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SELF-GRAVITATING ACCRETION DISKS

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Introduction

Accretion disks around compact objects have become popular models for some of the most luminous X-ray stars (Pringle and Rees 1972; Shakura and Sunyaev 1973; Novikov and Thorne 1973) and quasars (Lynden-Bell 1969). This is because they provide a method of redistributing angular momentum among the gas particles, orbiting around compact objects, in order to let some of them fall into the potential of these objects. The high rate of efficiency, $(10 \div 40)\%$, with which rest mass energy can be obtained, makes the accretion process in accretion disks an efficient converter of rest mass to radiation.

In dealing with accretion disks around compact objects we have to consider two gravity sources; one due to the compact object and another due the disk itself. It is, however, known that disk self-gravity is not significant around compact objects of stellar mass $(1 \div 10)M_0$, but it is for more massive compact objects $(10^6 \div 10^{10})M_0$. In the latter case the self-gravity of the disk becomes important in determining the vertical pressure support of the disk and, consequently, the temperature, the density and the energy generation rate in the accretion disk.

The early interest in the theory of self-gravitating plane symmetric configurations of matter was in modelling Saturn's rings and developing models for spiral galaxies. In recent years it has been suggested by many authors that disks of this kind can form in the nuclei of active galaxies.

Paczynski and his collaborators (Paczynski 1978a,b; Paczynski and

Jaroszynski 1978; Kozlowski, Wiita and Paczynski 1979) have examined the structure of self-gravitating accretion disks around supermassive black holes, giving arguments for their possible relevance as the powerhouse of quasars and active galaxies. In the first paper in this series Paczynski (1978a) developed a model where turbulence is driven by local gravitational instabilities due to self-gravity.

In this model the disk remains close to instability and heat is generated through turbulent viscosity which also makes possible the accretion of matter onto the central object. In this paper an analytic model is constructed using a polytropic equation of state together with a surface mass density-surface brightness relation (analogous to the mass-luminosity relation for stars) that is only valid when radiation pressure dominates over gas pressure and electron scattering provides the great bulk of the opacity. The above conditions are not totally satisfied, even for the extreme parameters used in a specific quasars model. In a following paper (Kozlowski, Wiita and Paczynski 1979) a more realistic equation of state and opacities is used which results in a complicated surface mass density-surface brightness relation.

All these models have been built with the fundamental assumption that in some circumstances, the disk may hover on the brink of gravitational instability. This is because in these disk models gravitationally unstable clumps are sufficiently opaque that collapse is possible only on a thermal time scale much longer than the disk's dynamical time scale. This physical regime gives rise to turbulence that, if violent enough, produces a feedback mechanism to heat the disk (increase the thickness) and switch off the instability. In this sense, it has been required that self-gravity be marginally important and the time averaged disk will have a thickness corresponding to marginal stability. Such a possibility has also been envisaged for galactic disks (Goldreich and Lynden-Bell 1965a,b; Marcus et al.1977; Toomre 1964; Toomre 1977; Hohl 1978).

A comparison of Paczynski's analytic results, and what we can expect from simple arguments of dynamic and thermal stability, shows a disagreement in the dependence of the surface mass density with the other disk parameters ($\Sigma = \Sigma(\dot{M})$), in a physical regime of marginal stability. This disagreement leads us to study the equilibrium and stability of self-gravitating accretion disks around compact objects. The main purpose to clarify the general criterion for the onset of instability of Paczynski's (1978a) model and thus to get a better idea of the regime of marginal stability.

In the first part of this thesis (Chapter I) we discuss briefly the models for self-gravitating accretions disks due to Paczynski (1978a) and Kozlowski, Wiita and Paczynski (1979). In the second part (Chapter II) starting from a discussion of these models we consider the stability of self-gravitating configurations, following in particular works by Hunter (1963) and Goldreich and Lynden-Bell (1965a,b).

In his paper Hunter (1963) gives a procedure for examining the stability of a self-gravitating disk of finite radial extent. He then applies this method to the case of a disk of infinitesimal thickness in the vertical direction.

In the last part of the thesis we discuss how to extend Hunter's stability analysis to the case of a disk of finite thickness around a compact object.

CHAPTER I

A MODEL FOR SELF-GRAVITATING DISKS AROUND COMPACT OBJECTS

Introduction

In this chapter we discuss a model of self-gravitating accretion disks (Paczynski 1978a ; Kozlowski, Wiita and Paczynski 1979) in which the accretion is driven by local gravitational instabilities. The main interest in this model is due to the possibility of obtaining, in particular physical conditions, analytical results. These turn out to be relevant only in models of accretion disks around supermassive black holes in centres of active galaxies.

In the first part of the chapter we outline the basic equations and assumptions of the model. Results obtained considering a disk model in which radiation pressure dominates over gas pressure and electron scattering is the dominant opacity mechanism. In the second part we discuss, in some detail, numerical results obtained using a more general equation of state, for a mixture of gas and radiation.

§ 1 Basic assumptions and relevant parameters

Let us consider an optically thick, self-gravitating accretion disk around a compact object of mass M , and follow its temporal evolution due to radiative losses. This cooling leads to contraction of the disk in the vertical or "z" direction (we will use polar cylindrical coordinates (z,r,φ) centered at the accreting compact object). This implies an increase of the volume mass density of the disk matter, until it becomes comparable with the value M/r^3 ; at this point local gravitational instabilities may develop and affect subsequent evolution.

Since the structure is optically thick, we assume the cooling time scale is longer than the rotational period. Thus the gravitational instability cannot produce local condensations, but only an increase in the velocity dispersion of the particles (Toomre 1964). As a consequence the disk height will increase and the instability will turn itself off. The switching on and off of the instability and the oscillation in disk height constitues a cycle.

Paczynski (1978a) considered an accretion disk with the above cycle, and studied a marginal stable stationary disk configuration. Here the turbulent viscosity, induced by the local gravitational instabilities, leads to accretion. In this case it is possible to adapt the "standard" theory of stationary turbulent accretion disks (Shakura and Sunyaev 1973) to obtain the usual surface brightness relation:

$$F_s = \dot{M} \frac{3}{8\pi} \frac{GM}{r^3} \left[1 - \left(\frac{r_0}{r} \right)^{1/2} \right] \qquad (1.1.1)$$

where $\dot{M}$ is the accretion rate and r_0 is the radius of the inner boundary of the disk.

To make explicit the effect of self-gravity on the disk quantities, Paczynski (1978a) defines a new parameter, A , which equals the ratio of the acceleration ,in the z direction, due to disk self-gravity, g_s , to the acceleration, in the z direction, due to the external compact object, g_z :

$$A = \frac{g_s}{g_z} = \frac{2\pi\Sigma_z r^3}{Mz} \tag{1.1.2a}$$

He also defines values of A in the central plane and on the disk surface:

$$A_c = \left(\frac{g_s}{g_z}\right)_{z=0} = \frac{4\pi\rho_c r^3}{M} \tag{1.1.2b}$$

$$A_s = \left(\frac{g_s}{g_z}\right)_{z=z_0} = \frac{2\pi\Sigma\, r^3}{Mz_0} \tag{1.1.2c}$$

where $\Sigma = 2\int_0^{z_0}\rho\, dz$ is the surface mass density of the disk. The quantity A defined by Eq. (1.1.2) is treated as a parameter constant for a particular disk model.This is a restrictive assumption since, in general, for a given mass M of the central object, we believe the influence of self-gravity increases with the radial distance from the central object, while the influence of gravity due to the central object decreases. Consequently it is often considered preferable to use equations for the conservation laws and vertical pressure balance, to solve for A as a function of the radial position within the disk (Sakimoto and Coroniti 1981; Shore and White 1982).

In Paczynski's (1978a) model we need to specify a physically relevant value for the parameter A. It is possible to obtain a range of possible values for A_s , in a regime of marginal stability, using the results of Goldreich and

Lynden-Bell (1965a,b). They drive the equation for the wave numbers associated with marginal stability which in the standard form reads

$$\frac{\pi G \rho_m}{4\Omega^2} = F(2kz_0) \qquad (1.1.3)$$

where k is the wave number, $\rho_m = \Sigma/2z_0$ is the mean density and Ω is the angular velocity of the matter in the disk configuration.
The functional dependence $F(2kz_0)$ has been calculated by Goldreich and Lynden-Bell (1965a) for three different values of the polytropic index γ : $\gamma = 1$, $\gamma = 2$, $\gamma = \infty$. These results are listed in the following table:

TABLE

	$\gamma = 1$	$\gamma = 2$		$\gamma = \infty$
		$Z = \pi/2$	$Z = 1$	
$\left(\frac{\pi G \rho_m}{4\Omega^2}\right)_{crit}$	0.73	1.11	1.31	1.75
$(2kz_0)_{crit}$	1.4	0.97	0.82	0.61

where Z is a dimensionless variable proportional to the disk thickness; $Z = \pi/2$ is the largest value for Z and corresponds to the case of a zero pressure halo over the disk structure.
In the case of Keplerian angular velocity of the disk matter we have:

$$\frac{\pi G \rho_m}{4\Omega^2} = \frac{\pi}{8} \frac{\Sigma r^3}{z_0 M} \simeq \frac{1}{16} A_s \qquad (1.1.4)$$

From values in the above table we can estimate $A_s \simeq (20 \div 30)$. In numerical calculations we can use $A_s = 25$.
From Eqs.(1.1.2) we have an interesting relation between A_c and A_s :

$$\frac{A_c}{A_s} = \frac{\rho_c}{\rho_m} \qquad (1.1.5)$$

To calculate A_c we need to know before the functional ratio ρ_c/ρ_m. This can be found by integrating equations describing the vertical structure of the disk:

$$\begin{cases} \dfrac{dP}{dz} = -\ (g_z+g_s)\ \rho = -\left(\dfrac{GM}{r^3}\ z + 2\pi G\Sigma_z\right)\ \rho \\ \left(d\Sigma_z/dz\right) = 2\rho \end{cases} \qquad (1.1.6)$$

With the boundary conditions:

$$\begin{aligned} \Sigma_z = 0 \qquad & \text{at} \quad z = 0 \\ P = 0\ ,\ \rho = 0 \qquad & \text{at} \quad z = z_0 \end{aligned} \qquad (1.1.7)$$

§ 2 Disk with polytropic equation of state

In general, Eqs. (1.1.6) with the boundary conditions (1.1.7) and a given equation of state have to be integrated numerically. In the case of a polytropic relation between pressure and density:

$$P = K\ \rho^{1+1/n} \qquad (1.2.1)$$

analytic solutions exist in the two limit cases of dominant and negligible self-gravity. In this cases it is possible to obtain useful relations between ρ_c/ρ_m , A_c and the polytropic index n ; it is found that $A_c \simeq 65$ for $A_s \simeq 25$ and $n = 3$.

