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CHIRAL SYMMETRY BREAKING BY INSTANTONS IN SUPERSYMMETRIC YANG-MILLS THEORY

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INTRODUCTION

Supersymmetric field theories have become a topic of intense interest in the past few years; revieuws of their general struc ture have been given in [1] . However most of that we know about these theories, coming from the perturbation theory, ap_ plies only to the weak coupling region. Recently considerable effort has been made with the aim of extending our perturbative knowledge to non perturbative phenomena. Because of non renorma lization theorems, relations between masses hold true at any order in perturbation theory, once they are set at the tree level. Only non perturbative phenomena avoid these theorems and may provide an explanation way these mass scales are so widely separated, if supersymmetry were broken dynamically. One possible mechanism could be provide by the non perturbati_ ve quantum fluctuations of the instanton type in the vacuum. In lower dimensions,[2] as for example in supersymmetric quantum mechanic , instantons have been shown to spontaneosly break supersymmetry. However in four dimensions, Witten has been pro_ ved that supersymmetry is unbroken in many interesting theories, as SU(N) gauge theories with any number of massive chiral mat_ ter fields, but nothing has been proved for massless quarks.[3] Instanton effects was studied in the contest of non supersymme_ tric gauge theories, a long time ago.[4] The interest in instantons arose because of the discovering of an exact finite action solution to the classical Yang-Mills equations in euclidean space-time and the realization that the existence of that finite action field configurations indicates that the structure of the vacuum in a gauge theory is much more complicate that one would have from straightforward perturbation theory. In a Yang-Mills gauge theory there exists an infinity of degenerate classical ground states, characterized by the integer topological charge. Instantons provide a description of quantum mechanical tunnelling between ground states of different topological charge, thereby contributing non-trivially to the vacuum energy density. However when massless fermions are present in the gauge theory, the picture changes drastically. Due to the zero modes of the Dirac operator in the topologically non-trivial background, the quantum tunnelling between classical vacua of different topological charge is completely suppressed. Therefore, in supersymmetric Yang-Mills theory, where we have massless fermions, single instantons (or anti-instantons) do not contribute to the vacuum energy. These cannot break supersymmetry and the vacuum energy stays at zero.

In ordinary QCD we are also interested on the problem of what are the global symmetries preserved by the instanton. In the presence of massless quarks the instanton effect can be repre sented in terms of an effective lagrangian that violates the chiral U(1) invariance and that can also provide a mechanism for the dynamical breaking of the chiral part of the $SU(N_f)xSU(N_f)$ symmetry group. Here the same analysės is applied to supersymmetric gauge theory: from the computation of certain Green's functions at short distances through a single instanton calculation,[5] we obtain a condensate which allows us to identify the patterns of the spontaneous symmetry breaking of the global symmetries of the original theory.

For definiteness we concentrate on pure supersymmetric Yang-Mills theory with SU(N) gauge group, where the gluino-gluino condensate implies the spontaneous breaking of the discrete Z_{2N} symmetry down to Z_2.[6]

However it has been clear that same degree of care must be taken with instanton analysės in supersymmetric theory. In many cases instanton effects would appear to produce supersymmetric breaking, even where such breaking is prevent by Witten's index theorem. For example, the 't Hooft effective action , which describes instanton effects, only involves fermions. If this was a complete representation of the instanton effect, then supersymmetry would be manifestly broken.[3,6] To recover supersymmetry it is absolutely necessary to integrate over all collective coordinates.[9] Since fixing the instanton size would violate supersymmetry, a supersymmetric gauge theory cannot be thought of as having only small instantons. This is a consequence of the conformal invariance of the theory. There is no analogue of this phenomenon in ordinary QCD and this might indicate that dynamics inherent to supersymmetric theories is quite specific.

The outline of this thesis is the following:

In section I the notation is fixed and a class of Green's functions suitable for instanton calculation is definite;

In section II instanton techniques are recalled and the differential instanton contribution to the vacuum energy is computed. The β-function for n-extended supersymmetric Yang-Mills theory is found here from a purly "classical" considerations.

In section III the computation of a simplest n-point Green function contribuited by instanton is performed. The result is not vanishing; the implication of it and comparison with the QCD result are discussed.

I. INSTANTON CONTRIBUTED GREEN FUNCTIONS

By supersymmetric QCD we will mean a supersymmetric theory with a SU(N) gauge group and with N_f flavours of quarks. The N_f quark flavours correspond to N_f chiral superfields Q^{ai} ($a=1,\dots,N$ $i=1,\dots,N_f$) in the N representation of the gauge group and N_f in the $\bar{N}$ representation, $\tilde{Q}_{ai}$ (higher and lower indices, a,b, i,j, are meant to indicate that the field belongs, respectively, to fundamental or its conjugate representation of the gauge group- or flavour symmetry group). In terms of component fields the N_f fermionic quarks and antiquarks, ψ^{ai} and $\bar{\psi}_{ai}$, are accompanied by scalar partners φ^{ai} and $\bar{\varphi}_{ai}$

$$\begin{aligned} Q^{ai} &= \varphi^{ai} + \sqrt{2}\,\theta^\alpha \psi_\alpha^{ai} + \theta^2 F^{ai} \qquad \text{the same for } \tilde{Q} \\ \bar{Q}_{ai} &= \varphi^*_{ai} + \sqrt{2}\,\bar{\theta}_{\dot\alpha} \bar{\psi}^{\dot\alpha}_{ai} + \bar{\theta}^2 \bar{F}_{ai} \qquad \text{" " " } \bar{\tilde{Q}} \end{aligned} \tag{1}$$

where $\alpha,\dot\alpha=1,2$ is a two valued spinorial index.

The gauge fields A_μ^A ($A=1,\dots,N^2-1$) are accompanied by gluinos λ^A, their fermionic partners. The strength tensor $F_{\mu\nu}^A$ and the spinor λ^A are embedded into the chiral superfield W_α :

$$W_\alpha = i\lambda_\alpha + i\left(F^{\mu\nu}\sigma_{\mu\nu\alpha}{}^{\beta} + D\delta_\alpha^\beta\right)\theta_\beta + \theta^2 \sigma^\mu{}_{\alpha\dot\alpha} D_\mu \bar{\lambda}^{\dot\alpha} \tag{2}$$

where we have used the notation $W_\alpha = W_\alpha{}^A T^A$; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$. The covariant derivative is defined as $D_\mu = \partial_\mu + i\,\tau^A A_\mu{}^A$, where τ^A are the SU(N) generators in the appropriate representation and they are called T^A in the adjoint one.

