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ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

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KALUZA-KLEIN COSMOLOGIES

Thesis submitted for the degree

of

Master of Philosophy

Candidate

P Y Thomas PANG

Supervisors

Professor Dennis W SCIAMA

Doctor Kei-ichi MAEDA

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獻給我的雙親

TO MY PARENTS

PREFACE

路漫漫其修遠兮
吾將上下而求索
——屈原

The road is prolonged and far-reaching
I will search by all means
Qu Yuan (~ 3rd Century B C)

Life is a long battle. The battle towards our understanding of Nature is even longer and involves a lot of people. Not much can be done in a short time. Physical science unveils the secrets of Nature with careful experimentation and rigorous analyses. Great scientists are also great artists. They tell fascinating stories and draw beautiful pictures. What is more important is that these stories and pictures stand on solid ground. The foundations are the accumulation of experimental facts from the laboratories and observational data of space over the centuries. Yes, this is also a kind of culture; a wealth that belongs to the whole of mankind.

Unification of all the forces of nature and the explanation of 'How God created the World' were the two aims of Albert Einstein, the greatest scientist of modern times, during his later years. They are also the dreams of many contemperary scientists. Particle Physics and Cosmology have had their own ways to address these two questions until recently when they started to help each other to solve a lot of problems. In the past decade, a large amount of work has been dedicated to the establishment of a consistent picture of the very early universe and a complete theory of interactions. Kaluza-Klein type theories are proposed in order to unify the electromagnetic, weak, strong and gravitational forces. At this point, the investigation of higher-dimensional cosmologies becomes important in two respects: to accomodate the new developments into a well founded picture and to derive constraints on the new theories. This dissertation is devoted to this course.

After reviewing some of the important ideas and main features of Kaluza-Klein theories, we go on to develope the framework of a higher-dimensional cosmology. A special class of models is discussed in further detail and a simple model is studied using numerical techniques. It is concluded that this model is very unlikely to be our present universe.

We do not expect every single attempt to be victorious. Rather, we gain something from every one of those, successful or not. The battles will go on and on. I am very happy being able to study the culture of science and join in the great battle. It would not be possible without the generosity and encouragement of one of my supervisors, Professor Dennis W Sciama, to whom I owe a lot. My gratitude is also due to Dr Kei-ichi Maeda for his careful and enthusiastic guidance. I would also like to take this opportunity to thank all my friends at the International School for Advanced Studies (SISSA-ISAS) and the International Centre for Theoretical Physics (ICTP) for making my two years' stay in Trieste so enjoyable.

Trieste, September 1985. P Y T Pang

A NOTE ON NOTATION

The sign conventions of Misner, Thorne and Wheeler (1973) are followed.

The unit $c = \hbar = k = 1$ is used. (In Chapter IV κ is also set to 1.)

Unless otherwise stated, the following symbols have the general meaning.

Greek letters stand for the general space-time coordinates, i.e., from 0 to 3 and from 5 to D+4.

Small latin letters stand for the external space and its coordinates, i.e., from 1 to 3.

Capital latin letters stand for the internal space and its coordinates, i.e., from 5 to D+4.

A bar ($\bar{\ }$) is used to denote the four-dimensional value of a quantity.

A head ($\hat{\ }$) is used to denote vielbein indices.

A tilde($\tilde{\ }$) is used to denote quantities due to the background field.

A subscribe ($_p$) is used to denote the Planck units.

F^4 is the four-dimensional Friedmann universe.

K^D is a D-dimensional constant internal space.

S^D is a D-dimensional sphere.

Λ is the cosmological constant while Λ_F is the fine-tuned value in order to achieve a Friedmann universe in four-dimension.

γ_{mn} and γ_{MN} are the metrics of constant curvature spaces.

Ω_d and Ω_D are the 'volume' of the subspaces as defined in eq (3.7)

k_d and k_D are the curvature constants of the two subspaces. They take the values -1, 0 or 1 for an open, flat or closed subspace.

CONTENTS

I INTRODUCTION

The dialogue between Particle Physics and Cosmology is no doubt getting more and more intense. In the past few years, the early universe has been being used as a test ground of particle theories. On the other hand, the application of quantum field theory in a cosmological context has helped to solve a lot of cosmological problems. This collaboration opens up a new, active and exciting field of research. We believe that important results will keep on being discovered, and our knowledge of the universe will get better and better, with a bit of hard work.

Unification of the fundamental forces has long been the aim of many scientists. Gauge theories play a very important role in this undertaking. They have been successfully applied to unify electromagnetic and weak interactions (Glashow 1961, Weinberg 1967, Salam 1968) and, independently, give a good description of strong interaction through Quantum Chromodynamics QCD[1]. It is generally believed that these gauge theories are unified in some Grand Unified Theory GUT at an energy scale of order 10^{15} GeV. There are also attempts to unify all the forces of nature including gravitation into a single theory in the local invariant version of a special kind of symmetry called supersymmetry which relates bosons and fermions. These supergravity theories[2] naturally point at higher-dimensions and is a prime reason of the recent revival of Kaluza-Klein theories.

