Physical Systems in the Shock Wave Backgrounds

Thesis Submitted for the Degree of Magister Philosophiae

Candidate: $C.Klim\check{c}ik$ Supervisor: $Prof.D.Amati$

Academic Year 1987/88

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Contents

Chapter 1

Introduction

Shock wave backgrounds have recently attracted an attention in field theory mainly because they play an important role in scattering processes at ultrahigh energies [1] and because they exhibit in some cases interesting focusing phenomena which may be related to gravitational singularities [2,3]. Two classes of these backgrounds are of particular interest namely the Aichelburg-Sexl one [4] and the homogeneous infinite planar shell shock wave [5], both parametrized by one real positive parameter. The former is an exact solution of the Einstein equations in D dimensions corresponding to a gravitational field generated by a massless particle of given energy, while the latter to that generated by a planar shell of null matter with constant surface energy density.

A behaviour of geodesics in these classes of shock waves was studied by T.Dray and G.t'Hooft [5] and by V.Ferrari, P.Pendenza and G.Veneziano [2]. In the case of the homogeneous infinite planar shell shock wave was found an interesting focusing of families of null geodesics occuring in location identical to that of the curvature singularity found in the head-on collision of two homogeneous infinite planar shell shock waves [6]. It is natural to believe [2] that focusing of geodesics is a sign of generation of singularities in general. This fact lead to an interest on what changes would suffer the geodesical picture at the quantum level and constituted itself one of motivations of a work [3] in which a quantum field theory in a generic shock wave background was exactly solved. It was found that in some cases the indicated geodesical singularity survives at the quantum level, but there are also cases in which the quantum effects cause a smearing of

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the singularity.

Having known the behaviour of the null geodesics in the Aichelburg Sexl metric t'Hooft was able to find an S matrix describing a scattering of a quantum particle in this metric [1]. This result was rigorously derived in ref. [3] and it is of a considerable importance, since it applies to an ultrahigh energy scattering of two particles due to gravitational interaction. Indeed in a reference system in which one particle is very hard, the motion of the softer one may be described as that of a particle in a gravitational field generated by the other one, which is nothing but the Aichelburg-Sexl metric. One then expects that the t'Hooft's S matrix should be obtained in some approximation from a correct theory of quantum gravity which is eventually supposed to be the string theory. A relevant string result is due to Amati, Ciafaloni and Veneziano [7] who found the S matrix for a superstring scattering at ultrahigh energies (in a flat spacetime theory). A connection between the works of t'Hooft and of Amati, Ciafaloni and Veneziano was established in a paper by Amati and the author [8] who solved the nonlinear supersymmetric σ-model that describes a superstring in a shock wave metric with an arbitrary profile. An exact operatorial expression for the S matrix (S_σ) was found. It coincides, for a profile given by a tree string amplitude in the impact parameter space, with the S matrix of Amati, Ciafaloni and Veneziano, indicating thus an emergence of a non trivial curved metric as an infinite genus effect of the flat spacetime treatment. The corresponding shock wave profile has the Aichelburg Sexl form for large distances but it differs from it for shorter ones. This short distance string generated softening eliminates - by the way - the direct channel poles in the t'Hooft S matrix.

It is worthwhile to note, that all three results are nonperturbative and, in the ultrahigh energy region, also exact. This fact may serve as a consistency test of the validity of the string theory predictions. Indeed, using three completely different languages and techniques the obtained results match perfectly each other showing that in the appropriate limit the string theory confirmes the expectations of the field theory. From the mathematical point of view to solve the field theory or the string motion in the shock wave backgrounds amounts essentially to solve a generally covariant field equation and the nontrivial quantum field theory (the σ-model) respectively. To provide an exact solution of such problems may be interesting by itself. For instance the generally covariant Klein-Gordon equation in

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an arbitrary shock wave background can be solved, rather remarkably, by using the path integral techniques, providing thus an interesting example of a situation in which a non-Gaussian continual integral can be exactly evaluated giving the same result as the more "conventional" procedures do.

In the second chapter of this thesis we show how to obtain the shock wave solutions of the Einstein equations, in particular the Aichelburg-Sexl metric. We provide two ways of solving this problem. One, more "mathematical", by directly solving the Einstein equations with properly chosen righthand side, the other in turn, more "physical", by boosting the Schwarzschild solution. Both methods help to elucidate physical interpretations of these backgrounds. We will finish the chapter by finding the null geodesics in the shock wave metrics, generalizing the results of the references [2,5] to nonaxisymmetric wave profiles.

The third chapter deals with the field theories in the shock wave backgrounds. A scalar massless theory is solved and expectation values of the energy-momentum tensor in the scattering states are explicitly provided. Then the rigorous derivation of the t'Hooft S matrix is given and the quantum focusing phenomena are discussed in the case of the homogeneous infinite planar shell shock wave and also for the sourceless waves [3].

The fourth chapter is devoted to the exact solution of the nonlinear supersymmetric σ-model. The bosonic and the supersymmetric versions are treated in detail and the comparison with the results of Amati, Ciafaloni and Veneziano and of t'Hooft respectively is provided. Finally we draw conclusions and provide a brief outlook as well.

Chapter 2

Shock Wave Solutions of the Einstein Equations

2.1 Boosting the Black Hole Solution

A gravitational field of a massive neutral point-like particle at rest is described by the Schwarzschild metric

$$ds^2 = -\left(1 - \frac{2m}{r'}\right)dt'^2 + \left(1 - \frac{2m}{r'}\right)^{-1}dr'^2 + r'^2 d\Omega'^2 \qquad (2.1.1)$$

Boosting the solution (2.1.1) one obtaines a field created by the particle moving with a constant velocity, which is, of course, smaller than that of light. To obtain a gravitational field generated by a light-like particle we may perform an infinite boost going with m to zero in such a way that the 3-momentum of the particle remains finite. Explicitly

$$m = 2p e^{-\beta}, \quad u' \equiv (t' - z') = e^{\beta} u, \quad v' \equiv (t' + z') = e^{-\beta} v$$

Since

$$p'^\mu = (m, 0, 0, 0)$$

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we have

$$p^\mu = \lim_{\beta \to \infty} (\cosh \beta \, 2p \, e^{-\beta}, 0, 0, \sinh 2p \, e^{-\beta}) = (p, 0, 0, p)$$

Now we boost the metric (2.1.1) to the unprimed coordinates. This may be done most easily by rewriting it in a form (see [11])

$$ds^2 = dx'^2 + \frac{4p \, e^{-\beta}}{\sqrt{x'^2 + (x' \omega')^2}} (\omega' dx')^2 +$$

$$+ \sum_{n=1}^{\infty} \left(\frac{4p \, e^{-\beta}}{\sqrt{x'^2 + (x' \omega')^2}} \right)^n \frac{1}{(x'^2 + (x' \omega')^2)} \quad \times$$

$$\times \left[(x' \, dx') + (x' \omega')(dx' \, \omega') \right]^2 \qquad (2.1.2)$$

where

$$\omega'^\mu \equiv (1, 0, 0, 0)$$

In fact

$$x'^2 + (x' \omega')^2 = r^2 \qquad\qquad (\omega' dx')^2 = dt^2$$

$$\left[(x' dx') + (x' \omega')(dx' \omega') \right]^2 = dr^2$$

Note, that the factor $(1 - 2m/r')^{-1}$ in (2.1.1) is expanded in the geo-metrical series, which is allowed, since $m \to 0$.

To boost this expression amounts to erasing the primes over x', dx' and ω'. It is convenient now to study the limit $\beta \to 0$ separately for cases $u \neq 0$ and $u = 0$. For $u \neq 0$ one has

$$ds^2 = dx^2 + \frac{4\rho}{|u|} du^2 \tag{2.1.3a}$$

whileas for $u = 0$

$$ds^2 = dx^2 + \frac{\rho e^3}{|\tilde{x}|} du^2 \tag{2.1.3b}$$

where $\tilde{x}$ denotes the transverse coordinates. To study the divergence in (2.1.3b) it is convenient to rewrite the expressions (2.1.3ab) in a form valid for all u, e.g.

$$ds^2 = \frac{4\rho}{\sqrt{u^2 + \alpha^2\tilde{x}^2}} du^2 - dudv + d\tilde{x}^2, \quad \alpha \to 0 \tag{2.1.4}$$

Perform now a transformation

$$\tilde{v} = v + \left(\theta_\alpha(-u) - \theta_\alpha(u)\right) 4\rho \ln \sqrt{u^2 + \alpha^2\tilde{x}^2} \tag{2.1.5}$$

where θ_α is some regularization of the usual step function. Going to $\alpha \to 0$ one gets the Aichelburg-Sexl metric

$$ds^2 = -dud\tilde{v} - 8\rho \ln(|\tilde{x}|/c)\delta(u)du^2 + d\tilde{x}^2 \tag{2.1.6}$$

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where C is an irrelevant scale.

2.2 General Shock Wave Solutions

Call the gravitational shock wave any metric of the form

$$ds^2 = -dudv + f(\tilde{x})\delta(u)du^2 + d\tilde{x}^2 \qquad (2.2.1)$$

with an obvious notation, but $\tilde{x}$ denotes now (D-2) transverse coordinates. The metric (2.2.1) corresponds to the shock wave moving along the positive z-axis with the wavefront $u = 0$. Following refs.[2,5] one finds the only nonvanishing component of the Ricci tensor to be

$$R_{uu} = -\frac{1}{2}\tilde{\Delta}f(\tilde{x})\delta(u) \qquad (2.2.2)$$

where $\tilde{\Delta}$ is the Laplacian in the transverse coordinates.Thus, unless the function f is "too singular"[1], the metric (2.2.1) is always a solution of the Einstein equations with a source given by

$$T_{uu} = \rho(\tilde{x})\delta(u) \qquad (2.2.3)$$

where

$$\rho(\tilde{x}) = -\frac{1}{16\pi G}\tilde{\Delta}f(\tilde{x}) \qquad (2.2.4)$$

and all other components of $T_{\mu\nu}$ being zero.(We restored also the gravitational constant G that was set $G = 1$ in the preceding section.) Given a particular wavefront energy density one obtains a corresponding profile modulo some sourceless shock wave, as may be called any solution of the

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[1] We do not need for our purposes to specify the allowed singularities of the function f. We note, however, that f need not be, in general, regular. This happens, for instance,for the Aichelburg Sexl metric.

