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STABLE COMPACTIFICATION TO CP(3) OF D=10 DIMENSIONAL EINSTEIN-YANG-MILLS SU(3) x U(1) THEORY

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I. Introduction

About 60 years ago Kaluza made an attempt to unify gravitation and electromagnetism by introducing the 5-dimensional space-time with the additional dimension taken to be a circle with a very small radius. In this way one can obtain the effective 4-dimensional theory in which U(1) gauge field is present. In a very similar way one can get an arbitrary nonAbelian gauge theory - just by enlarging a space-time by additional compact space on which Killing vectors act [1-4].

During last three years Kaluza-Klein type theories have been a subject of a growing interest since they represent the hope for unifying all the known interactions in a unique theory. This interest is connected with an idea of spontaneous compactifications which was put forward by Cremner, Scherk, Horvath and Palla [5] , and with a construction of supersymmetric models in more then four dimensions - especially D=11 supergravity. In what follows we will not discuss supergravity theories. We only want to mension an interesting paper written by Witten [6] in which he observed remarkable feature of number eleven which is simultaneously the upper bound for supergravity theories and the lower bound for models in which SU(3)xSU(2)xU(1) can be realized as a gauge symmetry. A lot of authors analyzed in detail similar models (e.g. seven-sphere compactification) [7] .

If we want to use a mechanism of spontaneous compactification and if we want to get a more interesting topology of the compact space (neither N-torus nor K3 [4]), we must introduce at a very begining some matter fields. There are at least two possibilities: either to start from supergravity theories, the best known example of which ($D=11$) contains three index totally antisymmetric field, or to start from Einstein-Yang-Mills theory. The second possibility at a first sight seems to be in contradiction with the original Kaluza-Klein idea, but it has its own very important advanteges: there is no cosmological constant problem and one can get massless chiral fermions in complex representations of gauge groups (Witten [14] showed that in purely Kaluza-Klein theories it is impossible).

We say that spontaneous compactification occurs if some theory in $D=4+N$ dimensions has a solution of equations of motion which factorizes the space-time into $M^4 x B^N$ where B^N is a compact space and M^4 is a Minkowski space or, as in supergravity theories an anti-deSitter space. We don`t consider a possible dynamical mechanism which could be responsible for a compactification. Our interest is concentrated on an effective theory which results from calculating fluctuations around the background solution. First of all we are interested whether the effective theory is classically stable ie whether there are no ghosts and tachyons in the mass spectrum. A semiclassical stability is discussed in [19].

The classical stability was for the first time investigated by Randjbar-Daemi, Salam and Strathdee [8] who made use of harmonic expansion technique developed by Salam and Strathdee [2] . It is also a basis for calculations presented in this report.

The problem of stability is not yet generally solved. A list of models which were investigated is not long and in all of them a compact space B is taken to be a N-sphere. The first such model was $D=6$ dimensional Einstein-Maxwell theory which compactify to S^2. . A background solution for a Maxwell field was taken as a magnetic monopole configuration [5]. It was shawn that the effective theory was stable and the massless modes were shawn to correspond to a graviton and SU(2)xU(1) gauge bosons. Later the same authors discovered that if a starting point was Einstein-Yang-Mills SU(3) theory, then stability could be lostdue to formation of tachons [9]. The next stable model to be found was D=8 dimensional Einstein-Yang-Mills SU(2) theory with 1-instanton configuration [10]. Recently Schellekens made an attempt to formulate some general conclusions for a class of similar models [11,12] - but again only for compactification to spheres. Firstly he investigated the compactification to N-spheres of Einstein-Yang-Mills SO(N) systems with standart background solution for SO(N) Yang-Mills fields equal to the connection 1-form on S^N [2,13]. His conclusion is that all such models except from N=3 are stable. Secondly he considered a compactification to a homogeneous space G/H with H embedded in a gauge group K. Schellekens obtained a classification of all the stable models for which $G/H=S^N$ [12] .

We see that there are no well defined rules for constructing stable Kaluza-Klein theories. However the chances are maximalized when we take K as equal to H (or a subgroup of H) and a background solution for K gauge fields to be topologically nontrivial. These two conditions are fulfilled in a model which will be discussed in this report: SU(3)xU(1) Yang-Mills fields plus gravitation in D=10 dimensions compactify to CP(3) taken as SU(4)/SU(3)xU(1). The answer for stability question is positive. It is verified that the massless modes in D=4 dimensional spectrum correspond to graviton and SU(4)xU(1) gauge fields. Moreover the effective SU(4) coupling constant is calculated.

If we want to get a realistic D=4 dimensional theory from some Kaluza-Klein model one of the main obstacles is connected with fermions which should appear as zero modes and correspond to complex representations of SU(3)xSU(2)xU(1). There is a general theorem [14] that in the absence of elementary gauge fields Dirac and Rarita-Schwinger operators on compact homogeneous spaces cannot have zero modes in complex representations of a gauge group. Also in any case a number of dimensions must be even. So the theories of a type which will be discussed have, due to topological properties of the background, remarkable advantage. In this report we don't introduce fermions since we are interested mostly in a stability problem. But an important motivation to investigate this particular model is connected with fermions. Witten [14] showed that the simplest anomaly free model is O(16) theory in D=10 dimensions. SU(3)xU(1) can be embedded in O(16) and one of possible compactification is to CP(3).

The report is organized as follows: In section II we introduce notation and try to describe a logic of all the calculation techniques involved. In the section III we decribe a geometric construction which enable us to get a very convenient background solution. In the section IV we discuss main steps of calculations and analize the particles' spectrum.

II. How the particles spectrum is calculated

The starting point is a gravitation and SU(3)xU(1) gauge fields in D=10 dimensions with a cosmological constant λ

$$\mathcal{A} = -\Omega \int d^{10}z \, (-g)^{1/2} \left(\frac{R}{\varkappa^2} + \lambda + \frac{1}{4e^2} B_{MN} B^{MN} + \frac{1}{4g^2} F^{i}_{MN} F^{iMN} \right) \tag{1}$$

We introduce an artificial constant Ω in order that gauge fields and constants $\varkappa^2, e^2, g^2$ have usual dimensionalities [15] .

$$B_{MN} = \partial_M B_N - \partial_N B_M$$

$$F^{i}_{MN} = \partial_M A^{i}_N - \partial_N A^{i}_M + f^{ikl} A^{k}_M A^{l}_N$$

e and g are U(1) and SU(3) coupling constants respectively.

f^{ikl} are SU(3) structure constants, they are taken to be totally antisymmetric $\quad i,k,l = 1,\ldots,8$

$M,N,\ldots$ run from 0 to 3 and from 5 to 10 - they are world indices

$A,B,\ldots$ run from 0 to 3 and from 5 to 10 - they are orthonormal frame indices

$A=(a,\alpha)$ $\quad a$ run from 0 to 3

α run from 5 to 10

$z^M = (x^m, y^\mu)$

Equations of motion are:

$$R_{MN} - \frac{1}{2} g_{MN} R = -\frac{\varkappa^2}{2} \left(-\lambda g_{MN} + T_{MN} \right)$$

where

$$T_{MN} = \frac{1}{e^2} \left(B_{MK} B_N{}^K - \frac{1}{4} g_{MN} B_{KL} B^{KL} \right) + \frac{1}{g^2} \left(F^{i}_{MK} F^{iK}_{N} - \frac{1}{4} g_{MN} F^{i}_{KL} F^{iKL} \right) \tag{2}$$

$$g^{LM} \nabla_L B_{MN} = g^{LM} \left(\partial_L B_{MN} - \Gamma^{P}_{LM} B_{PN} - \Gamma^{P}_{LN} B_{MP} \right) = 0$$

$$g^{LM} \nabla_L F^{\hat{o}}_{MN} = g^{LM} \left(\partial_L F^{\hat{o}}_{MN} - \Gamma^P_{LM} F^{\hat{o}}_{PN} - \Gamma^P_{LM} F^{\hat{o}}_{MP} + f^{\hat{o}\hat{k}\hat{l}} A^{\hat{k}}_L F^{\hat{l}}_{MN} \right) = 0$$

$$g_{AB} = (-,+,+,\dots,+) \quad ;$$

$$R^K{}_{LMN} = \partial_M \Gamma^K_{LN} - \partial_N \Gamma^K_{LM} + \Gamma^K_{PM} \Gamma^P_{LN} - \Gamma^K_{PN} \Gamma^P_{LM} \quad ;$$

$$R_{LM} = R^K{}_{LMK}$$

The first point is to find a solution of equations of motion which induces spontaneous compactification. We want the D=10 dimensional spacetime to factorize into $M^4 \times CP(3)$ where M^4 is a Minkowski space and $CP(3)$ is a homogeneous space $CP(3) = SU(4)/SU(3)\times U(1)$. ($CP(3)$ is a set of directions in 4-dimensional complex space ie. is a set of points (z,x,y,w) with equivalence relation $(z,x,y;w) \simeq (az,ax,ay,aw)$ for every $a \in \mathbb{C} - \{o\}$. It is easy to see that $SU(4)$ acts on $CP(3)$ transitivly and that stability group of a point (z,0,0,0) is $SU(3)\times U(1)$. Let us remark also that $CP(3)$ can also be viewed as other homogeneous spaces like eg. $SO(5)/SO(3)\times U(1)$)

Let us introduce $SU(4)$ algebra generators $Q_{\hat{\alpha}}; \hat{\alpha}=1,\dots,15$ with totally antisymmetric structure constants $f_{\hat{\alpha}\hat{\beta}\hat{\gamma}}$ such that $\mathrm{Tr}\, Q_{\hat{\alpha}} Q_{\hat{\beta}} = -\delta_{\hat{\alpha}\hat{\beta}}$

Let us also choose $SU(3)\times U(1)$ subalgebra

$$\{Q_{\delta}, Q_{15}\} \equiv \{Q_{\bar{\alpha}}\} \qquad \bar{\alpha} = 1,\dots,8,15$$

We divide all the $SU(4)$ generators into two sets

$$\{Q_{\hat{\alpha}}\} = \{Q_{\alpha}\} \cup \{Q_{\bar{\alpha}}\} \qquad \alpha = 9,\dots,14$$

They have the following properties:

$$[Q_{\bar{\alpha}}, Q_{\bar{\beta}}] = C_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}} Q_{\bar{\gamma}} \tag{3}$$

$$[Q_{\bar{\alpha}}, Q_{\beta}] = C_{\bar{\alpha}\beta}{}^{\gamma} Q_{\gamma} \quad (CP(3) \text{ is reductive})$$

$$[Q_{\alpha}, Q_{\beta}] = C_{\alpha\beta}{}^{\bar{\gamma}} Q_{\bar{\gamma}} \quad (CP(3) \text{ is symmetric})$$

For such spaces it is easy to find a so called standart solution of the equations of motion (2) [2,13,16]. We need to construct coset representatives ie functions $L: G/H \to G$ with a property $\pi \circ L = \mathrm{id}_{G/H}$ where π is a projection $G \to G/H$.

More details of the construction will be given in the section III. For a moment it is only important to know that symmetry properties induce the embedding of the H group in SO(N) group (in our case $SU(3) \times U(1) \subseteq SO(6)$).