The lagrangian of the theory in the case of massless quarks is given by

$$\mathcal{L} = \frac{1}{4g^2}\left[\int d^2\theta\, W^\alpha W_\alpha + \int d^2\bar\theta\, \bar W_{\dot\alpha} \bar W^{\dot\alpha}\right] + \int d^4\theta\, \bar Q e^{2V} Q + \tilde Q e^{-2V} \bar{\tilde Q} \qquad (3)$$

the superfiel V, in the Wess-Zumino gauge where we are working, is given by

$$V = -\theta^\alpha (\sigma_\mu)_{\alpha\dot\alpha} \bar\theta^{\dot\alpha} A^\mu + i\theta^2 \bar\theta_{\dot\alpha} \bar\lambda^{\dot\alpha} - i\bar\theta^2 \theta^\alpha \lambda_\alpha + \frac{1}{2}\theta^2\bar\theta^2 D$$

In this gauge, when the lagrangian (3) is written in components, it contains the usual gauge invariant kinetic terms for fermionic and scalar quarks and for gluons and gluinos:

$$\mathcal{L}_{kin} = -\frac{1}{2g^2} \mathrm{Tr}\, F_{\mu\nu} F_{\mu\nu} + \frac{1}{g^2} \bar\lambda \not{D} \lambda - D_\mu \varphi^* D_\mu \varphi - D_\mu \tilde\varphi^* D_\mu \tilde\varphi + \bar\psi \not{D} \psi + \bar{\tilde\psi} \not{D} \tilde\psi \qquad (4)$$

there is also the Yukawa coupling

$$\mathcal{L}_{Yukawa} = -i\sqrt{2}\left(\bar\psi \bar\lambda \varphi - \varphi^* \lambda \psi + \tilde\varphi \bar\lambda \bar{\tilde\psi} - \tilde\psi \lambda \tilde\varphi^*\right) \qquad (5)$$

The lagrangian (3) is gauge invariant and supersymmetric; the supersymmetry transformations for the gauge invariant fields are

$$\delta F_{\mu\nu} = \frac{i}{2}\left[\epsilon \sigma_\mu D_\mu \bar\lambda + \bar\epsilon \bar\sigma_\mu D_\mu\right] - (\mu \to \nu)$$

$$\delta\lambda = i\,\epsilon D + F_{\mu\nu}\sigma_{\mu\nu}\epsilon \qquad (6)$$

$$\delta D = \bar\epsilon \bar\sigma_\mu D_\mu \lambda - \epsilon \sigma_\mu D_\mu \bar\lambda$$

Classically this theory has an $U(N_f)_L \times U(N_f)_R \times U(1)_\chi$ global symmetry. The $U(N_f) \times U(N_f)$ symmetry is just like that of the usual QCD corresponding to separated rotations of the Q and $\tilde Q$ fields.

The $U(1)_\chi$ is an R-invariance, a symmetry under which the components of a given superfield transform differently. Its existence is related to the presence of the massless gluinos. Under this symmetry:

$$\lambda \longrightarrow e^{-i\alpha}\lambda$$
$$\varphi \longrightarrow e^{-i\alpha}\varphi$$
$$\tilde{\varphi} \longrightarrow e^{-i\alpha}\tilde{\varphi}$$

In terms of superfields this transformation corresponds to

$$\begin{aligned} W(\theta) &\longrightarrow e^{-i\alpha} W(\theta e^{i\alpha}) \\ Q(\theta) &\longrightarrow e^{-i\alpha} Q(\theta e^{i\alpha}) \quad \text{(same for } \tilde{Q}\text{)} \end{aligned} \qquad (7)$$

Since the anticommuting parameter θ is rotated under this symmetry, it does not commute with supersymmetry.

At the quantum level, however, both the axial U(1) current and the $U(1)_\chi$ are anomalous, but there is a subgroup of $U_A(1) \times U_\chi(1)$ which is not affected by anomaly; it is denoted by $\hat{U}(1) \times Z_{2N}$

$$\hat{U}(1): \quad \begin{aligned} W(\theta) &\longrightarrow e^{-i\alpha} W(\theta e^{i\alpha}) \\ Q(\theta) &\longrightarrow e^{i\alpha\frac{N-N_F}{N_F}} Q(\theta e^{i\alpha}) \\ \tilde{Q}(\theta) &\longrightarrow e^{i\alpha\frac{N-N_F}{N_F}} \tilde{Q}(\theta e^{i\alpha}) \end{aligned} \qquad (8)$$

or in components: $\lambda \rightarrow e^{-i\alpha}\lambda$, $\varphi \rightarrow e^{i\alpha(N-N_F)/N_F}\varphi$, $\underline{\psi} \rightarrow e^{i\alpha N/N_F}\underline{\psi}$ (same for $\tilde{\varphi}$ and $\underline{\tilde{\psi}}$), and Z_{2N} is the discrete subgroup of U (1) given by $\alpha = 2\ k/2N$ in eq. (7).

The Green's functions which are possible candidates for instan_ ton effects must satisfy the chirality selection rule imposed by the anomalous $U_\chi(1)$ symmetry and must be also invariant under the non anomalous $\hat{U}(1)$ symmetry.

Also, in order to avoid complications associated with the explicit breaking of supersymmetry due to gauge fixing terms and ghosts, we shall consider gauge-invariant n-point functions only. Moreover, in the classical case supersymmetry transformations are also assumed to be accompanied by some gauge transformations. Then from the transformation rules of the fields under $U_\chi(1)$ and $\hat{U}(1)$, we get

$$n_\lambda + n_\varphi = N\nu \qquad (9)$$
$$-N_f n_\lambda + N n_\psi + (N-N_f)\, n_\varphi = 0$$

where n_λ, n_φ, n_ψ are respectively the number of $\lambda\lambda$, $\tilde{\varphi}\varphi$, $\tilde{\psi}\psi$ pairs appearing in the Green's functions that we want to calculate and ν is the winding number.

In particular, for the supersymmetric Yang-Mills ($N_f=0$) a candidate for a single instanton contribution is

$$G(x_1, \dots x_N) = \langle 0|T(\lambda\lambda(x_1)\ \lambda\lambda(x_2)\ \dots\ \lambda\lambda(x_N))|0\rangle \qquad (10)$$

for the SU(N) gauge group.

Now we focalize on the property and on the computation of G. In fact this Green's function has the remarkable property of being space-time independent as a consequence of supersymmetry, since it is made of lowest components of a chiral superfield.

In fact, let $\Sigma_i = A_i + \sqrt{2}\theta\psi_i + \theta^2 F_i$ be chiral superfields, then

$$\bar{\sigma}^{\mu}_{\dot{\alpha}\alpha}\,\partial_{\mu}^{(x_i)}\langle 0|T(A_1(x_1)\dots A_i(x_i)\dots A_n(x_n))|0\rangle = \tag{11}$$
$$= \langle 0|T(A_1(x_1)\dots\{\bar{Q}_{\dot{\alpha}},\psi_{\alpha}(x_i)\}\dots A_n(x_n))|0\rangle = 0$$

if the A_i commute at equal time and $\bar{Q}|0\rangle = 0$ (recall that $[\bar{Q},A_i] = 0$). This property allows us to compute Green's functions of this type at short distances $|x_i - x_j| \ll \Lambda^{-1}$, where the semiclassical instanton approximation is reliable. In fact if at short distance we find for $G(x_1,\dots,x_N)$ a non zero constant, then the result will also be non zero at large $|x_i - x_j|$ separation because of eq. (11). In section III we will see that in fact $G(x_1,\dots,x_N)$ in the instanton approximation is non zero, finite and constant after the integration over the collective coordinates. It is very interesting that in this case the integration over the dilatation collective coordinate is, unlike the typical QCD case, infrared convergent.

The constancy of $G(x_1,\dots,x_N)$ allows us to use the clustering at large distances. The cluster decomposition leads to a condensate $\langle\lambda\lambda\rangle$ different from zero and finite. This of course cannot be explained directly by instantons, since the one-instanton contribution to $\langle\lambda\lambda\rangle$ is vanishing due to Z_{2N} symmetry preserved by the instanton (or in other terms, the fermions are not enough to cancel the fermionic zero modes).