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homogeneous equation (2.2.4).(It is easy to see that the profile of the sourceless shock wave is some traceless quadratic form.) If one puts

$$\rho(\tilde{x}) = \rho \delta^{(D-2)}(\tilde{x})$$ (2.2.5)

the solution of eq.(2.2.4) is proportional to the Green function of the Laplace equation, namely

$$f(\tilde{x}) = -8\rho G \ln(|\tilde{x}|/c), \quad D = 4$$ (2.2.6a)

$$f(\tilde{x}) = -\frac{16\rho G \pi}{(D-4)\Omega_{D-2}}, \quad D > 4, \quad \Omega_D \equiv \frac{2\pi^{D/2}}{\Gamma(D/2)}$$ (2.2.6b)

We see that for $D = 4$ one has the Aichelburg Sexl metric (2.1.6).

Another interesting class of the shock waves is that of homogeneous planar shells. In this case the wavefront energy density is given by

$$\rho(\tilde{x}) = \begin{cases} \rho & , \quad |\tilde{x}| < R \\ 0 & , \quad |\tilde{x}| \geq R \end{cases}$$ (2.2.7)

For R finite ($R \to \infty$) one has the finite (infinite) planar shell. In what follows we shall need the explicit form for the infinite case. One has immediately

$$f(\tilde{x}) = -a\tilde{x}^2 + f_s, \quad a = \frac{8\pi G \rho}{(D-2)}$$ (2.2.8)

For definiteness we put $f_s = 0$, where the subscript "s" indicates the f_s is a sourceless wave.

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2.3 Null Geodesics

We turn now to study the null geodesics. Their importance is at least twofold. Their knowledge provides an information, which enabled t'Hooft to find the phase shift of a wave function of the quantum relativistic test particle in the shock wave backgrounds. Supposing that this is the only change that the wave function suffers (which was shown in ref.[3]) one derives the correct S matrix for the scattering on the shock wave. The other reason lies on the fact that for some particular wave front profiles interesting focusing phenomena occur. As was argued in [2], it is natural to believe that focusing of geodesics is a sign of the generation of gravitational singularities, since the matter is getting concentrated to a point. This infinite energy density should generate a singularity of the curvature tensor. Indeed, in a head-on collision of two homogeneous infinite planar shells [6] the curvature singularity occurs, which lies exactly at the location of focal point of the family of null geodesics perpendicular to the wave front of one planar shell. One sees, quite remarkably, that nonlinear effects did not spoil the singularity picture found in the geodesical computations. We postpone a more detailed discussion to the chapter 3, in which the focusing phenomenon will be studied at the quantum level. Here we provide a classical picture.

The Christoffel symbols of the metric (2.2.1) read

$$\Gamma^i_{uu} = -\frac{1}{2} f_{,}{}^{i}(\tilde{x}) \delta(u) \tag{2.3.1a}$$

$$\Gamma^v_{uu} = -f(\tilde{x}) \delta'(u) \tag{2.3.1b}$$

$$\Gamma^v_{ui} = -f_{,i}(\tilde{x}) \delta(u) \tag{2.3.1c}$$

Thus the equation for null geodesics is

$$\frac{d^2 x^u}{d\tau^2} + \Gamma^u_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \,, \quad g_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} = 0 \tag{2.3.2}$$

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Since $d^2 u / d\tau^2 = 0$ and the geodesics are null we may choose directly u to be the affine parameter. So we have

$$\frac{d^2 x^i}{du^2} = \frac{1}{2} f_{,}{}^{i} (\hat{x}) \delta(u)$$

(2.3.3a)

$$\frac{d^2 v}{du^2} = f(\hat{x}) \delta'(u) + 2 f_{,i} (\hat{x}) \delta(u) \frac{dx^i}{du}$$

(2.3.3b)

and the constraint

$$\frac{dv}{du} - f(\hat{x}) \delta(u) - \left(\frac{dx^i}{du}\right)^2 = 0$$

(2.3.4)

Note that the equation (2.3.3b) is a consequence of eq.(2.3.3a) and (2.3.4). Since for $u \neq 0$ these equations are those for the flat space time one has immediately

$$x^i_{\lessgtr} (u) = x^i_{0\lessgtr} + p^i_{\lessgtr} u$$

(2.3.5a)

$$v_{\lessgtr} (u) = v_{\lessgtr} + p^v_{\lessgtr} u$$

(2.3.5b)

$$p^v_{\lessgtr} = (p^i_{\lessgtr})^2$$

(2.3.6)

where the subscripts $<, >$ indicate $u < 0$ and $u > 0$ respectively. The problem is to find out quantities ($u > 0$) in terms of the in ones. This is a simple matter. Since $x^i(u)$ is continuous at $u = 0$, integrating (2.3.3a) one has

$$p^i_> = p^i_< + \frac{1}{2} f_{,}{}^{i} (\hat{x}_0) , \quad x_0^i \equiv x_{0<}^i = x_{0>}^i$$

(2.3.7)

Integrating as well (2.3.4) we obtain

$$v_> = v_< + f(\hat{x}_0)$$

(2.3.8)

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Due to (2.3.6) we know also p^v and the problem is solved.

Let as consider now the homogeneous infinite planar shell shock wave (2.2.8) and a family of geodesics, for which at $u = -\omega$ ($\omega > 0$), $v(-\omega) = x^i(-\omega) = 0$. Thus the geodesics of this family are parametrized by p^i. From (2.3.5-8) one has for $u > 0$

$$x^i(u) = p^i \omega + p^i(1-a\omega)u \qquad (2.3.9a)$$

$$v(u) = p^2\omega(1-a\omega) + p^2(1-a\omega)^2 u \qquad (2.3.9b)$$

We see that for $u_F = \omega/(a\omega - 1)$ the focusing of geodesics occurs which is real or virtual depending on the sign of u_F. ω and u_F fulfil, quite remarkably, the perfect lense equation [2]

$$\frac{1}{u_F} + \frac{1}{\omega} = a \qquad (2.3.10)$$

Another focusing occurs when initially ($u < 0$) all geodesics in the family are colinear, e.g. v_0 is fixed, $p^i = 0$. Then for $u > 0$

$$x^i(u) = x_0^i - a x_0^i u \qquad (2.3.11a)$$

$$v(u) = -a \hat{x}_0^2 + a^2 \hat{x}_0^2 u \qquad (2.3.11b)$$

The focusing occur for $u_F = 1/a$.

We finish this section with a brief discussion of a "partial" focusing taking place for the sourceless shock waves e.g.

$$f(\hat{x}) = -\sum_i q_i x_i^2 \quad , \quad \sum_i q_i = 0 \qquad (2.3.12)$$

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Fixing v_0, $p^i = 0$ and all $x_0^i = 0$ except x_0^j one gets

$$\dot{x}^j(u) = \dot{x_0}^j - a_j \dot{x_0}^j u \qquad (2.3.13a)$$

$$V(u) = -a_j \dot{x_{0j}}^2 + a_j^2 \dot{x_{0j}}^2 u \qquad (2.3.13b)$$

The indicated subfamily gets focused for $u = 1/a^j$ hence some (milder) concentration of energy occurs.

Chapter 3

Field Theory in a Shock Wave Background

3.1 Introduction

In the previous section we have discussed in detail the focusing of geodesics in some special shock wave metrics. It is of obvious interest to obtain a field theory picture of corresponding phenomena i.e. the field theory in these background metrics. Such information would be obviously more complete simply because the geodesical picture should be reobtained in some appropriate limit. This is not the only reason, however, why to pursue such a program. Since the dynamics of the general relativity is described by the Einstein equations, the field theory language should provide more relevant information on the problem of creation of curvature singularities. It may also happen that one finds no energy density singularity at the quantum level (the indicated "geodesical" singularity would be smeared due to wave or quantum effects.) It cannot be said, however, that in these cases the creation of singularity is excluded. Large enough energy density at the scale of the Schwarzschild radius for the given energy may create a singularity as well. Nevertheless we shall see that in some cases the indicated geodesical singularities do survive in the framework of the field theory while in others are smeared by the quantum effects. As an example of the former there is the homogeneous infinite planar shell wave the latter is represented by the finite planar shells.

Another reason why to study the focusing phenomena at the quantum

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level is also a need for a quantitative expression for the energy density, which is obviously the energy momentum tensor. We may thus model the family of the classical geodesics by some quantum state and calculate expectation values of energy density in these states. The singularity of this density suggests an appearance of a curvature singularity via the Einstein equations, if we study in the first approximation the backreaction on the metric.

Besides the investigation of the focusing phenomena there is also a motivation to solve the problem of the scattering of the quantum relativistic particle on the Aichelburg Sexl metric [1]. Having supposed, that the only change which the relativistic wave function suffers in crossing the wavefront is the change of its phase (which may be found knowing the geodesics of the metric), t'Hooft found the S matrix. We shall show from the first principles that his assumption was correct.

3.2 General Formalism

To build a quantum field theory on a general curved background is somewhat intricated task (for a detailed discussion see f.i. ref.[9]). The main problems are connected with a possible nontrivial topology of the manifold (see f.i. the Hawking effect [10]), with a physical interpretation of the quantum modes if the curvature of the background is non-zero and also with renormalization of the energy momentum tensor, which, by the way, is the basic quantity needed for our discussion of the focusing phenomena. Fortunately enough we are not forced to enter these difficult problems here since the shock wave backgrounds are particularly simple. Indeed, they are topologically trivial and almost everywhere flat, so for instance the renormalization of the energy-momentum tensor outside the wavefront does not constitute any problem.

