theory computations with Liouville Gravity. App. 2.A contains the detailed derivation of the full $\mathcal{N} = 2$ supersymmetry generalized Killing equations discussed in sec.2.1. App. 2.B describes the solutions to Killing spinor equations in the general case of a four-manifold admitting a $U(1)$ isometry. App. 2.C describes a set of alternative Killing spinor solution on $S^2 \times S^2$ that we do not use elsewhere. App. 2.D contains our conventions on metric and spinors. App. 2.E contains our conventions on special functions.

In Chapter 3 we compute the partition functions of $\mathcal{N} = 2$ and $\mathcal{N} = 4$ theories on $\mathbb{P}^2$, which we identify with the generating function of equivariant Donaldson invariants and with the generating function of Euler characteristics of moduli space of instantons respectively. In sec. 3.1 we discuss the general features of $\mathcal{N} = 2$ gauge theories on complex four-manifolds and discuss equivariant observables. We then specialize to compact toric surfaces discussing the supersymmetric fixed points and the contour integral formula obtained by properly treating the fermionic zero-modes. The master formula for the generating function of local and surface observables is presented in equation (3.2.9), specializing to $U(2)$ gauge theories on $\mathbb{P}^2$. In sec. 3.2 we focus on $U(2)$ Super Yang-Mills on $\mathbb{P}^2$. We study in detail the analytic structure of the integrand by making use of Zamolodchikov's recursion relation for Virasoro conformal blocks. We then evaluate explicitly the contour integral. The main results are equations (3.2.43) and (3.2.70) for odd and even first Chern class respectively. We then proceed to the non-equivariant limit $\epsilon_1, \epsilon_2 \to 0$ and compare with the results in the mathematical literature. In subsection 3.2.8 we discuss the calculation of the pure partition function on $\mathbb{P}^2$ which implies remarkable cubic identities for the Nekrasov partition function. In sec. 3.3 we study the $\mathcal{N} = 2^\star$ theory and discuss the zero mass limit which we find to calculate the generating function of Euler characteristics of moduli spaces of rank-two sheaves. The main result is equation (3.3.20) which includes also the contribution of strictly semi-stable sheaves. We finally discuss the (mock-)modular properties of the $\mathcal{N} = 4$ partition function. App. 3.A describes the relation between the supersymmetric fixed point data and Klyachko's classification of semi-stable equivariant sheaves.

In Chapter 4 we compute the partition functions of $5d$ $\mathcal{N} = 1$ quiver gauge theories on $S^4 \times S^1$ and S^5, verifying their modular properties and comparing the result with the superconformal index of $6d$ $(1, 0)$ theory. In sec. 4.1 we describe the M-theory brane setup engineering the $5d$ quiver gauge theories, we discuss the presence of instanton operators therein and the enhancing of the global symmetries. In sec. 4.2 we evaluate the superconformal indices for $\mathcal{N} = (1, 0)$ $6d$ theories using the letter counting technique. In sec. 4.3 we compute the exact partition function for abelian linear quiver gauge theories on $S^4 \times S^1$ and we check the symmetry enhancing predicted by the S-duality property of the M-theory brane construction. In sec. 4.4 we compute the partition function for abelian circular quiver theories on S^5 and we compare it with the $(1, 0)$ superconformal index of section 4.2. In sec. 4.5 we discuss the conformal field theories AGT dual to the quiver gauge theories. In app. 4.A we recall the definition of the $5d$ Nekrasov partition function.

In Chapter 5 we report the conclusion of the thesis addressing open questions and further developments. Sec. 5.1 contains discussions about Chapters 2 and 3. Further directions related to Chapter 4 are in sec. 5.2.

Chapter 2

Supersymmetric theory on $S^2 \times S^2$ and Liouville gravity.

In this Chapter we study $\mathcal{N} = 2$ gauge theories on arbitrary Riemannian four manifolds, and show that the parameters generating the supersymmetry satisfy generalized Killing spinor equations arising from the requirement of closure of the superalgebra. For manifolds admitting an isometry, we show that these equations are solved by an equivariant version of the topological twist and we explicitly compute the gauge theory path-integral, which turns out to be given by an appropriate gluing of Nekrasov partition functions.

An interesting byproduct of the analysis is the natural appearance, in the $U(2)$ case, of three-point numbers and conformal blocks of Liouville gravity as building blocks of the $S^2 \times S^2$ partition function, related respectively to the one-loop and the instanton sectors. As we will discuss in section 2.4, a first hint to the relation with Liouville gravity can be obtained by considering the compactification of two M5-branes on $S^2 \times S^2 \times \Sigma$. The central charge of the resulting two-dimensional conformal field theory on Σ can be computed from the M5-branes anomaly polynomial [36, 4] and is indeed consistent with our findings.

This method applies in general to four-manifolds admitting a $U(1)$-action generated by a vector field V. The path integral localizes on flat connections when V has no zeros, for example Hopf surfaces or $S^1 \times M_3$, otherwise it localizes on (anti-)instantons on the zeros of V, as S^4 or compact toric manifolds. The cases $S^2 \times S^2$ and $\mathbb{P}^2$ discussed in detail in this Chapter and in the next one respectively belong to the latter class.

2.1 Supersymmetry on curved space

The algebras for $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supersymmetry on four dimensional curved spaces have been recently derived using supergravity considerations [65, 64, 82, 99]. In this section, we intend to re-derive the same results in a direct way building on the consistency of the supersymmetry algebra. For completeness and illustration of the method, we start by considering chiral $\mathcal{N} = 1$ supersymmetry and then we move to the full $\mathcal{N} = 2$ supersymmetry algebra.

2.1.1 $\mathcal{N} = 1$ Supersymmetry

We consider the case of supersymmetry algebra with one supercharge, parametrized by a (commuting) chiral spinor ξ_α of R-charge +1, and derive the algebra as realized on a vector multiplet, consisting of a gauge field $A_{\alpha\dot{\alpha}}$, gauginos λ_α and $\tilde{\lambda}_{\dot{\alpha}}$, and an auxiliary field D.

Supersymmetric variation of the gauge field and the gauginos is fixed by Lorentz covariance and R-charge conservation to be:

$$\begin{aligned} \delta A_{\alpha\dot{\alpha}} &= \xi_\alpha \tilde{\lambda}_{\dot{\alpha}}, \\ \delta \tilde{\lambda}_{\dot{\alpha}} &= 0, \\ \delta \lambda_\alpha &= i\xi_\alpha D + (F^+)_{\alpha\beta}\xi_\beta. \end{aligned} \tag{2.1.1}$$

Considering now the square of the supersymmetric variation of λ_α, we get

$$\begin{aligned} \delta^2 \lambda_\alpha &= i\xi_\alpha \delta D + [D_A(\xi\tilde{\lambda})]\xi_\beta \\ &= i\xi_\alpha \delta D + \nabla_{(\alpha\dot{\gamma}}\xi_{\beta)}\tilde{\lambda}_{\dot{\gamma}}\xi_\beta + (\xi_{(\alpha}D_{\beta)\dot{\gamma}}\tilde{\lambda}_{\dot{\gamma}})\xi_\beta \\ &= i\xi_\alpha \delta D + \nabla_{(\alpha\dot{\gamma}}\xi_{\beta)}\tilde{\lambda}_{\dot{\gamma}}\xi_\beta + (\xi_\alpha D_{\beta\dot{\gamma}}\tilde{\lambda}_{\dot{\gamma}})\xi_\beta \end{aligned} \tag{2.1.2}$$

where $\xi^2 = 0$ has been used to obtain the final term, ∇ is the covariant derivative containing the spin connection and $D = \nabla + A$. We now notice that for the final expression to vanish, the middle term should align in the direction of ξ_α, so that all the terms can be compensated by δD. For this to happen, we are forced to require that

$$\nabla_{(\alpha\dot{\gamma}}\xi_{\beta)} = i\hat{V}_{\alpha\dot{\gamma}}\xi_\beta + i\hat{V}_{\beta\dot{\gamma}}\xi_\alpha \tag{2.1.3}$$

for some background connection $\hat{V}$. We note that this is equivalent to the Killing spinor equation

$$\nabla_{\alpha\dot{\alpha}}\xi_\beta = iV_{\alpha\dot{\alpha}}\xi_\beta + iW_{\beta\dot{\alpha}}\xi_\alpha, \tag{2.1.4}$$

where $\hat{V} = V + W$. Requiring this allows us to set $\delta^2\lambda_\alpha = 0$ if we set

$$\delta D = iD_{\beta\dot{\gamma}}\tilde{\lambda}_{\dot{\gamma}}\xi_\beta - \hat{V}_{\beta\dot{\gamma}}\tilde{\lambda}_{\dot{\gamma}}\xi_\beta. \tag{2.1.5}$$

It follows from a routine calculation that $\delta^2 D = 0$.

Notice that (2.1.4) is the Killing spinor equation derived in [98, 65].

The same equation can be derived by considering the chiral multiplet in the following way. The supersymmetry variations of an anti-chiral multiplet $(\tilde{\phi}, \tilde{\psi}_{\dot{\alpha}}, F)$ generated by one supercharge of R-charge +1 are

$$\begin{aligned} \delta\tilde{\phi} &= 0, \\ \delta\tilde{\psi}_{\dot{\alpha}} &= i\xi_\alpha D_{\alpha\dot{\alpha}}\tilde{\phi}, \\ \delta\tilde{F} &= i\xi_\alpha D_{\alpha\dot{\alpha}}\tilde{\psi}_{\dot{\alpha}} + \xi_\alpha[\lambda_\alpha, \tilde{\phi}] + \xi_\alpha V_{\alpha\dot{\alpha}}\tilde{\psi}_{\dot{\alpha}} \end{aligned} \tag{2.1.6}$$

Consider first the square of the variation of $\tilde{\psi}_{\dot\alpha}$:

$$\begin{aligned}\delta^2\tilde{\psi}_{\dot\alpha} &= i\xi_\alpha\left(D_{\alpha\dot\alpha}\delta\tilde{\phi} + [\delta A_{\alpha\dot\alpha}, \tilde{\phi}]\right)\\ &= i\xi_\alpha\left(D_{\alpha\dot\alpha}\delta\tilde{\phi} + [\xi_\alpha\tilde{\lambda}_{\dot\alpha}, \tilde{\phi}]\right)\\ &= 0\end{aligned} \tag{2.1.7}$$

since $\delta\tilde{\phi} = 0$ and $\xi^2 = 0$. Consider similarly $\delta^2\tilde{F}$:

$$\begin{aligned}\delta^2\tilde{F} =& i\xi_\alpha[\delta A_{\alpha\dot\alpha}, \tilde{\psi}_{\dot\alpha}] + i\xi_\alpha D_{\alpha\dot\alpha}(i\xi_\beta D_{\beta\dot\alpha}\tilde{\phi})\\ &+ \xi_\alpha[i\xi_\alpha D + (F^+)_{\alpha\beta}\xi_\beta, \tilde{\phi}] + \xi_\alpha V_{\alpha\dot\alpha}(i\xi_\beta D_{\beta\dot\alpha}\tilde{\phi})\\ =& -\xi_\alpha\nabla_{\alpha\dot\alpha}\xi_\beta D_{\beta\dot\alpha}\tilde{\phi} - \xi_\alpha\xi_\beta D_{\alpha\dot\alpha}D_{\beta\dot\alpha}\tilde{\phi}\\ &+ \xi_\alpha\xi_\beta[(F^+)_{\alpha\beta}, \tilde{\phi}] + i\xi_\alpha V_{\alpha\dot\alpha}\xi_\beta D_{\beta\dot\alpha}\tilde{\phi}\\ =& -\xi_\alpha\nabla_{\alpha\dot\alpha}\xi_\beta D_{\beta\dot\alpha}\tilde{\phi} + i\xi_\alpha V_{\alpha\dot\alpha}\xi_\beta D_{\beta\dot\alpha}\tilde{\phi}.\end{aligned} \tag{2.1.8}$$

This is vanishing by equation (2.1.4).

2.1.2 $\mathcal{N} = 2$ Supersymmetry

We first consider the simpler case of chiral $\mathcal{N} = 2$ supersymmetry. Its straightforward (but tedious) generalization to the case with generators of both chiralities is treated next.

Chiral $\mathcal{N} = 2$ Supersymmetry

In this subsection, we derive the chiral $\mathcal{N} = 2$ algebra generated by a doublet of left-chirality spinors and the consistency conditions that the four manifold has to satisfy. We realize it on a vector multiplet. The derivation is based on the following considerations:

- The supersymmetry transformations of the scalar fields ϕ,$\bar{\phi}$ and the vector field A_μ are

$$\begin{aligned}\mathbf{Q}_L A_\mu &= i\xi^A\sigma_\mu\bar{\lambda}_A,\\ \mathbf{Q}_L\phi &= -i\xi^A\lambda_A,\\ \mathbf{Q}_L\bar{\phi} &= 0.\end{aligned} \tag{2.1.9}$$

- The chiral supersymmetry transformation squares to a gauge transformation on the vector multiplet. This implies the differential equations ("Killing spinor equations") satisfied by the transformation parameter ξ_A in order to the supersymmetry to hold. The specific form of the generator of the gauge transformation will be derived in the following.

- The scaling dimension of any background field is positive. The reason for this assumption is that we would like to recover the familiar algebra in the flat-space limit. Positivity of the scaling dimensions of background fields ensures that as the characteristic length scales of the manifold go to infinity (or equivalently, as we approach the flat metric), the background fields go to zero.

We recall below the $U(1)_R$ charges and scaling dimensions of the fields

Field	ϕ	$\bar{\phi}$	λ_A	$\bar{\lambda}_A$	A_μ	D_{AB}	ξ_A
$U(1)_R$ charge	2	-2	1	-1	0	0	1

Field	ϕ	$\bar{\phi}$	λ_A	$\bar{\lambda}_A$	D_{AB}	ξ_A
Scaling dimension	1	1	3/2	3/2	2	$-1/2$

As in the previous section, we now show how the closure of the supersymmetry algebra implies generalized Killing spinor equations with background fields. The most general variation of $\bar{\lambda}_A$ consistent with the considerations above is

$$\mathbf{Q}_L\bar{\lambda}_A = a\bar{\sigma}^\mu\xi_A D_\mu\bar{\phi} + b\bar{\sigma}^\mu D_\mu\xi_A\bar{\phi} \tag{2.1.10}$$

where a and b are complex numbers to be determined. Squaring supersymmetry, we get

$$\mathbf{Q}_L^2\bar{\lambda}_A = a\bar{\sigma}^\mu\xi_A[\xi^B\sigma_\mu\bar{\lambda}_B,\bar{\phi}] = i[i\xi^B\xi_B\bar{\phi},\bar{\lambda}_A] =: i[\Phi,\bar{\lambda}_A] \tag{2.1.11}$$

where the last equality defines the generator of gauge transformations $\Phi = i\xi^B\xi_B\bar{\phi}$. Consider now the square of the supersymmetry variation of the gauge field A_μ

$$\mathbf{Q}_L^2 A_\mu = i\xi^A\sigma_\mu\mathbf{Q}_L\bar{\lambda}_A = ia\xi^B\xi_B D_\mu\bar{\phi} + ib\bar{\phi}\xi^B\sigma_\mu\bar{\sigma}^\nu D_\nu\xi_B. \tag{2.1.12}$$

Since the supersymmetry squares to gauge transformation, and since the generator of gauge transformation is $\Phi = i\xi^B\xi_B\bar{\phi}$, we require that

$$ia\xi^B\xi_B D_\mu\bar{\phi} + ib\bar{\phi}\xi^B\sigma_\mu\bar{\sigma}^\nu D_\nu\xi_B = \mathbf{Q}_L^2 A_\mu = D_\mu(ia\xi^B\xi_B\bar{\phi}) \tag{2.1.13}$$

which gives

$$2a\xi^B D_\mu\xi_B = b\xi^B\sigma_\mu\bar{\sigma}^\nu D_\nu\xi_B. \tag{2.1.14}$$

We note that the above equation is satisfied when $a = 2b$ and $D_\mu\xi_A = \frac{1}{4}\sigma_\mu\sigma^\nu D_\nu\xi_A$ (or equivalently $D_\mu\xi_A = \sigma_\mu\bar{\xi}'_A$ for some right chirality spinor $\bar{\xi}'_A$). To see that this is indeed the conformal Killing equation, we consider the supersymmetry variation of λ_A. The most general expression possible is

$$\mathbf{Q}_L\lambda_A = \frac{1}{2}\sigma^{\mu\nu}\xi_A(kF_{\mu\nu} + \bar{\phi}T_{\mu\nu} + \phi W_{\mu\nu}) + c\,\xi_A[\phi,\bar{\phi}] + D_{AB}\xi^B \tag{2.1.15}$$

where k and c are complex numbers yet to be determined; $T_{\mu\nu}$ and $W_{\mu\nu}$ are anti self-dual background fields, both having mass dimension 1 and with $U(1)_R$ charge 2 and -2

respectively. Computing $\mathbf{Q}_L^2\phi$, we immediately see that $c = ia$. After some algebra, we find

$$\mathbf{Q}_L^2\lambda_A = i[ia\xi^B\xi_B\bar{\phi}, \lambda_A] + 2ik(\bar{\lambda}^B\bar{\sigma}^\mu\xi_A)\left(D_\mu\xi_B - \frac{1}{4}\sigma_\mu\bar{\sigma}^\nu D_\nu\xi_B\right) - \frac{i}{2}\sigma^{\mu\nu}W_{\mu\nu}(\xi^B\lambda_B)\xi_A + \Big[\mathbf{Q}_L D_{AB} - ik(\xi_A\sigma^\mu D_\mu\bar{\lambda}_B + \xi_B\sigma^\mu D_\mu\bar{\lambda}_A) - a[\bar{\phi}, \xi_A\lambda_B + \xi_B\lambda_A]\Big]\xi^B. \quad (2.1.16)$$

The right hand side has been arranged in a form that allows some immediate inferences. Firstly, the Killing spinor equation, as suggested earlier, is given by

$$D_\mu\xi_A = \sigma_\mu\bar{\xi}'_A, \quad (2.1.17)$$

which also confirms that $a = 2b$. Noting that $a = 2b = ic$ can be absorbed into $\bar{\phi}$, we will set $b = 1$. Secondly, the background field $W_{\mu\nu}$ has to be zero since it cannot be absorbed into the variation of the auxiliary field D_{AB}, which is symmetric in its indices and can not contain any term proportional to ϵ_{AB}. Thirdly, we can read off the expression for $\mathbf{Q}_L D_{AB}$ by equating the last parenthesis to zero:

$$\mathbf{Q}_L D_{AB} = ik(\xi_A\sigma^\mu D_\mu\bar{\lambda}_B + \xi_B\sigma^\mu D_\mu\bar{\lambda}_A) + 2[\bar{\phi}, \xi_A\lambda_B + \xi_B\lambda_A]. \quad (2.1.18)$$

Finally, we look at the square of the chiral supersymmetry transformation of the auxiliary field:

$$\mathbf{Q}_L^2 D_{AB} = i[2i\xi^C\xi_C\bar{\phi}, D_{AB}] + 4ikD^\mu\bar{\phi}\left\{\xi_A\left(D_\mu\xi_B - \frac{1}{4}\sigma_\mu\bar{\sigma}^\nu D_\nu\xi_B\right) + (A \leftrightarrow B)\right\} + ik\bar{\phi}\{\xi_A\sigma^\mu\bar{\sigma}^\nu D_\mu D_\nu\xi_B + (A \leftrightarrow B)\} = i[\Phi, D_{AB}]. \quad (2.1.19)$$

We recognize the first term to be the gauge transformation. The middle term in the curly brackets is once again a contraction of the main equation (2.1.17). The last piece in curly brackets is new: its vanishing is the additional condition on the Killing spinor

$$\xi_{(A}\sigma^\mu\bar{\sigma}^\nu D_\mu D_\nu\xi_{B)} = 0 \quad (2.1.20)$$

which implies

$$\sigma^\mu\bar{\sigma}^\nu D_\mu D_\nu\xi_A = M\xi_A \quad (2.1.21)$$

for some scalar background field M. We call (2.1.21) the auxiliary equation. The leftover parameter k can be set to one by a rescaling of λ_A, ϕ and D_{AB}. To summarize, the chiral supersymmetry transformation generated by a left chirality spinor ξ_A is given by

$$\begin{aligned}
\mathbf{Q}_L A_\mu &= i\xi^A\sigma_\mu\bar{\lambda}_A,\\
\mathbf{Q}_L\phi &= -i\xi^A\lambda_A,\\
\mathbf{Q}_L\bar{\phi} &= 0,\\
\mathbf{Q}_L\lambda_A &= \frac{1}{2}\sigma^{\mu\nu}\xi_A(F_{\mu\nu} + \bar{\phi}T_{\mu\nu}) + 2i\xi_A[\phi, \bar{\phi}] + D_{AB}\xi^B,\\
\mathbf{Q}_L\bar{\lambda}_A &= 2\bar{\sigma}^\mu\xi_A D_\mu\bar{\phi} + \bar{\sigma}^\mu D_\mu\xi_A\bar{\phi},\\
\mathbf{Q}_L D_{AB} &= i(\xi_A\sigma^\mu D_\mu\bar{\lambda}_B + \xi_B\sigma^\mu D_\mu\bar{\lambda}_A) + 2[\bar{\phi}, \xi_A\lambda_B + \xi_B\lambda_A]
\end{aligned} \quad (2.1.22)$$

where ξ_A satisfies

$$D_\mu \xi_A - \frac{1}{4}\sigma_\mu \bar{\sigma}^\nu D_\nu \xi_A = 0 \tag{2.1.23}$$

$$\sigma^\mu \bar{\sigma}^\nu D_\mu D_\nu \xi_A = M \xi_A \tag{2.1.24}$$

and M is a scalar background field.

Full $\mathcal{N} = 2$ Supersymmetry

We now turn to the case of $\mathcal{N} = 2$ supersymmetry with generators of both chiralities.

In this case we start from the supersymmetry transformations of the scalar fields and of the vector field

$$\begin{aligned} \mathbf{Q} A_\mu &= i\xi^A \sigma_\mu \bar{\lambda}_A - i\bar{\xi}^A \bar{\sigma}_\mu \lambda_A, \\ \mathbf{Q}\phi &= -i\xi^A \lambda_A, \\ \mathbf{Q}\bar{\phi} &= +i\bar{\xi}^A \bar{\lambda}_A, \end{aligned} \tag{2.1.25}$$

where the right chirality spinor $\bar{\xi}^A$ has $U(1)_R$ charge -1 and mass dimension $-1/2$.

Moreover we exploit the fact that the superconformal transformation squares to a sum of gauge transformation, Lorentz transformation, scaling, $U(1)_R$ transformation and $SU(2)_R$ transformation, generated by functions denoted by Φ, V, w, Θ and Θ_{AB} respectively, whose expressions we will determine in the following. Notice that the generator of scaling transformations w is related in four dimensions to the generator of coordinate translations V as $w = \frac{1}{4} D_\mu V^\mu$.

Using the masses and charges tabulated earlier, we can write the form of $\mathbf{Q}^2$ for all members of the vector multiplet. For example

$$\mathbf{Q}^2 \lambda_A = \left(iV^\nu D_\nu \lambda_A + \frac{i}{4}(\sigma^{\mu\nu} D_\mu V_\nu)\lambda_A \right) + i[\Phi, \lambda_A] + \frac{3}{2} w \lambda_A + \Theta \lambda_A + \Theta_{AB} \lambda^B \tag{2.1.26}$$

and for the gauge field

$$\mathbf{Q}^2 A_\mu = iV^\nu F_{\nu\mu} + D_\mu \Phi \tag{2.1.27}$$

and so on and so forth for the other members of the multiplet.

The explicit computations are reported in appendix 2.A and the final results are as follows. The spinor parameters have to satisfy the generalized Killing equation

$$D_\mu \xi_B + T^{\rho\sigma} \sigma_{\rho\sigma} \sigma_\mu \bar{\xi}_B - \frac{1}{4} \sigma_\mu \bar{\sigma}_\nu D^\nu \xi_B = 0 \tag{2.1.28}$$

and

$$D_\mu \bar{\xi}_B + \bar{T}^{\rho\sigma} \bar{\sigma}_{\rho\sigma} \bar{\sigma}_\mu \xi_B - \frac{1}{4} \bar{\sigma}_\mu \sigma_\nu D^\nu \bar{\xi}_B = 0 \tag{2.1.29}$$

and the auxiliary equations

$$\begin{aligned} \sigma^\mu \bar{\sigma}^\nu D_\mu D_\nu \xi_A + 4 D_\lambda T_{\mu\nu} \sigma^{\mu\nu} \sigma^\lambda \bar{\xi}_A &= M_1 \xi_A, \\ \bar{\sigma}^\mu \sigma^\nu D_\mu D_\nu \bar{\xi}_A + 4 D_\lambda \bar{T}_{\mu\nu} \bar{\sigma}^{\mu\nu} \bar{\sigma}^\lambda \xi_A &= M_2 \bar{\xi}_A. \end{aligned} \tag{2.1.30}$$

We summarize the supersymmetry algebra just derived for the vector multiplet

$$\begin{aligned}
\mathbf{Q}A_\mu &= i\xi^A\sigma_\mu\bar{\lambda}_A - i\bar{\xi}^A\bar{\sigma}_\mu\lambda_A,\\
\mathbf{Q}\phi &= -i\xi^A\lambda_A,\\
\mathbf{Q}\bar{\phi} &= +i\bar{\xi}^A\bar{\lambda}_A,\\
\mathbf{Q}\lambda_A &= \frac{1}{2}\sigma^{\mu\nu}\xi_A(F_{\mu\nu} + 8\bar{\phi}T_{\mu\nu}) + 2\sigma^\mu\bar{\xi}_A D_\mu\phi + \sigma^\mu D_\mu\bar{\xi}_A\phi + 2i\xi_A[\phi,\bar{\phi}] + D_{AB}\xi^B,\\
\mathbf{Q}\bar{\lambda}_A &= \frac{1}{2}\bar{\sigma}^{\mu\nu}\bar{\xi}_A(F_{\mu\nu} + 8\phi\bar{T}_{\mu\nu}) + 2\bar{\sigma}^\mu\xi_A D_\mu\bar{\phi} + \bar{\sigma}^\mu D_\mu\xi_A\bar{\phi} - 2i\bar{\xi}_A[\phi,\bar{\phi}] + D_{AB}\bar{\xi}^B,\\
\mathbf{Q}D_{AB} &= -i\bar{\xi}_A\bar{\sigma}^m D_m\lambda_B - i\bar{\xi}_B\bar{\sigma}^m D_m\lambda_A + i\xi_A\sigma^m D_m\bar{\lambda}_B + i\xi_B\sigma^m D_m\bar{\lambda}_A\\
&\quad - 2[\phi,\bar{\xi}_A\bar{\lambda}_B + \bar{\xi}_B\bar{\lambda}_A] + 2[\bar{\phi},\xi_A\lambda_B + \xi_B\lambda_A].
\end{aligned} \tag{2.1.31}$$

The square of the supersymmetry action is

$$\begin{aligned}
\mathbf{Q}^2A_\mu &= i\iota_V F + D\Phi,\\
\mathbf{Q}^2\phi &= i\iota_V D\phi + i[\Phi,\phi] + (w+2\Theta)\phi,\\
\mathbf{Q}^2\bar{\phi} &= i\iota_V D\bar{\phi} + i[\Phi,\bar{\phi}] + (w-2\Theta)\bar{\phi},\\
\mathbf{Q}^2\lambda_A &= i\iota_V D\lambda_A + i[\Phi,\lambda_A] + (\frac{3}{2}w+\Theta)\lambda_A + \frac{i}{4}(D_\rho V_\tau)\sigma^{\rho\tau}\lambda_A + \Theta_{AB}\lambda^B,\\
\mathbf{Q}^2\bar{\lambda}_A &= i\iota_V D\bar{\lambda}_A + i[\Phi,\bar{\lambda}_A] + (\frac{3}{2}w-\Theta)\bar{\lambda}_A + \frac{i}{4}(D_\rho V_\tau)\bar{\sigma}^{\rho\tau}\bar{\lambda}_A + \Theta_{AB}\bar{\lambda}^B,\\
\mathbf{Q}^2D_{AB} &= i\iota_V D(D_{AB}) + i[\Phi,D_{AB}] + 2wD_{AB} + \Theta_{AC}D^C{}_B + \Theta_{BC}D^C{}_A,
\end{aligned} \tag{2.1.32}$$

where the parameters of the bosonic transformations are

$$\begin{aligned}
V^\mu &= 2\bar{\xi}^A\bar{\sigma}^\mu\xi_A,\\
\Phi &= 2i\bar{\xi}_A\bar{\xi}^A\phi + 2i\xi^A\xi_A\bar{\phi},\\
\Theta_{AB} &= -i\xi_{(A}\sigma^\mu D_\mu\bar{\xi}_{B)} + iD_\mu\xi_{(A}\sigma^\mu\bar{\xi}_{B)},\\
w &= -\frac{i}{2}(\xi^A\sigma^\mu D_\mu\bar{\xi}_A + D_\mu\xi^A\sigma^\mu\bar{\xi}_A),\\
\Theta &= -\frac{i}{4}(\xi^A\sigma^\mu D_\mu\bar{\xi}_A - D_\mu\xi^A\sigma^\mu\bar{\xi}_A).
\end{aligned} \tag{2.1.33}$$

In general the spinorial parameters are sections of the corresponding vector bundles, namely $\xi \in \Gamma\left(S^+ \otimes \mathcal{R} \otimes \mathcal{L}_R\right)$ and $\bar{\xi} \in \Gamma\left(S^- \otimes \mathcal{R}^\dagger \otimes \mathcal{L}_R^{-1}\right)$ where $S^\pm$ are the spinor bundles of chirality $\pm$, $\mathcal{R}$ is the $SU(2)$ R symmetry vector bundle and $\mathcal{L}_R$ is the $U(1)$ R symmetry line bundle. The four manifold is subject to the condition that the above product bundles are well defined[1] and that a solution to the generalized Killing spinor equations exists and is everywhere well defined. These conditions differently constrain the space-time four manifold depending on the choice of $\mathcal{R}$ and $\mathcal{L}_R$. The choice leading to the topologically twisted theory is to set $\mathcal{L}_R = \mathcal{O}$ to be the trivial line bundle and

[1]Also the auxiliary field $D \in \Gamma_S(\mathcal{R} \otimes \mathcal{R})$ has to be well defined.

$\mathcal{R} = S^-$. Therefore, for this choice of the R-symmetry bundles, $S^+ \otimes S^- \sim T$ and $S^- \times S^- \sim \mathcal{O} + T^{(2,+)}$ with T the tangent bundle and $T^{(2,+)}$ the bundle of selfdual forms. In this case the four manifold has to be Riemannian and with a Killing vector in order to admit this realization of the $\mathcal{N} = 2$ super-algebra.

2.2 Spinor solutions on $S^2 \times S^2$

As derived in the previous section, the conformal Killing spinors satisfy two sets of equations: the main equations

$$\begin{aligned} D_\mu \xi_A &= -T^{\kappa\lambda}\sigma_{\kappa\lambda}\sigma_\mu \bar{\xi}_A - i\sigma_\mu \bar{\xi}'_A, \\ D_\mu \bar{\xi}_A &= -\bar{T}^{\kappa\lambda}\bar{\sigma}_{\kappa\lambda}\bar{\sigma}_\mu \xi_A - i\bar{\sigma}_\mu \xi'_A, \end{aligned} \tag{2.2.1}$$

and the auxiliary equations[2]

$$\begin{aligned} \sigma^\mu\bar{\sigma}^\nu D_\mu D_\nu \xi_A + 4D_\lambda T_{\mu\nu}\sigma^{\mu\nu}\sigma^\lambda \bar{\xi}_A &= M\xi_A \\ \bar{\sigma}^\mu\sigma^\nu D_\mu D_\nu \bar{\xi}_A + 4D_\lambda \bar{T}_{\mu\nu}\bar{\sigma}^{\mu\nu}\bar{\sigma}^\lambda \xi_A &= M\bar{\xi}_A, \end{aligned} \tag{2.2.2}$$

with

$$\begin{aligned} D_\mu \xi_A &= \nabla_\mu \xi_A + iG_{\mu A}{}^B \xi_B = \partial_\mu \xi_A + \frac{1}{4}\omega_\mu^{ab}\sigma_{ab}\xi_A + iG_{\mu A}{}^B \xi_B, \\ D_\mu \bar{\xi}_A &= \nabla_\mu \bar{\xi}_A + iG_{\mu A}{}^B \bar{\xi}_B = \partial_\mu \bar{\xi}_A + \frac{1}{4}\omega_\mu^{ab}\bar{\sigma}_{ab}\bar{\xi}_A + iG_{\mu A}{}^B \bar{\xi}_B, \end{aligned} \tag{2.2.3}$$

where $T^{\kappa\lambda}$ and $\bar{T}^{\kappa\lambda}$ are anti self-dual and self-dual background fields respectively, M is a scalar background field and $A, B, \ldots$ are the $SU(2)_R$ doublet indices. The covariant derivative involves spin connection ω_μ^{ab} and a background $SU(2)_R$ gauge field $G_{\mu A}{}^B$. We are looking for a solution that satisfies the following reality condition:

$$(\xi_{\alpha A})^\dagger = \xi^{A\alpha} = \epsilon^{\alpha\beta}\epsilon^{AB}\xi_{\beta B}, \quad (\bar{\xi}_{\dot{\alpha} A})^\dagger = \bar{\xi}^{A\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}}\epsilon^{AB}\bar{\xi}_{\dot{\beta} B}. \tag{2.2.4}$$

If the spinors $\xi'_A, \bar{\xi}'_A$ on the r.h.s. of (2.2.1) are orthogonal to $\xi_A, \bar{\xi}_A$, i.e. if

$$\xi'^{A\alpha}\xi_{A\alpha} = 0, \qquad \bar{\xi}'^{A\alpha}\bar{\xi}_{A\alpha} = 0, \tag{2.2.5}$$

then they can be parametrized as follows

$$\xi'_A = -iS^{\kappa\lambda}\sigma_{\kappa\lambda}\xi_A, \qquad \bar{\xi}'_A = -i\bar{S}^{\kappa\lambda}\bar{\sigma}_{\kappa\lambda}\bar{\xi}_A, \tag{2.2.6}$$

where S, $\bar{S}$ are respectively anti self-dual and self-dual tensors. If this happens, equation (2.2.1) can be written as

$$\begin{aligned} D_\mu \xi_A &= -T^{\kappa\lambda}\sigma_{\kappa\lambda}\sigma_\mu \bar{\xi}_A - \bar{S}^{\kappa\lambda}\sigma_\mu\bar{\sigma}_{\kappa\lambda}\bar{\xi}_A, \\ D_\mu \bar{\xi}_A &= -\bar{T}^{\kappa\lambda}\bar{\sigma}_{\kappa\lambda}\bar{\sigma}_\mu \xi_A - S^{\kappa\lambda}\bar{\sigma}_\mu\sigma_{\kappa\lambda}\xi_A. \end{aligned} \tag{2.2.7}$$

[2]Here and in the following we consider the particular case $M_1 = M_2 = M$. This choice reproduces the auxiliary equations considered in [82].

2.2.1 Twisting solutions

Witten twisting solutions

The problem of finding solutions to (2.2.1) simplifies a lot if we turn on the background $SU(2)_R$ gauge field ${G_\mu}_A{}^B$ in equation (2.2.3). Turning on only the $U(1)_R \subset SU(2)_R$ component ${G_\mu}_A{}^B = G_\mu \, {\sigma_3}_A{}^B$ means twisting the Euclidean rotation group as $SO(4)' \subset SO(4) \times U(1)_R$. The twisted theory is obtained gauging $SO(4)'$ by the spin connection. In this way the spinors of the untwisted theory become sections of different bundles.

We now derive the simplest solution of (2.2.1) performing the following twist:

$$ {G_\mu}_A{}^B = G_\mu \, {\sigma_3}_A{}^B, \qquad G_\mu = -\frac{1}{2}(\omega_\mu + \omega'_\mu), \tag{2.2.8} $$

where ω_μ, ω'_μ are the components of the spin connection (see appendix 2.D.2, equation (2.D.9))

$$ \begin{aligned} \omega_\mu &= \omega_\mu^{12} = -2i\omega_{\mu 1 \bar{1}}, \\ \omega'_\mu &= \omega_\mu^{34} = -2i\omega_{\mu 2 \bar{2}}. \end{aligned} \tag{2.2.9} $$

The right hand side of equation (2.2.1) becomes

$$ \begin{aligned} D_\mu \xi_1 &= (\partial_\mu - i\omega_\mu P_- - i\omega'_\mu P_+)\xi_1, \\ D_\mu \xi_2 &= (\partial_\mu + i\omega_\mu P_+ - i\omega'_\mu P_-)\xi_2, \\ D_\mu \bar{\xi}_1 &= (\partial_\mu - i\omega_\mu P_- - i\omega'_\mu P_+)\bar{\xi}_1, \\ D_\mu \bar{\xi}_2 &= (\partial_\mu + i\omega_\mu P_+ - i\omega'_\mu P_-)\bar{\xi}_2, \end{aligned} \tag{2.2.10} $$

where P_+ and P_- are respectively the projectors in the first and in the second component of the two components Weyl spinor

$$ P_\pm := \frac{\mathbb{1} \pm \sigma_3}{2}. \tag{2.2.11} $$

It is easy to check that

$$ \xi_A = 0, \qquad \bar{\xi}_1 = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad \bar{\xi}_2 = \begin{pmatrix} 0 \\ \bar{a} \end{pmatrix}, \quad a \in \mathbb{C} \tag{2.2.12} $$

(where the bar over a means taking the complex conjugate) is a solution to the equations (2.2.1), (2.2.2) and (2.2.4) with

$$ T_{\mu\nu} = \bar{T}_{\mu\nu} = 0, \qquad \xi'_A = \bar{\xi}'_A = 0, \qquad M = 0. \tag{2.2.13} $$

The theory invariant under the supersymmetry generated by the solution (2.2.12) coincides with Witten's topologically twisted version of Super Yang-Mills [144]. The corresponding path integral localizes on the moduli space of anti-instantons on $S^2 \times S^2$. The integration over this moduli space is however a difficult task, and can be simplified further by exploiting the isometry of the base manifold $S^2 \times S^2$ by considering a new supersymmetry generator which closes on a $U(1)$ isometry. To this end, one has to find another set of solutions where $\xi_A \neq 0$, as we will show in the next subsection.

Equivariant twisting solutions

We will follow the procedure described in [47] to obtain a more general solution[3] for the twist (2.2.8). This procedure is actually available more in general for a generic Riemannian four-manifold admitting a $U(1)$ isometry. We report the general result in appendix 2.B.

We would like to find a supersymmetry generator which squares on an isometry of the base manifold, in order to localize the path integral to its fixed points. To obtain this we have to turn on the left chirality solution ξ_A. Indeed the vector generating the isometry is proportional to

$$\frac{1}{2}V_a = \bar{\xi}^A \bar{\sigma}_a \xi_A, \tag{2.2.14}$$

where $\bar{\sigma}_a = (i\sigma_1, i\sigma_2, i\sigma_3, \mathbb{1})$, or using complex coordinate in the orthonormal frame

$$\begin{aligned}
\bar{\sigma}_1 &= \frac{1}{2}i\sigma_1 + \frac{1}{2i}i\sigma_2 = i\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad &\bar{\sigma}_{\bar{1}} &= \frac{1}{2}i\sigma_1 - \frac{1}{2i}i\sigma_2 = i\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \\
\bar{\sigma}_2 &= \frac{1}{2}i\sigma_3 + \frac{1}{2i}\mathbb{1} = i\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \qquad &\bar{\sigma}_{\bar{2}} &= \frac{1}{2}i\sigma_3 - \frac{1}{2i}\mathbb{1} = i\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\end{aligned} \tag{2.2.15}$$

The vector field that we will consider for $S^2 \times S^2$ is

$$\frac{1}{2}V = \epsilon_1(iz\partial_z - i\bar{z}\partial_{\bar{z}}) + \epsilon_2(iw\partial_w - i\bar{w}\partial_{\bar{w}}) \tag{2.2.16}$$

where z and w are complex coordinates on the two S^2s and $\epsilon_1 = 1/r_1$, $\epsilon_2 = 1/r_2$ their radii [4].

We want to find a solution $(\xi_A, \bar{\xi}_A)$ that satisfies (2.2.14), where V is as given in (2.2.16). Expanding (2.2.14) and denoting the two components of a Weyl spinor $\psi = (\psi^+, \psi^-)$ we obtain the following four equations

$$\begin{aligned}
V_1 &= 2i\bar{\xi}_2^+\xi_1^+ - 2i\bar{\xi}_1^+\xi_2^+, \\
V_{\bar{1}} &= 2i\bar{\xi}_1^-\xi_2^- - 2i\bar{\xi}_2^-\xi_1^-, \\
V_2 &= 2i\bar{\xi}_1^+\xi_2^- - 2i\bar{\xi}_2^+\xi_1^-, \\
V_{\bar{2}} &= 2i\bar{\xi}_1^-\xi_2^+ - 2i\bar{\xi}_2^-\xi_1^+.
\end{aligned} \tag{2.2.17}$$

Let us fix $a = 1$ in equation (2.2.12), then we turn on the zero components of the *real* spinors ξ_A, $\bar{\xi}_A$ as

$$\xi_1 = \begin{pmatrix} b \\ c \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} -\bar{c} \\ \bar{b} \end{pmatrix}, \quad \bar{\xi}_1 = \begin{pmatrix} 1 \\ d \end{pmatrix}, \quad \bar{\xi}_2 = \begin{pmatrix} -\bar{d} \\ 1 \end{pmatrix}. \tag{2.2.18}$$

[3] See also [99].

[4] The isometry generated by V is actually a diagonal combination of the two isometries of the maximal torus $U(1)^2 \subset SO(4)$. To obtain separately the action of the two $U(1)$ one can consider complexified version of (2.2.16) with complex parameters ϵ_1, ϵ_2. One can obtain solutions generating such an isometry relaxing the condition of reality of the spinors (2.2.4).