In ref. [6] Rossi and Veneziano claim that contributions with $\nu \neq 1$ and/or other non-perturbative effects[10] are needed to have consistency with the clustering.
This situation is not so different from the massless QCD.
This theory has an high symmetry: it is formally invariant under $U_V(N_f) \times U_A(N_f)$ but the axial U(1) is broken by the Adler-Bell-Jackiw anomaly and hence by instanton effects.
This is an explicit breaking since the instanton contribution to $\langle \prod_{i}^{N_f} \bar{\Psi}_i(x_i) \Psi_i(x_i) \rangle$ is different from zero already at finite volume.[11] The remaining symmetries are preserved unless spontaneous breaking occurs; so at finite volume we expect that $SU(N_f) \times SU(N_f)$ is unbroken. In order to see if the spontaneous symmetry breaking of the axial part of SU(N) occurs, we have to formulate the theory on a finite volume, then the symmetry breaking is determined by the dependence of the correlation functions on the boundary conditions in the infinite volume limit. This is the same as introducing an external field which breaks the symmetry explicitely as a mass term and taking the limit $m \to 0$ after the thermodynamical limit has been taken:

$$\lim_{m \to 0} \lim_{V \to \infty} \langle \bar{\Psi} \Psi \rangle_m$$

where $\langle \bar{\Psi} \Psi \rangle_m$ is calculated in the perturbation theory.
An other possible procedure is to consider correlation functions

which are different from zero already at finite volume, since they are symmetric, as $\langle \prod_i^{N_f} \bar{\psi}_i(x_i)\psi_i(x_i)\rangle$, and define the fermionic condensate from the limit $|x_i - x_j| \to \infty$ of it in the infinite volume limit:

$$\lim_{|x_i - x_j| \to \infty} \lim_{V \to \infty} \langle \prod_i^{N_f} \bar{\psi}_i(x_i)\psi_i(x_i)\rangle = \langle \bar{\psi}\psi\rangle^{N_f}$$

Otherwise if one computes directly the condensate in a completely symmetric situation one gets zero and obviously it remains zero also in the $V \to \infty$ limit. This is due to the fact that we are computing the expectation value of $\bar{\psi}\psi$ in a state which is a mixture of the possible vacua of the theory, just like in a ferromagnetic system in a symmetrical situation $\langle s_x \rangle = 0$, i.e. there is no spontaneous magnetization also below the critical temperature. Even here in order to define the spontaneous magnetization we have to follow a well defined procedure[12] such as, for example, introducing an external magnetic field H

$$m_s = \lim_{H \to 0^+} \lim_{V \to \infty} \frac{1}{V} \langle \sum_x s_x \rangle_H$$

or, equivalently, without magnetic field, from the correlation function

$$m_c^2 = \lim_{r_{ij} \to \infty} \lim_{V \to \infty} \langle s_i s_j \rangle_0$$

for spins separated by distances going to infinity when V goes to infinity.

II. INSTANTON MEASURE

Following t'Hooft,[4] we compute the one-instanton contribution to the effective lagrangian in a supersymmetric Yang-Mills theory ($N_f=0$), with SU(2) as a gauge group.

The classical solution

$$A_\mu(x) = A_\mu^{inst.}(x) = \frac{2}{g}\,\frac{\eta_{a\mu\nu}(x-z)_\nu}{[(x-z)^2+\rho^2]^2} \qquad (1)$$

$$\lambda = \bar{\lambda} = 0$$

is a saddle point of the euclidean action; z_μ and ρ are the center and the size of the instanton; $\eta_{a\mu\nu}$ are the t'Hooft coefficients.

We consider the vacuum-vacuum amplitude in the presence of the instanton normalized by the same amplitude in the absence of it:

$$W = \frac{\int (dA)(d\lambda)(d\bar{\lambda})\ldots e^{-S[A,\lambda,\ldots]}}{\int (dA^\circ)(d\lambda^\circ)(d\bar{\lambda}^\circ)\ldots e^{-S[A^\circ,\lambda^\circ,\ldots]}} \qquad (2)$$

where the dots stand for the gauge-fixing term and the ghosts. The one loop approximation to W is obtained in the same way of non supersymmetric theories. Here we expand the action S in the numerator around the classical minimum $A_\mu = A_\mu^{inst}$, $\Psi = 0$ and S in the denominator around the normal vacuum $A_\mu=0$, retaining terms which are almost quadratic in the quantum fluctuations and performing the resulting gaussian integrals. This gives W as a product of determinants of the operators $\mathcal{M}^{V,G,F,S}$, describing the quantum fluctuations in the classical solution, raised to variou powers depenging on the statistic of the field involved, divided by the same expression using the operator $\mathcal{M}^{(o)}$

$$W^{(1)} = e^{-8\pi^2/g^2}\left[\frac{\det \mathcal{M}^V}{\det \mathcal{M}^V_o}\right]^{-1/2} \cdot \frac{\det \mathcal{M}^G}{\det \mathcal{M}^G_o} \cdot \frac{\det \mathcal{M}^F}{\det \mathcal{M}^F_o} \cdot \frac{\det \mathcal{M}^S}{\det \mathcal{M}^S_o} \qquad (3)$$

V, G, F, S stand for vector, ghost, fermion and scalar respectively. The determinants can be computed by solving the corresponding eigenvalue equations. It has been shown that these equations reduce to the eigenvalue equation for the quadratic Casimir of the total O(5) group and the eigenvalues can be computed using group theoretical methods. The determinants in eq. (3) are actually infinite and one must regularize them. S me of them also contain zero modes which must be carefully treated. A very interesting way of regularizing the determinants in eq. (3) is the ζ - function regularization procedure. It consist of defining the generalized ζ - function associated with the operator $\mathcal{M}$ as follows

$$\zeta_s(\mathcal{M}) = \sum_n \lambda_n^{-s}$$

where λ_n are the non-zero eigenvalues of $\mathcal{M}$. Then one can definesthe following regularized formula for the determinant of $\mathcal{M}$ to be:

$$\det(\mu\rho\,\mathcal{M}) = (\mu\rho)^{\zeta_0(\mathcal{M})+N}\, e^{-\zeta_0'(\mathcal{M})}\, I$$

where μ is the momentum scale subtraction point and ρ is the instanton size; N is the ~~the number of zero~~ modes and the factor I

$$I = \lim_{\epsilon_m \to 0} \prod_{m=1}^{N} \epsilon_m$$

is a carefully defined limit over the product of the zero eigenvalues and is to be replaced by a suitable integration over collective coordinates.

Computing ζ_0 and ζ_0' for the operators $\mathcal{M}$, one gets the following expression for $W^{(1)}$ [13]:

$$W^{(1)}_{QCD} = \int dC\, e^{-8\pi^2/g^2} (\mu\rho)^A\, e^B \tag{4}$$

Where $\int dC$ is the integral over the collective coordinates,

$$\begin{aligned} A &= 8 - C(T)N_F + N_S\,\tilde{\zeta}_0^{(S)}(T_S) + \tilde{\zeta}_0^{(V-G)}(1) - 2N_F\,\tilde{\zeta}_0^{(F)}(T_F) \\ B &= \tilde{\zeta}_0'^{(V-G)}(1) - 2N_F\,\tilde{\zeta}_0'^{(F)}(T_F) + N_S\,\tilde{\zeta}_0'^{(S)}(T_S) \end{aligned} \tag{5}$$

$\tilde{\zeta}_0(T)$ is the difference between the ζ-function in the back_ground instanton field and that in the vacuum; eight is the number of bosonic zero-modes (five for the fluctuations around the instanton obtained by infinitesimal traslations and dila_tation on the classical solution; the remaining three zero modes are due to the fluctuations of the ghost around the vacuum and they are a direct consequence that the gauge condition $\Delta \cdot A = 0$ does not fix the gauge uniquely).