Having the backgroud solution we can construct the effective D=4 dimensional theory by expanding all the fields around their background values and collecting bilinear terms in the action.

We put

$$
\begin{aligned}
g_{MN} &= \eta_{MN} + \varkappa h_{MN} \\
A_M^j &= \bar{A}_M^j + W_M^j \\
B_M &= \bar{B}_M + V_M
\end{aligned}
\tag{4}
$$

In orthonormal basis our bilinear action has now the following form:

$$
\begin{aligned}
\mathcal{A} = \Omega \int d^{10}z \; \Big\{ & -\frac{1}{4} \nabla_C h_{AB} \nabla_C h_{AB} + \frac{1}{4} \nabla_C h_{AA} \nabla_C h_{BB} - \frac{1}{2} \nabla_A h_{AB} \nabla_B h_{CC} \\
& + \frac{1}{2} \nabla_A h_{AB} \nabla_D h_{BD} - \frac{1}{2e^2} \nabla_A V_B \nabla_A V_B - \frac{1}{2g^2} \nabla_A W_B^j \nabla_A W_B^j + \frac{1}{2e^2} \nabla_A V_A \nabla_B V_B \\
& + \frac{1}{2g^2} \nabla_A W_A^j \nabla_B W_B^j + \frac{1}{4} \left(h_{AB} h_{AB} - \frac{1}{2} h_{AA} h_{BB} \right) \left(\varkappa^2 \lambda + \bar{R} + \frac{\varkappa^2}{4e^2} \bar{B}^2 + \frac{\varkappa^2}{4g^2} \bar{F}^2 \right) \\
& - \frac{1}{2} \bar{R}_{BC} \left(h_{AB} h_{AC} - h_{AA} h_{BC} \right) - \frac{\varkappa^2}{2} \left(h_{AB} h_{AC} - \frac{1}{2} h_{AA} h_{BC} \right) \left(\frac{1}{e^2} \bar{B}_{AB} \bar{B}_{AC} + \frac{1}{g^2} \bar{F}^j_{AB} \bar{F}^j_{AC} \right) \\
& + \frac{1}{2} R_{AB} \left(\frac{1}{e^2} V_A V_B + \frac{1}{g^2} W_A^j W_B^j \right) - f^{jkl} \frac{1}{g^2} \bar{F}^j_{AB} W^k_A W^l_B \\
& + \frac{\varkappa}{e^2} \left(\nabla_A V_C - \nabla_C V_A \right) \left(\bar{B}_{AB} h_{BC} - \frac{1}{4} \bar{B}_{AC} h_{BB} \right) \\
& + \frac{\varkappa}{g^2} \left(\nabla_A W_C^j - \nabla_C W_A^j \right) \left(\bar{F}^j_{AB} h_{BC} - \frac{1}{4} \bar{F}^j_{AC} h_{BB} \right) \\
& + \frac{1}{2} h_{AB} h_{CD} \left(\bar{R}_{ACBD} - \frac{\varkappa^2}{2e^2} \bar{B}_{AC} \bar{B}_{BD} - \frac{\varkappa^2}{2g^2} \bar{F}^j_{AC} \bar{F}^j_{BD} \right) \Big\}
\end{aligned}
\tag{5}
$$

Now we make use of the background solution (details in the section III), we explore SU(3)xU(1) embedding in SO(6) and choose a light cone gauge. After all this is done we arrive at the bilinear action in which all the fields belong to some irreducible SU(3)xU(1) representation.

$$
\begin{aligned}
\mathcal{A} = \int d^{10}z \Big\{ & \tfrac{1}{4} h_{jk} (\partial^2 + \nabla^2) h_{\dot{j}\dot{k}} + \tfrac{1}{2} h_{rs} \big(\partial^2 + \nabla^2 - m^2 - \tfrac{m^2}{2}(1-Y)\big) h_{\dot{r}\dot{s}} \\
& + \tfrac{1}{2} h_{\dot{r}s} \big(\partial^2 + \nabla^2 - m^2 - \tfrac{3}{4} m^2 (1-Y)\big) h_{r\dot{s}} + h_{j\dot{r}} \big(\partial^2 + \nabla^2 - \tfrac{m^2}{2}\big) h_{\dot{j}r} + \\
& + \tfrac{1}{6} h_{j\dot{j}} \big(\partial^2 + \nabla^2 - \tfrac{m^2}{4}\big) h_{k\dot{k}} + \tfrac{1}{2e^2} V_j (\partial^2 + \nabla^2) V_{\dot{j}} + \tfrac{1}{2g^2} W_{j\dot{r}s} (\partial^2 + \nabla^2) W_{\dot{j}r\dot{s}} \\
& + \tfrac{1}{e^2} V_r \big(\partial^2 + \nabla^2 - \tfrac{m^2}{2} - \tfrac{m^2}{3}(3-2Y)\big) V_{\dot{r}} + \tfrac{1}{g^2} W_{rst} \big(\partial^2 + \nabla^2 - \tfrac{3}{4} m^2\big) W_{\dot{t}\dot{r}\dot{s}} \\
& + \tfrac{2}{g^2} W_{pr} \big(\partial^2 + \nabla^2 - \tfrac{m^2}{4}\big) W_{\dot{p}\dot{r}} + \tfrac{3}{8g^2} W_r \big(\partial^2 + \nabla^2 + \tfrac{1}{4} m^2 - m^2 \tfrac{2}{3} Y\big) W_{\dot{r}} + \\
& + \tfrac{iY}{\varkappa\sqrt{2}} \big(h_{j\dot{r}} \nabla_{\dot{s}} W_{js\dot{r}} - h_{\dot{j}\dot{r}} \nabla_s W_{\dot{j}ri} + h_{rs} \nabla_{\dot{t}} W_{\dot{t}\dot{r}\dot{s}} - h_{\dot{r}\dot{s}} \nabla_t W_{rst} + \\
& \qquad + \tfrac{1}{8} h_{rs} (\nabla_{\dot{r}} W_{\dot{s}} + \nabla_{\dot{s}} W_{\dot{r}}) - \tfrac{1}{8} h_{\dot{r}\dot{s}} (\nabla_r W_s + \nabla_s W_r) + \\
& \qquad + h_{\dot{r}s} \nabla_{\dot{t}} W_{rt\dot{s}} - h_{r\dot{t}} \nabla_s W_{t\dot{r}\dot{s}} + \epsilon_{rtp} h_{\dot{r}s} \nabla_{\dot{t}} W_{p\dot{s}} - \epsilon_{rsp} h_{r\dot{t}} \nabla_s W_{pt} \qquad (6) \\
& \qquad + \tfrac{1}{8} h_{r\dot{s}} (\nabla_s W_{\dot{r}} - \nabla_{\dot{r}} W_s) + \tfrac{1}{6} h_{j\dot{j}} (\nabla_s W_{\dot{s}} - \nabla_{\dot{s}} W_s) \big) + \\
& i\sqrt{\tfrac{2}{3}}\, \tfrac{3-2Y}{\varkappa\xi} \big(h_{jr} \nabla_{\dot{r}} V_{\dot{j}} - h_{j\dot{r}} \nabla_r V_{\dot{j}} + \tfrac{1}{2} h_{rs} (\nabla_{\dot{s}} V_{\dot{t}} + \nabla_{\dot{t}} V_{\dot{s}}) - \\
& \qquad - \tfrac{1}{2} h_{\dot{r}\dot{s}} (\nabla_r V_s + \nabla_s V_r) + h_{\dot{r}s} (\nabla_{\dot{s}} V_t - \nabla_t W_{\dot{s}}) \\
& \qquad + \tfrac{1}{6} h_{j\dot{j}} (\nabla_t V_{\dot{t}} - \nabla_{\dot{t}} V_t) \big) + \\
& - \tfrac{1}{\sqrt{3}} \tfrac{Y(3-2Y)}{\varkappa^2 \xi} (W_{\dot{r}} V_r + W_r V_{\dot{r}}) \Big\}
\end{aligned}
$$

Indices j,k run 1,2

Indices $r,\dot{r}$ run 1,2,3

$Y \equiv \frac{m^2 \varkappa^2}{2g^2}$ is a free parameter in the theory. It has a range $[0,3/2]$ and its appearence will be discussed in the next section.

ξ is a U(1) magnetic monopole number on CP(3), which values are restricted to be 3n/4 where n is arbitrary integer;

m is the inverse of a lenght scale on CP(3).

The transformation properties of fields present in (6) are the following:

$h_{\dot\delta\dot\delta}, h_{\dot\delta\mu}, V_{\dot\delta}$ transform like	$\underline{1}_0$
$h_{\dot\delta r}, V_r, W_r$	$\underline{3}_1$
$h_{\dot\delta\dot r}, V_{\dot r}, W_{\dot r}$	$\underline{3}^*_{-1}$
$W^T_{\dot\delta r\dot s}, h^T_{r\dot s}$	$\underline{8}_0$
h_{rs}	$\underline{6}_2$
$h_{\dot r\dot s}$	$\underline{6}^*_{-2}$
W_{rs}	$\underline{6}_{-1}$
$W_{\dot r\dot s}$	$\underline{6}^*_1$
W_{rst}	$\underline{15}_1$
$W_{\dot r\dot s\dot t}$	$\underline{15}^*_{-1}$

These are all the SU(3)xU(1) representations present. According to the general rule [2] we can expand all the fields in series of harmonics on CP(3). Expansions will contain all the irreducible SU(4) representations equal to a given SU(3)xU(1) representation on a restriction to it. In order to find all such SU(4) representations in a systematic way Gelfand-Zetlin notation is very useful [17] . In this notation SU(4) representations are described by four integers (m_1, m_2, m_3, m_4) such that $m_1 \geq m_2 \geq m_3 \geq m_4$ and a representation $(m_1+k, m_2+k, m_3+k, m_4+k)$ is equivalent to (m_1, m_2, m_3, m_4) for k integer.

In the section IV we will discuss a problem of SU(3)xU(1) embedding in SU(4) in the language of Gelfand-Zetlin patterns.

Now we only want to state results. There are ten series of SU(4) representations in which SU(3)xU(1) representations given in (7) are contained. In the table I we summerize their field content. We classified all the fields according to their helicity. The classification given in the table I is a natural one since the symmetry of the background is SU(4) x Poincaré.