The covariant derivatives of b, c, d have the following expressions due to the twist (2.2.8)

$$\begin{aligned} D_\mu b &= (\partial_\mu - i\omega'_\mu)b, \\ D_\mu c &= (\partial_\mu - i\omega_\mu)c, \\ D_\mu d &= (\partial_\mu - i\omega_\mu - i\omega'_\mu)d. \end{aligned} \tag{2.2.19}$$

Putting (2.2.18) in (2.2.17) we obtain the system

$$\begin{aligned} V_1 &= 2(i\bar{c} - i\bar{d}b), \\ V_{\bar{1}} &= 2(-ic + id\bar{b}), \\ V_2 &= 2(i\bar{b} + i\bar{d}c), \\ V_{\bar{2}} &= 2(-ib - id\bar{c}), \end{aligned} \tag{2.2.20}$$

where of course $\overline{(V_1)} = V_{\bar{1}}$ and $\overline{(V_2)} = V_{\bar{2}}$ due to the reality of the spinor, (which is equivalent to reality of the vector V). The simplest choice for b, c, d is

$$b = \frac{i}{2}V_{\bar{2}}, \qquad c = \frac{i}{2}V_{\bar{1}}, \qquad d = 0. \tag{2.2.21}$$

Now we have to show that this is actually a solution to equation (2.2.1) for some values of the background fields. We can rewrite (2.2.19) using (2.2.9) in terms of V_1 and V_2 as

$$\begin{aligned} (\partial_\mu + i\omega_\mu)V_1 &= (\partial_\mu + \omega_{\mu 1}{}^{1})V_1 = \nabla_\mu V_1, \\ (\partial_\mu - i\omega_\mu)V_{\bar{1}} &= (\partial_\mu + \omega_{\mu \bar{1}}{}^{\bar{1}})V_{\bar{1}} = \nabla_\mu V_{\bar{1}}, \\ (\partial_\mu + i\omega'_\mu)V_1 &= (\partial_\mu + \omega_{\mu 2}{}^{2})V_2 = \nabla_\mu V_2, \\ (\partial_\mu - i\omega'_\mu)V_{\bar{2}} &= (\partial_\mu + \omega_{\mu \bar{2}}{}^{\bar{2}})V_{\bar{2}} = \nabla_\mu V_{\bar{2}}. \end{aligned} \tag{2.2.22}$$

Using the properties of Killing vectors and the factorization of the metric (see equations (2.D.15), (2.D.4) in appendix 2.D.2), it's easy to show that the only non-zero components of $\nabla_\mu V_\nu$ are

$$\nabla_z V_{\bar{1}} = e_{\bar{1}}^{\bar{z}} \nabla_z V_{\bar{z}} = e_{\bar{1}}^{\bar{z}} \nabla_{[z} V_{\bar{z}]} = e_{\bar{1}}^{\bar{z}} \partial_{[z} V_{\bar{z}]} = \frac{1}{2}\left(\frac{\partial_z V_{\bar{z}} - \partial_{\bar{z}} V_z}{\sqrt{g_1}}\right) g_1^{1/4} =: \mathcal{H}_1\, g_1^{1/4} \tag{2.2.23}$$

and similarly[5]

$$\begin{aligned} \nabla_{\bar{z}} V_1 &= -\mathcal{H}_1\, g_{(1)}^{1/4}, \\ \nabla_w V_{\bar{2}} &= +\mathcal{H}_2\, g_{(2)}^{1/4}, \\ \nabla_{\bar{w}} V_2 &= -\mathcal{H}_2\, g_{(2)}^{1/4}, \end{aligned} \tag{2.2.24}$$

[5] g_1, g_2 are respectively the determinants of the metric in the first and in the second sphere, $\sqrt{g_1} := 2g_{z\bar{z}}$ and $\sqrt{g_2} := 2g_{w\bar{w}}$.

where $\mathcal{H}_1$ and $\mathcal{H}_1$ are proportional to the height functions on the first and the second sphere respectively. Indeed considering the Killing vector $V = \frac{1}{r_1}(iz\partial_z - i\bar{z}\partial_{\bar{z}}) + \frac{1}{r_2}(iw\partial_w - i\bar{w}\partial_{\bar{w}})$ we have

$$\begin{aligned} \mathcal{H}_1 &:= \frac{\partial_z V_{\bar{z}} - \partial_{\bar{z}} V_z}{2\sqrt{g_{(1)}}} = \frac{i}{r_1}\frac{1-|z|^2}{1+|z|^2} = \frac{i}{r_1}\cos\theta_1, \\ \mathcal{H}_2 &:= \frac{\partial_w V_{\bar{w}} - \partial_{\bar{w}} V_w}{2\sqrt{g_{(2)}}} = \frac{i}{r_2}\frac{1-|w|^2}{1+|w|^2} = \frac{i}{r_2}\cos\theta_2. \end{aligned} \tag{2.2.25}$$

Using these facts and recalling the form of the candidate solution

$$\xi_1 = \frac{i}{2}\begin{pmatrix} V_{\bar{2}} \\ V_{\bar{1}} \end{pmatrix}, \quad \xi_2 = \frac{i}{2}\begin{pmatrix} V_1 \\ -V_2 \end{pmatrix}, \quad \bar{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.2.26}$$

we get the following equations for the left chirality spinor ξ_A:

$$\begin{aligned} D_z\xi_1 &= \tfrac{i}{2}\mathcal{H}_1\, g_{(1)}^{1/4}\begin{pmatrix} 0 \\ 1 \end{pmatrix}, & D_z\xi_2 &= 0, \\ D_w\xi_1 &= \tfrac{i}{2}\mathcal{H}_2\, g_{(2)}^{1/4}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, & D_w\xi_2 &= 0, \\ D_{\bar{z}}\xi_1 &= 0, & D_{\bar{z}}\xi_2 &= -\tfrac{i}{2}\mathcal{H}_1\, g_{(1)}^{1/4}\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ D_{\bar{w}}\xi_1 &= 0, & D_{\bar{w}}\xi_2 &= \tfrac{i}{2}\mathcal{H}_2\, g_{(2)}^{1/4}\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{2.2.27}$$

This can be rewritten in a clever way as

$$\begin{aligned} D_z\xi_A &= -\frac{1}{2}\mathcal{H}_1\,\sigma_z\,\bar{\xi}_A, \\ D_w\xi_A &= -\frac{1}{2}\mathcal{H}_2\,\sigma_w\,\bar{\xi}_A, \\ D_{\bar{z}}\xi_A &= +\frac{1}{2}\mathcal{H}_1\,\sigma_{\bar{z}}\,\bar{\xi}_A, \\ D_{\bar{w}}\xi_A &= +\frac{1}{2}\mathcal{H}_2\,\sigma_{\bar{w}}\,\bar{\xi}_A, \end{aligned} \tag{2.2.28}$$

where $\sigma_z = e_z^1\,\sigma_1 = g_{(1)}^{1/4}\,\sigma_1$ etc., and $\sigma_1, \sigma_{\bar{1}}, \sigma_2, \sigma_{\bar{2}}$ are defined analogously to (2.2.15).

The last thing to do now is to express the background $T, \bar{S}$ of (2.2.7) in terms of $\mathcal{H}_1, \mathcal{H}_2$ since we already know

$$D_\mu\bar{\xi}_A = 0 \quad \Rightarrow \quad \bar{T}, S = 0. \tag{2.2.29}$$

Staring at (2.2.28) one can notice that we need to associate $\mathcal{H}_1$ to the coordinates $z, \bar{z}$ of the first sphere and $\mathcal{H}_2$ to the coordinates $w, \bar{w}$ of the second sphere. To reproduce this in (2.2.7) we need the combinations $T^{\kappa\lambda}\sigma_{\kappa\lambda}$, $\bar{S}^{\kappa\lambda}\bar{\sigma}_{\kappa\lambda}$ to be proportional to σ_3, since this matrix has the property

$$\{\sigma_3, \sigma_z\} = \{\sigma_3, \sigma_{\bar{z}}\} = 0, \qquad [\sigma_3, \sigma_w] = [\sigma_3, \sigma_{\bar{w}}] = 0. \tag{2.2.30}$$

Therefore the only possibility is

$$T = t(\omega(1) - \omega(2)), \qquad \bar{S} = s(\omega(1) + \omega(2)), \tag{2.2.31}$$

where t and s are two real scalar functions and $\omega(1), \omega(2)$ are respectively the volume forms on the first and on the second sphere. Indeed from (2.2.31) we have

$$T^{\kappa\lambda}\sigma_{\kappa\lambda} = 4it\sigma_3, \qquad \bar{S}^{\kappa\lambda}\bar{\sigma}_{\kappa\lambda} = 4is\sigma_3. \tag{2.2.32}$$

Inserting these two in equations (2.2.7) and using the further property

$$\sigma_z\sigma_3 = \sigma_z, \quad \sigma_{\bar{z}}\sigma_3 = -\sigma_{\bar{z}}, \quad \sigma_w\sigma_3 = \sigma_w, \quad \sigma_{\bar{w}}\sigma_3 = -\sigma_{\bar{w}}, \tag{2.2.33}$$

we obtain

$$\begin{aligned} D_z\xi_A &= 4i(t-s)\sigma_z\bar{\xi}_A, \\ D_{\bar{z}}\xi_A &= 4i(-t+s)\sigma_{\bar{z}}\bar{\xi}_A, \\ D_w\xi_A &= 4i(-t-s)\sigma_w\bar{\xi}_A, \\ D_{\bar{w}}\xi_A &= 4i(t+s)\sigma_{\bar{w}}\bar{\xi}_A. \end{aligned} \tag{2.2.34}$$

Finally comparing with (2.2.28) we get

$$t - s = \frac{i}{8}\mathcal{H}_1, \quad t + s = -\frac{i}{8}\mathcal{H}_2 \quad \Rightarrow \quad t = \frac{i}{16}(\mathcal{H}_1 - \mathcal{H}_2), \quad s = -\frac{i}{16}(\mathcal{H}_1 + \mathcal{H}_2). \tag{2.2.35}$$

It remains to evaluate the background field M in (2.2.2). From the second of (2.2.2) it trivially follows

$$M = 0 \tag{2.2.36}$$

due to $D_\mu\bar{\xi}_A = 0$ and $\bar{T} = 0$. It is of course possible to check this result also using the first of (2.2.2) and inserting the values of $D_\mu\xi_A$ and T.

Summary of the results

We summarize here the results of the previous subsection

$$\xi_1 = \frac{i}{2}\begin{pmatrix} V_{\bar{2}} \\ V_{\bar{1}} \end{pmatrix}, \quad \xi_2 = \frac{i}{2}\begin{pmatrix} V_1 \\ -V_2 \end{pmatrix}, \quad \bar{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.2.37}$$

satisfying

$$\begin{aligned} D_\mu\xi_A &= -T^{\kappa\lambda}\sigma_{\kappa\lambda}\sigma_\mu\bar{\xi}_A - \bar{S}^{\kappa\lambda}\sigma_\mu\bar{\sigma}_{\kappa\lambda}\bar{\xi}_A, \\ D_\mu\bar{\xi}_A &= 0, \end{aligned} \tag{2.2.38}$$

with

$$T = t(\omega(1) - \omega(2)), \quad \bar{S} = s(\omega(1) + \omega(2)), \tag{2.2.39}$$

and

$$\begin{aligned} t &= \frac{i}{16}(\mathcal{H}_1 - \mathcal{H}_2), & s &= -\frac{i}{16}(\mathcal{H}_1 + \mathcal{H}_2), \\ \mathcal{H}_1 &= \frac{\partial_z V_{\bar{z}} - \partial_{\bar{z}} V_z}{2\sqrt{g_{(1)}}}, & \mathcal{H}_2 &= \frac{\partial_w V_{\bar{w}} - \partial_{\bar{w}} V_w}{2\sqrt{g_{(2)}}}. \end{aligned} \tag{2.2.40}$$

In polar coordinates of the two spheres these are

$$\xi_1 = -\frac{1}{2}\begin{pmatrix} \sin\theta_2 \\ \sin\theta_1 \end{pmatrix}, \quad \xi_2 = \frac{1}{2}\begin{pmatrix} \sin\theta_1 \\ -\sin\theta_2 \end{pmatrix}, \quad \bar{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (2.2.41)$$

and

$$t = \frac{1}{16}\Big(-\frac{\cos\theta_1}{r_1} + \frac{\cos\theta_2}{r_2}\Big), \qquad s = \frac{1}{16}\Big(\frac{\cos\theta_1}{r_1} + \frac{\cos\theta_2}{r_2}\Big). \qquad (2.2.42)$$

The square norms of the spinors are

$$\xi^2 := \xi^A \xi_A = \frac{1}{2}(\sin^2\theta_1 + \sin^2\theta_2), \qquad \bar{\xi}^2 := \bar{\xi}_A \bar{\xi}^A = 2. \qquad (2.2.43)$$

Instead in complex coordinates

$$\xi_1 = -\begin{pmatrix} \frac{w}{1+|w|^2} \\ \frac{z}{1+|z|^2} \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} \frac{\bar{z}}{1+|z|^2} \\ -\frac{\bar{w}}{1+|w|^2} \end{pmatrix}, \quad \bar{\xi}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar{\xi}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad (2.2.44)$$

and

$$t = \frac{1}{16}\left(-\frac{1}{r_1}\frac{1-|z|^2}{1+|z|^2} + \frac{1}{r_2}\frac{1-|w|^2}{1+|w|^2}\right), \qquad s = \frac{1}{16}\left(\frac{1}{r_1}\frac{1-|z|^2}{1+|z|^2} + \frac{1}{r_2}\frac{1-|w|^2}{1+|w|^2}\right). \qquad (2.2.45)$$

2.3 Partition function on $S^2 \times S^2$

In this section we proceed to the computation of the partition function of $\mathcal{N} = 2$ SYM theory on $S^2 \times S^2$. At the end of the section we will present the extension of this result in presence of matter fields in the (anti)fundamental representation.

The strategy we follow consists of performing a change of variables in the path integral to an equivariant extension of the Witten's topologically twisted theory [144]. As we will see this maps the supersymmetry algebra to an equivariant BRST algebra which is the natural generalization of the supersymmetry algebra of the Nekrasov Ω-background [119, 16] on $S^2 \times S^2$. Since this is a toric manifold, the partition function reduces to copies of the Nekrasov partition functions, glued together in a way that will be explained below. The result we obtain is indeed in agreement with the one conjectured by Nekrasov in [120] for toric compact manifolds.

2.3.1 Change to twisted variables

The starting supersymmetry algebra for the vector multiplet is (2.1.31) and the square of the supersymmetry action is (2.1.32), where the parameters $w = 0$ and $\Theta = 0$ due to the orthogonality of our solution (2.2.5).

Now we are going to make a change of variables in the supersymmetry algebra that will simplify the localization procedure in the path integral. We re-organize the eight components of the fermions $\lambda_{A\alpha}, \bar{\lambda}_A^{\dot{\alpha}}$ as a fermionic scalar η, a vector Ψ^μ and a self dual tensor $\chi^{(+)\mu\nu}$:

$$\begin{aligned} \eta &:= -i(\xi^A \lambda_A + \bar{\xi}^A \bar{\lambda}_A), \\ \Psi_\mu &:= i(\xi^A \sigma_\mu \bar{\lambda}_A - \bar{\xi}^A \bar{\sigma}_\mu \lambda_A), \\ \chi^+_{\mu\nu} &:= 2\bar{\xi}^A \bar{\sigma}_{\mu\nu} \bar{\xi}^B (\bar{\xi}_A \bar{\lambda}_B - \xi_A \lambda_B). \end{aligned} \tag{2.3.1}$$

We also redefine the scalars in a suitable way to simplify the supersymmetry algebra:

$$\begin{aligned} \bar{\Phi} :=& \phi - \bar{\phi} \\ \Phi :=& 2i\bar{\xi}^2 \phi + 2i\xi^2 \bar{\phi} \\ B^+_{\mu\nu} :=& 2(\bar{\xi}^2)^2 (F^+_{\mu\nu} + 8\phi \bar{T}_{\mu\nu} - 8\bar{\phi}\bar{S}_{\mu\nu}) \\ &- (\bar{\xi}^A \bar{\sigma}_{\mu\nu} \bar{\xi}^B)(\xi_A \sigma^{\kappa\lambda} \xi_B)(F_{\kappa\lambda} + 8\bar{\phi} T_{\kappa\lambda} - 8\phi S_{\kappa\lambda}) \\ &- 4\bar{\xi}^2 V_{[\mu} D^+_{\nu]} \bar{\Phi} + \frac{1}{2}(\xi^2 + \bar{\xi}^2)(\bar{\xi}^A \bar{\sigma}_{\mu\nu} \bar{\xi}^B) D_{AB}. \end{aligned} \tag{2.3.2}$$

where ξ^2 and $\bar{\xi}^2$ are the square norms of the spinors (2.2.43).

The inverse of the relation (2.3.1) is given by

$$\begin{aligned} \lambda_A &= \frac{1}{\xi^2 + \bar{\xi}^2}(i\xi_A \eta - i\sigma^\mu \bar{\xi}_A \Psi_\mu + \xi^B \Xi_{BA}) \\ \bar{\lambda}_A &= \frac{1}{\xi^2 + \bar{\xi}^2}(-i\bar{\xi}_A \eta - i\bar{\sigma}^\mu \xi_A \Psi_\mu + \bar{\xi}^B \Xi_{BA}) \end{aligned} \tag{2.3.3}$$

where

$$\Xi_{AB} = \frac{1}{2(\bar{\xi}^2)^2} \bar{\xi}_A \bar{\sigma}^{\mu\nu} \bar{\xi}_B \chi^+_{\mu\nu}. \tag{2.3.4}$$

It is immediate to verify the relation (2.3.3) by inserting it back in (2.3.1); or conversely by inserting (2.3.1) in (2.3.3) and using the following non-trivial spinor identity

$$(\psi^B \cdot \lambda_B)\xi_A - (\psi_A \cdot \lambda_B)\xi^B - (\psi_B \cdot \lambda_A)\xi^B = 0, \tag{2.3.5}$$

for two-components spinors ψ, λ, ξ.

A comment is now important: this change of variables is everywhere invertible. Indeed, its Jacobian is given by

$$\begin{aligned} \mathcal{J} &= \frac{\mathcal{J}_{bos}}{\mathcal{J}_{ferm}} = 1 \\ \mathcal{J}_{bos} &= \mathcal{J}_{ferm} \sim \left(\bar{\xi}^2 + \xi^2\right)^4 (\bar{\xi}^2)^3. \end{aligned} \tag{2.3.6}$$

Notice that this change of variables is everywhere well defined due to the nature of the solution derived in section 2.2.1. Indeed, neither of the factors $\bar{\xi}^2$ and $\xi^2 + \bar{\xi}^2$ in the Jacobian ever vanish.

The supersymmetry algebra in terms of the new variables, computed from (2.1.31) and (2.1.32), is

$$\begin{aligned} &\mathcal{Q}A = \Psi, && \mathcal{Q}\Psi = i\iota_V F + D\Phi, && \mathcal{Q}\Phi = i\iota_V \Psi, \\ &\mathcal{Q}\bar{\Phi} = \eta, && \mathcal{Q}\eta = i\,\iota_V D\bar{\Phi} + i[\Phi,\bar{\Phi}], && \\ &\mathcal{Q}\chi^+ = B^+, && \mathcal{Q}B^+ = i\mathcal{L}_V\chi^+ + i[\Phi,\chi^+]. && \end{aligned} \tag{2.3.7}$$

These are the equivariant extension of the twisted supersymmetry considered in [144] and we finally got rid of all the indices in our formulas by passing to the differential form notation.

In (2.3.7) ι_V is the contraction with the vector V and $\mathcal{L}_V = D\iota_V + \iota_V D$ is the covariant Lie derivative.

Let us notice that the supercharge (2.3.7) manifestly satisfies $\mathcal{Q}^2 = i\mathcal{L}_V + \delta_\Phi^{gauge}$. There is still a consistency condition on the last line, that is the action has to preserve the self-duality of B^+ and χ^+. This is satisfied iff $L_V\star = \star L_V$, where $\star$ is the Hodge-$\star$ and $L_V = d\iota_V + \iota_V d$ is the Lie derivative. This condition coincides with the requirement that V is an isometry of the four manifold. Therefore, we have proved that for any four-manifold with a $U(1)$ isometry, once the R-symmetry bundle is chosen to fit the equivariant twist, there is a consistent realization of the corresponding $\mathcal{N} = 2$ supersymmetry algebra[6], explicit formulae for the generators of supersymmetry and background fields in this general case are reported in appendix 2.B.

2.3.2 Localizing action and fixed points

In terms of the new variables (2.3.1), we consider the following supersymmetric Lagrangian

$$L = \frac{i\tau}{4\pi}\,\mathrm{Tr}\,F\wedge F + \omega\wedge\mathrm{Tr}\,F + \mathcal{Q}\mathcal{V} \tag{2.3.8}$$

where τ is the complexified coupling constant, $\omega \in H^2\left(S^2\times S^2,\mathbb{R}\right)$ and

$$\begin{aligned} \mathcal{V} = &-\mathrm{Tr}\big[\chi^+\wedge\star F + \star i\bar{\Phi}(-\star D\star\Psi + \mathcal{L}_V\eta) + \star\eta(i\,\mathcal{L}_V\bar{\Phi} + i[\Phi,\bar{\Phi}])^\dagger\big] \\ &-\mathrm{Tr}\big[\chi^+\big]\wedge\star\mathrm{Tr}\big[B^+\big]. \end{aligned} \tag{2.3.9}$$

Proceeding to discuss the localization of the gauge field, we will split the calculation between the $u(1)$ and the $su(N)$ sector which must be differently treated. This is due to the fact that we want to allow gauge vector bundles with non trivial and unrestricted first Chern class $c_1 = \frac{1}{2\pi}TrF$. The usual δ-gauge fixing $F^+ = 0$ in the whole $u(N)$ Lie algebra would be then incompatible with the previous request. Therefore we split the gauge fixing of the two sectors with the additional term in the last line of (2.3.9), keeping a Gaussian gauge fixing in the $u(1)$ sector and a δ-gauge fixing in the $su(N)$ sector. If the manifold is Kähler, then an equivalent procedure would be to localize on

[6] In terms of the vector field, the Jacobian factors above read $\mathcal{J}_{bos} = \mathcal{J}_{ferm} \sim (2 + \frac{1}{8}V^2)$ which is positive.

Hermitian-Yang-Mills connections, namely those satisfying the equation $\omega \wedge F = c\, \omega \wedge \omega \mathbb{1}$ and $F^{(2,0)} = 0$, where ω is the Kähler form and c is a constant.

We look at the fixed points of the supersymmetry (2.3.7). On setting the fermions to zero, the fixed points of the supercharge read

$$\begin{aligned} \iota_V D\bar{\Phi} + [\Phi, \bar{\Phi}] = 0, \\ i\iota_V F + D\Phi = 0. \end{aligned} \tag{2.3.10}$$

The integrability conditions of the second equation are

$$\begin{aligned} \iota_V D\Phi = 0, \\ \mathcal{L}_V F = [F, \Phi]. \end{aligned} \tag{2.3.11}$$

We choose the following reality condition for the scalars fields $\bar{\Phi} = -\Phi^\dagger$, then the first of (2.3.10) splits as

$$\iota_V D\bar{\Phi} = 0 \quad \text{and} \quad [\Phi, \bar{\Phi}] = 0. \tag{2.3.12}$$

which imply that Φ and $\bar{\Phi}$ lie in the same Cartan subalgebra.Moreover, since we consider $F^\dagger = F$, we can split similarly the second of (2.3.11), obtaining that also the curvature lies along the Cartan subalgebra

$$[F, \Phi] = [F, \bar{\Phi}] = 0. \tag{2.3.13}$$

Therefore the second of (2.3.10) can be rewritten as

$$\iota_V F = i\, d\Phi \tag{2.3.14}$$

since the extra term $[A, \Phi]$ is different from zero only outside the Cartan subalgebra. (2.3.14) means that Φ is the moment map for the action of V on F.

The gauge fixing condition comes by integrating out the auxiliary field B^+ from (2.3.9). As anticipate we obtain different gauge conditions for the $u(1)$ and $su(N)$ sector[7]

$$d \star (F_{u(1)}) = 0, \qquad (F_{su(N)})^+ = 0. \tag{2.3.15}$$

In particular $d \star F = 0$ in the whole $u(N)$. This, together with the Bianchi identity $dF = 0$ and the fact that F lies in the Cartan subalgebra of $u(N)$ (2.3.13), implicates that the curvature must be a harmonic 2-form with values on the Cartan subalgebra and integer periods. Namely for each elements in the Cartan subalgebra, labeled by $\alpha = 1, \ldots, N$

$$(c_1)_\alpha \equiv \frac{iF_\alpha}{2\pi} \in H^2(S^2 \times S^2, \mathbb{Z}). \tag{2.3.16}$$

So

$$\frac{iF_\alpha}{2\pi} = m_\alpha\, \omega(1) + n_\alpha\, \omega(2), \qquad m_\alpha, n_\alpha \in \mathbb{Z}. \tag{2.3.17}$$

[7] For the $u(1)$ sector, define $f := F_{u(1)}$ and $b^+ := B^+_{u(1)}$. From (2.3.9) we have $\mathcal{Q}\mathcal{V}_{u(1)} = -b^+ \wedge \star f^+ - b^+ \wedge \star b^+$. Integrating out b^+ we get the condition $b^+ = -f^+/2$ and inserting this back, we obtain $\mathcal{Q}\mathcal{V}_{u(1)} = \frac{1}{4} f^+ \wedge \star f^+ = \frac{1}{8}(f \wedge f + f \wedge \star f)$ that give the equation of motion $d \star f = 0$.

where a basis of normalized harmonic 2-forms for $S^2 \times S^2$ is given by

$$\omega(i) = \frac{1}{4\pi} \sin\theta_i d\theta_i \wedge d\varphi_i \quad i = 1, 2. \tag{2.3.18}$$

Replacing this expression for F in (2.3.14) we get

$$\iota_V(m_\alpha\, \omega(1) + n_\alpha\, \omega(2)) = -2\, d\Phi_\alpha \tag{2.3.19}$$

Since $S^2 \times S^2$ is simply connected the closed 2-forms $\iota_V\omega(1)$, $\iota_V\omega(2)$ are also exact and the equation (2.3.17) can be integrated

$$\iota_V\omega(1) = 2\, dh_1, \qquad \iota_V\omega(2) = 2\, dh_2, \tag{2.3.20}$$

the solution being given respectively by the height functions on the two spheres[8]

$$h_1 = \epsilon_1 \cos\theta_1, \qquad h_2 = \epsilon_2 \cos\theta_2. \tag{2.3.21}$$

Finally, integrating equation (2.3.17) we obtain

$$\Phi_\alpha = -m_\alpha\, h_1 - n_\alpha\, h_2 + a_\alpha, \qquad m_\alpha, n_\alpha \in \mathbb{Z}. \tag{2.3.22}$$

where a_α are integration constants. On top of fluxes, the complete solution contains also point-like instantons located at the zeroes of the vector field V. These do not contribute to the equations above since $i_V F^{\text{point}} = 0$.

We are then reduced to a sum over point-like instantons and an integration over the constant Cartan valued variable Φ. Let us notice that the above arguments are quite general and apply to more general four-manifolds than $S^2 \times S^2$.

Before proceeding to the computation, let us notice that on compact manifolds one has to take care of normalizable fermionic zero modes of the Laplacian, counted by the Betti numbers. If the manifold is simply connected, as we assume, then the field Ψ doesn't display such zero modes, while η will display one zero mode and χ^+ will display b_2^+ of them (for each element in the $su(N)$ Cartan[9] labeled by $\rho = 1, \ldots, N-1$). On $S^2 \times S^2$, and in general on any toric manifold, $b_2^+ = 1$ and therefore the (η, χ^+) zero modes come in pairs, one pair for each element in the $su(N)$ Cartan. One can soak-up those zero modes by adding the exact term $S_{\text{zm}} = s\mathcal{Q} \int \sum_\rho \bar{\Phi}_\rho \chi_\rho^+ B_\rho$ to the action whose effect, after the integration over the (η, χ^+) zero mode pairs and the B-zero modes, is to insert a derivative with respect to $\bar{a}$ for each $su(N)$ Cartan element. This reduces the integration over the a-plane to a contour integral around the diagonals where $a_\alpha - a_\beta = 0$.

2.3.3 Computation of the partition function

Due to the results of the previous section, the integration over the instanton moduli space is reduced to instanton counting. In particular due to supersymmetry, the instanton

[8] The vector generated by the supersymmetry generators (2.2.41) is $V = 2\epsilon_1\partial_{\varphi_1} + 2\epsilon_2\partial_{\varphi_2}$

[9] The ones in the $U(1)$ are gauge fixed as a BRST quartet.

configurations have to be equivariant under the action of $U(1)^2 \times U(1)^N$ which is the maximal torus of the isometry group of $S^2 \times S^2$ times the constant gauge transformations. It is well known that the fixed points are classified by Young diagrams [119], so that for each fixed point we have to consider a contribution given by the Nekrasov instanton partition function with the proper torus weights.

Let us underline the important difference between the compact and non-compact case, namely that in the former the gluing of Nekrasov partition functions also involves the integration over the Cartan subalgebra of the gauge group. This appears as a contour integral as explained at the end of the previous subsection.

We regard the manifold $S^2 \times S^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$ as a complex toric manifold described in terms of four patches. The weights $(\epsilon_1^{(\ell)}, \epsilon_2^{(\ell)})$ of the $(\mathbb{C}^*)^2$ torus action in each patch are

$$\begin{array}{c|cccc} \ell & 1 & 2 & 3 & 4 \\ \hline \epsilon_1^{(\ell)} & \epsilon_1 & -\epsilon_2 & -\epsilon_1 & \epsilon_2 \\ \epsilon_2^{(\ell)} & \epsilon_2 & \epsilon_1 & -\epsilon_2 & -\epsilon_1 \end{array} \tag{2.3.23}$$

where in our case $\epsilon_1 = \frac{1}{r_1} > 0$ and $\epsilon_2 = \frac{1}{r_2} > 0$ are the inverse radii of the two spheres.

The fixed point data on $S^2 \times S^2$ will be described in terms of a collection of Young diagrams $\{\vec{Y}_\ell\}$, and of integers numbers $\vec{m}, \vec{n}$ describing respectively the $(\mathbb{C}^*)^{N+2}$-invariant point-like instantons in each patch (localized at the fixed points p_ℓ) and the magnetic fluxes of the gauge field on the spheres which correspond to the first Chern class $c_1(E)$ of the gauge bundle E. More explicitly, for a gauge bundle with $c_1 = n\omega_1 + m\omega_2$ and $ch_2 = K$, the fixed point data satisfy

$$n = \sum_{\alpha=1}^{N} n_\alpha, \qquad m = \sum_{\alpha=1}^{N} m_\alpha, \qquad K = \sum_{\alpha,\ell} |Y_\alpha^{(\ell)}|. \tag{2.3.24}$$

The full partition function on $S^2 \times S^2$ is given by

$$Z_{\text{full}}^{S^2\times S^2}(q, z_1, z_2, \epsilon_1, \epsilon_2) = \sum_{\{\vec{m},\vec{n}\}} \oint_{\mathfrak{t}} d\vec{a}\, t_1^m t_2^n \prod_{\ell=1}^{4} Z_{\text{full}}^{\mathbb{C}^2}(q, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}, \vec{a}^{(\ell)}) \tag{2.3.25}$$

where $q = \exp(2\pi i\tau)$ is the gauge coupling, $t_1 = z_1^{\frac{1}{2}}$ and $t_2 = z_2^{\frac{1}{2}}$ are the source terms corresponding to $\omega = v_2\omega_1 + v_1\omega_2$ in (2.3.8) so that $z_i = e^{2\pi v_i}$ for $i = 1, 2$.

Moreover, $\vec{a}^{(\ell)} = \{a_\alpha^{(\ell)}\}$, $\alpha = 1, \dots, N$ are the v.e.v.'s of the scalar field Φ calculated at the fixed points p_ℓ

$$a_\alpha^{(\ell)} = \langle \Phi(p_\ell) \rangle. \tag{2.3.26}$$

Explicitly, by (2.3.22)

$$\begin{array}{c|l} \ell & \vec{a}^{(\ell)} \\ \hline 1 & \vec{a} + \vec{m}\epsilon_1 + \vec{n}\epsilon_2 \\ 2 & \vec{a} + \vec{m}\epsilon_1 - \vec{n}\epsilon_2 \\ 3 & \vec{a} - \vec{m}\epsilon_1 - \vec{n}\epsilon_2 \\ 4 & \vec{a} - \vec{m}\epsilon_1 + \vec{n}\epsilon_2 \end{array} \tag{2.3.27}$$

The factors appearing in (2.3.25) are the Nekrasov partition functions

$$Z^{\mathbb{C}^2}_{\text{full}}(q,\epsilon_1,\epsilon_2,\vec{a}) = Z^{\mathbb{C}^2}_{\text{classical}}(q,\epsilon_1,\epsilon_2,\vec{a}) Z^{\mathbb{C}^2}_{\text{1-loop}}(\epsilon_1,\epsilon_2,\vec{a}) Z^{\mathbb{C}^2}_{\text{instanton}}(q,\epsilon_1,\epsilon_2,\vec{a}). \quad (2.3.28)$$

whose explicit expressions we report below.

Classical action

Let us first of all consider the contribution to (2.3.25) of the classical partition function

$$\prod_{\ell=1}^{4} Z^{\mathbb{C}^2}_{\text{classical}}(\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\vec{a}^{(\ell)}). \quad (2.3.29)$$

For each patch this is given by

$$Z^{\mathbb{C}^2}_{\text{classical}}(\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\vec{a}^{(\ell)}) = \exp\left[-\pi i\tau \sum_{\alpha=1}^{N} \frac{(a_\alpha^{(\ell)})^2}{(\epsilon_1^{(\ell)}\epsilon_2^{(\ell)})}\right] \quad (2.3.30)$$

Inserting the values of the equivariant weights (2.3.23) and (2.3.27) we obtain

$$\prod_{\ell=1}^{4} Z^{\mathbb{C}^2}_{\text{classical}}(\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\vec{a}^{(\ell)}) = \exp\left[-\pi i\tau \sum_{\alpha=1}^{N} 8 m_\alpha n_\alpha\right] = q^{-4\sum_{\alpha=1}^{N} m_\alpha n_\alpha} \quad (2.3.31)$$

with $q = \exp(2i\pi\tau)$.

One-loop

The one-loop contribution in (2.3.25) is given by

$$Z^{S^2\times S^2}_{\text{1-loop}}(\vec{a},\epsilon_1,\epsilon_2) = \prod_{\ell=1}^{4} Z^{\mathbb{C}^2}_{\text{1-loop}}(\vec{a}^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}) = \prod_{\ell=1}^{4} \exp\Big[-\sum_{\alpha\neq\beta} \gamma_{\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}}(a_{\alpha\beta}^{(\ell)})\Big] \quad (2.3.32)$$

where $a_{\alpha\beta}^{(\ell)} := a_\alpha^{(\ell)} - a_\beta^{(\ell)}$.

Inserting the values of the equivariant weights (2.3.23) and (2.3.27) and using the definition of $\gamma_{\epsilon_1,\epsilon_2}$ (appendix 2.E equation (2.E.4)) we can rewrite the exponent in (2.3.32) as

$$-\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{-ta_{\alpha\beta}}}{(1-x)(1-y)} p(x,y), \quad (2.3.33)$$

where we have defined $x := e^{-\epsilon_1 t}$ and $y := e^{-\epsilon_2 t}$ and $p(x,y)$ is a polynomial in x and y given by

$$p(x,y) = x^{-m}y^{-n} - x^{-m}y^{n+1} - x^{m+1}y^{-n} + x^{m+1}y^{n+1} \quad (2.3.34)$$

where it is understood that $m \equiv m_{\alpha\beta}$ and $n \equiv n_{\alpha\beta}$. The residues of this polynomial at $x = 1$ and $y = 1$ are zero, this means that in those points $p(x,y)$ has zeros which cancel the poles $(1-x)^{-1}, (1-y)^{-1}$ in (2.3.33).

If $m = n = 0$

$$p(x, y) = 1 - y - x + xy = (1 - x)(1 - y), \tag{2.3.35}$$

and integrating (2.3.33) we obtain

$$Z^{S^2\times S^2}_{\text{1-loop}}(\vec{a}, \epsilon_1, \epsilon_2)\Big|_{m=n=0} = \prod_{\alpha\neq\beta} a_{\alpha\beta} = \prod_{\alpha>\beta} (-a^2_{\alpha\beta}). \tag{2.3.36}$$

In general, for every choice of $\{m, n\}$, one can factorize $(1-x)(1-y)$ out of the polynomial using the expansion $1 - x^N = (1 - x) \sum_{j=0}^{N-1} x^j$. Then $p(x, y)$ can be written as follows

$$p(x,y) = \begin{cases} (1-x)(1-y) \displaystyle\sum_{j=-m}^{m} \sum_{k=-n}^{n} x^j y^k & \text{if } m \geq 0,\ n \geq 0 \\ (1-x)(1-y) \displaystyle\sum_{j=m+1}^{-m-1} \sum_{k=n+1}^{-n-1} x^j y^k & \text{if } m < 0,\ n < 0 \\ -(1-x)(1-y) \displaystyle\sum_{j=-m}^{m} \sum_{k=n+1}^{-n-1} x^j y^k & \text{if } m \geq 0,\ n < 0 \\ -(1-x)(1-y) \displaystyle\sum_{j=m+1}^{-m-1} \sum_{k=-n}^{n} x^j y^k & \text{if } m < 0,\ n \geq 0 \end{cases} \tag{2.3.37}$$

Using this result in (2.3.33) we obtain for fixed m and n the following result:

$$Z^{S^2\times S^2}_{\text{1-loop}}(a, \epsilon_1, \epsilon_2, m, n)\Big|_{\alpha\beta} = \\ = \begin{cases} \displaystyle\prod_{k=-m}^{m} \prod_{j=-n}^{n} (a + k\epsilon_1 + j\epsilon_2) & \text{if } m \geq 0, n \geq 0 \\ \displaystyle\prod_{k=m+1}^{-m-1} \prod_{j=n+1}^{-n-1} (a + k\epsilon_1 + j\epsilon_2) & \text{if } m < 0, n < 0 \\ \displaystyle\prod_{k=-m}^{m} \prod_{j=n+1}^{-n-1} (a + k\epsilon_1 + j\epsilon_2)^{-1} & \text{if } m \geq 0, n < 0 \\ \displaystyle\prod_{k=m+1}^{-m-1} \prod_{j=-n}^{n} (a + k\epsilon_1 + j\epsilon_2)^{-1} & \text{if } m < 0, n \geq 0 \end{cases} \tag{2.3.38}$$

where $a \equiv a_{\alpha\beta}$, $m \equiv m_{\alpha\beta}$ and $n \equiv n_{\alpha\beta}$.

Instantons

The instanton contribution in (2.3.25) is given by

$$\prod_{\ell=1}^{4} Z^{\mathbb{C}^2}_{\text{instanton}}(\epsilon^{(\ell)}_1, \epsilon^{(\ell)}_2, \vec{a}^{(\ell)}). \tag{2.3.39}$$

where $Z^{\mathbb{C}^2}_{\text{instanton}}$ is the Nekrasov partition function defined as follows.

Let $Y = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ be a Young diagram, and $Y' = \{\lambda'_1 \geq \lambda'_2 \geq \dots\}$ its transposed. λ_i is the length of the i-column and λ'_j the length of the j-row of Y. For a given box $s = \{i, j\}$ of the diagram we define respectively the arm and leg length functions

$$A_Y(s) = \lambda_i - j, \qquad L_Y(s) = \lambda'_j - i. \tag{2.3.40}$$

and the arm and leg co-length functions

$$A'_Y(s) = j - 1, \qquad L'_Y(s) = i - 1. \tag{2.3.41}$$

The fixed points data for each patch are given by a collection of Young diagrams $\vec{Y}^{(\ell)} = \{Y^{(\ell)}_\alpha\}$. and the instanton contribution is

$$Z^{\mathbb{C}^2}_{\text{instanton}}(\epsilon_1, \epsilon_2, \vec{a}) = \sum_{\{Y_\alpha\}} q^{|\vec{Y}|} z_{\text{vec}}(\epsilon_1, \epsilon_2, \vec{a}, \vec{Y}) \tag{2.3.42}$$

where $q = \exp(2i\pi\tau)$ and

$$\begin{aligned} z_{\text{vec}}(\epsilon_1, \epsilon_2, \vec{a}, \vec{Y}) = \prod_{\alpha,\beta=1}^{N} \Bigg[\prod_{s \in Y_\alpha} \Big(a_{\alpha\beta} - L_{Y_\beta}(s)\epsilon_1 + (A_{Y_\alpha}(s) + 1)\epsilon_2 \Big) \\ \prod_{t \in Y_\beta} \Big(a_{\alpha\beta} + (L_{Y_\beta}(t) + 1)\epsilon_1 - A_{Y_\alpha}(t)\epsilon_2 \Big) \Bigg]^{-1} . \end{aligned} \tag{2.3.43}$$

2.3.4 Adding matter fields

The above formulae are easily modified in presence of matter fields. In the following we discuss the contribution of matter in the (anti)fundamental representation, which will be used in the last section when comparing with Liouville gravity. The contribution to the classical action is vanishing, so we concentrate on one-loop and instanton terms.

One-loop

When considering matter one has to modify the formula for the one-loop partition function (2.3.32) as

$$Z^{\mathbb{C}^2}_{\text{1-loop}}(\vec{a}^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) = \exp\left[-\sum_{\alpha \neq \beta} \gamma_{\epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}}(a^{(\ell)}_{\alpha\beta}) + \sum_{f=1}^{N_f} \sum_{\rho \in R_f} \gamma_{\epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}}\left(a^{(\ell)}_\rho + \mu_f - \frac{Q^{(\ell)}}{2} \right) \right] \tag{2.3.44}$$

where ρ are the weights of the representation R_f of the hypermultiplet with mass μ_f and R-charge one, while $Q^{(\ell)} := \epsilon_1^{(\ell)} + \epsilon_2^{(\ell)}$.

The computation goes as in the previous section, the additional contribution for each hypermultiplet in the exponential being

$$\frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{-t(a_\rho + \mu_f + \frac{Q}{2})}}{(1-x)(1-y)} p(x, y). \tag{2.3.45}$$

where again $x := e^{-\epsilon_1 t}$ and $y := e^{-\epsilon_2 t}$, with

$$p(x,y) = x^{-m}y^{-n} - x^{-m}y^{n} - x^{m}y^{-n} + x^{m}y^{n}. \tag{2.3.46}$$

Eventually, each hypermultiplet contributes as

$$Z^{S^2\times S^2}_{\text{1-loop}}(a,\epsilon_1,\epsilon_2,m,n,\mu_f)\Big|_{\text{hyp},f,\rho} = \\
= \begin{cases} 1 & \text{if } m\cdot n = 0 \\ \prod_{k=-m}^{m-1}\prod_{j=-n}^{n-1}\left(a+\mu_f+\frac{Q}{2}+k\epsilon_1+j\epsilon_2\right)^{-1} & \text{if } m>0, n>0 \\ \prod_{k=m}^{-m-1}\prod_{j=n}^{-n-1}\left(a+\mu_f+\frac{Q}{2}+k\epsilon_1+j\epsilon_2\right)^{-1} & \text{if } m<0, n<0 \\ \prod_{k=-m}^{m-1}\prod_{j=n}^{-n-1}\left(a+\mu_f+\frac{Q}{2}+k\epsilon_1+j\epsilon_2\right) & \text{if } m>0, n<0 \\ \prod_{k=m}^{-m-1}\prod_{j=-n}^{n-1}\left(a+\mu_f+\frac{Q}{2}+k\epsilon_1+j\epsilon_2\right) & \text{if } m<0, n>0 \end{cases} \tag{2.3.47}$$

where $a \equiv a_\rho$, $m \equiv m_\rho$ and $n \equiv n_\rho$.

Instantons

The modification of the instanton partition function due to the presence of matter in the (anti)fundamental representation is

$$Z^{\mathbb{C}^2}_{\text{instanton}}(\epsilon_1,\epsilon_2,\vec{a}) = \sum_{\{Y_\alpha\}} q^{|\vec{Y}|} z_{\text{vec}}(\epsilon_1,\epsilon_2,\vec{a},\vec{Y}) \prod_{f=1}^{N_F} z_{\text{(anti)fund}}(\epsilon_1,\epsilon_2,\vec{a},\vec{Y},\mu_f) \tag{2.3.48}$$

where

$$\begin{aligned} z_{\text{fund}}(\epsilon_1,\epsilon_2,\vec{a},\vec{Y},\mu_f) &= \prod_\alpha \prod_{s\in Y_\alpha} \left(a_\alpha + L'(s)\epsilon_1 + A'(s)\epsilon_2 + Q - \mu_f\right) \\ z_{\text{antifund}}(\epsilon_1,\epsilon_2,\vec{a},\vec{Y},\mu_f) &= \prod_\alpha \prod_{s\in Y_\alpha} \left(a_\alpha + L'(s)\epsilon_1 + A'(s)\epsilon_2 + \mu_f\right) \end{aligned} \tag{2.3.49}$$

where $L'(s)$ and $A'(s)$ are the co-length functions defined in (2.3.41).

2.4 Liouville Gravity

We now proceed to the discussion of a possible two-dimensional Conformal Field Theory (CFT) interpretation of our results, prompted by AGT correspondence. We focus on the $N = 2$ case. A natural viewpoint to start with is the calculation of the expected central charge via reduction of the anomaly polynomial of two M5-branes theory [36, 4]. Upon compactification on the manifold $S^2 \times S^2 \times \Sigma$, the central charge of the resulting two-dimensional CFT on Σ is easily computed via localization formulae from the weights of the $U(1)^2$ torus action, see Table (2.3.23), to be

$$\sum_{\ell=1}^{4} \left(1 + 6 \left(b^{(\ell)} + \frac{1}{b^{(\ell)}} \right)^2 \right) = 52 = 26 + 26 \tag{2.4.1}$$

were $b^{(\ell)} = \sqrt{\frac{\epsilon_1^{(\ell)}}{\epsilon_2^{(\ell)}}}$. Notice that in passing from one patch to the other only one of the epsilons change sign so that from real b one passes to imaginary one and viceversa. This will play a relevant role in the subsequent discussion. It was observed in [28] that (2.4.1) suggests a link to Liouville gravity. In the following we will show that indeed three-point number and conformal blocks of this CFT arise as building blocks of the supersymmetric partition function of $\mathcal{N} = 2$ $U(2)$ gauge theory on $S^2 \times S^2$.