$C(T) = \frac{2}{3} T(T+1)(2T+1)$ is the number of the zero eigenstates of the fermionic operator; T labels the representations of the SU(2) gauge group.

Supersymmetric Yang-Mills theory is the particular situation where there is only one fermion which belongs to the adjoint representation as the gluon (T = 1). The real difference is that here the spinor is a Weyl (or Majorana) spinor not Dirac as in QCD.

In euclidean space a Weyl spinor λ ,as well its hermitian conjugate λ^+, trasforms according to the same representation of the Lorentz group and even the kinetic energy cannot be constructed without introducing additional fields. To circumvent this difficulty we will use the fermion doubling procedure[7] . We complete the Weyl spinor λ performing a linear change of variables:

$$\lambda_\alpha^A = (\sigma_2)_{\alpha\beta} \, \tilde{\lambda}^{*A}_\beta$$

so as to obtain a four components Dirac spinor $\Psi^A = \begin{pmatrix} \lambda^A \\ \hat{\lambda}^A \end{pmatrix}$

then the fermion determinant is defined as

$$Z_{Fermion} = \left\{ \int \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \, e^{-\int d^4x \, \bar{\Psi} \bar{\gamma}_\mu \cdot \bar{D}_\mu \Psi} \right\}^{1/2}$$

where $\bar{\gamma}_\mu$ are the euclidean Dirac matrices

$$\gamma_4 \tilde{\gamma}_\mu = \begin{pmatrix} & \bar{\sigma}_\mu \\ \sigma_\mu & \end{pmatrix} \qquad \begin{array}{l} \sigma_\mu = (i\vec{\sigma}, \ 1) \\ \bar{\sigma}_\mu = (-i\vec{\sigma}, \ 1) \end{array}$$

In the instanton calculation we obtain the fermion contribution first considering the spinor as a Dirac spinor; this gives the usual $N_f = 1$ QCD fermion factor, then we have to extract the square root of it. This amounts to make the rescaling $N_f \rightarrow \frac{1}{2} N_f$ in eq. (5). (6)

As it was shown by D'Adda and Di Vecchia [14], the eigenvalue equations for the fluctuations of the scalars, fermions, gluons have the same spectrum of non zero eigenvalues, if the fields trasform according to the same representation of the gauge group; in this situation one gets:

$$\zeta^{(F)}(T) = 2\,\zeta^{(S)}(T) \qquad \zeta^{(V)}(1) = 2\,\zeta^{(S)}(1) \qquad (7)$$

there is a factor two because the fermion eigenvalue are twice degenerate with respect to the scalar. Also each vector eigenvalue is four times degenerate, but two components are killed by the Faddeev-Popov ghost, so that only the two physical components of the gauge field give actually a contribution. Inserting eqs. (6) and (7) in eq. (5) we obtain for the supersymmetric Yang-Mills theory ($N_S=0$, $T_f=1$)

$$B = 0 \qquad A = 8 - 2 = 6 \qquad (8)$$

therefore the coefficient of the term containing the subtraction point μ is given only in terms of the zero modes of the gluon, fermion and ghost. In the supersymmetric case the partition function becomes extremely simple

$$W^{(1)}_{SYM} = \int d\zeta\, e^{-8\pi^2/g^2} (\mu\rho)^6 \qquad (9)$$

The collective coordinates that specify the eight zero modes associated with the traslational invariance, scale invariance and the residual gauge freedom are the center, the size and the gauge orientation of the instanton. For the four fermionic zero modes we introduce anticommuting collective coordinates θ_α, $\bar\beta_{\dot\alpha}$ $(\alpha, \dot\alpha = 1,2)$, since they are associated with the supersymmetry and superconformal invariance of the theory

$$\begin{aligned} \delta\lambda_\alpha^{a\,ss} &= F_{\mu\nu} (\sigma_{\mu\nu})_\alpha{}^\beta\, \theta_\beta \\ \delta\lambda_\alpha^{a\,sc} &= F_{\mu\nu} (\sigma_{\mu\nu})_\alpha{}^\beta (x-z)^\mu (\sigma_\mu)_{\beta\dot\alpha}\, \bar\beta^{\dot\alpha} \end{aligned} \qquad (10)$$

where $\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar\sigma_\nu - \sigma_\nu \bar\sigma_\mu)$ and θ is the parameter of the supersymmetry trasformation and $(x-z)\bar\beta$ is the x-dependent parameter of the superconformal one. In fact when the ~~se~~ trasformations are applied to the classical solution

$$\begin{aligned} F^a_{\mu\nu} &= -\frac{4}{g}\, \eta_{a\mu\nu} \frac{\rho^2}{[(x-z)^2+\rho^2]^2} \\ \lambda &= 0 \end{aligned} \qquad (11)$$

we get

$$\lambda^{a\,SS}_{\alpha}(x) = -i\,\frac{2^3}{g}\,\frac{\rho^2}{[(x-z)^2+\rho^2]^2}\,(\sigma^a)_\alpha{}^\beta\,\theta_\beta$$

$$\lambda^{a\,SC}_{\alpha}(x) = -i\,\frac{2^2}{g}\,\frac{\rho^2}{[(x-z)^2+\rho^2]^2}\,(\sigma^a)_\alpha{}^\beta\,(x-z)^\mu\,(\sigma_\mu)_{\beta\dot\alpha}\,\bar\beta^{\dot\alpha} \qquad (12)$$

Indeed they are annihilated by the Dirac operator

$$\bar\sigma^{\mu\dot\alpha\alpha} D_\mu \lambda^{SS}_\alpha = (\bar\sigma^\mu\,\sigma_{\rho\sigma})^{\dot\alpha\beta} D_\mu F_{\rho\sigma}\,\theta_\beta =$$

$$= 2\,(\delta^{\mu\rho}\bar\sigma^\sigma - \delta^{\mu\sigma}\bar\sigma^\rho - \epsilon^{\mu\rho\sigma\nu}\bar\sigma^\nu)^{\dot\alpha\beta} D_\mu F_{\rho\sigma}\,\theta_\beta =$$

$$= 4\,[D_\rho F_{\rho\sigma}(\bar\sigma^\sigma)^{\dot\alpha\beta} - D_\mu \tilde F_{\mu\nu}\,\sigma^{\nu\,\dot\alpha\beta}]\,\theta_\beta =$$

$$= 0$$

since for a self-dual field $D_\mu F_{\mu\nu}=0$ by Jacoby identity.

In the same way one can prove again that $\lambda^{a\,SC}_\alpha$ satisfies the Dirac operator: $\bar\sigma D_\mu \lambda^{SC}=0$. Also, from the $\bar Q_{\dot\alpha}$ generators we do not obtain new zero modes, in fact

$$\delta\bar\lambda_{\dot\alpha} = \bar\epsilon_{\dot\beta}\,(\bar\sigma^{\mu\nu})^{\dot\beta}{}_{\dot\alpha}\,F^{\mu\nu} = 0$$

because of the self-duality of the instanton[15] (as it must be since the degeneration of the fermionic zero modes is fixed by C(T=1) = 4):

$$\bar\sigma^{\mu\nu}F^{\mu\nu} = \frac{1}{2}\,\bar\sigma^{\mu\nu}\,\epsilon^{\mu\nu\rho\sigma}F^{\rho\sigma} = -\bar\sigma^{\rho\sigma}F^{\rho\sigma} = 0$$

Assembling all this considerations, the one instanton contri_ bution to the vacuum-vacuum amplitude is given, in terms of the collective coordinates, by

$$W^{(1)}_{SYM} = N\, g^{-8} g^{4} \int \frac{d^4 z\, d\rho}{\rho^5}\, e^{-8\pi^2/g^2} (\rho\mu)^6\, d^2\theta\, d^2\bar{\beta} \quad (13)$$

where N is a numerical factor.