Table I

SU(4) representation	Helicity 0	Helicities 1 and 2	
(n,0,0,-n)	$h_{\delta\delta}, V_{\underline{3}}\ V_{\underline{3}*}\ W_{\underline{3}}\ W_{\underline{3}*}\ h_{\underline{8}}\ W_{\underline{6}}\ W_{\underline{6}*}\ W_{\underline{15}}\ W_{\underline{15}*}$	$V_{\delta}\ h_{\delta\underline{3}}\ h_{\delta\underline{3}*}\ W_{\delta\underline{8}}$	$h_{\delta\kappa}$
(n,1,0,-1-n)	$h_{\underline{8}}\ V_{\underline{3}}\ W_{\underline{3}}\ h_{\underline{6}}\ W_{\underline{6}*}\ W_{\underline{15}}\ W_{\underline{15}*}$	$h_{\delta\underline{3}}\ W_{\delta\underline{8}}$	
(n+1,0,-1,-n)	$h_{\underline{8}}\ V_{\underline{3}*}\ W_{\underline{3}*}\ h_{\underline{6}*}\ W_{\underline{6}}\ W_{\underline{15}}\ W_{\underline{15}*}$	$h_{\delta\underline{3}*}\ W_{\delta\underline{8}}$	
(n+1,2,0,-3-n)	$h_{\underline{6}}\ , W_{\underline{15}}$		
(n+3,0,-2,-n)	$h_{\underline{6}*}, W_{\underline{15}*}$		
(n,1,-1,-n)	$h_{\underline{8}}, W_{\underline{6}}, W_{\underline{6}*}\ W_{\underline{15}}\ W_{\underline{15}*}$	$W_{\delta\underline{8}}$	
(n+2,-1,-1,-n)	$W_{\underline{6}}$		
(n,1,1,-2-n)	$W_{\underline{6}*}$		
(n+1,2,-1,-2-n)	$W_{\underline{15}}$		
(n+2,1,-2,-1-n)	$W_{\underline{15}*}$		

Now we have almost everything to be able to calculate particles spectrum. We only must to learn how to calculate covariant derivatives. According to [2,16] covariant derivatives are equivalent to algebraic manipulations with Gelfand-Zetlin patterns. Similarly - the values of Laplace operator are given as differences of two quadratic Casimir operator eigenvalues for some representations of SU(4) and SU(3)xU(1).

Due to orthogonal properties of SU(4) harmonics equations of motion factorize for all the sectors (different SU(4) representations) and it is enough to discuss them separatly. Finally we must diagonalize some mass metrices. From technical point of view it may be quite difficult since in the sector (n,0,0,-n) we work with 10x10 matrix. Instead of calculating precise values of masses we rather formulated a general criterion for nonnegativity of eigenvalues of a given matrix. We showed that this criterion is always fulfilled. It this way the stability of the D=4 dimensional theory is proved. We recognized massless helicity-1 modes corresponding to SU(4)xU(1) symmetry as well as helicity-2 modes corresponding to a graviton. No massless scalars are present. Moreover, there are a lot of towers of massive modes. All of them are of order of m^2. Since effective coupling constant for SU(4) theory is of order $m\tilde{\varkappa}$ where $\tilde{\varkappa}^2 = 16\pi G$; G - gravitational constant, we deduce that m is very large (if coupling constant is to be of order of 1).

III. Background solution

In this section we want to discuss in more detail a geometry of a background solution of classical equations of motion (2) and its symmetry properties. In our exposition we will follow papers [2,13,20] .

There is a very simple way of construction of a set of covariant basis on the homogeneous space G/H .

Let L(y) be a function L(y): $G/H \to G$ such that $\pi \circ L = id_{G/H}$ where π is a canonocal projection $\pi : G \to G/H$ (we take G/H to be a set of left cosets).

Let us construct Lie algebra valued 1-form

$$e(y) = L^{-1}(y)\, dL(y) = e^{\hat{\alpha}}(y)\, Q_{\hat{\alpha}} \tag{8}$$

where $Q_{\hat{\alpha}}$ are generators of a Lie algebra of G with structure constants $C_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}}$

$$[Q_{\hat{\alpha}}, Q_{\hat{\beta}}] = C_{\hat{\alpha}\hat{\beta}}{}^{\hat{\gamma}}\, Q_{\hat{\gamma}} \tag{9}$$

The 1-form e(y) fulfills a Cartan-Maurer equation

$$de(y) = -\, e(y) \wedge e(y) \tag{10a}$$

$$de^{\hat{\alpha}} = -\frac{1}{2}\, e^{\hat{\beta}} \wedge e^{\hat{\gamma}}\, C_{\hat{\beta}\hat{\gamma}}{}^{\hat{\alpha}} \tag{10b}$$

Let us divide $Q_{\hat{\alpha}}$ into two subsets: $Q_{\bar{\alpha}}$ corresponding to the Lie algebra of H and Q_{α} - the rest.

To be specific: In our case G=SU(4) and H=SU(3)xU(1). Our choice of generators is

$$Q_1 = \frac{1}{4}\begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad Q_2 = \frac{1}{4}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \ldots \quad Q_8 = \frac{1}{4\sqrt{3}}\begin{pmatrix} i & & & 0 \\ & i & & \\ & & -2i & \\ 0 & & & 0 \end{pmatrix}$$

$$Q_9 = \frac{1}{4}\begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \quad \ldots \quad Q_{15} = \frac{1}{4\sqrt{6}}\begin{pmatrix} i & & & 0 \\ & i & & \\ & & i & \\ 0 & & & -3i \end{pmatrix} \tag{11}$$

$Q_{\hat\alpha}$ are taken as antihermitian. Normalization is such that $\mathrm{Tr}\, Q_{\hat\alpha} Q_{\hat\beta} = -\delta_{\hat\alpha\hat\beta}$ for $(Q_{\hat\alpha})_{\hat\beta\hat\gamma} \equiv C_{\hat\alpha\hat\beta\hat\gamma}$; structure constant are totally antisymmetric. SU(3)xU(1) embedding in SU(4) is chosen in such a way that $Q_{\bar\alpha},\ \bar\alpha = 1,\ldots,8$ are genarators of SU(3) and Q_{15} is a generator of U(1).

We can check that

$$\begin{aligned} [Q_{\bar\alpha}, Q_{\bar\beta}] &= C_{\bar\alpha\bar\beta}{}^{\bar\gamma}\, Q_{\bar\gamma} \\ [Q_{\alpha}, Q_{\beta}] &= C_{\alpha\beta}{}^{\bar\gamma}\, Q_{\bar\gamma} \\ [Q_{\alpha}, Q_{\bar\beta}] &= C_{\alpha\bar\beta}{}^{\gamma}\, Q_{\gamma} \end{aligned} \tag{12}$$

We see that CP(3) is symmetric. For such spaces there exist a standart background solution:

$$\begin{aligned} & g_{MN}\, dz^M dz^N = g_{mn}(x)\, dx^m dx^n + g_{\mu\nu}(y)\, dy^\mu dy^\nu \\ & R_{mn} = 0 \\ & g_{\mu\nu}(y) = m^2 \delta_{\alpha\beta}\, e_\mu{}^\alpha e_\nu{}^\beta \qquad (m^{-1} \text{ is a length scale of CP(3)}) \\ & A_m^{\bar\alpha} = 0 \qquad A_\mu^{\bar\alpha}(y) = e_\mu^{\bar\alpha}(y) \end{aligned} \tag{13}$$

$\{e^\alpha\}$ constitute an orthonormal coframe on G/H. If we look now for a torsion free connection on G/H we find

$$de^\alpha = e^\beta \wedge B_\beta^\alpha$$

$$B_\beta^\alpha = C_{\bar\gamma\beta}^{\alpha}\, e^{\bar\gamma} \tag{14a}$$

Curvature tensor is

$$R^{\alpha}_{\ \beta} = dB^{\alpha}_{\ \beta} + B^{\alpha}_{\ \gamma} \wedge B^{\gamma}_{\ \beta}$$

$$R^{\alpha}_{\ \beta\gamma\sigma} = m^2 C^{\alpha}_{\ \beta\bar{\delta}} \ C^{\bar{\delta}}_{\ \gamma\sigma} \tag{14b}$$

$$R_{\alpha\beta} = -\frac{m^2}{2} g_{\alpha\beta} \quad ; \quad R = -\frac{m^2}{2} N \qquad (N\text{-dimensionality of } G/H)$$

m^2 is proportional to the scalar curvature of G/H.

Also in orthonormal frame $\quad F^{\bar{\alpha}}_{\ \alpha\beta} = -m^2 \ C^{\bar{\alpha}}_{\ \alpha\beta}$

We have taken $A^{\bar{\alpha}}_{\mu}$ equal to $e^{\bar{\alpha}}_{\mu}$ since it fulfills equations of motion for a Yang-Mills field. But at this point we must be more carefull. Our coset representatives $L(y): G/H \to G$ are only local functions on G/H. It is well known however that there are manifolds which cannot be covered by a single map. The example of such manifold is CP(3)-we need four maps! We must show that connections introduced by four different patches differ from each other by some U(1) transformation. So we write $A^{15}_{\mu}(y) = \xi \, e^{15}_{\mu}(y)$ and look for a constraint on values of ξ. Values of ξ clearly depend on the normalization of U(1) charge. Later on we will use a normalization in which $\underline{4} = \underline{3}_{1/4} + \underline{1}_{-3/4}$. In this normalization one can show that $\xi = \frac{3}{4} n$ (I wish to thank S.Randjbar-Daemi for showing me his calculations of magnetic monopole number on CP(N)).

Our background solution in orthonormal basis has a form

$$R^{\alpha}_{\ \beta\gamma\delta} = m^2 \left(C_{\dot{\delta}\alpha\beta} \, C_{\dot{\delta}\gamma\delta} + C_{15\,\alpha\beta} \, C_{15\,\gamma\delta} \right)$$

$$F^{\dot{\delta}}_{\ \alpha\beta} = m^2 C_{\dot{\delta}\beta\alpha} \qquad \overline{F}^2 = 2m^4 \tag{15}$$

$$\overline{B}_{\alpha\beta} = \xi m^2 C_{15\,\beta\alpha} \qquad \overline{B}^2 = \xi^2 m^4$$

Einstein equation give us now:

$$\lambda = \frac{3}{2}\frac{m^2}{\varkappa^2}$$

$$m^2\varkappa^2 = \frac{6}{\frac{2}{g^2} + \frac{\xi^2}{e^2}} \qquad (16)$$

When we put back back all these equations to (5) a lot of terms will drop out. However

$$R_{\alpha\beta\gamma\delta} - \frac{\varkappa^2}{2}\left(\frac{\bar{B}_{\alpha\gamma}\bar{B}_{\beta\delta}}{e^2} + \frac{\bar{F}^{o}_{\alpha\delta}\bar{F}^{o}_{\beta\delta}}{g^2}\right) \neq 0 \qquad (17)$$

There are new terms in bilinear action due to nonconnectivity of the gauge group. These terms contain a free parameter of the theory - we call it Y

$$Y \equiv \frac{m^2\varkappa^2}{2g^2} = \frac{3}{2+(\frac{\xi g}{e})^2} \qquad (18)$$

We see that range of Y is $0 \leqslant Y \leqslant 3/2$ and

$Y \to 0$ correspond to limit in which $\xi g \gg e$

$Y \to 3/2$ correspond to limit in which $\xi g \ll e$

Finally we want to discuss some symmetry properties of the background solution.