Liouville Gravity (LG) [156, 19] is a well-known two-dimensional theory of quantum gravity composed of three CFT sectors

1. **Liouville theory sector**, which has a central charge

$$c_L = 1 + 6Q^2, \qquad Q = b + b^{-1}, \tag{2.4.2}$$

and a continuous family of primary fields parametrized by a complex parameter α as

$$V_\alpha = e^{2\alpha\varphi(x)} \tag{2.4.3}$$

with conformal dimension

$$\Delta_\alpha^L = \alpha(Q - \alpha). \tag{2.4.4}$$

2. **Matter sector**, a generalized CFT with central charge

$$c_M = 1 - 6q^2, \quad q = b^{-1} - b. \tag{2.4.5}$$

and generic primary fields, labeled by a continuous parameter α, Φ_α with dimension

$$\Delta_\alpha^M = \alpha(\alpha - q). \tag{2.4.6}$$

3. **Ghost sector** needed to gauge fix the conformal symmetry. This is described by a fermionic bc system of spin $(2, -1)$ of central charge

$$c_{\text{gh}} = -26. \tag{2.4.7}$$

The fact that

$$c_L + c_M = 26 \tag{2.4.8}$$

allows the construction of a BRST complex.

The vertex operators of the complete system are built out of primary operators in the Liouville plus matter sector as

$$U_\alpha = \Phi_{\alpha-b} V_\alpha \tag{2.4.9}$$

which are $(1,1)$-forms with ghost number zero, and can be integrated on the space. This is ensured by the condition

$$\Delta^M_{\alpha-b} + \Delta^L_\alpha = 1. \tag{2.4.10}$$

We are mainly interested in three-point numbers and conformal blocks. The former have been computed in [156] (eq.7.9) for three generic dressed operators U_{α_i} and can be written in terms of γ function, see eq. (2.E.11), as

$$\begin{aligned} C^{\mathrm{LG}}(\alpha_1,\alpha_2,\alpha_3) &= C^L(\alpha_1,\alpha_2,\alpha_3) C^M(\alpha_1-b,\alpha_2-b,\alpha_3-b) \\ &= \left(\pi\mu\gamma(b^2)\right)^{(Q-\sum_{i=1}^3\alpha_i)/b} \left[\frac{\gamma(b^2)\gamma(b^{-2}-1)}{b^2}\right]^{1/2} \prod_{i=1}^3 \left[\gamma(2\alpha_i b - b^2)\gamma(2\alpha_i b^{-1} - b^{-2})\right]^{1/2}. \end{aligned} \tag{2.4.11}$$

Let us remark that the ghost sector does not play any role in our considerations. Indeed this is suited to produce a proper measure on the moduli space of the Riemann surface over which the CFT is formulated. On the gauge theory side this would correspond to the quite unnatural operation of integrating over the gauge coupling.

2.4.1 LG three-point function versus one-loop in gauge theory

Let us now compare the results of the one-loop gauge theory partition function with the above three-point number of LG. We consider the sector with zero magnetic fluxes $\vec{m} = \vec{n} = 0$ of $U(2)$ gauge theory with $N_f = 4$. The contribution of the one-loop partition function is given by equation (2.3.36), and setting $a_{12} =: 2a$, we have

$$Z^{S^2\times S^2}_{\text{1-loop}}(\vec{a},\epsilon_1,\epsilon_2) = \prod_{\alpha\neq\beta} a_{\alpha\beta} = \prod_{\pm} \pm 2a = -4a^2. \tag{2.4.12}$$

Indeed one can show that in the sector $\vec{m} = \vec{n} = 0$ the contribution of hypermultiplets in the four patches (2.3.47) cancel each other.

The above result can be compared with the product of Liouville gravity three point numbers (2.4.11). Indeed, if we consider

$$\alpha = \frac{Q}{2} + a, \qquad \alpha_i = \frac{Q}{2} + p_i, \qquad a, p_i \in i\mathbb{R} \tag{2.4.13}$$

we get for the product of two three point numbers

$$C^{\mathrm{LG}}(\alpha_1,\alpha_2,\alpha) C^{\mathrm{LG}}(\bar{\alpha},\alpha_3,\alpha_4) = N \left(\prod_{i=1}^4 f(\alpha_i)\right) \left(4a^2\right), \tag{2.4.14}$$

where

$$N = \Big(\pi\mu\gamma(b^2)\Big)^{1+b^{-2}} \frac{\gamma(b^2)\gamma(b^{-2}-1)}{b^2},$$
$$f(\alpha_i) = \Big(\pi\mu\gamma(b^2)\Big)^{-\alpha_i/b} \sqrt{\gamma(2\alpha_i b - b^2)\gamma(2\alpha_i b^{-1} - b^{-2})}. \tag{2.4.15}$$

The dependence on a of (2.4.12) and (2.4.14) is the same. Moreover one can check [6] that the contribution of the two patches with $\epsilon_1^{(\ell)} \cdot \epsilon_2^{(\ell)} > 0$ naturally compares to the product of Liouville theory three-point numbers ([156], eq. (2.2)). On the other hand, the contribution of the two patches with $\epsilon_1^{(\ell)} \cdot \epsilon_2^{(\ell)} < 0$ naturally compares to the product of generalized minimal model three-points functions ([156], eq. (5.1) with $\beta = b$ and $\alpha = a - b$), which is the matter sector of the LG. Explicitly

$$\left| Z^{S^2\times S^2}_{\text{1-loop}}(\vec{a}, \epsilon_1^{(\ell)} \cdot \epsilon_2^{(\ell)} > 0) \right| = C^L(\alpha_1, \alpha_2, \alpha) C^L(\bar{\alpha}, \alpha_3, \alpha_4),$$

$$\left| Z^{S^2\times S^2}_{\text{1-loop}}(\vec{a}, \epsilon_1^{(\ell)} \cdot \epsilon_2^{(\ell)} < 0) \right| = C^M(\alpha_1 - b, \alpha_2 - b, \alpha - b) C^M(\bar{\alpha} - b, \alpha_3 - b, \alpha_4 - b), \tag{2.4.16}$$

up to renormalization of the vertices analogously to (2.4.15), once the v.e.v. a and μ_f are assumed to be purely imaginary.

We expect the gauge theory sectors with non-vanishing magnetic fluxes $\vec{m}$ and $\vec{n}$ to be related to the insertions of degenerate fields. Indeed the same comment applies to the results on the conformal blocks obtained in the next subsection.

2.4.2 Conformal blocks versus instantons

It is a well known fact that the instanton contribution to the partition function (2.3.48) for $U(2)$ gauge theory with $N_f = 4$ on $\mathbb{C}^2$ can be matched with the four point conformal block on the sphere, up to a $U(1)$ factor [6],

$$Z^{\mathbb{C}^2, U(2)}_{\text{instanton}}(\epsilon_1, \epsilon_2, \vec{a}, \mu_1, \mu_2, \mu_3, \mu_4) \simeq \mathcal{F}^L{}_{\alpha_1}{}^{\alpha_2}{}_{\alpha}{}^{\alpha_3}{}_{\alpha_4}(\tau) \tag{2.4.17}$$

where $\mu_1 = p_1 + p_2$, $\mu_2 = p_1 - p_2$, $\mu_3 = p_3 + p_4$, $\mu_4 = p_3 - p_4$ and α_i are defined in (2.4.13).

Moreover contrary to three-point correlators, the conformal blocks of the matter sector in LG are the analytic continuation of those of Liouville theory under $b \to ib$. These two facts allow us to interpret the instanton partition function (2.4.18) of $U(2)$ gauge theory with $N_f = 4$ on $S^2 \times S^2$ in the sector $\vec{m} = \vec{n} = 0$ as two copies of four point conformal blocks of LG on the sphere. Indeed, by using $b^{(\ell)} = \sqrt{\epsilon_1^{(\ell)}/\epsilon_2^{(\ell)}}$, we have from (2.4.18) and (2.3.23)

$$\begin{aligned} &Z^{S^2\times S^2}_{\text{instanton}}(b, b^{-1}, a, \mu_f) \\ &= \prod_{\ell=1}^{4} Z^{\mathbb{C}^2}_{\text{instanton}}(b^{(\ell)}, (b^{(\ell)})^{-1}, a, \mu_f) \\ &= \Big[Z^{\mathbb{C}^2}_{\text{instanton}}(b, b^{-1}, a, \mu_f) Z^{\mathbb{C}^2}_{\text{instanton}}(ib, (ib)^{-1}, a, \mu_f) \Big]^2 \end{aligned} \tag{2.4.18}$$

where $\mu_f = \{\mu_1, \mu_2, \mu_3, \mu_4\}$. From the discussion above, these are two copies of four points conformal blocks of the two sectors of LG: Liouville and matter

$$Z^{S^2\times S^2}_{\text{instanton}}(q,b,b^{-1},a,\mu_f) \simeq \left[\mathcal{F}^L{}_{\alpha_1}{}^{\alpha_2}{}_{\alpha}{}^{\alpha_3}{}_{\alpha_4}(\tau)\mathcal{F}^M{}_{\alpha_1}{}^{\alpha_2}{}_{\alpha}{}^{\alpha_3}{}_{\alpha_4}(\tau)\right]^2 = \left[\mathcal{F}^{\text{LG}}{}_{\alpha_1}{}^{\alpha_2}{}_{\alpha}{}^{\alpha_3}{}_{\alpha_4}(\tau)\right]^2. \tag{2.4.19}$$

The full partition function (2.3.25) in the sector $\vec{m} = \vec{n} = 0$ is then expressible as

$$Z^{S^2\times S^2}_{\text{full}}(q,b,b^{-1},a,\mu_f) \propto \int d\alpha\, C^{\text{LG}}(\alpha_1,\alpha_2,\alpha)C^{\text{LG}}(\bar{\alpha},\alpha_3,\alpha_4)\left[\mathcal{F}^{\text{LG}}{}_{\alpha_1}{}^{\alpha_2}{}_{\alpha}{}^{\alpha_3}{}_{\alpha_4}(\tau)\right]^2. \tag{2.4.20}$$

Few remarks are in order here. First of all, the holomorphicity in its arguments of the supersymmetric partition function under scrutiny is reflected in the holomorphic gluing of building blocks of the corresponding CFT, in contrast to the one appearing in correlation functions of Liouville gravity. Moreover, we underline that the three-point numbers of the matter sector C^M naturally arising in the gauge theory context are strictly speaking not the ones of generalised minimal model. Indeed they do not obey the selections rules of this model, and were introduced in [156] only as a technical tool to solve the relevant bootstrap equations. Indeed to get the physical three-point functions one has to multiply them by a suitable non-analytic term which takes into account the selection rule (see eq(3.16) in [156] and also [19]). We remark that a CFT which consistently makes use of the analytic three-point correlator C^M appearing in the gauge theory can be formulated [129].

2.A Full $\mathcal{N} = 2$ Supersymmetry

In this appendix we give the detailed calculations of the results stated in subsection 2.1.2.

We proceed by writing the most general form for supersymmetric variation for the gauginos, consistent with the properties of positivity of mass of background fields, their gauge neutrality, and balancing of masses and $U(1)_R$ charges. We have

$$\begin{aligned}\mathbf{Q}\lambda_A &= \frac{k_1}{2}\sigma^{\mu\nu}\xi_A(F_{\mu\nu} + 8\bar{\phi}T_{\mu\nu} + 8\phi W_{\mu\nu}) + a_1\sigma^{\mu}\bar{\xi}_A D_\mu\phi + b_1\sigma^\mu D_\mu\bar{\xi}_A\phi + c_1\xi_A[\phi,\bar{\phi}] + d_1 D_{AB}\xi^B,\\ \mathbf{Q}\bar{\lambda}_A &= \frac{k_2}{2}\bar{\sigma}^{\mu\nu}\bar{\xi}_A(F_{\mu\nu} + 8\phi\bar{T}_{\mu\nu} + 8\bar{\phi}\bar{W}_{\mu\nu}) + a_2\bar{\sigma}^{\mu}\xi_A D_\mu\bar{\phi} + b_2\bar{\sigma}^\mu D_\mu\xi_A\bar{\phi} - c_2\bar{\xi}_A[\phi,\bar{\phi}] + d_2 D_{AB}\bar{\xi}^B.\end{aligned} \tag{2.A.1}$$

Consider now the square of the supersymmetry transformation acting on the scalar fields

$$\begin{aligned}\mathbf{Q}^2\phi &= -i\xi^A(\mathbf{Q}\lambda_A) = -ia_1\xi^A\sigma^\mu\bar{\xi}_A D_\mu\phi - ib_1\xi^A\sigma^\mu D_\mu\bar{\xi}_A\phi - ic_1\xi^A\xi_A[\phi,\bar{\phi}],\\ \mathbf{Q}^2\bar{\phi} &= -i\bar{\xi}^A(\mathbf{Q}\bar{\lambda}_A) = +ia_2\bar{\xi}^A\bar{\sigma}^\mu\xi_A D_\mu\bar{\phi} + ib_2\bar{\xi}^A\bar{\sigma}^\mu D_\mu\xi_A\bar{\phi} - ic_2\bar{\xi}^A\bar{\xi}_A[\phi,\bar{\phi}].\end{aligned} \tag{2.A.2}$$

From the above, it clearly follows that $V^\mu = ia_1\bar{\xi}^A\bar{\sigma}^\mu\xi_A = ia_2\bar{\xi}^A\bar{\sigma}^\mu\xi_A$. We define $a \equiv a_1 = a_2$. We also infer that $\Phi = c_1\xi^A\xi_A\bar{\phi} - c_2\bar{\xi}^A\bar{\xi}_A\phi$. We will return to $\mathbf{Q}^2\phi$ and $\mathbf{Q}^2\bar{\phi}$ momentarily to investigate the scaling and the $U(1)_R$ terms.

Consider now the $\mathbf{Q}^2 A_\mu$

$$
\begin{aligned}
\mathbf{Q}^2 A_\mu =& i\xi^A \sigma_\mu (\mathbf{Q}\bar{\lambda}_A) - i\bar{\xi}^A \bar{\sigma}_\mu (\mathbf{Q}\lambda_A) \\
=& \frac{i}{2} F^{\rho\sigma} \left(k_2 \xi^A \sigma_\mu \bar{\sigma}_{\rho\sigma} \bar{\xi}_A - k_1 \bar{\xi}^A \bar{\sigma}_\mu \sigma_{\rho\sigma} \xi_A \right) \\
&+ 4i \left(k_2 \phi \bar{T}^{\rho\sigma} \xi^A \sigma_\mu \bar{\sigma}_{\rho\sigma} \bar{\xi}_A - k_1 \bar{\phi} T^{\rho\sigma} \bar{\xi}^A \bar{\sigma}_\mu \sigma_{\rho\sigma} \xi_A \right) \\
&+ 4i \left(k_2 \bar{\phi} \bar{W}^{\rho\sigma} \xi^A \sigma_\mu \bar{\sigma}_{\rho\sigma} \bar{\xi}_A - k_1 \phi W^{\rho\sigma} \bar{\xi}^A \bar{\sigma}_\mu \sigma_{\rho\sigma} \xi_A \right) \\
&+ ia \left(\xi^A \sigma_\mu \bar{\sigma}_\nu \xi_A D^\nu \bar{\phi} - \bar{\xi}^A \bar{\sigma}_\mu \sigma_\nu \bar{\xi}_A D^\nu \phi \right) \\
&+ i \left(b_2 \bar{\phi} \xi^A \sigma_\mu \bar{\sigma}_\nu D^\nu \xi_A - b_1 \phi \bar{\xi}^A \bar{\sigma}_\mu \sigma_\nu D^\nu \bar{\xi}_A \right) \\
&- i[\phi, \bar{\phi}] \left(c_1 \xi^A \sigma_\mu \bar{\xi}_A + c_2 \bar{\xi}^A \bar{\sigma}_\mu \xi_A \right) \\
&- i D_{AB} \left(d_2 \xi^A \sigma_\mu \bar{\xi}^B - d_1 \bar{\xi}^A \bar{\sigma}_\mu \xi^B \right)
\end{aligned}
\tag{2.A.3}
$$

The commutator term must vanish by the assumptions on the nature of $\mathbf{Q}^2$, which implies $c_1 = c_2 \equiv c$. Similarly the vanishing of the D_{AB} requires $d_1 = d_2$, which can now be absorbed in D_{AB}, and we will therefore set $d_1 = d_2 = 1$. We want the term with $F^{\rho\sigma}$ to equal $V^\nu F_{\nu\mu}$, which forces $k_1 = k_2 = (a/2) \equiv k$ as can be seen after some algebraic manipulations of the spinor products. The terms that remain are the ones with the background fields and the ones with the derivatives of the scalar field:

$$
\begin{aligned}
& 4i \left(k_2 \phi \bar{T}^{\rho\sigma} \xi^A \sigma_\mu \bar{\sigma}_{\rho\sigma} \bar{\xi}_A - k_1 \bar{\phi} T^{\rho\sigma} \bar{\xi}^A \bar{\sigma}_\mu \sigma_{\rho\sigma} \xi_A \right) \\
&+ 4i \left(k_2 \bar{\phi} \bar{W}^{\rho\sigma} \xi^A \sigma_\mu \bar{\sigma}_{\rho\sigma} \bar{\xi}_A - k_1 \phi W^{\rho\sigma} \bar{\xi}^A \bar{\sigma}_\mu \sigma_{\rho\sigma} \xi_A \right) \\
&+ ai \left(\xi^A \sigma_\mu \bar{\sigma}_\nu \xi_A D^\nu \bar{\phi} - \bar{\xi}^A \bar{\sigma}_\mu \sigma_\nu \bar{\xi}_A D^\nu \phi \right).
\end{aligned}
\tag{2.A.4}
$$

We require these terms to be equal to the gauge variation

$$
D_\mu \Phi = c\xi^A \xi_A D_\mu \bar{\phi} - c\bar{\xi}^A \bar{\xi}_A D_\mu \phi + 2c\bar{\phi} \xi^A D_\mu \xi_A - 2c\phi \bar{\xi}^A D_\mu \bar{\xi}_A. \tag{2.A.5}
$$

Equating the terms with the derivatives of the scalar field on the two sides gives $c = ia$ while equating the terms in $\bar{\phi}$ we get

$$
2ia\xi^A D_\mu \xi_A = -4ik(T^{\rho\sigma} - W^{\rho\sigma}) \xi^A \sigma_{\rho\sigma} \sigma_\mu \bar{\xi}_A + ib_2 \xi^A \bar{\sigma}_\nu D^\nu \xi_A. \tag{2.A.6}
$$

We note that this is satisfied when

$$
aD_\mu \xi_A = -2k(T^{\rho\sigma} - W^{\rho\sigma}) \sigma_{\rho\sigma} \sigma_\mu \bar{\xi}_A + \frac{b_2}{2} \sigma_\mu \bar{\sigma}_\nu D^\nu \xi_A. \tag{2.A.7}
$$

Contracting either side with σ^μ we find that $a = 2b_2$. Similarly starting with the equation for ϕ we find that $a = 2b_1$. We also find analogously the equation

$$
aD_\mu \bar{\xi}_A = -2k(\bar{T}^{\rho\sigma} - \bar{W}^{\rho\sigma}) \bar{\sigma}_{\rho\sigma} \bar{\sigma}_\mu \xi_A + \frac{b_2}{2} \bar{\sigma}_\mu \sigma_\nu D^\nu \bar{\xi}_A. \tag{2.A.8}
$$

Define $b \equiv b_1 = b_2$. We now return to expressions for $\mathbf{Q}^2\phi$, $\mathbf{Q}^2\bar{\phi}$ and $\mathbf{Q}^2 A_\mu$ and identify the remaining terms. Consider first the equation for $\mathbf{Q}^2\phi$ and $\mathbf{Q}^2\bar{\phi}$. Since we have identified the Lie derivative term and the gauge transformation term, the remaining terms must combine to give the scaling and the $U(1)_R$ terms. Therefore

$$\begin{aligned} 4ik(D_\mu\bar{\xi}^A\bar{\sigma}^\mu\xi_A + \bar{\xi}^A\bar{\sigma}^\mu D_\mu\xi_A) + 2\Theta = ib_1 D_\mu\bar{\xi}^A\bar{\sigma}^\mu\xi_A \\ 4ik(D_\mu\bar{\xi}^A\bar{\sigma}^\mu\xi_A + \bar{\xi}^A\bar{\sigma}^\mu D_\mu\xi_A) - 2\Theta = ib_2\bar{\xi}^A\bar{\sigma}^\mu D_\mu\xi_A \end{aligned} \tag{2.A.9}$$

which gives

$$\Theta = \frac{ib}{4}(D_\mu\bar{\xi}^A\bar{\sigma}^\mu\xi_A - \bar{\xi}^A\bar{\sigma}^\mu D_\mu\xi_A). \tag{2.A.10}$$

Consider now $\mathbf{Q}^2 A_\mu$. We are left with the following terms.

$$\begin{aligned} &4i\left(k_2\phi\bar{T}^{\rho\sigma}\xi^A\sigma_\mu\bar{\sigma}_{\rho\sigma}\bar{\xi}_A - k_1\bar{\phi}T^{\rho\sigma}\bar{\xi}^A\bar{\sigma}_\mu\sigma_{\rho\sigma}\xi_A\right) \\ &+ 4i\left(k_2\bar{\phi}\bar{W}^{\rho\sigma}\xi^A\sigma_\mu\bar{\sigma}_{\rho\sigma}\bar{\xi}_A - k_1\phi W^{\rho\sigma}\bar{\xi}^A\bar{\sigma}_\mu\sigma_{\rho\sigma}\xi_A\right) \\ &+ ia\left(\xi^A\sigma_\mu\bar{\sigma}_\nu\xi_A D^\nu\bar{\phi} - \bar{\xi}^A\bar{\sigma}_\mu\sigma_\nu\bar{\xi}_A D^\nu\phi\right). \end{aligned} \tag{2.A.11}$$

We require that these combine to give the appropriate gauge transformation term

$$D_\mu[2ik(\xi^A\xi_A\phi - \bar{\xi}^A\bar{\xi}_A\bar{\phi})] \tag{2.A.12}$$

which happens when equations (2.A.7) and (2.A.8) are satisfied.

Note that we can rescale the gauginos and the auxiliary field to get rid to the normalization k (or equivalently a, b or c). We therefore set $k = 1$. We summarize the expressions for the generators of the bosonic symmetries that we have found till now:

$$\begin{aligned} V^\mu &= 2i\bar{\xi}^A\bar{\sigma}^\mu\xi_A, \\ w &= \frac{1}{4}D_\mu v^\mu \\ \Theta &= \frac{i}{4}(D_\mu\bar{\xi}^A\bar{\sigma}^\mu\xi_A - \bar{\xi}^A\bar{\sigma}^\mu D_\mu\xi_A) \\ \Phi &= 2i\bar{\phi}\xi^A\xi_A - 2i\phi\bar{\xi}^A\bar{\xi}_A. \end{aligned} \tag{2.A.13}$$

We now study $\mathbf{Q}^2\lambda_A$. The case of $\mathbf{Q}^2\bar{\lambda}_A$ is analogous and will not be detailed. In doing so, will find the expression for $\mathbf{Q}D_{AB}$ and also show that $W_{\mu\nu}$ vanishes.

$$\begin{aligned}
\mathbf{Q}^2\lambda_A =&\frac{1}{2}\sigma^{\mu\nu}\xi_A\left(\mathbf{Q}F_{\mu\nu}+8(\mathbf{Q}\bar{\phi})T_{\mu\nu}+8(\mathbf{Q}\phi)W_{\mu\nu}\right)\\
&+2\sigma^{\mu}\bar{\xi}_A(D_\mu(\mathbf{Q}\phi)-i[\mathbf{Q}A_\mu,\phi])+\sigma^\mu D_\mu\bar{\xi}_A(\mathbf{Q}\phi)\\
&+2i\xi_A[\mathbf{Q}\phi,\bar{\phi}]+2i\xi_A[\phi.\mathbf{Q}\bar{\phi}]+\mathbf{Q}D_{AB}\xi^B\\
=&\frac{i}{2}\sigma^{\mu\nu}\xi_A\Big[D_\mu\xi^B\sigma_\nu\bar{\lambda}_B+\xi^B\sigma_\nu\bar{D}_\mu\lambda_B\\
&-D_\mu\bar{\xi}^B\bar{\sigma}_\nu\lambda_B-\bar{\xi}^B\bar{\sigma}_\nu D_\mu\lambda_B\\
&-D_\nu\xi^B\sigma_\mu\bar{\lambda}_B-\xi^B\sigma_\mu\bar{D}_\nu\lambda_B\\
&+D_\nu\bar{\xi}^B\bar{\sigma}_\mu\lambda_B+\bar{\xi}^B\bar{\sigma}_\mu D_\nu\lambda_B\Big]\\
&+4i\sigma^{\mu\nu}\xi_AT_{\mu\nu}(\bar{\xi}^B\bar{\lambda}_B)-4i\sigma^{\mu\nu}\xi_AW_{\mu\nu}(\xi^B\lambda_B)\\
&-2i\sigma^\mu\bar{\xi}_A(D_\mu\xi^B\lambda_B)-2i\sigma^\mu\bar{\xi}_A(\xi^BD_\mu\lambda_B)\\
&+2\sigma^\mu\bar{\xi}_A[\xi^B\sigma_\mu\bar{\lambda}_B-\bar{\xi}^B\bar{\sigma}_\mu\lambda_B,\phi]-i\sigma^\mu D_\mu\bar{\xi}_A(\xi^B\lambda_B)\\
&-2\xi_A[\bar{\phi},\xi^B\lambda_B]-2\xi_A[\phi,\bar{\xi}^B\bar{\lambda}_B]\\
&+(\mathbf{Q}D_{AB})\xi^B.
\end{aligned}\tag{2.A.14}$$

Since we know what form we should force $\mathbf{Q}^2\lambda_A$ to take, we can rearrange the above terms to obtain them. We try to cancel all the offending terms by postulating the form for the supersymmetric variation of the auxiliary field (as we did in the chiral case). We find that

$$\mathbf{Q}D_{AB}=-2i\bar{\xi}_{(A}\bar{\sigma}^\mu D_\mu\lambda_{B)}+2i\xi_{(A}\sigma^\mu D_\mu\bar{\lambda}_{B)}-4\left[\phi,\bar{\xi}_{(A}\bar{\lambda}_{B)}\right]+4\left[\bar{\phi},\xi_{(A}\lambda_{B)}\right].\tag{2.A.15}$$

We also find the form of Θ_{AB} as follows:

$$\Theta_{AB}=-i\xi_{(A}\sigma^\mu D_\mu\bar{\xi}_{B)}+iD_\mu\xi_{(A}\sigma^\mu\bar{\xi}_{B)}.\tag{2.A.16}$$

We rediscover the main Killing equation as the co-efficient of the terms with $\bar{\lambda}$

$$2i(\bar{\lambda}^B\bar{\sigma}^\mu\xi_A)\left(D_\mu\xi_B+T^{\rho\sigma}\sigma_{\rho\sigma}\sigma_\mu\bar{\xi}_B-\frac{1}{4}\sigma_\mu\bar{\sigma}_\nu D^\nu\xi_B\right).\tag{2.A.17}$$

And finally, consider the term with $W_{\mu\nu}$:

$$-4i\sigma_{\mu\nu}W^{\mu\nu}(\xi^B\lambda_B)\xi_A=-4i\sigma_{\mu\nu}W^{\mu\nu}\left[(\xi^B\xi_B)\lambda_A+(\xi_A\lambda_B+\xi_B\lambda_A)\xi^B\right].\tag{2.A.18}$$

We see that while the second parenthesis can possibly be absorbed into $\mathbf{Q}D_{AB}$, the first term remains and does not fit the desired form for $\mathbf{Q}^2\lambda_A$ which implies that $W_{\mu\nu}=0$. Finally, let us consider $\mathbf{Q}^2D_{AB}$. After a routine calculation, and using the main Killing equation we recover the expression written above, except one failure term:

$$\begin{aligned}
&-i\phi(\bar{\xi}_{(A}\bar{\sigma}^\mu\sigma^\nu D_\mu D_\nu\bar{\xi}_{B)}-4\xi_{(A}\sigma^\mu\bar{\sigma}^{\rho\sigma}\bar{\xi}_{B)}D_\mu\bar{T}_{\rho\sigma})\\
&+i\bar{\phi}(\xi_{(A}\sigma^\mu\bar{\sigma}^\nu D_\mu D_\nu\xi_{B)}-4\bar{\xi}_{(A}\bar{\sigma}^\mu\sigma^{\rho\sigma}\bar{\xi}_{B)}D_\mu T_{\rho\sigma})
\end{aligned}$$

which yield the following two auxiliary equations

$$\begin{aligned} &\sigma^\mu \bar\sigma^\nu D_\mu D_\nu \xi_A + 4 D_\lambda T_{\mu\nu} \sigma^{\mu\nu} \sigma^\lambda \bar\xi_A = M_1 \xi_A, \\ &\bar\sigma^\mu \sigma^\nu D_\mu D_\nu \bar\xi_A + 4 D_\lambda \bar T_{\mu\nu} \bar\sigma^{\mu\nu} \bar\sigma^\lambda \xi_A = M_2 \bar\xi_A. \end{aligned} \tag{2.A.19}$$

We note that there is no reason, at the level of the supersymmetry algebra from the above approach, to have a single scalar background field M.

2.B Generic twisting solutions

It is possible to derive solutions for (2.2.7) (2.2.2) of the same kind of (2.2.37) for a generic four-manifold admitting a $U(1)$ isometry generated by a Killing vector V.

Such a solution will generate the vector V as in (2.2.14).

In this general setting we have to turn on the whole $SU(2)_R$ bundle and the Witten twist (2.2.8) becomes

$$G_{\mu A}{}^{B} = \sum_{k=1}^{3} G_\mu^k \, \sigma^{(k)B}_{A} \tag{2.B.1}$$

with

$$G^1 = -\frac{1}{2}(\omega^{14} + \omega^{23}), \quad G^2 = -\frac{1}{2}(\omega^{13} - \omega^{24}), \quad G^3 = -\frac{1}{2}(\omega^{12} + \omega^{34}), \tag{2.B.2}$$

where ω^{ab} denote the components of the spin connection one-form. This twist admits the following solution for equations (2.2.7) (2.2.2)

$$\xi_1 = \frac{i}{4}\begin{pmatrix} V_3 + iV_4 \\ V_1 + iV_2 \end{pmatrix}, \quad \xi_2 = \frac{i}{4}\begin{pmatrix} V_1 - iV_2 \\ -V_3 + iV_4 \end{pmatrix}, \quad \bar\xi_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \bar\xi_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{2.B.3}$$

(where $V_a = e_a^\mu V_\mu$) and the background fields are chosen as

$$T = -\frac{1}{32}(d\zeta)^-, \quad \bar S = \frac{1}{32}(d\zeta)^+, \quad \bar T = 0, \quad S = 0, \quad M = 0, \tag{2.B.4}$$

where the superscripts $-$ and $+$ denote the anti self-dual and the self-dual part respectively and $\zeta = \star\iota_V \star 1$.

2.C Untwisted solutions

In this appendix we summarize solutions to equations (2.2.1),(2.2.2) that follow from assumptions of the vanishing of $SU(2)_R$ gauge field and the direct product decomposition of ξ_A in terms of conformal Killing spinors on 2-spheres. However, these solutions have the disadvantage of not being real in the sense of equation (2.2.4). We are not using the solutions derived in this appendix in the rest of the Chapter, but we summarize them here for possible future applications.

To solve the equations we consider the following ansatze:

- The $SU(2)_R$ gauge field is zero
$$G_{\mu\,A}^{\ \,B} = 0. \tag{2.C.1}$$
Since this condition implies there is no mixing between different $SU(2)_R$ components, we will drop the indices $A, B, \ldots$ for the remainder of this subsection.

- The background fields $T_{\mu\nu}$, $\bar{T}_{\mu\nu}$ are respectively anti self-dual and self-dual combination of the two-dimensional volume forms in the two sphere $\omega_{(1)}$, $\omega_{(2)}$.
$$\begin{aligned} T &\equiv \frac{1}{2} T_{\mu\nu}\, dx^\mu \wedge dx^\nu = t(\omega_{(1)} - \omega_{(2)}), \\ \bar{T} &\equiv \frac{1}{2} \bar{T}_{\mu\nu}\, dx^\mu \wedge dx^\nu = \bar{t}(\omega_{(1)} + \omega_{(2)}). \end{aligned} \tag{2.C.2}$$
where t and $\bar{t}$ are complex numbers (the bar does not imply that they are complex conjugates).

- The candidate solution ξ is a tensor product of two-dimensional Killing spinors on each sphere
$$\xi = \epsilon_{(1)} \otimes \epsilon_{(2)} \tag{2.C.3}$$
where $\epsilon_{(1)} = \epsilon_{(1)}(\theta_1, \varphi_1)$ and $\epsilon_{(2)} = \epsilon_{(2)}(\theta_2, \varphi_2)$. The spinor on the right hand side, ξ' has an analogous decomposition $\xi' = \epsilon'_{(1)} \otimes \epsilon'_{(2)}$.

Through these assumptions, we intend to decompose (2.2.1) and (2.2.2) into tensor products of equations on either spheres. To do this we use the following representation for the gamma matrices
$$\begin{aligned} &\gamma_1 = \sigma_1 \otimes \mathbb{1}, \qquad \gamma_2 = \sigma_2 \otimes \mathbb{1}, \\ &\gamma_3 = \sigma_3 \otimes \sigma_1, \qquad \gamma_4 = \sigma_3 \otimes \sigma_2, \qquad \gamma_5 = \sigma_3 \otimes \sigma_3 = -\gamma_1\gamma_2\gamma_3\gamma_4. \end{aligned} \tag{2.C.4}$$
Using these facts and the ansatze at the beginning of this section for the background fields and spinors we obtain
$$\begin{aligned} &(D_{(1)}\epsilon_{(1)} \otimes \epsilon_{(2)}) + (\epsilon_{(1)} \otimes D_{(2)}\epsilon_{(2)}) \\ &+ i(\bar{t} + t)\big[(\sigma_3\, \sigma_{(1)}\epsilon_{(1)} \otimes \epsilon_{(2)}) + (\epsilon_{(1)} \otimes \sigma_{(2)}\epsilon_{(2)})\big] \\ &+ i(\bar{t} - t)\big[(\sigma_{(1)}\epsilon_{(1)} \otimes \sigma_3\, \epsilon_{(2)}) + (\sigma_3\, \epsilon_{(1)} \otimes \sigma_3\, \sigma_{(2)}\epsilon_{(2)})\big] \\ &\quad = -i(\sigma_{(1)}\epsilon'_{(1)} \otimes \epsilon'_{(2)}) - i(\sigma_3\, \epsilon'_{(1)} \otimes \sigma_{(2)}\epsilon'_{(2)}) \end{aligned} \tag{2.C.5}$$
where the labels (1) and (2) mean "relative of the first and the second sphere" respectively, and[10]
$$\begin{aligned} D_{(1)} &= d_{(1)} + \frac{1}{2}\Omega^{12}_{(1)}\sigma_{12} & D_{(2)} &= d_{(2)} + \frac{1}{2}\Omega^{34}_{(2)}\sigma_{12} \\ d_{(1)} &= \partial_{\theta_1}\, d\theta_1 + \partial_{\varphi_1}\, d\varphi_1 & d_{(2)} &= \partial_{\theta_2}\, d\theta_2 + \partial_{\varphi_2}\, d\varphi_2 \\ \sigma_{(1)} &= e^1\sigma_1 + e^2\sigma_2 & \sigma_{(2)} &= e^3\sigma_1 + e^4\sigma_2 \\ \Omega^{ab}_{(1)} &= \Omega^{ab}_{\varphi_1}\, d\varphi_1 \quad a, b = 1, 2 & \Omega^{a-2,b-2}_{(2)} &= \Omega^{ab}_{\varphi_2}\, d\varphi_2 \quad a, b = 3, 4. \end{aligned} \tag{2.C.6}$$

[10] In this subsection we use the symbol Ω for the spin connection, this is to avoid confusion with the volume forms.

The vielbein are

$$e^1 = r_1 d\theta_1 \quad e^2 = r_1 \sin\theta_1 d\varphi_1 \quad e^3 = r_2 d\theta_2 \quad e^4 = r_2 \sin\theta_2 d\varphi_2. \tag{2.C.7}$$

The conformal Killing spinors in two dimension for the S^2 metric are already known [21]. These are spanned by the solutions to the following equations

$$D\epsilon_\pm = \pm\frac{i}{2r} e^a \sigma_a \epsilon_\pm. \tag{2.C.8}$$

One can find an alternate basis for the Killing spinors, where the elements of the basis satisfy

$$D\hat{\epsilon}_\pm = \pm\frac{1}{2r} e^a \sigma_a \sigma_3 \hat{\epsilon}_\pm. \tag{2.C.9}$$

The two basis are related by

$$\hat{\epsilon}_\pm = (\mathbb{1} + i\sigma_3)\epsilon_\pm. \tag{2.C.10}$$

Corresponding to each sign, in either of the two equations, there are two linearly independent solutions. For example, the solutions to equation (2.C.8) are given (up to normalization) by

$$\begin{aligned} \epsilon^{+,1} &= e^{-(i/2)\varphi}\begin{pmatrix} \sin\theta/2 \\ -i\cos\theta/2 \end{pmatrix}, \quad \epsilon^{+,2} = e^{(i/2)\varphi}\begin{pmatrix} \cos\theta/2 \\ i\sin\theta/2 \end{pmatrix}, \\ \epsilon^{-,1} &= e^{-(i/2)\varphi}\begin{pmatrix} \sin\theta/2 \\ i\cos\theta/2 \end{pmatrix}, \quad \epsilon^{-,2} = e^{(i/2)\varphi}\begin{pmatrix} \cos\theta/2 \\ -i\sin\theta/2 \end{pmatrix}. \end{aligned} \tag{2.C.11}$$

The linearly independent solutions to (2.C.9) may be found using the above solutions and equation (2.C.10). It must be noted that the sign of ϵ does not indicate its chirality, and indeed solutions of definite "positivity" do not have definite chirality.

We use the existence of these solutions to rewrite (2.C.5) into an algebraic equation. To do so, let us notice that on the left hand side of equation (2.C.5), we have the terms $(D_{(1)}\epsilon_{(1)} \otimes \epsilon_{(2)})$ as well as of the form $(\sigma_3\,\sigma_{(1)}\epsilon_{(1)} \otimes \epsilon_{(2)})$. This suggests that we should take $\epsilon_{(1)}$ to be a solution of equation (2.C.9), and $\epsilon'_{(1)}$ to be proportional to $\sigma_3\epsilon_{(1)}$ (with $r = r_1$). Similarly, because the left hand side of (2.C.5) contains $\epsilon_{(1)} \otimes D_{(2)}\epsilon_{(2)}$ and $\epsilon_{(1)} \otimes \sigma_{(2)}\epsilon_{(2)}$, we are compelled to choose $\epsilon_{(2)}$ as a solution of (2.C.8) (with $r = r_2$) and $\epsilon'_{(2)}$ to be proportional to $\epsilon_{(2)}$. Equation (2.C.5) then decomposes into two algebraic equations for Killing spinors on either spheres if we take the coefficient $(\bar{t} - t) = 0$ (as the terms with this coefficient do not conform to the pattern of the other terms and to equations (2.C.8) and (2.C.9)).

Incorporating these observations in (2.C.5), we get:

$$\begin{aligned} &(D_{(1)}\epsilon_{(1)} \otimes \epsilon_{(2)}) + (\epsilon_{(1)} \otimes D_{(2)}\epsilon_{(2)}) \\ &+ 2it\big[(\sigma_3\,\sigma_{(1)}\epsilon_{(1)} \otimes \epsilon_{(2)}) + (\epsilon_{(1)} \otimes \sigma_{(2)}\epsilon_{(2)})\big] \\ &= -iC\big[(\sigma_{(1)}\sigma_3\epsilon_{(1)} \otimes \epsilon_{(2)}) - i(\epsilon_{(1)} \otimes \sigma_{(2)}\epsilon_{(2)})\big] \end{aligned} \tag{2.C.12}$$

where C is a proportionality constant; $\epsilon_{(1)}$ and $\epsilon_{(2)}$ are solutions of (2.C.9) and (2.C.8) respectively. It is obvious that up on using equations (2.C.9) and (2.C.8) we are left with purely algebraic equations that can be easily solved for t and C in terms of r_1 and r_2. We have four families of solutions in all, corresponding to four choices of signs that can be made. The solutions can be summarized as:

$$\xi = \hat{\epsilon}^{\pm}_{(1)} \otimes \epsilon^{\pm'}_{(2)}; \qquad C = \frac{1}{4}\left(\pm\frac{i}{r_1} \mp' \frac{1}{r_2}\right); \qquad t = \bar{t} = \frac{1}{16}\left(\mp\frac{i}{r_1} \pm' \frac{1}{r_2}\right). \quad (2.C.13)$$

The auxiliary equation (2.2.2) may be decomposed in a similar manner. It turns out that the value of the scalar background field M is the same for all four families of solutions and is given by

$$M = -\left(\frac{1}{r_1^2} + \frac{1}{r_2^2}\right). \quad (2.C.14)$$

Note that since for each choice of sign in either equations (2.C.9) or (2.C.8), we have a 2-complex dimensional family of solutions, each family of solutions is the complex span of four linearly independent spinors.

Finally we would like to return to the standard Clifford algebra representation

$$\gamma_a = \begin{pmatrix} 0 & \sigma_a \\ \bar{\sigma}_a & 0 \end{pmatrix} \qquad a = 1, \dots, 4. \quad (2.C.15)$$

To do that we use the following unitary transformation

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \end{pmatrix} \quad (2.C.16)$$

We may then use two of the four solutions in any one given family, and put them in an $SU(2)_R$ doublet.

2.D Conventions

2.D.1 Notation

Latin indices $\{a, b, \dots\}$ are used for flat space coordinates, and are used for both real coordinates $a, b = 1, 2, 3, 4$ and complex coordinates $a, b = 1, \bar{1}, 2, \bar{2}$. Greek indices $\{\mu, \nu, \dots\}$ are used for curved space coordinates, real $\mu, \nu = \theta_1, \varphi_1, \theta_2, \varphi_2$ or complex $\mu, \nu = z, \bar{z}, w, \bar{w}$. Any ambiguities in this notation should be clarified from the context.