Let us mention that the integration over the fermionic collective coordinates makes the vacuum energy exactly zero, according to the standard rules of integration over the Grassmann variables $\int d^2\theta\, d^2\bar{\beta} = 0$ (it is a zero of the fourth order that reflects the existence of four fermionic zero modes).

The factor g^4 in eq. (13) requires same explanation.[16] The expression (4) holds for normalized eigenfunctions. The zero modes obtained by supersymmetry trasformation, instead, are proportional to $1/g$ and when we make the expansion of the fermionic fields on unnormalized eigenfunction ψ_n the measure for the functional integration becomes:

$$[d\psi][d\bar{\psi}] \longrightarrow \prod_i \frac{da_i\, da_i^+}{\|\psi_i\|}$$

$$\psi = \sum_n a_n \psi_n \qquad \bar{\psi} = \sum_n a_n^+ \bar{\psi}_n$$

where $\|\psi_i\|$ is the norm of ψ_i. Then we get a factor g^2 for any fermionic zero mode. Since there are four of them and since the result must be raised to power $\frac{1}{2}$, the total factor is g^4.

As we have stressed before at one-loop order the power of is determined by the zero modes alone, since in a supersymmetric theory the degeneration of fermionic and bosonic zero modes is the same. Then we can assume that the exact cancellation between non zero modes holds for all orders of perturbation theory so that the explicit μ-dependence of $W^{(1)}$ is given exactly by the factor μ^6. This allows us to find the exact Gell-Mann-Low function of the supersymmetric Yang-Mills theory from the one-instanton contribution to $W^{(1)}$.[16]
In fact g and μ are not independent parameters: the renormalizability of the theory implies that the explicit μ dependence may be absorbed in g, so that the observable quantities, like $W^{(1)}$, are independent on μ ; as usual the renormalization group fixes the μ-dependence of the coupling constant.

Taking the derivative with respect to logμ of the logarithm of $W^{(1)}$ we get:

$$0 = -\frac{4}{g}\frac{\partial g}{\partial \lg \mu} + \frac{16\pi^2}{g^3}\frac{\partial g}{\partial \lg \mu} + 6$$

giving the Gell-Mann-Low function

$$\beta(g) = \frac{\partial g}{\partial \lg \mu} \qquad (14)$$

for the SU(2) supersymmetric Yang-Mills

$$\beta(g) = -\frac{6g^3}{(4\pi)^2}\frac{1}{1-\frac{g^2}{4\pi^2}} \qquad (15)$$

More generally, we can apply the same analysis to the n-extended supersymmetric Yang-Mills theory (n= 1,2,4) with SU(N) gauge group: due to supersymmetry also here there is an exact cancellation between non zero modes and the total dependence of the vacuum energy on μ and g is given by the zero modes. In this case these numbers are:

$$n_b = 4N \qquad n_f = 2Nn \tag{16}$$

and the μ-dependence of the vacuum energy is given by the factor

$$g^{-4N+2Nn} \, e^{-8\pi^2/g^2} \, \mu^{4N-\frac{2Nn}{2}}$$

Requiring that the explicit μ-dependence may be absorbed in g we get:

$$(-4N+2Nn)\frac{1}{g}\beta(g) + \frac{16\pi^2}{g^3}\beta(g) + (4N-Nn) = 0$$

then

$$\beta(g) = \frac{(n-4)N}{(4\pi)^2} g^3 \cdot \frac{1}{1-(2-n)\frac{g^2}{8\pi^2}} \tag{17}$$

which reproduces the vanishing of the β-function for the n=4 case. For n=2 it reduces a single term; this result is in agreement with the explicit calculation, in fact the two- and three-loop contribution to β vanish. For n=1 the eq. (17) can be expressed as a series in g^2 in order to make a comparison with the perturbative calculation; they agree only up to the second term, but the other terms are renormalization scheme dependent. The formula (17) for β has also recently been obtained in Ref.[17] by the axial and the trace anomaly together with the Adler-Bardeen theorem.

The instanton calculation is also supersymmetric[9], although the instanton contribution to the vacuum energy (13) can be considered as an effective interaction with four fermion legs and therefore it apparently violate supersymmetry. This can be proved explicitly from the change of the collective coordinates under supersymmetry trasformations. Fermionic and bosonic solutions of the classical equations of motion can be embedded into a vector superfield, making a traslation in the superspace of the 'initial' superfield containing only the bosonic solution (11) and no fermion components

$$V(z,\rho,\theta\,\bar{\beta}) = e^{iPz}\, e^{-i(\theta Q+\bar{S}\bar{\beta})}\, V(z=0,\rho)_{init}\, e^{i(\theta Q+\bar{S}\bar{\beta})}\, e^{-iPz} \tag{18}$$

where Q and $\bar{S}$ are the generators of supersymmetric and superconformal trasformation, respectively, and P is the generator of traslations. Using the supersymmetric algebra, we find that supersymmetry trasformations with parameters ϵ and $\bar{\epsilon}$ on the instanton superfield induce the following change of the z, ρ, θ, $\bar{\beta}$ parameters

$$
\begin{aligned}
z &\longrightarrow z - 2i\,(\theta\sigma^{\mu}\bar{\epsilon}) \\
\theta &\longrightarrow \theta + \epsilon \\
\rho &\longrightarrow \rho\,(1 + 2i(\bar{\beta}\bar{\epsilon})) \\
\bar{\beta} &\longrightarrow \bar{\beta}\,(1 + 4i\,(\bar{\beta}\bar{\epsilon}))
\end{aligned}
\qquad (19)
$$

Then the instanton measure that appears in W

$$\mu(z, \rho\, \theta, \bar{\beta}) = d^4 z\, d\rho^2\, d^2\theta\, d^2\bar{\beta}$$

is invariant under this shift since

$$d^2\bar{\beta} \longrightarrow (1 - 4i\,\bar{\beta}\,\bar{\epsilon})\, d^2\bar{\beta}$$
$$d^2\rho^2 \longrightarrow (1 + 4i\,\bar{\beta}\bar{\epsilon})\, d\rho^2$$

Note that the invariance of the measure is due to a simul_ taneous change of ρ and $\bar{\beta}$ that make the product $d\rho^2 d^2\bar{\beta}$ invariant. Then the notion of an instanton of fixed size cannot be supersymmetric, since the $(\bar{Q}\bar{\epsilon})$ transformation induces also a change of ρ.