Under the action of the group G

$$gL(y)=L(y')h \qquad g \in G \qquad h \in H \qquad (19)$$

h and y' depend on g and y

$$\begin{aligned} L^{-1}(y')dL(y') &= e^{\hat{a}}(y')Q_{\hat{a}} = hL^{-1}(y)g^{-1}\,(dg\,L(y)h^{-1} + g\,dL(y)g^{-1} + \\ &\quad + gL(y)\,dh^{-1}) = \\ &= hL^{-1}(y)g^{-1}dg\,L(y)h^{-1} + h\,e^{\hat{a}}(y)Q_{\hat{a}}h^{-1} + h\,dh^{-1} \end{aligned} \qquad (20)$$

We introduce matrices of the adjoint representation of G

$$g^{-1} Q_{\hat{\alpha}}\, g = D_{\hat{\alpha}}{}^{\hat{\beta}}(g)\, Q_{\hat{\beta}} \tag{21}$$

$$e^{\hat{\alpha}}(y') Q_{\hat{\alpha}} = (g^{-1}dg)^{\hat{\beta}} D_{\hat{\alpha}}{}^{\hat{\gamma}} (Lh^{-1})\, Q_{\hat{\gamma}} + e^{\hat{\alpha}} D_{\hat{\alpha}}{}^{\hat{\gamma}} (h^{-1})\, Q_{\hat{\gamma}} + (hdh^{-1})^{\hat{\alpha}} Q_{\hat{\alpha}}$$

$$e^{\alpha}(y') = e^{\beta}(y)\, D_{\beta}{}^{\alpha}(h^{-1}) + (g^{-1}dg)^{\hat{\beta}} D_{\hat{\beta}}{}^{\alpha}(Lh^{-1})$$

$$e^{\bar{\alpha}}(y') = (hdh^{-1})^{\bar{\alpha}} + e^{\bar{\beta}} D_{\bar{\beta}}{}^{\bar{\alpha}}(h^{-1}) + (g^{-1}dg)^{\hat{\beta}} D_{\hat{\beta}}{}^{\bar{\alpha}}(Lh^{-1}) \tag{22}$$

Let us consider x dependent (local) transformations.

$$y' = y'(x,y) \qquad dy'^{\mu} = dy^{\nu} \frac{\partial y'^{\mu}}{\partial y^{\nu}} + dx^{m} \frac{\partial y'^{\mu}}{\partial x^{m}}$$

So

$$e_{\mu}^{\alpha}(y') = \frac{\partial y^{\nu}}{\partial y'^{\mu}}\, e_{\nu}^{\beta}(y)\, D_{\beta}{}^{\alpha}(h^{-1}) \tag{23}$$

$$e_{\mu}^{\bar{\alpha}}(y') = \frac{\partial y^{\nu}}{\partial y'^{\mu}} \left(e_{\nu}^{\bar{\beta}}(y)\, D_{\bar{\beta}}{}^{\bar{\alpha}}(h^{-1}) + (h\, \partial_{\nu} h^{-1})^{\bar{\alpha}} \right) \tag{24}$$

We can see that e^{α} is invariant under the action of G up to tangent space rotation induced by h and that $e_{\mu}^{\bar{\alpha}}$ really transforms like a connection form on G/H.

Equation(23) is particulary important becouse it fixes embedding $H \subseteq SON$.

Infinitesimally we have

$$h \simeq 1 + \delta h^{\bar{\delta}} Q_{\bar{\delta}}$$

$$D_{\beta}^{\alpha}(h^{-1}) = \delta_{\beta}^{\alpha} + \delta h^{\bar{\delta}} C_{\alpha\bar{\delta}\beta} = \delta_{\beta}^{\alpha} + \omega_{\beta}^{\alpha} \tag{25}$$

Since it is SO(N) transformation it follows that

$$Q_{\bar{\delta}} = -\tfrac{1}{2}\, C_{\bar{\delta}\alpha\beta}\, \Sigma^{\alpha\beta} \tag{26}$$

$$\left(\tfrac{1}{2}\, \omega_{\alpha\beta}\, \Sigma^{\alpha\beta} = \delta h^{\bar{\delta}} Q_{\bar{\delta}} = \tfrac{1}{2}\, \delta h^{\bar{\delta}}\, C_{\alpha\bar{\delta}\beta}\, \Sigma^{\alpha\beta} \right)$$

We have performed explicite calculations in order to get SU(3)xU(1) content of SO(6) vector. Results are

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\phi_9 - i\,\phi_{10}\\ \phi_{11} - i\,\phi_{12}\\ \phi_{13} - i\,\phi_{14}\end{pmatrix} \quad \text{transforms like } \underline{3}^{*} \text{ with U(1) charge } -1$$

$$\frac{1}{\sqrt{2}}\begin{pmatrix}\phi_9 + i\phi_{10}\\ \phi_{11} + i\phi_{12}\\ \phi_{13} + i\phi_{14}\end{pmatrix} \quad \text{transforms like } \underline{3} \text{ with U(1) charge } +1$$

(remember: the normalization is since now such, that the fundamental SU(4) representation $\underline{4}$ transforms like $\underline{4} = \underline{3}_{1/4} + \underline{1}_{-3/4}$.

For the sake of completness we can also identify the set of Killing vectors for the right action of G on G/H.

$$L_{y+\delta y} = (1+\delta g^{\hat{\alpha}} Q_{\hat{\alpha}})\, L_y\, (1-\delta h^{\bar{\alpha}} Q_{\bar{\alpha}})$$

$$\delta y^{\mu} \partial_{\mu} L_y = \delta g^{\hat{\alpha}} Q_{\hat{\alpha}} L_y - L_y\, \delta h^{\bar{\alpha}} Q_{\bar{\alpha}}$$

$$\delta y^{\mu} e_{\mu}{}^{\hat{\alpha}} Q_{\hat{\alpha}} = \delta g^{\hat{\alpha}} L_y^{-1} Q_{\hat{\alpha}} L_y - \delta h^{\bar{\alpha}} Q_{\bar{\alpha}} = \delta g^{\hat{\alpha}} D_{\hat{\alpha}}{}^{\hat{\beta}}(L_y) Q_{\hat{\beta}} - \delta h^{\bar{\alpha}} Q_{\bar{\alpha}} \qquad (28)$$

$$\delta y^{\mu} e_{\mu}{}^{\alpha} = \delta g^{\hat{\beta}} D_{\hat{\beta}}{}^{\alpha}(L_y) \qquad \delta y^{\mu} = \delta g^{\hat{\beta}} D_{\hat{\beta}}{}^{\alpha}(L_y)\, e_{\alpha}{}^{\mu}(y) = \delta g^{\hat{\beta}} K_{\hat{\beta}}{}^{\mu}(y)$$

$$K_{\hat{\beta}}{}^{\mu}(y) = D_{\hat{\beta}}{}^{\alpha}(L_y)\, e_{\alpha}{}^{\mu}(y)$$

In general a metric which was already shown to be left invariant can also have symmetry connected with the action on cosets from the right. The group of additional symmetry is N(H)/H where N(H) is a normalizer of H in G [18]

$$g \in N \qquad gHg^{-1} = H$$

For CP(3) N(H) = H so nothing new is obtained.

IV. Details of calculations

In this section we want to explain in detail calculation techiques used and analisis of the particles spectrum.

A. Gelfand - Zetlin patterns [17]

In the language of Gelfand - Zetlin all the irreducible representations of SU(4) are described by four integers (m_1, m_2, m_3, m_4) such that $m_1 \geqslant m_2 \geqslant m_3 \geqslant m_4$. (m_1, m_2, m_3, m_4) and $(m_1+k, m_2+k, m_3+k, m_4+k)$ decribe the same representation. Similarly three integers (n_1, n_2, n_3) correspond to a irreducible representation of SU(3) .

Dimensionalities of these representations are given by formulas

$$N_1 = \frac{1}{2}(n_1 - n_2 + 1)(n_1 + 2 - n_3)(n_2 - n_3 + 1) \qquad \text{for } SU(3)$$

$$N_2 = \frac{1}{12}(m_1 - m_2 + 1)(m_1 - m_3 + 2)(m_1 - m_4 + 3)(m_2 - m_3 + 1)(m_2 - m_4 + 2)(m_3 - m_4 + 1) \qquad \text{for } SU(4) \tag{29}$$

(for a genaral formula look [17])

We see that (1,0,0) is a triplet and (1,0,0,0) is a quadret representation.

A given SU(4) representation contains as basic vectors all the possible Gelfand-Zetlin patterns which can be built according to the following rules

$$\begin{bmatrix} m_1 & & m_2 & & m_3 & & m_4 \\ & n_1 & & n_2 & & n_3 & \\ & & k_1 & & k_2 & & \\ & & & l_1 & & & \end{bmatrix} \qquad \begin{array}{l} k_1 \geqslant l_1 \geqslant k_2 \\ n_1 \geqslant k_1 \geqslant n_2 \geqslant k_2 \geqslant n_3 \\ m_1 \geqslant n_1 \geqslant m_2 \geqslant n_2 \geqslant m_3 \geqslant n_3 \geqslant m_4 \end{array} \tag{30}$$

A problem of finding all the SU(4) representations which contain a given SU(3) representation is trivial. For a given SU(3) representation (n_1, n_2, n_3) one easly finds all the (m_1, m_2, m_3, m_4) which fulfill (30). Additional restriction is given by a value of a U(1) charge. The problem arises how to describe U(1) charge in terms of Gelfand-Zetlin patterns.

Again, a solution is easy to find. As we remember we normalized U(1) charge in such a way that $\underline{4} = \underline{3}_{1/4} + \underline{1}_{-3/4}$

so $$Q_{15} = \frac{1}{4} \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & 1 & \\ 0 & & & -3 \end{pmatrix}$$

There are simple rules for values of A_{jj} generators of GL(4,R) while acting on Gelfand -Zetlin patterns. Using notation(30)

$$\begin{aligned}
A_{11}\,(m) &= l_1\,(m) \\
A_{22}\,(m) &= (k_1 + k_2 - l_1)\,(m) \\
A_{33}\,(m) &= ((n_1 + n_2 + n_3) - (k_1 + k_2))\,(m) \\
A_{44}\,(m) &= ((m_1 + m_2 + m_3 + m_4) - (n_1 + n_2 + n_3))\,(m) \\
Q_{15} &= \frac{1}{4}\,(A_{11} + A_{22} + A_{33} - 3A_{44})
\end{aligned} \tag{31}$$

(Having a representation of GL(4;R) we can construct a representation of SU(4))

The action of U(1) generator is hence

$$Q_{15}\,(m) = \frac{1}{4}\;\left((n_1 + n_2 + n_3) - \frac{3}{4}\,(m_1 + m_2 + m_3 + m_4)\right)(m) \tag{32}$$

Using this formula the table I from chapter II can be systematically reproduced.