The metric in the flat space δ_{ab} is link with the metric in curved space $g_{\mu\nu}$ via the vierbein e^a_μ

$$g_{\mu\nu} = e^a_\mu e^b_\nu \delta_{ab}. \quad (2.D.1)$$

2.D.2 Metrics

The metric of $S^2 \times S^2$ in real coordinates is

$$\begin{aligned} ds^2 &= g_{\mu\nu}dx^\mu dx^\nu = \delta_{ab}e^a e^b \\ &= r_1^2(d\theta_1^2 + \sin^2\theta_1 d\varphi_1^2) + r_2^2(d\theta_2{}^2 + \sin^2\theta_2 d\varphi_2{}^2). \end{aligned} \tag{2.D.2}$$

Therefore the vierbein 1-forms $e^a = e^a_\mu dx^\mu$ are

$$e^1 = r_1 d\theta_1, \quad e^2 = r_1 \sin\theta_1 d\varphi_1, \quad e^3 = r_2 d\theta_2, \quad e^4 = r_2 \sin\theta_2 d\varphi_2. \tag{2.D.3}$$

As a complex manifold ($\mathbb{P}^1 \times \mathbb{P}^1$) the metric is written as two copies of the Fubini-Study metric

$$\begin{aligned} ds^2 &= 2g_{z\bar{z}}(z,\bar{z})dzd\bar{z} + 2g_{w\bar{w}}(w,\bar{w})dwd\bar{w} \\ &= 4r_1^2\frac{dzd\bar{z}}{(1+|z|^2)^2} + 4r_2^2\frac{dwd\bar{w}}{(1+|w|^2)^2}, \end{aligned} \tag{2.D.4}$$

The change of variables from real to complex coordinates is

$$z = \tan(\theta_1/2)e^{i\varphi_1}, \qquad w = \tan(\theta_2/2)e^{i\varphi_2}. \tag{2.D.5}$$

The flat metric in complex coordinate has the following nonzero components $\delta_{1\bar{1}} = \delta_{2\bar{2}} = \frac{1}{2}$, $\delta^{1\bar{1}} = \delta^{2\bar{2}} = 2$. Then defining $\sqrt{g_1} := 2g_{z\bar{z}}$ and $\sqrt{g_2} := 2g_{w\bar{w}}$ we can write rewrite the metric (2.D.4) using complex vierbein 1-forms

$$ds^2 = e^1 e^{\bar{1}} + e^2 e^{\bar{2}} \tag{2.D.6}$$

where

$$e^1 = g_1^{1/4}dz, \quad e^{\bar{1}} = g_1^{1/4}d\bar{z}, \quad e^2 = g_2^{1/4}dw, \quad e^{\bar{2}} = g_2^{1/4}d\bar{w}. \tag{2.D.7}$$

Moreover we can write the non-zero Christoffel symbols of the Levi-Civita connection as

$$\begin{aligned} \Gamma^z_{zz} &= \frac{1}{2}\partial_z \log g_1, \quad \Gamma^{\bar{z}}_{\bar{z}\bar{z}} = \frac{1}{2}\partial_{\bar{z}} \log g_1, \\ \Gamma^w_{ww} &= \frac{1}{2}\partial_w \log g_2, \quad \Gamma^{\bar{w}}_{\bar{w}\bar{w}} = \frac{1}{2}\partial_{\bar{w}} \log g_2, \end{aligned} \tag{2.D.8}$$

and the relative spin connection[11] $\omega_\mu := -2i\omega_{\mu 1\bar{1}}$, $\omega'_\mu := -2i\omega_{\mu 2\bar{2}}$ as

$$\begin{aligned} \omega_z &= \frac{i}{4}\partial_z \log g_1, \quad \omega_{\bar{z}} = -\frac{i}{4}\partial_{\bar{z}} \log g_1, \\ \omega'_w &= \frac{i}{4}\partial_w \log g_2, \quad \omega'_{\bar{w}} = -\frac{i}{4}\partial_{\bar{w}} \log g_2. \end{aligned} \tag{2.D.9}$$

The explicit expression for the non zero components of the spin connection in complex coordinates are

$$\omega_z = -i\frac{\bar{z}}{1+|z|^2}, \quad \omega_{\bar{z}} = i\frac{z}{1+|z|^2}, \quad \omega'_w = -i\frac{\bar{w}}{1+|w|^2}, \quad \omega'_{\bar{w}} = i\frac{w}{1+|w|^2}. \tag{2.D.10}$$

[11] The convention for the spin connection is $\omega^{ab}_\mu = e^{[a}_\nu \partial_\mu e^{b]\nu} + e^{[a}_\nu e^{b]\rho}\Gamma^\nu_{\mu\rho}$.

If one prefers to work in polar coordinates, the spin connection ($\omega_\mu = \omega_{\mu 12}$, $\omega'_\mu = \omega_{\mu 34}$) is

$$\omega_{\theta_1} = \omega'_{\theta_2} = 0, \quad \omega_{\varphi_1} = -\cos\theta_1, \quad \omega'_{\varphi_2} = -\cos\theta_2. \tag{2.D.11}$$

The Riemann tensor $R_{\mu\nu\rho\sigma}$ has two independent components

$$R_{z\bar{z}z\bar{z}} = -\frac{1}{2} g_{z\bar{z}} g_{z\bar{z}} \mathcal{R}_1, \qquad R_{w\bar{w}w\bar{w}} = -\frac{1}{2} g_{w\bar{w}} g_{w\bar{w}} \mathcal{R}_2, \tag{2.D.12}$$

where $\mathcal{R}_1$ and $\mathcal{R}_2$ are the summands of the Riemann scalar $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$ related respectively to the first and second sphere, which are expressed as

$$\mathcal{R}_1 = -\frac{2}{\sqrt{g_1}} \partial_z \partial_{\bar{z}} \log g_1 = \frac{2}{r_1^2}, \qquad \mathcal{R}_2 = -\frac{2}{\sqrt{g_2}} \partial_w \partial_{\bar{w}} \log g_2 = \frac{2}{r_2^2}. \tag{2.D.13}$$

Finally using the spin connection is possible to write the action of the covariant derivative on spinors

$$\nabla_\mu \psi = (\partial_\mu + \frac{i}{2}(\omega_\mu - \omega'_\mu)\sigma_3)\psi, \qquad \nabla_\mu \bar{\psi} = (\partial_\mu + \frac{i}{2}(\omega_\mu + \omega'_\mu)\sigma_3)\bar{\psi}. \tag{2.D.14}$$

And on 1-forms

$$\nabla_\mu X_a = e_a^\nu \nabla_\mu X_\nu \tag{2.D.15}$$

where ∇_μ on the r.h.s. is the Levi-Civita connection.

2.D.3 Spinor convention

Left and right chirality spinors are denoted $\xi_{A\alpha}$ and $\bar{\xi}_A{}^{\dot{\alpha}}$. The multiplication of spinors is usually implicit as $\xi^A \xi_A = \xi^{A\alpha}\xi_{A\alpha} = \epsilon^{AB}\epsilon^{\alpha\beta}\xi_{B\beta}\xi_{A\alpha}$ and $\bar{\xi}_A \bar{\xi}^A = \bar{\xi}_{A\dot{\alpha}}\bar{\xi}^{A\dot{\alpha}} = \epsilon^{AB}\epsilon_{\dot{\alpha}\dot{\beta}}\bar{\xi}_A{}^{\dot{\beta}}\bar{\xi}_B{}^{\dot{\alpha}}$. The invariant antisymmetric tensors are $\epsilon^{\alpha\beta}$, $\epsilon_{\alpha\beta}$ for left chirality spinors, $\epsilon^{\dot{\alpha}\dot{\beta}}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$ for right chirality ones, with $\epsilon^{12} = 1$ $\epsilon_{12} = -1$. Our choice for the two set of matrices $(\sigma_a)_{\alpha\dot{\alpha}}$, $(\bar{\sigma}_a)^{\dot{\alpha}\alpha}$ $(a = 1, 2, 3, 4)$ is

$$\begin{aligned} \sigma_a &= \{-i\sigma_j, \mathbb{1}\}, \quad j = 1, 2, 3, \\ \bar{\sigma}_a &= \{+i\sigma_j, \mathbb{1}\}, \quad j = 1, 2, 3, \end{aligned} \tag{2.D.16}$$

where σ_j are the Pauli matrices. Their expression with curved indices is derived using vierbein:

$$\sigma_\mu = e_\mu^a \sigma_a, \qquad \bar{\sigma}_\mu = e_\mu^a \bar{\sigma}_a. \tag{2.D.17}$$

The generators of rotations for left and right chirality spinors are respectively

$$\sigma_{ab} = \frac{1}{2}(\sigma_a\bar{\sigma}_b - \sigma_b\bar{\sigma}_a), \qquad \bar{\sigma}_{ab} = \frac{1}{2}(\bar{\sigma}_a\sigma_b - \bar{\sigma}_b\sigma_a). \tag{2.D.18}$$

Note that σ_{ab} is anti self-dual and $\bar{\sigma}_{ab}$ is self-dual.

Some useful identities of the sigma matrices are

$$\begin{aligned}
&\sigma_a \bar{\sigma}_b + \sigma_b \bar{\sigma}_a = 2\delta_{ab}, \\
&\bar{\sigma}_a \sigma_b + \bar{\sigma}_b \sigma_a = 2\delta_{ab}, \\
&\sigma_a \bar{\sigma}_b \sigma_c = \delta_{ab}\sigma_c + \delta_{bc}\sigma_a - \delta_{ac}\sigma_b + \epsilon_{abcd}\sigma^d, \\
&\bar{\sigma}_a \sigma_b \bar{\sigma}_c = \delta_{ab}\bar{\sigma}_c + \delta_{bc}\bar{\sigma}_a - \delta_{ac}\bar{\sigma}_b - \epsilon_{abcd}\bar{\sigma}^d.
\end{aligned} \tag{2.D.19}$$

The last two identities imply

$$\sigma_{ab}\sigma_c = -4P^-_{abcd}\sigma^d, \qquad \bar{\sigma}_{ab}\bar{\sigma}_c = -4P^+_{abcd}\bar{\sigma}^d, \tag{2.D.20}$$

where P^- and P^+ are given by

$$P^{\pm}_{abcd} = \frac{1}{4}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc} \pm \epsilon_{abcd}). \tag{2.D.21}$$

and are projectors on the anti-self dual and self dual forms respectively.

Others useful identities are

$$\begin{aligned}
&(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}(\sigma^\mu)_{\beta\dot{\beta}} = 2\delta^\alpha_\beta \delta^{\dot{\alpha}}_{\dot{\beta}}, \\
&(\sigma_\mu)_{\alpha\dot{\alpha}}(\sigma^\mu)_{\beta\dot{\beta}} = 2\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}, \\
&(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}(\bar{\sigma}^\mu)^{\dot{\beta}\beta} = 2\epsilon^{\alpha\beta}\epsilon^{\dot{\alpha}\dot{\beta}}.
\end{aligned} \tag{2.D.22}$$

2.E Special functions

The Barnes' double zeta function ζ_2 has the following integral representation:

$$\zeta_2(x; s|\epsilon_1, \epsilon_2) = \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{-tx}}{(1 - e^{-\epsilon_1 t})(1 - e^{-\epsilon_2 t})}. \tag{2.E.1}$$

This integral is well-defined if $\mathrm{Re}\,\epsilon_1 > 0$, $\mathrm{Re}\,\epsilon_2 > 0$, $\mathrm{Re}\,x > 0$ and can be analytically continued to all complex values of ϵ_1 and ϵ_2 except when $\frac{\epsilon_1}{\epsilon_2} \neq a$ with $a \in \mathbb{R}_{<0}$. The following series expansion of ζ_2 for $\epsilon_1 > 0$, $\epsilon_2 > 0$ is convergent if $\mathrm{Re}\,s > 2$:

$$\zeta_2(x; s|\epsilon_1, \epsilon_2) = \sum_{m,n \geq 0} (x + m\epsilon_1 + n\epsilon_2)^{-s}. \tag{2.E.2}$$

The Barnes' double Gamma function Γ_2, defined by

$$\log \Gamma_2(x|\epsilon_1, \epsilon_2) = \frac{d}{ds}\Big|_{s=0} \zeta_2(x; s|\epsilon_1, \epsilon_2), \tag{2.E.3}$$

is analytic in x except at the poles at $x = -m\epsilon_1 - n\epsilon_2$ with $m, n \in \mathbb{Z}$. Define

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \frac{d}{ds}\Big|_{s=0} \frac{1}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \frac{e^{-tx}}{(1 - e^{\epsilon_1 t})(1 - e^{\epsilon_2 t})}. \tag{2.E.4}$$

Then

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \log \Gamma_2(x|-\epsilon_1, -\epsilon_2). \tag{2.E.5}$$

The function Γ_2 has the following infinite-product representations:

$$\Gamma_2(x|\epsilon_1,\epsilon_2) = \begin{cases} \prod_{m,n\geq 0} (x + m\epsilon_1 + n\epsilon_2)^{-1} & \text{if } \epsilon_1 > 0,\ \epsilon_2 > 0, \\ \prod_{m,n\geq 0} (x + m\epsilon_1 - (n-1)\epsilon_2) & \text{if } \epsilon_1 > 0,\ \epsilon_2 < 0, \\ \prod_{m,n\geq 0} (x - (m-1)\epsilon_1 + n\epsilon_2) & \text{if } \epsilon_1 < 0,\ \epsilon_2 > 0, \\ \prod_{m,n\geq 0} (x - (m-1)\epsilon_1 - (n-1)\epsilon_2)^{-1} & \text{if } \epsilon_1 < 0,\ \epsilon_2 < 0. \end{cases} \tag{2.E.6}$$

The function Γ_2 satisfies the following multiplicative identity

$$\Gamma_2(x+\epsilon_1|\epsilon_1,\epsilon_2)\Gamma_2(x+\epsilon_2|\epsilon_1,\epsilon_2) = x\Gamma_2(x|\epsilon_1,\epsilon_2)\Gamma_2(x+Q|\epsilon_1,\epsilon_2) \tag{2.E.7}$$

where $Q = \epsilon_1 + \epsilon_2$, the shift identities

$$\begin{aligned} \Gamma_2(x+\epsilon_1|\epsilon_1,\epsilon_2) &= \frac{\sqrt{2\pi}\epsilon_2^{1/2-x/\epsilon_2}}{\Gamma(x/\epsilon_2)}\Gamma_2(x|\epsilon_1,\epsilon_2), \\ \Gamma_2(x+\epsilon_2|\epsilon_1,\epsilon_2) &= \frac{\sqrt{2\pi}\epsilon_1^{1/2-x/\epsilon_1}}{\Gamma(x/\epsilon_1)}\Gamma_2(x|\epsilon_1,\epsilon_2). \end{aligned} \tag{2.E.8}$$

The Upsilon function is defined as

$$\Upsilon_{\epsilon_1,\epsilon_2}(x) = \frac{1}{\Gamma_2(x|\epsilon_1,\epsilon_2)\Gamma_2(Q-x|\epsilon_1,\epsilon_2)} = \Upsilon_{\epsilon_1,\epsilon_2}(Q-x) \tag{2.E.9}$$

It exhibits the shift property

$$\begin{aligned} \Upsilon_{\epsilon_1,\epsilon_2}(x+\epsilon_1) &= \epsilon_2^{2x/\epsilon_2-1}\gamma(x/\epsilon_2)\Upsilon_{\epsilon_1,\epsilon_2}(x), \\ \Upsilon_{\epsilon_1,\epsilon_2}(x+\epsilon_2) &= \epsilon_1^{2x/\epsilon_1-1}\gamma(x/\epsilon_1)\Upsilon_{\epsilon_1,\epsilon_2}(x), \end{aligned} \tag{2.E.10}$$

where

$$\gamma(x) = \frac{\Gamma(x)}{\Gamma(1-x)}. \tag{2.E.11}$$

Chapter 3

Supersymmetric theory on $\mathbb{P}^2$ and Donaldson invariants.

In this Chapter we generalize the result of Chapter 2 to general complex compact toric surfaces.

Using localization we show that the path integral is again governed by point-like instantons sitting at the fixed points of the toric action. The contribution related to each of these fixed points is given by a copy of the Nekrasov partition function on the tangent space to the point, that is a copy of the Omega-background with equivariant parameters determined by the fan of the toric manifold. Some of these parameters, called a_ρ, describe the change of framing at infinity of the point-like instanton, they correspond to the v.e.v. of the classical solution of the supersymmetric minima equation for the scalar field Φ and, as such, they have to be integrated over.

We work out this integration for $U(2)$ theories on the complex projective plane $\mathbb{P}^2$ by specifying the contour and by spelling out the conditions imposed on the fixed point data by the stability conditions on the equivariant sheaves.

We treat separately the case of odd and even first Chern class. In the first case we show that the partition function with the insertion of local and surface observables reproduce the generating function of equivariant Donaldson invariants obtained in [75]. In the second case we obtain a similar formula that we conjecture to be the generating function of equivariant Donaldson invariants in the presence of reducible connections. Since this last case has not been calculated in [75], we match the non-equivariant limit of our result with the $SU(2)$ ordinary Donaldson polynomials computed in [66]. This result gives a robust confirmation of a conjecture made by Nekrasov [120].

Finally we consider $\mathcal{N} = 2^*$ gauge theory, that is Super-Yang-Mills theory in presence of a hypermultiplet of mass M in the adjoint representation. This theory interpolates between pure $\mathcal{N} = 2$ in the decoupling limit $M \to \infty$ and $\mathcal{N} = 4$ for $M \to 0$. In the latter case the partition function is expected to be the generating function of the Euler characteristic of the moduli spaces of *unframed* sheaves. We check this for $U(2)$ gauge theories on $\mathbb{P}^2$. For odd first Chern class we get results in agreement with [102], and for even first Chern class we compare with the results obtained by Yoshioka using finite field methods [152, 142].

3.1 $\mathcal{N}=2$ theory on complex surfaces and Hermitian Yang Mills bundles

In this section we discuss $U(N)$ $\mathcal{N}=2$ gauge theories on complex surfaces and specify the results of [17] to toric surfaces.

Four dimensional $\mathcal{N}=2$ gauge theories can be considered on any orientable four manifold M upon a proper choice of the $\mathcal{R}$-symmetry bundle [144]. The sum over the physical vacua contributing to the supersymmetric path-integral depends of course on the specific gauge group at hand. In the case of $SU(N)$ gauge theories, these are completely described in terms of anti-selfdual connections $F^+=0$, once the orientation on M is chosen. In the $U(N)$ case extra contributions arise from gauge bundles with non trivial first Chern class. Indeed, beyond anti-instantons, one has to consider gauge bundles with first Chern class aligned along $H^+(X,\mathbb{Z})$. This led in [17] to consider the gauge fixing of the supersymmetric path-integral in a split form, where the $U(1)$ sector is treated separately. If M is an hermitian manifold , an equivalent procedure is given by gauge fixing the path-integral to Hermitian-Yang-Mills (HYM) connections

$$\begin{aligned} &F^{(2,0)}=0\\ &g^{i\bar{j}}F_{i\bar{j}}=\lambda\mathbb{1} \end{aligned} \tag{3.1.1}$$

where $F^{(2,0)}$ is the $(2,0)$ component of the gauge curvature in a given complex structure, g is the hermitian metric on M and λ is a real parameter.

If the manifold M is Kähler, then (3.1.1) reads

$$\begin{aligned} &F^{(2,0)}=0\\ &\omega\wedge F=\lambda\,\omega\wedge\omega\mathbb{1} \end{aligned} \tag{3.1.2}$$

where $\lambda=\frac{2\pi\int_M c_1(E)\wedge\omega}{r(E)\int_M\omega\wedge\omega}=\frac{2\pi\mu(E)}{\int_M\omega\wedge\omega}$ and $\mu(E)$ is the *slope* of the vector bundle. Here $r(E)=N$ is the rank of E and $c_1(E)=\frac{1}{2\pi}\mathrm{Tr}F_E$ its first Chern class.

In the rest of the Chapter we consider Kähler four manifolds admitting a $U(1)$ action with isolated fixed points. In this case, as shown in [17], one can improve the supersymmetric localization technique by making it equivariant with respect to such a $U(1)$ action and localize on point-like instantons. The resulting partition function is obtained by a suitable gluing of Nekrasov partition functions which includes the sum over fluxes and the integration over the Coulomb parameters.

In the twisted variables, the supersymmetry reads as

$$\begin{aligned} &\mathcal{Q}A=\Psi, &&\mathcal{Q}\Psi=i\iota_V F+D\Phi, &&\mathcal{Q}\Phi=i\iota_V\Psi,\\ &\mathcal{Q}\bar{\Phi}=\eta, &&\mathcal{Q}\eta=i\,\iota_V D\bar{\Phi}+i[\Phi,\bar{\Phi}], &&\\ &\mathcal{Q}\chi^+=B^+, &&\mathcal{Q}B^+=i\mathcal{L}_V\chi^++i[\Phi,\chi^+]. && \end{aligned} \tag{3.1.3}$$

In (3.1.3) ι_V is the contraction with the vector field V and $\mathcal{L}_V=D\iota_V+\iota_V D$ is the covariant Lie derivative. On a Kähler four manifold self-dual forms split as

$$\chi^+=\chi^{(2,0)}\oplus\chi^{(0,2)}\oplus\chi\,\omega \ \text{ and } \ B^+=B^{(2,0)}\oplus B^{(0,2)}\oplus b\,\omega. \tag{3.1.4}$$

Let us notice that the supercharge (3.1.3) manifestly satisfies $\mathcal{Q}^2 = i\mathcal{L}_V + \delta_\Phi^{\text{gauge}}$. Consistency of the last line implies that the V-action preserves the self-duality of B^+ and χ^+, that is $L_V\star = \star L_V$, where $\star$ is the Hodge-$\star$ and $L_V = d\iota_V + \iota_V d$ is the Lie derivative. This condition coincides with the requirement that V generates an isometry of the four manifold.

The supersymmetric Lagrangian we consider is

$$L = \frac{i\tau}{4\pi}\Big(\operatorname{Tr} F\wedge F - c\operatorname{Tr} F\wedge \operatorname{Tr} F\Big) + \gamma\wedge\operatorname{Tr} F + \mathcal{Q}\mathcal{V} \tag{3.1.5}$$

where c is a constant[1], τ is the complexified coupling constant, $\gamma\in H^2(M)$ is the source for the c_1 of the vector bundle and $\mathcal{V}$ is a gauge invariant localizing term, chosen in order to implement the Hermitean-Yang-Mills equations, namely

$$\mathcal{V} = -\operatorname{Tr}\big[i\chi^{(0,2)}\wedge F^{(2,0)} + i\chi\,(\omega\wedge F - \lambda\,\omega\wedge\omega\mathbb{1}) + \Psi\wedge\star(\mathcal{Q}\Psi)^\dagger + \eta\wedge\star(\mathcal{Q}\eta)^\dagger\big]. \tag{3.1.6}$$

The integration over $B^{(0,2)}$ and b in (3.1.5) implies the Hermitean Yang-Mills equations (3.1.2) as delta-gauge conditions. In particular, the path integral over the field b ensures the semi-stability of the bundle[2]. Recall that [101] a bundle E is said to be (slope) semistable if for every proper sub-bundle $G\subset E$, the slope of the bundle $\mu(E)$, defined below (3.1.2), is greater or equal than the slope of the sub-bundle $\mu(G)$. If it is stricly greater E is said to be stable. If the bundle E admits a sub-bundle G, then the b field has an integration mode proportional to the projector onto G, namely $ib_0\Pi_G$. The connection splits as

$$A_E = \begin{pmatrix} A_G & n \\ n^\dagger & \star \end{pmatrix} \tag{3.1.7}$$

and the curvature accordingly as

$$F_E = \begin{pmatrix} F_G + n\wedge n^\dagger & \star \\ \star & \star \end{pmatrix}. \tag{3.1.8}$$

Let us focus on the integral along the above integration mode. The corresponding term in the action comes from

$$\int_M \operatorname{Tr}\left[b\,(\omega\wedge F_E - \lambda\,\omega\wedge\omega\mathbb{1}_E)\right] \tag{3.1.9}$$

and reads

$$ib_0\int_M \operatorname{Tr}\left[\Pi_G\,(\omega\wedge F_E - \lambda\,\omega\wedge\omega\mathbb{1}_E)\right] = ib_0\left[2\pi r(G)\,(\mu(G)-\mu(E)) + \int_M |n|^2\right] \tag{3.1.10}$$

[1] Different values of c in (3.1.5) produce different expansion in the final formula. The usual choice is $c=0$, which produces an expansion in the instaton number, or equivalently in the second Chern character $-ch_2 = c_2^2 - \frac{1}{2}c_1^2$ of the bundle. The choice $c=1$ produces an expansion in the second Chern class c_2 and the choice $c=\frac{1}{2}$ produces an expansion on the discriminant D of the bundle. In comparing our result with the literature we will use the last two choices.

[2] The semi-stability of the bundle and HYM condition are actually equivalent. This is the so called Hitchin-Kobayashi correspondence, that was proven in [54, 139, 140].

Therefore the path integral includes the term

$$\int db_0 e^{ib_0\left[2\pi r(G)(\mu(G)-\mu(E))+\int_M |n|^2\right]} \sim \delta\left(2\pi r(G)\left(\mu(G)-\mu(E)\right)+\int_M |n|^2\right) \quad (3.1.11)$$

which, because of $\int_M |n|^2 \geq 0$, implies that the partition function is supported on vector bundles E such that

$$\mu(E) \geq \mu(G) \quad (3.1.12)$$

for any sub-bundle G, that is on semi-stable vector bundles. Notice that this condition depends on the point in the Kähler cone defining the polarization ω.

3.1.1 Equivariant observables

In this subsection we discuss equivariant observables in the topologically twisted gauge theory. These are obtained by the equivariant version of the usual descent equations.

The scalar supercharge action can be written as the equivariant Bianchi identity for the curvature $\mathbf{F}$ of the universal bundle as [16]

$$\mathbf{DF} \equiv \left(-Q+D+i\iota_V\right)\left(F+\psi+\Phi\right)=0, \quad (3.1.13)$$

where D is the covariant derivative. Therefore, for any given ad-invariant polynomial $\mathcal{P}$ on the Lie algebra of the gauge group, we have

$$Q\mathcal{P}(\mathbf{F}) = \left(d+i\iota_V\right)\mathcal{P}(\mathbf{F}) \quad (3.1.14)$$

and the observables are obtained by intersection of the above with elements of the equivariant cohomology of the manifold, $\mathbf{\Omega} \in H_V^\bullet(M)$ as

$$\mathcal{O}\left(\mathbf{\Omega},\mathcal{P}\right) \equiv \int \mathbf{\Omega} \wedge \mathcal{P}(\mathbf{F}). \quad (3.1.15)$$

As far as the $U(N)$ gauge theory is concerned, we can consider the basis of single trace observables $\mathcal{P}_n(x) = \frac{1}{n}\operatorname{Tr} x^n$ with $n = 1,\ldots N$.

The equivariant cohomology splits in even and odd parts which can be discussed separately. We focus on the relevant observables corresponding to the even cohomology. The two cases to discuss in the $U(2)$ theory are $n = 1, 2$. The first $\int_M \operatorname{Tr}\mathbf{F}\wedge\Omega$ is the source term for the first Chern class and for the local observable $\operatorname{Tr}\Phi(P)$, where P is a fixed point of the vector field V. The second is

$$\frac{1}{2}\int_M \Omega^{[\text{even}]} \wedge \operatorname{Tr}\mathbf{F}^2 \quad (3.1.16)$$

This generates

- the second Chern character of the gauge bundle $\int_M \operatorname{Tr}(F\wedge F)$ for $\mathbf{\Omega} = 1$ (the Poincaré dual of M),

- surface observables for $\mathbf{\Omega} = \omega + H$, where ω is a V-equivariant element in $H^2(M)$ and H a linear polynomial in the weights of the V-action satisfying $dH = \iota_V \omega$. Namely
$$\int_M \omega \wedge \mathrm{Tr}\left(\Phi F + \Psi^2\right) + H\,\mathrm{Tr}(F \wedge F) \tag{3.1.17}$$
- for $\mathbf{\Omega} = (\omega + H) \wedge (\omega' + H') + K$, with $\omega + H$ and $\omega' + H'$ as in the previous item and K a quadratic, coordinate independent, polynomial in the weights of the V-action, we get
$$\int_M \omega \wedge \omega'\,\mathrm{Tr}\,\Phi^2 + (\omega H' + H'\omega) \wedge \mathrm{Tr}\left(\Phi F + \frac{1}{2}\Psi^2\right) + (HH' + K)\,\mathrm{Tr}(F \wedge F) \tag{3.1.18}$$
- local observables at the fixed points $\mathrm{Tr}\,\Phi^2(P)$, for $\mathbf{\Omega} = \delta_P$ the Poincaré dual of any fixed point P under the V-action.

Let us remark that local observables in the equivariant case depend on the insertion point via the equivariant weights of the fixed point. This is due to the fact that the equivariant classes of different fixed points are distinct. From the gauge theory viewpoint one has
$$\mathrm{Tr}\,\Phi^2(P) - \mathrm{Tr}\,\Phi^2(P') = \int_{P'}^{P} \iota_V\,\mathrm{Tr}\left(\Phi F + \frac{1}{2}\Psi^2\right) + Q[\ldots] \tag{3.1.19}$$
so that the standard argument of point location independence is flawed by the first term in the r.h.s.

Indeed the set of equivariant observables is richer than the set of non-equivariant ones. Also the observables in (3.1.18) reduce in the non equivariant limit to local observables up to a volume factor.

The mathematical meaning of these facts is that the equivariant Donaldson polynomials give a finer characterization of differentiable manifolds. The physical one is that the Ω-background probes the gauge theory via a finer BPS structure.

3.1.2 Gluino zero modes and contour integral prescription

An issue that we have not analyzed till now is the existence of gluino zero modes and its consequences in the evaluation of the path integral.

The fermionic fields are the scalar η, the 1-form Ψ and the selfdual 2-form χ^+. The number of zero modes is given by the respective Betti numbers $b_0 = 1$, $b_1 = 0$ and $b_2^+ = 1$ times the rank of the gauge group[3]. Specifically, the χ^+ zero mode is proportional to the Kähler form ω.

The discussion on the integration on the zero-modes for the complete $U(N)$ theory is naturally split in the $U(1)$ sector and the $SU(N)$ sector. Actually, the two sectors are different in nature. The first is related to a global symmetry of the theory while the second to the structure of the moduli space at the fixed points of the supercharge of the microscopic theory.

[3]We remind the reader that $b_2^+ = 1$ for all toric surfaces.

The zero modes in the $U(1)$ sector

The zero modes in the $U(1)$ sector come as a quartet of symmetry parameters of the whole twisted super-algebra. The c-number BRST charge implementing this shift symmetry is given by

$$\begin{array}{llll} \mathsf{q}A = 0, & \mathsf{q}\Psi = 0, & \mathsf{q}\Phi = \kappa_\Phi \mathbb{1}, & \mathsf{q}\kappa_\Phi = 0, \\ \mathsf{q}\bar{\Phi} = \kappa_{\bar{\Phi}} \mathbb{1}, & \mathsf{q}\kappa_{\bar{\Phi}} = 0, & \mathsf{q}\eta = \kappa_\eta \mathbb{1}, & \mathsf{q}\kappa_\eta = 0, \\ \mathsf{q}\chi = \kappa_\chi \omega \mathbb{1}, & \mathsf{q}\kappa_\chi = 0, & \mathsf{q}B = 0, & \end{array} \tag{3.1.20}$$

and the action of $\mathcal{Q}$ on the c-number parameters above is given by

$$\mathcal{Q}\kappa_\Phi = 0, \quad \mathcal{Q}\kappa_{\bar{\Phi}} = -\kappa_\eta, \quad \mathcal{Q}\kappa_\eta = 0, \quad \mathcal{Q}\kappa_\chi = 0, \tag{3.1.21}$$

so that $\{\mathcal{Q}, \mathsf{q}\} = 0$. The κ-ghosts have to be supplemented by their corresponding anti-ghosts $\bar{\kappa}_I$ and Lagrange multipliers λ_I, with $I \in \left\{\Phi, \bar{\Phi}, \eta, \chi\right\}$ and $\mathsf{q}\bar{\kappa}_I = \lambda_I$ and $\mathsf{q}\lambda_I = 0$. It is needless to say that $\mathcal{Q}\bar{\kappa}_I = 0$ and $\mathcal{Q}\lambda_I = 0$.

Notice that $\mathsf{q}\mathcal{V} = 0$. The gauge fixing fermion for the $U(1)$ zero modes then reads

$$\nu = \sum_I \bar{\kappa}_I \int_M \mathrm{Tr}(I) e^\omega \tag{3.1.22}$$

so that the gauge fixing action $(\mathcal{Q} + \mathsf{q})\nu$ gives a suitable measure to integrate out these modes as a perfect quartet.

The only $U(1)$ zero mode who survives is that of the B field which is still playing as a Lagrange multiplier for the HYM equations.

Zero modes in the $SU(N)$ sector and integration contour prescription

In this subsection we show that by correctly treating the issue of gaugino zero modes in the $SU(N)$ sector we get precise instructions about the integration on the leftover $N-1$ Cartan parameters $a_\rho = a_\alpha - a_\beta$.

The presence of gaugino zero modes implies a ghost number anomaly that has to be compensated by the insertion of appropriate supersymmetric terms which cancel the ghost number excess and soak-up the fermionic zero modes. The path integral as it stands is indeed undefined and its measure has to be improved. In order to do this we add to the localizing action the further term

$$S_{\text{gauginos}} = s\mathcal{Q} \int_M \mathrm{Tr}\, \bar{\Phi}_0 \chi_0 \omega = s \int_M \mathrm{Tr} \left\{ \eta_0 \chi_0 \omega + \bar{\Phi}_0 b_0 \omega \right\} . \tag{3.1.23}$$

where s is a complex parameter and only the zero modes of the fields enter. The final result does not depend on the actual value of s as long as $s \neq 0$. The first term in the r.h.s. of (3.1.23) contributes to the ghost number anomaly by one insertion per element in the Cartan subalgebra of $su(N)$. Once the integral over the $N-1$ couples of gluino zero modes (η_0, χ_0) is taken, we stay with an insertion of b-field zero mode per $su(N)$ Cartan element as

$$\prod_\rho \left(\int da\, d\bar{a}\, db_0\, (s\omega)\, e^{s\bar{a}b_0\omega} \right)_\rho e^{\mathcal{Q}\mathcal{V}} \tag{3.1.24}$$

where ρ spans the $su(N)$ Cartan subalgebra. By renaming $\bar{a} \to \bar{a}/s$ and letting $s \to \infty$ we then get

$$\prod_{\rho} \left(\int da \, d\bar{a} \frac{\partial}{\partial \bar{a}} \int \frac{db_0}{b_0} \, e^{\bar{a} b_0 \omega} \right)_{\rho} e^{\mathcal{QV}|_{\bar{a}=0}} \,. \tag{3.1.25}$$

Similar arguments appeared in the evaluation of the low-energy effective Seiberg-Witten theory [107]. The integrals over the $N-1$ zero modes of b are taken by evaluating at $b = 0$ by Cauchy theorem. This implies that the leftover integral over the Cartan parameters is a total differential in the $\bar{\Phi}$ zero-mode variables, namely in $\bar{a}_\rho$, so that it gets reduced to a contour integral along the boundary of the moduli space of solutions of the fixed points equations that will be discussed in the next subsection.

Let us notice that the way in which we have soaked up the (η, χ) fermionic zero modes in (3.1.23) implies that the path integral localizes on configurations satisfying a more general condition than the Hermitian Yang-Mills equation. This is due to the fact that the b-field zero modes along the Cartan of $su(N)$ are not playing the role of Lagrange multipliers anymore. Therefore the gauge fixing condition results to be $F^+ = \omega \mathtt{t}$, where $\mathtt{t}$ is a constant Cartan element in $u(N)$, instead of (3.1.2). The former is indeed the condition satisfied by the supersymmetric fixed points that we will discuss in the next subsection.

3.1.3 Localization onto the fixed points

The localization proceeds as follows: by setting the fermions to zero, the fixed points of the supercharge read

$$\begin{aligned} &\iota_V D\bar{\Phi} + [\Phi, \bar{\Phi}] = 0, \\ &i\iota_V F + D\Phi = 0, \end{aligned} \tag{3.1.26}$$

and their integrability conditions

$$\begin{aligned} &\iota_V D\Phi = 0, \\ &\mathcal{L}_V F = [F, \Phi]. \end{aligned} \tag{3.1.27}$$

By using the reality condition for the scalar fields $\bar{\Phi} = -\Phi^\dagger$ and the first of (3.1.27), the first of (3.1.26) splits in two, that is

$$\iota_V D\bar{\Phi} = 0 \quad \text{and} \quad [\Phi, \bar{\Phi}] = 0 \tag{3.1.28}$$

which imply that Φ and $\bar{\Phi}$ lie in the same Cartan subalgebra. By reasoning in an analogous way on the second equation in (3.1.27), we get that the gauge curvature too is aligned along the Cartan subalgebra.

We now describe the solution in detail for compact toric manifolds. These latter are described by their toric fan [71]. The supersymmetry algebra is equivariant with respect to the maximal torus $U(1)^{N+2}$, where the first factor is the Cartan torus of the gauge group and the second is the isometry V of the four manifold[4]. In components, labeled

[4]We remind the reader that for toric surfaces V generates a $(\mathbb{C}^*)^2$-action, which correspond to a complexification of the Ω-background parameters.

by $\alpha = 1, \ldots, N$, we have

$$(F+\Phi)_\alpha = F_\alpha^{\text{point}} + a_\alpha + \sum_\ell k_\alpha^{(\ell)} \omega^{(\ell)} \tag{3.1.29}$$

that is, $F+\Phi$ is the $U(1)^{N+2}$ equivariant curvature of the bundle. The a_α parameters generate the $U(1)^N$-action. Moreover $\omega^{(\ell)} \in H_V^2(M)$ is the V-equivariant two-form Poincaré dual of the equivariant divisor D_ℓ corresponding to the ℓ-th vector of the fan (see figure 3.1).

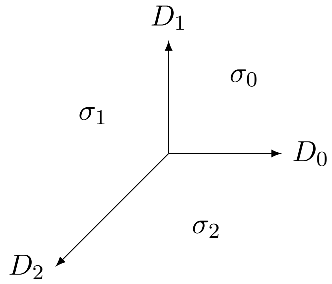


Figure 3.1: Toric fan of $\mathbb{P}^2$. σ_ℓ labels the cone of dimension two relative to the ℓ-th $\mathbb{C}^2$ coordinates patch.

Let us denote by $H^{(\ell)}$ the zero-form part of $\omega^{(\ell)}$. We get

$$\Phi_\alpha = a_\alpha + \sum_\ell k_\alpha^{(\ell)} H^{(\ell)} \, . \tag{3.1.30}$$

The values of Φ_α at each fixed point $P_{(\kappa)}$ will be denoted by

$$a_\alpha^{(\kappa)} \equiv \Phi_\alpha \left(P_{(\kappa)}\right) \, . \tag{3.1.31}$$

In (3.1.29), F^{point} is the contribution of point-like instantons located at the fixed points of the $U(1)^2$-action. For each of these fixed points we have then an independent contribution given by the Nekrasov partition function associated to the affine patch where the fixed point is sitting. In this framework, the contribution of point-like instantons correspond to the one of ideal sheaves on $\mathbb{C}^2$ supported at the fixed points of the $U(1)^2$-action, labeled by Young diagrams $\{Y_\alpha^{(\ell)}\}$ [5]. We remind the reader that the Chern classes of the point-like instantons are given by

$$\begin{aligned} c_1^{(\ell)} &= \sum_{\alpha=1}^N k_\alpha^{(\ell)}, \\ -ch_2^{(\ell)} &= \sum_{\alpha=1}^N |Y_\alpha^{(\ell)}|. \end{aligned} \tag{3.1.32}$$

[5] Locally this compactification can be regarded as a non-commutative deformation in the affine patch of M.

Summarizing, we find that the localization procedure implies that the partition function is written as a product of copies of the Nekrasov partition function in the appropriate shifted variables glued by the integration over the Cartan parameters $\{a_{\alpha\beta}\}$.

The integration contour is specified according to the discussion in the previous subsection as follows. Solving the fixed point equations we bounded the field theory phase to the deep Coulomb branch by declaring Φ and $\bar{\Phi}$ to lie at a generic point in the Cartan subalgebra where the gauge symmetry is maximally broken as $U(N) \to U(1)^N$. This implies the integral over $(a, \bar{a})$ to be in $\mathbb{C}^{N-1} \setminus \mathcal{T}$ where $\mathcal{T}$ is a tubular neighborhood of the hyperplanes set $\Delta = \{a_\alpha - a_\beta = 0\}$. This choice guarantees maximal gauge symmetry breaking. Henceforth, by using Stokes theorem in formula (3.1.25), we find that the complete partition function is given by a contour integral around the above regions of the leftover terms in the path integral evaluation. In particular, for $N = 2$ we find a single contour integral around the origin in $\mathbb{C}$.

Moreover, the stability condition on the equivariant *unframed* sheaves induces constraints on the allowed values of the fixed points data $\{k^{(\ell)}_{\alpha\beta} := k^{(\ell)}_\alpha - k^{(\ell)}_\beta\}$. We will describe in section 3.2 the details of all this for $U(2)$ gauge theories on $\mathbb{P}^2$.

3.2 Exact partition function on $\mathbb{P}^2$ and equivariant Donaldson Invariants

Let us denote the homogeneous coordinates of $\mathbb{P}^2$ by $[z_0 : z_1 : z_2]$. The $(\mathbb{C}^*)^2$ torus action, generated by the vector[6], acts on homogeneus coordinates as $[z_0 : e^{\epsilon_1} z_1 : e^{\epsilon_2} z_2]$. In local coordinates $(x^{(\ell)}, y^{(\ell)})$ in the three coordinates patches $(z_\ell \neq 0)$ the action is $(e^{\epsilon_1^{(\ell)}} x^{(\ell)}, e^{\epsilon_2^{(\ell)}} y^{(\ell)})$ with weights

ℓ	$\epsilon_1^{(\ell)}$	$\epsilon_2^{(\ell)}$
0	ϵ_1	ϵ_2
1	$\epsilon_2 - \epsilon_1$	$-\epsilon_1$
2	$-\epsilon_2$	$\epsilon_1 - \epsilon_2$

(3.2.1)

ordered so that $\epsilon_1^{(\ell)} = -\epsilon_2^{(\ell+1)}$. The fixed points under the V-action are denoted by

$$P_{(0)} = [1 : 0 : 0], \qquad P_{(1)} = [0 : 1 : 0], \qquad P_{(2)} = [0 : 0 : 1]. \tag{3.2.2}$$

The generators of the global gauge transformation $(\mathbb{C}^*)^N$ are denoted by $\vec{a} = \{a_\alpha\}$, $\alpha = 1, \ldots, N$. The v.e.v. of the scalar field Φ is given by specifying (3.1.30) and (3.1.31) to $\mathbb{P}^2$. The equivariant extensions of the Fubini-Study two-form $\omega = i\partial\bar{\partial} \log(|z_0|^2 + |z_1|^2 + |z_2|^2)$

[6] In local coordinates $x^{(0)} = z_1/z_0, y^{(0)} = z_2/z_0$ in the patch $z_0 \neq 0$ the vector has the following expression $V = i\epsilon_1(x^{(0)}\partial_{x^{(0)}} - \bar{x}^{(0)}\bar{\partial}_{\bar{x}^{(0)}}) + i\epsilon_2(y^{(0)}\partial_{y^{(0)}} - \bar{y}^{(0)}\bar{\partial}_{\bar{y}^{(0)}})$.

are

$$\begin{aligned} \omega^{(0)} &= \omega + \frac{\epsilon_1 |z_0|^2 + (\epsilon_1 - \epsilon_2)|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \\ \omega^{(1)} &= \omega + \frac{\epsilon_2 |z_0|^2 + (\epsilon_2 - \epsilon_1)|z_1|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \\ \omega^{(2)} &= \omega + \frac{-\epsilon_1 |z_1|^2 - \epsilon_2 |z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2} \end{aligned} \tag{3.2.3}$$

and satisfy $(\iota_V - d)\omega^{(\ell)} = 0$. So that

$$a_\alpha^{(\ell)} = a_\alpha + k_\alpha^{(\ell)} \epsilon_1^{(\ell)} + k_\alpha^{(\ell+1)} \epsilon_2^{(\ell)} \tag{3.2.4}$$

and, setting $k_\alpha^{(0)} \equiv k_\alpha^{(3)} = p_\alpha$, $k_\alpha^{(1)} = q_\alpha$ and $k_\alpha^{(2)} = r_\alpha$, we have explicitly, by (3.2.4) and (3.2.1)

$$\begin{aligned} \vec{a}^{(0)} &= \vec{a} + \vec{p}\epsilon_1 + \vec{q}\epsilon_2 \\ \vec{a}^{(1)} &= \vec{a} + \vec{q}(\epsilon_2 - \epsilon_1) + \vec{r}(-\epsilon_1) \\ \vec{a}^{(2)} &= \vec{a} + \vec{p}(\epsilon_1 - \epsilon_2) + \vec{r}(-\epsilon_2). \end{aligned} \tag{3.2.5}$$

The fixed point data on $\mathbb{P}^2$ are described in terms of a collection of Young diagrams $\{\vec{Y}_\ell\}$, and of integer numbers $\{\vec{k}^{(\ell)}\}$ $\ell = 0, 1, 2$ describing respectively the $(\mathbb{C}^*)^{N+2}$-invariant point-like instantons in each patch and the magnetic fluxes of the gauge field, which correspond to the first Chern class c_1 as prescribed by (3.1.32).