III. INSTANTON COMPUTATION OF GREEN FUNCTIONS

In a supersymmetric Yang-Mills theory we consider the correlation function

$$\Phi(x_1 x_2) = \langle 0|\, T\left(g\lambda(x_1)\, g\lambda(x_1)\, g\lambda(x_2)\, g\lambda(x_2)\right)|0\rangle \qquad (1)$$

where $\lambda\lambda(x)$ is a shorthand notation for $\epsilon^{\alpha\beta}\lambda_\beta\, \lambda_\alpha(x)$. We expect this correlation function to be a constant, since it is made of lowest components of the chiral superfield W_α (see sect. I); it receives a contribution from the one instanton sector since it contains all spinorial fields necessary to eliminate the fermionic zero modes. We compute a Green's function made of the combination $g\lambda$ since it stays unrenormalized. This can be shown in the background field method.[18] In this approach it is not necessary to renormalize the quantum fields (including the ghosts) since their renormalization constants cancel between propagator and vertices so that there is only a wave function renormalization for the background gauge field (Z) and a coupling constant renormalization (Z_g). Also manifest background gauge invariance is maintained at the quantum level, so that the only counter term needed is the background field action (for the n=1 supersymmetric Yang-Mills this action can be written as as a full superspace integral so this counterterm is allowed). For example $F^a_{\mu\nu}$ is renormalized by

$$(F^a_{\mu\nu})_R = Z^{1/2}\left[\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g Z_g Z^{1/2} f^{abc} A^b_\mu A^c_\nu\right]$$

This will only take on the gauge-covariant form of a constant time if

$$Z_g = Z_A^{-1/2}$$

and the combination $g\lambda$ stays unrenormalized. Then we compute Green's functions made of this combination since they are good candidates to obtain a renormalization group invariant result. A result of this kind is not trivial in fact in the operator product expansion of the composite operator $g\lambda(x) g\lambda(x)$ other combinations will appear which may destroy such property. From the instanton calculation, we shall see that it does not happen.

Using the instanton measure (Sect. II, (13)), the explicit calculation of (1) is

$$\Phi(x_1,x_2) = 2^{10}\pi^6 g^{-4} \int \frac{d^4z\, d\rho}{\rho^5}\, e^{-\frac{8\pi^2}{g^2}} (\rho\mu)^6\, d^2\theta\, d^2\bar\beta$$
$$\times\ g^4\rho^2 \sum_{\text{Perm}} (-)^P \lambda_0^{ss} \lambda_0^{sc}(x_1)\, \lambda_0^{sc} \lambda_0^{sc}(x_2) \qquad (2)$$

where the numerical factor is the same as one finds in Ref. [19] for the pure SU(2) Yang-Mills theory; λ_{ss} and

and λ_{sc} are the fermionic zero modes. They differ from those of the Sect. II of a factor defined so as to normalize lize them to $1/g$:

$$\lambda_\alpha^{a\,ss}(x) = -i\,\frac{\sqrt{2}}{\pi g}\,\frac{\rho^2}{[(x-z)^2+\rho^2]^2}\,\sigma_\alpha^{a\,\beta}\,\theta_\beta$$

(3)

$$\lambda_\alpha^{a\,sc}(x) = -i\,\frac{1}{g\pi}\cdot\frac{\rho}{[(x-z)^2+\rho^2]^2}\,\sigma_\alpha^{a\,\beta}(x-z)^\mu(\sigma_\mu)_{\beta\dot\alpha}\,\bar\beta^{\dot\alpha}$$

and the sum is over the possible permutation of these modes. We need then to compute gauge invariant composite operator $\lambda(x)\lambda(x)$ for each pair of zero modes

$$\epsilon^{\alpha\beta}\lambda_\beta^{a\,ss}(x)\,\lambda_\alpha^{a\,ss}(x) = \frac{-2}{\pi^2 g^2}\,\frac{\rho^4}{[(x-z)^2+\rho^2]^4}\cdot\epsilon^{\alpha\beta}\,\sigma_\beta^{a\,\gamma}\,\sigma_\alpha^{a\,\delta}\,\theta_\gamma\,\theta_\delta$$

$$= \frac{2}{\pi^2 g^2}\,\frac{\rho^4}{[(x-z)^2+\rho^2]^4}\cdot 3\,\theta^2.$$

$$\epsilon^{\alpha\beta}\lambda_\beta^{a\,sc}(x)\,\lambda_\alpha^{a\,sc}(x) = -\frac{1}{\pi^2 g^2}\cdot\frac{\rho^2(x-z)^\mu(x-z)^\nu}{[(x-z)^2+\rho^2]^4}\,\epsilon^{\alpha\beta}\sigma_\beta^{a\,\gamma}\sigma_\alpha^{a\,\delta}\,\sigma_{\mu\,\gamma\dot\gamma}\,\sigma_{\nu\,\delta\dot\delta}\,\bar\beta^{\dot\gamma}\bar\beta^{\dot\delta}$$

$$= -\frac{1}{\pi^2 g^2}\,\frac{\rho^2(x-z)^\mu(x-z)^\nu}{[(x-z)^2+\rho^2]^4}\cdot 3\,(\sigma_\mu)_{\gamma\dot\gamma}(\bar\sigma_\nu)^{\dot\gamma\gamma}\,\frac{1}{2}\,\bar\beta^2$$

$$= -\frac{1}{\pi^2 g^2}\,\frac{\rho^2(x-z)^2}{[(x-z)^2+\rho^2]^4}\,3\,\bar\beta^2$$

$$\epsilon^{\alpha\beta}\lambda^{a\,ss}_{\beta}(x)\lambda^{a\,sc}_{\alpha}(x) = -\frac{\sqrt{2}}{\pi^2 g^2}\frac{\rho^3(x-z)^{\mu}}{[(x-z)^2+\rho^2]^4}\cdot\epsilon^{\alpha\beta}\sigma^{a\,\gamma}_{\beta}\sigma^{a\,\rho}_{\alpha}\sigma_{\mu\,\rho\dot{\rho}}\,\theta_{\gamma}\bar{\beta}^{\dot{\rho}}$$

$$= \frac{\sqrt{2}}{\pi^2 g^2}\cdot\frac{\rho^3(x-z)^{\mu}}{[(x-z)^2+\rho^2]^4}\;3(\theta\sigma_{\mu}\bar{\beta})$$

then the different terms of the sum are

$$\lambda^{ss}(x_1)\lambda^{ss}(x_2)\lambda^{sc}(x_2)\lambda^{sc}(x_2) = -\frac{2\cdot 3^2}{\pi^4 g^4}\frac{\rho^6(x_2-z)^2}{[(x_1-z)^2+\rho^2]^4[(x_2-z)^2+\rho^2]^4}\,\theta^2\bar{\beta}^2$$

plus another similar term with $x_1 \longrightarrow x_2$;

$$\lambda^{ss}(x_1)\lambda^{sc}(x_1)\;\lambda^{ss}(x_2)\lambda^{sc}(x_2) = \frac{2\cdot 3^2}{\pi^4 g^2}\frac{\rho^6(x_1-z)^{\mu}(x_2-z)^{\nu}\;(\theta\sigma_{\mu}\bar{\beta})(\theta\sigma_{\nu}\bar{\beta})}{[(x_1-z)^2+\rho^2]^4\,[(x_2-z)^2+\rho^2]^4}$$

$$= \frac{2\cdot 3^2}{\pi^4 g^2}\frac{\rho^6(x_1-z)^{\mu}(x_2-z)^{\nu}}{[(x_1-z)^2+\rho^2]^4[(x_2-z)^2+\rho^2]^4}\cdot\frac{1}{2}\theta^2\bar{\beta}^2\delta_{\mu\nu}$$

$$= \frac{3^2}{\pi^4 g^2}\rho^6\frac{(x_1-z)\cdot(x_2-z)}{[(x_1-z)^2+\rho^2]^4[(x_2-z)^2+\rho^2]^4}\,\theta^2\bar{\beta}^2.$$

there are also the permutation $\lambda^{ss}(x_1)\lambda^{sc}(x_1)\lambda^{sc}(x_2)\lambda^{ss}(x_2)$

and the ~~two~~ other terms corresponding to $\lambda^{ss} \longrightarrow \lambda^{sc}$

which give an analogous contribution.