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1 The theory of QCD was developed by Fritzsch and Gell-Mann (1972), Fritzsch, Gell-Mann and Leutwyler (1973), Gross and Wilczek (1973) and Weinberg (1973) following some related earlier work by Nambu (1966).

2 For example, N=8 supergravity in four-dimension can be represented by N=1 supergravity in eleven-dimension with the extra bonus that the Lagrangian is unique, but this does not say that higher dimensions are absolutely necessary.

Kaluza (1921)-Klein (1926) theory was, in fact, another approach to the unification of gravitation and other interactions. A five-dimensional extension of Einstein's theory of general relativity was proposed in the original theory so that electromagnetism and gravity could be treated on a similar footing. This idea was generalized into even more dimensions to accomodate the other 'fundamental' interactions by DeWitt (1965). A complete higher-dimensional theory that describes all sorts of interactions is being pursued. There are prescriptions on calculating the gauge coupling constants in terms of the circumferences of the compact subspaces (Weinberg 1983). Since the low-energy interactions are described by $SU(3) \times SU(2) \times U(1)$ gauge symmetry with three different coupling constants, the internal symmetry must have undergone some symmetry breaking process somehow. Some authors have demonstrated that a round higher-dimensional sphere is unstable against deformation in one form or another, in certain cases, leading to a spontaneous symmetry breaking (Lim 1984, Okada 1985). Indeed, these speculations are beautiful, but we cannot test them directly since the energy to excite the internal space may be of Planck scale — a long way beyond terrestrial accelerators can be built and experiment on!

At this point, we have to turn to cosmology. The extreme conditions implied by the now standard Hot Big Bang Model is an inexhaustible source of energy, so it is a possible place to test the high-energy theories. A lot of information has been supplied to particle physics from observations of the remnants from this heat bath. Calculations of the light elements formed in the cosmological nucleosynthesis has served as a very good place to restrain the number of light neutrino types (Steigman, Schramm and Gunn 1977). By requiring the contribution of a stable particle (life-time $\gtrsim 10^9$ years) to the energy density of the universe to be less than the critical density ρ_c (the density that separates an open and a close universe), a useful relation between its mass and its coupling to other particles can be found (Lee and Weinberg 1977). Baryon asymmetry is serving as a good test ground for CP violation (Sakharov 1967). In the 'new inflationary' scenario (more on inflation in the next paragraph), the scale-independent density perturbation that leads to the formation of galaxies and the like is explained by the quantum fluctuation during the slow roll-over of an order parameter down a potential curve (Hawking 1982, Starobinsky 1982, Guth and Pi 1982, Bardeen, Steinhardt and Turner 1983). In this manner, the form of the Lagrangian leading to this effective potential can be tested indirectly.

This is not a one-way street. The beautiful paper on 'inflation' by Alan Guth (1981) has shown that some cosmological problems could be solved if there was an inflationary period due to a phase transition of the right kind. For instance, if the universe stayed for some time in a state where the cosmological constant is large compared to the energy density, there would be some de Sitter-like exponential expansion of the scale factor. Provided this period is long enough, the horizon puzzle can be explained since the size of causally related regions becomes very large. The flatness problem is softened since this period drives the value of Ω (defined as ρ / ρ_c at any time) very close to one. Contribution of monopoles to the energy density of the universe will not be unacceptable any more if inflation dilutes its number density enough. However, since we do not want baryon asymmetry to be swept away as well, the timing of inflation is very important: it has to be after monopole formation and before baryons dominate over antibaryons. In the original version, inflation happens when the universe is trapped in a false vacuum as the temperature of the universe drops below a critical temperature at which the potential at the symmetric minimum becomes higher than that at an asymmetric true vacuum. This model did not work due to the inhomogeneity produced at the end of the phase transition. Linde (1982), Albrecht and Steinhardt (1982) then proposed the 'new inflation' model in which a bubble is formed and then inflated as the order parameter slowly rolls down the potential curve. The Coleman-Weinberg (1973) mechanism of symmetry breaking was used. This model explains the large scale homogeneity and the scale-independent spectrum of density although there are still other problems. Perhaps it is not an overstatement to say that new break throughs in Cosmology depend on new ideas from particle theories.