The explicit expression at the three fixed points $P_{(\ell)}$ of the V-equivariant local and surface observables introduced in section 3.1.1 is given as follows. By calling for brevity

$$\alpha = \omega + H, \qquad p = \alpha' \wedge \alpha'' + K \tag{3.2.6}$$

where H was defined in formula (3.1.17), we can write the most general equivariant extension α as

$$\alpha = \omega + \frac{h|z_0|^2 + (h - \epsilon_1)|z_1|^2 + (h - \epsilon_2)|z_2|^2}{|z_0|^2 + |z_1|^2 + |z_2|^2}\,, \tag{3.2.7}$$

where ω is the Fubini-Study form of $\mathbb{P}^2$ and h a linear, coordinate independent, polynomial in the weights of the V-action. The evaluation at the fixed points of the observables α, p, with fugacities z, x is[7]

$$\begin{aligned} \imath^*_{P_{(0)}}(z\alpha + xp) &= zh + x\tilde{K} \\ \imath^*_{P_{(1)}}(z\alpha + xp) &= z(h - \epsilon_1) + x(\tilde{K} - \tilde{h}\epsilon_1 + \epsilon_1^2) \\ \imath^*_{P_{(2)}}(z\alpha + xp) &= z(h - \epsilon_2) + x(\tilde{K} - \tilde{h}\epsilon_2 + \epsilon_2^2). \end{aligned} \tag{3.2.8}$$

The full $U(2)$ partition function on $\mathbb{P}^2$ is given by

$$Z^{\mathbb{P}^2}_{\text{full}}(\mathbf{q}, x, z, y\,; \epsilon_1, \epsilon_2) = \sum_{\{k_\alpha^{(\ell)}\}|\text{semi-stable}} \oint_\Delta da \prod_{\ell=0}^{2} Z^{\mathbb{C}^2}_{\text{full}}(\mathbf{q}^{(\ell)}\,; a^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)})\, y^{c_1^{(\ell)}} \tag{3.2.9}$$

[7] We defined $\tilde{h} = h' + h''$, $\tilde{K} = K + h'h''$ some new, coordinate independent, polynomial in ϵ_1, ϵ_2 of degree one and two respectively.

where $\mathbf{q} = \exp(2\pi i\tau)$ is the exponential of the gauge coupling and $\mathbf{q}^{(\ell)} = \mathbf{q}\, e^{i^*_{P_{(\ell)}}(\alpha z + px)}$ is the one shifted by the observable (3.2.8) evaluated at the fixed points $P_{(\ell)}$ of $\mathbb{P}^2$. Finally y is the source term corresponding to the Kähler form $t\omega$ with t the complexified Kälher parameter, so that $y = e^{2\pi t}$.

The integration in (3.2.9) realizes an isomorphism between the fixed points of the *unframed* moduli space of equivariant rank two sheaves on $\mathbb{P}^2$ and copies of the fixed points of the *framed* moduli space on $\mathbb{P}^2$. Details of this isomorphism are presented in the explicit computation below and, in the case of odd c_1, reproduce exactly the results of [75].

The stability conditions constraining the fixed point data $\{k_\alpha^{(\ell)}\}$'s are obtained by mapping these latter to the data describing unframed equivariant sheaves in terms of filtrations as in [100]. More details are provided in appendix 3.A.

The factors appearing in (3.2.9) are the Nekrasov full partition functions

$$Z^{\mathbb{C}^2}_{\text{full}}(\mathbf{q}\,; a, \epsilon_1, \epsilon_2) = Z^{\mathbb{C}^2}_{\text{class}}(\mathbf{q}\,; a, \epsilon_1, \epsilon_2) Z^{\mathbb{C}^2}_{\text{1-loop}}(a, \epsilon_1, \epsilon_2) Z^{\mathbb{C}^2}_{\text{inst}}(\mathbf{q}\,; a, \epsilon_1, \epsilon_2) \tag{3.2.10}$$

whose explicit expressions we report below.

In the following we will compute the integral (3.2.9) with $x = z = 0$ (so $\mathbf{q}^{(\ell)} = \mathbf{q}$) and $y = 1$. The case with $x, z \neq 0, y \neq 1$ is a straightforward modification of the calculations below. In particular if one keeps $x, z \neq 0$ the result of the integration will give the generating function for equivariant Donaldson invariants for $\mathbb{P}^2$.

3.2.1 Classical action

The classical part of the partition function coming from (3.2.10) is given by evaluating (3.1.5) on the supersymmetric minima (3.1.29)

$$\begin{aligned} Z^{\mathbb{P}^2}_{\text{class}}(\mathbf{q}\,; \vec{a}, \epsilon_1, \epsilon_2) &= \prod_{\ell=0}^{2} Z^{\mathbb{C}^2}_{\text{class}}(\mathbf{q}\,; \vec{a}^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) \\ &= \prod_{\ell=0}^{2} \exp\left[-\pi i\tau \frac{\sum_{\alpha=1}^2 \left(a_\alpha^{(\ell)}\right)^2 - c\left(\sum_{\alpha=1}^2 a_\alpha^{(\ell)}\right)^2}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}\right]. \end{aligned} \tag{3.2.11}$$

Inserting the values of the equivariant weights (3.2.1) and (3.2.5) we obtain

$$Z^{\mathbb{P}^2}_{\text{class}}(\mathbf{q}\,; \vec{a}, \epsilon_1, \epsilon_2) = \exp\left[-\pi i\tau \left(\sum_{\alpha=1}^{2} (p_\alpha + q_\alpha + r_\alpha)^2 - c\left(\sum_{\alpha=1}^{2} p_\alpha + q_\alpha + r_\alpha\right)^2\right)\right]. \tag{3.2.12}$$

Since $\mathbf{q} = \exp[2\pi i\tau]$ we have

$$Z^{\mathbb{P}^2}_{\text{class}}(\mathbf{q}\,; \vec{a}, \epsilon_1, \epsilon_2) = \mathbf{q}^{-\frac{1}{2}\left(\sum_{\alpha=1}^2 (p_\alpha+q_\alpha+r_\alpha)^2 - c\left(\sum_{\alpha=1}^2 p_\alpha+q_\alpha+r_\alpha\right)^2\right)} = \mathbf{q}^{-\frac{1}{4}\left((1-2c)c_1^2 + (p+q+r)^2\right)} \tag{3.2.13}$$

where we defined

$$p = p_1 - p_2, \quad q = q_1 - q_2, \quad r = r_1 - r_2, \tag{3.2.14}$$

and $c_1 = \sum_{(\ell)} c_1^{(\ell)}$ with $c_1^{(\ell)}$ defined in (3.1.32).

The sum in front of the full partition function can be rewritten as

$$\sum_{\{\vec{p},\vec{q},\vec{r}\}\in(\mathbb{Z}^2)^3} = \sum_{c_1\in\mathbb{Z}} \sum_{\substack{\{p,q,r\}\in\mathbb{Z}^3 \\ p+q+r+c_1=\text{even}}} \tag{3.2.15}$$

where we have performed a zeta function regularization of the sum over two integers, since the full partition function will depend only on p, q, r, c_1. Moreover is enough to consider only the cases $c_1 = \{0, 1\}$, because we are considering a rank two bundle, therefore the moduli spaces of two bundles with both $c_1 = 0$ (or 1) mod 2 are isomorphic after the twist by a line bundle.[8]

As discussed in section 3.1 the Hermitian-Yang-Mills equation implies semi-stability of the bundle. This in turn consists in some restrictions on the integers $\{k\}$ in the summation of (3.2.9) which will be discussed in subsections 3.2.5, 3.2.6 and in appendix 3.A.

3.2.2 One-loop contribution

The one-loop contribution in (3.2.9) is given by

$$Z^{\mathbb{P}^2}_{\text{1-loop}}(\vec{a},\epsilon_1,\epsilon_2) = \prod_{\ell=0}^{2} Z^{\mathbb{C}^2}_{\text{1-loop}}(\vec{a}^{(\ell)},\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}) = \prod_{\ell=0}^{2} \exp\left[-\sum_{\alpha\neq\beta}\gamma_{\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}}(a^{(\ell)}_{\alpha\beta})\right] \tag{3.2.16}$$

where $a_{\alpha\beta} := a_\alpha - a_\beta$ and the double gamma-function is defined as

$$\gamma_{\epsilon_1,\epsilon_2}(x) = \frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}\frac{e^{-tx}}{(1-e^{\epsilon_1 t})(1-e^{\epsilon_2 t})}\,, \tag{3.2.17}$$

with $\mathrm{Re}(\epsilon_1)$ and $\mathrm{Re}(\epsilon_2)$ positive. We have $a_{\alpha\beta} = \{a_{12}, a_{21}\} =: \{a, -a\}$ and similarly $p_{\alpha\beta} =: \{p, -p\}$ etc.[9] Inserting the values of the equivariant weights (3.2.1), (3.2.5) and using the definition of $\gamma_{\epsilon_1,\epsilon_2}$ (3.2.17) we can write

$$Z^{\mathbb{P}^2}_{\text{1-loop}} = \prod_{\pm}\exp\left[-\frac{d}{ds}\Big|_{s=0}\frac{1}{\Gamma(s)}\int_0^\infty dt\, t^{s-1}e^{-t(\pm a)}\frac{x^{\pm(q+r)}y^{\pm(p+r)}}{(1-x)(1-y)(x-y)}P_\pm(x,y)\right], \tag{3.2.18}$$

where we defined[10] $x := e^{\epsilon_1 t}$ and $y := e^{\epsilon_2 t}$, and $P_\pm(x, y)$ is a rational function in x and y

$$P_\pm(x,y) = x^{\mp N}y^{\mp N}(x-y) + x^{\mp N}y^2(1-x) - x^2y^{\mp N}(1-y) \tag{3.2.19}$$

with $N := p+q+r$ an integer with the same parity of c_1 (3.2.15). The values of $P_\pm(x, y)$ on $x = 1$, $y = 1$ and $x = y$ are zero, this means that in those points $P_\pm(x, y)$ has zeros

[8] The case $c_1 = 0$ or equivalently $c_1 =$ even hides some subtleties since the bundle can be reducible and the moduli space becomes singular [55]. We will in fact treat this case separately.

[9] Note that this differs from the usual convention $a_{\alpha\beta} =: \{2a, -2a\}$.

[10] This choice of analytic continuation implies that $\gamma_{\epsilon_1\epsilon_2}(x)$ has a branch cut for $x > 0$.

which cancel the denominators $(1-x)^{-1}, (1-y)^{-1}, (x-y)^{-1}$ in (3.2.18). Making use of the identity

$$x^N - y^N = (x-y)\sum_{i=0}^{N-1} x^i y^{N-1-j} \tag{3.2.20}$$

we arrive at the following expression for $P_\pm(x,y)$:

- $N \geq 0$.

$$\begin{aligned} &P_+(x,y) = x^{-N}y^{-N}(1-x)(1-y)(x-y)\sum_{i=0}^{N} y^i \sum_{j=0}^{N-i} x^j, \\ &P_-(x,y) = \begin{cases} (1-x)(1-y)(x-y) & N=0 \\ 0 & N=1,2 \\ x^{N-1}y^{N-1}(1-x)(1-y)(x-y)\sum_{i=0}^{N-3} y^{-i} \sum_{j=0}^{N-3-i} x^{-j} & N>2 \end{cases} \end{aligned} \tag{3.2.21}$$

- $N < 0$.

$$\begin{aligned} &P_+(x,y) = \begin{cases} 0 & N=-1,-2 \\ x^{|N|-1}y^{|N|-1}(1-x)(1-y)(x-y)\sum_{i=0}^{|N|-3} y^{-i} \sum_{j=0}^{|N|-3-i} x^{-j} & N<-2 \end{cases} \\ &P_-(x,y) = x^{-|N|}y^{-|N|}(1-x)(1-y)(x-y)\sum_{i=0}^{|N|} y^i \sum_{j=0}^{|N|-i} x^j. \end{aligned} \tag{3.2.22}$$

Inserting this result back in (3.2.18) and using the definition of the Gamma function:

$$\Gamma(s) = \int_0^\infty dt\, t^{s-1}e^{-t} \tag{3.2.23}$$

we obtain for $Z^{\mathbb{P}^2}_{\text{1-loop}}$ of (3.2.16) the following results

- $N = 0$

$$Z^{\mathbb{P}^2}_{\text{1-loop}} = -\left(a + p\epsilon_1 + q\epsilon_2\right)^2 \tag{3.2.24}$$

- $N > 0$

$$\begin{aligned} Z^{\mathbb{P}^2}_{\text{1-loop}} = &\prod_{i=0}^{N}\prod_{j=0}^{N-i}(a+(p-j)\epsilon_1+(q-i)\epsilon_2)\cdot \\ \times &\prod_{i=0}^{N-3}\prod_{j=0}^{N-3-i^\diamond} -(a+(p-1-j)\epsilon_1+(q-1-i)\epsilon_2) \end{aligned} \tag{3.2.25}$$

- $N < 0$

$$
\begin{aligned}
Z^{\mathbb{P}^2}_{\text{1-loop}} = & \prod_{i=0}^{|N|} \prod_{j=0}^{|N|-i} -(a + (p+j)\epsilon_1 + (q+i)\epsilon_2) \cdot \\
& \times \prod_{i=0}^{|N|-3} \prod_{j=0}^{|N|-3-i^{\diamond}} (a + (p+1+j)\epsilon_1 + (q+1+i)\epsilon_2)
\end{aligned}
\tag{3.2.26}
$$

where the symbol $\diamond$ over the products in the second lines of formulas (3.2.25), (3.2.26) mean that those products are equal to 1 if $|N| < 3$. The only relevant case is actually that with $p, q, r \in \mathbb{Z}_{\geq 0}$. This can be seen by a direct computation which shows that the final result does depend on the absolute values of p, q, r only. Therefore from now on we assume $N \geq 0$.

3.2.3 Instanton contribution

The instanton contribution in (3.2.9) is given by

$$
\prod_{\ell=0}^{2} Z^{\mathbb{C}^2}_{\text{inst}}(\mathbf{q}\,; \vec{a}^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) \tag{3.2.27}
$$

where $Z^{\mathbb{C}^2}_{\text{inst}}$ is the Nekrasov partition function defined as follows. Let $Y = \{\lambda_1 \geq \lambda_2 \geq \dots\}$ be a Young diagram, and $Y' = \{\lambda'_1 \geq \lambda'_2 \geq \dots\}$ its transposed. λ_i is the length of the i-column and λ'_j the length of the j-row of Y. For a given box $s = \{i, j\}$ we define respectively the arm and leg length functions

$$
A_Y(s) = \lambda_i - j, \qquad L_Y(s) = \lambda'_j - i. \tag{3.2.28}
$$

Note that these quantities can also be negative when s does not belong to the diagram Y. The fixed points data for each patch are given by a collection of Young diagrams $\vec{Y}^{(\ell)} = \{Y_\alpha^{(\ell)}\}$, and the instanton contribution is [119, 69, 42]

$$
Z^{\mathbb{C}^2}_{\text{inst}}(\mathbf{q}\,; \vec{a}, \epsilon_1, \epsilon_2) = \sum_{\{Y_\alpha\}} \mathbf{q}^{|\vec{Y}|} z_{\text{vec}}(\vec{a}, \vec{Y}, \epsilon_1, \epsilon_2) \tag{3.2.29}
$$

where $\mathbf{q} = \exp(2i\pi\tau)$ and

$$
\begin{aligned}
z_{\text{vec}}(\vec{a}, \vec{Y}, \epsilon_1, \epsilon_2) = & \prod_{\alpha,\beta=1}^{N} \prod_{s \in Y_\alpha} \Big(a_{\beta\alpha} - L_{Y_\beta}(s)\epsilon_1 + (A_{Y_\alpha}(s)+1)\epsilon_2\Big)^{-1} \\
& \times \Big(a_{\alpha\beta} + (L_{Y_\beta}(s)+1)\epsilon_1 - A_{Y_\alpha}(s)\epsilon_2\Big)^{-1}.
\end{aligned}
\tag{3.2.30}
$$

3.2.4 Analytic structure of the integrand

In order to integrate the full partition function (3.2.9) along a we need to study the analytic structure of the integrand.

The instanton partition function (3.2.29) has simple poles at

$$a \equiv a_{12} = m\epsilon_1 + n\epsilon_2, \quad m,n \in \mathbb{Z}, \quad m \cdot n > 0. \tag{3.2.31}$$

This behavior can be displayed explicitly by the Zamolodchikov's recursion relation [155] which was analyzed for gauge theories in [127]. In the evaluation of the integral it will be very useful to write it as

$$Z_{\text{inst}}(\mathbf{q}; a, \epsilon_1, \epsilon_2) = 1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R_{m,n}\, Z_{\text{inst}}(\mathbf{q}; m\epsilon_1 - n\epsilon_2, \epsilon_1, \epsilon_2)}{(a - m\epsilon_1 - n\epsilon_2)(a + m\epsilon_1 + n\epsilon_2)} \tag{3.2.32}$$

where

$$R_{m,n} = 2 \underbrace{\prod_{i=-m+1}^{m} \prod_{j=-n+1}^{n}}_{(i,j)\neq\{(0,0),(m,n)\}} \frac{1}{(i\epsilon_1 + j\epsilon_2)}. \tag{3.2.33}$$

Therefore the product of the three instanton partition functions coming from the three patches

$$Z_{\text{inst}}(\mathbf{q}; a^{(0)}, \epsilon_1, \epsilon_2) Z_{\text{inst}}(\mathbf{q}; a^{(1)}, -\epsilon_2, \epsilon_1 - \epsilon_2) Z_{\text{inst}}(\mathbf{q}; a^{(2)}, \epsilon_2 - \epsilon_1, -\epsilon_1) \tag{3.2.34}$$

displays a polar structure as depicted in figure 3.2. The lattice[11] $(x, y) = (i\epsilon_1, j\epsilon_2)$

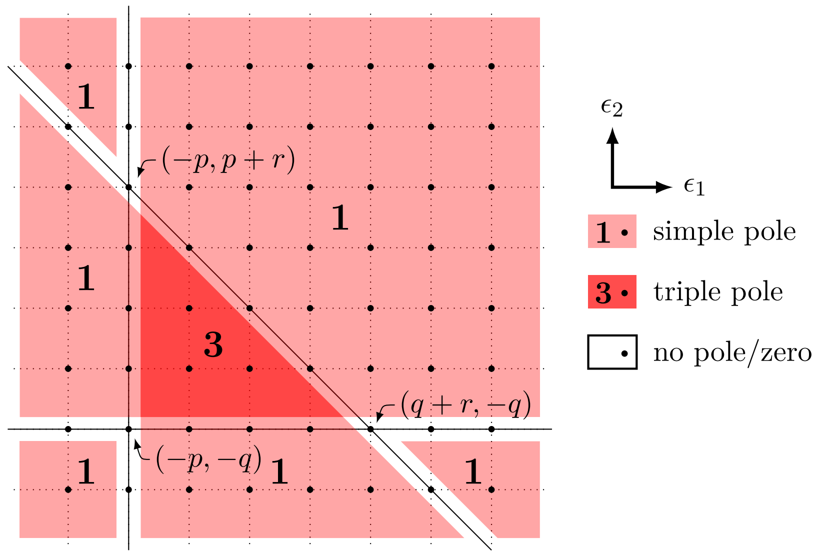


Figure 3.2: Poles of instanton partition function.

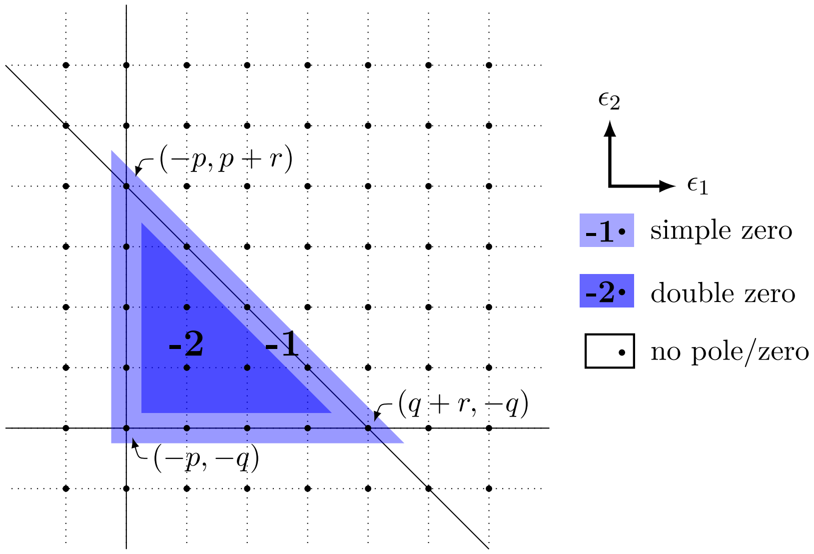


Figure 3.3: Poles of one-loop partition function.

$i, j \in \mathbb{Z}$ is separated in seven regions by three straight lines

$$x = -p, \quad y = -q, \quad y = -x + r. \tag{3.2.35}$$

In the interior of the triangle $T_I = \{(-p, -q), (q + r, -q), (-p, p + r)\}$ formed by these three lines there are triple poles. Along the three lines there are simple poles *only* in the segment strictly contained between two vertices of the triangle. In all the other points of the lattice there are simple poles.

In the analysis of the one-loop contribution one can see[12] that the only relevant case is $N > 0$. Looking at (3.2.25) one can see that this contributes with double zeros in the interior of the triangle T_I (which cancel the multiplicity of the poles of the instanton part) and simple zeros along the perimeter of T_I (which cancel the simple poles of the instanton part on the edges of the triangle)[13]. The positions of the zeroes of the one-loop part is described in figure 3.3. The overall polar structure of the full partition function is drawn in figure 3.4: there are simple poles in all the points of the lattice that are not along the three straight lines (3.2.35). This implies that the integration of Z_{full} will be given by the sum of the residues of simple poles inside the contour of integration $\Delta = \partial C$ given in (3.2.9)

$$\begin{aligned}\oint_{\partial C} Z_{\text{full}}(\mathbf{q}\,; a, \epsilon_1, \epsilon_2) da \propto & \sum_{(i,j)\in C} \text{Res}(Z_{\text{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2)|a = i\epsilon_1 + j\epsilon_2) \\ = & \sum_{(i,j)\in C} \lim_{a\to i\epsilon_1 + j\epsilon_2} (a - i\epsilon_1 - j\epsilon_2) Z_{\text{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2),\end{aligned} \tag{3.2.36}$$

[11] We consider ϵ_1, ϵ_2 to be incommensurable.
[12] Indeed in the case $N = 0$ the integrand in (3.2.9) does not display any pole at the origin.
[13] Of course if $N < 3$ there is none interior of the triangle, so only simple poles.

and from the discussion in section 3.1.2 the only residue to evaluate is the one relative to the pole at the origin.

3.2.5 Exact results for odd c_1

Now we can perform the integration by residues evaluation as anticipated in (3.2.36). We are focusing on the case with $c_1 = 1$, the other case $c_1 = 0$ is more subtle and will be studied in a separate section.

From the analysis of the previous section we know that the full partition function has a pole at the origin only if the integers $p = p_{12}$, $q = q_{12}$, $r = r_{12}$ are strictly positive. Moreover we have to impose the stability conditions, which are discussed in appendix 3.A, see (3.A.13). These, together with $p + q + r + c_1 = even$ imply that the integers p, q, r have to satisfy strict triangle inequalities, namely

$$p + q > r > 0, \quad p + r > q > 0, \quad q + r > p > 0. \tag{3.2.37}$$

Using the expressions for the classical (3.2.13), one-loop (3.2.25) and instanton (3.2.32) partition functions, we can put all together (details are given in section 3.2.5) obtaining as the final result of the integration

$$\begin{aligned} &Z^{\mathbb{P}^2}_{\mathcal{N}=2}(\mathbf{q}\,; \epsilon_1, \epsilon_2)\big|_{c_1=1} = \\ &= \mathbf{q}^{-\frac{1}{4}(1-2c)} \sum_{\{p,q,r\}} \mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \prod_{\{(i,j)\}} \frac{1}{i\epsilon_1 + j\epsilon_2} \\ &\quad \times Z_{\text{inst}}\big(\mathbf{q}; a^{(0)}_{\text{res}}, \epsilon_1, \epsilon_2\big) Z_{\text{inst}}\big(\mathbf{q}; a^{(1)}_{\text{res}}, \epsilon_2 - \epsilon_1, -\epsilon_1\big) Z_{\text{inst}}\big(\mathbf{q}; a^{(2)}_{\text{res}}, -\epsilon_2, \epsilon_1 - \epsilon_2\big) \end{aligned} \tag{3.2.38}$$

where

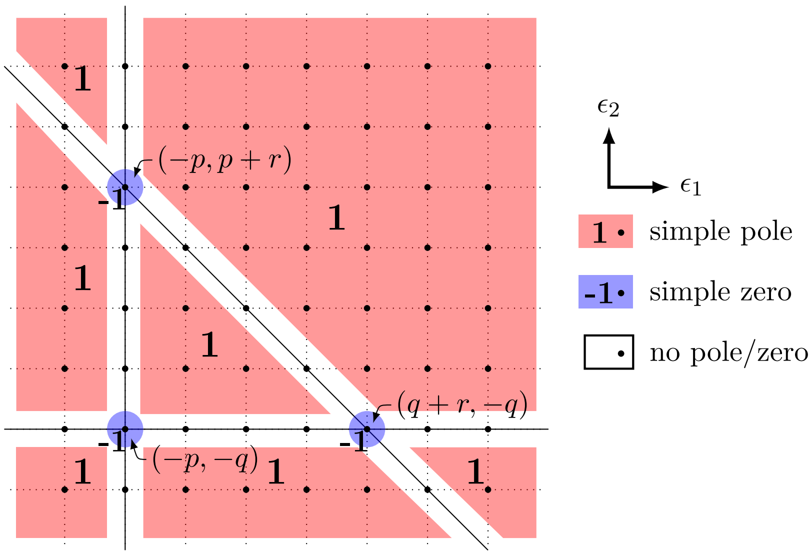


Figure 3.4: Poles of the full partition function.

- the sum is over positive integers p, q, r satisfying the triangle inequality (3.2.37) and also $p+q+r = odd$,
- the product is over the points of the lattice $(i,j) \in (D^{(p,q,r)} \cap \mathbb{Z}^2) \setminus (0,0)$; where the regions $D^{(p,q,r)}$ are the intersections of two triangles T_1 and T_2, one of side $p+q+r$ and the other of side $p+q+r-3$:

$$\begin{aligned} T_1 &= \{(-p,-q),(q+r,-q),(-p,p+r)\}, \\ T_2 &= \{(p-1,q-1),(-q-r+2,q-1),(p-1,-p-r+2)\}. \end{aligned} \tag{3.2.39}$$

  T_1 is delimited by the three straight lines

$$x = -p, \quad y = -q, \quad y = -x + r. \tag{3.2.40}$$

  T_2 is delimited by the three straight lines

$$x = p-1, \quad y = q-1, \quad y = -x - r + 1. \tag{3.2.41}$$

- we used the following notation

$$\begin{aligned} a^{(0)}_{\text{res}} &= p\epsilon_1 - q\epsilon_2, \\ a^{(1)}_{\text{res}} &= q(\epsilon_2 - \epsilon_1) - r(-\epsilon_1), \\ a^{(2)}_{\text{res}} &= r(-\epsilon_2) - p(\epsilon_1 - \epsilon_2). \end{aligned} \tag{3.2.42}$$

We can compare the expression (3.2.38) with theorem 6.15 in [75]. Indeed, (3.2.38) coincide with the formula in [75] with x, z set to zero. Indeed the region $D^{(p,q,r)}$ defined above coincides with the one in Lemma 6.12 of [75].

To reproduce the full generating function of equivariant Donaldson invariant in [75] one should repeat the computation and the integration of $Z^{\mathbb{P}^2}_{\text{full}}$ with $x, z \neq 0$ in (3.2.9). This implies a light modification in the calculations, namely one should replace $\mathbf{q}$ with $\mathbf{q}^{(\ell)}$ in every copy of $Z^{\mathbb{C}^2}_{\text{full}}$, with $\mathbf{q}^{(\ell)}$ defined below (3.2.9). Moreover we need to expand in the discriminant of the bundle (see (3.A.9) in appendix 3.A), that is choosing $c = \frac{1}{2}$ in (3.1.5). The result in this case is

$$\begin{aligned} &Z^{\mathbb{P}^2}_{\mathcal{N}=2}(\mathbf{q}, x, z, \epsilon_1, \epsilon_2)\big|_{c_1=1} = \\ &= \sum_{\{p,q,r\}} \mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \exp\left(-\frac{1}{4}\sum_{\ell=0}^{2} \frac{(a^{(\ell)}_{\text{res}})^2\, \imath^*_{P_{(\ell)}}(\alpha z + px)}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}} \right) \prod_{\{(i,j)\}} \frac{1}{i\epsilon_1 + j\epsilon_2} \\ &\times Z_{\text{inst}}(\mathbf{q}^{(0)}; a^{(0)}_{\text{res}}, \epsilon_1, \epsilon_2) Z_{\text{inst}}(\mathbf{q}^{(1)}; a^{(1)}_{\text{res}}, \epsilon_2 - \epsilon_1, -\epsilon_1) Z_{\text{inst}}(\mathbf{q}^{(2)}; a^{(2)}_{\text{res}}, -\epsilon_2, \epsilon_1 - \epsilon_2) \end{aligned} \tag{3.2.43}$$

where sum and product are the same of (3.2.38). Since $q = \Lambda^4$, formula (3.2.43) matches completely with the theorem 6.15 of [75].[14]

[14] To be meticulous in [75] there is also an extra factor Λ^{-3} because that is a generating function in the dimension of the moduli space of *unframed* instantons, that for a generic metric is precisely $\dim = 2pq + 2pr + 2qr - p^2 - q^2 - r^2 - 3$.

Proof of (3.2.38)

We evaluate the residue of Z_{full} at $a = 0$, namely

$$\begin{aligned} a^{(0)} &= p\epsilon_1 + q\epsilon_2 \\ a^{(1)} &= q(\epsilon_2 - \epsilon_1) + r(-\epsilon_1) \\ a^{(2)} &= p(\epsilon_1 - \epsilon_2) + r(-\epsilon_2). \end{aligned} \tag{3.2.44}$$

We know from section 3.2.4 that p, q, r are strictly positive. Therefore we see from (3.2.31) and (3.2.34) that the three instanton partition functions have a simple pole each, which identifies the region with triple poles in figure 3.2. Moreover

$$p, q, r \geq 1 \quad \Rightarrow \quad N = p + q + r \geq 3 \tag{3.2.45}$$

so we get a double zero from the one-loop part. Using (3.2.32) the instanton part is

$$\begin{aligned} Z_{\text{inst}}^{\mathbb{P}^2} = & \left(1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R_{m,n}^{(0)} Z_{\text{inst}}\left(\mathbf{q}; m\epsilon_1 - n\epsilon_2, \epsilon_1, \epsilon_2\right)}{\left(a^{(0)} - m\epsilon_1 - n\epsilon_2\right)\left(a^{(0)} + m\epsilon_1 + n\epsilon_2\right)}\right) \\ & \times \left(1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R_{m,n}^{(1)} Z_{\text{inst}}\left(\mathbf{q}; m(\epsilon_2 - \epsilon_1) - n(-\epsilon_1), \epsilon_2 - \epsilon_1, -\epsilon_1\right)}{\left(a^{(1)} - m(\epsilon_2 - \epsilon_1) - n(-\epsilon_1)\right)\left(a^{(1)} + m(\epsilon_2 - \epsilon_1) + n(-\epsilon_1)\right)}\right) \\ & \times \left(1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R_{m,n}^{(2)} Z_{\text{inst}}\left(\mathbf{q}; m(-\epsilon_2) - n(\epsilon_1 - \epsilon_2), -\epsilon_2, \epsilon_1 - \epsilon_2\right)}{\left(a^{(2)} - m(-\epsilon_2) - n(\epsilon_1 - \epsilon_2)\right)\left(a^{(2)} + m(-\epsilon_2) + n(\epsilon_1 - \epsilon_2)\right)}\right) \end{aligned} \tag{3.2.46}$$

where similarly to (3.2.33)

$$R_{m,n}^{(\ell)} = 2 \underbrace{\prod_{i=-m+1}^{m} \prod_{j=-n+1}^{n}}_{(i,j) \neq \{(0,0),(m,n)\}} \frac{1}{\left(i\epsilon_1^{(\ell)} + j\epsilon_2^{(\ell)}\right)}. \tag{3.2.47}$$

The triple pole is obtained by picking respectively from the three sums the terms $(m = p, n = q)$, $(m = q, n = r)$, $(m = r, n = p)$ giving

$$Z_{\text{inst}}^{\mathbb{P}^2} = -\frac{1}{a^3}\, \mathbf{q}^{pq+pr+qr}\, \tilde{R}_{p,q}^{(0)}\, \tilde{R}_{q,r}^{(1)}\, \tilde{R}_{r,p}^{(2)}\, Z_{\text{Res}} + O\left(\frac{1}{a^2}\right) \tag{3.2.48}$$

where

$$\tilde{R}_{m,n}^{(\ell)} = \frac{1}{a^{(\ell)} + m\epsilon_1^{(\ell)} + n\epsilon_2^{(\ell)}} R_{m,n}^{(\ell)} \tag{3.2.49}$$

and we defined

$$\begin{aligned} Z_{\text{Res}} = & Z_{\text{inst}}\left(\mathbf{q}; p\epsilon_1 - q\epsilon_2, \epsilon_1, \epsilon_2\right) Z_{\text{inst}}\left(\mathbf{q}; q(\epsilon_2 - \epsilon_1) - r(-\epsilon_1), \epsilon_2 - \epsilon_1, -\epsilon_1\right) \\ & \times Z_{\text{inst}}\left(\mathbf{q}; r(-\epsilon_2) - p(\epsilon_1 - \epsilon_2), -\epsilon_2, \epsilon_1 - \epsilon_2\right). \end{aligned} \tag{3.2.50}$$

Note that Z_{Res} is equal to the last line of (3.2.38).

When calculated at the point $a = 0$ the three factors $\tilde{R}^{(\ell)}$ can be rewritten as

$$\tilde{R}^{(\ell)} = \prod_{(i,j)\in U_\ell \setminus (0,0)} \frac{1}{(i\epsilon_1 + j\epsilon_2)}, \tag{3.2.51}$$

where the three regions U_ℓ are depicted in figure 3.5 and are defined as:

- U_0 is a rectangle $2p - 1 \times 2q - 1$ delimited by the four straight lines

$$x = -p + 1, \quad x = p, \quad y = -q + 1, \quad y = q. \tag{3.2.52}$$

- U_1 is a parallelogram delimited by the four straight lines

$$y = -q + 1, \quad y = q, \quad y = -x - r, \quad y = -x + r - 1. \tag{3.2.53}$$

- U_2 is a parallelogram delimited by the four straight lines

$$x = -p + 1, \quad x = p, \quad y = -x - r, \quad y = -x + r - 1. \tag{3.2.54}$$

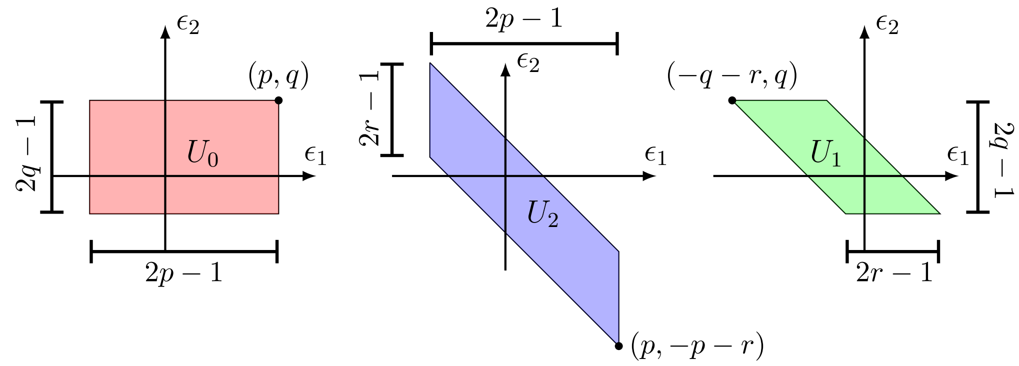


Figure 3.5: Regions U_ℓ.

Since $N \geq 3$ (3.2.45), from (3.2.25) we get for the one-loop part

$$Z^{\mathbb{P}^2}_{\text{1-loop}} = \prod_{i=0}^{N}\prod_{j=0}^{N-i} (a + (p-j)\epsilon_1 + (q-i)\epsilon_2) \prod_{i=0}^{N-3}\prod_{j=0}^{N-3-i} -(a + (p-1-j)\epsilon_1 + (q-1-i)\epsilon_2). \tag{3.2.55}$$

The double zero in $a = 0$ is hidden in the products

$$Z^{\mathbb{P}^2}_{\text{1-loop}} = -a^2 \underbrace{\prod_{i=0}^{N}\prod_{j=0}^{N-i}}_{(i,j)\neq(q,p)} (a+(p-j)\epsilon_1+(q-i)\epsilon_2) \underbrace{\prod_{i=0}^{N-3}\prod_{j=0}^{N-3-i}}_{(i,j)\neq(q-1,p-1)} -(a+(p-1-j)\epsilon_1+(q-1-i)\epsilon_2). \tag{3.2.56}$$

When evaluated in $a = 0$ the two products in (3.2.56) can be rewritten as

$$\prod_{(i,j)\in V_1\setminus(0,0)} (i\epsilon_1 + j\epsilon_2) \prod_{(i,j)\in V_2\setminus(0,0)} (i\epsilon_1 + j\epsilon_2) \tag{3.2.57}$$

where V_1, V_2 are two triangles depicted in figure 3.6 and defined as:

- V_1 is the triangle with vertices $\{(p,q), (-q-r,q), (p,-p-r)\}$. It is delimited by the three straight lines

$$x = p, \quad y = q, \quad y = -x - r. \tag{3.2.58}$$

- V_2 is the triangle with vertices $\{(-p+1,-q+1), (q+r-2,-q+1), (-p+1,p+r-2)\}$. It is delimited by the three straight lines

$$x = -p + 1, \quad y = -q + 1, \quad y = -x + r - 1. \tag{3.2.59}$$

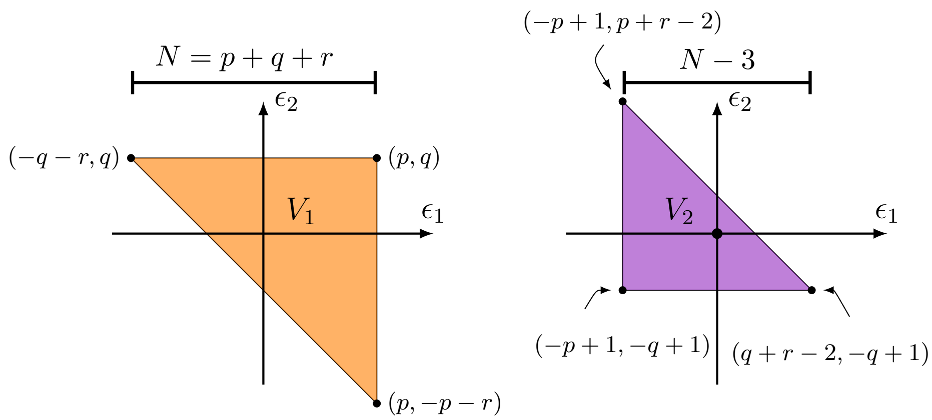


Figure 3.6: Regions V_1, V_2.

The residue evaluation is therefore

$$\begin{aligned} &\mathrm{Res}\big(Z_{\mathrm{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2)\big|a = 0\big) = \lim_{a\to 0} a\, Z_{\mathrm{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2) \\ &= \mathbf{q}^{-\frac{1}{4}(1-2c)-\frac{1}{4}(p+q+r)^2+pq+pr+qr} \prod_{(i,j)\in V_1\setminus(0,0)} (i\epsilon_1 + j\epsilon_2) \prod_{(i,j)\in V_2\setminus(0,0)} (i\epsilon_1 + j\epsilon_2) \\ &\quad\times \prod_{(i,j)\in U_0\setminus(0,0)} \frac{1}{(i\epsilon_1 + j\epsilon_2)} \prod_{(i,j)\in U_1\setminus(0,0)} \frac{1}{(i\epsilon_1 + j\epsilon_2)} \prod_{(i,j)\in U_2\setminus(0,0)} \frac{1}{(i\epsilon_1 + j\epsilon_2)}\, Z_{\mathrm{Res}}(\mathbf{q}). \end{aligned} \tag{3.2.60}$$

Comment: it is simple to verify that the number of points different from $(0,0)$ in the regions $U_\ell \cap \mathbb{Z}^2$ and $V_{1,2} \cap \mathbb{Z}^2$ sum together to an *even* number. This means that the total product over these regions in (3.2.60) is invariant under the reflection $(i,j) \to (-i,-j)$.

The final result (3.2.38) is recovered by imposing the stability conditions (3.2.37) on (3.2.60). The detailed derivation of these conditions is performed in appendix 3.A. Due to the strict triangle inequality we have

$$U_0 \cap U_1 \cap U_2 = U_0 \cap U_1 = U_0 \cap U_2 = U_1 \cap U_2 = V_1 \cap V_2; \tag{3.2.61}$$

and

$$(U_0 \cup U_1 \cup U_2) \cap \mathbb{Z}^2 = (V_1 \cup V_2) \cap \mathbb{Z}^2. \tag{3.2.62}$$

This means that (3.2.60) reduces to

$$\begin{aligned} &\operatorname{Res}(Z_{\text{full}}(\mathbf{q}; a, \epsilon_1, \epsilon_2)|a=0) = \\ &= \mathbf{q}^{-\frac{1}{4}(1-2c)-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \prod_{(i,j)\in[(V_1\cap V_2)\cap\mathbb{Z}^2]\setminus(0,0)} \frac{1}{(i\epsilon_1 + j\epsilon_2)} Z_{\text{Res}}(\mathbf{q}) \end{aligned} \tag{3.2.63}$$

Moreover we see from (3.2.39),(3.2.40),(3.2.41) and (3.2.58),(3.2.59) that $V_1 = \overline{T}_1$, $V_2 = \overline{T}_2$ where the bar indicates the reflection of the two axis highlighted above. Therefore the intersection $V_1 \cap V_2$ is precisely the region $D^{(p,q,r)}$ mirrored through the origin, and from the above comment this means that (3.2.63) is equal to (3.2.38) once summed over all the (proper) integers p, q, r.

Finally we show (3.2.61) (3.2.62). Eq.(3.2.61) comes directly from the construction of the five regions. Indeed each U_i shares a couple of “delimitation” parallel straight lines with another U_j and the other parallel couple with the remaining U_k. Moreover each U_i shares a couple of consecutive non-parallel lines with one V_i and the other couple with the other V_j. See figure 3.7. In formulae, we define the region $\langle r_i, r_j, r_k \dots \rangle$ as the convex hull of the intersection points of all the straight lines $r_i, r_j, r_k \dots$ and call

$$\begin{aligned} r_1 &= \{x = -p+1\}, & r_2 &= \{x = p\}, \\ r_3 &= \{y = -q+1\}, & r_4 &= \{y = q\}, \\ r_5 &= \{y = -x + r - 1\}, & r_6 &= \{y = -x - r\}. \end{aligned} \tag{3.2.64}$$

Then we have

$$\begin{aligned} &U_0 = \langle r_1, r_2, r_3, r_4 \rangle, \quad U_1 = \langle r_3, r_4, r_5, r_6 \rangle, \quad U_2 = \langle r_1, r_2, r_5, r_6 \rangle, \\ &V_1 = \langle r_2, r_4, r_6 \rangle, \quad V_2 = \langle r_1, r_3, r_5 \rangle, \end{aligned} \tag{3.2.65}$$

from which (3.2.61) directly follows.

We will now show that (3.2.62) is equivalent to the triangle inequality. Indeed in general $(V_1 \cup V_2) \cap \mathbb{Z}^2$ can exceed $(U_0 \cup U_1 \cup U_2) \cap \mathbb{Z}^2$, (causing the appearance of terms $(i\epsilon_1 + j\epsilon_2)^{+1}$ in (3.2.63)). This does not happen if the following three conditions are satisfied:

1. the segment between the vertex $(p, -q+1)$ of U_0 and the vertex $(p, r-p-1)$ of U_2 has distance strictly less than 2 (so that it cannot contain points of the lattice), so

$$-q + 1 - (r - p - 1) < 2 \iff -q - r + p + 2 < 2 \iff q + r > p; \tag{3.2.66}$$

   see figure 3.8.