Putting everything together we find

$$\sum_P (-)^P \lambda(x_1)\,\lambda(x_1)\,\lambda(x_2)\,\lambda(x_2) = -\frac{2\cdot 3^2}{\pi^4 g^4}\,\frac{\rho^6\left[(x_2-z)^2+(x_1-z)^2-\frac{1}{2}(x_1-z)(x_2-z)\cdot 4\right]}{\left[(x_1-z)^2+\rho^2\right]^4\left[(x_2-z)^2+\rho^2\right]^4}\,\theta^2\bar\beta^2$$

$$= -\frac{2\cdot 3^2}{\pi^4 g^4}\cdot\frac{\rho^6\,(x_1-x_2)^2}{\left[(x_1-z)^2+\rho^2\right]^4\left[(x_2-z)^2+\rho^2\right]^4}\;\theta^2\bar\beta^2 \qquad (4)$$

Using the integration rules over Grassmann variables

$$\int d^2\theta\, d^2\bar\beta\;\theta^2\bar\beta^2 = 1$$

the Green's function (2) becomes

$$\phi(x_1 x_2) = 2^{11}\cdot 3^2\pi^2 g^{-4}\int \frac{d^4z\,d\rho}{\rho^5}\, e^{-\frac{8\pi^2}{g^2}}\,(\rho\mu)^6\,\rho^8$$

$$\times\;\frac{(x_1-x_2)^2}{\left[(x_1-z)^2+\rho^2\right]^4\left[(x_2-z)^2+\rho^2\right]^4} \qquad (5)$$

The power of the subtraction point mass scale μ is the first coefficient of the β-function, $b_1 = 3N = 6$.

When this factor is combined with the classical action $\exp -8\pi^2/g^2$ and with the factor g^{-4}, we obtain the renormalization group invariant combination

$$g^{-2/3}\,\mu\, e^{-8\pi^2/g^2} = \Lambda$$

(Λ has not to be confused with a cutoff: it is the in_ tegration constant of eq. (I, 4) with β given by eq. (II,15); it reflects the fundamental momentum scale of the theory). This gives a factor Λ^6 in eq. (5) which accounts completely for the physical dimension of $\Phi(x_1, x_2)$; then the rest must be a dimensionless function and, since the integrand is a sign definite function, the result of the integration is not vanishing.

The integration over the instanton position can easily be evaluated by looking at $[(x_1 - z)^2 + \rho^2]^{-1}$ as a particle propagator and introducing the Feynman parameters

$$\int d\rho^2 \rho^8 \int d^4 z \frac{1}{[(x_1 - z)^2 + \rho^2]^4 \, [(x_2 - z)^2 + \rho^2]^4} =$$

$$= \int d\rho^2 \rho^8 \int d^4 z \frac{\Gamma(8)}{\Gamma(4)\Gamma(4)} \int_0^1 d\alpha \frac{\alpha^3 (1-\alpha)^3}{\{[(x_1 - z)^2 + \rho^2]\alpha + [(x_2 - z)^2 + \rho^2](1-\alpha)\}^8}$$

$$= \frac{\pi^2 \Gamma(6)}{\Gamma(4)\Gamma(4)} \int d\rho^2 \rho^8 \int_0^1 d\alpha \frac{\alpha^3 (1-\alpha)^3}{[\alpha(1-\alpha)(x_1 - x_2)^2 + \rho^2]^6}$$

$$= \frac{10}{3}\pi^2 \int_0^1 d\alpha \, \alpha^3 (1-\alpha)^3 \cdot \frac{1}{5} \frac{1}{\alpha(1-\alpha)\,(x_1 - x_2)^2} =$$

$$= \frac{\pi^2}{45} \frac{1}{(x_1 - x_2)^2}. \qquad (6)$$

Then the final result is

$$\phi(x_1 x_2) = \frac{2^{10}}{5} \pi^6 \Lambda^6 \qquad (7)$$

Of the previous calculation we want stress that the integration over the ρ collective coordinate is fully convergent, with the most important contribution coming from instanton size $\rho \sim |x_2 - x_1|$. This gives us some confidence on the instanton calculation since it is perfectly consistent by it self; it respects supersymmetry; is both ultraviolet and infrared stable and stands true to any order in perturbation theory.

The result (7) prove not only that the combination is renormalization group invariant, but also that the composite operator $g\lambda g\lambda(x)$ have this property. This is a very special result but not so surprising since one of the most striking features of supersymmetric field theories is that they are less ultraviolet divergent that their non-supersymmetric counterparts.

Following the general discussion given in Sect.I, when the eq. (7) is combined with the cluster decomposition at large distances $|x_1 - x_2| \rightarrow \infty$ it implies a non zero finite expectation value for the order parameter

$$\langle g\lambda g\lambda \rangle = \left[\lim_{x-y \to \infty} \langle g^2 \lambda\lambda(x)\, g^2 \lambda\lambda(y) \rangle \right]^{1/2} \qquad (8)$$

$$\langle g\lambda g\lambda\rangle = \left[\Phi(x_1, x_2)\right]^{1/2} = \text{[illegible]} \tag{8}$$

As it was recalled in Sect. I, the theory possesses a Z_{2N} invariance (the discrete subgroup of the $U_\chi(1)$ symmetry preserved by instantons). $\langle g^2\lambda\lambda\rangle \neq 0$ breaks the Z_4 symmetry spontaneosly down to Z_2; this implies the existence of two degenerate vacua characterized by plus or minus the value ($\mathcal{S}$) for the gluino condensate. This degeneration is in agreement with the value of the Witten's index [3] $\Delta = N = 2$. The instanton calculation (7) being Z_4 invariant corresponds to averaging over the two vacua.

We want also stress that the gluino condensate does not imply the supersymmetry breaking . Veneziano and Yankielowicz have argued that $\langle\lambda\lambda\rangle \neq 0$ can be embedded in a supersymmetric effective description of the supersymmetric Yang-Mills theory. This effective theory is constructed in terms of an effective chiral multiplet whose components are gauge invariant composite operators. Since $\lambda\lambda$ is the lowest component of this multiplet, it can never be obtained as a commutator of the supersymmetry charge with same other components, then it can have non-zero expectation value, without break supersymmetry.