The general strategy for formulating a quantum field theory on the shock wave background is in our case simple. One must find two complete sets of solutions of a generally covariant field equation, which look like the free ones for $u < 0$ (in-region) and $u > 0$ (out-region) respectively. The Bogoliubov transformation which connects these two sets then follows and the dynamical content of the theory is fixed e.g the S matrix elements and expectation values of observables may be found in terms of the Bogoliubov

coefficients. Our shock wave treatment will be a simple illustration of this procedure.

For simplicity we will consider the scalar massless theory, since we wish to mimick the classical picture of null geodesics. We note, however, that generalization of our results to the massive case is trivial as will be clear from what follows. The Klein-Gordon equation in our background reads

$$-\frac{\partial^2}{\partial u \partial v}\varphi - \delta(u) f(\hat{x})\frac{\partial^2}{\partial v^2}\varphi + \frac{1}{4}\tilde{\Delta}\varphi = 0 \qquad (3.2.1)$$

The in-modes must look like the free ones

$$u_{free\ k_-,\tilde{k}}(\hat{x}, u, v) = N_k \exp\left[i\left(-k_- v - k_+ u + \tilde{k}\tilde{x}\right)\right] \qquad (3.2.2)$$

for $u < 0$. Here $k_{D-1}, \tilde{k}$ are the components of the (D-1)-momentum, k_0 is the energy,

$$k_{\pm} \equiv \frac{1}{2}\left(k_0 \pm k_{D-1}\right) \qquad (3.2.3)$$

and

$$N_k \equiv \left((2\pi)^{D-1} 2k_-\right)^{-1/2} \qquad (3.2.4)$$

The normalization factor N_k ensures the usual normalization of the modes and, hence, of the annihilation and creation operators with respect to the measure $dk_- d\tilde{k}$ (note that $dk_- d\tilde{k} = (k_-/k_0)dk_{D-1}d\tilde{k}$) so that

$$\varphi(x) = \int dk_- d\tilde{k}\left(a_{k_-,\tilde{k}}\, u_{k_-,\tilde{k}}(x) + a_{k_-,\tilde{k}}^+\, u_{k_-,\tilde{k}}^*(x)\right)$$

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with

$$\left[a_{k,\tilde{k}} \, , \, a^{+}_{\ell,\tilde{\ell}} \right] = \delta(\tilde{k} - \tilde{\ell}) \, \delta(k - \ell)$$ (3.2.5)

If we happen to know how the in-modes look in the out-region it is easy to decompose them in terms of the out-modes, to compute the corresponding Bogoliubov transformation between a_{in}, a^{+}_{in} and a_{out}, a^{+}_{out} operators and, consequently, the S matrix elements and various expectation values of the field observables.

We will look for the in-solutions of (3.2.1) of the form

$$u_{k,in}(u, v, \tilde{x}) = N_{k} \, exp(-ik.v) \, \Psi_{k,in}(u, \tilde{x})$$ (3.2.6)

From (3.2.1) one sees that the functions $\Psi_{k,in}$ must fulfil the Schrödinger equation

$$i \frac{\partial}{\partial u} \Psi_{k,in} = \left(-\frac{\tilde{\Delta}}{4k_{-}} - f(\tilde{x}) \, \delta(u) \, k_{-} \right) \Psi_{k,in}$$ (3.2.7)

All information about the dynamical evolution of a quantum mechanical system is contained in the kernel $G(\tilde{x}'', u'', \tilde{x}', u')$ of the equation (3.2.7). Knowing this kernel it is a simple matter to continue a solution from the in- to the out-region. Moreover, since the evolution is that of the free system unless $u = 0$ in fact we need to know just $G(\tilde{x}'', 0^{+}, \tilde{x}', 0^{-})$ We provide two simple ways of determination of this quantity, using the operatorial language and the path integral formalism. This is quite remarkable in view of a fact that the integrand is not Gaussian. We shall see that both methods give the same result, thus having a new example of an exact result obtained by means of continual integration. Let us start with the "classical" operatorial approach. One regularizes the δ-function in (3.2.7) by an expression

$$\delta_{\varepsilon}(u) = \frac{1}{2\varepsilon} \left(\theta(u + \varepsilon) - \theta(u - \varepsilon) \right)$$ (3.2.8)

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thus having

$$i\frac{\partial}{\partial u}\,\mathcal{Y}_{k,in} = \overset{1}{H}_{\varepsilon}(u)\,\mathcal{Y}_{k,in} \equiv \left(-\frac{\widetilde{\Delta}}{4k_-} - f(\tilde{x})\,\delta_{\varepsilon}(u)k_-\right)\mathcal{Y}_{k,in} \quad (3.2.9)$$

A solution ot (3.2.9) must be a continuous function, since the right hand side is bounded. Now the only nontrivial propagation occurs in the interval $-\epsilon < u < \epsilon$ and one has

$$\mathcal{Y}_{k,in}(+\varepsilon,\tilde{x}) = \int d\tilde{q}\,\langle \tilde{x}|\exp[-iH_{\varepsilon}(0)(2\varepsilon)]|\tilde{q}\rangle\,\mathcal{Y}_{k,in}(-\varepsilon,\tilde{q}) \quad (3.2.10)$$

because in this interval the Hamiltonian is time-independent. Performing a limit $\epsilon \to 0$ we obtain

$$\mathcal{Y}_{in}(0^+,\tilde{x}) = \mathcal{Y}_{in}(0^-,\tilde{x})\,\exp\,ik_-\,f(\tilde{x}) \quad (3.2.11)$$

or

$$G(\tilde{x}'',0^+,\tilde{x}',0^-) = \delta(\tilde{x}''-\tilde{x}')\,\exp ik_-f(\tilde{x}) \quad (3.2.12)$$

We turn now to the continual integration formalism starting with the well-known Feynman formula for the kernel i.e.

$$G(\tilde{x}'',u'',\tilde{x}',u') = \int_{\substack{\tilde{q}(u')=\tilde{x}'\\\tilde{q}(u'')=\tilde{x}''}}\mathcal{D}\tilde{q}(u)\,\exp\left(i\int_{u'}^{u''}du[k_-\dot{\tilde{q}}^2 + f(\tilde{q})k_-\delta(u)]\right) =$$

$$= \int \mathcal{D}\tilde{q}(u)\,\exp\,ik_-f(\tilde{q}(0))\,\exp\,i\,S_{free}[\tilde{q}(u)] =$$

$$= c\int d\tilde{x}\,\exp[ik_-f(\tilde{x})]\,G_{free}(\tilde{x}'',u'',\tilde{x},0)\,G_{free}(\tilde{x},0,\tilde{x}',u')$$

$$(3.2.13)$$

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where $c = 1$ in order to recover a correct expression for the $f = 0$ case. Thus

$$G\left(\tilde{r}'',0^+,\tilde{r}',0^-\right) = \delta(\tilde{r}''-\tilde{r}')\exp ik_-f(\tilde{r}')$$

that coincides with (3.2.12).
Substitute now to (3.2.11) in-mode $\Psi_{k,in}$ i.e.

$$\Psi_{k,in}\left(0^+,\tilde{r}\right) = \exp i\left(\tilde{k}\tilde{r} + k_-f(\tilde{r})\right) \qquad (3.2.14)$$

It is not difficult to see from (3.2.6) and (3.2.14) that the energy $k_- + k_+$ remains positive for all out-modes in the decomposition of the in-mode. We can therefore conclude that no particle production is seen, or in other words, the in- and out-vacua are identical. It is easy now to find the Bogoliubov coefficients. Indeed, we look for a function (or a distribution) $\Phi(k_-,\tilde{k},l_-,\tilde{l})$ with a property (for $u > 0$)

$$\int dl_- d\tilde{l}\ \Phi(k,l)\ u_{l_-,\tilde{l}\ out} = u_{k_-,\tilde{k}\ in} \qquad (3.2.15)$$

and hence

$$a_{l_-,\tilde{l}\ out} = \int dk_- d\tilde{k}\ \Phi(k,l)\ a_{k_-,\tilde{k}\ in} \qquad (3.2.16)$$

Clearly

$$\Phi(k,l) = \delta(k_- - l_-)\tilde{\Phi}(k,l) \qquad (3.2.17)$$

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Thus

$$\int d\tilde{l}\, \phi(k,l)\, exp\, i\tilde{l}\tilde{x} = exp\, i[\tilde{k}\tilde{x} + k_- f(\tilde{x})] \qquad (3.2.18)$$

and

$$\tilde{\phi}(k,l) = \frac{1}{(2\pi)^{D-2}} \int d\tilde{x}\, exp\, i[(\tilde{k}-\tilde{l})\tilde{x} + k_- f(\tilde{x})] \qquad (3.2.19)$$

Knowing $\Phi(k,l)$ one may compute the S-matrix elements. For example

$$\langle 0|a_{k_{0-l},\tilde{l}\,out}\, a^+_{k_{0-l},\tilde{k}\,in}|0\rangle = \frac{k_-}{\sqrt{k_0 l_0}}\, \tilde{\phi}(k,l)\, \delta(k_- - l_-) \qquad (3.2.20)$$

Note that the factor $k_-/\sqrt{k_0 l_0}$ is needed to make a transition from the light-cone formalism to the usual one.

As an example let us shortly discuss the Aichelburg-Sexl metric in $D = 4$ i.e.

$$f(\tilde{x}) = -4\rho G\, ln\, (\tilde{x}/c)^2 \qquad (3.2.21)$$

The basic quantity $\bar{\Phi}_{AS}(k,l)$ reads

$$\tilde{\phi}_{AS}(k,l) = \frac{1}{4\pi}\, \frac{\Gamma(1-i4\rho k_- G)}{\Gamma(i4\rho k_- G)}\, \left(\frac{4}{(\tilde{k}-\tilde{l})^2}\right)^{1-i4\rho k_- G} \qquad (3.2.22)$$

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which is the t'Hooft's result.

Thus we solved the field theory on an arbitrary shock wave background reducing the problem to a mere integration. In the next section we shall study the expectation values of the energy-momentum tensor.