§ 3 Disk with radiation pressure >> gas pressure

At this point it is interesting to discuss results obtained (Paczynski 1978a) by making the additional assumption that electron scattering dominates the opacity and radiation pressure is much larger than gas pressure. This regime known to exist in supermassive spherical stars (Wagoner 1969), and in the innermost region of the "standard" (Shakura and Sunyaev 1973) accretion disk model, it is assumed by Paczynski (1978a) to characterize the whole disk structure. The self-consistency of this assumption can be seen later by estimating the value of $\beta = P_{gas}/P_{tot}$ and comparing the implications of this result with those for spherical stars (Chandrasekhar 1939; Paczynski and Rozyczka 1977).

If $P_{rad} \gg P_{gas}$ and $x = x_{es} = 0.2(1+X)$ it is possible to adapt for the heat flux from the disk surface:

$$F = bT_e^4 = \frac{c}{x} g =$$

$$= \frac{c}{x} (A_s+1) \frac{GM}{r^3} z_0 = \left[\frac{2\pi Gc}{x} \frac{(A_s+1)}{A_s} \right] \Sigma \qquad (1.3.1)$$

where T_e is the effective temperature of the disk photosphere, c is the speed of light and g is the surface gravity.

A comparison of Eq. (1.3.1) with Eq. (1.1.1) permits to obtain useful analytical expressions for the disk quantities: z_0 , Σ , T_e , F , g .

$$\Sigma = \frac{x}{2\pi Gc} \frac{A_s}{A_s+1} F_s =$$

$$= \dot{m} \frac{GM}{r^3} \frac{3A_s}{2\pi(A_s+1)} \left(\frac{r_0}{r} \right)^3 \left[1 - \left(\frac{r_0}{r} \right)^{1/2} \right] \qquad (1.3.2)$$

where F_s is given by relation (1.1.1); $\dot{m}$ is a dimensionless parameter defined as the ratio $\dot{M}/\dot{M}_{crit}$ with $\dot{M}_{crit}$ equal to the accretion rate necessary for the disk luminosity to be equal to the Eddington critical luminosity: $L_{edd} = \frac{4\pi Gc}{\chi} \dot{M}$

From Eq. (1.3.2) we have $\Sigma \propto \dot{M}$. This is the important result of Paczynski's (1978a) model, showing a dependence of Σ on $\dot{M}$ even when the disk is marginally stable. However in the next Chapter, we see that the condition of *marginal stability* for the disk is $\left(d\ln\Sigma/d\ln\dot{M} \right)_{h=const} = 0$ This disagreement with the above result $\Sigma \propto \dot{M}$ highlights the need to derive carefully the criterion for the onset of instability and study the stability of self-gravitating disks extended around compact objects.

We test, following Paczynski (1978a) approach, the goodness of the used assumption about radiation pressure.The idea is to estimate the quantity β starting from a polytropic equation of state for the gas plus radiation system:

$$P = \frac{k}{\mu H} \rho T + \frac{a}{3} T^4 = K \rho^{4/3}$$

$$K = \left[\frac{3}{a} \left(\frac{k}{\mu H} \right)^4 \frac{1-\beta}{\beta^4} \right]^{1/3} \qquad (1.3.3)$$

Then the value of $K = P_c/\rho_c^{4/3}$ (with P_c and ρ_c the pressure and the mass density on the plane $z = 0$) estimated with the disk quantities permits us to obtain an expression for $(1-\beta)/\beta^4$ that can be compared with the corresponding one known to exist for spherical stars:

$$\left(\frac{1-\beta}{\beta^4} \right)_{max} = 5.2\ 10^{-12}\ \dot{m}^6\ m^2 \quad \text{for a disk}$$

$$(1.3.4)$$

$$\left(\frac{1-\beta}{\beta^4} \right) = 4.51\ 10^{-4}\ m^4 \quad \text{for a spherical star (Chandrasekhar 1939)}$$

We see that gas pressure is equal to radiation pressure ($\beta = 0.5$)

in a spherical star with a mass $m = M/M_0 = 133$ (M_0 is the sun mass). For a self-gravitating disk with $\dot{m} = 0.1$ we have $\beta = 0.5$ for $m = M/M_0 = 1.2\ 10^9$, of the central black hole.

It is clear that the only possible application of such models is to active galaxies or quasars. However even if we consider: $M = 10^{10} M_0$ and $\dot{m} = 0.1$ it is possible to obtain a minimum value of β equal to 0.2; such value is obtained in a spherical star of $M = 10^3\ M_0$ (Eq. (1.3.4)).

From Paczynski and Rozyczka (1977), we have that such a supermassive star with $M = 10^3\ M_0$ has surface luminosity a factor of three below the critical value. Hence this extreme star is not massive and luminous enough to justify the use of the asymptotic surface mass density-surface brightness relation given by Eq. (1.3.1).

§ 4 Equation of state for gas plus radiation: results

We discuss important results which can be obtained using a more general equation of state for a mixture of gas plus radiation; in this case we follow a numerical integration of equations describing the vertical structure of the disk. This has been done by Kozlowski, Wiita and Paczynski (1979) considering an equation of state of the form:

$$P = n_t kT + aT^4/3 \qquad (1.4.1)$$

where n_t is the total number of particles for unit volume.

In this case we cannot make use of the simple relationship, given by Eq.(1.2.1), for the surface brightness, F , and the surface mass density, Σ ; on the contrary we have to solve for the equations of the vertical disk structure supplemented with:

$$\begin{aligned} dL_z &= \epsilon \, d\Sigma_z \\ \frac{dT}{dr} &= - \frac{3}{4ac} \frac{x\rho}{T^3} L_z \end{aligned} \qquad (1.4.2)$$

The first equation relates the luminosity, L_z , with the energy deposition rate, ϵ ; the second gives the radiative temperature gradient. The opacity, x , is taken from Paczynski (1970).

Along with the boundary conditions given by Eq.(1.1.7) we add:

$$L_z(z_0) = F \qquad T(z_0) = (4F/ac)^{1/4} \qquad (1.4.3)$$

Now we can solve Eqs.(1.1.6) and (1.4.2), given Eq.(1.4.1) and the opacity law, for Σ_z , $P(z)$, L_z and $T(z)$. The presence, however, of five boundary conditions allows us to consider this as an eigenvalue problem for the emergent flux, F . Results of this procedure are summarized in **Fig.1** ; from this figure we can see how the critical regime, marked by the curve $F = F_{edd}$,is approached by each curve (characterized by a given value of $D = M/r^3$) only for large value of Σ . A decrease of Σ is followed by an increase of the distance between each curve and the Eddington limit, until for very low values of Σ, the shape of each curve is characterized by the onset of a deep convection.

Since the curves are similar, Kozlowski, Wiita and Paczynski (1979) have derived from them an empirical surface mass density – surface brightness relation:

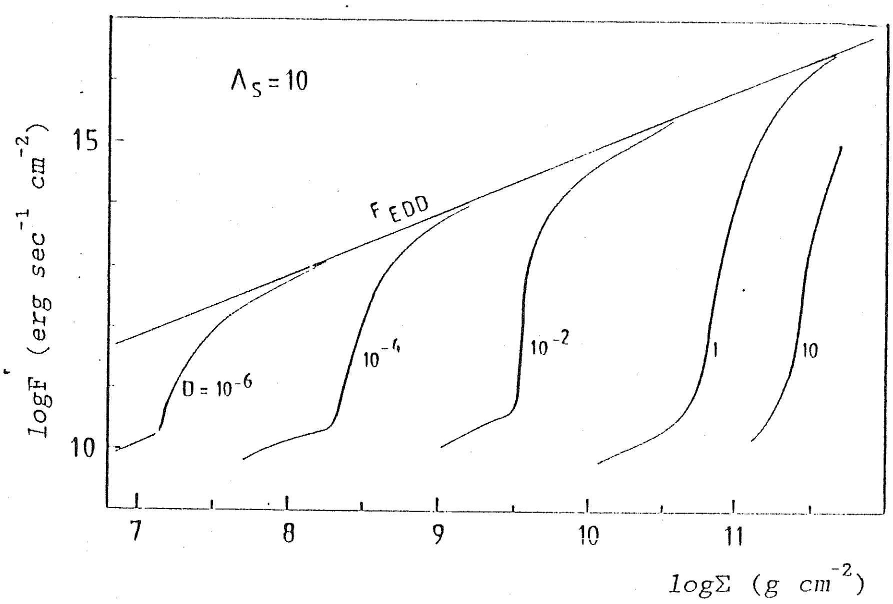


Fig.1 (Kozlowski, Wiita and Paczynsky 1979)