It is, thus, incomplete to develope Kaluza-Klein theories without considering their cosmological implications. There is also the hope of getting more insights and alternatives in the scenarios of the early universe. This dissertation is written for this purpose. Without spending too much time on the complicated non-Abelian structure of a generalized Kaluza-Klein theory, the beauty of Kaluza-Klein theories is illustrated by a five-dimensional example in Chapter II. This is followed by a discussion of some general properties of a higher-dimensional theory, especially on the features that have important cosmological significance. In Chapter III, the conventional Friedmann Universe is extended into a higher-dimensional setting. The dynamics of a special class of models in which the effective potential is a function of only the scale factor of the internal space is investigated. Cosmological constraints are imposed to restrain and provide a framework to 'test' this class of models. Chapter IV is devoted to the detail studies of a toy model using a Candelas-Weinberg potential.

II KALUZA-KLEIN THEORIES

Two major concepts lie at the foundation of modern physics: local coordinate invariance and local internal symmetry. The former leads to the theory of gravity while the latter leads to the gauge theories of strong, weak and electromagnetic interactions. Merely a few years after Einstein published his celebrated papers on General Theory of Relativity, Kaluza (1921) came up with the marvellous observation that the then known forces, gravitation and electromagnetism, could be unified by means of a five-dimensional metric. His theory was expanded and made connected to quantum theory by Klein (1926). Although the attempt was not completely successful, one lesson is learnt—local gauge symmetry can be derived from local coordinate invariance if a higher-dimensional world is necessary.

Over the decades, the theory has been developed by a number of people. Einstein and Bergmann (1938) gave physical meaning to the fifth dimension. DeWitt (1965) extended the gauge group to non-Abelian. Despite its elegance, the Kaluza-Klein concept has remained outside the mainstream of physics until recently. With the success of gauge theories, people march on for a theory that unifies gravitation with the other forces of nature by means of supergravity. These theories naturally lead people to consider more space-time dimensions. This area of research is very active and fast-developing. This review is not intended to cover the recent developments of Kaluza-Klein theories. However, in this chapter, some basic features of modern Kaluza-Klein theories that are important to a consistent picture of the universe are discussed.

2.1 The five-dimension theory

Following history, and also as a simple example of how the Kaluza-Klein idea works, we start with the five-dimension theory. The metric is written in a 4 + 1 form

$$(2.1) \qquad g_{\mu\nu} = \begin{pmatrix} g_{mn} & g_{m5} \\ g_{n5} & g_{55} \end{pmatrix}$$

where m,n run from 0 to 3 in this chapter, otherwise the sign conventions on page iii are followed.

Consider an infinitesimal coordinate transformation

$$(2.2) \qquad \begin{cases} x^m \to x'^m = x^m \\ x^5 \to x'^5 = x^5 - \phi(x^m), \end{cases}$$

the metric tensor transforms like a rank two covariant tensor; in particular, the components g_{m5} transform like

$$\begin{aligned} g_{m5} \to g'_{m5} &= g_{\mu\nu} \frac{\partial x^\mu}{\partial x'^m} \frac{\partial x^\nu}{\partial x'^5} \\ &= g_{\mu\nu} [\delta^\mu{}_m + \delta^\mu{}_5 (\frac{\partial \phi}{\partial x^m})] \delta^\nu{}_5 \\ &= g_{m5} + g_{55} \frac{\partial \phi}{\partial x^m} \end{aligned}$$

which is the transformation law for a gauge potential

$$A_m \to A_m' = A_m + \partial_m \phi .$$

So it is possible to include a gauge field in the theory by means of an extended metric.

A Killing vector ∂_5 is assumed in Kaluza-Klein theory. It is better to use the Horizontal Lift Basis (HLB). The basis one-forms are

(2.4) $$\begin{cases} dx^{\hat{m}} = dx^m \\ dx^{\hat{5}} = dx^5 + A_m dx^m , \end{cases}$$

and the dual basis is

(2.5) $$\begin{cases} e_{\hat{m}} = \partial_m - A_m \partial_5 \\ e_{\hat{5}} = \partial_5 . \end{cases}$$

In terms of this basis, we have the following metric components

$$\begin{aligned} g(e_{\hat{m}}, e_{\hat{n}}) &= g(\partial_m - A_m \partial_5, \partial_n - A_n \partial_5) \\ &= g(\partial_m, \partial_n) - A_m g(\partial_5, \partial_n) - A_n g(\partial_5, \partial_n) \\ &\quad + A_m A_n g(\partial_5, \partial_5) \\ &= g_{mn} - A_m g_{n5} - A_n g_{m5} + A_m A_n g_{55} \end{aligned}$$

$$\begin{aligned} g(e_{\hat{m}}; e_{\hat{5}}) &= g(\partial_m - A_m \partial_5, \partial_5) \\ &= g_{m5} - g_{55} A_m \end{aligned}$$

$$g(e_{\hat{5}}, e_{\hat{5}}) = g(\partial_5, \partial_5) = g_{55} .$$

If we define $g_{m5} = g_{55} A_m$, we get the metric in this new basis

(2.6) $$g_{\hat{\mu}\hat{\nu}} = \begin{pmatrix} g_{mn} - g_{55} A_m A_n & 0 \\ 0 & g_{55} \end{pmatrix} .$$

This is to be identified with

$$\begin{pmatrix} \bar{g}_{mn} & 0 \\ 0 & g_{55} \end{pmatrix}$$

It is, however, important to realize that the HLB is anholonomic:

$$[e_{\hat{m}}, e_{\hat{n}}] = -F_{mn}\partial_5 = C_{\hat{m}\hat{n}}^{\ \ \hat{5}} e_{\hat{5}}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$.