2. the distance between the vertex $(-p+1, q)$ of U_0 and the vertex $(r-q-1, q)$ of U_1 must be strictly less than 2

$$-p+1-(r-q-1)<2 \iff -p-r+q+2<2 \iff p+r>q; \quad (3.2.67)$$

3. the distance between the vertex $(-p+1, -r+p-1)$ of U_2 and the vertex $(-r+q-1, -q+1)$ of U_1 must be strictly less than $2\sqrt{2}$

$$-p+1-(-r+q-1)<2 \iff -p-q+r+2<2 \iff p+q>r. \quad (3.2.68)$$

3.2.6 Exact results for even c_1

The case with even first Chern class is subtle because it allows for reducible connections. Namely the bundle can be written as a direct sum of line bundles, and the presence of this kind of connections makes the moduli space singular ([55] section 4.2).

Indeed one can saturate one of the three inequalities, and so define a strict *semi*-stable bundle, only if the sum of the three integers p, q, r is even

$$p+q \geq r, \qquad p+r \geq q, \qquad q+r \geq p, \qquad (3.2.69)$$

e.g. $p+q=r$. From the discussion about the supersymmetric fixed point locus of section 3.1 we know that we should consider also this kind of configurations in the construction of the partition function.

Technically nothing changes in the calculation since we already noticed that the full partition function $Z^{\mathbb{P}^2}_{\text{full}}$ has a pole at the origin only if $p, q, r > 0$. We have only to add the contribution saturating (3.2.69). These kind of configurations have non trivial automorphism group, that is the action of a $\mathbb{Z}_2$-group.[15] Therefore in counting gauge invariant configurations one has to divide by the order of the automorphism group, namely $\sharp\mathbb{Z}_2 = 2$. This appears as a coefficient 1/2 on the sum over stricly semi-stable configurations in the final result. Henceforth the gauge theoretical conjecture for the

[15] A reducible $U(2)$-bundle splits in the sum of two line bundles as $E = L_1 \oplus L_2$. There is a $\mathbb{Z}_2$ gauge symmetry exchanging the two line bundles as $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} L_1 & 0 \\ 0 & L_2 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} L_2 & 0 \\ 0 & L_1 \end{pmatrix}$.

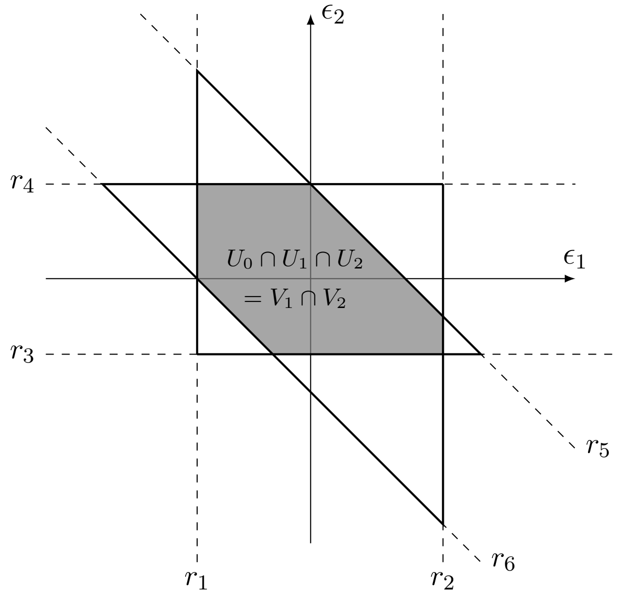


Figure 3.7: Intersections of the regions U_ℓ, V_1, V_2.

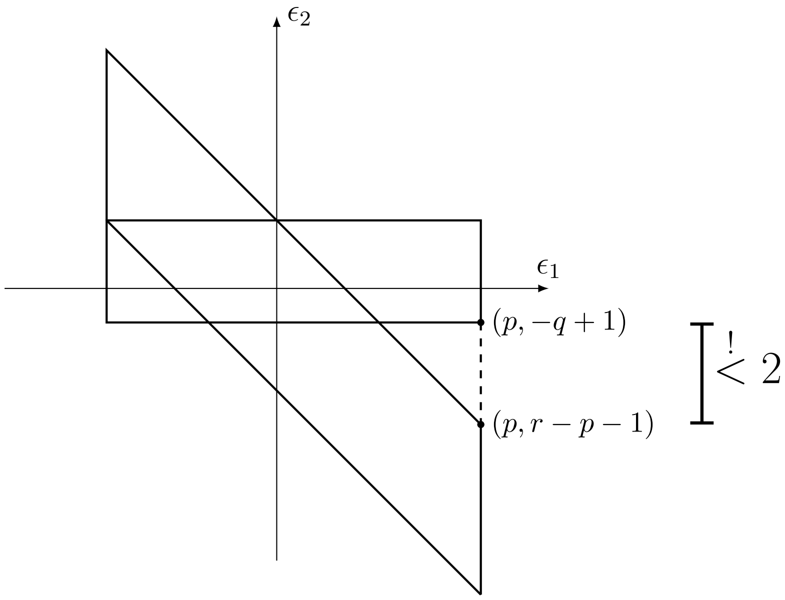


Figure 3.8: The union $V_1 \cup V_2$ exceed the union $U_0 \cup U_1 \cup U_2$ iff the strict triangle inequality is not satisfied.

generating function of equivariant Donaldson invariants reads[16],

$$
\begin{aligned}
Z^{\mathbb{P}^2}_{\mathcal{N}=2}(\mathbf{q},x,z,\epsilon_1,\epsilon_2)\big|_{c_1=0} = &\left(\sum_{\substack{\{p,q,r\} \\ \text{strictly stable}}} + \frac{1}{2}\sum_{\substack{\{p,q,r\} \\ \text{strictly semi-stable}}} \right) \mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \\
&\times \exp\left(-\frac{1}{4}\sum_{\ell=0}^{2} \frac{(a^{(\ell)}_{\text{res}})^2\, \imath^*_{P_{(\ell)}}(\alpha z + p x)}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}} \right) \prod_{(i,j)\in V_1\setminus(0,0)} (i\epsilon_1 + j\epsilon_2) \prod_{(i,j)\in V_2\setminus(0,0)} (i\epsilon_1+j\epsilon_2) \\
&\times \prod_{(i,j)\in U_0\setminus(0,0)} (i\epsilon_1+j\epsilon_2)^{-1} \prod_{(i,j)\in U_1\setminus(0,0)} (i\epsilon_1+j\epsilon_2)^{-1} \prod_{(i,j)\in U_2\setminus(0,0)} (i\epsilon_1+j\epsilon_2)^{-1} \\
&\times Z_{\text{inst}}(\mathbf{q}^{(0)}; a^{(0)}_{\text{res}}, \epsilon_1, \epsilon_2) Z_{\text{inst}}(\mathbf{q}^{(1)}; a^{(1)}_{\text{res}}, \epsilon_2-\epsilon_1, -\epsilon_1) Z_{\text{inst}}(\mathbf{q}^{(2)}; a^{(2)}_{\text{res}}, -\epsilon_2, \epsilon_1-\epsilon_2)
\end{aligned}
\tag{3.2.70}
$$

where $p+q+r = even$, $a^{(\ell)}_{\text{res}}$ are defined in (3.2.42), $(i,j)\in\mathbb{Z}^2$ and the regions U, V are defined in (3.2.52)–(3.2.53) and (3.2.58),(3.2.59). As (3.2.43), expression (3.2.70) is obtained taking $c=\frac{1}{2}$ in (3.1.5). For the stricly stable configurations the products in (3.2.70) can be rewritten as the product over the regions $D^{(p,q,r)}$ described below (3.2.43), but this is no more true for the strictly semi-stable ones (see the discussion at the end of subsection 3.2.5).

The result (3.2.70) provides a conjecture for equivariant $SU(2)$ Donaldson invariants. These are not known in the mathematical literature. In the next section we show that in the limit $\epsilon_1,\epsilon_2\to 0$ the formula (3.2.70) reproduces the $SU(2)$ Donaldson invariants for $\mathbb{P}^2$.

Let us underline that imposing the stability condition is crucial in order to get a finite $\epsilon_1,\epsilon_2\to 0$ limit for the gauge theory partition function. Indeed we checked that removing the stability condition from (3.2.43) and (3.2.70) would produce partition functions which are diverging in that limit.

3.2.7 Non equivariant limit

In this section we will compare our results in the limit $\epsilon_1,\epsilon_2\to 0$ with Donaldson invariants.

We start with the example of formula (3.2.43), that is known [75] to be the generating function of equivariant Donaldson invariants in the case of $U(2)$-bundle with $c_1=1$. This bundle can be reduced to a projective unitary group bundle $PU(2)=SU(2)/\mathbb{Z}_2=SO(3)$. Therefore, in the limit $\epsilon_1,\epsilon_2\to 0$ (3.2.43) should produce $SO(3)$-Donaldson invariants on $\mathbb{P}^2$. Indeed expanding (3.2.43) in series, before in $\mathbf{q}$ and then in x, z, and

[16] To obtain the partition function on $\mathbb{P}^2$ is enough to put to zero x and z in (3.2.70) so that also $\mathbf{q}^{(\ell)}\to\mathbf{q}$.

performing the limit[17] $\epsilon_1, \epsilon_2 \to 0$, we obtain

$$
\begin{aligned}
&\lim_{\epsilon_1,\epsilon_2\to 0} Z^{\mathbb{P}^2}_{\text{full}}(\mathbf{q}, x, z, \epsilon_1, \epsilon_2)\big|_{c_1=1} = \\
&= 1 + \mathbf{q}\frac{1}{16}\left(19\frac{x^2}{2!} + 5\frac{xz^2}{2!} + 3\frac{z^4}{4!}\right) + \mathbf{q}^2\frac{1}{32}\left(85\frac{x^4}{4!} + 23\frac{x^3z^2}{2!\,3!} + 17\frac{x^2z^4}{2!\,4!} + 19\frac{xz^6}{6!} + 29\frac{z^8}{8!}\right) \\
&\quad + \mathbf{q}^3\frac{1}{4096}\left(29557\frac{x^6}{6!} + 8155\frac{x^5z^2}{2!\,5!} + 6357\frac{x^4z^4}{4!\,4!} + 7803\frac{x^3z^6}{3!\,6!} + 12853\frac{x^2z^8}{2!\,8!} + \right. \\
&\qquad \left. + 26907\frac{xz^{10}}{10!} + 69525\frac{z^{12}}{12!}\right) + O(\mathbf{q}^4)
\end{aligned} \tag{3.2.71}
$$

this result is in perfect agreement with the literature [66] Theorem 4.4.

In the case $c_1 = 0$ we obtained expression (3.2.70), in this case the $U(2)$-bundle can be reduced to the $SU(2)$-bundle. With the same procedure as before we can check that the limit $\epsilon_1, \epsilon_2 \to 0$ produces $SU(2)$-Donaldson invariants on $\mathbb{P}^2$. Indeed we get

$$
\begin{aligned}
&\lim_{\epsilon_1,\epsilon_2\to 0} Z^{\mathbb{P}^2}_{\text{full}}(\mathbf{q}, x, z, \epsilon_1, \epsilon_2)\big|_{c_1=0} = \\
&= \mathbf{q}\left(-\frac{3}{2}z\right) + \mathbf{q}^2\left(-\frac{13}{8}\frac{x^2z}{2!} - \frac{xz^3}{3!} + \frac{z^5}{5!}\right) \\
&+ \mathbf{q}^3\left(-\frac{879}{256}\frac{x^4z}{4!} - \frac{141}{64}\frac{x^3z^3}{3!\,3!} - \frac{11}{16}\frac{x^2z^5}{2!\,5!} + \frac{15}{4}\frac{xz^7}{7!} + 3\frac{z^9}{9!}\right) \\
&+ \mathbf{q}^4\left(-\frac{36675}{4096}\frac{x^6z}{6!} - \frac{1515}{256}\frac{x^5z^3}{5!\,3!} - \frac{459}{128}\frac{x^4z^5}{4!\,5!} + \frac{51}{16}\frac{x^3z^7}{3!\,7!} + \frac{159}{8}\frac{x^2z^9}{2!\,9!} + 24\frac{xz^{11}}{11!} + 54\frac{z^{13}}{13!}\right) \\
&+ \mathbf{q}^5\left(-\frac{850265}{32768}\frac{x^8z}{8!} - \frac{143725}{8192}\frac{x^7z^3}{7!\,3!} - \frac{3355}{256}\frac{x^6z^5}{6!\,5!} - \frac{5}{16}\frac{x^5z^7}{5!\,7!} + \frac{2711}{64}\frac{x^4z^9}{4!9!} + \right. \\
&\qquad \left. + \frac{2251}{16}\frac{x^3z^{11}}{3!\,11!} + \frac{487}{2}\frac{x^2z^{13}}{2!\,13!} + 694\frac{xz^{15}}{15!} + 2540\frac{z^{17}}{17!}\right) + O(\mathbf{q}^6)
\end{aligned} \tag{3.2.72}
$$

and we again have agreement with the literature [66] Theorem 4.2. This show that formula (3.2.70) is indeed a good candidate for the generating function of equivariant Donaldson invariants for an $SU(2)$-bundle, even in the cases where reducible connections are present.

3.2.8 Remarkable identities from the evaluation of the partition function

In this subsection we specify our computation to the partition functions without any inserion of observables.

[17] The limit sets to zero also $h, \tilde{h}, \tilde{K}$ in (3.2.8), being these polynomials in ϵ_1, ϵ_2.

It was noticed in [144] that the partition function of twisted $\mathcal{N}=2$ Super Yang-Mills theory on a differentiable oriented four manifold is vanishing, due to the presence of ψ-zero modes. These span the tangent space of the instanton moduli space. Therfore the only case in which the partition function is non vanishing correspondes to zero-dimensional components of the moduli space. The partition function is a topological invariant counting, with signs dictated by their relative orientation, the number of the above connected components.

By inspecting our results on the pure partition functions, we obtain results in agreement with the above observation. This in turn implies some remarkable cubic identities on the Nekrasov partition function that we display below.

More explicitly, by computing the coefficients of the power series in $\mathbf{q}$ of the partition function (i.e. formula (3.2.38) for $c_1=1$ and formula (3.2.70) in the limit $x,z\to 0$ for $c_1=0$), one can see that they are almost all equal to zero! Actually only one term survives, namely $p=q=r=1$ that contributes to the $c_1=1$ case. So we can rewrite the partition function for the pure $\mathcal{N}=2$ theory as

$$Z^{\mathbb{P}^2}_{\mathcal{N}=2}(\mathbf{q})\big|_{c_1=1}=\mathbf{q}^{(1+c)/2},\qquad Z^{\mathbb{P}^2}_{\mathcal{N}=2}(\mathbf{q})\big|_{c_1=0}=0. \tag{3.2.73}$$

This result is in full agreement with the expected behavior of the equivariant partition function in the limit $\epsilon_1,\epsilon_2\to 0$. In this limit the partition function is expected to be a finite function of the gauge coupling. Indeed, looking at (3.2.38) at fixed power in the expansion in $\mathbf{q}$, all the dependence on ϵ_1,ϵ_2 appears in the product and in the $Z^{(\ell)}_{\text{inst}}$, the latter depending on ϵ_1,ϵ_2 in the denominators only. So, to obtain a finite limit for $\epsilon_1,\epsilon_2\to 0$, these terms should sum up to zero but for the term $p=q=r=1$ in which case both the product and the instanton partition functions contribute as 1. A similar argument holds for the case with $c_1=0$. As expected, the non zero term is the contribution of the zero dimensional moduli space components, since $\dim\mathcal{M}=D-3$ (where the discriminant D is given in (3.A.9)).

These results imply the following cubic identities for the Nekrasov partition function

$$\begin{aligned}\mathbf{q}^{-\frac{3}{4}}\sum_{\substack{\{p,q,r\}\\ \text{strictly stable}}}&\Bigg[\mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)}\prod_{\{(i,j)\}}\frac{1}{i\epsilon_1+j\epsilon_2}\\ &\times Z_{\text{inst}}\big(\mathbf{q};p\epsilon_1-q\epsilon_2,\epsilon_1,\epsilon_2\big)Z_{\text{inst}}\big(\mathbf{q};q(\epsilon_2-\epsilon_1)+r\epsilon_1,\epsilon_2-\epsilon_1,-\epsilon_1\big)\\ &\times Z_{\text{inst}}\big(\mathbf{q};-r\epsilon_2-p(\epsilon_1-\epsilon_2),-\epsilon_2,\epsilon_1-\epsilon_2\big)\Bigg]=1\end{aligned} \tag{3.2.74}$$

and

$$\begin{aligned}&\left(\sum_{\substack{\{p,q,r\}\\ \text{strictly stable}}}+\frac{1}{2}\sum_{\substack{\{p,q,r\}\\ \text{strictly semi-stable}}}\right)\Bigg[\mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)}\prod_{\substack{\{(i,j)\}\\ \{(k,l)\}}}\frac{i\epsilon_1+j\epsilon_2}{k\epsilon_1+l\epsilon_2}\\ &\times Z_{\text{inst}}\big(\mathbf{q};p\epsilon_1-q\epsilon_2,\epsilon_1,\epsilon_2\big)Z_{\text{inst}}\big(\mathbf{q};q(\epsilon_2-\epsilon_1)+r\epsilon_1,\epsilon_2-\epsilon_1,-\epsilon_1\big)\\ &\times Z_{\text{inst}}\big(\mathbf{q};-r\epsilon_2-p(\epsilon_1-\epsilon_2),-\epsilon_2,\epsilon_1-\epsilon_2\big)\Bigg]=0\end{aligned} \tag{3.2.75}$$

where the product on $\{i, j\}$ and $\{k, l\}$ in (3.2.74) and (3.2.75) can be read from (3.2.38) and (3.2.70) respectively.

3.3 $\mathcal{N} = 2^\star$ theory and Euler characteristics

In this section we extend our results to the presence of a hypermultiplet in the adjoint representation with mass M, namely to the so-called $\mathcal{N} = 2^\star$ theory. In the limit $M \to 0$, one gets $\mathcal{N} = 4$ gauge theory whose partition function is the generating function of the Euler characteristics of the moduli spaces of unframed semi-stable equivariant torsion free sheaves [142].

In the following we will compute the full $U(2)$ partition function of the $\mathcal{N} = 2^\star$ theory on $\mathbb{P}^2$ and, after an integration over the v.e.v. of the scalar field, analogous to the one performed in the previous section, we will take the massless limit checking the relation with the Euler characteristics computed in [102, 152, 142]. The insertion of the hypermultiplet modifies both the one-loop and the instanton part of the partition function. The one-loop partition function has the extra factor

$$Z^{\mathbb{P}^2}_{\text{1-loop,hyp}}(\vec{a}, M, \epsilon_1, \epsilon_2) = \prod_{\ell=0}^{2} \exp\left[\sum_{\alpha\neq\beta} \gamma_{\epsilon_1^{(\ell)},\epsilon_2^{(\ell)}}(a^{(\ell)}_{\alpha\beta} + M)\right]. \tag{3.3.1}$$

Following the same steps as in section 3.2.2, and assuming again $N > 2$ as in (3.2.45), we obtain similarly to (3.2.55)

$$\begin{aligned} Z^{\mathbb{P}^2}_{\text{1-loop,hyp}}(\vec{a}, M, \epsilon_1, \epsilon_2) = &\prod_{i=0}^{N}\prod_{j=0}^{N-i} (a + M + (p-j)\epsilon_1 + (q-i)\epsilon_2)^{-1} \times \\ &\prod_{i=0}^{N-3}\prod_{j=0}^{N-3-i} -(a - M + (p-1-j)\epsilon_1 + (q-1-i)\epsilon_2)^{-1}, \end{aligned} \tag{3.3.2}$$

where $N = p + q + r$ with p, q, r defined in (3.2.14). For the instanton part we should consider the appropriate recursion relation in the presence of an adjoint hypermultiplet that generalizes (3.2.32). The instanton partition function on $\mathbb{C}^2$ (3.2.29) in the presence of an adjoint hypermultiplet becomes

$$Z^{\mathbb{C}^2}_{\text{inst,adj}}(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2) = \sum_{\{Y_\alpha\}} \mathbf{q}^{|\vec{Y}|} z_{\text{adj}}(a, M, \vec{Y}, \epsilon_1, \epsilon_2) \tag{3.3.3}$$

where $\mathbf{q} = \exp(2i\pi\tau)$ and

$$z_{\text{adj}} = \prod_{\alpha,\beta=1}^{2}\prod_{s\in Y_\alpha} \frac{\Big(a_{\beta\alpha} - M - L_{Y_\beta}(s)\epsilon_1 + (A_{Y_\alpha}(s)+1)\epsilon_2\Big)\Big(a_{\alpha\beta} - M + (L_{Y_\beta}(t)+1)\epsilon_1 - A_{Y_\alpha}(t)\epsilon_2\Big)}{\Big(a_{\beta\alpha} - L_{Y_\beta}(s)\epsilon_1 + (A_{Y_\alpha}(s)+1)\epsilon_2\Big)\Big(a_{\alpha\beta} + (L_{Y_\beta}(t)+1)\epsilon_1 - A_{Y_\alpha}(t)\epsilon_2\Big)}. \tag{3.3.4}$$

A recursion relation for (3.3.4) similar to (3.2.32) is also reported in [127], and has the form

$$Z^{\mathbb{C}^2}_{\text{inst,adj}}(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2) = \left(\hat{\eta}(\mathbf{q})\right)^{-2\frac{(M-\epsilon_1)(M-\epsilon_2)}{\epsilon_1\epsilon_2}} H(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2), \tag{3.3.5}$$

where $\hat{\eta}(q) = \prod_{n=1}^{\infty}(1-q^n)$ and

$$H(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2) = 1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R^{\text{adj}}_{m,n}\, H\left(\mathbf{q};\, m\epsilon_1 - n\epsilon_2, M, \epsilon_1, \epsilon_2\right)}{(a - m\epsilon_1 - n\epsilon_2)(a + m\epsilon_1 + n\epsilon_2)} \tag{3.3.6}$$

with

$$R^{\text{adj}}_{m,n} = 2\left(\prod_{i=-m+1}^{m}\prod_{j=-n+1}^{n}(M - i\epsilon_1 - j\epsilon_2)\right) \Big/ \Bigg(\underbrace{\prod_{i=-m+1}^{m}\prod_{j=-n+1}^{n}}_{(i,j)\neq\{(0,0),(m,n)\}}(i\epsilon_1 + j\epsilon_2)\Bigg). \tag{3.3.7}$$

The instanton partition function for $\mathbb{P}^2$ is obtained by multiplying (3.3.5) over the three patches

$$\begin{aligned} Z^{\mathbb{P}^2}_{\text{inst,adj}}(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2) &= \prod_{\ell=0}^{2} Z^{\mathbb{C}^2}_{\text{inst,adj}}(\mathbf{q};\, a^{(\ell)}, M, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}) \\ &= \left(\hat{\eta}(\mathbf{q})\right)^{-6} \prod_{\ell=0}^{2}\left(1 - \sum_{m,n=1}^{\infty} \frac{\mathbf{q}^{mn} R^{\text{adj},(\ell)}_{m,n}\, H\left(\mathbf{q};\, m\epsilon_1^{(\ell)} - n\epsilon_2^{(\ell)}, M, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)}\right)}{(a^{(\ell)} - m\epsilon_1^{(\ell)} - n\epsilon_2^{(\ell)})(a^{(\ell)} + m\epsilon_1^{(\ell)} + n\epsilon_2^{(\ell)})}\right). \end{aligned} \tag{3.3.8}$$

Before discussing the limit $M \to 0$ let us make a preliminary comment. First of all notice that, where z_{adj} (3.3.4) is regular, we have

$$\lim_{M\to 0} z_{\text{adj}}(a, M, \vec{Y}, \epsilon_1, \epsilon_2) = 1. \tag{3.3.9}$$

Since

$$\sum_{\{Y_\alpha\}} \mathbf{q}^{|\vec{Y}|} = \left(\hat{\eta}(\mathbf{q})\right)^{-2} \tag{3.3.10}$$

we get from (3.3.3), (3.3.9) and (3.3.5) that

$$\lim_{M\to 0} H\left(\mathbf{q};\, m\epsilon_1 - n\epsilon_2, M, \epsilon_1, \epsilon_2\right) = 1, \tag{3.3.11}$$

because in $a = m\epsilon_1 - n\epsilon_2$ we are away from the poles of H.

We will now compute the residue of Z_{full} in the origin as we did in section 3.2.5. We assume $M > 0$ and, since we want to take eventually the massless limit, M small enough not to meet poles of $Z_{\text{1loop,hyp}}$. We recall that

$$Z^{\mathcal{N}=2^\star}_{\text{full}} = Z_{\text{class}}\, Z_{\text{1loop}}\, Z_{\text{1loop,hyp}}\, Z_{\text{inst,adj}} \tag{3.3.12}$$

with components reported in (3.2.13), (3.2.55), (3.3.2) and (3.3.8) respectively. At the origin:

- Z_{class} and $Z_{\text{1loop,hyp}}$ have neither poles nor zeros,
- Z_{1loop} has a double zero,
- $Z_{\text{inst,adj}}$ has a triple pole.

Indeed we can write

$$Z^{\mathbb{P}^2}_{\text{1-loop}}(a,\epsilon_1,\epsilon_2)=a^2\prod_{(i,j)\in V_1\setminus(0,0)}(a+i\epsilon_1+j\epsilon_2)\prod_{(i,j)\in V_2\setminus(0,0)}(-a+i\epsilon_1+j\epsilon_2).$$

$$Z^{\mathbb{P}^2}_{\text{1-loop,hyp}}(a,M,\epsilon_1,\epsilon_2)=\prod_{(i,j)\in V_1}(a+M+i\epsilon_1+j\epsilon_2)^{-1}\prod_{(i,j)\in V_2}(-a+M+i\epsilon_1+j\epsilon_2)^{-1}. \tag{3.3.13}$$

where the region V_1 and V_2 are described in (3.2.58) and (3.2.59) respectively. Similarly to (3.2.48)

$$Z^{\mathbb{P}^2}_{\text{inst,adj}}=(\hat\eta(\mathbf{q}))^{-6}\frac{1}{a^3}\,\mathbf{q}^{pq+pr+qr}\,\tilde R^{\text{adj},(0)}_{p,q}\,\tilde R^{\text{adj},(1)}_{q,r}\,\tilde R^{\text{adj},(2)}_{r,p}\,H_{\text{Res}}(\mathbf{q};M)+O\left(\frac{1}{a^2}\right) \tag{3.3.14}$$

where

$$\tilde R^{\text{adj},(\ell)}_{m,n}=\frac{1}{a^{(\ell)}+m\epsilon_1^{(\ell)}+n\epsilon_2^{(\ell)}}R^{\text{adj},(\ell)}_{m,n} \tag{3.3.15}$$

and

$$\begin{aligned}H_{\text{Res}}(\mathbf{q};M)=&H(\mathbf{q};\,p\epsilon_1-q\epsilon_2,M,\epsilon_1,\epsilon_2)H(\mathbf{q};\,q(\epsilon_2-\epsilon_1)-r(-\epsilon_1),M,\epsilon_2-\epsilon_1,-\epsilon_1)\\&\times H(\mathbf{q};\,r(-\epsilon_2)-p(\epsilon_1-\epsilon_2),-\epsilon_2,M,\epsilon_1-\epsilon_2).\end{aligned} \tag{3.3.16}$$

By calculating the factors $R^{\text{adj},(\ell)}$ in $a=0$ we get

$$\tilde R^{(\ell)}=\frac{\prod_{(i,j)\in U_\ell}(M-i\epsilon_1-j\epsilon_2)}{\prod_{(i,j)\in U_\ell\setminus(0,0)}(i\epsilon_1+j\epsilon_2)}, \tag{3.3.17}$$

with U_ℓ defined in (3.2.52), (3.2.54), (3.2.53).

All in all, $Z^{\mathcal{N}=2^\star}_{\text{full}}$ has a simple pole located at the origin whose residue is[18]

$$\begin{aligned}&M^{-1}\text{Res}(Z^{\mathcal{N}=2^\star}_{\text{full}}(\mathbf{q};\,a,M,\epsilon_1,\epsilon_2)|a=0)=M^{-1}\lim_{a\to0}a\,Z^{\mathcal{N}=2^\star}_{\text{full}}(\mathbf{q};\,a,M,\epsilon_1,\epsilon_2)\\&=M^{-1}\mathbf{q}^{-\frac14(1-2c)c_1^2}\mathbf{q}^{-\frac14(p+q+r)^2}\\&\quad\times\frac{\prod_{(i,j)\in V_1\setminus(0,0)}(i\epsilon_1+j\epsilon_2)}{\prod_{(i,j)\in V_1}(M+i\epsilon_1+j\epsilon_2)}\frac{\prod_{(i,j)\in V_2\setminus(0,0)}(i\epsilon_1+j\epsilon_2)}{\prod_{(i,j)\in V_2}(M+i\epsilon_1+j\epsilon_2)}\\&\quad\times M^3\prod_{(i,j)\in U_0\setminus(0,0)}\frac{(M-i\epsilon_1-j\epsilon_2)}{(i\epsilon_1+j\epsilon_2)}\prod_{(i,j)\in U_1\setminus(0,0)}\frac{(M-i\epsilon_1-j\epsilon_2)}{(i\epsilon_1+j\epsilon_2)}\prod_{(i,j)\in U_2\setminus(0,0)}\frac{(M-i\epsilon_1-j\epsilon_2)}{(i\epsilon_1+j\epsilon_2)}\\&\quad\times(\hat\eta(\mathbf{q}))^{-6}\mathbf{q}^{pq+pr+qr}\,H_{\text{Res}}(\mathbf{q};M).\end{aligned} \tag{3.3.18}$$

[18] We normalize the integrated partition function with M^{-1} to get dimensionless quantities.

Taking the limit $M \to 0$, and using the fact that from (3.3.11) $H_{\text{Res}}(\mathbf{q}; M) \to 1$, we obtain

$$\lim_{M\to 0} \frac{1}{M} \text{Res}(Z_{\text{full}}^{\mathcal{N}=2^\star}(\mathbf{q};\, a, M, \epsilon_1, \epsilon_2)|a=0) = \left(\hat{\eta}(\mathbf{q})\right)^{-6} \mathbf{q}^{-\frac{1}{4}c_1^2} \mathbf{q}^{-\frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)}, \tag{3.3.19}$$

where $6 = \chi(\mathbb{P}^2) \cdot \text{rank}\,(U(2))$.

The complete result holds with both $c_1 = 0, 1$, once the contribution of the stricly semi-stable bundles (the ones allowing for reducible connections) are weighed with the factor 1/2 as in (3.2.70)

$$\begin{aligned} &Z_{\mathcal{N}=4}^{\mathbb{P}^2}(\mathbf{q}) = \left(\hat{\eta}(\mathbf{q})\right)^{-6} \times \\ &\times \sum_{c_1=0,1} \Bigg(\sum_{\substack{\{p,q,r\} \\ \text{strictly stable}}} + \frac{1}{2} \sum_{\substack{\{p,q,r\} \\ \text{strictly semi-stable}}} \Bigg) \mathbf{q}^{-\frac{1}{4}(1-2c)c_1^2 - \frac{1}{4}(p^2+q^2+r^2-2pq-2pr-2qr)} \end{aligned} \tag{3.3.20}$$

where p, q, r are positive integers with $p+q+r+c_1 = even$, and they satisfy respectively strict triangle inequalities in the stable case and large triangle inequalities in the semi-stable one. In the case with only strictly stables configurations this result reduce to the one computed by Kool in [102] when we take the expansion in the second Chern class c_2 $(c = 1)$.

Moreover we have checked up to high orders in the power series that for both $c_1 = 0, 1$ (3.3.20) is in agreement with the mock-modular form of [142]

$$\begin{aligned} Z_0(\mathbf{q}) &= \left(\hat{\eta}(\mathbf{q})\right)^{-6} \sum_{n=0}^{\infty} 3H(4n)\mathbf{q}^n \qquad & c_1 = 0 \\ Z_1(\mathbf{q}) &= \left(\hat{\eta}(\mathbf{q})\right)^{-6} \sum_{n=0}^{\infty} 3H(4n-1)\mathbf{q}^n \qquad & c_1 = 1 \end{aligned} \tag{3.3.21}$$

where $H(n)$ is the Hurwitz class number [49].

3.A Stability conditions for equivariant vector bundles

In this Appendix we make a dictionary between Klyachko's classification of semi-stable equivariant vector bundles on $\mathbb{P}^2$ [100] (for a review see [101], section 4) and the gauge theory fixed point data we sum over in the partition function, in order to discover the constraints to be imposed because of the stability conditions. Klyachko's main result is that equivariant vector bundles on $\mathbb{P}^2$ can be completely described by sets of decreasing filtrations of vector spaces $E_\ell(i)$, one filtration for each open subset of the standard cover $\mathcal{U}_\ell$ $(\ell = 0, 1, 2)$. Explicitly

$$E = E_\ell(I_\ell) \supsetneq E_\ell(I_\ell+1) \supset \cdots \supset E_\ell(I_\ell+n_\ell) \supsetneq E_\ell(I_\ell+n_\ell+1) = 0 \tag{3.A.1}$$

where $E \simeq \mathbb{C}^N$ is the fiber of the bundle (N is the rank of the bundle) at the ℓ-th point and $E_\ell(i) = E$, $\forall i \leq I_\ell$ and $E_\ell(i) = 0$, $\forall i > I_\ell + n_\ell$. The explicit form of the vector subspaces $E_\ell(i)$ in the filtration (3.A.1) for a given equivariant bundle is reported in [100]. Starting from the filtration (3.A.1) it is possible to compute the Chern classes of the vector bundle by the following formula

$$c_1(E) = \sum_{\ell=0}^{2}\sum_i i \dim\left(E_\ell(i)/E_\ell(i+1)\right),$$
$$-\,ch_2(E) \equiv c_2 - \frac{1}{2}c_1^2 = -\frac{1}{2}\sum_{\ell=0}^{2}\sum_i i^2 \dim\left(E_\ell(i)/E_\ell(i+1)\right) - \sum_{\ell<\ell'}\sum_{i,j} ij \dim E^{[\ell\ell']}(i,j), \tag{3.A.2}$$

where

$$E^{[\ell\ell']}(i,j) := E_\ell(i)\cap E_{\ell'}(j)/\left(E_\ell(i+1)\cap E_\ell(j) + E_\ell(i)\cap E_\ell(j+1)\right). \tag{3.A.3}$$

Let us consider in detail the case of $N=2$. The relevant steps of the filtration are the ones where the dimension of the subspaces jumps. In the rank two case these are two of them: $i = I_\ell$ in which the dimension jumps from 2 to 1, and $i = I_\ell + n_\ell$ when it jumps from 1 to 0. In particular $n_\ell = \sharp\{i|\dim E_\ell(i) = 1\}$. We then obtain

$$\begin{aligned} c_1(E) &= \sum_{\ell=0}^{2}(2I_\ell + n_\ell), \\ -\,ch_2(E) \equiv c_2 - \frac{1}{2}c_1^2 &= -\frac{1}{2}\sum_{\ell=0}^{2}\left(I_\ell^2 + (I_\ell+n_\ell)^2\right) - \sum_{\ell\neq\ell'} I_\ell(I_{\ell'}+n_{\ell'}). \end{aligned} \tag{3.A.4}$$

To compare with the gauge theory it is more convenient to use the discriminant D, that for $N=2$ is

$$\frac{1}{4}D(E) := c_2 - \frac{1}{4}c_1^2 \equiv -ch_2 + \frac{1}{4}c_1^2 = -\frac{1}{4}\left(\sum_{\ell=0}^{2} n_\ell^2 - \sum_{\ell<\ell'} 2n_\ell n'_\ell\right). \tag{3.A.5}$$

Actually this quantity D has a more fundamental geometric interpretation, indeed it completely determines the isomorphism class of the moduli space $\mathcal{M}(c_1,c_2)$ of the equivariant bundles with given Chern classes c_1 and c_2. In the gauge theory parametrization the first Chern class is

$$c_1(\mathcal{E}) = \sum_{\ell=0}^{2}\sum_{\alpha=1}^{2} k_\alpha^{(\ell)}. \tag{3.A.6}$$

To extract the ch_2 for unframed sheaves $\mathcal{E}_0$ we just expand

$$Z_{\text{full}} = \mathbf{q}^{-ch_2(\mathcal{E}_0)} \times \left(\cdots\right) \tag{3.A.7}$$

so that $ch_2(\mathcal{E}_0)$ can be directly obtained from (3.2.60)

$$\begin{aligned} -ch_2(\mathcal{E}_0) &= \sum_{\ell=0}^{2}|\vec{Y}^{(\ell)}| - \frac{1}{4}\left[\left(\sum_{\ell=0}^{2} k_1^{(\ell)} + k_2^{(\ell)}\right)^2 + \sum_{\ell=0}^{2}(k^{(\ell)})^2 - \sum_{\ell<\ell'} 2k^{(\ell)}k^{(\ell')}\right], \\ &= \sum_{\ell=0}^{2}|\vec{Y}^{(\ell)}| - ch_2(E) \end{aligned} \tag{3.A.8}$$

where $k^{(\ell)} := k_1^{(\ell)} - k_2^{(\ell)}$ and we isolated in the second line the vector bundle contribution from the one of the ideal sheaves. The discriminant of the vector bundle E is then

$$\frac{1}{4}D(E) := -ch_2(E) + \frac{1}{4}c_1(E)^2 = -\frac{1}{4}\left(\sum_{\ell=0}^{2}(k^{(\ell)})^2 - \sum_{\ell<\ell'} 2k^{(\ell)}k^{(\ell')}\right). \tag{3.A.9}$$

Comparing (3.A.2) and (3.A.5) with (3.A.6) and (3.A.9) is immediately clear what the dictionary between gauge theory and Klyachko's parameters is

$$I_\ell = \min(k_1^{(\ell)}, k_2^{(\ell)}), \quad I_\ell + n_\ell = \mathrm{Max}(k_1^{(\ell)}, k_2^{(\ell)}), \quad n_\ell = k^{(\ell)} = |k_1^{(\ell)} - k_2^{(\ell)}|. \tag{3.A.10}$$

Namely the $k_\alpha^{(\ell)}$ are labeling the positions of the jumps in the filtration. Then by making use of Weyl symmetry one can always assume $k_1^{(\ell)} \geq k_2^{(\ell)}$, which we used in the main text.

By using the dictionary (3.A.10) it is possible to finally read the stability conditions for the equivariant vector bundles directly from the following

Theorem (Klyachko[100]): *The equivariant vector bundle on* $\mathbb{P}^2$ *defined by the filtrations* (3.A.1) *is slope-stable iff for any proper subspace* $0 \subsetneq F \subsetneq E$ *one has for* $\tilde{\imath} \ll 0$

$$\sum_{\ell=0}^{2}\sum_{i>\tilde{\imath}} \frac{\dim(E_\ell(i)\cap F)}{\dim F} < \sum_{\ell=0}^{2}\sum_{i>\tilde{\imath}} \frac{\dim(E_\ell(i))}{\dim E}. \tag{3.A.11}$$

The slope-semi-stable case has a large inequality in (3.A.11).

We work out explicitly the case of $N = 2$. The three filtrations for $\mathbb{P}^2$ are of this form

$$E = \mathbb{C}^2 \supsetneq W_\ell \supset \cdots \supset W_\ell \supsetneq 0 \tag{3.A.12}$$

for each $\ell = 0, 1, 2$. Here W_ℓ is a line in $\mathbb{C}^2$, so $W_\ell \in Gr(1,2) \simeq \mathbb{P}^1$ and appears n_ℓ time in the filtration since $n_\ell = \sharp\{i|\dim E_\ell(i) = 1\}$.

We can assume that all W_ℓ ($\ell = 0,1,2$) are distinct[19] and also that $n_\ell > 0$, $\forall\ell$. Indeed it turns out that this is the only relevant case for stability. Either if two or more W_ℓ are equal, or if at least one $n_\ell = 0$, the bundle described by such a filtration does not admit stability, i.e. the strict inequalities (3.A.11) are mutually incompatible.

Finally we apply the theorem $\forall\, F \subsetneq E = \mathbb{C}^2$. The relevant conditions come from the choices $F = W_\ell$, $\ell = 0,1,2$. The only contribution in (3.A.11) that is not equal on the r.h.s. and l.h.s. of the inequality is the one relative to the one-dimensional n_ℓ subspaces W_ℓ of the filtrations. Eventually we obtain conditions on n_0, n_1, n_2, namely they have to satisfy strict triangle inequalities

$$n_\ell + n_{\ell'} > n_{\ell''}, \quad \text{for all the choices } \{\ell,\ell',\ell''\} = \{0,1,2\}. \tag{3.A.13}$$

The dictionary (3.A.10) implies that the gauge parameters $k^{(0)}, k^{(1)}, k^{(2)}$ (often called p, q, r in the main text) have to satisfy the same inequalities.

[19] We have actually used this assumption when computing (3.A.4).

Chapter 4

Supersymmetric theory on $5d$ and $6d$ superconformal index.

In this Chapter we study five-dimensional gauge theories on compact manifolds. We focus on certain classes of quiver gauge theories obtained via circle compactification of systems of M5-branes probing a transverse ALE orbifold singularity.

We compute the partition function for the abelian theories analyzing the relevant basic five-dimensional instanton operators which generate the full infinite tower of non-perturbative corrections. We focus on the theories on $S^4 \times S^1$ and S^5, whose partition functions present a factorization into bulding blocks similarly to the four-dimensional case. These building blocks are glued together using their modular properties under transformation of the equivariant parameters.

We then verify the symmetry enhancement for these theories predicted by the S-duality of the corresponding pq-web systems of five-branes, writing their partition functions in terms of characters of the expected enhanced global symmetries.

Our main result is the identification of the partition function for a circular quiver gauge theory with k abelian nodes with the $(1,0)$ six-dimensional super conformal index for a tensor multiplet and k^2 hypermultiplets, which we compute indipendently via letter counting of BPS states. This generalizes the result of [106] for $k = 1$.

Finally we give the interpretation of these results in terms of representation theory of q-deformed infinite-dimensional Lie algebrae. Five-dimensional quiver gauge theories partition functions can be interpreted as correlators of vertex operators of $q\mathcal{W}$-algebrae. The pq-web S-duality suggests relations among correlators of different $q\mathcal{W}$-algebrae, which we check in some examples in Section 4.5.

4.1 M5 branes on $\mathbb{C}^2/\mathbb{Z}_k$: 6d and 5d gauge theory descriptions

We start from N M5 branes sitting at the tip of the orbifold $\mathbb{C}^2/\mathbb{Z}_k$ singularity. This is an interacting superconformal $(1,0)$ field theory that we call $\mathcal{T}^{6d}_{k,A_{N-1}}$.

We can gain some knowledge about this class of theories reducing to Type IIA on a circle inside the $\mathbb{C}^2/\mathbb{Z}_k$: the M5's become NS5 branes and the orbifold geometry $\mathbb{C}^2/\mathbb{Z}_k$ becomes k D6 branes. So we end up with the following brane setup [39, 40, 84]: N NS5 branes sitting on top of a stack of k D6's.

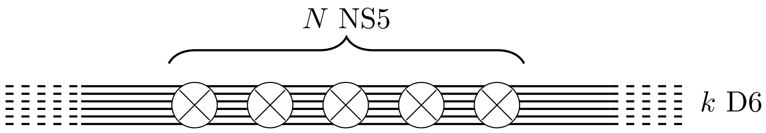


Figure 4.1: N NS5 branes on top of k D6 branes.

If we separate the NS5 branes we go on the nowadays called tensor branch [51], which gives a Lagrangian IR description of the deformed SCFT. On the tensor branch the field theory is a linear quiver $SU(k)^{N-1}$ of the form

$$\boxed{k} - (k) - (k) - \cdots\cdots - (k) - (k) - \boxed{k}\ . \tag{4.1.1}$$

There are also N tensor multiplets parameterizing the positions of the N NS5 branes. Both from the Type IIA brane setup and from the quiver, it is easy to see the global symmetry $SU(k)^2$. There is also an additional global $U(1)$ symmetry, that acts on all the N bifundamentals hypers with charge 1. For $N = 1$ the theory is free, we will compute its superconformal index and compare it with $5d$ partition functions. Notice that it is not the usual gauge theory orbifold of the $(2, 0)$ free supermultiplet.