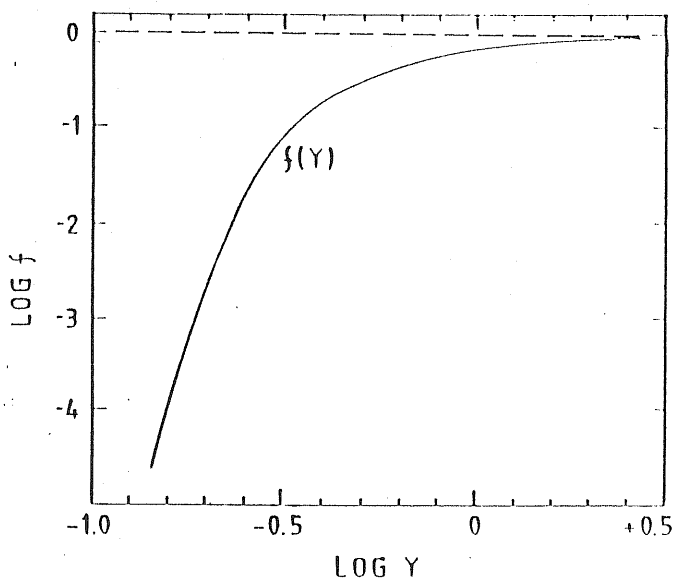


Fig.2 (Kozlowski, Wiita and Paczynsky 1979)

$$F = \max \begin{cases} \dfrac{4\pi Gc\left(A_s+1\right)\Sigma}{2xA_s} f(y)=F_{edd}f(y) \\ \\ \dfrac{3\Sigma}{2(A_s D)^{0.53}} \end{cases} \qquad (1.4.5)$$

where $$y = 10^{-11} \frac{\Sigma/2}{(A_s D)^{0.65}} \qquad (1.4.6)$$

and the function $f(y)$ is shown in **Fig.2** . We note that all the above results have been obtained for the flux emitted by each ring of the disk , considered as a single entity; to build a complete disk model, it is necessary to put together all these rings. Then we have to replace, in the above expressions, the quantity D with the mass M of the central object. A comparison of Eq.(1.4.5) with (1.1.1), for the emergent flux from the disk surface will give expressions for Σ and z_0 .

Differences between these results and those of Paczynski (1978a) are shown in **Fig.3** . It is evident that the contribution of gas pressure to the equation of state produces an increase either of the the surface mass density either of the disk thickness even when the flux and surface temperature of both model are identical.

The most important result of all this procedure is that the regime of dominant radiation pressure can be obtained only for large values of Σ (for constant D and A_s). Otherwise there is a substantial difference between F_{edd} and the flux obtained using $P = P_{gas} + P_{rad}$.

A lower bound on possible values for Σ comes from the limits of validity of expression (1.4.5); if $\Sigma < 2\ 10^5 gm\ cm^{-2}$ this formula no longer holds since $f(y)$ deviates from plot of **Fig.2** . This requirement together with constraints arising from the basic assumptions (cooling time scale ≫ rotational period; $v_r \ll v_\varphi$ and disk mass ≪ mass of the central object) defines the range of applicability of the model:

$$(10^6 < M < 10^{10})M_0 \qquad (10^{-4} < \dot{M} < 10^{-1})\dot{M}_{crit}$$

This more general and realistic model appears to be of limited relevance to massive objects like quasars and AGNs .

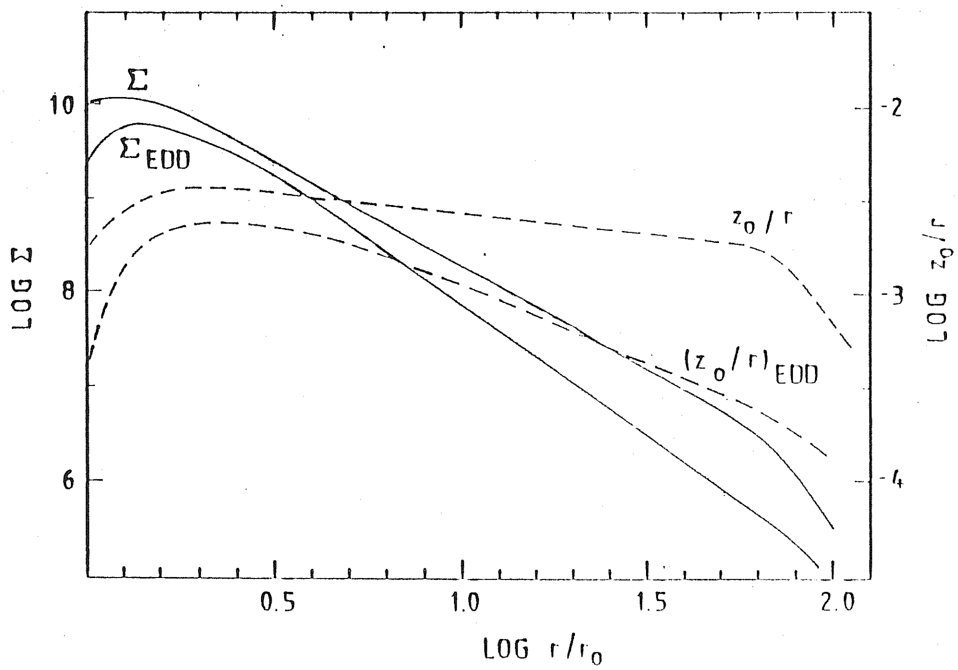


Fig.3 (Kozlowski, Wiita and Paczynsky 1979)

CHAPTER II

STABILITY PROBLEMS IN ACCRETION DISKS

Introduction

We now consider the method and some important results on the stability of non self-gravitating accretion disks and wish to extent this to self-gravitating disks.

The presence of instabilities in accretion disks has been extensively examined in order to explain the periodic, quasi-periodic or chaotic variability observed in some binary X-ray sources, quasars and galactic nuclei. For binary X-ray sources, interest has been focused on interpreting the cyclic outburst behavior of cataclysmic variables in terms of disk instability mechanisms (Smak 1982; Cannizzo *et al*. 1982; Papaloizou *et al*. 1983). In these works a cyclic behavior is reproduced in which the disk alternates between short lived radiative stages of high mass flow and long lived convective states of low mass flow (Meyer and Meyer-Hofmeister 1981 and 1982).

Among possible instabilities of accretion disks, pulsational instability has been considered particularly relevant in explaining periodic optical luminosity variations occurring in a large number of quasar like objects (Ozernoy and Usov 1977).In this direction Kato (1978) has developed a detailed study on the pulsational instability of a non self-gravitating disk considering, primarily, effects of the shear motion on the growth rate of the instabilities. The contribution of self-gravity to the study of pulsations of a disk like configuration has been taken into account by Vila (1979). This work is of interest because it provides results in good agreement with observations and a good approach to investigate stability of a self-gravitating disk in the direction perpendicular to its symmetry plane.

However, much work has been done on the stability of self-gravitating configurations primarily to interpret the Saturnian ring phenomenon (Laplace 1789; Maxwell 1859; Cook and Franklin 1964; Ostriker 1964; Ginzburg *et.al.* 1972; Shukhman 1983), and to develop models for spiral galaxies (Randers 1942; Goldreich and Lynden-Bell 1965b).The explanation of the spiral structure of many galaxies has been one of the outstanding problems of cosmology. The galaxy is thought of as a star-gas mixture, supported by rotation and stellar motion. The spiral arms show up due to the high luminosity of stars in them suggesting that these structure are short lived.

Star formation in spiral arms as the end-product of Jeans' gravitational instability (Toomre 1964) arises the question whether the arms themselves can be due to gravitational instability on a larger scale. To answer this question Goldreich and Lynden-Bell (1965b) have studied how gravitational instability occurs in a differentially rotating structure of finite thickness. The usual approach to stability is the normal-mode one. This consists of considering small perturbations of the system and studying the subsequent linear time-dependent equations of hydrodynamics. Since the problem is linear we can separate the space variable r from time; in this case we search for normal-mode solution for the displacement of the form:

$$\xi(r,t) = \xi(r)\ e^{iwt}$$

Then the equations will provide a dispersion relation from which it is possible to study the sign of ω^2 . It turns out that if $\omega^2<0$ the system will be unstable since perturbations will increase in time; if $\omega^2>0$ the system will be stable. The onset of instability occurs when $\omega^2=0$, the modes corresponding to these values of ω are called "neutral modes".

The system is called dynamically unstable if the characteristic time scale of the instability of the order of the free fall time, $(G\rho)^{-1/2}$, thermally unstable if the time scale is of the order of the Kelvin time, GM^2/LR , (ρ is the volume mass density, M,R and L are respectively the mass, the radial extent and the luminosity of the configuration). The main condition governing the rotation law is the Hoiland criterion which states that a general non-isentropic configuration is dynamically unstable if on each surface of constant entropy, S , the specific angular momentum, $l = |r^2\Omega|$ must be increasing function of r : $dl/dr > 0$.

In the next section we consider stability problems in not self-gravitating geometrically thin accretion disk models. Then we will give a short discussion of Vila's (1979) approach and relevant results. In the last part of the chapter we will consider Hunter's (1963) general approach to the stability of an infinitesimally thin and self-gravitating accretion disk, primarily seen in relation to our problem.

§ 1. **Stability of the "standard" accretion disk model: hints**

The "standard" model (Shakura and Sunyaev 1973; Novikov and Thorne 1973) considers a geometrically thin accretion disk around a Schwarzschild black hole. Self-gravity effects are not taken into account. The disk is usually divided into three different regions. Of these the innermost one is dominated by radiation pressure and electron scattering opacity; in the middle and outer region gas pressure dominates over radiation pressure. Electron scattering opacity dominates in the middle region and free-free absorption in the outer region.

Thin accretion disks are usually considered to be dynamically stable since they can be seen as a collection of stable circular Keplerian orbits. Thus the stability analysis is directed to their thermal plus secular properties. Many researchers (Pringle, Rees and Pacholczyk 1973; Lightman and Eardley 1974; Lightman 1974; Shakura and Sunyaev 1976; Piran 1978) have considered the stability of the "standard" model coming to the same conclusion that the innermost region (where $P_{rad} \gg P_{gas}$) is thermally unstable while the outer regions (where $P_{rad} \ll P_{gas}$) are thermally stable.