B. Covariant derivatives and harmonic expansion

There is well known for mathematicians Peter-Weyl theorem stating that functions on a compact Lie group G may be expanded in a series of all the matrix elements of all the unitary irreducible representations of G. In our problem we have functions on a homogeneous space G/H belonging to some irreducible representation of H (embedding of H in SO(N)). For such functions the expansion into all the matrix elements of all the irreduble representations of G must be constrained. Relevant representations of G are only those containing a given representation of H on a restriction to it. [2,16]

$$\begin{aligned} \Phi_i(g) &= \sum_n \sum_q \sqrt{\frac{d_n}{d_D}}\, D^{(n)}_{iq}(g)\, \Phi^{(n)}_q \\ \Phi_i(hg) &= \mathbb{D}_{ij}(h)\, \Phi_j(g) \end{aligned} \tag{33}$$

n numerates G representations

i is H representation index

d_n, d_D are dimensionalities of n and $\mathbb{D}$ representations of G and H respectively

A given $g \in G$ may be written as $g = hL_y^{-1}$

$$\begin{aligned} \Phi_i(g) &= \mathbb{D}_{ij}(h)\, \Phi_j(L_y^{-1}) \\ \Psi_i(x,y) &= \sum_n \sum_q \sqrt{\frac{d_n}{d_D}}\, D^n_{iq}(L_y^{-1})\, \Psi^n_q(x) \end{aligned} \tag{34}$$

Now we must learn how to calculate covariant derivatives.

In the action there are full covariant derivatives containing Riemanian connection on G/H and Yang-Mills connection for H (of H is nonAbelian). Due to embedding $H \subseteq SO(N)$ however these two connections may be substituted by a single one.

$$\nabla_\alpha \Psi_i(x,y) = e_\alpha{}^\mu \partial_\mu \Psi_i - \frac{1}{2} B_{\alpha[\beta\gamma]} D_{ij}(\Sigma^{\beta\gamma}) \Psi_j \tag{35}$$

We know that $\quad Q_{\bar\gamma} = \frac{1}{2} C_{\alpha\bar\gamma\beta} \Sigma^{\alpha\beta}$

$$\nabla_\alpha \Psi_i = e_\alpha{}^\mu \partial_\mu \Psi_i - e_\alpha{}^\mu e_\mu{}^{\bar\gamma} D_{ij}(Q_{\bar\gamma}) \Psi_j$$

Since $\quad \partial_\mu D^n(L_y^{-1}) = - e_\mu{}^{\bar\gamma} D^n(Q_{\bar\gamma}) D^n(L_y^{-1})$

$$\nabla_\alpha D^n_{ip}(L_y^{-1}) = - D^n_{iq}(Q_\alpha) D^n_{qp}(L_y^{-1}) \tag{36}$$

The problem is reduced to a calculation of $D^n_{iq}(Q_\alpha)$ and it is purely algebraic exercise.

Explicitè calculations may be performed knowing the action of generators of GL(4;R) on Gelfand-Zetlin patterns [17] ;

Let us introduce a notation like in [17] .

$$(m) = \begin{bmatrix} m_{1n} & m_{2n} & \cdots & & \cdots & m_{n-1\,n} & m_{nn} \\ & m_{2n-1} & \cdots & & \cdots & m_{n-1\,n-1} & \\ & & & \ddots & & & \\ & & & m_{12} \quad m_{22} & & & \\ & & & m_{11} & & & \end{bmatrix} \tag{37}$$

Let A_{kl} be nxn matrix defined as $(A_{kl})_{ij} = \delta_{ki}\delta_{lj}$

$$A_{k,k-1}(m) = \sum_j^{k-1} a^j_{k-1}(m)\, m^j_{k-1}$$

$$A_{k-1,k}(m) = \sum_j^{k-1} b^j_{k-1}(m)\, \hat m^j_{k-1} \tag{38}$$

where m^{j}_{k-1} is a pattern in which a number m_{jk-1} is replaced by $m_{jk-1}-1$ and $\hat{m}^{j}_{k-1}$ is a pattern in which a number m_{jk-1} is replaced by $m_{jk-1}+1$.

$$
\begin{aligned}
a^{j}_{k-1}(m) &= \left[- \frac{\prod_{i=1}^{k}(l_{ik}-l_{jk-1}+1)\prod_{i=1}^{k-2}(l_{ik-2}-l_{jk-1})}{\prod_{i\neq j}(l_{ik-1}-l_{jk-1}+1)\,(l_{ik-1}-l_{jk-1})} \right]^{1/2} \\
b^{j}_{k-1}(m) &= \left[- \frac{\prod_{i=1}^{k}(l_{ik}-l_{jk-1})\prod_{i=1}^{k-2}(l_{ik-2}-l_{jk-1}-1)}{\prod_{i\neq j}(l_{ik-1}-l_{jk-1})\,(l_{ik-1}-l_{jk-1}-1)} \right]^{1/2}
\end{aligned}
\qquad (39)
$$

Now we must find a correspondance between generators of irreduble SU(3) representations and Gelfand-Zetlin patterns. Result is the following:

$$
\Phi_1 = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 1 & & \end{bmatrix} \qquad
\Phi_2 = \begin{bmatrix} 1 & & 0 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \qquad
\Phi_3 = \begin{bmatrix} 1 & & 0 & & 0 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}
$$

$$
\Phi_{\dot{1}} = \begin{bmatrix} 0 & & 0 & & -1 \\ & 0 & & -1 & \\ & & -1 & & \end{bmatrix} \qquad
\Phi_{\dot{2}} = -\begin{bmatrix} 0 & & 0 & & -1 \\ & 0 & & -1 & \\ & & 0 & & \end{bmatrix} \qquad
\Phi_{\dot{3}} = \begin{bmatrix} 0 & & 0 & & -1 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}
$$

$$
\Phi_{1\dot{3}} = \begin{bmatrix} 1 & & 0 & & -1 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \qquad
\Phi_{2\dot{3}} = \begin{bmatrix} 1 & & 0 & & -1 \\ & 1 & & 0 & \\ & & 0 & & \end{bmatrix} \qquad \cdots
$$

$$
\Phi_{1\dot{1}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & & 0 & & -1 \\ & 1 & & -1 & \\ & & 0 & & \end{bmatrix} - \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & & 0 & & -1 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix} \qquad (40)
$$

$$
\Phi_{2\dot{2}} = -\frac{1}{\sqrt{2}}\begin{bmatrix} 1 & & 0 & & -1 \\ & 1 & & -1 & \\ & & 0 & & \end{bmatrix} - \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & & 0 & & -1 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}
$$

$$
\Phi_{3\dot{3}} = \sqrt{\frac{2}{3}}\begin{bmatrix} 1 & & 0 & & -1 \\ & 0 & & 0 & \\ & & 0 & & \end{bmatrix}
$$

$$
\Phi_{11} = \sqrt{2}\begin{bmatrix} 2 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 2 & & \end{bmatrix} \qquad
\Phi_{12} = \begin{bmatrix} 2 & & 0 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{bmatrix} \qquad \cdots
$$

$$
\Phi_{23\dot{1}} = \begin{bmatrix} 3 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 0 & & \end{bmatrix} \qquad
\Phi_{11\dot{3}} = \sqrt{2}\begin{bmatrix} 3 & & 1 & & 0 \\ & 3 & & 1 & \\ & & 0 & & \end{bmatrix} \qquad \cdots
$$

We can express generators of SU(4) algebra in terms of generators of GL(4;R) algebra.

We need Q_α 9,...,14 or rather after the change of variables

$$
\begin{aligned}
Q_1 &= \tfrac{1}{\sqrt{2}}\,(Q_9 + i\,Q_{10}) \qquad & Q_{\bar{1}} &= \tfrac{1}{\sqrt{2}}\,(Q_9 - i\,Q_{10}) \\
Q_2 &= \tfrac{1}{\sqrt{2}}\,(Q_{11} + i\,Q_{12}) \qquad & Q_{\bar{2}} &= \tfrac{1}{\sqrt{2}}\,(Q_{11} - i\,Q_{12}) \\
Q_3 &= \tfrac{1}{\sqrt{2}}\,(Q_{13} + i\,Q_{14}) \qquad & Q_{\bar{3}} &= \tfrac{1}{\sqrt{2}}\,(Q_{13} - i\,Q_{14})
\end{aligned}
\tag{41}
$$

Looking at our choice of SU(4) generators

$$
\begin{aligned}
Q_1 &= \frac{i}{2\sqrt{2}}\,A_{14} \qquad & Q_{\bar{1}} &= \frac{i}{2\sqrt{2}}\,A_{41} \\
Q_2 &= \frac{i}{2\sqrt{2}}\,A_{24} \qquad & Q_{\bar{2}} &= \frac{i}{2\sqrt{2}}\,A_{42} \\
Q_3 &= \frac{i}{2\sqrt{2}}\,A_{34} \qquad & Q_{\bar{3}} &= \frac{i}{2\sqrt{2}}\,A_{43}
\end{aligned}
\tag{42}
$$

We are now able to define covariant derivatives - but we must remember about dimensionalities and introduce a length scale of G/H namely m^{-1}.

Finally

$$
\begin{aligned}
\nabla_{r} &= \frac{im}{2\sqrt{2}}\,A_{r4} \\
\nabla_{\bar{r}} &= \frac{im}{2\sqrt{2}}\,A_{4r}
\end{aligned}
\tag{43}
$$

Eigenvalues of Laplace operator are given by the difference between values of two quadratic Casimir operators for appropriate represantations of SU(4) and SU(3)xU(1). The values are given in the table II.

SU(4) representation	$\mathcal{C}_2$	SU(3) representation	$\mathcal{C}_2$
$(n,0,0,-n)$	$-\frac{m^2}{4}(n^2+3n)$	$\underline{3}_1$	$-\frac{1}{2}m^2$
$(n,1,0,-1-n)$	$-\frac{m^2}{4}(n^2+4n+3)$	$\underline{8}_0$	$-\frac{3}{4}m^2$
$(n+1,2,-1,-2-n)$	$-\frac{m^2}{4}(n^2+6n+11)$	$\underline{6}_2$	$-\frac{6}{4}m^2$
$(n,1,1,-n-2)$	$-\frac{m^2}{4}(n^2+5n+7)$	$\underline{6}_{-1}$	$-m^2$
$(n+1,2,0,-3-n)$	$-\frac{m^2}{4}(n^2+7n+14)$	$\underline{15}_1$	$-\frac{6}{4}m^2$
$(n,1,-1,-n)$	$-\frac{m^2}{4}(n^2+3n+2)$		

Let us consider an example of calculations.