The formalism and basic results needed to work in this basis are summarized in Appendix A, and the calculation in this particular case is performed in Appendix B. The curvature scalar is found to be

$$R = \bar{R} - \frac{1}{4} g_{55} F^{mn} F_{mn} \tag{B6}$$

where $\bar{R}$ is the curvature scalar in a four-dimensional theory.

The five-dimensional Einstein-Hilbert action is given by

$$S = \frac{1}{16\pi G} \int d^5x \sqrt{-g}\, R \tag{2.7}$$

where g is the determinant of the metric.
When the metric (2.6) is used, we can easily perform the integration in the fifth dimension, giving

$$S = \frac{L}{16\pi G} \int d^4x \sqrt{-\bar{g}}\, (\bar{R} - \frac{1}{4} g_{55} F^{mn} F_{mn}) \tag{2.8}$$

where $\bar{g}$ is the determinant of the four-dimensional metric $\bar{g}_{mn}$, and
L is the circumference of the extra dimension given by

$$L = \int \sqrt{g_{55}}\, dx^5 = 2\pi \sqrt{g_{55}} \tag{2.9}$$

In order to obtain the Maxwell Lagrangian, we rescale the potential $A_m \rightarrow r a_m$ where a_m is the electromagnetic four-potential. Equation (2.8) is compared to the four-dimensional Einstein-Maxwell action, we get

$$\begin{cases} \dfrac{L}{16\pi G} = \dfrac{1}{16\pi \bar{G}} \\ g_{55} r^2 = 4\bar{G} \end{cases} \tag{2.10}$$

where $\bar{G}$ is the four-dimensional gravitation constant.

To see how the quantization of electric charge is related to L we consider the wave equation of a massless complex field $\psi(x^\mu)$:

$$\Box\, \psi(x^\mu) = 0$$

which when expanded gives

$$\left[g^{mn} (\partial_m - r a_m \partial_5)(\partial_n - r a_n \partial_5) + \frac{1}{g_{55}} \partial_5 \partial_5 \right] \psi(x^\mu) = 0.$$

Consider the Fourier expansion on the circle[1]

$$\psi(x^\mu) = \sum_{k=-\infty}^{\infty} \psi_k(x^m)\, e^{\frac{ikx^5}{2\pi}},$$

the wave equation becomes

(2.11) $$\left[g^{mn} \left(\partial_m - i a_m \frac{kr}{2\pi}\right)\left(\partial_n - i a_n \frac{kr}{2\pi}\right) - \frac{k^2}{g_{55}(2\pi)^2} \right] \psi_k(x^m) = 0.$$

Now we can see the electric charge and mass of this 'particle' in four-dimension to be

(2.12) $$\begin{cases} e_k = k\,e = k\dfrac{r}{2\pi} = \dfrac{k}{2\pi}\sqrt{\dfrac{4\bar{G}}{g_{55}}} \\ m_k = \dfrac{k}{2\pi\sqrt{g_{55}}} \end{cases}$$

which are determined solely by geometry[2]!

From these equations, it is easy to realize that if we want to identify e with the unit of electric charge, $\sqrt{g_{55}}$ has to be of Planck scale

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1 The group space of the U (1) symmetry of electromagnetism is a circle, S^1, which is tacitly assumed as the topology of x^5.

2 This is not describing particles and interactions by geometry.

(which is good anyway: to make sure that the extra dimension is not observed). However, the mass unit is also of Planck scale. Thus, the mass spectrum of the 'ordinary' particles cannot be explained by Kaluza-Klein theory on its own. These particles have important significance in the calculation of quantum effect in four-dimension as well as in cosmology, as will be discussed in the later sections.

It is also interesting to note that if we took the fifth fimension to be time-like (g_{55} negative), both the Maxwell Lagrangian and the wave equation would have a wrong sign, leading to negative electromagnetic energy and tachyonic behaviour. From now on, all the extra dimensions are taken to be space-like and have equal physical meaning as the ordinary space.