Below we look at the general stability problem using the Piran (1978) criterion.

Let us define the following phenomenological parameters:

$$K = \left(\frac{\partial \ln Q^-}{\partial \ln H} \right)_\Sigma \qquad L = \left(\frac{\partial \ln Q^-}{\partial \ln \Sigma} \right)_H$$

$$M = \left(\frac{\partial \ln \nu}{\partial \ln H} \right)_\Sigma \qquad N = \left(\frac{\partial \ln \nu}{\partial \ln \Sigma} \right)_H \qquad (2.1.1)$$

Where H is the thickness of the disk, Σ is the surface mass density, ν is the kinematic viscosity and Q^- is the vertically integrated cooling rate. All these quantities are defined and related each others by the following equations:

1) $P = P(\rho, T)$	pressure	material equations
2) $x = x(\rho, T)$	opacity	
3) $\nu = \alpha v_s H$	viscosity	
4) $H/r = v_s/v_k$	thickness	structure equations
5) $\rho = \Sigma/H$		
6) $v_s^2 = P/\rho$		
7) $F^+ = \frac{1}{2} Q^+ = \frac{3GM\dot{M}}{8\pi r^3} \left[1 - \left(\frac{r_{in}}{r} \right)^{1/2} \right]$		thermal balance
8) $F^- = \frac{1}{2} Q^- = \frac{acT^4}{3\rho H x}$		
9) $\dot{M} = 2\pi r \Sigma (-v_r)$		conservation of mass and momentum
10) $\nu\Sigma = \frac{\dot{M}}{3\pi} \left[1 - \left(\frac{r_{in}}{r} \right)^{1/2} \right]$		

It is possible to study the temporal behavior of the disk from the two time equations:

$$\frac{\partial \Sigma}{\partial t} = \frac{9}{r} \frac{\partial}{\partial r} (\Sigma\nu) + 6 \frac{\partial^2}{\partial r^2} (\Sigma\nu) \qquad (2.1.2)$$

$$\frac{\partial E}{\partial t} = \frac{P}{\rho^2} \frac{d\rho}{dt} + \frac{Q^+}{\Sigma} - \frac{Q^-}{\Sigma} \qquad (2.1.3)$$

where E is the internal energy per gram of matter; Eq.(2.1.2) expresses combined mass and angular momentum conservation; while Eq.(2.1.3) is due to thermal energy conservation.

These equations are linearized assuming perturbations with wavelength λ : $H_0 < \lambda < r$ (subscript zero denotes equilibrium values; r is the radial distance from the central compact object) so as to keep only terms of order $(H_0/\lambda)^2$. We find the dispersion relation from which the necessary conditions for a stable disk are:

$$M < K \qquad (2.1.4a)$$

$$M\,L < (N + 1)\,K \qquad (2.1.4b)$$

condition (2.1.4b) is general and condition (2.1.4a) holds only in the long-wavelength limit.

Piran (1978) has given a physical interpretation of these conditions, showing mathematically that the first equation gives a condition for thermal stability, while the second gives a condition for the secular L-E instability. This latter oscillates even when $Q^+ = Q^-$ and produces the disk to break up into a set of concentric dense rings.

In the "standard" model: $M = 2$, $N = 0$ so conditions (2.1.4) become:

$$K > 2 \quad \text{and} \quad K > 2L \qquad (2.1.5)$$

a computation of K and L gives:

$$K = 4(1+\beta)/(4-3\beta)$$
$$L = -\beta/(4-3\beta) \qquad (2.1.6)$$

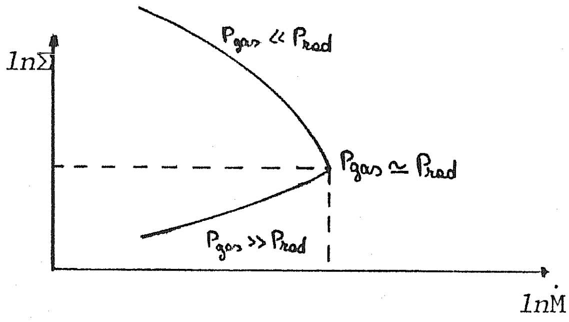
with $\beta = P_{gas}/P_{tot}$.

In the innermost region ($P_{rad} \gg P_{gas}$) we have $\beta = 0$ and $K = 1$ so that condition (2.1.5) does not hold. In the outer region $\beta \approx 1$ and so $K = 8$; condition (2.1.5) is satisfied.

It is interesting to see the dependence of Σ on $\dot{M}$ in the stable and unstable regions; solving Eqs.1)-10) we find:

$$\Sigma \propto \dot{M}^{(3/5)} \qquad P_{gas} \gg P_{rad}$$
$$\Sigma \propto \dot{M}^{-1} \qquad P_{gas} \ll P_{rad} \tag{2.1.7}$$

This dependence is shown in **Fig.4** , in which a sequence of solutions are represented for different disk models, but with the same (r, α, M).



In the turning point

$$\frac{\partial ln\Sigma}{\partial ln\dot{M}} = 0 \tag{2.1.8}$$

Fig.4

It is possible to show that near the turning point we have:

$$\frac{d ln\Sigma}{d ln\dot{M}} = 5\,\frac{\beta - 2/5}{2+3\beta} \tag{2.1.9}$$

Using Eqs.(2.1.6) and the stability condition (2.1.4) we obtain stability when $\beta > 2/5$; this result together with (2.1.9) gives instability where $P_{rad} \gg P_{gas}$.

The turning point, corresponding to a situation of marginal stability, is located at $\beta = 2/5$.

§ 2.Stability of self-gravitating systems

The theory of gravitational stability of a compressible medium was developed to describe large scale objects in the Universe. Jeans (1928) was the first to investigate the stability of an infinite homogeneous distributions of matter. The homogeneous case is simple since the spatial dependence of perturbations may be chosen in the form of a plane wave with a wave vector k (every perturbed quantity $\propto e^{ikr}$). The initial differential equations then become a set of algebraic equations. By equating the determinant of this system to zero, we obtain the following dispersion equation:

$$\omega^2 = -\omega_0^2+k^2c^2 \tag{2.2.1}$$

where $c = (\partial P_0/\partial \rho_0)^{1/2}$ is the sound velocity in the medium.
From Eq.(2.2.1) it follows that there is a critical wavelength (also called Jeans wavelength)

$$\lambda_{cr} = \frac{2\pi}{k_{cr}} = c\left(\frac{\pi}{G\rho_0}\right)^{1/2} \tag{2.2.2}$$

at ($\lambda < \lambda_{cr}$) we have stable perturbations and at ($\lambda > \lambda_{cr}$) we have the unstable ones.

For $\lambda \ll \lambda_{cr}$, the resulting small wavelength perturbations propagate like sound wave:

$$\omega^2 \approx k^2c^2 \tag{2.2.3}$$

For an infinite uniform, uniformly-rotating system Chandrasekhar (1961) proved that Jeans' criterion is unaffected by rotation except for a set of modes with wave-vectors exactly perpendicular to the rotation axis. For this "singular mode" his criterion is

$$k^2c^2 > 4\pi G\rho-4\Omega^2 \tag{2.2.4}$$

Goldreich and Lynden-Bell (1965a,b) have made a detailed study of the stability of a stratified self-gravitating rotating medium, discussing the two cases of uniform (1965a) and differential (1965b) angular velocities. In the first paper they have shown how Chandrasekhar's (1961) result can be modified considering a disk with a finite thickness. In this case there are two critically unstable modes (if rotation is not too fast) for a uniformly rotating disk. We see from Fig.5 that disturbances with wave vectors between the two critical values are unstable, while others are stable.

To see how strongly dependent this behavior is on the finitenes of disk thickness, consider a uniformly rotating isothermal disk with scale height H . A region of this disk with horizontal scale k^{-1} is gravitationally unstable if its self-gravitational energy is greater then the sum of its internal plus its rotational energies. The internal energy per unit mass is approximately:

$$c^2 = (\partial P/\partial \rho) \quad , \qquad \text{the sound velocity of the gas}$$

The rotational energy per unit mass is approximately:

$$\frac{1}{2}\frac{I}{M}\Omega^2 \approx \frac{1}{2}k^{-2}\Omega^2$$

The gravitational energy per unit mass is given by the gravitational potential ψ which can be obtain integrating the Poisson equation, $\nabla^2\psi = -4\pi G\rho$:

$$\psi = \frac{4\pi G\rho}{k^2+H^{-2}}$$

Set

$$\frac{4\pi G\rho}{k^2+H^{-2}} = \frac{\Omega^2}{k^2} + c^2$$

Then in the limit for $H \Rightarrow \infty$ we obtain, apart from a numerical coefficient Chandrasekhar's (1961) result.