Compactification to $5d$: S-duality for the rectangular pq-web.

Reducing N M5 branes sitting at the tip of the orbifold $\mathbb{C}^2/\mathbb{Z}_k$ singularity on a circle transverse to $\mathbb{C}^2/\mathbb{Z}_k$ and along the M5's, we get N D4 branes sitting at the tip of the orbifold $\mathbb{C}^2/\mathbb{Z}_k$ singularity, so the gauge theory can be understood using the methods of [61]. One alternative description is given in terms of a pq-web of 5 branes in type IIB: the branes are on a cilinder $\mathbb{R} \times S^1$, k D5 branes along $\mathbb{R}$ and N NS5 branes along the S^1. If this pq-web was on $\mathbb{R}^2$ the gauge theory would have been the $SU(N)^{k-1}$ linear quiver, while putting the pq-web on the cilinder the field theory becomes the 5d $\mathcal{N} = 1$ circular quiver $SU(N)^k$ gauge theory, that we call $\mathcal{T}^{5d}_{k,A_{N-1}}$

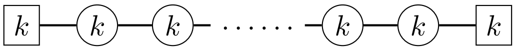
$$|| - (N) - (N) - \cdots\cdots - (N) - (N) - || \tag{4.1.2}$$

The pq-web on $\mathbb{R}^2$ and 5d S-duality.

Let us first analyze the brane setup on $\mathbb{R}^2$, a pq-web of N D5's intersecting k NS5's [83, 3]. The gauge theory is a linear quiver $SU(N)^{k-1}$ with N flavors at both ends:

$$\boxed{N} - (N) - (N) - \cdots\cdots - (N) - (N) - \boxed{N}\ . \tag{4.1.3}$$

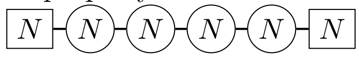
It is also possible to perform a type IIB S-duality on the brane setup, getting the pq-web of k D5's intersecting N NS5's. The gauge theory in this case is the linear quiver $SU(k)^{N-1}$ with k flavors at both ends:


. (4.1.4)

The right way to think about this 5d "duality" is that there is a strongly coupled 5d SCFT corresponding to the completely unresolved pq-web, where all branes are on top of each other, emanating from a single point and respecting a rescaling symmetry. This UV SCFT admits relevant deformations that can lead to either IR Lagrangian QFT: $SU(k)^{N-1}$ or $SU(N)^{k-1}$, which are clearly perturbatively different. However, if we are able to perform computations in the IR QFT's that can be uplifted to the strongly coupled UV SCFT, like the partition function on $S^4 \times S^1$, then the results of the two computations should agree [14, 26, 113].

Instanton operators and global symmetry

5d gauge theories contain non perturbative operators $\mathfrak{I}$, charged under the topological symmetries whose currents are $*tr(F^2)$. When inserted at a point in space-time, the flux of $tr(F^2)$ on the sphere S^4 surrounding the point measures the instanton charges of the operators [150].

This is analogous to the 3d case, where monopole operators carry a flux for $tr(F)$ on an S^2. In a balanced linear quiver, there is a special set of 'minimal' monopole or instanton operators, see [25] for a recent discussion in the case of $3d\,\mathcal{N}=2$ quivers. Their topological charges are 0 or 1 and are defined by the property that the non vanishing charges are contiguous. For instance for a quiver  we can organize the $4+3+2+1$ basic instanton states as

$$\begin{pmatrix} \mathfrak{I}^{1,0,0,0} & \mathfrak{I}^{1,1,0,0} & \mathfrak{I}^{1,1,1,0} & \mathfrak{I}^{1,1,1,1} \\ & \mathfrak{I}^{0,1,0,0} & \mathfrak{I}^{0,1,1,0} & \mathfrak{I}^{0,1,1,1} \\ & & \mathfrak{I}^{0,0,1,0} & \mathfrak{I}^{0,0,1,1} \\ & & & \mathfrak{I}^{0,0,0,1} \end{pmatrix} \tag{4.1.5}$$

where the superscripts denote the topological charges of the operator $\mathfrak{I}$ under each gauge group.

In 3d for $\mathcal{N}=4$ theories it is known that these non perturbative operators form a supermultiplet whose primary component has scaling dimension 1. Such supermultiplets contain conserved currents with scaling dimension 2. For a quiver $U(N)^{k-1}$, putting together these $k(k-1)/2$ basic monopoles, the correponding $k(k-1)/2$ anti-monopoles and the $k-1$ topological $U(1)$ currents, we get the k^2-1 currents of the enhanced $SU(k)$.

In 5d with $\mathcal{N}=1$ supersymmetry the story should be similar, but it is not much discussed in the literature. Symmetry enhancements of this type have been studied in [135], see also [153, 151, 50].

One important feature of these instanton operators is their baryonic charge spectrum: the instanton operators are charged under the Abelian factors of the global symmetries, which are usually called baryonic symmetries. The charges can in principle be computed studying fermionic zero modes around the instanton background. It turns out that a basic instanton state whose topological charges are 1 from node i to node j is charged precisely under the symmetry that rotates the i^{th} and the $(j+1)^{th}$ bifundamentals. This is the case both in $\mathcal{N}=2$ 3d theories [2, 25] and in $\mathcal{N}=1$ 5d theories. Let us define i^{th} baryonic symmetry $U(1)_{bar;i}$ to act with charge $+1$ and -1 on the $(i-1)^{th}$ and the i^{th} bifundamental, respectively. Then the basic instantons have baryonic charges equal to N times the topological charges. So, denoting as in [135],

$$U(1)_{i,\pm} \equiv \frac{1}{2}\left(U(1)_{top;\,i} \pm \frac{U(1)_{bar;i}}{N}\right) \tag{4.1.6}$$

the basic instantons are charged under $U(1)_{i,+}$ and neutral under $U(1)_{i,-}$. The corresponding anti-instantons are neutral under $U(1)_{i,+}$ and are charged under $U(1)_{i,-}$.

Armed with these results we can study the global symmetries that can be inferred from the low energy Lagrangian description. In the gauge theory $SU(N)^{k-1}$, each gauge group $U(N)$ gives a $U(1)$ "topological" or "instantonic" global symmetry, whose current is $*tr(F^2)$. Each bifundamental hypermultiplet is charged under a standard $U(1)$ "baryonic" symmetry. The theory enjoys a $U(1)^{k-1}_{top} \times U(1)^k_{bar} \times SU(N)^2$ global symmetry. This global symmetry is actually enhanced in the UV SCFT. In [135] it is shown how topological and baryonic symmetries can enhance to a non Abelian group: if we have a IR quiver (or a sub quiver) where every node is balanced, then the global symmetry of the UV SCFT is the square of the group whose Dynkin diagram is the quiver in question. A $U(N)$ node with zero Chern-Simon coupling is balanced if the total number of flavors is precisely $2N$. Here the quiver has the shape of the $A_{k-1} = SU(k)$ Dyinkin diagram so the global symmetry enhancement in the UV is

$$U(1)^{k-1}_{+} \times U(1)^{k-1}_{-} \to SU(k)_+ \times SU(k)_- \tag{4.1.7}$$

Starting from the S-dual gauge theory $SU(k)^{N-1}$, we can repeat the same arguments. In both models one concludes that the total global symmetry in the UV is

$$SU(k)^2 \times SU(N)^2 \times U(1) \tag{4.1.8}$$

This is a well known first check of the pq-web S-duality.

pq-web on the cilinder: 6d/5d duality

In the case $k=1$, a well known conjecture [60, 105] relates the 5d $\mathcal{N}=2$ field theory $\mathcal{T}^{5d}_{1,A_N}$ to the 6d $(2,0)$ type A_N $\mathcal{T}^{6d}_{1,A_N}$.

Here we are adding the orbifold $\mathbb{C}^2/\mathbb{Z}_k$, and it is natural to conjecture a relation between $\mathcal{T}^{5d}_{k,A_N}$ and $\mathcal{T}^{6d}_{k,A_N}$.

Compactifying the pq-web on a circle we are gauging the $SU(N)$ symmetries together, so the quiver is now the Dynkin diagram of $\widehat{A_{k-1}}$, the affine extension of $SU(k)$.

The two $SU(N)$ global symmetries are lost, but we gain one additional topological $U(1)$ Also, the sum of all baryonic symmetries acts trivially on the theory. The remaining $U(1)^k_{top} \times U(1)^{k-1}_{bar}$ symmetry is enhanced to the infinite dimensional group $\widehat{A_{k-1} \times A_{k-1}}$, as argued in [135]. There is also a $U(1)$ symmetry acting on all bifundamentals with charge $+1$.

For the circular quiver there are $k(k-1)$ basic instanton operators with the property that at least one topological charge is zero. We call these 'non-wrapping' instantons. There are k 'non-wrapping' instantons of length l, with $l = 1, 2, \ldots, k-1$. However thare are also 'wrapping' instantons of the form

$$\mathfrak{I}^{(1,1,\ldots,1)} \tag{4.1.9}$$

In section 4.4 we will show explicitly how, for $N = 1$, wrapping instantons corresponds to Kaluza-Klein modes, summing all of them reproduces a $6d$ $(1,0)$ superconformal index.

4.2 6d $(1,0)$ superconformal index

In this section we derive the superconformal indices for the $6d$ free $(1,0)$ supermultiplets, using letter counting.

The 6d $(1,0)$ and $(2,0)$ superconformal indices are discussed in [31]. Here we only need the $(1,0)$ case: the superconformal algebra is $osp(6,2|2)$ with R-symmetry $Sp(2) \simeq SU(2)$. The supercharges Q^i_α transform in the $(2,4)$ of $SU(2)_R \times SO(6)$.

Picking an appropriate supercharge Q and its conjugate $Q^\dagger$, it is possible to define a Witten index, with fugacities associated to the symmetries that commute with Q and $Q^\dagger$. The index reads

$$\mathcal{I} = tr(-1)^F q_0^{J_{12}+R} q_1^{J_{34}+R} q_2^{J_{56}+R}. \tag{4.2.1}$$

Only states with $\{Q, Q^\dagger\} = \delta = 0$ contribute to the superconformal index, where

$$\delta = \Delta - J_{12} - J_{34} - J_{56} - 4R. \tag{4.2.2}$$

Δ is the scaling dimension of the states, $J_{i\,i+1}$ are the angular momenta on the three hortogonal planes in $\mathbb{R}^6$, R is the $SU(2)_R$ spin.

Let us study explicitly the free superconformal multiplets: hypermultiplet $\{\Phi, \psi\}$ and self-dual tensor multiplet $\{H^+, \eta, \phi\}$.

Letter	Δ	J_{12}	J_{34}	J_{56}	R	$SO(6)$ irrep	$\delta=\Delta-\sum J-4R$	Index
Φ	2	0	0	0	$\pm1/2$	1	2 ± 2	$\sqrt{q_0q_1q_2}$
ψ	5/2	1/2	1/2	$-1/2$	0	4	2	0
ψ	5/2	1/2	$-1/2$	1/2	0	4	2	0
ψ	5/2	$-1/2$	1/2	1/2	0	4	2	0
ψ	5/2	$-1/2$	$-1/2$	$-1/2$	0	4	4	0
ϕ	2	0	0	0	0	1	2	0
η	5/2	1/2	1/2	$-1/2$	$\pm1/2$	4	2 ± 2	$-q_0q_1$
η	5/2	1/2	$-1/2$	1/2	$\pm1/2$	4	2 ± 2	$-q_0q_2$
η	5/2	$-1/2$	1/2	1/2	$\pm1/2$	4	2 ± 2	$-q_1q_2$
η	5/2	$-1/2$	$-1/2$	$-1/2$	$\pm1/2$	4	4 ± 2	0
H^+	3	1	1	1	0	10	0	$q_0q_1q_2$
H^+	3	-1	1	1	0	10	2	0
H^+	3	1	-1	1	0	10	2	0
H^+	3	1	1	-1	0	10	2	0
H^+	3	±1	0	0	0	10	3 ± 1	0
H^+	3	0	±1	0	0	10	3 ± 1	0
H^+	3	0	0	±1	0	10	3 ± 1	0
$\partial_{1,2}$	1	±1	0	0	0	6	1 ± 1	q_0
$\partial_{3,4}$	1	0	±1	0	0	6	1 ± 1	q_1
$\partial_{5,6}$	1	0	0	±1	0	6	1 ± 1	q_2

The two free supermultiplets are simple cases in the list of all possible short unitary representations of the $6d$ minimal susy superconformal algebra $osp(6,2|2)$ [110, 52]. The full classification is given in terms of the $SO(6) \simeq SU(4)$ Dynkin labels of the superconformal primary of the entire superconformal multiplet.

In the case of the half hypermultiplet the superconformal primary is the $\Delta = 2$ complex scalar Φ, transforming in the 2 of $SU(2)_R$. Acting with the supercharges Q^i_α, we obtain a $SU(2)_R$ singlet fermion ψ with $\Delta = 5/2$, transforming in the 4 of $SO(6)$, while the $SU(2)_R$-triplet is a null state.

In the case of the self-dual tensor multiplet, the superconformal primary is the $\Delta = 2$ real scalar ϕ, an $SU(2)_R$-singlet. Acting with the supercharges Q^i_α, we obtain a $SU(2)_R$-doublet fermion η with $\Delta = 5/2$, transforming in the 4 of $SO(6)$. Acting on η there is a null state (recall that for $SO(6)$, $4 \otimes 4 = 6 \oplus 10$) in the 6 of $SO(6)$ and the self-dual tensor in the 10 of $SO(6)$.

Using Table 4.2, the two indices are computed

$$\mathcal{I}_{1/2hyper} = \frac{\sqrt{q_0q_1q_2}}{(1-q_0)(1-q_1)(1-q_2)} \tag{4.2.3}$$

$$\mathcal{I}_{SDtensor} = \frac{q_0q_1q_2 - q_0q_1 - q_1q_2 - q_0q_2}{(1-q_0)(1-q_1)(1-q_2)} \tag{4.2.4}$$

In section 4.4 it will be easy, using these results, to write down the superconformal index of the $(1,0)$ SCFT corresponding to 1 M5 brane at the $\mathbb{C}^2/\mathbb{Z}_k$ orbifold, that on the tensor branch is simply k^2 free hypers plus 1 self-dual tensor.

4.3 Exact partition functions: 5d Abelian linear quiver

For $N = 1$ it is possible to compute the Nekrasov instanton partition [119] function explicitly to all orders in the instanton fugacities. We review the definition of the Nekrasov partition function in appendix 4.A. We first consider the simpler case of the linear quiver and we postpone the discussion on the circular quiver, which is the main result of this Chapter, to the next section. Although the 5d Nekrasov partition function of the linear quiver can be inferred from to the topological string amplitudes computed in [88, 89, 136, 14, 113], here we perform a direct gauge theory calculation.

We need to take the multi-particle partition function generated by the basic instantons we described in section 4.1. This is done by using the so called Plethystic Exponential $PE[f]$ [24]:

$$PE[f(t_1, t_2, \ldots, t_K)] = \exp\left[\sum_{n=1}^{\infty} \frac{f(t_1^n, t_2^n, \ldots, t_K^n)}{n}\right]. \tag{4.3.1}$$

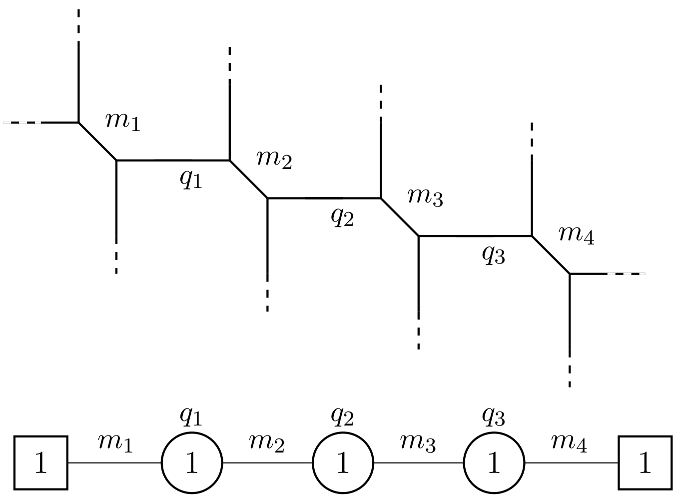


Figure 4.2: 1 D5 and $k = 4$ NS5 on the plane.

For the Abelian linear quiver with $k-1$ nodes, there are $k(k-1)/2$ basic instantons with topological charges

$$\mathfrak{I}^{(0,0,\ldots,0,1,1,\ldots,1,0,\ldots,0)}.$$

The formula is thus the PE of a sum of $k(k-1)/2$ terms:

$$\mathcal{Z}^{\mathbb{R}^4\times S^1}_{linear,inst} = PE\left[\frac{\sum_{i=1}^{k-1}\sum_{l=0}^{k-1}(\prod_{s=i}^{i+l} q_s)(\prod_{r=i}^{i+l-1} m_r)(1-m_{i-1})(1-m_{i+l}t_1t_2)}{(1-t_1)(1-t_2)}\right] \tag{4.3.2}$$

$l = 0, 1, \ldots, k-1$ is the length of the basic instantons, i.e. the number of non zero topological charges.

It is easy to check that this formula correctly reproduces the terms proportional to one single q_i and to $q_i q_{i+1}$ in the Nekrasov partition function. (4.3.2) can also be checked to high orders in the instanton fugacities q_i with Mathematica against the Nekrasov partition function. (4.3.2) must be supplemented by the perturbative contribution of k hypers:

$$\mathcal{Z}^{\mathbb{R}^4\times S^1}_{linear,pert} = PE\left[\frac{t_1 t_2 \sum_{i=1}^k m_i}{(1-t_1)(1-t_2)}\right] \tag{4.3.3}$$

We get, for $\mathcal{Z}^{\mathbb{R}^4\times S^1}_{linear,pert+inst}$

$$PE\left[\frac{(\sum_{i=1}^k m_i)t_1 t_2 + \sum_{i,I=1}^{k-1}(\prod_{s=i}^{i+I-1} q_s m_s)(1-m_{i-1})(m_{i+I-1}^{-1} - t_1 t_2)}{(1-t_1)(1-t_2)}\right] \tag{4.3.4}$$

Each of the $k(k-1)/2$ instantonic terms decomposes in two positive terms and two negatives terms, so in the double sum in the numerator we can collect

- k^2 positive terms, $k(k-1)$ instanton plus k perturbative terms. We will show shortly that the positive terms transform like the bifundamental of the enhanced global symmetries $SU(k) \times SU(k)$:

$$\sum_{i=1}^k m_i t_1 t_2 + \sum_{i,I=1}^{k-1}(\prod_{s=i}^{i+I-1} q_s m_s)(m_{i-1}t_1 t_2 + m_{i+I-1}^{-1}) \tag{4.3.5}$$

- $k(k-1)/2$ negative terms of the form $-q_i m_{i-1}, -q_i m_{i-1} q_{i+1} m_i, \ldots$. These terms transform like 'half' adjoints (positive roots) of one of the two $SU(k)$ factors.

- $k(k-1)/2$ negative terms of the form $-q_i m_i t_1 t_2, -q_i m_i q_{i+1} m_{i+1} t_1 t_2, \ldots$. These terms transform like 'half' adjoints (positive roots) of the other $SU(k)$ factor.

The latter $k(k-1)$ negative terms, beside being negative, are not invariant under $t_i \to 1/t_i$, they must be removed. We call these negative terms *spurious*. The spurious contributions come from D1 branes stretching between parallel external NS5's in the pq-web, that can slide off to infinity, as was understood in [27, 13, 85, 26, 113], see Fig. 4.3.

We now want to show that (4.3.2), after removing the spurious contributions, reproduces precisely the partition function of k^2 free hypers, a expected from S-duality. We construct the $S^4 \times S^1$ partition functions multiplying the contribution from the North and South poles, that are the $\mathbb{R}^4 \times S^1$ Nekrasov partition functions. We analyze the cases $k = 1$ (which is trivial) and $k = 2$ first.

$k = 1$

In this case we just have the perturbative contribution of a free hyper

$$\mathcal{Z}^{\mathbb{R}^4\times S^1}_{k=1,full} = \mathcal{Z}^{\mathbb{R}^4\times S^1}_{k=1,pert} = PE\left[\frac{\tilde{m}_0\sqrt{t_1t_2}}{(1-t_1)(1-t_2)}\right]. \tag{4.3.6}$$

where $\tilde{m}_0 = m_0\sqrt{t_1t_2}$. On $S^4 \times S^1$ we get

$$\begin{aligned}\mathcal{Z}^{S^4\times S^1}_{k=1,full} =& PE\left[\frac{\tilde{m}_0\sqrt{t_1t_2}}{(1-t_1)(1-t_2)} + \frac{\tilde{m}_0^{-1}\sqrt{t_1^{-1}t_2^{-1}}}{(1-t_1^{-1})(1-t_2^{-1})}\right] \\ =& PE\left[\frac{(\tilde{m}_0+\tilde{m}_0^{-1})\sqrt{t_1t_2}}{(1-t_1)(1-t_2)}\right].\end{aligned} \tag{4.3.7}$$

This is the $S^4 \times S^1$ 5d index of a free hyper with mass $\tilde{m}_0$.

$k = 2$

In this case the linear quiver is $\boxed{1}-\!\!(1)\!\!-\boxed{1} = (1)\!\!-\boxed{2}$

$$\mathcal{Z}^{\mathbb{R}^4\times S^1}_{k=2,pert+inst} = PE\left[\frac{(m_0+m_1)t_1t_2 + q(1-m_0)(1-m_1t_1t_2)}{(1-t_1)(1-t_2)}\right] \tag{4.3.8}$$

Where $\mathcal{Z}^{\mathbb{R}^4\times S^1}_{k=2,pert+inst} = \mathcal{Z}^{\mathbb{R}^4\times S^1}_{k=2,pert}\mathcal{Z}^{\mathbb{R}^4\times S^1}_{k=2,inst}$. Changing variables to

$$m_0 = \frac{xA}{y\sqrt{t_1t_2}} \qquad m_1 = \frac{yA}{x\sqrt{t_1t_2}} \qquad q = \frac{xy\sqrt{t_1t_2}}{A} \tag{4.3.9}$$

with inverse

$$x = \sqrt{m_0q} \qquad A = \sqrt{m_0m_1t_1t_2} \qquad y = \sqrt{qm_1} \tag{4.3.10}$$

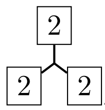
we get

$$\mathcal{Z}^{\mathbb{R}^4\times S^1}_{k=2,pert+inst} = PE\left[\frac{(xA/y + Ay/x + xy/A + xyA)\sqrt{t_1t_2} - x^2 - y^2t_1t_2}{(1-t_1)(1-t_2)}\right] \tag{4.3.11}$$

On $S^4 \times S^1$ we get

$$\mathcal{Z}^{S^4\times S^1}_{k=2,pert+inst} = PE\left[\frac{(x+1/x)(A+1/A)(y+1/y)\sqrt{t_1t_2} - (x^2+1/y^2+(1/x^2+y^2)t_1t_2)}{(1-t_1)(1-t_2)}\right] \tag{4.3.12}$$

In the 8 positive terms we recognise the trifundamental of $SU(2)^3$, with fugacities x, A, y. The 4 negative terms are spurious and must be removed. Recall that in the case $k = 2$

$\boxed{2}-\boxed{2}$, with $SU(2)^2 \times U(1)$ global symmetry, is actually a trifundamental . So we recover the partition function of 4 free hypers, as expected from S-duality.

Removing the spurious negative terms on $\mathbb{R}^4 \times S^1$ amounts to multiply the partition function by a factor

$$\mathcal{Z}^{\mathbb{R}^4\times S^1}_{k=2,spurious} = PE\left[\frac{qm_0 + qm_1t_1t_2}{(1-t_1)(1-t_2)}\right]. \tag{4.3.13}$$

Generic k

Let us change variables in (4.3.2) from the k masses m_i and $k-1$ couplings q_i to x_i, y_i ($i = 1, \ldots, k-1$, $x_0 = y_0 = x_k = y_k = 1$) and A:

$$m_i = \frac{x_{i+1}y_iA}{x_iy_{i+1}\sqrt{t_1t_2}} \qquad q_i = \frac{x_iy_i\sqrt{t_1t_2}}{x_{i+1}y_{i-1}A} \tag{4.3.14}$$

which implies

$$\prod_{s=j}^{j+I-1} q_sm_s = \frac{y_j}{y_{j-1}}\frac{y_{j+I-1}}{y_{j+I}} \qquad \prod_{s=j}^{j+I-1} q_sm_{s-1} = \frac{x_j}{x_{j-1}}\frac{x_{j+I-1}}{x_{j+I}} \tag{4.3.15}$$

The x_i and y_i will be the chemical potentials of the enhanced $SU(k)\times SU(k)$ symmetry, and A is the chemical potential of the $U(1)$ symmetry. Recalling that our formula for generic k is

$$PE\left[\frac{(\sum_{i=1}^k m_i)t_1t_2 + \sum_{i,I=1}^{k-1}(\prod_{s=i}^{i+I-1} q_sm_s)(1-m_{i-1})(m_{i+I-1}^{-1} - t_1t_2)}{(1-t_1)(1-t_2)}\right] \tag{4.3.16}$$

under the above change of variables, the positive terms in the numerator become

$$\sum_{j=1}^{k}\frac{x_{j+1}y_j}{x_jy_{j+1}}A\sqrt{t_1t_2} + \sum_{j=1}^{k-1}\sum_{I=1}^{k-j}\Big(\frac{x_jy_{j+I-1}}{x_{j-1}y_{j+I}}A + \frac{y_jx_{j+I-1}}{y_{j-1}x_{j+I}}A^{-1}\Big)\sqrt{t_1t_2}\,, \tag{4.3.17}$$

while the negative terms become

$$\sum_{j=1}^{k-1}\sum_{I=1}^{k-j}\prod_{s=j}^{j+I-1} q_sm_s(m_{j-1}m_{j+I-1}^{-1} + t_1t_2) = \sum_{j=1}^{k-1}\sum_{I=1}^{k-j}\Big(\frac{x_j}{x_{j-1}}\frac{x_{j+I-1}}{x_{j+I}} + \frac{y_j}{y_{j-1}}\frac{y_{j+I-1}}{y_{j+I}}t_1t_2\Big)\,. \tag{4.3.18}$$

The negative terms are associated to strings stretching between external D5 branes, so they must be removed. Their flavour fugacities are precisely the ones expected from the pq-web, and transform like a ‘half-adjoint’ of the two $SU(k)$ groups, while they are not charged under the $U(1)$.

We are then left with k^2 positive terms, $k(k+1)/2$ terms with A-charge ‘+1’, $k(k-1)/2$ terms with A-charge ‘-1’.

When we consider the gauge theory on $S^4\times S^1$ we need to sum, inside the PE, the contribution from the North pole and the South pole. The South pole contribution has all the chemical potentials, and t_1, t_2, inverted, so we get k^2 terms with A-charge ‘+1’, k^2

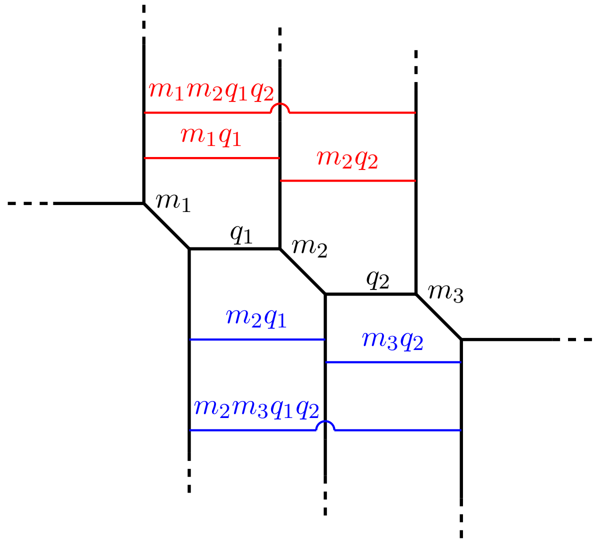


Figure 4.3: Graphical representation of the $k(k-1)$ spurious contributions. These are D1 branes that can slide off to infinity. In the picture $k=3$.

terms with A-charge '-1'. The full $S^4\times S^1$ partition function $\mathcal{Z}_{full}=\mathcal{Z}_{pert}\mathcal{Z}_{inst}\mathcal{Z}_{spurious}$ can be simplified to

$$\begin{aligned}\mathcal{Z}^{S^4\times S^1}_{linear,full} &= PE\left[\frac{((\sum x_{i+1}/x_i)(\sum y_j/y_{j+1})A+(\sum x_i/x_{i+1})(\sum y_{j+1}/y_j)A^{-1})\sqrt{t_1t_2}}{(1-t_1)(1-t_2)}\right]\\ &= PE\left[\frac{(\chi^{SU(k)}_{fund}[x_i]\chi^{SU(k)}_{antifund}[y_i]A+\chi^{SU(k)}_{antifund}[x_i]\chi^{SU(k)}_{fund}[y_i]A^{-1})\sqrt{t_1t_2}}{(1-t_1)(1-t_2)}\right]\end{aligned} \tag{4.3.19}$$

where $\chi^{SU(k)}_{(anti)fund}[x_i]$ is the character of the (anti)fundamental representation of $SU(k)$. Eq. (4.3.19) is the same partition function of k^2 free hypers in the bifundamental $\boxed{k}-\boxed{k}$. Summarizing, we proved that

$$\mathcal{Z}_{[1]-(1)^{k-1}-[1]}(q_i,m_j)=\mathcal{Z}_{[k]-[k]}(x_i,y_j,A) \tag{4.3.20}$$

where the two sets of variables are related by

$$m_i=\frac{x_{i+1}y_iA}{x_iy_{i+1}\sqrt{t_1t_2}} \qquad q_i=\frac{x_iy_i\sqrt{t_1t_2}}{x_{i+1}y_{i-1}A} \tag{4.3.21}$$

4.4 Exact partition functions: $5d$ Abelian circular quiver

In this section we write down the exact Nekrasov instanton partition function for the abelian necklace quiver, we study the spurious terms, we construct the S^5 partition function and we compare it with the 6d $(1,0)$ index.

The topological string partition for this pq-web has been studied in [81, 91].

For the abelian necklace quiver (figures 4.4 and 4.5) the partition function (see appendix 4.A) receives contribution from $k(k-1)$ non wrapping instantons, but we also need to consider the wrapping instantons of the form $\mathfrak{I}^{(1,1,\ldots,1)}$. It turns out that, in order to reproduce the Nekrasov partition function, we need to sum over a full tower of wrapping instantons $\mathfrak{I}^{(n,n,\ldots,n)}$ for all $n \geq 1$ [1]. From the index computation point of view the instanton quantum number corresponds to the Kaluza-Klein charge on the circle [94]. We are thus led to propose the following formula:

$$\mathcal{Z}^{\mathbb{R}^4\times S^1}_{inst} = PE\left[\frac{Q}{1-Q} + \frac{\sum_{i,I=1}^{k}(\prod_{s=i}^{i+I-1} q_s m_s)(1-m_{i-1})(m_{i+I-1}^{-1} - t_1 t_2)}{(1-Q)(1-t_1)(1-t_2)}\right] \tag{4.4.1}$$

where $Q = \prod_{i=1}^{k} q_i m_i$ is the lenght of the circle the pq-web lives on. We checked this result to high orders with Mathematica.

There is also the perturbative contribution of the k hypermultiplets

$$\mathcal{Z}^{\mathbb{R}^4\times S^1}_{pert} = PE\left[\frac{\sum_{i=1}^{k} m_i t_1 t_2}{(1-t_1)(1-t_2)}\right], \tag{4.4.2}$$

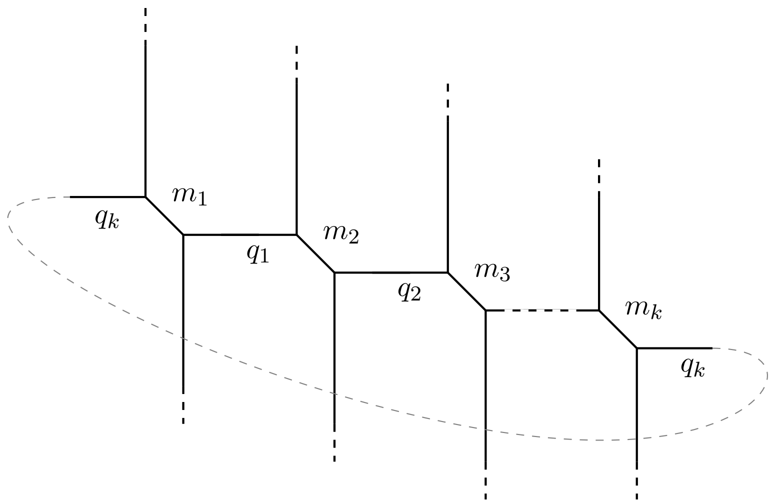


Figure 4.4: 1 D5 and k NS5 on the cylinder

Let us focus on the numerator inside the PE and split the instanton sum into wrap-

[1] Notice that this aspect looks different from the $3d$ $\mathcal{N}=2$ case. In [25] it is shown how in circular quivers with flavors at each node there is only one wrapping monopole in the chiral ring, with all topological charges $+1$. The wrapping monopoles with all charges $+n$ are simply the n^{th} power of the basic one.

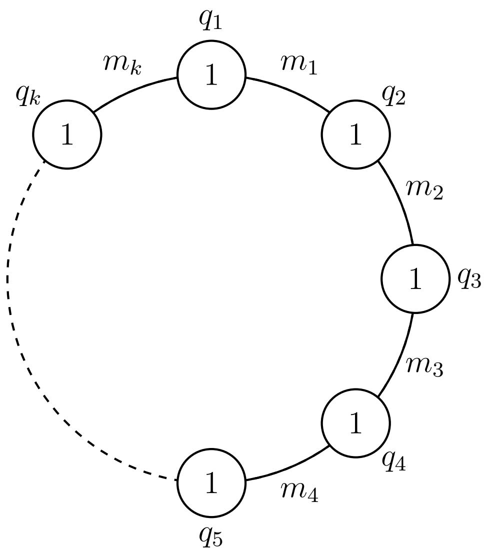


Figure 4.5: Circular quiver with k $U(1)$ nodes.

ping instantons and non wrapping instantons:

$$(1-Q)(\sum_{i=1}^{k} m_i t_1 t_2) + Q(1-t_1)(1-t_2) + Q\sum_{i=1}^{k}(1-m_i)(m_i^{-1} - t_1 t_2) + \\ + \sum_{i=1}^{k}\sum_{I=1}^{k-1}(\prod_{s=i}^{i+I-1} q_s m_s)(1-m_{i-1})(m_{i+I-1}^{-1} - t_1 t_2)$$

Which can be rewritten as

$$\begin{aligned} &Q(1-t_1)(1-t_2) - kQt_1t_2 - kQ + \sum_{i=1}^{k}(m_i t_1 t_2 + m_i^{-1}Q) + \\ &+\sum_{i=1}^{k}\sum_{I=1}^{k-1}(\prod_{s=i}^{i+I-1} q_s m_s)(m_{i+I-1}^{-1} + m_{i-1}t_1t_2 - t_1t_2 - m_{i+I-1}^{-1}m_{i-1}) \end{aligned} \tag{4.4.3}$$

We see that

$$\mathcal{Z}_{pert+inst}^{\mathbb{R}^4\times S^1} = PE\left[\frac{Q(1-t_1)(1-t_2) - kQt_1t_2 - kQ + N(q_i, m_i)}{(1-Q)(1-t_1)(1-t_2)}\right] \tag{4.4.4}$$

with

$$\begin{aligned} N(q_i, m_i) = &\sum_{i=1}^{k}(m_i t_1 t_2 + m_i^{-1}Q) \\ &+\sum_{i=1}^{k}\sum_{I=1}^{k-1}(\prod_{s=i}^{i+I-1} q_s m_s)(m_{i+I-1}^{-1} + m_{i-1}t_1t_2 - t_1t_2 - m_{i+I-1}^{-1}m_{i-1} \end{aligned} \tag{4.4.5}$$

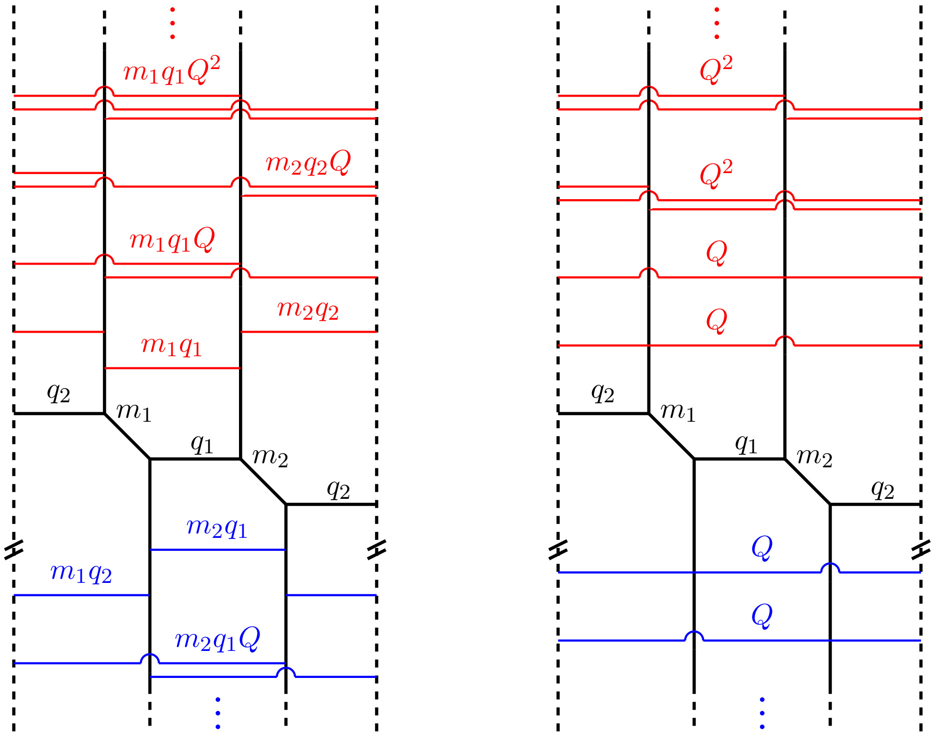


Figure 4.6: Spurious contributions on the cylinder. Here we display $k = 2$. On the left the terms dependent on q_i and m_i, associated to non-wrapping instantons, second line of (4.4.3). On the right the terms dependent only on Q, associated to wrapping instantons, first line of (4.4.3).

The formula for $N(q_i, m_i)$ displays $2k^2$ positive terms and $2k(k-1)$ negative terms.

The $2k^2$ positive terms transform in the bifundamental representation of $SU(k) \times SU(k)$, except that they are paired as $x + Q/x$ instead of $x + 1/x$. We will see in the next subsection that the correct character of the bifundamental is reproduced once we build the S^5 partition function, with a proper analytic continuation. Let us underline that this is a non-trivial check that it is really the S^5 partition function which matters for comparison with the M5 brane index.

The $2k(k-1)$ negative terms are all spurious terms to be factored out (Fig. 4.6). Notice that, beacuse of the factor $(1-Q)$ in the denominator, we are really claiming that there is an infinite tower of spurious terms. This fact has a natural interprete-tation in the pq-web: for each pair of semi-infinite NS5 branes, we can strech a D1 brane, of length, say, $m_1 q_1$, but we can also strech a D1 going around the circle the other way, of length $Q/(m_1 q_1)$. As shown graphically in Figure 4.6, there are also D1 branes going around the circle more than once, of length $m_1 q_1 Q, m_1 q_1 Q^2, m_1 q_1 Q^3, \ldots$ or $Q^2/(m_1 q_1), Q^3/(m_1 q_1), Q^4/(m_1 q_1), \ldots$. This explains the $2k(k-1)$ towers of spurious states.

The first part of the numerator in (4.4.4) contains the negative terms $\frac{-kQt_1t_2 - kQ}{1-Q}$. We interpret these as towers of spurious contributions associated to D1 branes going from one NS5 to itself and wrapping the circle an integer number of times. See the right part of Figure 4.6. There are k such contributions, one for every NS5 brane. It looks like these spurious terms are not independent, so to avoid overcounting we have to add back to the partition function a term

$$-\frac{Q + Qt_1t_2}{1-Q} + t_1t_2 = -\frac{Q + t_1t_2}{1-Q} \tag{4.4.6}$$

We also added the t_1t_2 term, which is just a McMahon function, that is $PE[\frac{t_1t_2}{(1-t_1)(1-t_2)}]$, in agreement with [106] for $k = 1$.

The final result for $\mathcal{Z}^{\mathbb{R}^4\times S^1}_{pert+inst+spurious}$ for the k-nodes circular quiver is

$$PE\left[\frac{-Q - t_1t_2 + \sum_{i=1}^{k}(m_i t_1 t_2 + m_i^{-1}Q) + \sum_{i=1}^{k}\sum_{I=1}^{k-1}(\prod_{s=i}^{i+I-1} q_s m_s)(m_{i+I-1}^{-1} + m_{i-1}t_1t_2)}{(1-Q)(1-t_1)(1-t_2)}\right] \times PE\left[\frac{Q}{1-Q}\right] \tag{4.4.7}$$

4.4.1 Modularity and partition function on S^5

We now patch together 3 copies of $\mathcal{Z}^{\mathbb{R}^4\times S^1}$ to form $\mathcal{Z}^{S^5}$. First we rewrite $\mathcal{Z}^{\mathbb{R}^4\times S^1}$ in terms of modular forms G_2 [68], where for $\mathrm{Im}(\omega_i) > 0$

$$G_2(z;\omega_0,\omega_1,\omega_2) = PE\left[\frac{-e^{2\pi i z} - e^{2\pi i(-z+\omega_0+\omega_1+\omega_2)}}{(1-e^{2\pi i\omega_0})(1-e^{2\pi i\omega_1})(1-e^{2\pi i\omega_2})}\right] \tag{4.4.8}$$

The exponentiated variables are $t_1 = e^{-\beta\epsilon_1}, t_2 = e^{-\beta\epsilon_2}, m_i = e^{\beta\mu_i}, q_i = e^{\beta\tau_i}$ and we defined $Q = \prod_{i=1}^{k} q_i m_i = e^{\beta\sum_{i=1}^{k}(\tau_i+\mu_i)} =: e^{\beta\Omega}$. It's non trivial that our result (4.4.7) for $\mathcal{Z}_{full}^{\mathbb{R}^4\times S^1} = \mathcal{Z}_{pert}^{\mathbb{R}^4\times S^1}\mathcal{Z}_{inst}^{\mathbb{R}^4\times S^1}\mathcal{Z}_{spurious}^{\mathbb{R}^4\times S^1}$ can be written in terms of G_2 functions as

$$\mathcal{Z}_{full}^{\mathbb{R}^4\times S^1} = \frac{G_2'(0;\frac{\beta\Omega}{2\pi i},\frac{\beta\epsilon_1}{2\pi i},\frac{\beta\epsilon_2}{2\pi i})}{\eta(\frac{\beta\Omega}{2\pi i})\prod_{i=1}^{k}\Big(G_2(\frac{\beta(\Omega-\mu_i)}{2\pi i};\frac{\beta\Omega}{2\pi i},\frac{\beta\epsilon_1}{2\pi i},\frac{\beta\epsilon_2}{2\pi i})\prod_{l=0}^{k-2}G_2(\frac{\beta(\sum_{s=i}^{i+l}(\tau_s+\mu_s)-\mu_{i+l})}{2\pi i};\frac{\beta\Omega}{2\pi i},\frac{\beta\epsilon_1}{2\pi i},\frac{\beta\epsilon_2}{2\pi i})\Big)} \tag{4.4.9}$$

The partition function on S^5 is obtained taking the product of three copies of $\mathcal{Z}_{full}^{\mathbb{R}^4\times S^1}$

$$\mathcal{Z}^{S^5} = \prod_{\ell=1}^{3}\mathcal{Z}_{full}^{\mathbb{R}^4\times S^1}(\epsilon_1^{(\ell)},\epsilon_2^{(\ell)},\beta^{(\ell)},\vec{q},\vec{m}) \tag{4.4.10}$$

These parameters take the following values in the 3 patches of S^5 [125]

ℓ	$\epsilon_1^{(\ell)}$	$\epsilon_2^{(\ell)}$	$\beta^{(\ell)}$
1	ω_2	ω_3	$2\pi i/\omega_1$
2	ω_3	ω_1	$2\pi i/\omega_2$
3	ω_1	ω_2	$2\pi i/\omega_3$

(4.4.11)

and the double elliptic gamma functions in the $\mathbb{R}^4\times S^1$ partition function are of the form $G_2(z;\frac{\beta\Omega}{2\pi i},\frac{\beta\epsilon_1}{2\pi i},\frac{\beta\epsilon_2}{2\pi i})$. They satisfy the modularity property

$$\begin{aligned}&G_2\Big(\frac{z}{\omega_1}|\frac{\Omega}{\omega_1},\frac{\omega_2}{\omega_1},\frac{\omega_3}{\omega_1}\Big)G_2\Big(\frac{z}{\omega_2}|\frac{\Omega}{\omega_2},\frac{\omega_3}{\omega_2},\frac{\omega_1}{\omega_2}\Big)G_2\Big(\frac{z}{\omega_3}|\frac{\Omega}{\omega_3},\frac{\omega_1}{\omega_3},\frac{\omega_2}{\omega_3}\Big)\\&= e^{-\frac{\pi i}{12}B_{4,4}(z|\omega_1,\omega_2,\omega_3,\Omega)}G_2\Big(\frac{z}{\Omega}|\frac{\omega_1}{\Omega},\frac{\omega_2}{\Omega},\frac{\omega_3}{\Omega}\Big)^{-1}\end{aligned} \tag{4.4.12}$$

The modular properties of the topological string partition function, in the case of all the masses equal, have been studied in [91].