A typical expansion is (for SU(3) x U(1) scalar)

$$V_\delta(x,y) = \sum_{n=1}^{\infty} \frac{n(n+1)}{2} \sqrt{\frac{2n+1}{3}} \sum_m \overset{(n-1,0,0,1-n)}{D_{0m}} (L_y^{-1}) \overset{(n-1,0,0,1-n)}{V_{\delta m}}(x) \tag{44}$$

$\overset{(n-1,0,0,1-n)}{D_{0m}}$ means a matrix element corresponding to the vector

$$\begin{bmatrix} n-1 & & 0 & & 0 & & 1-n \\ & 0 & & 0 & & 0 & \\ & & 0 & & 0 & & \\ & & & 0 & & & \end{bmatrix}$$

and some other vector labelled by m .

$$\begin{aligned} \nabla_\nu V_\delta(x,y) &\sim \sum_q \overset{(n,0,0,-n)}{D_{0q}} (A_{\nu 4}) \overset{(n,0,0,-n)}{D_{qm}} (L^{-1}) \overset{(n,0,0,-n)}{V_{\delta m}}(x) = \\ &= \overset{(n,0,0,-n)}{D_{0\nu}} (A_{\nu 4}) \overset{(n,0,0,-n)}{D_{\nu m}} (L^{-1}) \overset{(n,0,0,-n)}{V_{\delta m}}(x) \end{aligned} \tag{45}$$

$\overset{(n,0,0,-n)}{D_{0\nu}} (A_{\nu 4})$ may be calculated to be $\sqrt{\frac{n(n+3)}{3}}$.

$$\nabla_\nu V_\delta(x,y) \approx -\sum_{n=1}^{\infty} \frac{n(n+1)}{2} \sqrt{\frac{2n+1}{3}} \sqrt{\frac{n(n+3)}{3}} \sum_m \overset{(n,0,0,-n)}{D_{\nu m}} (L^{-1}) \overset{(n,0,0,-n)}{V_{\delta, m}}(x) \tag{46}$$

C. Equations of motion and spectrum analysis

Now we can write down equations of motions, perform harmonic expansion and calculate derivatives. After doing it we will arrive at series of equations of motion for different harmonic components [8,10] . We will analyze them systematically sector by sector (by a sector we mean irreducible SU(4) representation.). Before discussing our CP(3) model

we will give a simple example in order to ilustrate how the particles spectrum is obtained.

E. Example. D=5 dimensional Kaluza-Klein model

In the original Kaluza-Klein model we start from gravitation in D=5 dimensions

$$\mathcal{A} = - \int d^5 z \quad (-g)^{1/2} \quad \frac{R}{\varkappa^2} \tag{47}$$

$$g_{MN} = \eta_{MN} + \varkappa h_{MN} \quad ; \quad \eta_{MN} = (-,+,+,\dots,+) \quad ;$$

The background solution is

$$\bar{R}_{ABCD} = 0$$

The actin bilinear in fluctuations reads:

$$\mathcal{A} = - \int d^4 x\, dy \; \Big(\tfrac{1}{4} \nabla_C h_{AB} \nabla_C h_{AB} - \tfrac{1}{4} \nabla_C h_{AA} \nabla_C h_{BB} + \\ + \tfrac{1}{2} \nabla_A h_{AB} \nabla_B h_{CC} - \tfrac{1}{2} \nabla_A h_{AB} \nabla_C h_{CB} \Big) \tag{48}$$

$A = (a, 5)$

In the light-cone gauge $\quad h_{A-} = 0$

$$\mathcal{A} = \int d^4 x\, dy \quad \Big\{ \tfrac{1}{4} h^T_{jk} (\partial^2 + \nabla_5^2) h^T_{jk} + \tfrac{1}{2} h_{j5} (\partial^2 + \nabla_5^2) h_{j5} + \\ + \tfrac{3}{8} h_{jj} (\partial^2 + \nabla_5^2) h_{kk} \Big\} \tag{49}$$

$j, k = 1, 2$

In this case harmonic expansion is trivial

$$h(x, y) = \sum_n h^{(n)}(x) \exp \frac{iny}{2\pi r} \qquad h(x,0) = h(x, 2\pi r) \tag{50}$$

Equations of motions for h_{jk}, h_{j5} and h_{jj} are the same (what is important: there are series of equations of motion for all the harmonic components of them).

$$\forall_n \qquad \Big(\partial^2 - \frac{n^2}{(2\pi r)^2}\Big) h^{(n)}_{jk}(x) = 0 \tag{51}$$

$$\text{the same for } h_{j5} \text{ and } h_{jj}$$

We see that for n=0 there are zero modes for h_{jk} (helicity 2), h_{j5} (helicity 1) and h_{jj} (helicity 0). For $n \neq 0$ there is a tower of massive modes - the same for helicities 0,1,2. The particles spectrum is: massless particles ie graviton, gauge boson and Brans-Dicke scalar; massive particles - a tower of spin 2 particles (they 'eat' degrees of freedom with helicities 0 and 1.

C. Continuation

Sectors (n+1,2,-1,-2-n) and (n+2,1,-2,-1-n) $n \geq 1$

Only one field, namely $W_{\underline{15}}$ (or $W_{\underline{15}*}$) is present. Equation of motion is

$$\left(\partial^2 - \frac{m^2}{4}(n^2+6n+8)\right) W^{(n)}_{\underline{15}}(x) = 0 \qquad (52)$$

Masses are $\frac{m^2}{4}(n^2+6n+8)$ for $n \geq 1$.

Sectors (n,1,1,-2-n) and (n+2,-1,-1,-n) $n \geq 1$

Only one field is present, namely $W_{\underline{6}}$ (or $W_{\underline{6}*}$)

Equation of motion is

$$\left(\partial^2 - \frac{m^2}{4}(n^2+5n+4)\right) W^{(n)}_{\underline{6}*}(x) = 0 \qquad (53)$$

Masses are $\frac{m^2}{4}(n^2+5n+4)$ for $n \geq 1$

Sectors (n+1,2,0,-3-n) and (n+3,0,-2,-1-m) $n \geq 1$

Two fields are present, namely $h_{\underline{6}*}$, $W_{\underline{15}*}$ (or $h_{\underline{6}}$, $W_{\underline{15}}$)

Equations of motion are

$$\left(\partial^2 - \frac{m^2}{4}(n^2+7n+10+2Y)\right) h^{(n)}_{\underline{6}}(x) + \frac{mY_3}{2\sqrt{2}\,\kappa}\sqrt{n^2+7n+12}\; W^{(n)}_{\underline{15}}(x) = 0$$

$$\left(\partial^2 - \frac{m^2}{4}(n^2+7n+11)\right) W^{(n)}_{\underline{15}}(x) + \frac{mY_3}{2\sqrt{2}\,\kappa}\sqrt{n^2+7n+12}\; h^{(n)}_{\underline{6}}(x) = 0 \qquad (54)$$

One can easly check that for $Y \in (0,3/2)$ two eigenvalues of the mass matrix are positive. In this sector it is trivial to calculate them because the mass matrix has dimensionality two. But for other sectors we will be only able to obtain qualitative results that eigenvalues of the mass matrix are all positive without calcutating their values. So for this sector we don't calculate M^2_{ph} . It is enough for us to know that M^2_{ph} are of order of m^2 where m is the inverse of the length scale of CP(3).

Sector (n,1,-1,-n) n ≥ 1

In this sector there are helicity 0 as well as helicity 1 states. Fields which are present may be read from the table I (page 11). Equations of motion are:

$$\left(\partial^2 - \frac{m^2}{4}(n^2+3n-1)\right) W^{(n)}_{0\underline{8}}(x) = 0$$

$$\begin{aligned}
&\left(\partial^2 - \frac{m^2}{4}(n^2+3n-1)\right) W^{(n)}_{\underline{15}}(x) - \frac{mY_g}{4\sqrt{2}\kappa}\sqrt{n^2+3n-4}\; h^{(n)}_{\underline{8}}(x) = 0\\
&\left(\partial^2 - \frac{m^2}{4}(n^2+3n-1)\right) W^{(n)}_{\underline{15}^*}(x) + \frac{mY_g}{4\sqrt{2}\kappa}\sqrt{n^2+3n-4}\; h^{(n)}_{\underline{8}}(x) = 0\\
&\left(\partial^2 - \frac{m^2}{4}(n^2+3n-1)\right) W^{(n)}_{\underline{6}^*}(x) + \frac{mY_g}{8\kappa}\sqrt{2(n^2+3n-2)}\; h^{(n)}_{\underline{8}}(x) = 0\\
&\left(\partial^2 - \frac{m^2}{4}(n^2+3n-1)\right) W^{(n)}_{\underline{6}}(x) + \frac{mY_g}{8\kappa}\sqrt{2(n^2+3n-2)}\; h^{(n)}_{\underline{8}}(x) = 0\\
&\left(\partial^2 - \frac{m^2}{4}(n^2+3n+6-3Y)\right) h^{(n)}_{\underline{8}}(x) + \frac{mY_g}{8\kappa}\sqrt{2(n^2+3n+2)}\left(W^{(n)}_{\underline{6}} + W^{(n)}_{\underline{6}^*}\right) +\\
&\qquad + \frac{mY_g}{4\sqrt{2}\kappa}\sqrt{n^2+3n-4}\left(W^{(n)}_{\underline{15}^*} - W^{(n)}_{\underline{15}}\right) = 0
\end{aligned} \tag{55}$$

Masses of helicity 1 states are

$\frac{m^2}{4}(n^2+3n-$ [illegible] $)$

The mass matrix for helicity 0 fields is

$$M = \begin{bmatrix}
\partial^2 - M_1^2 & A & A & B & -B\\
A & \partial^2 - M_2^2 & 0 & 0 & 0\\
A & 0 & \partial^2 - M_2^2 & 0 & 0\\
B & 0 & 0 & \partial^2 - M_2^2 & 0\\
-B & 0 & 0 & 0 & \partial^2 - M_2^2
\end{bmatrix} \tag{56}$$

where $A = \frac{mYg}{8\kappa}\sqrt{2(n^2+3n+2)}$; $B = \frac{mYg}{4\sqrt{2}\kappa}\sqrt{n^2+3n-4}$

$$\mathrm{Det}\, M = (\partial^2 - M_2^2)^3 \left((\partial^2 - M_1^2)(\partial^2 - M_2^2) - 2A^2 - 2B^2 \right)$$

Values of ∂^2 for which $\mathrm{Det}\, M = 0$ are for sure real (M is hermitian). If we write

$$(\partial^2 - \tilde{M}_1^2)(\partial^2 - \tilde{M}_2^2) = \left((\partial^2 - M_1^2)(\partial^2 - M_2^2) - 2A^2 - 2B^2 \right)$$

$$\tilde{M}_1^2 \tilde{M}_2^2 = M_1^2 M_2^2 - 2A^2 - 2B^2 \geq 0$$

$$\tilde{M}_1^2 + \tilde{M}_2^2 = M_1^2 + M_2^2 \geq 0$$

These are in fact necessary and sufficient conditions for all the solutions to be positive. Conditions of that kind may be generalized for matrices of higher order and are very useful in proving the stability of a spectrum.

In our case

$$M_1^2 M_2^2 = \frac{m^4}{16}(n^2+3n-1)(n^2+3n+6-3Y)$$

$$2A^2 + 2B^2 = \frac{m^4}{32}(2n^2+6n-2)$$

So $M_1 M_2 \geq 2A^2 + 2B^2$ for $Y \leq 4$ what is always true.