If, on the other hand, H is finite we have:

$$(k^2c^2)^2 + k^2c^2\left[\frac{c^2}{H^2} + \Omega^2 - 4\pi G\rho\right] + \frac{\Omega^2c^2}{H^2} = 0 \qquad (2.2.5)$$

If Ω is sufficiently small there two positive roots of this equation for k^2c^2 . This is just the Goldreich and Lynden-Bell

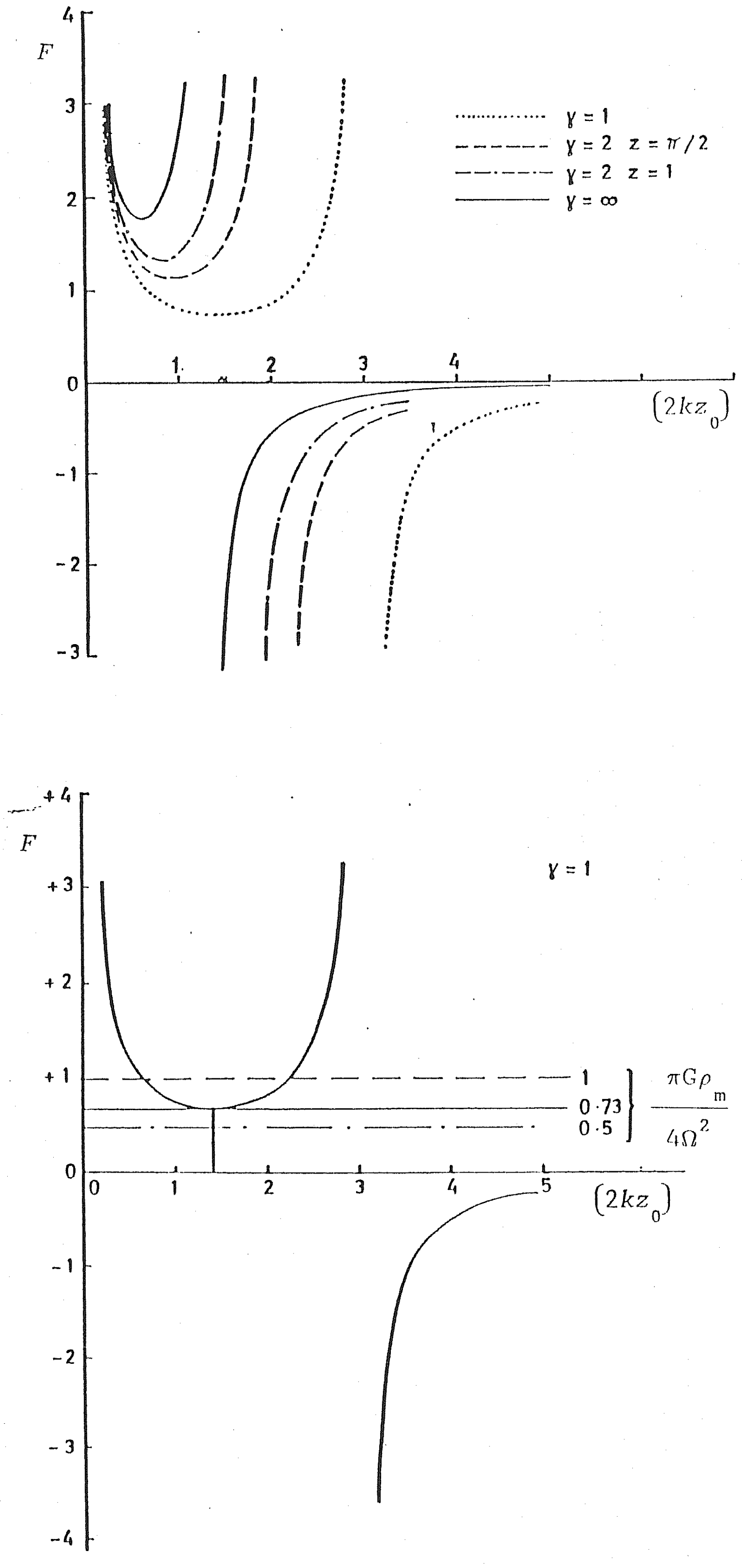


Fig.5 (Goldreich and Lynden-Bell 1965a)

result apart for numerical corrections to the coefficients. They have found that in all cases pressure effects stabilize short waves, while long waves are stabilized by rotation. A useful dimensional analysis of the stability of an infinitesimally thin stellar disk has been given by Toomre (1964) too. He has considered the combined effects of random motions and rotation.

Rotation effects

Following Toomre (1964) consider, in the disk, a local condensation of length scale L , involving changes of relative magnitude ϵ.

If Σ_{loc} is the local surface density, the mass involved in the condensation is $O(\Sigma L^2)$ and the particles displacement is $O(\epsilon L)$ so that the produced extragravitational forces per unit mass are:

$$O(G\ \Sigma L^2 \epsilon L/L^3) = O(\Sigma G \epsilon)$$

On the other hand, in the zero temperature approximation with pressure neglected, the only possible forces to prevent a condensation from growing are the inertial forces. They are $O(\omega_{loc}$ x perturbed velocity$) = O(\epsilon L \omega^2_{loc})$, where ω_{loc} is the angular velocity of the disk matter at a given distance from the rotation axis.

We have stability if:

$$\Sigma G \epsilon \ll \epsilon L \omega^2_{loc}$$

$$L \gg L_{crit} = (\Sigma/\Sigma_{mean})(\omega/\omega_{loc})^2 R \qquad (2.2.6)$$

where Σ_{mean} is the mean surface mass density of a configuration of radius R and angular velocity ω of revolution around its center. Eq.(2.2.6) indicates that, when all stabilizing influences other than rotation are absent the length scale, which divides the unstable disturbances from those that are stable, is apparently of the same order of magnitude as the radius of the configuration. So the disk must be unstable with respect to perturbations of all wavelengths.

Random motion

The random movements of particles inside the disk configuration can be thought either as a diffusive mechanism or as pressure. In any, case there is little doubt that these motions will on the whole tend to suppress any given instability. In absence of rotation and random motions, the characteristic time of growth of a disturbance is of the order of

$$(L/G\Sigma)^{1/2} > \langle u^2\rangle^{-1/2} L$$

$$L < L_j = \langle u^2\rangle/G\Sigma \qquad (2.2.7)$$

We now see that the combined influence of a rotation and a superposed velocity dispersion confines the unstable disturbances to a certain intermediate range of wavelengths which disappears as the velocity dispersion is increased. The minimum velocity dispersion for complete stability roughly corresponds to $L_{crit} = L_j$

$$\langle u^2\rangle^{+1/2}_{min} = G\Sigma/\omega_{loc}$$

$$\approx \omega R\ (\Sigma/\Sigma_{mean})(\omega/\omega_{loc}) \qquad (2.2.8)$$

This is the same order of magnitude as the typical (linear) velocity of revolution. The purpose of our work is to do a stability analysis of a self-gravitating accretion disk taking into account both the above discussed effects.

In the next section using Vila's (1979) work, we decide how to investigate stability of the vertical structure of self-gravitating disks. Then we discuss Hunter's (1963) approach to the stability of self-gravitating configurations, which are infinitesimally thin ($z = 0$) and cold ($P = 0$).

In the last section of this chapter we will show how it is possible to use all the above knowledge to formulate a more general stability problem for a geometrically thin ($0 < z < r$) hot ($P \neq 0$) self-gravitating accretion disk around a compact object.

§ 3. Pulsations in self-gravitating accretion disks

Many quasars and AGN_s , are generally characterized by luminosity variations over a large range of time scales: from years to days (Kellermann and Pauliny-Toth 1968) . This rapid variability cannot be explained using Paczynski's (1978a) model, in which local gravitational instabilities may give rise to light variations on a thermal time scale:

$$\tau_{th} = \left(\frac{3}{2} \frac{K}{\mu H} T(z_0) \Sigma \right) \Big/ F \qquad (2.3.1)$$

$$= 2\ 10^2 \text{ years}$$

This time scale, of hundreds of years, is too long compared with observations. Vila (1979) tried to explain this observed rapid variability using a self-pulsating accretion disk model characterized by pulsations perpendicular to the symmetry plane of the disk. We relate this to Paczynski's (1978a) model where the physical regime is one of marginal stability between expansion and contraction of the disk configuration. This is because Vila studied pulsations in the z direction of a self-gravitating accretion disk completely described by Paczynski's (1978a) model. Vila made the following assumption: he considered adiabatic oscillations of a plane-parallel configuration, assuming that it could be considered locally a plane-polytropic layer:

$$P = K\,\rho^{(1+1/n)} \qquad (2.3.2)$$

In this case the hydrostatic equilibrium equation in the z direction takes the form

$$\frac{d^2\vartheta}{d\xi^2} = -a\ -\vartheta^n \qquad (2.3.3)$$

where we have used the relations:

$$z = a\xi = \left[\frac{(n+1)\ K}{4\pi G\lambda^{(1+1/n)}} \right] \xi$$

$$\rho = \lambda\vartheta^n \qquad (2.3.4)$$

$$a = \frac{M_a}{4\pi r^3 \rho_c}$$

The boundary conditions are:

$$\text{at}\quad \xi = 0 \qquad \vartheta = 1 \qquad \frac{d\vartheta}{d\xi} = 0$$

$$(2.3.5)$$

$$\text{at}\quad \xi = \xi_0 \qquad \vartheta = 0 \qquad \left(\frac{d\vartheta}{d\xi} \right) = \left(\frac{d\vartheta}{d\xi} \right)_{\xi=\xi_0}$$

Pulsation can be driven by vertical variations. To analyse this, Vila started from the equation of motion

$$\frac{d^2 z}{dt^2} = -\ g - \frac{1}{\rho}\frac{dP}{dz} \qquad (2.3.6)$$

where $g = 2\pi G\Sigma$

In equilibrium:

In equilibrium:

$$g_0 + \frac{1}{\rho_0}\frac{dP_0}{dz_0} = 0$$

where P_0 is given by the adiabatic relation: $P_0 = K\rho_0^{\gamma}$

The variation of Eq.(2.3.6) can be made taking into account the relation:

$$\Delta = \delta + z_0 z_1 \frac{\partial}{\partial z_0} \qquad (2.3.7)$$

with $\Delta z = z_0 z_1$ = lagrangian displacement

$$\Delta\rho = \rho - \rho_0 = \rho_0\rho_1 \qquad \Delta P = P - P_0 = P_0 P_1$$