3.3 Expectation Values of the Energy Density

We shall be interested in the expectation values of the energy density in the scattering states. In other words, having prepared the system in a one-particle state with a sharp value of the momentum $k_-, \bar{k}$ we want to know the mean value of the energy momentum density in the out-region. As we already have mentioned we do not have the problem of the renormalization of this quantity. Indeed, the energy momentum tensor as a local quantity must be given outside the wavefront by the same expression as that valid in the flat spacetime background. Hence we adopt the usual normal ordering renormalization (which in our case is the same with respect to in- as to out-operators). In what follows we shall always consider a situation outside of the wavefront, so our formulas will be those of the flat spacetime. We have

$$T_{\mu\nu}(x) = \; : \partial_\mu \varphi(x) \partial_\nu \varphi(x) - \frac{1}{2} \eta_{\mu\nu} \eta^{\sigma\tau} \partial_\sigma \varphi(x) \partial_\tau \varphi(x) : \qquad (3.3.1)$$

and

$$\varphi(x) = \int dk_- \, d\tilde{k} \left(a_{k \; in(out)} \, u_{k \; in(out)}(x) + a^+_{k \; in(out)} \, u^*_{k \; in(out)}(x) \right) \qquad (3.3.2)$$

Putting (3.3.2) into (3.3.1) and taking the expectation value in the scattering state $\mid k_-, \bar{k} >$ in the in-region one obtains

$$\langle 0 \mid a_{k \; in} \, T_{\mu\nu}(x) \, a^+_{k \; in} \mid 0 \rangle = \left(\delta_\mu^\sigma \delta_\nu^\rho - \frac{1}{2} \eta_{\mu\nu} \eta^{\sigma\rho} \right) \times$$

$$\times \left(u_{k \; in, \sigma}(x) \, u^*_{k \; in, \rho}(x) + c.c. \right) \qquad (3.3.3)$$

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We know the in-modes look in the out-region from the formula (3.2.15). We see that all information is again contained in the basic quantity $\bar{\Phi}(k, l)$ given by (3.2.19).

We provide explicit computations for the case of infinite homogeneous planar shell shock wave and for the sourceless wave, since in both cases we may observe an interesting phenomenon of the focusing of geodesics (see section 1.3). Thus we put

$$f(\tilde{r}) = A_{ij}\, x_i\, x_j \tag{3.3.4}$$

where A is some symmetric matrix. If $TrA = 0$ the wave is sourceless, if $A_{ij} = -a\delta_{ij}$ the profile (3.3.4) is that of the infinite planar shell, with the wavefront energy density given by

$$\rho = \frac{(D-2)a}{16\pi G} \tag{3.3.5}$$

We shall always work in a coordinate frame in which the matrix A is diagonal. Moreover we shall understand $detA \neq 0$ since otherwise the integration in (3.2.19) would give rise trivially the δ-functions in the transverse momenta corresponding to the zero eigenvalues of A. Thus put

$$A_{ij} = a_i\, \delta_{ij} \tag{3.3.6}$$

and compute

$$\hat{\Phi}(k, \ell) \equiv \frac{1}{(2\pi)^{D-2}} \int d\tilde{r}\, exp\left(i\left[(\hat{k}-\hat{\ell})\tilde{r} + \ell\, a_i\, r_i^2\right]\right) =$$

$$= (4\pi\ell)^{-(D-2)/2}\, \sqrt{det\, iA^{-1}}\, exp\left(-\frac{i a_i^{-1}(k-\ell)_i^2}{4\ell_-}\right)$$

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Consequently, for $u > 0$

$$u_{k,in}(\hat{x}, u, v) = N_k \, e^{-ik_- v} (4\pi k_-)^{-(D-2)/2} \sqrt{\det i A^{-1}} \, \times$$

$$\times \int d\hat{\ell} \exp\left[-\frac{i\ell_j^2}{4k_-}(u - u_{F_j}) + \frac{i u_{F_j} k_j^2}{4k_-} - i \sum_j \left(\frac{k_j u_{F_j}}{2k_-} - \hat{x}_j\right)\ell_j\right]$$

$$(3.3.7)$$

where $1/a_i = -u_{F_i}$.

The integration in (3.3.7) is simple. Unless $u = u_{F_j}$ for some j it gives

$$u_{k,in}(\hat{x}, u, v) = N_k \, e^{-ik_- v}\left(\sqrt{\det i A^{-1}} \Big/ \sqrt{\det i R(u)}\right) \exp\left(i u_{F_j} k_j^2 / 4k_-\right) \times$$

$$\times \exp\left[ik_- \sum_j \frac{1}{u - u_{F_j}}\left(\hat{x}_j - u_{F_j} \frac{k_j}{2k_-}\right)^2\right] \quad (3.3.8)$$

here $R(u)_{ij} = (u - u_{F_j})\delta_{ij}$.

If $u = u_{F_j}$ for some j, the integration is again trivial, nonetheless the result is somewhat cumbersome, therefore we do not list it here in the general case. One gets typically a product of $\delta(\bar{x}_j - \bar{k}_j u_{F_j}/2k_-)$ (no summing) and of expressions of the kind (3.3.8). There is a particular case, however, which we do present, because it is directly connected to the very purpose of these computations, namely to the focusing phenomena. It is the case of the full degeneracy i.e. $u_{F_j} = u_F$ for all j. Then for $u = u_F$ one has

$$u_{k,in}(\hat{x}, u_F, v) = N_k \, e^{-ik_- v}\left(\frac{k_-}{\pi}\right)^{-(D-2)/2} \sqrt{\det i A^{-1}} \, \times$$

$$\times \exp\left[i \frac{u_F \tilde{k}^2}{4k_-}\right] \delta\left(\hat{x} - \frac{\hat{k} u_F}{2k_-}\right) \quad (3.3.9)$$

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Loosely speaking we observe a "focusing" of the in-mode on the line $u = u_F$, $\bar{x} = \bar{k}u_F/2k_-$ and v arbitrary. Finally having the explicite expression for the $u_{k,in}$ we can compute the expectation values of the energy-momentum tensor from (3.3.3). Though being straightforward the computation is a bit tedious. Our interest lies mainly in the head-on scattering state $| k_-, \bar{k} = 0 >$ since for the family of the geodesics perpendicular to the wavefront we obtained the full focusing (see (2.3.11ab),(2.3.13ab)). The result reads

$$
\langle 0| \, a_{k_-, \bar{k}=0, in} \; T_{00} \, (x) \, a^+_{k_-, \bar{k}=0, in} \; |0\rangle \; = \tag{3.3.10}
$$

$$
= 2N_k^{\,2} \left| \det R(u) A \right|^{-1} \left[\frac{1}{4} \, Tr^2 R^{-1}(u) + k_-^{\,2} \left(\sum_j \frac{\bar{x}_j^{\,2}}{(u-u_{F_j})^2} + 1 \right)^2 \right]
$$

unless $u = u_{F_j}$ for some j. We note also that for $A \to 0$ (3.3.10) gives $2N_k^2 k_-^2$ e.g. the free field result. The formula (3.3.10) will be our main tool in exploring the focusing phenomenon at the quantum level which we shall do in the next section.

3.4 Quantum focusing

Before a discussion of coincidences and differences between the classical and the quantum pictures let us first use (3.3.10) to provide an explicit formula for some concrete examples. We start with the homogeneous infinite planar shell shock wave i.e.

$$
f(\bar{x}) \; = \; - \, \bar{x}^2/u_F \; , \quad u_F \; = \; (D-2)(8\pi G\rho)^{-1} \tag{3.4.1}
$$

Then

$$
\langle k_-, 0 | T_{00} \, (x) | k_-, 0 \rangle \; = \; 2N_k^{\,2} \frac{u_F^{\,D-2}}{|u-u_F|^{D-2}} \; \times
$$

$$
\times \left[\frac{(D-2)^2}{4(u-u_F)^2} + k_-^{\,2} \left(\frac{\bar{x}^2}{(u-u_F)^2} + 1 \right)^2 \right] \tag{3.4.2}
$$

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for $u \neq u_F$, $u > 0$. Using (3.3.3) and (3.3.7) we may obtain also a formula for $u = u_F$, explicitly

$$
\langle k_{-,}0 \mid T_{00}(x) \mid k_{-,}0 \rangle = \begin{cases} 0 & , \ \tilde{r} \neq 0 \\ \\ \infty & , \ \tilde{r} = 0 \end{cases} \tag{3.4.3}
$$

In the derivation of this formula one must take care of the usual damping regulator needed for computing the Gaussian integrals. The infinity in (3.4.3) is an ill-behaved expression due to multiplication of distributions.

As another example we consider a sourceless shock wave in $D = 4$ with the polarization matrix A given by

$$
A = \begin{pmatrix} a & 0 \\ 0 & -a \end{pmatrix} , \ a > 0 \tag{3.4.4}
$$

Putting now $u_F = 1/a$ one gets from (3.3.10)

$$
\langle k_{-,}0 \mid T_{00}(x) \mid k_{-,}0 \rangle = \frac{|u_F|^2}{|(u-u_F)(u+u_F)|} \times
$$

$$
\times \left[\frac{u_F^2}{(u-u_F)^2(u+u_F)^2} + k_-^2 \left(\frac{r_1^2}{(u+u_F)^2} + \frac{r_2^2}{(u-u_F)^2} + 1 \right)^2 \right] \tag{3.4.5}
$$

for $u > 0$, $u = u_F$.

We see that the divergence in $u = u_F$ is milder than in the previous

case of the infinite planar shell, or, in other words, the focusing is only partial. Such result, of course, might be anticipated from the behaviour of geodesics.It is interesting to note as well, that in (3.4.5) we observe a quantum virtual focusing. Indeed, the pole $u = -u_F$ is not physical, since the formula is valid only for $u > 0$. Speaking generally, for all sourceless waves must exist these virtual focal points, as may be seen directly from (3.3.10), since $TrA = 0$ and some u_{F_j} are negative. This fact may be understood also physically. TrA measures in a sense global attraction or repulsion of the matter scattered by the wave. For example since the planar energy density ρ in (3.4.1) is positive, the infinite planar shell wave attracts the matter, as it should. For sourceless waves $TrA = 0$ thus the focusing in some direction must be balanced by a defocusing in some other, which is precisely the state of things which we obtained.