2.2 Some features of the higher-dimensional theories

The Kaluza-Klein concept is not interesting at all if it cannot be extended into dimensions higher than five and generalized from Abelian Maxwell field to non-Abelian Yang-Mills field. Indeed, if we replace the circle at each space-time point by some general manifold M which permits the action of a group G on it, a gauge theory based on G can be obtained. In general, M could be smaller than the group manifold G —it could be the coset space G/H where H is a subgroup of G. Neither the structure of the groups and manifolds that are favoured by particle physicists nor the vielbein formalism is discussed here. (the interested readers are asked to read the book edited by Lee 1984, and Salam and Strathdee, 1982.) In this section, some properties of the higher-dimensional theories that are important in a cosmological consideration are summarized.

2.2.1 Compactification and the mass spectrum

In the last section, we have seen that the beauty of Kaluza-Klein theories is closely related to the fact that the extra dimensions are space-like. The fact that the observed world is apparently four-dimensional suggests that the extra dimensions are highly compactified and geometry of the internal space is somehow decoupled from that of the four-dimensional space-time:

$$g_{\mu\nu} = \begin{pmatrix} g_{mn}(x^l) & 0 \\ 0 & g_{MN}(x^L) \end{pmatrix} \quad (2.13)$$

Because of the form of the metric, the D'Alembertian in the higher-dimensional space-time can be expressed as the sum of a four-dimensional D'Alembertian and a D-dimensional Laplacian

$$\Box = \bar{\Box} + \Delta_D . \quad (2.14)$$

When a scalar field ϕ of mass M is introduced in the higher-dimensional manifold, we have the wave equation

$$(\Box + M^2)\phi = 0 . \quad (2.15)$$

Assume that ϕ can be expressed as a sum of harmonic eigen-functions ϕ_l with eigen-value f_l in the compactified internal manifold, with degeneracy d_l, i.e.,

$$\phi = \sum_l d_l \phi_l , \text{ and} \quad (2.16)$$

$$\Delta_D \phi_l = f_l \phi_l . \quad (2.17)$$

Then we can see that this massive scalar field (massless if we take $M = 0$) becomes an infinite tower of massive scalar fields of mass

$(M^2 + f_1)$ with degeneracy d_1, in the four-dimensional world:

$$[\bar{\Box} + (M^2 + f_1)]\,\phi_1 = 0 . \tag{2.18}$$

It is important to note, however, that this mass spectrum cannot explain the observed mass spectrum of quarks and leptons because it is too heavy. To see this, we just need to consider the Laplacian

$$\Delta_D = g^{MN}\partial_M \partial_N$$

and

$$\| g^{MN} \| = \frac{1}{A^2} ,$$

where A is the scale factor of the internal space.
Since A is very very small, the eigen-value f_1 is very large.

2.2.2 The Casimir effect or the one-loop quantum correction

In 1948, Casimir (1948) showed that an attractive force exists between two uncharged conducting plates as a consequence of the quantum mechanical vacuum fluctuations of electromagnetic field. Similar forces are expected to arise whenever periodic boundary conditions are imposed on a quantum field. Since the extra dimensions of Kaluza-Klein theories are compactified, we would expect the Casimir effect to take place. This Casimir energy is given by the one-loop effective potential obtained by expanding the action about a suitable background configuration and then summing up the first order contributions.

In Kaluza-Klein theories, this quantum effect is very important because the length scales of the internal space are not too far from Planck length. On the other hand, the balance of this effect and other effects like the curvature effect enables the size of the internal space, A, to be determined. (See the next chapter.) The gauge coupling constant g in a Kaluza-Klein theory is proportional to $\bar{l}_p/A$ (Weinberg 1983). This scheme enables predictions of Kaluza-Klein theories to be compared with measured quantities.

Contributions to this Casimir energy due to various species of matter fields and gravitational field are discussed by Candelas and Weinberg (1984) and Ordóñez and Rubin (1984), Sarmadi (1984) and Chodos and Myers (1984, 1985) respectively.

The effective potential is given in the general form

$$V = \frac{C_N}{A^4} \tag{2.19}$$

where C_N is a constant in the case of matter field and has further dependence on A in the case of gravitons.

The Candelas-Weinberg model is briefly described in section 4.1, and a cosmological model using this model is studied in Chapter IV.

III HIGHER-DIMENSIONAL COSMOLOGIES

No one will deny that the Kaluza-Klein approach to unify all known forces is appealing, if not imperative, yet it is hard to perceive that we are living in a world that has more than four dimensions. The explanation is that space-time is actually a product of a four-dimensional manifold and a D-dimensional compact manifold. The size of this 'internal' space is so small that it is not revealed in everyday kinematics. Recall that high energy is needed to probe small scales. The ideal place to look for traces of evidence is the very early universe, if we believe in a Hot Big Bang Cosmology.