Then assuming that the configuration oscillates with a constant local frequency σ so that: $z_1 = \xi_1(z)e^{i\sigma t}$

from Eq.(2.3.6) it is possible to obtain:

$$\frac{d^2\xi_1}{dz^2} = \frac{2-\mu}{z}\frac{d\xi_1}{dz} + \left(\frac{\rho}{\gamma P}\sigma^2 - \frac{\mu}{z^2}\right)\xi_1 = 0 \qquad (2.3.8)$$

where μ depends only on equilibrium values:

$$\mu = \frac{g_0\rho_0 z_0}{P_0}$$

Now applying Eq.(2.3.8) to plain polytropes, making use of the polytropic variables:

$$P = P_c\vartheta^{n+1} \quad , \quad \rho = \rho_c\vartheta^n \quad , \quad z = a\xi$$

it is possible to obtain:

$$\frac{d^2\xi_1}{d\xi^2} + \left[2 + \frac{(n+1)\xi}{\vartheta}\frac{d\vartheta}{d\xi}\right]\frac{1}{\xi}\frac{d\xi_1}{d\xi} + \left[\frac{\omega^2}{\vartheta} + \frac{(n+1)}{\vartheta\xi}\frac{d\vartheta}{d\xi}\right]\xi_1 = 0 \qquad (2.3.9)$$

where $$\omega^2 = \frac{(n+1)\sigma^2}{4\pi\gamma G\rho_e} \qquad (2.3.10)$$

Eq.(2.3.9) with the boundary conditions:

$$z\xi_1 = 0 \qquad \text{at} \quad z = 0 \qquad (2.3.11a)$$

$$\frac{1}{\xi_1}\left(\frac{d\xi_1}{d\xi}\right) = \frac{1}{\xi}\left[-\frac{\omega^2}{(n+1)}\frac{\xi}{\vartheta_1} - 1\right] \qquad \text{at} \quad z = z_0 \qquad (2.3.11b)$$

$$\vartheta_1 = \frac{d\vartheta}{d\xi}$$

Have been integrated by Vila for two values of A_c ($A_c = 65$ and $A_c = \infty$) and for different values of the polytropic index, n. The integration method used consists of integrating Eqs.(2.3.9) and (2.3.3) for a given value of ω^2, in the neighborhoods of two points $\xi = 0$ and $\xi = \xi_R$. The difference in the values of the quantity $\frac{1}{\xi_1}\frac{d\xi_1}{d\xi}$ are evaluated at an intermediate point. The value of ω^2 is then changed to zero this difference. From Eq.(2.3.10) we can then find the frequency σ or, equivalently, the period $\tau = 2\pi/\sigma$ of the pulsation.

In fact once we have obtained a value of ω^2, for a given value of A_c and n, it is possible to get a relation between the period τ and the central density ρ_c of the configuration. Now consider a polytropic equation of state of a system of gas plus radiation. We have $n = 3$ and $\gamma = 4/3$. Then using Paczynski (1978a) estimate: $A_c = 65$, we have from Vila's results:

$$\omega^2 = 2.9712$$

and so

$$\tau = 6.9\, 10^8 \rho_c^{-1/2} \ sec \qquad (2.3.12)$$

This is an important result because it allows us to obtain periods in the observed range, from days to years, by changing only the value of the central density ρ_c. In the case of $A_c = 65$ (Paczynski 1978a):

$$\rho_c = 5.2 \frac{M}{r^3} \qquad (2.3.13)$$

Then different periods can be obtained varying the mass M of the central object and the distance r from it. The possibility of having different periods at different radii can be used to explain quasi-periodic or non-periodic variations observed in many cases through a superposition of the different independent pulsations.

§ 4. Hunter stability-analysis: hints

Hunter (1963) has presented a method for relating the law of rotation and the density distribution of a plane disk of self-gravitating material. The method involves the use of oblate spherical coordinates (ζ, μ, φ) and the superposition of elementary solutions of Poisson's equation due to the distribution of matter in the disk. In this way he has found general expressions for the law of rotation and the density distribution as related series of Legendre functions.

Simple disk models can be obtained considering only a finite number of terms in these series. The simplest model of this type (only the first two terms in the series) has a surface density Σ :

$$\Sigma = \frac{3M_d}{2\pi R_0^2} \left(1 - \frac{r^2}{R_0^2} \right)^{1/2} \qquad (2.4.1)$$

where R_0 is the outer boundary of the disk and M_d is the total mass of the disk:

$$M_d = \int_0^{R_0} 2\pi r \Sigma dr \qquad (2.4.2)$$

This model has a constant angular velocity Ω:

$$\Omega = \left(\frac{3\pi G M_d}{4R_0^3} \right)^{1/2} \qquad (2.4.3)$$

Hunter (1963) has also developed a small-perturbation stability analysis for this particular model by solving the zero pressure hydrodynamics equations of compressible flow. He has found the disk to be unstable to an infinite number of axisymmetric and non-axisymmetric modes of oscillation.

We are interested in Hunter's stability analysis method since, independent of the particular disk model used it provides us with a general procedure able to obtain the gravitational potential on the disk surface. The possible presence of a compact central object would add an important new term in this expression for the potential, and would modify substantially the disk geometry: However the Hunter approach still holds.

He considered the particular case of an infinitesimally thin circular disk, only limited by its outer edge, R_0 ; this configuration is considered to come from a compression in the z direction of a primitive uniform spheroid:

$$\frac{r^2}{c^2+R_0^2} + \frac{z^2}{c^2} = 1 \qquad (2.4.4)$$

where c is a constant.

Hunter supposed the outer edge to undergo small perturbations so that:

$$R = R_0 + \epsilon(\varphi,t) \qquad (2.4.5)$$
$$|\epsilon(\varphi,t)| \ll R_0$$

and considered a perturbed density field of the form:

$$\Sigma = \frac{3M_d}{2\pi R_0^2}\left[1-\frac{r^2}{R^2}\right]^{1/2} + \sigma^* \qquad (2.4.6)$$

where σ^* is small compared with the unperturbed density. Considering R instead of R_0 in Eq.(2.4.6) it is possible to avoid singularities in solutions for σ^* otherwise occurring at the outer edge of the disk.

To obtain an expression for the gravitational potential due to the density distribution:

$$\Sigma = \frac{3M_d}{2\pi R_0^2}\left[1-\frac{r^2}{R^2}\right]^{1/2} \quad \text{for the disk} \quad r = R(\varphi,t)$$
$$(2.4.7)$$
$$\Sigma = 0 \quad \text{outside}$$

Hunter considered a body of uniform density ρ with outer boundary:

$$\frac{r^2}{(c^2+R^2)} + \frac{z^2}{c^2} = 1 \qquad (2.4.8)$$

and, to first order, he regarded it as the uniform spheroid (2.4.4) with a surface distribution of matter of density ρh per unit area; h is the normal displacement (Hunter 1963 Eq.(2.6)) of the body (2.4.8) from the spheroid (2.4.4). This latter can be seen as a level surface, $\zeta = c/R_0$, of the spheroidal coordinates. The gravitational potential due to a distribution of matter on it can be obtained by matching two expansions of separable harmonic functions which represent the gravitational potentials inside, ψ_{int} , and outside, ψ_{ext} , the spheroid (2.4.4).

In this way Hunter obtained:

$$\psi_{int} = \sum_{1,k} B_{1,k}\ p_1^k(\zeta)\ P_1^k(\mu)\ e^{ik\varphi} \tag{2.4.9}$$

with

$$B_{1,k} = \frac{4\pi\rho c G R_0^2 q_1^k(c/R_0) D_{1k}(t)}{(c^2+R_0^2)\{P_1^{k'}(c/R_0) q_1^k(c/R_0) - P_1^k(c/R_0) q_1^{k'}(c/R_0)\}} \tag{2.4.10}$$

where the function $P_1^k(\mu)$ is an associated Legendre function solution of the second order differential equation for $H(\mu)$:

$$\frac{1}{H(\mu)}\frac{\partial}{\partial\mu}\left(1-\mu^2\right)\frac{\partial H(\mu)}{\partial\mu} - \frac{m^2}{1-\mu^2} + n(n+1) = 0$$

$p_1^k(\zeta)$ and $q_1^k(\zeta)$ are two independent solutions of the second order differential equation:

$$\frac{1}{X(\zeta)}\frac{\partial}{\partial\zeta}\left(\zeta^2+1\right)\frac{\partial X(\zeta)}{\partial\zeta} + \frac{m^2}{1+\zeta^2} - n(n+1) = 0$$

In these two equations $H(\mu)$ and $X(\zeta)$ come from searching for separable solutions of Laplace's equation, in oblate spheroidal coordinates (ζ,μ,φ), of the form: $\phi = X(\zeta)H(\mu)e^{im\varphi}$.

Coefficients $D_{1k}(t)$ come from the series expansion:

$$\left(1 - \mu^2\right)\epsilon(\varphi,t) = \sum_{1,k} D_{1k}(t) P_1^k(\mu) e^{ik\varphi}$$

where the left hand side comes from the matching condition:

$$-4\pi G\rho h = \left\{\frac{1}{h_\zeta}\left[\frac{\partial\psi_{ext}}{\partial\zeta} - \frac{\partial\psi_{int}}{\partial\zeta}\right]\right\}_{\zeta=c/R_0}$$

with $h_\zeta = R_0\left(\frac{\zeta^2+\mu^2}{\zeta^2+1}\right)^{1/2}$ and h the normal displacement (Hunter 1963).

The total interior potential in the body (2.3.8) obtains by adding to (2.4.9) the potential due to the uniform spheroid (2.4.4), which on the equatorial plane $z = 0$ is well known to be (Lamb 1945):

$$\psi_{us} = -\pi G\rho\left\{r^2\left[\left(\frac{c^2}{R_0^2}+1\right)\frac{c}{R_0}cot^{-1}\left(\frac{c}{R_0}\right)-\frac{c^2}{R_0^2}\right]-\frac{2c}{R_0}(c^2+R_0^2)cot^{-1}\left(\frac{c}{R_0}\right)\right\} \tag{2.4.11}$$

The final step consists of flattening the body (2.4.8) by making $c \Rightarrow 0$ and $\rho \Rightarrow \infty$ so that

$$\rho c \Rightarrow (3M/4\pi R_0^2) \tag{2.4.12}$$

In this way the spheroid becomes just the disk (2.4.7) since its surface density is

$$\Sigma = \lim_{c\Rightarrow 0} (2\rho z) = \frac{3M}{2\pi R_0^2}\left(1-\frac{r^2}{R^2}\right)^{1/2} \tag{2.4.13}$$

Taking the same limit, Hunter obtained that the gravitational potential in the plane of the disk is given by:

$$\psi = \frac{-3\pi GM(r^2-2R_0^2)}{8R_0^3} + \psi_i \tag{2.4.14}$$

where

$$\psi_i(\xi,\varphi,t) = \sum_{\substack{(l-k)\,even \\ l>k}} \frac{3\pi GM\gamma_l^k D_{lk}(t)}{R_0^2} P_l^k(\xi)\, e^{ik\varphi} \tag{2.4.15}$$

with

$$\xi = \left(1-\frac{r^2}{R^2}\right)^{1/2} \tag{2.4.16}$$

and γ_l^k are constants for given values of l and k (Hunter 1963) .