Now we want make a comparison of the supersymmetric result with the fermionic condensate of the non-supersymmetric QCD.[11]

In QCD when we focus on the problem of chiral symmetry breaking, we look at correlation functions which are finite, as far as the ultraviolet behavoir is concerned; this implies that they depend only on the renormalization group running parameters, such as the running masses $\overline{m}_i(\mu)$ and coupling $\bar{g}(\mu)$. In the massive QCD the combination $m_i \bar{\psi}_i \psi_i$ has a finite expectation value and $\langle \prod_i^{N_F} m_i \bar{\psi}_i(x_i) \psi_i(x_i) \rangle$ goes to zero linearly with the masses; therefore we define

$$\chi = \langle \prod_i^{N_F} \bar{\psi}_i(x_i) \psi_i(x_i) \rangle = \frac{\langle \prod_i^{N_F} m_i \bar{\psi}_i(x_i) \psi_i(x_i) \rangle}{\prod_i^{N_F} \overline{m}_i(\mu)} \qquad (10)$$

in the limit $m_i \longrightarrow 0$. Eq.(10) is expected to be dependent only on the running coupling constant and it is the correlation function which we are interested, since it is symmetryc under $SU(N_f) \times SU(N_f)$. This correlation function receives contribution from the one-instanton sector. In the semiclassical instanton approximation it is given by an integral over the instanton size and position of the instanton measure multiplied by the product of the

N_f fermionic zero modes

$$\chi \sim \int \frac{d^4 y \, d\rho}{\rho^5} (\rho\mu)^{b_1} e^{-8\pi^2/g^2} \rho^{N_F} \prod_i^{N_F} \bar{\psi}_0(x_i)\psi_0(x_i) \tag{11}$$

where ψ_0 is the zero mode

$$\psi_0(x) = \frac{\rho}{[(x-y)^2+\rho^2]^{3/2}} u \tag{12}$$

μ is the subtraction point mass scale and b_1 is the fir[st] coefficient of the QCD β-function

$$\beta = -\frac{g^2}{(4\pi)^2} b_1 - \frac{g^5}{(4\pi)^4} b_2 \tag{13}$$

$$b_1 = \frac{11}{3} N - \frac{2}{3} N_F$$

$$b_2 = \frac{1}{3}(34N - 13N_F)N + \frac{N_F}{N}$$

However the integral over the instanton size cannot be done, due to infrared divergences. A way to avoid these divergences is to formulate the theory on a finite volume as for example on a sphere of radius R. This makes the computation possible and gives for finite R a finite result.

Let us briefly describe the actual computation of the one-instanton contribution to the eq. (10) on the sphere. Really the computation on the sphere follows in the same way in of the flat space one, except that the eigenvalues of the quantum fluctuation ~~operators are those~~ computing on the sphere of fixed radius R. We can express this calculation in terms of the flat one making a Weyl transformation of the metric

$$g_{\mu\nu} = \Omega^2(x)\, \hat{g}_{\mu\nu} = \left(\frac{4R^2}{x^2+4R^2}\right)^2 \delta_{\mu\nu}$$

to a flat metric $\hat{g}_{\mu\nu} = \delta_{\mu\nu}$ together with a suitable trasnformation of the fields. Since the Weyl symmetry is broken by quantum effects, the flat instanton measure becomes multiplied by a factor

$$e^{\frac{b_1}{16\pi^2}\int d^4x \, \mathrm{Tr}\left(F_{\mu\nu}^{\,inst}\right)^2 \lg \Omega(x)}$$

Then

$$\left\langle \prod_i^{N_F} \bar{\Psi}_i(x_i)\, \Psi_i(x_i) \right\rangle_R = c\, g^{-4N} e^{-\frac{8\pi^2}{g^2}} \int \frac{d^4y\, d\rho}{\rho^5} (\rho\mu)^{b_1}$$

$$\times\, e^{\frac{b_1}{16\pi^2}\int d^4x \, \mathrm{Tr}\left(F_{\mu\nu}^{\,inst}\right)^2 \lg \Omega(x)} \rho^{N_F} \prod_i^{N_F}\left[\bar{\Psi}_0(x_i)\, \Psi_0(x_i)\, \Omega^{-3}(x_i)\right]$$

where Ψ_0 is the flat zero mode (12). Then we introduce a set of O(5) covariant parameters

$$q^\mu = \frac{-4R^2 y^\mu}{(y^2+\rho^2+4R^2)}$$

$$q^5 = \frac{(y^2+\rho^2-4R^2)}{(y^2+\rho^2+4R^2)}\, R.$$

In terms of these variables the measure becomes

$$\frac{d^4 y\, d\rho}{\rho^5} = 2\,dz\,\frac{(1-z)^6}{z^3}\,d\Omega$$

where z is

$$z = \frac{R-\rho}{R+\rho}.$$

and $d\Omega$ is the element of the solid angle in five dimensions.

$$\frac{1}{16\pi^2}\int d^4x\,\mathrm{Tr}\left(F_{\mu\nu}^{\,inst.}\right)^2 \lg \Omega(x) = -\frac{1}{(1-z)^3}(3z-1)\lg z + 2z(1-z)$$

$$\bar{\Psi}_0(x_i)\,\Psi_0(x_i) = \frac{z^2}{2R^6}\left[z+1+(1-z)\cos(\theta-\theta_i)\right]^{-3}$$

where we have chosen the position x_i on a common great circle and the θ_i parametrize these positions. Putting everything together after same algebra, we find

$$\left\langle \prod_i^{N_F} \bar{\Psi}_i(x_i)\,\Psi_i(x_i)\right\rangle = C\,\bar{g}^{-4N} e^{-\frac{8\pi^2}{\bar{g}^2}} R^{-3N_F}\int dz\,\frac{(1-z)^4}{z^3}\,4\pi^2$$
$$\times\, e^{\frac{b_0}{2}F(z)}\, z^{3N_F/2}\int_0^\pi \sin^3\theta\,d\theta \prod_i^{N_F}\left[z+1+(1-z)\cos(\theta-\theta_i)\right]^{-3} \qquad (14)$$

where

$$F(z) = \frac{1}{(1-z)^3}\left[(z^3-3z^2-3z+1)\lg z + 4z(z-1)\right]$$

the integrand in eq. (14) can be done numerically for fixed positions, for example chosing the θ_i equally spaced on the circle, since it is slightly dependent on the positions x_i.

$\bar{g}^2(\Lambda_{QCD}R)$ is the running coupling constant

$$\frac{16\pi^2}{\bar{g}^2} = -2b_1 \lg(\Lambda_{QCD}R) + \frac{b_2}{b_1} \lg\left(-2\lg(\Lambda_{QCD}R)\right) \qquad (15)$$

and Λ_{QCD} is the momentum scale involved in the renormalization procedure which must be done in order to cure the ultraviolet divergences.

In particular when $N_f = N$, b_1 is equal to $3N$ and, at the one loop order the R-dependence of $\bar{g}^2$ coming from eq. (15) cancels exactly the factor R^{-3N} in eq. (14), so the result becomes indepentent on R, i.e. in this case there are not infrared divergences at the one-loop order and the result

$$\langle \prod_i^{N_f} \bar{\psi}_i(x_i)\, \psi_i(x_i) \rangle_R = \text{Factor} \times \Lambda_{QCD}^{3N}$$

remains the same also in the limit $R \longrightarrow \infty$, making possible a definition of the fermionic condensate as

$$\langle \bar{\psi}\psi \rangle = \left[\lim_{R\to\infty} \langle \prod_i^{N_f} \bar{\psi}\psi \rangle\right]^{1/N_f}.$$

However when we go at two loop order we have logarithmic corrections coming from the substitution of the expression of $\bar{g}^2$ at the two-loop order (15) in the classical term as well as from the R-dependence of $\bar{g}^{-4N}$. Unlike the supersymmetric case where these two contributions cancel and the gluino-gluino condensate is stable at all higher orders corrections (indeed it is exact), in this case they do not cancel and there is a dependence of the quark condensate on log R

$$\langle \prod_i^{N_f} \bar{\Psi}_i(x_i)\, \Psi_i(x_i) \rangle = \text{Factor} \times \Lambda_{QCD}^{3N} \left[\lg (R \Lambda_{QCD}) \right]^{-2N - \frac{7N^2}{6N}}$$

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