Bearing in mind that the four-dimensional cosmological constant and the gauge coupling parameters are not fundamental constants in a higher-dimensional theory (Candelas and Weinberg 1984), we may like to ask whether the compact dimensions are always so small and much less than the space-time dimensions . It is also important to make sure that the configuration of a constant internal space is stable, and the variations of the above mentioned quantities are small.

The Hot Big Bang Model is generally accepted as the standard model and gives a picture that is consistent with 'known' physics up to as early as nucleosynthesis (Weinberg 1972). If we push further backward in time, the temperature gets higher and higher. Then we shall possibly hit a quark-hadron phase transition and then baryosynthesis, followed by an inflation, before we arrive at a Grand Unification era. Experts are investigating these extensions of the standard picture. At this point, we would like to ask the question: what about if the universe has more than four dimensions as is assumed in the above picture? The following scenario may be considered. After the universe emerges from an quantum era, a topological compactification occurs by an unknown mechanism. At this time, the scale of the internal space is nearly the same as that of the three-space. Then the universe evolves according to the higher dimensional Einstein equations. The internal space compactifies dynamically whereas the external space expands. This shrinking is stopped by some other effects and the universe is described by the Friedmann equations with a constant internal space. The massive particles due to the compactification is diluted in an inflation afterwards.

3.1 The higher-dimensional Einstein equations

The Einstein equations are generalized into a higher-dimensional form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa^2 T_{\mu\nu} \tag{3.1}$$

where $\kappa^2 = 8\pi G$ and the sign conventions used are listed on page iii. Assume that the universe consists of one time-like and two constant curvature space-like subspaces of d and D dimensions. The metric is then written in the form

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & a^2(t)\gamma_{mn} & 0 \\ 0 & 0 & A^2(t)\gamma_{MN} \end{pmatrix} \tag{3.2}$$

where a and A are the scale factors of the two subspaces, and γ_{mn} and γ_{MN} are the metrics of the subspaces.
Usually, d is taken to be three, but in this chapter, it is unspecified to allow a general discussion.

From the Einstein equations, the stress-energy tensor $T_{\mu\nu}$ is istropic in each subspace

$$(3.3) \qquad T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 \\ 0 & p_d \gamma_{mn} & 0 \\ 0 & 0 & p_D \gamma_{MN} \end{pmatrix}.$$

The higher-dimensional Einstein equations then become

$$(3.4) \qquad \frac{d(d-1)}{2}\left[\left(\frac{\dot a}{a}\right)^2 + \frac{k_d}{a^2}\right] + \frac{D(D-1)}{2}\left[\left(\frac{\dot A}{A}\right)^2 + \frac{k_D}{A^2}\right] + dD\frac{\dot a \dot A}{aA} - \Lambda = \kappa^2 \rho$$

$$(3.5) \qquad \begin{cases} (d-1)\dfrac{\ddot a}{a} + D\dfrac{\ddot A}{A} + \dfrac{(d-1)(d-2)}{2}\left[\left(\dfrac{\dot a}{a}\right)^2 + \dfrac{k_d}{a^2}\right] \\ \quad + \dfrac{D(D-1)}{2}\left[\left(\dfrac{\dot A}{A}\right)^2 + \dfrac{k_D}{A^2}\right] + D(d-1)\dfrac{\dot a \dot A}{aA} - \Lambda = -\kappa^2 p_d \\ (D-1)\dfrac{\ddot A}{A} + d\dfrac{\ddot a}{a} + \dfrac{(D-1)(D-2)}{2}\left[\left(\dfrac{\dot A}{A}\right)^2 + \dfrac{k_D}{A^2}\right] \\ \quad + \dfrac{d(d-1)}{2}\left[\left(\dfrac{\dot a}{a}\right)^2 + \dfrac{k_d}{a^2}\right] + d(D-1)\dfrac{\dot a \dot A}{aA} - \Lambda = -\kappa^2 p_D . \end{cases}$$

Here we have taken the internal space to be maximally symmetric, and the curvature constants k_d and k_D take the value 1, 0 or -1 according as the space is close, flat or open respectively. After some rearrangment of this set of equations, the system can be rewritten in the form of a constraint equation (3.4) and two dynamical equations

$$(3.6) \qquad \begin{cases} \dfrac{\ddot a}{a} + (d-1)\left[\left(\dfrac{\dot a}{a}\right)^2 + \dfrac{k_d}{a^2}\right] + D\dfrac{\dot a \dot A}{aA} = \dfrac{2\Lambda}{d+D-1} + \kappa^2\left(-\dfrac{T}{d+D-1} + p_d\right) \\ \dfrac{\ddot A}{A} + (D-1)\left[\left(\dfrac{\dot A}{A}\right)^2 + \dfrac{k_D}{A^2}\right] + d\dfrac{\dot a \dot A}{aA} = \dfrac{2\Lambda}{d+D-1} + \kappa^2\left(-\dfrac{T}{d+D-1} + p_D\right) \end{cases}$$