§ 5 Stability problems in Paczynski's model

In Paczynski's (1978a) analytical model we are concerned with stability problems which relate to two important characteristics of the model. One is that we deal with a self-gravitating accretion disk in which turbulence is driven by local gravitational instabilities which develop when the surface mass density exceeds a critical value, $\Sigma > \Sigma_{crit}$ where $\Sigma_{crit} = A_s (H/r)\ M/2\pi r^2$. The second is that the disk configuration is one of *marginal stability* between the contraction of the disk in the *z* direction, as a result of strong cooling, and a regime of expansion due to the enhanced velocity dispersion of the particles.

The disagreement found between Paczynski's obtained function $\Sigma = \Sigma(\dot{M})$, and what we expect to have in a condition of *marginal stability* leads us to study this regime more carefully and, in general, the stability of the whole model. In particular it is necessary to test Paczynski's basic assumption of the model that the disk structure is radiation pressure supported. This assumption allow us to adopt the Eddington expression , $F = \frac{x}{c} g$, for the surface brightness. This uniquely determines the relation $\Sigma = \Sigma(\dot{M})$; again the above assumption of radiation pressure support provides stringent limits on the mass of the central compact object ($M > 10^9\ M_0$).

§ 6. A tentative of solution of the problem

The main purpose of this work is to obtain stability criterion to understand the above results of Paczynski's model and the curve shown in **Fig. 4** in a more complete frame-work. To proceed in this direction we consider Hunter's (1963) approach to the stability of a self-gravitating disk like configuration since this work is about an infinitesimally thin disk of finite radial extent, R_0 . This structure can be identified with a level surface of spheroidal coordinates, and can be seen as the final product of a contraction in the z direction of a primitive spheroid surface. This identification is possible because Hunter (1963) in his model doesn't consider the presence of a central compact object which strongly would modify the equations and the geometry of the problem. This is just the main idea of our work: To apply Hunter's stability analysis to a self-gravitating accretion disk extended around a compact object.

We consider a disk delimited by an inner edge located at $r = R_{1n}$ together with the outer edge at $r = R_0$. In this case the disk delimited by the two rings $r = R_{in}$ and $r = R_0$ it will be no more a level surface of spheroidal coordinates and should be seen as the final product of contraction of a toroidal structure in the z direction.

Let us consider a torus with major radius R_0 and an elliptical cross section described by the equation:

$$\frac{[r-(R_0-a)]}{a^2} + \frac{z^2}{c^2} = 1 \qquad (2.5.1)$$

where r is the cylindrical radial coordinate measured from the axis of revolution, z ,of the torus; the major azimuth of the torus is taken as the third coordinate, φ .Since we are interested in studying the stability of this thin accretion disk to short wavelength (λ) perturbations: $H \ll \lambda \ll r$, we can use the approximation that the semi-axis a and c of the ellipse are much smaller than R_0 .

Let us consider an infinitesimally thin accretion disk, $R_{in} < r < R_0$, around a compact object of mass M ; this mass will produce an external gravitational potential:

$$\psi_{bh} = \frac{GM}{\left(r^2+z^2\right)^{1/2}} \qquad (2.6.1)$$

where r is the distance from the rotation axis.

To obtain the gravitational potential of the disk itself we will follow Hunter's treatment. We adopt oblate spheroidal coordinates (ζ, μ, φ) with $0 \le \zeta \le \infty$ and $-1 \le \mu \le 1$. The presence of a symmetry plane enables us to take, in the series, Legendre polynomials of even order only.

Consider an axisymmetric distribution of matter in this plane. The surface mass density can be represented by a series of the form:

$$\Sigma = \sum_{i=0}^{\infty} \frac{C_{2i} P_{2i}(\xi)}{\xi} \qquad (2.6.2)$$

where
$$\xi = |\mu| = \left(1 - r^2/R_0\right)^{1/2} \qquad (2.6.3)$$

We have $\xi = 1$ when $r = 0$, but this never happens in our disk model in which r is confined between R_{in} and R_0. In our model the highest value of ξ , is $\xi_{in} = \left[1 - \left(R_{in}^2/R_0^2\right)\right]^{1/2} < 1$ so we have to restrict the range of interest of the variable ξ to: $0 < \xi < \xi_{in}$.

In Eq.(2.6.2) we have constant coefficients C_{2i} which, from the continuity of Σ at $\xi = 0$, satisfy:

$$\sum_{i=0}^{\infty} C_{2i} P_{2i}(0) = 0 \qquad (2.6.3)$$

For polynomials of even order only we can define coefficients C_{2i} as (Lamb 1945):

$$C_{2i} = \int_0^{\xi_{in}} \xi\ \Sigma(\xi)\ P_{2i}(\xi)\ d\xi \qquad (2.6.5)$$

Using Eqs.(2.6.2) and (2.6.5) we can obtain an expression for the disk mass and for the first coefficient C_0:

$$M_d = \int_{R_{in}}^{R_0} 2\pi\Sigma r dr = 2\pi R_0 \int_0^{\xi_{in}} \xi\Sigma d\xi = 2\pi R_0^2 C_0 \qquad (2.6.6)$$

$$C_0 = \frac{M_d}{2\pi R_0^2} \qquad (2.6.7)$$

The gravitational potential on the disk is given by:

$$\psi = 2\pi^2 G R_0 \sum_{i=0}^{\infty} C_{2i}\ \gamma_{2i}^0\ P_{2i}(\xi) \qquad (2.6.8)$$

where γ_{2i}^0 are constant for each value of i .

We have already mentioned that the simplest disk model can be obtained considering only the first two coefficients, C_0 and C_2 . In this case from Eq.(2.6.4) we have $C_2 = 2C_0$ and so:

$$\Sigma = \frac{3M_d}{2\pi R_0^2}\left(1 - \frac{r^2}{R_0^2}\right)^{1/2} \qquad (2.6.9)$$

Moreover this particular model is characterized by a constant angular velocity (Hunter 1963). The possibility of a disk model with differential angular velocity is contained in Hunter's work. It is sufficient to take three non-zero coefficients C_0, C_2, C_4 , in the above series, to obtain it. However, in this case the expressions for Σ and the angular velocity Ω contain a free parameter , α .

To test the stability of a self-gravitating accretion disk around a compact object, we use Hunter's simplest disk model since we also consider the effect on the rotation law, due to the presence of the compact object.

The basic idea is as follows: assume a self-gravitating accretion disk embedded in the strong gravitational potential (Eq.(2.6.1)) due to a compact object of mass M . Consider the effect produced by the compact object to be stronger than the one due to the disk self-gravity. Then the motion in the disk is primarily Keplerian even if influenced by the disk self-gravity.

Let us continue our stability analysis supposing like in Hunter's work that the outer edge of the disk undergoes small perturbations:

$$R = R_0 + \epsilon(\varphi, t) \qquad (2.6.10)$$

with

$$|\epsilon(\varphi, t)| \ll R_0$$

Now we try to obtain an expression for the disk gravitational potential due to the density distribution:

$$\begin{cases} \Sigma = \dfrac{3M_d}{2\pi R_0^2}\left(1 - \dfrac{r^2}{R^2}\right)^{1/2} & \text{for } R_{in} < r < R \\ \Sigma = 0 & \text{outside} \end{cases} \qquad (2.6.11)$$

Then we consider a spheroid of uniform density ρ with outer boundary:

$$\frac{[r - (R-a)]^2}{a^2} + \frac{z^2}{c^2} = 1 \qquad (2.6.12)$$

This body can be regarded, to first order, as the uniform toroid (2.5.1) with a surface mass density ρh ; where h is the normal displacement of the torus boundary:

$$h = n_r \xi_r + n_z \xi_z \qquad (2.6.13)$$

where ξ_r and ξ_z are the components of the displacement of some fluid particle, while n_r and n_z are the components of the unit vector n normal to the surface $\prod$ (Eq.(2.6.1)).