After this short digression about the sourceless waves we shall discuss in detail the infinite planar shell wave, in particular we will study an interplay between the classical and the quantum pictures of the focusing phenomenon. Thus we remind the result from the section 2.3. If one chooses u as the affine parameter for the family G of geodesics perpendicular to the wavefront there was obtained

$$ r(u) = b - b\left(u/u_F\right)\theta(u) \tag{3.4.6a} $$

$$ V(u) = -\left(b^2/u_F\right)\theta(u) + \left(b^2/u_F^2\right)u\,\theta(u) + a \tag{3.4.6b} $$

$$ \varphi(u) = \varphi_0 \tag{3.4.6c} $$

Here b ($b \equiv x_0$ in the previous notation) is the "impact" parameter, r the transverse radial coordinate, ϕ the set of angular coordinates, a is the "initial position" on the v-axis and $\theta(u)$ the usual step function. If we now fix a and vary b from (3.4.6) follows that such subfamily of G gets focused at

$$ u = u_F \quad , \quad V = a \quad , \quad \tilde{r} = 0 \tag{3.4.7} $$

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Varying also a one spreads the focal point (3.4.7) over the v-axis. Moreover, no geodesics from G touch the points $u = u_F$, $\bar{x} = 0$, but one sees an "accumulation" of geodesics with large impact parameter b near the plane $u = u_F$. We observe that the described geodesical picture corresponds very well to the quantum results (3.4.2-3). We must clarify, however, in what sense the family of geodesics G is a classical limit of the state $\mid k_-, 0 >$.

Generally speaking the classical limit of the quantum field theory is the classical field theory and the geodesical picture is geometrical optics' approximation to the latter. So, in the strict sense, to study the geometrical optics limit of the field theory we should consider an expectation value of the quantum field in some many-particle state with a property that this mean value would look locally like a plane wave with an eikonal giving rise to the family G. This is obviously not our case since we have considered only one-particle state in which mean value of the field operator is simply zero. There are other reasons , however, why the quantum results (3.4.2-3) are analogous to the classical ones (2.3.11). Probably it is not very suprising at the first sight since the state $\mid k_-, 0 >$ is a translationally invariant state and as such it models at the quantum level a family of colinear geodesics. Yet there is a deeper connection between both pictures and relies on the important role played by the Schrödinger equation (3.2.7) in our analysis. From (3.2.2), (3.3.8) and (3.3.9) one sees that the u-dependent potential term in (3.2.7) has an amusing property, namely it changes in the course of evolution the state with a sharp value of the momentum into the state with a sharp value of the coordinate $\bar{x}$ at the "time" $u = u_F$. One has for $u < 0$

$$\Psi_{k,in}(u,\tilde{x}) = exp\left(-i\frac{\tilde{k}^2}{2m}u + i\hat{k}\hat{x}\right) \equiv exp\, i\, S(\tilde{x},u,\hat{k}) \quad (3.4.8)$$

and for $u > 0$

$$\Psi_{k,in}(u,\tilde{x}) = A\, exp\, i\frac{m}{2}\frac{(\hat{x}-(\tilde{k}/m)u_F)^2}{u-u_F} \equiv A\, exp\, i\, S'(\tilde{x},u,\tilde{q}) \quad (3.4.9)$$

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here $m = 2k_-$, $\tilde{q} = (\bar{k}/m)u_F$ and A does not depend on $\bar{x}$.

The point is, that the functions $S(\bar{x}, u, \bar{k})$ and $S'(\bar{x}, u, \tilde{q})$ are both the full integrals of the Hamilton-Jacobi equation (HJE) for a free particle with a mass m (not to be confused with the mass of the quantum field,which we set zero at the very beginning). Having some full integral of the HJE $\Sigma(\bar{x}, u, \tilde{\alpha})$ it is easy to find trajectories by expressing $\bar{x}$ as the function of u, $\tilde{\alpha}$ and $\tilde{\beta}$ from the equation

$$\frac{\partial \Sigma}{\partial \tilde{\alpha}} (\tilde{x}, u, \tilde{\alpha}) = \tilde{\beta} \tag{3.4.10}$$

where $\tilde{\beta}$ is canonically conjugated variable of $\tilde{\alpha}$ Hence giving a full integral of HJE, fixing $\tilde{\alpha}$ and varying the canonically conjugated variable $\tilde{\beta}$ the family of the classical trajectories is defined. Thus one may say loosely, at least in our special case, that a quantum eigenstate of the observable $\bar{x}$ "contains" all classical states with $\tilde{\alpha}$ fixed and $\tilde{\beta}$ varying. If, in particular, one considers the full integrals S, S' (given by (2.4.8),(2.4.9) respectively) for $\bar{k} = 0$ (and, consequently, $\tilde{q} = 0$) one finds that the corresponding family of classical trajectories is precisely the family (3.4.6) with varying impact parameter b and ϕ_0. Thus the behaviour of the phase of the in-mode Ψ_{kin} is dictated by the behaviour of the beam of all classical trajectories with the incident momentum $\bar{k}$. In the same spirit the "geodesical content" of the state $\mid k_-, 0 >$ would be the full family G with varying position in both v and $\bar{x}$-axis, namely varying a, b and ϕ_0 in (3.4.6). In this sense, the scattering state $\mid k_-, 0 >$ corresponds to the family G, therefore our results may said to be the quantum version of the geodesical focusing obtained before.

This just performed analysis finds another immediate applicition, namely it enables us to say something about the status of the singularity for the homogeneous *finite* planar shell shock wave ($\rho = 0$ for $\mid \bar{x} \mid > R$). For $u > 0$ trajectories with large transverse momenta are not present in the family G since the trajectories with large impact parameters ($b > R$) are only slightly deflected (slightly because they still "feel" the null matter in the domain $\mid \bar{x} \mid < R$). Therefore the corresponding quantum state cannot be a true position state in which, as we have seen, all momenta must be present

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(and for $u > 0$ the momenta here play the role of the parameter $\tilde{\beta}$ in eq. (3.4.10)). Thus from the Heisenberg uncertainty principle one expects a spreading of the focal point over the scale h/Λ, where Λ is the momentum cut-off.

We conclude this section with several remarks connecting the subject with the string theory. Though not transparent at the first sight the interest in the phenomena occuring in the shock wave backgrounds was motivated by the recent active research in this field, in particular in connection with the gravity limit from the strings. In principle the string theory should account for the quantum gravity phenomena, in its present form, however, one may calculate only scattering processes, so if one wants to make a link between the "old" and the "new" physics there is an obvious motivation to try to compute the scattering process at the level of the field theory. This was, in fact, t'Hooft's approach, which uses explicitly the shock wave background. Remarkably enough, a "track" of the shock wave background was found directly in the relevant string computations, namely Amati, Ciafaloni and Veneziano [7] have found that for large impact parameters the wave function suffers the Aichelburg-Sexl phase-shift, as was assumed by t'Hooft. In a more detailed study of the shock wave backgrounds by Veneziano [12] and by Ferrari, Pendenza and Veneziano [2] was discovered the geodesical focusing in the shock wave backgrounds which we described in the second chapter (see also ref.[13]). This fact may lead back to an interesting problem in the string theory, namely : "What can the string theory say about status and character of the gravitational singularities? Hopefully they should be smeared at the distances comparable with the string scale. It is difficult, however, to treat the problem from the technical (and, perhaps, conceptual as well) point of view. Therefore it is very important to have an example of a classical curvature singularity which can be relatively easily controlled technically and which occurs in a process admitting a treatment within the present days' framework of the string theory. The scattering singularities occuring in collisions of shock waves ([6,14,15,16,17,18,19,20,21,22]) may be good candidates because scattering processes, at least in some kinematic domain, can be addressed in the string picture ([7,23,24]). As we have seen some information about the status of singularity can be obtained even without entering the full collision problem, simply by exploring the behaviour of the physical system in the shock wave backgrounds. Therefore a motivation is arising to find the string motion in these metrics [8]. As we will see

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in the next chapter, the result turns out to make an interesting connection between the works of Amati, Ciafaloni, Veneziano and t'Hooft. It may be considered as interesting by itself since the solution of the (supersymmetric) nonlinear σ-model is provided.

Chapter 4

The Nonlinear σ-model

4.1 The Bosonic String

In this section we shall treat the string (both bosonic and superstring) in an arbitrary shock wave background metric in D-dimensional spacetime. We shall show that this theory may be exactly solved and provides - at the quantum level - a nontrivial S-matrix explicitly given in an operatorial form in terms of the shock wave profile. First we shall find a classical motion of the string in the shock wave background. If g_{mn} is a background metric the action S is that of the nonlinear σ-model i.e.

$$S = -\frac{1}{2\pi} \int d\sigma d\tau \sqrt{h}\, h^{\alpha\beta} \partial_\alpha X^m \partial_\beta X^n g_{mn}(X) \tag{4.1.1}$$

To be closer to the usual free string formalism we change the normalization of the light-cone coordinates and set the shock wave metric in the form

$$ds^2 = -2\,du\,dv + f(\tilde{x})\delta(u)\,du^2 + d\tilde{x}^2 \tag{4.1.2}$$

where

$$u \equiv (t - z)/\sqrt{2}$$

(4.1.3)

$$v \equiv (t + z)/\sqrt{2}$$

with a corresponding redefinition of $f(\tilde{x})$. The action (4.1.1) then becomes

$$S = -\frac{1}{2\eta} \int d\sigma d\tau \sqrt{h}\, h^{\alpha\beta} \left(\partial_\alpha X^m \partial_\beta X^n \eta_{mn} + f(\tilde{x}) \delta(v) \partial_\alpha v \partial_\beta v \right)$$

(4.1.4)