where T is the trace of the stress-energy tensor

$$T = T^{\mu}{}_{\mu} = -\rho + d\,p_d + D\,p_D .$$

If we differentiate the constraint equation (3.4) with respect to t and use the two dynamical equations, we can arrive at a conservation equation of the form

$$(3.7) \qquad \frac{d}{dt}(\Omega_d \Omega_D \rho) + p_d \Omega_D \frac{d}{dt}\Omega_d + p_D \Omega_d \frac{d}{dt}\Omega_D = 0$$

where Ω_d and Ω_D are the 'volume' of the two subspaces

$$\Omega_D(A) = \int d^D x \sqrt{\gamma_D} \qquad \gamma_D \text{ is the determinant } \gamma_{MN}$$

$$= \frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})} A^D \quad \text{in the case of spheres.}$$

This energy momentum conservation equation is just the consequence of Bianchi identity applied to the Einstein equations.

The set of equations (3.4) and (3.6) are good for investigating the individual scale factors a and A and see how they are coupled to each other through the term $\frac{\dot{a}\dot{A}}{aA}$.

The Einstein equations can also be rewritten in terms of the variables

$$(3.8) \qquad \begin{cases} y = \ln R \\ z = \ln(A/a) \\ \eta = \int^t R^{-1}(t')\,dt' \end{cases}$$

where $R = (a^d A^D)^{\frac{1}{d+D}}$ is the geometrical mean of the scale factors and η is the 'conformal' time.

After some algebraic calculations, the following system of equations is obtained

$$
(3.9)\quad \begin{cases}
(d+D)(d+D-1)\,y'^2 - \dfrac{dD}{d+D}\,z'^2 + F_1(z) = 2e^{2y}(\kappa^2\rho + \Lambda) \\[2ex]
y'' + \dfrac{d+D+1}{2}\,y'^2 + \dfrac{dD}{2(d+D)^2}\,z'^2 + \dfrac{1}{2(d+D)}\,F_1(z) \\[2ex]
\quad = e^{2y}\left[\dfrac{d+D+1}{(d+D)(d+D-1)}\Lambda - \kappa^2 \dfrac{T}{(d+D)(d+D-1)}\right] \\[2ex]
z'' + (d+D-1)\,y'z' + F_2(z) = -\kappa^2 e^{2y}(p_d - p_D)
\end{cases}
$$

where a prime (') denotes differentiation with respect to η, and

$$F_1(z) = d(d-1)\,k_d \exp\left(\frac{2D}{d+D}z\right) + D(D-1)\,k_D \exp\left(\frac{-2d}{d+D}z\right)$$

$$F_2(z) = -(d-1)\,k_d \exp\left(\frac{2D}{d+D}z\right) + (D-1)\,k_D \exp\left(\frac{-2d}{d+D}z\right).$$

The equations (3.9) are used in the numerical integrations in Chapter IV to see how the higher-dimensional universe evolves in time η.

3.2 The stress-energy tensor

"Geometry tells matter how to move, and matter tells geometry how to curve." Stress-energy tensor is the device to measure the distribution of matter which effects the dynamics of space-time. In general, one can write down the Lagrangian $\mathcal{L}$ of the system and then uses a variational method to find the stress-energy tensor. The Lagrangian may usually be separated into two parts:

$$(3.10)\qquad \mathcal{L} = \mathcal{L}_g + \mathcal{L}_m$$

where $\mathcal{L}_g = \sqrt{-g}\,R$ is the gravitational part, and

$\mathcal{L}_m$ is the matter Lagrangian which includes the various particles present and being created during the cosmological evolution, quantum fluctuations of the fields, etc.

It is well known that by using variational principle, the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$ emerges from $\mathcal{L}_g$ whereas the stress-energy tensor is given by

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_m}{\delta g^{\mu\nu}} . \tag{3.11}$$

The matter Lagrangian $\mathcal{L}_m$ is model-dependent. In the present discussion, two contributions to $\mathcal{L}_m$ are important:

$$\mathcal{L}_m = \mathcal{L}_{BG} + \mathcal{L}_{rad} \tag{3.12}$$

where $\mathcal{L}_{BG}$ is the effective Lagrangian of a background field which may be due to the quantum fluctuations of matter or gravitation fields, or the vacuum expectation value of some matter field, and

$\mathcal{L}_{rad}$ is the Lagrangian of radiation which is important when the temperature of the universe is very high.