In our case we have $\xi_z = 0$ and $\xi_r = \delta R \left(\frac{\partial r}{\partial R}\right)_{R=R_0} = \epsilon$ while:

$$n_r = \frac{1}{|n|}\frac{\partial \prod}{\partial r} = \frac{c[r - R_0]}{\left[(r-R_0)^2(c^2+a^2) + a^4\right]^{1/2}}$$

So we obtain

$$h = \frac{c[r - R_0]\epsilon}{\left[(r-R_0)^2(c^2+a^2) + a^4\right]^{1/2}} \tag{2.6.14}$$

Following Hunter's basic idea to obtain the gravitational potential on the disk surface, we compute the gravitational potential inside the body (2.6.12), and then take the limit $c \Rightarrow 0$ and $\rho \Rightarrow \infty$ in such a way that $\rho c \Rightarrow 3M/4\pi R_0^2$ and the body becomes just the disk (2.6.11). For the gravitational potential inside the body (2.6.12) we have to consider two contributions: one due to the distribution of matter of density ρh per unit area, the other due to the uniform toroid (2.5.1). The first contribution is (Hunter 1963):

$$\psi_{int} = \sum_{l,k} B_{l,k}\, p_l^k(\zeta)\, P_l^k(\mu) e^{ik\varphi} \tag{2.6.15}$$

$$B_{l,k} = \frac{4\pi G c\rho R_0\, q_l^k(\bar{\zeta})\, p_l^k(\bar{\zeta})\, D_{lk}(t)}{\left\{ q_l^k(\bar{\zeta}) p_l^{k'}(\bar{\zeta}) - p_l^k(\bar{\zeta}) q_l^{k'}(\bar{\zeta}) \right\}} \tag{2.6.16}$$

Here the variable $\bar{\zeta}$ comes from the fact that in our case the toroidal surface (2.5.1) is not a level surface $\zeta = const$ of the spheroidal coordinates even if we write (2.5.1) in spheroidal coordinates. We have called $\bar{\zeta}$ and $\bar{\mu}$ the spheroidal coordinates which describes the surface (2.5.1). Then the potential (2.6.15) with coefficients (2.6.16) is obtained by matching at the surface $\bar{\zeta} = \bar{\zeta}(\bar{\mu})$ two expansions of separable harmonic functions which represent the gravitational potential inside, ψ_{int} , and outside, ψ_{ext} , of the toroid (2.5.1).

The gravitational potential inside the toroid (2.5.1) can be obtained (Randers 1942; Shukhman 1983) as the limit of the potential due to an elliptic cylinder. On the equatorial plane, $z = 0$,we have:

$$\psi_{tor} = -2\pi G\rho ac\left\{ log\frac{16\left(R_0 - a\right)}{a + c} + \frac{1}{2}\left[1 - \frac{2\left[r - \left(R_0 - a\right)\right]^2}{a(a^2 + c^2)}\right]\right\} \tag{2.6.17}$$

Taking now the limit $c \Rightarrow 0$ $\rho \Rightarrow \infty$,as already mentioned, we have:

$$\psi_{tor} = -\frac{3GM_d}{2R_0^2}\left\{ a\ log\frac{16(R_0 - a)}{a} + \frac{a}{2} - \frac{[r(R_0 - a)]^2}{a^2}\right\} \tag{2.6.18}$$

$$\psi_{int} = \sum_{l,k} \frac{3\pi GM_d\ \gamma_l^k\ D_l^k(t)}{R_0} P_l^k(\xi)\ e^{ik\varphi} \tag{2.6.19}$$

So the total gravitational potential on the plane of our self-gravitating accretion disk is found to be:

$$\psi = \psi_{bh} + \psi_{sg} \tag{2.6.20}$$

with $\psi_{sg} = \psi_{int} + \psi_{tor}$

At this point we make this fundamental assumption: we consider a self-gravitating disk of finite thickness, 2h , much less than the radial distance, r . This thickness can be seen as the result of an hydrostatic equilibrium, in the z direction, between pressure force and acceleration, in that direction, due to the disk self-gravity and to the central compact object.

A finite disk thickness can also be described as a perturbation, in the z direction, of the Hunter's (1963) infinitesimally thin disk configuration; this perturbation is present in terms of a perturbed pressure, P^*, and of a perturbed surface mass density, σ^* .

To see this better let us write the hydrodynamic equations in polar cylindrical coordinates (r,z,φ) referred to the rotating axis ($r = 0$) perpendicular to the symmetry plane ($z = 0$):

$$\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_r}{\partial \varphi} - \frac{v_\varphi}{r} v_\varphi = - \frac{\partial \psi_{sg}}{\partial r} - \frac{\partial \psi_{bh}}{\partial r} \qquad (2.6.21)$$

$$\frac{\partial v_\varphi}{\partial t} + v_r \frac{\partial v_\varphi}{\partial r} + \frac{v_\varphi}{r}\frac{\partial v_\varphi}{\partial \varphi} + \frac{v_\varphi}{r} v_r = - \frac{\partial \psi_{sg}}{r\partial \varphi} - \frac{\partial \psi_{bh}}{r\partial \varphi} \qquad (2.6.22)$$

$$\frac{1}{\rho}\frac{\partial P}{\partial z} = - \frac{\partial \psi_{sg}}{\partial z} - \frac{\partial \psi_{bh}}{\partial z} \qquad (2.6.23)$$

$$\frac{\partial \Sigma}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\Sigma v_r\right) + \frac{1}{r}\frac{\partial}{\partial \varphi}\left(\Sigma v_\varphi\right) = 0 \qquad (2.6.24)$$

The first three equations represent the Euler equation, while the last one is the continuity equation.

Now we can arrange Eq.(2.6.23) considering the finite thickness of the disk; in this case taking the thickness as a free parameter along z , we can write:

$$\left[\frac{1}{\rho}\frac{\partial P}{\partial z}\right]_{z=h} = - \left[\frac{\partial \psi_{sg}}{\partial z}\right]_{z=h} - \left[\frac{\partial \psi_{bh}}{\partial z}\right]_{z=h} \qquad (2.6.23)$$

We use the subscript $_0$ to denote values at $z = 0$ (Hunter's (1963) model with $P_0 = 0$ and $\frac{\partial P_0}{\partial z} = 0$) and the subscript $*$ to denote perturbed values. Expanding around $z = 0$, we have:

$$\left[\frac{1}{\rho}\frac{\partial P^*}{\partial z}\right]_{z=h} \simeq \left[\frac{1}{\rho_0}\frac{\partial P^*}{\partial z}\right]_{z=0} \simeq \frac{1}{\rho_0}\frac{P^*}{h} \qquad (2.6.24)$$

$$- \left[\frac{\partial \psi_{sg}}{\partial z}\right]_{z=h} - \left[\frac{\partial \psi_{bh}}{\partial z}\right]_{z=h} = - \left[\frac{\partial \psi_{sg}}{\partial z}\right]_{z=0} - \left[\frac{\partial \psi_{bh}}{\partial z}\right]_{z=0} +$$

$$- \left[\frac{\partial^2 \psi_{sg}}{\partial z^2}\right]_{z=0} h - \left[\frac{\partial^2 \psi_{bh}}{\partial z^2}\right]_{z=0} h =$$

$$= 2\pi G\Sigma_0 + 4\pi G\rho_0 h + \frac{GM}{r^3} h \qquad (2.6.25)$$

Now we look at the second term of Eq.(2.6.25) as a perturbation in the density produced by the finite thickness of the disk configuration, so we can put: $2\rho_0 h = \sigma^*$ (2.6.26)

The hydrostatic equilibrium equation becomes:

$$2 \frac{P^*}{\sigma^*} = 2\pi G\Sigma_0 + 4\pi G\sigma^* + \frac{GM}{r^3} h \qquad (2.6.27)$$

We express P^* using the polytropic equation of state for a mixture of gas plus radiation:

$$P^* = K \left(\rho^*\right)^{4/3} = K \left(\frac{\sigma^*}{2h}\right)^{4/3} \qquad (2.6.28)$$

where K is given by Eq.(1.3.3).

Then a use of Eq.(2.6.28) in Eq.(2.6.27) gives:

$$2K \frac{(\sigma^*)^{1/3}}{(2h)^{4/3}} = 2\pi G\Sigma_0 + 4\pi G\sigma^* + \frac{GM}{r^3} h \qquad (2.6.29)$$

This is a very interesting relation, between σ^* and h, which allow us to have a value of the perturbed density once given a value for te finite disk thickness.

However the right choice for $\sigma^* = \sigma^*(h)$ is not trivial since Eq.(2.6.29) can be written as the following cubic equation:

$$z^3 - \frac{K}{\pi G h^{4/3}} z + \Sigma_0 + \frac{M}{2\pi r^3} h = 0 \qquad (2.6.30)$$

where: $z = \left(\sigma^*\right)^{1/3}$

We have three possible expressions for $\left(\sigma^*\right)^{1/3}$, of which the right one for physical conditions in which we are interested can be determined by a careful study of Eq.(2.6.30).

Once the appropriate $\sigma^* = \sigma^*(h)$ is found we go to look for solutions of the form:

$$\begin{cases} \sigma = \dfrac{3M_d \eta}{2\pi R_0^2} + \sigma^*(\eta,\varphi,t) \\ \phi = \psi_{sg} + \psi_{bh} + \psi^*(\eta,\varphi,t) \\ v_r = v_r^*(\eta,\varphi,t) \\ v_\varphi = \Omega r + v_\varphi^*(\eta,\varphi,t) \end{cases} \tag{2.6.30}$$

where $$\Omega = \left(\frac{GM}{r^3}\right)^{1/2} + \left(\frac{3\pi GM_d}{4R_0^3}\right)^{1/2} \tag{2.6.31}$$

$$\eta = \left(1 - \frac{r^2}{R^2}\right)^{1/2}$$

and M_d is the disk mass, given by Eq.(2.6.6).

Putting these relations in Eqs.(2.6.21), (2.6.22) and (2.6.24), a stability analysis can be done following Hunter (1963). This would lead to: 1) a dispersion relation, from the study of which conditions for marginal stability can be obtained; 2) solutions of Eqs. (2.6.21) to (2.6.24) which should provide us with the expression for the surface mass density of the disk that we need.

§ 7. Conclusions

Self-gravity in accretion disks may be an important source of providing viscosity through self-gravitating instabilities, as originally suggested by Paczynski (1978a) and Kozlowski, Wiita and Paczynski (1979). However much work must be done and these ideas put on much stronger theoretical basis.

Quite recently (last summer) Goodman and Narayan (1987, unpublished preprint) discussed the problem of self-gravity versus stability of accretion disks from quite different point of view in connection with the important, newly discovered by Papaloizou and Pringle (1984), global non-axially symmetric instability of accretion tori and disks. Although the instabilities discussed in this thesis act on thermal time scale, and Papaloizou and Pringle on dynamical, it seems that the work of Goodman and Narayan gives an opportunity to a very powerfull approach to the general problem of stability of disks, different than suggested in this thesis.

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