We must be careful since for n=1 only three fields are present: $h_{\underline{8}}$, $w_{\underline{6}}$, $w_{\underline{6}*}$. Mass matrix is now

$$M = \begin{bmatrix} \partial^2 - M_1^2 & A & A \\ A & \partial^2 - M_2^2 & 0 \\ A & 0 & \partial^2 - M_2^2 \end{bmatrix} \qquad (57)$$

Stability condition is $M_1 M_2 \geq 2A^2$ what means $Y \leq 2.5$

Sector (n,1,0,-1-n)

The following fields are present: $h_{\delta\underline{3}}$ $W_{\delta\underline{8}}$,

$V_{\underline{3}}$, $W_{\underline{3}}$, $h_{\underline{8}}$, $W_{\underline{6}*}$, $W_{\underline{15}*}$, $W_{\underline{15}}$, $h_{\underline{6}}$

Equations of motion are:

$$\left(\partial^2 - \frac{m^2}{4}(n^2+4n+3)\right) h_{\delta\underline{3}} + \frac{Ymg}{4\varkappa}\sqrt{n^2+4n+3}\; W_{\delta\underline{8}} = 0$$

$$\left(\partial^2 - \frac{m^2}{4}(n^2+4n)\right) W_{\delta\underline{8}} + \frac{Ymg}{4\varkappa}\sqrt{n^2+4n+3}\; h_{\delta\underline{3}} = 0$$

$$\left(\partial^2 - \frac{m^2}{4}(n^2+4n+7) - \frac{m^2}{3}2Y\right) V_{\underline{3}} - \frac{egY(3-2Y)}{3\varkappa^2\xi} W_{\underline{3}} -$$
$$- \frac{m(3-2Y)e}{2\sqrt{3}\,\varkappa\xi}\sqrt{n^2+4n+3}\; h_{\underline{8}} + \frac{m(3-2Y)e}{2\sqrt{3}\,\varkappa\xi}\sqrt{n^2+4n-5}\; h_{\underline{6}} = 0$$

$$\left(\partial^2 - \frac{m^2}{4}(n^2+4n) - \frac{2}{3}m^2Y\right) W_{\underline{3}} - \frac{egY(3-2Y)\sqrt{8}}{3\varkappa^2\xi} V_{\underline{3}} + \frac{mYg}{8\sqrt{6}\varkappa}\sqrt{n^2+4n+3}\; h_{\underline{8}} + \frac{mYg}{4\sqrt{6}\varkappa}\sqrt{n^2+4n-5}\; h_{\underline{6}} = 0 \qquad (58)$$

$$\left(\partial^2 - \frac{m^2}{4}(n^2+4n+7) + \frac{3}{4}m^2Y\right) h_{\underline{8}} - \frac{m(3-2Y)e}{2\sqrt{3}\,\varkappa\xi}\sqrt{n^2+4n+3}\; V_{\underline{3}} + \frac{mYg}{8\sqrt{6}\varkappa}\sqrt{n^2+4n+3}\; W_{\underline{3}}$$
$$- \frac{mYg}{8\varkappa}\sqrt{n^2+4n+3}\; W_{\underline{6}*} - \frac{mYg}{4\sqrt{2}\varkappa}\sqrt{2(n^2+4n)}\; W_{\underline{15}*} - \frac{mYg}{4\sqrt{2}\varkappa}\sqrt{\frac{3}{4}(n^2+4n+5)}\; W_{\underline{15}} = 0$$

$$\left(\partial^2 - \frac{m^2}{4}(n^2+4n+1) - \frac{m^2}{2}Y\right) h_{\underline{6}} + \frac{m(3-2Y)e}{2\sqrt{3}\,\varkappa\xi}\sqrt{n^2+4n-5}\; h_{\underline{6}} + \frac{mYg}{4\sqrt{6}\varkappa}\sqrt{n^2+4n-5}\; W_{\underline{3}} +$$
$$+ \frac{mYg}{2\sqrt{2}\varkappa}\sqrt{\frac{3}{4}(n^2+4n+3)}\; W_{\underline{15}} = 0$$

$$\left(\partial^2 - \frac{m^2}{4}(n^2+4n)\right) W_{\underline{6}*} - \frac{mYg}{8\varkappa}\sqrt{n^2+4n+3}\; h_{\underline{8}} = 0$$

The mass matrix has a form:

$$M = \begin{bmatrix} \partial^2 - M_1^2 & M_{12} & M_{13} & 0 & 0 & 0 & M_{17} \\ & \partial^2 - M_2^2 & M_{23} & 0 & 0 & 0 & M_{27} \\ & & \partial^2 - M_3^2 & M_{34} & M_{35} & M_{36} & 0 \\ & & & \partial^2 - M_4^2 & 0 & 0 & 0 \\ & & & & \partial^2 - M_5^2 & 0 & 0 \\ & & & & & \partial^2 - M_6^2 & M_{67} \\ & & & & & & \partial^2 - M_7^2 \end{bmatrix}$$

$M^t = M$; $M_1^2, M_2^2, M_3^2, M_4^2, M_5^2, M_6^2, M_7^2 \geqslant 0$

$$M_2 = \begin{bmatrix} \partial^2 - \frac{m^2}{4}(n^2+4n+3) & ; & \frac{Ymg}{4\varkappa}\sqrt{n^2+4n+3} \\ \frac{Ymg}{4\varkappa}\sqrt{n^2+4n+3} & ; & \partial^2 - \frac{m^2}{4}(n^2+4n) \end{bmatrix} ;$$

For M_2 stability condition is $Y \leq 10$

$$M_{12}^2 = \frac{2m^4 Y(3-2Y)}{9}$$

$$M_{13}^2 = (n^2+4n+3)\,\frac{m^4(3-2Y)}{24}$$

$$M_{17}^2 = (n^2+4n-5)\,\frac{m^4(3-2Y)}{24}$$

$$M_{23}^2 = (n^2+4n+3)\,\frac{m^4 Y}{12\cdot 64}$$

$$M_{27}^2 = (n^2+4n-5)\,\frac{m^4 Y}{16\cdot 12}$$

$$M_{34}^2 = (n^2+4n+3)\,\frac{m^4 Y}{128}$$

$$M_{35}^2 = (n^2+4n)\,\frac{m^4 Y}{32}$$

$$M_{36}^2 = (n^2+4n-5)\,\frac{m^2 Y\cdot 3}{256}$$

$$M_{67}^2 = (n^2+4n+3)\,\frac{m^4 Y\cdot 3}{64}$$

$$M_1^2 = \frac{m^2}{4}(n^2+4n+7) - \frac{2}{3}m^2 Y$$

$$M_2^2 = \frac{m^2}{4}(n^2+n) + \frac{2}{3}m^2 Y$$

$$M_3^2 = \frac{m^2}{4}(n^2+4n+7) - \frac{3}{4}Y m^2$$

$$M_4^2 = \frac{m^2}{4}(n^2+4n)$$

$$M_5^2 = M_6^2 = \frac{m^2}{4}(n^2+4n) \tag{60}$$

$$M_7^2 = \frac{m^2}{4}(n^2+4n-1) + \frac{m^2}{2}Y$$

It is not easy to calculate det M. But as mentioned before we are only interested in showing that there are no tachyons in the spectrum. Necessary and sufficient condition for it is that

$$\mathrm{Det}\; M = (\partial^2)^7 - (\partial^2)^6 A + (\partial^2)^5 B - \ldots - G \tag{61}$$

$$A, B, \ldots, G > 0 \tag{62}$$

Let us now observe that Det M contains a term

$(\partial^2 - M_1^2)(\partial^2 - M_2^2)\ \ldots\ (\partial^2 - M_7^2)$ all $M_j^2 > 0$

In all the other terms two or more $(\partial^2 - M_i^2)$ factors are replaced by M_{jk} terms with <u>possible</u> opposite sign. We see now that if we show that $M_i^2 M_j^2 \gg M_{ij}^2$, we can be sure that the condition (62) is fulfilled.

We have checked that $M_{ij}^2 / M_i^2 M_j^2$ are at most of order 1/10 what proves the stability in this sector.

Sector (n,0,0,n)

The following fields of helicity 0,1,2 are present:

There is only one helicity 2 field, equation of motion for which is:

$$\left(\partial^2 - \frac{m^2}{4}(n^2+3n)\right) h_{\mu\nu}^{(n)T}(x) = 0 \tag{63}$$

In the sector (0,0,0,0) there is a masless mode which we interpret as graviton.

In the sector (n,0,0,-n) for $n \geq 1$ there are massive helicity 2 particles with masses

$$\frac{m^2}{4}(n^2+3n)$$

The mass matrix for helicity 1 fields is (for $n \geq 1$)

$$M = \begin{bmatrix} \partial^2 - \frac{m^2}{4}(n^2+3n) & 0 & \frac{m}{2\sqrt{3}}\frac{e(3-2\gamma)}{\varkappa\xi}\sqrt{n^2+3n} & , -\frac{m}{2\sqrt{3}}\frac{e(3-2\gamma)}{\varkappa\xi}\sqrt{n^2+3n} \\ 0 & \partial^2 - \frac{m^2}{4}(n^2+3n) & \frac{m}{4}\frac{g\gamma}{\varkappa}\sqrt{\frac{2}{3}(n^2+3n)} & -\frac{m}{4}\frac{g\gamma}{\varkappa}\sqrt{\frac{2}{3}(n^2+3n)} \\ \frac{m}{2\sqrt{3}}\frac{e(3-2\gamma)}{\varkappa\xi}\sqrt{n^2+3n}, & \frac{m}{4}\frac{g\gamma}{\varkappa}\sqrt{\frac{2}{3}(n^2+3n)} & \partial^2 - \frac{m^2}{4}(n^2+3n) & 0 \\ -\frac{m}{2\sqrt{3}}\frac{e(3-2\gamma)}{\varkappa\xi}\sqrt{n^2+3n}, & -\frac{m}{4}\frac{g\gamma}{\varkappa}\sqrt{\frac{2}{3}(n^2+3n)} & 0 & \partial^2 - \frac{m^2}{4}(n^2+3n) \end{bmatrix} \tag{64}$$

$$\begin{aligned} \text{Det } M = \left(\partial^2 - \frac{m^2}{4}(n^2+3n)\right)\Big(& \left(\partial^2 - \frac{m^2}{4}(n^2+3n)\right)^2 \left(\partial^2 - \frac{m^2}{4}(n^2+3n-3)\right) - \\ & - \left(\partial^2 - \frac{m^2}{4}(n^2+3n-3)\right)\frac{m^4}{12}(3-2\gamma)(n^2+3n) \\ & - \left(\partial^2 - \frac{m^2}{4}(n^2+3n)\right)\frac{m^2\gamma}{2a}(n^2+3n)\Big) \qquad n \geq 1 \end{aligned}$$

One can see that for n=1 (adjoint representation of SU(4)) there is a massless mode. We interpret it as the SU(4) gauge boson.

In the sector (0,0,0,0) there is only one massless field $V_\mu^{(0,0,0,0)}(x)$ which is the gauge boson of the unbroken U(1) symmetry.