We fix the conformal gauge $h^{\alpha\beta} = \eta^{\alpha\beta}$. The equations of motion read

$$\Box v = 0$$

(4.1.5a)

$$\Box X_m = \frac{1}{2} \partial_m f(\tilde{x}) \delta(v) \partial_\alpha v \partial^\alpha v$$

(4.1.5b)

$$\Box V = \frac{1}{2} f(\tilde{x}) \delta'(v) \partial_\alpha v \partial^\alpha v + 2 \partial_m f(\tilde{x}) \partial_\alpha X^m \partial^\alpha v \delta(v)$$

(4.1.5c)

and the constraints (vanishing of the world sheet energy momentum tensor)

$$-2\dot{u}\dot{v} - 2u'v' + \dot{\tilde{x}}^2 + \tilde{x}'^2 + f(\tilde{x})\delta(v)(\dot{u}^2 + v'^2) = 0 \quad (4.1.6a)$$

$$-\dot{u}v' - \dot{v}u' + \dot{\tilde{x}}\tilde{x}' + \dot{v}v' f(\tilde{x})\delta(v) = 0 \qquad (4.1.6b)$$

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where, as usual, dot and prime represent derivatives with respect to τ and σ respectively. One may simplify considerably these equations by noting that U fulfils the free equation, hence we are allowed to go to the light-cone gauge by setting

$$U(\sigma,\tau) = p^u \tau \tag{4.1.7}$$

The remaining equations of motion then become

$$\Box X_i (\sigma,\tau) = -\frac{1}{2} p^u \partial_i f(\tilde{X}) \delta(\tau) \tag{4.1.8a}$$

$$\Box V (\sigma,\tau) = -\frac{1}{2} f(\tilde{X}) \delta'(\tau) - \partial_i f(\tilde{X}) \dot{X}^i \delta(\tau) \tag{4.1.8b}$$

and the constraints

$$2p^u \dot{V} = \dot{\tilde{X}}^2 + \tilde{X}'^2 + p^u f(\tilde{X}) \delta(\tau) \tag{4.1.9a}$$

$$p^u V' = \dot{\tilde{X}} \tilde{X}' \tag{4.1.9b}$$

where, from now on, $f(\tilde{x}) \equiv \text{sign}(p^u) f(\tilde{x})$.

Since (4.1.8a) and (4.1.9) imply the V equation of motion (4.1.8b), we shall solve (4.1.8a) for the transverse coordinates X^i and then obtain V from the constraints (4.1.9). For $\tau \neq 0$ the problem is the same as in the usual flat space-time, and we shall denote modes and solutions in terms of the free ones for $\tau < 0$ and $\tau > 0$ with subscripts $<$ and $>$ respectively. Thus for open string, for instance,

$$X^i_{\lessgtr} (\sigma,\tau) = x^i_{\lessgtr} + p^i_{\lessgtr} \tau + i\sum_{n\neq 0} \frac{1}{n} \alpha^i_{n\lessgtr} e^{-in\tau} \cos n\sigma \tag{4.1.10a}$$

$$V_{\lessgtr} (\sigma,\tau) = v_{\lessgtr} + p^v_{\lessgtr} \tau + i\sum_{n\neq 0} \frac{1}{n} \alpha^v_{n\lessgtr} e^{-in\tau} \cos n\sigma \tag{4.1.10b}$$

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where

$$\alpha_{n \lessgtr}^{\nu} = \frac{1}{2p^u} \sum_m \alpha_{n-m \lessgtr}^i \, \alpha_{m \lessgtr}^i \quad , \quad \alpha_{0 \lessgtr}^{i,\nu} \equiv p_{\lessgtr}^{i,\nu} \tag{4.1.11}$$

The problem is to find the out-quantities $p_{>}^u, x_{>}^i, \alpha_{n>}^i, v_{>}$ as functions of the in- ones $p_{<}^u, x_{<}^i, \alpha_{n<}^i$ and $v_{<}$. We start by inserting in (4.1.8a) the following ansatz

$$X^i(\sigma,\tau) = x^i(\tau) + i \sum_{n \neq 0} \frac{1}{n} \alpha_n^i(\tau) \cos n\sigma =$$

$$= x^i(\tau) + i \sum_{n > 0} \frac{1}{n} (\alpha_n^i - \alpha_{-n}^i)(\tau) \cos n\sigma \tag{4.1.12}$$

The equation for the zero mode $x^i(\tau)$ becomes

$$-\ddot{x}^i(\tau) = -\frac{1}{2}p^u \left(\frac{1}{\eta} \int_0^{\eta} \partial_i f\,(\tilde{X}(\sigma,0))\, d\sigma \right) \delta(\tau) \tag{4.1.13a}$$

and for other modes $n > 0$

$$-\frac{i}{n}(\ddot{\alpha}_n^i - \ddot{\alpha}_{-n}^i) - in(\alpha_n^i - \alpha_{-n}^i) =$$

$$= -\frac{1}{2}p^u \left(\frac{2}{\eta} \int_0^{\eta} \partial_i f\,(\tilde{X}(\sigma,0))\, \cos n\sigma\, d\sigma \right) \tag{4.1.13b}$$

From these equations immediately follows that

$$x_{>}^i = x_{<}^i \tag{4.1.14}$$

and

$$(\alpha_{n>}^i - \alpha_{-n>}^i) = (\alpha_{n<}^i - \alpha_{-n<}^i) \, , \, n > 0 \tag{4.1.15}$$

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Then integrating the equations (4.1.13ab) one has

$$\dot{\rho}_>^i = \dot{\rho}_<^i + \frac{\rho^u}{2\eta} \int \partial_i^{\,\prime} f\left(\tilde{X}(\sigma,0)\right) d\sigma \tag{4.1.16}$$

and

$$(\dot{\alpha}_{n>}^i + \dot{\alpha}_{-n>}^i) - (\dot{\alpha}_{n<}^i + \dot{\alpha}_{-n<}^i) = \frac{\rho^u}{\eta} \int_0^{\eta} \partial_i^{\,\prime} f\left(\tilde{X}(\sigma,0)\right) \cos n\sigma \, d\sigma \tag{4.1.17}$$

Putting together (4.1.15,16,17) we get for all n

$$\alpha_{ni>} = \alpha_{ni<} + \frac{\rho^u}{2\eta} \int_0^{\eta} \partial_i f\left(\tilde{X}(\sigma,0)\right) \cos n\sigma \, d\sigma \tag{4.1.18}$$

Then we obtain α_n^v from the equation (4.1.11). There remains to determine v . One uses the constraint (4.1.9a) that may be integrated directly yielding

$$V_> (\sigma,0) = V_< (\sigma,0) + \frac{1}{2} f\left(\tilde{X}(\sigma,0)\right) \tag{4.1.19}$$

For the zero mode of (4.1.19) we thus obtain

$$V_> = V_< + \frac{1}{2\eta} \int_0^{\eta} d\sigma \, f\left(\tilde{X}(\sigma,0)\right) \tag{4.1.20}$$

It is not difficult to check that the nonzero modes of (4.1.19) are satisfied by α_n^v given by (4.1.11) and (4.1.18).

For the closed string case we proceed analogously having

$$X_{\lessgtr}^i (\sigma,\tau) = x_{\lessgtr}^i + \rho_{\lessgtr}^i \tau + \frac{i}{2} \sum_{n \neq 0} \frac{1}{n} \left(\tilde{\alpha}_{n\lessgtr}^i e^{-2in\tau} - \alpha_{n\lessgtr}^{i} e^{2in\tau}\right)\!/e^{-2in\sigma} \tag{4.1.21a}$$

and

$$V_{\lessgtr}^i(\sigma,\tau) = V_{\lessgtr}^i + P_{\lessgtr}^\nu \tau +$$

$$+ \frac{i}{2} \sum_{n\neq 0} \frac{1}{n} \left(\tilde{\alpha}_{n\lessgtr}^\nu \, e^{-2in\tau} - \alpha_{n\lessgtr}^{\nu +} \, e^{2in\tau} \right) e^{-2in\sigma} \tag{4.1.21b}$$

The ansatz reads

$$X^i(\sigma,\tau) = x^i(\tau) + \frac{i}{2} \sum_{n\neq 0} \frac{1}{n} \left(\tilde{\alpha}_n^i(\tau) - \alpha_{-n}^i(\tau) \right) e^{-2in\sigma} \tag{4.1.22}$$

Then the equation (4.1.8a) becomes

$$- \ddot{x}^i(\tau) = - \frac{1}{2} P^u \left(\frac{1}{\pi} \int \partial_i f(\tilde{X}(\sigma,0)) \, d\sigma \right) \delta(\tau) \tag{4.1.23a}$$

and for $n \neq 0$

$$-\frac{i}{2n} \left(\ddot{\tilde{\alpha}}_{in} - \ddot{\alpha}_{in} \right) - 2ni \left(\dot{\tilde{\alpha}}_{in} - \dot{\alpha}_{i-n} \right) = -\frac{1}{2} P^u \frac{1}{\pi} \int \partial_i f \, e^{2in\sigma} d\sigma$$

$$\tag{4.1.23b}$$

As before we find

$$x_>^i = x_<^i$$

$$\alpha_{ni>} = \alpha_{ni<} + \frac{P^u}{4\pi} \int_0^\pi \partial_i f(\tilde{X}(\sigma,0)) e^{-2in\sigma} d\sigma$$

$$\tilde{\alpha}_{ni>} = \tilde{\alpha}_{ni<} + \frac{P^u}{4\pi} \int_0^\pi \partial_i f(\tilde{X}(\sigma,0)) e^{2in\sigma} d\sigma$$

$$V_> = V_< + \frac{1}{2\pi} \int_0^\pi f(\tilde{X}(\sigma,0)) \, d\sigma \tag{4.1.24}$$

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At this point we finish the classical treatment. After solving the super-string case in the next section we shall provide a quantum version of the model.