Using this equation for every triple of G_2's in the S^5 partition function we get[2]

$$\mathcal{Z}^{S^5} = \frac{\prod_{i=1}^{k}\prod_{l=0}^{k-1}G_2\Big(\frac{1}{\Omega}\big(\sum_{s=i}^{i+l}(\tau_s+\mu_s)-\mu_{i+l}\big)|-\sigma_1,-\sigma_2,-\sigma_3\Big)}{\eta(-\sigma_1^{-1})\eta(-\sigma_2^{-1})\eta(-\sigma_3^{-1})G_2'(0|-\sigma_1,-\sigma_2,-\sigma_3)} \tag{4.4.13}$$

where $\sigma_i = -\omega_i/\Omega$ and the convention $\prod_{i=1}^{0}\equiv 1$ is used. Using the modular properties of the G_r [68]

$$G_r(-z;-\vec{\tau}) = \frac{1}{G_r(z;\vec{\tau})}, \qquad \eta(\tau^{-1}) = \sqrt{i\tau}\,\eta(-\tau) \tag{4.4.14}$$

and exponentiated variables $\mathtt{q_i} = e^{-2\pi i\omega_i/\Omega}, \tilde{q}_s = e^{-2\pi i\tau_s/\Omega}, \tilde{m}_s = e^{-2\pi i\mu_s/\Omega}$ we have[3]

$$\mathcal{Z}^{S^5} = \frac{G_2'(0|-\sigma_1,-\sigma_2,-\sigma_3)}{\eta(\sigma_1)\eta(\sigma_2)\eta(\sigma_3)\prod_{i=1}^{k}\prod_{l=0}^{k-1}G_2\big(-(\sum_{s=i}^{i+l}\tau_s\mu_s-\mu_{i+l})/\Omega;\sigma_1,\sigma_2,\sigma_3\big)} \tag{4.4.15}$$

[2]This is computed up to Bernoulli polynomials and $q^{-1/24}$: harmless terms which do not play any role in our analysis.

[3] $G_2(0)$ contains a zero mode that can be regularized replacing it with $G_2'(0;\sigma_1,\sigma_2,\sigma_3) = G_2(0;\sigma_1,\sigma_2,\sigma_3)PE[1] = PE\Big[\frac{-\mathtt{q}_1-\mathtt{q}_2-\mathtt{q}_3+\mathtt{q}_1\mathtt{q}_2+\mathtt{q}_1\mathtt{q}_3+\mathtt{q}_2\mathtt{q}_3-2\mathtt{q}_1\mathtt{q}_2\mathtt{q}_3}{(1-\mathtt{q}_1)(1-\mathtt{q}_2)(1-\mathtt{q}_3)}\Big]$.

The variables $\tilde{q}_s, \tilde{m}_s$ satisfy $\prod_{s=1}^k \tilde{q}_s \tilde{m}_s = e^{-2\pi i \frac{1}{\Omega}\sum_{s=1}^k \tau_s + \mu_s} = e^{-2\pi i \frac{1}{\Omega}\Omega} = 1$. Let us now change variables to $x_1, x_2, \dots, x_{k-1}$, $y_1, y_2, \dots, y_{k-1}$, A

$$\tilde{m}_i = \frac{x_i y_{i-1}}{x_{i-1} y_i} \frac{A}{\sqrt{\mathsf{q}_1\mathsf{q}_2\mathsf{q}_3}} \qquad \tilde{t}_i = \frac{x_{i-1} y_{i-1}}{x_i y_{i-2}} \frac{\sqrt{\mathsf{q}_1\mathsf{q}_2\mathsf{q}_3}}{A} \tag{4.4.16}$$

and notice that, for $\mathrm{Im}(\sigma_i) > 0$, we can rewrite (4.4.13) as

$$\begin{aligned}\mathcal{Z}^{S^5} =& PE\Bigg[\frac{1}{\prod_{i=1}^3(1-\mathsf{q}_i)}\Big(\mathsf{q}_1\mathsf{q}_2\mathsf{q}_3 - \mathsf{q}_1\mathsf{q}_2 - \mathsf{q}_1\mathsf{q}_3 - \mathsf{q}_2\mathsf{q}_3 + \\ &+ \sqrt{\mathsf{q}_1\mathsf{q}_2\mathsf{q}_3}\left[A\left(\sum_{l=1}^k \frac{x_{l+1}}{x_l}\right)\left(\sum_{i=1}^k \frac{y_i}{y_{i+1}}\right) + A^{-1}\left(\sum_{l=1}^k \frac{x_l}{x_{l+1}}\right)\left(\sum_{i=1}^k \frac{y_{i+1}}{y_i}\right)\right]\Big)\Bigg]\end{aligned}$$

Using the charachters of the (anti)fundamental of $SU(k)$, $\chi^{SU(k)}_{(anti)fund}[x_i]$, our final result is

$$\begin{aligned}\mathcal{Z}^{S^5} =& PE\Bigg[\frac{1}{\prod_{i=1}^3(1-\mathsf{q}_i)}\Big(\mathsf{q}_1\mathsf{q}_2\mathsf{q}_3 - \mathsf{q}_1\mathsf{q}_2 - \mathsf{q}_1\mathsf{q}_3 - \mathsf{q}_2\mathsf{q}_3 + \\ &+ \sqrt{\mathsf{q}_1\mathsf{q}_2\mathsf{q}_3}\left[\chi^{SU(k)}_{fund}[x_i]\chi^{SU(k)}_{antifund}[y_i]\,A + \chi^{SU(k)}_{antifund}[x_i]\chi^{SU(k)}_{fund}[y_i]\,A^{-1}\right]\Big)\Bigg].\end{aligned} \tag{4.4.17}$$

It is easy to recognize the $6d$ $(1,0)$ superconformal index of a free self-dual tensor (4.2.4) (first line) plus k^2 free hypers (4.2.3) (second line).

4.5 pq-webs and $q\mathcal{W}$ algebrae

Five dimensional gauge theories describing the dynamics of the above pq-webs are expected to have a relation with representation theory of $q\mathcal{W}$ algebrae [67, 11], generalising to five dimensions [12, 124] the known AGT relation for four-dimensional class $\mathcal{S}$ theories [6].

The most interesting consequence of this relation relies in the fact the S-duality in superstring theory, dubbed fiber-base duality in the subclass of topological string amplitudes, predicts a duality between $k+2$-point correlators of $q\mathcal{W}_N$ algebrae and $N+2$-point correlators of $q\mathcal{W}_k$ algebrae. Indeed pq-webs on $\mathbb{R}^2$ are described by five-dimensional gauge theories associated to linear quivers of the kind depicted in Fig. 4.7. The brane system on the left-hand side consists in N parallel D4 branes (horizontal black lines) suspended between k NS5 branes (vertical red lines). As described in Sect. 4.1, the effective field theory living on the D4 system is a five-dimensional $SU(N)^{k-1}$ linear quiver with N-flavors at both ends. The S-dual system on the right of Fig. 4.7 corresponds to a linear quiver $SU(k)^{N-1}$ with k flavors at both the ends. As depicted in Fig. 4.7 one expects the $S^4 \times S^1$ supersymmetric partition function of the first linear quiver to compute the $k+2$-point correlator of $q\mathcal{W}_N$ algebra on the sphere with k simple punctures corresponding to semi-degenerate vertex operators of q-Toda, and 2 full N-punctures, corresponding to full vertex operators. Analogously, the $S^4 \times S^1$ partition function of the S-dual theory is expected to compute the correlator of $q\mathcal{W}_N$ algebra with N semi-degenerate and two k-full insertions (figure 4.7).

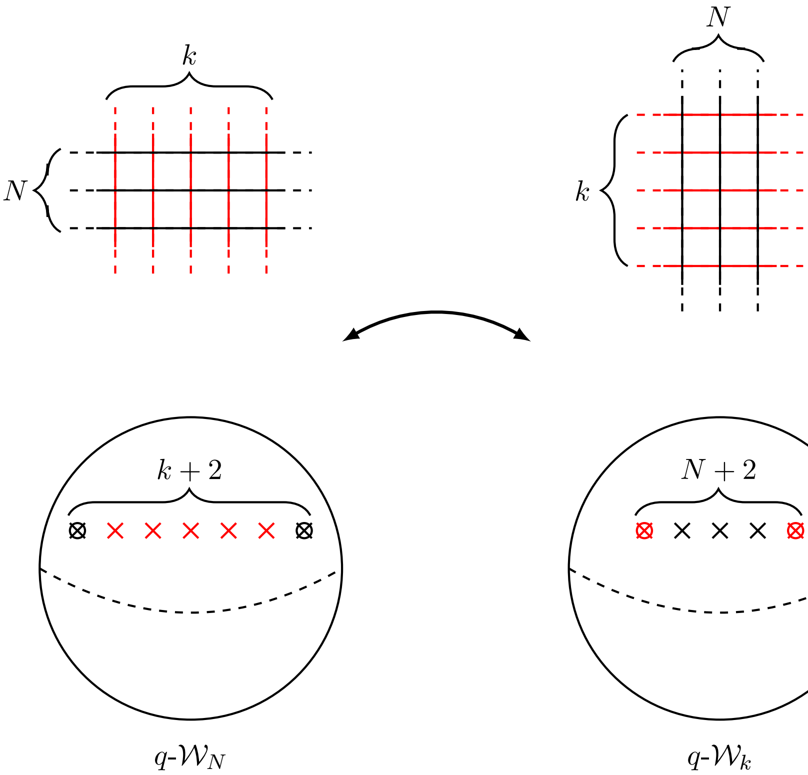


Figure 4.7: Linear quiver: a cross is a simple puncture, a cross with a circle is a full puncture.

A first check of this duality is the matching of the dimensions of the space of parameters the correlation functions depend on. Indeed, k simple punctures on the sphere count k positions of the vertex operator insertions and the corresponding k momenta. On the other hand, the two full N-punctures count each $N-1$ momenta and one position. Overall, taking into account $PSL(2,\mathbb{C})$ symmetry, this amounts to $2(N+k)-3$ parameters. The counting for the dual correlators is obtained by simply swapping k and N.

A more explicit check can be made in the simplest case $N=1$ by making use of the explicit computations of the supersymmetric partition functions displayed in the previous sections. In this case the left-hand side of the story reduces to q-Heisenberg algebra. A correspondence between this vertex algebra and five-dimensional gauge theories has been discussed in [46]. According to the duality stated above, the $k+2$-point correlator of q-Heisenberg vertex operators should capture the three-point correlator of $q\mathcal{W}_k$ algebra with two k-full and one semi-degenerate insertion [103, 37, 112, 92]. Studies of the non Abelian cases appeared in [1].

Let us now proceed to the comparison of the two dual correlators.

We saw in section 4.3 that the partition function for the $U(1)^{k-1}$ linear quiver theory has three contributions: perturbative (1-loop), instanton and spurious (due to semi-infinite parallel branes). The 1-loop and the instanton part can be written as

$$\begin{aligned}\mathcal{Z}^{S^4\times S^1}_{pert+inst} = PE\Big[&\frac{1}{(1-t_1)(1-t_2)}\Big\{-(k-1)(t_1+t_2)\\ &-\sum_{i<j}\Big[(\frac{x_i}{x_{i-1}})(\frac{x_{j-1}}{x_j})+t_1t_2(\frac{x_{i-1}}{x_i})(\frac{x_j}{x_{j-1}})+(\frac{y_{i-1}}{y_i})(\frac{y_j}{y_{j-1}})+t_1t_2(\frac{y_i}{y_{i-1}})(\frac{y_{j-1}}{y_j})\Big]\\ &+\sqrt{t_1t_2}\Big[A(\sum_{l=1}^k\frac{x_l}{x_{l-1}})(\sum_{i=1}^k\frac{y_{i-1}}{y_i})+A^{-1}(\sum_{l=1}^k\frac{x_{l-1}}{x_l})(\sum_{i=1}^k\frac{y_i}{y_{i-1}})\Big]\Big\}\Big]\end{aligned} \tag{4.5.1}$$

where we included the 1-loop contribution of the $N-1$ vector multiplets that played no role in the identification of the partition function with the M5 brane SCFT index [4]

$$\mathcal{Z}^{S^4\times S^1}_{\text{pert,vector}} = PE\left[\frac{-1-t_1t_2}{(1-t_1)(1-t_2)}+1\right] = PE\left[\frac{-t_1-t_2}{(1-t_1)(1-t_2)}\right]. \tag{4.5.2}$$

By introducing the new variables $\alpha_i, \tilde{\alpha}_i, \varkappa$ defined by

$$\begin{aligned} x_1 &= \prod_{i=1}^{k-1}\frac{e^{-\frac{\beta}{k}(\alpha_i-\alpha_{i+1})(i-k)}}{t_1t_2}, \qquad y_1 = \prod_{i=1}^{k-1}\frac{e^{+\frac{\beta}{k}(\tilde{\alpha}_i-\tilde{\alpha}_{i+1})(i-k)}}{t_1t_2},\\ x_n &= x_1^n\prod_{i=1}^{n-1}\frac{e^{-\beta(\alpha_i-\alpha_{i+1})(n-i)}}{t_1t_2}, \qquad y_n = y_1^n\prod_{i=1}^{n-1}\frac{e^{+\beta(\tilde{\alpha}_i-\tilde{\alpha}_{i+1})(n-i)}}{t_1t_2}, \qquad n=2,\dots,k-1\\ A^k &= \frac{e^{-\beta\varkappa}}{(t_1t_2)^{k/2}}\end{aligned} \tag{4.5.3}$$

we can rewrite (4.5.1) in a form which is more suitable for the comparison with the $q\mathcal{W}_k$ correlator

$$\mathcal{Z}^{S^4\times S^1}_{pert+inst}(\alpha,\tilde{\alpha},\varkappa) = \frac{\Upsilon_q'(0)^{k-1}\prod_{e>0}\Upsilon_q(\langle Q-\alpha,e\rangle)\Upsilon_q(\langle Q-\tilde{\alpha},e\rangle)}{\prod_{i,j=1}^k\Upsilon_q(\frac{\varkappa}{k}+\langle\alpha-Q,h_i\rangle+\langle\tilde{\alpha}-Q,h_j\rangle)} \tag{4.5.4}$$

where e are the positive roots of the A_{k-1} gauge group, $Q=(\epsilon_1+\epsilon_2)\rho$, ρ is the Weyl vector (half the sum of all positive roots), h_i are the weights of the fundamental representation and $\langle\cdot,\cdot\rangle$ denotes the scalar product on the root space. The Υ_q function, with $q=e^{-\beta}$ can be defined as follows

$$\Upsilon_q(x|\epsilon_1,\epsilon_2) = (1-q)^{-\frac{1}{\epsilon_1\epsilon_2}\left(x-\frac{Q}{2}\right)^2}PE\left[\frac{-q^x-q^{-x}t_1t_2}{(1-t_1)(1-t_2)}\right]. \tag{4.5.5}$$

[4] The -1 term in the PE is needed to remove a zero mode, for $SU(N)$ gauge group it corresponds to the Haar measure on the $S^4\times S^1$ [95].

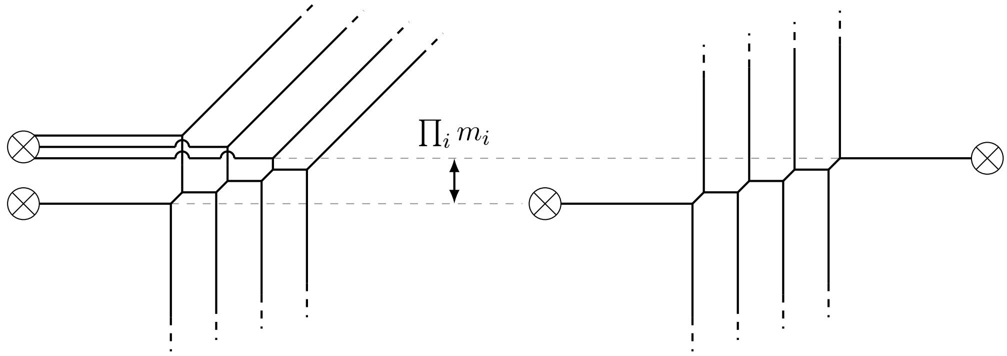


Figure 4.8: On the left a generalized pq-web which can be obtained from Higgling the $T_{N=4}$ SCFT [20], the global symmetry is $S(U(1)\times U(1))\times SU(N)\times SU(N)$ (if $N>2$, for $N=2$ it is a standard pq-web with $SU(2)^3$ symmetry). On the right the standard pq-web, one D5 intersecting k NS5's, the global symmetry is the same, $SU(N)\times SU(N)\times U(1)$. The symbols $\otimes$ represent D7 branes. The two pq-webs are related by a sequence of N Hanany-Witten transitions, that, starting from the right, create the $N-1$ D5's and also bend the upper part of the N NS5's. The vertical displacement between the two D7 branes attached to the semi-infinite D5's does not change, and it equals the product of all the masses in the linear quiver $\prod_i m_i = A^k$. The difference between the two partition functions is just that the left diagram has one more spurious term, which is precisely $PE[\frac{-A^{-k}-A^k t_1 t_2}{(1-t_1)(1-t_2)}]=\Upsilon_q(\varkappa)$.

For A_{k-1} the above can be written in term of k-dimensional vectors u_i, whose ith entry is one and all others zero, as

$$e = u_i - u_j, \qquad 1\leq i<j\leq k \tag{4.5.6}$$

$$\rho = \frac{1}{2}\sum_{i=1}^{k}(k+1-2i)u_i, \qquad h_i = -u_i + \frac{1}{k}\sum_{j=1}^{k} u_j \tag{4.5.7}$$

A $(k-1)$-dimensional vector of fields ϕ can be expanded on the base of the simple roots $\hat{e}_i$ of the A_{k-1}

$$\phi = \sum_{i=1}^{k-1}\phi_i\hat{e}_i = \sum_{i=1}^{k-1}\phi_i(u_i-u_{i+1}). \tag{4.5.8}$$

Formula (4.5.4) can be compared to the three-point correlation function $C_q(\alpha,\tilde{\alpha},\varkappa h_{k-1})$ with one degenerate insertion (parallel to the highest weight of antifundamental representation h_{k-1}) of the k-qToda theory with central charge $c=k-1+12\langle Q,Q\rangle$, that has been conjectured in [112, 92]. The two formulae are different by a factor $\Upsilon_q(\varkappa)$ in the numerator, which is due to the fact that the computation in [112] corresponds to the generalized pq-web [20] diagram on left side of Fig. 4.8, while ours corresponds to the standard pq-web on the right hand side. The two diagrams are related by moving one D7 brane all the way through the k NS5-branes keeping track of the Hanany-Witten brane creation effect, see [20, 97, 96]. From the viewpoint of the index computation, on which we focus in this Chapter, this extra factor is a spurious one due to the contribution of strings stretching between the two sets of parallel horizontal branes, which are present only in the left-hand side brane diagram. From the viewpoint of q-deformed CFT, the left-hand side diagram is more natural and has also the correct four-dimensional limit.

Let us now make some comments on the case of pq-webs on the cylinder. As explained in Sect. 4.1 this is described in terms of circular quivers. In this case one expects a relation with correlators of $q\mathcal{W}$ algebrae on a torus. For the case $N = 1$ it has indeed been shown in [46] that the one point chiral correlator of q-Heisenberg on the torus computes the $k = 1$ circular quiver partition function on $\mathbb{R}^4 \times S^1$. The results of Sect. 4.4 should correspond to the k-point chiral correlator of the same vertices, see Fig. 4.9. It would be interesting to check this relation explicitly.

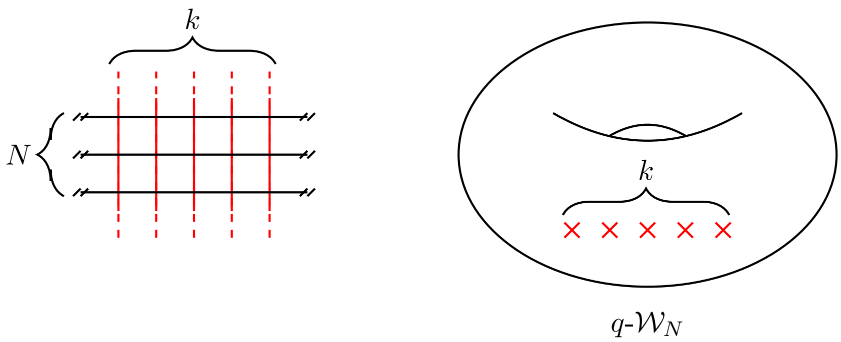


Figure 4.9: Circular quiver: a cross is a simple puncture.

4.A Nekrasov partition function

In this appendix we recall the definition of the Nekrasov partition function [119, 69, 42].

The 1-loop part for the circular $U(N)$ quiver with k nodes is

$$Z^{\text{1-loop}}_{U(N)^k}(\{\vec{a}_i\},\{\mu_i\}) = \prod_{i=1}^{k} z^{\text{1-loop}}_{vec}(\vec{a}_i) z^{\text{1-loop}}_{bif}(\vec{a}_i,\vec{a}_{i+1},\mu_i) \tag{4.A.1}$$

where

$$\begin{aligned} z^{\text{1-loop}}_{vec}(\vec{a}) &= \prod_{\alpha,\gamma=1}^{N} \prod_{m,n>1} \sinh \frac{\beta}{2}(a_\alpha - a_\gamma + m\epsilon_1 + n\epsilon_2) \\ z^{\text{1-loop}}_{bif}(\vec{a},\vec{b},\mu) &= \prod_{\alpha,\gamma=1}^{N} \prod_{m,n>1} \sinh \frac{\beta}{2}(a_\alpha - b_\gamma - \mu + m\epsilon_1 + n\epsilon_2)^{-1}. \end{aligned} \tag{4.A.2}$$

The instanton partition function for the circular $U(N)$ quiver with k nodes is

$$Z_{U(N)^K}(\{\vec{a}_i\},\{\mathbf{q}_i\},\{\mu_i\}) = \sum_{\{\vec{Y}_1,\dots\vec{Y}_k\}} \prod_{i=1}^{k} \mathbf{q}_i^{|\vec{Y}_i|} z_{vec}(\vec{a}_i,\vec{Y}_i)\, z_{bif}(\vec{a}_i,\vec{a}_{i+1},\vec{Y}_i,\vec{Y}_{i+1},\mu_i) \quad (4.A.3)$$

where $\vec{a}_{k+1} \equiv \vec{a}_1$, $\mathbf{q}_{k+1} \equiv \mathbf{q}_1$, $\mu_{k+1} \equiv \mu_1$. And

$$\begin{aligned} z_{vec}(\vec{a},\vec{Y}) = \prod_{\alpha,\gamma=1}^{N} \prod_{s\in Y_\alpha} & \sinh\frac{\beta}{2}\left[a_\alpha - a_\gamma - \epsilon_1 L_{Y_\gamma}(s) + \epsilon_2\left(A_{Y_\alpha}(s)+1\right)\right]^{-1} \\ & \sinh\frac{\beta}{2}\left[a_\gamma - a_\alpha + \epsilon_1\left(L_{Y_\gamma}(s)+1\right) - \epsilon_2 A_{Y_\alpha}(s)\right]^{-1} \end{aligned} \quad (4.A.4)$$

$$\begin{aligned} z_{bif}(\vec{a},\vec{b},\vec{Y},\vec{W},\mu) = \prod_{\alpha,\gamma=1}^{N} \prod_{s\in Y_\alpha} & \sinh\frac{\beta}{2}\left[a_\alpha - b_\gamma - \epsilon_1 L_{W_\gamma}(s) + \epsilon_2\left(A_{Y_\alpha}(s)+1\right) - \mu\right] \\ \prod_{t\in W_\gamma} & \sinh\frac{\beta}{2}\left[a_\alpha - b_\gamma + \epsilon_1\left(L_{Y_\alpha}(t)+1\right) - \epsilon_2 A_{W_\gamma}(t) - \mu\right] \end{aligned} \quad (4.A.5)$$

These expressions can be substituted in (4.A.3) with[5]

$$\begin{aligned} z'_{vec}(\vec{a},\vec{Y}) &= \prod_{\alpha,\gamma=1}^{N} \prod_{s\in Y_\alpha} \left(1 - e^{-\beta(a_\alpha - a_\gamma)} t_1^{-L_{Y_\gamma}(s)} t_2^{A_{Y_\alpha}(s)+1}\right)^{-1} \times \\ &\qquad \left(1 - e^{-\beta(a_\gamma - a_\alpha)} t_1^{L_{Y_\gamma}(s)+1} t_2^{-A_{Y_\alpha}(s)}\right)^{-1} \\ z'_{bif}(\vec{a},\vec{b},\vec{Y},\vec{W},m) &= \prod_{\alpha,\gamma=1}^{N} \prod_{s\in Y_\alpha} \left(1 - e^{-\beta(a_\alpha - b_\gamma)} t_1^{-L_{W_\gamma}(s)} t_2^{A_{Y_\alpha}(s)+1} m\right) \times \\ &\qquad \prod_{t\in W_\gamma} \left(1 - e^{-\beta(a_\alpha - b_\gamma)} t_1^{L_{Y_\alpha}(t)+1} t_2^{-A_{W_\gamma}(t)} m\right) \end{aligned} \quad (4.A.6)$$

redefining $q_i = \mathbf{q}_i/\sqrt{m_i m_{i+1}}$. The 5D exponentiated variables are $t_1 = e^{-\beta\epsilon_1}$, $t_2 = e^{-\beta\epsilon_2}$, $m_i = e^{\beta\mu_i}$ and $q_i = e^{\beta\tau_i}$.

To consider the instanton contribution of the linear $U(N)$ quiver with $k-1$ nodes it is sufficient to take the limit $q_k \to 0$. This correspond to freeze the k-th gauge group, indeed $q_k = \exp(\beta\tau_k)$ where $\beta = 2\pi i R_{S^1}$ and $\tau = \frac{4\pi i}{g_{YM}^2}$. So $q_k \sim \exp(-1/g_{YM}^2) \to 0$ when $g_{YM} \to 0$.

We obtain a linear quiver gauge theory with $k-1$ $U(N)$ gauge groups, $k-2$ massive bifundamentals and $2k$ massive flavour at the endpoints of the quiver: k in the fundamental representation at one endpoint and k in the antifundamental at the other

[5] It is easy to check this analytically when $k=2$. We have checked it also for $k=3,4$ with Mathematica up to instanton number 10.

endpoint. This is because

$$z_{bif}(\vec{a},\vec{b},\vec{Y},\vec{\varnothing},\mu)=\prod_{\gamma=1}^{N}z_{fund}(\vec{a},\vec{Y},\mu{+}b_\gamma),\qquad z_{bif}(\vec{a},\vec{b},\vec{\varnothing},\vec{W},\mu)=\prod_{\gamma=1}^{N}z_{antif}(\vec{b},\vec{W},\mu{-}a_\gamma),\tag{4.A.7}$$

where

$$z_{fund}(\vec{a},\vec{Y},\mu)=\prod_{\alpha=1}^{N}\prod_{(i,j)\in Y_\alpha}\sinh\frac{\beta}{2}\left[a_\alpha+i\epsilon_1+j\epsilon_2-\mu\right]=-z_{antif}(\vec{a},\vec{Y},\mu-a_\alpha).\tag{4.A.8}$$

Chapter 5

Conclusion and outlook

Here we present some discussion about the results of this thesis and some related open questions which would be interesting to answer.

In this thesis we calculated exactly partition functions for supersymmetric theories defined on four and five dimensional compact manifolds. The main ingredient was the use of supersymmetric localization that allowed us to reduce the infinite dimensional path integral into an integral over the finite dimensional moduli space of instantons. We obtained explicit results in some examples, and we could identify them with different objects of interest both in mathematics and in physics: as correlator of vertex operators of infinite dimensional Lie algebrae, generating functions of topological invariants and indices of higher dimensional superconformal field theories.

It is natural to split the discussion into two parts. A first part in section 5.1, related to Chapters 2 and 3, is about the results on four dimensional manifolds and their relations with two-dimensional conformal field theories and the evaluation of topological invariants. In section 5.2 we report open questions related to Chapter 4, about the results on five-dimensional manifolds and superconformal indices.

5.1 About theories in four dimensions

In Chapter 2 and 3 we give a contour integral expression for the partition function of the $\mathcal{N} = 2$ twisted theory on a compact toric surface.

We started in Chapter 2 by deriving the generalized Killing spinor equations for the $\mathcal{N} = 2$ algebra by demanding the consistency of the supersymmetry algebra on the manifold. The result is a slight generalization of the equations obtained in [82] by coupling with off-shell supergravity in the spirit of [65].

We derived spinor solutions to these equations realizing a version of Witten's topological twist which is equivariant with respect to a $U(1)$ isometry of the manifold, and then exploited them to construct a supercharge localizing on the fixed points of the $U(1)$ isometry.

The resulting partition function for the theory on $S^2 \times S^2$ is defined by gluing Nekrasov partition functions and integrating the v.e.v. of the scalar field Φ of the twisted vector multiplet over a suitable contour.

This result is generalized in Chapter 3 to a general compact toric surface M. The supersymmetric minima governing the path integral are given by two contributions: a generalization of instantons called Hermitian Yang-Mills connections localized in the fixed points of the manifolds under the toric action, plus magnetic fluxes $\vec{k}^{(\ell)}$ wrapping equivariant two-cycles. The final formula for the partition function, with also the insertion of equivariant local p and surface α observables, is given in section 3.1 and reads

$$Z^M(\mathbf{q}, x, z, y\,; \epsilon_1, \epsilon_2) = \sum_{\{k_\alpha^{(\ell)}\}|\text{semi-stable}} \oint_\Delta d\vec{a} \prod_{\ell=1}^{\chi(M)} Z_{\text{full}}^{\mathbb{C}^2}(\mathbf{q}^{(\ell)}\,; \vec{a}^{(\ell)}, \epsilon_1^{(\ell)}, \epsilon_2^{(\ell)})\, y^{c_1^{(\ell)}} \tag{5.1.1}$$

where $\mathbf{q}^{(\ell)} = \mathbf{q}\, e^{\imath^*_{P^{(\ell)}}(\alpha z + p x)}$, and the sum over $k_\alpha^{(\ell)}$s has to be constrained by suitable stability conditions.

The result (5.1.1) is carried out explicitly in two examples. In Chapter 2 we showed that the partition function in the case $M = S^2 \times S^2$ exhibits the three-point correlators and conformal blocks of Liouville gravity as building blocks. However these are glued in a different way with respect to Liouville gravity correlators. In particular our partition function is holomorphic in the momenta of the vertices and in their positions. It would be interesting to investigate further if there exists a chiral conformal field theory interpretation of the gauge theory result. Notice that chiral correlation functions for Liouville theory can be defined for some special values of the central charge by using the relation with super Liouville suggested by gauge theory [33, 34] and further investigated in [18, 131, 78]. In this case the field theory is defined over the resolution of $\mathbb{C}^2/\mathbb{Z}_2$, whose projective compactification is the second Hirzebruch surface $\mathbb{F}_2$.

In Chapter 3 we completely solve the contour integral in (5.1.1) for the $U(2)$ theory on $\mathbb{P}^2$ for both the values of the first Chern class $c_1 = 0, 1$. In case of odd first Chern class the expression of the partition function corresponds to the generating function of equivariant Donaldson invariants computed in [75, Theorem 6.15] by equivariant localization in the moduli space of unframed sheaves on $\mathbb{P}^2$, proving in this specific case a conjecture by Nekrasov [120]. In the case of even first Chern class we obtained a similar expression. We conjecture it to be the generating function of equivariant Donaldson invariants in the presence of strictly semi-stable bundles. This formula does not appear yet in the literature and so, as a test, we check that in the non equivariant limit it reproduces ordinary $SU(2)$ Donaldson invariants [66].

Let us notice that crucial tools for the evaluation of the contour integral (5.1.1) are Zamolodchikov's recursion relation for the Virasoro conformal blocks and AGT correspondence, which together allow us to locate the poles of the integrand and to easily compute the residues at all instanton numbers. The discovery of similar recursion relation for $\mathcal{W}$-algebrae, since these are conjectured to be dual to $SU(N)$ gauge theories [149], could be of great importance. In our context it should give a computational tool for the evaluation of Donaldson invariants in higher rank where wall-crossing formulas are notoriously difficult.

We also considered the $\mathcal{N} = 4$ theory on $\mathbb{P}^2$, taking the massless limit of the $\mathcal{N} = 2^\star$. In this case we reproduced the generating function of the Euler characteristics of the moduli space of *unframed* sheaves studied in [152, 142, 102].

Let us notice that the path integral computation directly evaluates the generating functions of Donaldson invariants. Whereas to obtain the generating function via wall-crossing requires the sum over a infinite number of contributions. This makes our approach a powerful way to study the structure of the invariants.

Finally we would like to make a comment regarding the comparison of our result with the result of [130], where localization on Kähler manifolds was considered using the explicit example of $\mathcal{N} = 2$ theory on $\mathbb{P}^2$. The result of the localization is similar to ours but not identical, since the sector of quantized magnetic fluxes whitin the supersymmetric minima is not considered therein. The reason is that these contributions are claimed to have no finite energy action on the local Omega-background around the three fixed points of the manifold. This is true, since their energies are proportional to the equivariant volume of $\mathbb{C}^2$ which diverges when the Omega-background parameters are turned off. However when the three local Omega-backgrounds contributions to the energy are summed together they produce the volume of $\mathbb{P}^2$ that is finite (since the manifold is compact) and depends no more on Omega-background parameters. So eventually these solutions have a global finite energy action and, in our opinion, they should be considered.

In the following we point out some interesting open questions related to this part of the project, separating them according to their affinity to mathemathics and physics. Some future directions in *mathematics* are

- The four manifolds studied in this thesis are all toric surfaces for which $b_2^+ = 1$. In this case, as described in the introduction (section 1.3.3), Donaldson invariants exhibit a wall crossing phenomenon that is reflected in the same effect for the partition function. The standard evaluation of Donaldson invariants in a specific chamber is usually done via wall crossing starting from an empty chamber. In these cases, (5.1.1) should reproduce the wall crossing terms as computed in [75]. Proving this for a general toric manifolds would prove completely the conjecture of [120].

- The case of $\mathbb{P}^2$ is quite special since it presents a single chamber and therefore it is not possible to use wall-crossing formula to compare our result. Indeed we checked our expression directly with the generating function calculated ad hoc in [75]. Something missing in the analysis therein is the reduction of the generating function, in the non-equivariant limit, to the contour integral formulas of Moore and Witten [114] in terms of modular forms. This should be easier starting from (5.1.1) that is already in a contour integral representation.

- Our result suggests a relation between fixed point locus of the moduli space of unframed instantons on $\mathbb{P}^2$ and copies of fixed point locus of the moduli space of framed instantons on $\mathbb{C}^2$. The investigation of these topics should give a rigorous confirmation of the results of Chapter 3 and new interesting insights in the geometry of the moduli space of unframed torsion-free sheaves. In turn this could help to give a mathematical proof of Witten's conjecture, that relates Donaldson invariants to Seiberg-Witten invariants [146].

- The generalization of our techniques to 6d manifolds could allow the construction of the generating function of Euler characteristic of the moduli space of torsion-free sheaves and therefore the evaluation of Donaldson-Thomas invariants. In the case of an elliptic fibration over a complex toric surface should be possible to write the partition function as a product of elliptic Virasoro conformal blocks [123, 90, 111].

- In Chapter 3 the quantized charges of the magnetic fluxes are assumed to be in one to one correspondence with the Klyachko parameters describing GIT stability conditions for equivariant vector bundles on compact toric manifolds [100]. The stability of the vector bundle is indeed equivalent to the Hermitian Yang-Mills condition and, therefore, it has to be taken into account in the evaluation of the path integral.

  The gauge theory interpretation of this conditions on the magnetic fluxes is an interesting topic of study. In a BPS state counting perspective it seems to put some constraint on states. Also its interpretation in the dual CFT is interesting, as it may be seen as some special relation in the vertex operator algebra of the theory, as the one which arises from Nakajima-Yoshioka blow-up formula [116, 29].

Some future directions in *physics* are

- A subtlety mentioned in Chapter 2, and already described in [144], is the existence of two consistent choices of reality conditions for the fields $\Phi, \bar{\Phi}$ in gauge theory. These can be either real and independent or $\bar{\Phi} = -\Phi^\dagger$. This leads to different localization schemes which would be interesting to compare and to investigate in the conformal field theory counterpart.

- In appendix 2.C we also discussed another solution to the Killing spinor equation on $S^2 \times S^2$ which are composed by the spinorial solutions on S^2 discussed in [21, 59]. These solutions correspond to trivial R-symmetry bundle and therefore imply the Witten topological twist only locally, more precisely they localize on instantons on two of the fixed points of $S^2 \times S^2$ and on anti-instantons on the other two. The partition function in this case is neither real nor holomorphic and it would be interesting to further discuss the supersymmetric path integral induced by these solutions and its conformal field theory dual.

- Another related subject to investigate is the reduction of the partition function on $S^2 \times S^2$ to spherical partition functions in the zero volume limit of one of the two spheres. This analysis would help in shedding light on the relation between instanton and vortex partition functions. One could consider the insertion of surface operators [5, 62] on one of the two spheres. Actually, surface operators on $\mathbb{C}^2$

are related to the moduli space of instantons on $S^2 \times S^2$ framed on one of the two spheres [10] so that a nice interplay could arise among these partition functions.

- In the same spirit of Chapter 2 one can try to find a general pattern for a AGT-like correspondence of the $\mathcal{N} = 2$ partition function of the four dimensional $SU(2)$ gauge theories on a general compact toric manifold. Our result suggests to read the gauge theory partition function in terms of a chiral CFT whose sectors are in one-to-one correspondence with the toric patches. The contribution of each sector to the correlation number is given by a copy of Virasoro conformal block with central charge $c^{(\ell)} = 1 + 6\frac{\left(\epsilon_1^{(\ell)}+\epsilon_2^{(\ell)}\right)^2}{\epsilon_1^{(\ell)}\epsilon_2^{(\ell)}}$ in the ℓ-th sector and three point functions related to the corresponding one-loop contributions of the gauge theory. The change of $(\epsilon_1^{(\ell)}, \epsilon_2^{(\ell)})$ under change of patch is related to the intersection of the corresponding divisors. Investigations in similar directions for Hirzebruch surfaces have been pioneered in [28].

- Our analysis suggests that Hermitian-Yang-Mills connections are the natural fixed points for the supersymmetric localization of equivariant twisted theories. This is in agreement with [142]. These configurations are a slight generalization of anti-self-dual connections, as they allow the curvature to have a self-dual component along the Kähler form of the toric variety. Since instantons and anti-instantons are counted with complex conjugated fugacities $\tau, \bar{\tau}$ respectively, the localization on HYM connections could solve the issue of holomorphic anomaly opened in [142] and could allow the reconstruction of the anti-holomorphic completion of the mock-modular forms generating Euler characteristic of the moduli space of instantons. The study of $\mathcal{N} = 2^\star$ theory in the massless limit as done in section 3.3, but keeping track of the anti-instanton counting parameter $\bar{\tau}$, could answer this question.

- The twisted $\mathcal{N} = 4$ theory studied in section 3.3 can also realize E-strings BPS state counting in terms of elliptic genera [109, 32, 79]. These partition functions enjoy interesting and non-trivial modular properties [108]. It would be useful to explore if and how these properties are realized for non-vanishing mass $M \neq 0$ of the $\mathcal{N} = 2^\star$ theory.

- It would be very interesting to look for the existence of holographic dual theories to equivariant twisted gauge theories (with possibly a mass deformation) on compact manifolds. The dual supergravity theories in these cases should involve anti-symmetric tensor fields, differently from other cases already studied, due to the background anti-symmetric tensor fields necessary to construct the generalized Killing spinors generating the supersymmetry on the boundary manifold.

5.2 About theories in five dimensions

In Chapter 4 we studied circle compactification of systems of M5-branes probing a transverse ALE orbifold singularity. In the case of a single M5-brane we computed the six dimensional $(1,0)$ super conformal indices of these systems via letter counting of BPS states, and showed that they coincide with partition functions of suitable five-dimensional circular quiver gauge theories.

We furthermore exposed modular properties of these theories encoded by the S-duality of the corresponding pq-web systems of five-branes by showing the global symmetry enhancement induced by instanton operators of the quiver gauge theories.

The partition functions of these theories can be constructed in terms of a K-theoretic version of the Nekrasov partition function. We rewrote the latter in a plethystic exponential form by studying five-dimensional instanton operators generating non perturbative corrections.

A crucial issue was the presence of *spurious* terms, these are associated to the presence of parallel external legs in the corresponding pq-web system. We identified and removed properly these spurious contributions. After this step the building blocks could be written in terms of double elliptic gamma functions [68]. Using the modular properties of the latter, we built the S^5 partition function, which we indentified with the $6d$ $(1,0)$ index previously calculated.

We finally discussed how the S-duality of these theories, suggests relations among correlators of different $q\mathcal{W}$-algebrae obtained by five-dimensional AGT duality. We spelled out only the known case of linear quiver, whereas what happens in the more interesting case of the circular quiver, displayed in figure 4.9, is still an open problem.

Some interesting questions related to this topic are as follows

- A natural extension of our work, would be to fully compactify the pq-web brane diagram on the torus. This amounts to compactify the D6 branes in Fig. 4.1 on a circle, and has a link to topological string amplitudes on elliptically fibered Calabi-Yau's which are relevant for the classification and study of $\mathcal{N} = (1,0)$ superconformal field theories [80]. From the viewpoint of the index computation this set-up would also be useful to analyze the issue of spurious factors due to the strings stretched between semi-infinite branes, which should appear in the decompactification limit. The analysis of the S-duality of pq-web diagrams on the torus should be useful to investigate elliptic $\mathcal{W}$-algebrae as considered in [123, 90, 111].

- It would be very interesting to analyze the case of non-abelian theories, which would provide information about interacting M5 brane systems. This implies the integration over the Coulomb branch parameters and a full control of the polar structure of non-abelian Nekrasov partition functions in five and six dimensions. In the simplest example of non-abelian theory, $SU(2)$, similar techniques to the ones of Chapter 3 could be used.

- The study of S-duality of linear and circular pq-webs and its relation with q$\mathcal{W}$-algebrae has to be further investigated, in order to have an independent proof of the plethystic formulae for the supersymmetric partition functions derived in this Chapter as well as their interpretation in terms of dualities of q-deformed correlators.

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