The mass matrix for scalar fields is: $(M = M^T)$

$$M = \begin{bmatrix}
\partial^2 - M_1^2 & 0 & 0 & 0 & M_{15} & -M_{15} & M_{17} & -M_{17} & 0 & 0 \\
 & \partial^2 - M_2^2 & 0 & 0 & M_{25} & -M_{25} & M_{27} & -M_{27} & M_{29} & -M_{29} \\
 & & \partial^2 - M_3^2 & 0 & M_{35} & 0 & M_{37} & 0 & M_{39} & 0 \\
 & & & \partial^2 - M_4^2 & 0 & M_{35} & 0 & -M_{37} & 0 & -M_{39} \\
 & & & & \partial^2 - M_5^2 & 0 & M_{57} & 0 & 0 & 0 \\
 & & & & & \partial^2 - M_6^2 & 0 & -M_{57} & 0 & 0 \\
 & & & & & & \partial^2 - M_7^2 & 0 & 0 & 0 \\
 & & & & & & & \partial^2 - M_8^2 & 0 & 0 \\
 & & & & & & & & \partial^2 - M_9^2 & 0 \\
 & & & & & & & & & \partial^2 - M_{10}^2
\end{bmatrix} \qquad (65)$$

where

$$M_{15}^2 = \frac{m^4 (3-2Y)}{12 \cdot 24} (n^2+3n)$$

$$M_{17}^2 = \frac{m^4 Y}{4 \cdot 36} (n^2+3n)$$

$$M_{25}^2 = \frac{m^4 (3-2Y)}{36} (n^2+3n)$$

$$M_{27}^2 = \frac{m^4 Y}{24 \cdot 48} (n^2+3n)$$

$$M_{29}^2 = \frac{3}{32} m^4 Y (n^2+3n-4)$$

$$M_{35}^2 = \frac{m^4 (3-2Y)}{12} (n^2+3n-4)$$

$$M_{37}^2 = \frac{m^4 Y}{96} (n^2+3n-4)$$

$$M_{39}^2 = \frac{m^4 Y}{32} (n^2+3n)$$

$$M_{57}^2 = \frac{Y(3-2Y)2}{9} m^4$$

$$M_1^2 = \frac{m^2}{4} (n^2+3n+1)$$

$$M_2^2 = \frac{m^2}{4} (n^2+3n+4-3Y)$$

$$M_3^2 = M_4^2 = \frac{m^2}{4} (n^2+3n+4+2Y)$$

$$M_5^2 = M_6^2 = \frac{m^2}{4} (n^2+3n+4) - \frac{2}{3} m^2 Y$$

$$M_7^2 = M_8^2 = \frac{m^2}{4} (n^2+3n+3) + \frac{2}{3} m^2 Y$$

$$M_9^2 = M_{10}^2 = \frac{m^2}{4} (n^2+3n-3)$$

Using the same argument as before, by evaluating $M_{ij}^2 / M_i^2 M_j^2$ which are all $\leqslant 2/7$ (the value $2/7$ is for $i=3$ and $j=5$ in the limit $Y \to 0$) we proved the stability of the spectrum.

Effective SU(4) coupling constant

We have calculated the effective SU(4) coupling constant using the method described in [8] (zero-mode limit)

It is a good moment to emphesize an important feature of the similar class of models. In original Kaluza-Klein theory gauge fields are obtained from the metric tensor in D dimensions, more precisely from its nondiagonal part by putting $g_{m\mu}(x,y) = A_m^{\hat{a}}(x)\, K_\mu^{\hat{a}}(y)$; $K_\mu^{\hat{a}}$ is a Killing vector for the symmetry group G. In the models similar to one which is investigated here SU(4) gauge group fields are some linear combinations of fluctuations of all the helicity 1 fields which are present. It is clear from the discussion of the sector (1,0,0,-1).

Let us write the zero mode ansatz

$$
\begin{aligned}
E^a &= dx^m\, E_m^{\ a}(x) \\
E^\alpha &= \frac{1}{m}\, dy^\mu\, e_\mu^{\ \alpha} - \varkappa\, dx^n\, Z_n^{\hat{a}}\, D_{\hat{a}}^{\alpha} \\
A^{15} &= dx^n V_n + \xi \left(e^{15} - \varkappa m\, dx^n\, U_n^{\hat{a}}\, D_{\hat{a}}^{15} \right) \\
A^{\dot{\delta}} &= dx^n W_n^{\dot{\delta}} + e^{\dot{\delta}} - \varkappa m\, dx^n\, S_n^{\hat{a}}\, D_{\hat{a}}^{\dot{\delta}}
\end{aligned}
\tag{66}
$$

There are three SU(4) vectors $Z_n^{\hat{a}}$, $U_n^{\hat{a}}$, $S_n^{\hat{a}}$ U(1) vector V_n and SU(3) vector $W_n^{\dot{\delta}}$. The motivation for such ansatz are transformation properties of under SU(4) which result from it.

We remember that $gL(y) = L(y')h$

Let $h = \tilde{h}\, e^{\varphi Q_{15}}$ $\quad \tilde{h} \in SU(3)$

$$
\begin{aligned}
E^\alpha(x,y) &\to E'^\alpha(x,y') = E^\beta(x,y)\, D_\beta^{\ \alpha}(h^{-1}) \\
A^{\dot{\delta}}(x,y) &\to A'^{\dot{\delta}}(x,y') = A^\kappa(x,y)\, D_\kappa^{\ \dot{\delta}}(\tilde{h}^{-1}) + (\tilde{h}\, d\tilde{h}^{-1})^{\dot{\delta}} \\
A^{15}(x,y) &\to A'^{15}(x,y') = A^{15}(x,y) - \xi\, d\varphi
\end{aligned}
\tag{67}
$$

$$
\begin{aligned}
U'^{\hat\alpha}(x) &= U^{\hat\beta}(x)\, D_{\hat\beta}^{\hat\alpha}(g^{-1}) - \frac{1}{m\varkappa}\,(g\,dg^{-1})^{\hat\alpha} \\
Z'^{\hat\alpha}(x) &= Z^{\hat\beta}(x)\, D_{\hat\beta}^{\hat\alpha}(g^{-1}) - \frac{1}{m\varkappa}\,(g\,dg^{-1})^{\hat\alpha} \\
S'^{\hat\alpha}(x) &= S^{\hat\beta}(x)\, D_{\hat\beta}^{\hat\alpha}(g^{-1}) - \frac{1}{m\varkappa}\,(g\,dg^{-1})^{\hat\alpha}
\end{aligned}
\qquad (68)
$$

In the action we have:

$$
\begin{aligned}
\frac{R}{\varkappa^2} \equiv \frac{R_4}{\varkappa^2} &+ \frac{1}{4} D_{\hat\alpha}^{\alpha} D_{\hat\beta}^{\alpha}\,(\partial_m Z_n^{\hat\alpha} - \partial_n Z_m^{\hat\alpha} - m\varkappa\, Z_m^{\hat\beta} Z_n^{\hat\gamma}\, C_{\hat\alpha\hat\beta\hat\gamma}) \times \\
&\times (\partial_m Z_n^{\hat\beta} - \partial_n Z_m^{\hat\beta} - m\varkappa\, Z_m^{\hat\sigma} Z_n^{\hat\varepsilon}\, C_{\hat\beta\hat\sigma\hat\varepsilon})
\end{aligned}
\qquad (69)
$$

$$
\begin{aligned}
dA^{15} = \frac{1}{2} dx^m \wedge dx^n\, \big(& V_{nm} + m^2\varkappa^2\xi\, C_{\hat\delta\hat\alpha\hat\beta}\, Z_m^{\hat\alpha} Z_n^{\hat\beta}\, D_{\hat\delta}^{15} - \varkappa m\xi\, U_{nm}^{\hat\alpha} D_{\hat\alpha}^{15} \\
& - 2m^2\varkappa^2\xi\, C_{\hat\delta\hat\alpha\hat\beta}\, U_m^{\hat\alpha} Z_n^{\hat\beta} D_{\hat\delta}^{15}\big) \\
&+ \frac{1}{2} E^\alpha \wedge E^\beta\, m^2\xi\, C_{15\alpha\beta} + dx^m \wedge E^\beta\, C_{15\alpha\beta}\, m^2\varkappa\xi\, D_{\hat\alpha}^{\alpha}(Z_m^{\hat\alpha} - U_m^{\hat\alpha})
\end{aligned}
\qquad (70)
$$

$$
\begin{aligned}
dA^{j} - \frac{1}{2}[A,A]^{j} = \frac{1}{2} dx^m \wedge dx^n\, \big(& W_{nm}^{j} - \varkappa m\, S_{nm}^{\hat\alpha} D_{\hat\alpha}^{j} - 2m^2\varkappa^2\, C_{\hat\delta\hat\alpha\hat\beta}\, S_m^{\hat\alpha} Z_n^{\hat\beta} D_{\hat\delta}^{j} + \\
& + m^2\varkappa^2\, C_{\hat\delta\hat\alpha\hat\beta}\, Z_m^{\hat\alpha} Z_n^{\hat\beta} D_{\hat\delta}^{j} - C_{jkl} W_m^k W_n^l - m^2\varkappa^2 C_{jkl}\, S_m^{\hat\alpha} S_n^{\hat\beta} D_{\hat\alpha}^{k} D_{\hat\beta}^{l} + \\
& + 2\varkappa m\, C_{jkl} W_m^k S_n^{\hat\alpha} D_{\hat\alpha}^{l} - 2m^2\varkappa^2 C_{jkl}\, S_m^{\hat\alpha} Z_n^{\hat\beta} D_{\hat\alpha}^{k} D_{\hat\beta}^{l} - m^2\varkappa^2 Z_m^{\hat\alpha} Z_n^{\hat\beta} D_{\hat\alpha}^{k} D_{\hat\beta}^{l} \\
& + \frac{1}{2} E^\alpha \wedge E^\beta\, m\, C_{j\alpha\beta} + dx^m \wedge E^\beta\, (C_{j\alpha\beta}\, m^2\varkappa\, (Z_m^{\hat\alpha} - S_m^{\hat\alpha}) D_{\hat\alpha}^{\alpha}) \\
& - C_{jkl}\, e^k \wedge dx^m\, W_m^l
\end{aligned}
\qquad (71)
$$

Like in [8] we expact that among the fields Z,S,U Z-S and Z-U are massive and some other independent (orthogonal) linear combination of them is massless.

We take

$$
\begin{aligned}
B_m &\equiv Z_m - S_m \\
C_m &\equiv Z_m - U_m \\
G_m &= Z_m + a S_m + b U_m
\end{aligned}
\qquad (72)
$$

solve it for Z,S,U and put back into (69-71).

We calculated a coefficient standing at

$$
(\partial_a G_b^{\hat\alpha} - \partial_b G_a^{\hat\alpha} - \varkappa m\, C_{\hat\alpha\hat\beta\hat\gamma}\, G_a^{\hat\beta} G_b^{\hat\gamma})^2
$$

Then by recaling G we obtained a canonical form of a gauge field kinematical term with a coupling constant which was calculated to be

$$f_{eff} = \tilde{\varkappa} m \frac{\sqrt{5}}{2\sqrt{1+4}} \qquad \tilde{\varkappa}^2 = 16\pi G \tag{73}$$

G is 4-dimensional gravitational constant

Resumè

As an example of a mechanism of spontaneous compactification Einstein-Yang-Mills SU(3)xU(1) theory compactifying to M^4x CP(3) was investigated. It was shawn that for a canonical compactifying solution of equations of motion and for any value of a magnetic monopole number for U(1) field on CP(3) no tachyons are present in the spectrum of particles. SU(4)x U(1) gauge bosons are identified and effective SU(4) coupling constant is calculated.

Introduction of fermions is possible (because CP(3) admits a spinor structure) and will be investigated in a future.

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