4.2   The Superstring

In the covariant gauge the action of the superstring moving in a generic shock wave backgroung is that of the nonlinear supersymmetric σ-model, e.g.

$$ S = \frac{i}{4\pi} \int d\sigma \, d\tau \, d^2\theta \; \bar{D} Y^m D Y^n g_{mn}(Y) \tag{4.2.1} $$

where

$$ Y^m(\sigma,\tau,\theta) \equiv X^m(\sigma,\tau) + \bar{\theta} \psi^m(\sigma,\tau) + \frac{1}{2} \bar{\theta}\theta B^m(\sigma,\tau) \tag{4.2.2} $$

θ and ψ^m being the 2-dimensional Majorana spinors, B^m the auxiliary field and $D, \bar{D}$ the usual covariant derivatives i.e.

$$ D = \frac{\partial}{\partial\bar{\theta}} - i \rho^\alpha \theta \partial_\alpha \quad , \quad \bar{D} = -\frac{\partial}{\partial\theta} + i \bar{\theta} \rho^\alpha \partial_\alpha \tag{4.2.3} $$

with the 2-dimensional Dirac matrices

$$ \rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \quad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \tag{4.2.4} $$

In our specific case we have

$$S = \frac{i}{4\pi} \int d\sigma \, d\tau \, d^2\theta \left(\bar{D} Y^m D Y^n \eta_{mn} + \bar{D} Y^u D Y^u_{\cdot} f(Y^i) \delta(Y^u) \right) \qquad (4.2.5)$$

and the equations of motion read

$$\left(D_A \bar{D}^A - \bar{D}^A D_A \right) Y^u = 0$$

$$\left(D_A \bar{D}^A - \bar{D}^A D_A \right) Y^i + \bar{D} Y^u D Y^u f'(Y^i) \delta(Y^u) = 0 \qquad (4.2.6a)$$

$$\left(D_A \bar{D}^A - \bar{D}^A D_A \right) Y^v - \left(D_A [f(Y^i) \delta(Y^u)] \right) \bar{D}^A Y^u \qquad (4.2.6b)$$

$$+ \left(\bar{D}^A [f(Y^i) \delta(Y^u)] \right) D_A Y^u - \bar{D} Y^u D Y^u f(Y^i) \delta'(Y^u) = 0 \qquad (4.2.6c)$$

Since we obtained for the components of the Y^u supermultiplet the free equations of motion we can go to the light cone gauge for which

$$U(\sigma,\tau) = p^u \tau, \quad \psi^u(\sigma,\tau) = 0, \quad B^u(\sigma,\tau) = 0 \qquad (4.2.7)$$

The equations (4.2.6bc) now written in the components become

$$\Box X^i = -\frac{1}{2} \partial_i f(\tilde{X}) p^u \delta(\tau) \qquad (4.2.8a)$$

$$\Box V = -\frac{1}{2} f(\tilde{X}) \delta'(\tau) - \partial_i f(\tilde{X}) \dot{X}^i \delta(\tau) \qquad (4.2.8b)$$

$$i \rho^\alpha \partial_\alpha \psi^i = 0 \qquad (4.2.8c)$$

$$i \rho^\alpha \partial_\alpha \psi^v = \frac{1}{2} i \partial_i f(\tilde{X}) \rho^0 \psi^i \delta(\tau) \qquad (4.2.8d)$$

$$B^i = B^v = 0 \qquad (4.2.8e)$$

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We observe that, rather amazingly, the bosonic equations of motion remain untouched by the presence of the fermionic modes. There remains to compute the (super)constraints. In the light-cone gauge they are

$$2p^u \dot{v} = \dot{X}^2 + \dot{X}'^2 - \frac{i}{2} \bar{\Psi}'(S_0 \partial_0 + S_1 \partial_1)\Psi' + f(\dot{X})p^u \delta(\tau)$$ (4.2.9a)

$$2p^u v' = 2\dot{\hat{X}}\hat{X}' - \frac{i}{2} \bar{\Psi}'(\partial_0 S_1 + S_0 \partial_1)\Psi'$$ (4.2.9b)

$$p^u S^0 p^\alpha \psi^v = S^1 p^\alpha \psi' \partial_s X'$$ (4.2.9c)

Again, as before, having solved the transverse equations of motion (4.2.8ac) and the (super)constraints (4.2.9), the Y^v equations of motion i.e. (4.2.8bd) are automatically satisfied. We observe that the transverse fermionic modes are not influenced by the presence of the shock wave and the equation (4.1.20) for the zero mode v by the presence of the fermionic degrees of freedom in the constraint equation (4.2.9ab), from which was derived. Finally the superconstraint (4.2.9c) gives us ψ^v in terms of the in-modes. With these remarks we finish the classical discussion and shall turn to the quantization at the next section.

4.3   Quantization and Interpretation

To quantize the classical theories discussed in the previous two sections is now a simple matter. We may do it in terms of the in- or the out-modes, both satisfying the canonical (anti)commutation relations e.g.

$$[x^i, p^j] = i\delta^{ij} \quad , \quad [v, p^u] = -i$$

$$[\alpha_m^{(\sim)i}, \alpha_n^{(\sim)j}] = \delta^{ij} m \, \delta_{m+n}$$

$$\{\psi_A^i(\sigma,0), \psi_B^j(\sigma,0)\} = \eta \delta^{ij} \delta_{AB} \, \delta(\sigma - \sigma')$$ (4.3.1)

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These two sets of modes should be related by a unitary transformation which is by definition the S-matrix. For all cases previously considered i.e.bosonic, super, both open and closed, the S-matrix is given by

$$S = exp\left[\frac{i}{2\eta} p^u \int_0^\eta d\sigma f(\tilde{X}(\sigma,0))\right] \qquad (4.3.2)$$

We observe firstly that the S-matrix does not contain the fermionic degrees of freedom. This is so because the transverse fermionic modes fulfil the free equation of motion (4.2.8c). We shall formally prove that the S-matrix (4.3.2) correctly transforms the in- into the out-modes without entering possible ultraviolet problems in its definition. For the open string one has directly

$$S^+ p^u S = p^u, \quad S^+ x^i S = x^i, \quad S^+ v S = v + \frac{1}{2\eta} \int_0^\eta d\sigma f(\tilde{X}(\sigma,0)) \qquad (4.3.3)$$

To compute the transformation of the α_n^i modes is slightly more involved. We start with a trivial formula

$$S^+ \dot{\alpha}_n^i S = \dot{\alpha}_n^i - [\dot{\alpha}_n^i, S^+] S \qquad (4.3.4)$$

There remains to evaluate $[\alpha_n^i, S^+]$. First one realizes that

$$[\alpha_n^i, f(\tilde{X}(\sigma,0))] = -\partial^i f(\tilde{X}(\sigma,0)) i \cos n\sigma \qquad (4.3.5)$$

Now from $[\partial^i f, f] = 0$ it follows $[[\alpha_n^i, f], f] = 0$. This information enables us to write

$$\left[\alpha_n^i, exp -\frac{i}{2\eta} p^u \int_0^\eta d\sigma f(\tilde{X}(\sigma,0))\right] = \qquad (4.3.6)$$

$$= \left[\alpha_n^i, -\frac{i}{2\eta} p^u \int_0^\eta d\sigma f(\tilde{X}(\sigma,0))\right] exp -\frac{ip^u}{2\eta} \int_0^\eta d\sigma f(\tilde{X}(\sigma,0))$$

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Thus

$$S^{\dagger} \alpha_n{}^{i} S = \alpha_n{}^{i} - \left[\alpha_n{}^{i} , -\frac{i}{2\pi} p^\mu \int_0^\eta d\sigma f(\tilde{X}(\sigma,0)) \right] S^{\dagger} S =$$

$$= \alpha_n{}^{i} + \frac{1}{2\pi} p^\mu \int_0^\eta \partial_i f(\tilde{X}(\sigma,0)) \cos n\sigma \, d\sigma \qquad (4.3.7)$$

Putting together (4.3.3) and (4.3.7) we observe that the equations (4.1.14a), (4.1.18) and (4.1.20) are reproduced as they should. For the closed string one has in the same spirit

$$\left[\tilde{\alpha}_n{}^{i} , f(\tilde{X}(\sigma,0)) \right] = -\frac{i}{2} e^{2in\sigma} \partial_i f(\tilde{X}(\sigma,0)) \qquad (4.3.8a)$$

$$\left[\alpha_n{}^{i} , f(\tilde{X}(\sigma,0)) \right] = -\frac{i}{2} e^{-2in\sigma} \partial_i f(\tilde{X}(\sigma,0)) \qquad (4.3.8b)$$

and

$$S^{\dagger} \alpha_{ni} S = \alpha_{ni} + \frac{p^\mu}{4\pi} \int_0^\eta \partial_i f(\tilde{X}(\sigma,0)) e^{-2in\sigma} d\sigma \qquad (4.3.9a)$$

$$S^{\dagger} \tilde{\alpha}_{ni} S = \tilde{\alpha}_{ni} + \frac{p^\mu}{4\pi} \int_0^\eta \partial_i f(\tilde{X}(\sigma,0)) e^{2in\sigma} d\sigma \qquad (4.3.9b)$$

This result reproduces the formulas (3.1.24). We notice also that at the quantum level an ambiguity in the operator ordering arises. For instance, it is not difficult to see that the normal ordering in the exponent of (4.3.2) causes the normal ordering in (4.3.3),(4.3.7) and (4.3.8). This fact will find an application soon in a discussion of a physical interpretation of our

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result. Before this, however, we provide several particular examples of the general formula (4.3.2). To make a connection with the ref.[3] we give the exact S-matrix for the homogeneous infinite planar shell shock wave (e.g. $f(x) = -ax^2$, $a > 0$)

$$S = exp\left[-\frac{i}{2}p^u a \left(\tilde{x}^2 + \sum_{n \neq 0} \frac{1}{4n^2} (\tilde{\alpha}_n^{i'} - \alpha_n^{i'+})(\tilde{\alpha}_n^{+i'} - \alpha_n^{i'})\right)\right] \quad (4.3.10)$$

We have considered for concreteness the closed string case. It is interesting to discuss (4.3.2) in a regime, in which we consider the string zero mode x,i.e. the string position, to be large with respect to the string scale. In this regime one may decompose $f(X(\sigma,0))$ in (4.3.2) in a Taylor series in $\hat{X}$, where $\hat{X}$ denotes the non-zero modes' contribution. At the lowest nontrivial approximation one has (again for the closed strings)

$$S^{(2)} = exp\left\{\frac{i}{2}p^u\left[f(\tilde{x}) + \right.\right.$$

$$\left.\left. + \frac{1}{2} \partial_i \partial_j f(\tilde{x}) \sum_{n \neq 0} \frac{1}{4n^2} (\tilde{\alpha}_n^{i'} - \alpha_n^{i'+})(\tilde{\alpha}_n^{+j} - \alpha_n^{j'})\right]\right\} \quad (4.3.11)$$

where the superscript (2) indicates that the approximation is quadratic in the non-zero modes. For them one obtains

$$S^{(2)+} \tilde{\alpha}_n^{i'} S^{(2)} = \tilde{\alpha}_n^{i'} - \frac{ip^u}{8n} \partial^i \partial^j f(\tilde{x})(\tilde{\alpha}_{nj}^{+} - \alpha_{nj}^{\,}), \quad (4.3.12)$$

$$\tilde{\tilde{\alpha}}_n^{i'} \equiv \tilde{\alpha}_n^{i'}$$

i.e. a Bogoliubov-like transformation. The S-matrix in the quadratic approximation was found for other backgrounds as well [25] but without a knowledge of an exact expression. Indeed, to provide the exact solution is an unusual feature of the shock wave backgrounds.

In ref.[7] Amati, Ciafaloni and Veneziano (in what follows ACV) have

computed the S-matrix (S_{ACV}) for the direct channel scattering of two superstring low level excitations.It is given (in the impact parameter space) by

$$S = exp\left(2i\int_0^\pi :a\left(s, b + \hat{X}^u(\sigma_u, 0) - \hat{X}^d(\sigma_d, 0)\right): \frac{d\sigma_u d\sigma_d}{\pi^2}\right) \quad (4.3.13)$$

where a(s,b) is the string tree amplitude

$$a(s,b) = \frac{s}{\pi^{D/2-2}}\frac{G}{2}b^{4-D}\int_0^\infty dt\, e^{-t} t^{\frac{D}{2}-3} e^{-\frac{1}{t}b^2/(\gamma - i\pi/2)}, \quad \gamma \equiv \log s \quad (4.3.14)$$

Here $\sqrt{s}$ is the energy of the process, b the impact parameter, $\hat{X}^u(\sigma_u, 0)$ and $\hat{X}^d(\sigma_d, 0)$ the nonzero modes transverse position operators of the strings (denoted up and down) participating in the process. We remark that the S-matrix of eq.(4.3.13) coincides with the one obtained by a noninteracting string moving in a shock wave metric generated by the other string with a profile f(y) related to the function a(s,y) of (3.3.14). Explicitly

$$f(y) = q^v \int_0^\pi \frac{\beta}{s} : a\left(s, y - \hat{X}^d(\sigma_d, 0)\right): \frac{d\sigma_d}{\pi} \quad (4.3.15)$$

where q^v is the momentum of the other string (or the other incoming particle) impinging in the v-direction, that creates the shock wave. In other words, the string "up" moves in a shock wave profile f given by (4.3.15), where $b = x^u - x^d$ is the difference of the zero modes and $s = 2p^u q^v$. That this should have been the case was already anticipated in ref.[26]. For very large y the profile (4.3.15) becomes

$$f(y) = q^v \frac{16\pi G}{(D-4)\Omega_{D-2}}\frac{1}{|y|^{D-4}} \quad (4.3.16)$$

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where

$$\Omega_d \equiv 2\eta^{d/2}/\Gamma(d/2)$$

which is the Aichelburg-Sexl metric. Moreover the S-matrix (4.3.2) in the particle limit (no string modes excited) gives precisely the t'Hooft S-matrix. For smaller y, however, the metric differs from the A.S. one. It developes an imaginary part thus indicating the presence of inelastic channels. The real part for small y avoids the singular A.S. metric behaviour and, consequently, the poles found by t'Hooft [1] in his scattering amplitude. Finally let us remark that the S_{ACV}-matrix does not contain the fermionic modes even though it was computed in the framework of the superstring theory. This fact fits well the irrelevance of the shock wave background for the fermionic degrees of freedom in the σ-model treatment. Thus we may say that coincidence of both S-matrices i.e. S_{ACV} and S_σ shows the generation of a nontrivial metric in superstring collisions at ultrahigh energies in the flat spacetime background.

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Chapter 5

Conclusions and Outlook

In this thesis we have discussed in detail the behaviour of various dynamical systems in the shock wave backgrounds of an arbitrary profile. The results found to have applications in discussing the ultrahighenergy scattering of particles or strings and in the study of interesting focusing phenomena which may be related to the generation of gravitational singularities. There is a pleasing feature of these results namely they are all exact. This is quite a rare situation in field theory on a curved background and even rarer for the quantum string dynamics in curved target space where a nonlinear σ-model has to be solved.

From the point of view of physical interpretation of our results we obtained the remarkable coincidence of the S-matrix of the shock wave supersymmetric σ-model with that obtained by Amati, Ciafaloni and Veneziano from the full-fledged string theory . This coincidence holds for the particular profile of the shock wave which for large impact parameters has the Aichelburg-Sexl form. Since the ACV result was obtained by exploring the infinite loop superstrings amplitudes in the flat spacetime background the coincidence of both S-matrices indicates the generation of curved geometry as an infinite genus effect of the flat spacetime string theory. Moreover the particle limit from the σ-model S-matrix (e.g. no string modes excited) gives precisely the t'Hooft S-matrix for the quantum relativistic particle scattering on the Aichelburg-Sexl shock wave. This shows that, as expected, the string theory results coincide with the field theoretical ones in the appropriate limit. The other application of our results concerns the focusing phenomena in the planar shell shock waves discovered by Penrose,

Veneziano and Ferrari, Pendenza and Veneziano who studied the behaviour of null geodesics in these backgrounds.They had a motivation that such geodesical focusing would indicate the generation of curvature singularity if the dynamical effect of the "test" matter, which would follow the geodesics, were taken into account. That such generation of curvature singularity does occur in the the head-on collision of two infinite planar shells was shown by Dray and t'Hooft. The field theory on the shock wave background provides the quantum information about the focusing phenomenon. We have shown that at the quantum level the indicated geodesical singularity does survive for the infinite planar shell but is smeared for the finite one suggesting thus that the curvature singularity may be avoided in this case.

There remain open important questions, however. First of all the conformal anomaly problem in the non-linear σ-model. We have solved the σ-model in the light-cone gauge. In the case of the flat background the light-cone gauge theory exhibit the Lorentz anomaly unless $D = 26$ (we speak for concreteness about the bosonic string). In the curved background, however, no Lorentz symmetry is present even at the classical level so we cannot judge the consistency of the theory from this point of view. Hence the conformal symmetry plays a fundamental role in exploring the consistency of the quantum string motion so we canot avoid the problem. The difficulty is that the shock wave σ-model is "more nonlinear" in the covariant gauge exhibiting very complicated ultraviolet behaviour. One may try to obtain soma relevant information about the conformal anomaly working directly in the light-cone gauge. There is a promising hint [27] that such program may be carried out at least for a particular subclass of the considered σ-models. We conclude this chapter by noting that another interesting problem may be a formulation of the string field theory in the shock wave background with the possibility to study the focusing phenomenon in this dynamical framework.

Bibliography

[1] G.t'Hooft, Phys.Lett.198B (1987) 61

[2] V.Ferrari,P.Pendenza and G.Veneziano, CERN preprint TH-4973/88

[3] C.Klimčík, Phys.Lett.208B (1988) 373

[4] P.C.Aichelburg and R.U.Sexl, Gen.Rel.Grav.2 (1971) 303

[5] T.Dray and G.t'Hooft, Nucl.Phys.B253 (1985) 173

[6] T.Dray and G.t'Hooft, Class.Quantum Grav.3 (1986) 825

[7] D.Amati,M.Ciafaloni and G.Veneziano, Phys.Lett.197B (1987)81

[8] D.Amati and C.Klimčík, Phys.Lett.B210 (1988) 92

[9] N.D.Birrell and P.C.W.Davies, Quantum Fields in Curved Space, Cambridge University Press 1982

[10] S.Hawking, Com.Math.Phys 43 (1975) 199

[11] G.t'Hooft, Quantum Gravity and Black Holes, Cargèse School on Nonperturbative Field Theory, July 1987

[12] G.Veneziano, Mod.Phys.Lett.2A (1987) 899

[13] R.Penrose, Rev.Mod.Phys.37 (1965) 215

[14] P.Szekeres, Nature 228 (1970) 1183

[15] K.Khan and R.Penrose, Nature 229 (1971) 185

[16] P.Szekeres, J.Math.Phys. 13 (1972) 286

[17] Y.Nutku and M.Halil, Phys.Rev.Lett. 39 (1977) 1379

[18] G.E.Curtis, Gen.Rel.Grav. 9 (1978) 987

[19] F.J.Tipler, Phys.Rev. D22 (1980) 2929

[20] R.Matzner and F.J.Tipler, Phys.Rev. D29 (1984) 1575

[21] S.Chandrasekhar and B.Xanthopoulos, Proc.R.Soc.London A414 (1987) 1 and references therein

[22] V.Ferrari,J.Ibanez and M.Bruni, Phys.Rev. D36 (1987) 1053

[23] D.Gross and P.F.Mende, Phys.Lett. B197 (1987) 129

[24] I.Muzinich and M.Soldate, Santa Barbara preprint NSF-ITP-87-93 (1987)

[25] H.J.Vega and N.Sanchez, PAR-LPTHE 88/06,07

[26] G.Veneziano, CERN-preprint TH-5019/88

[27] C.Klimčík, in preparation