A lot of Kaluza-Klein theories and models of their compactification have been built. In order to investigate the cosmological aspect of these theories in a systematic way, Maeda (1985) has classified these theories or models into three types by the effective Lagrangian $\mathcal{L}_{eff}$. The first class (class I theories) are those $\mathcal{L}_{eff}$ is a function of only the scale factor A of the internal space, i.e., $\mathcal{L}_{eff} = \mathcal{L}_{eff}(A)$. The second class (class II theories) are those $\mathcal{L}_{eff}$ is a function of two or more variables, A and some 'scalar' fields σ_i. The last class are those the Lagrangian contains the curvature squared terms such as $R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma}$.

This dissertation is mainly concerned with class I models. (For examples, see Maeda 1985.) In this class of models, we may write the effective potential of the background field, V_{BG}, as defined by

(3.13) $$\mathcal{L}_{BG} = - \sqrt{-g}\, V_{BG} .$$

In general, the effective potential may have contributions from a number of various sources (including a cosmological constant, Λ), and depends only on the scale factor A of the internal space

(3.14) $$V_{BG} = V_{BG}(A) = \sum_i V_i(A).$$

We can separate the contributions of radiation and that of the background field to the energy density and principal pressures by

(3.15) $$\begin{cases} \rho = \tilde{\rho} + \rho_{rad} \\ p_d = \tilde{p}_d + p_{rad} \\ p_D = \tilde{p}_D + p_{RAD} \end{cases}$$

where all quantities due to the background field are denoted by a tilde ($\tilde{}$), and

p_{rad} and p_{RAD} are the principal pressures in the two subspaces due to the radiation.

From equation (3.10), we have

(3.16) $$\delta \mathcal{L}_{BG} = - \delta\sqrt{-g}\, V_{BG}(A) - \sqrt{-g}\, \delta V_{BG}(A) .$$

We know that

$$\delta\sqrt{-g} = -\frac{\sqrt{-g}}{2} g_{\mu\nu}\, \delta g^{\mu\nu}$$

and from

$$\delta g^{MN} = \delta\left(\frac{1}{A^2}\gamma^{MN}\right) = -2\frac{\delta A}{A} g^{MN}$$

that

$$\delta V_{BG}(A) = V'_{BG}\,\delta A = -\frac{A V'}{2 D} g_{MN}\, \delta g^{MN} .$$

Therefore, from the above relations, we can work out the quantities

$$(3.17)\quad \begin{cases} \tilde{T}_{\mu\nu} = - g_{\mu\nu} V_{BG}(A) - \frac{1}{D} A V'_{BG}(A)\, g_{MN}\, \delta_{\mu}^{M}\, \delta_{\nu}^{N} \\ \tilde{\rho} = - \tilde{T}^{0}{}_{0} = V_{BG}(A) \\ \tilde{p}_{d} = \tilde{T}^{m}{}_{m} / d = - V_{BG}(A) \\ \tilde{p}_{D} = \tilde{T}^{M}{}_{M} / D = - V_{BG}(A) - \frac{1}{D} A V'_{BG}(A) \\ \tilde{T} = - \tilde{\rho} + d\, \tilde{p}_{d} + D\, \tilde{p}_{D} \\ \quad = - [(d + D + 1) V_{BG}(A) + A V'_{BG}(A)]. \end{cases}$$

3.3 The cosmological constant

Ever since the time of Einstein, the cosmological constant has been a mystery and poses challenge to any ambitious model in cosmology. From astronomical observations, the upper bound of the energy density of the universe is 2 ρ_c, which implies the limit on the cosmological constant

$$|\bar{\Lambda}| / \bar{m}_p^{\,2} < 10^{-120}$$

where $\bar{\Lambda}$ and $\bar{m}_p$ are the four-dimensional cosmological constant and Planck mass respectively.

On the other hand, one might expect that the zero point energies of quantum fluctuations would produce an effective or induced $\bar{\Lambda} / \bar{m}_p^{\,2}$ of order one. In Kaluza-Klein theories, since the extra dimensions are regarded as physical reality and are very small, as a result of this 'compactification', the four-dimensional cosmological constant becomes large and negative ($\sim -\bar{m}_p^{\,2}$). As the observed universe is so flat, we usually introduce a higher-dimensional cosmological constant Λ in the higher-dimensional Einstein equations to make the four-dimensional world free of cosmological constant after the compactification of the internal space. However, as the four-dimensional cosmological constant is so close to zero, Λ has to be fine-tuned to Λ_F, the value to achieve a Friedmann universe in four dimensions.

Let us see how this works. When we substitute the components of the stress-energy tensor from equations (3.15) and (3.17) into the Einstein equations (3.4) and (3.6), we obtain the following equations

$$(3.18)\quad \begin{cases} \frac{d(d-1)}{2}\left[\left(\frac{\dot a}{a}\right)^2+\frac{k_d}{a^2}\right]+\frac{D(D-1)}{2}\left[\left(\frac{\dot A}{A}\right)^2+\frac{k_D}{A^2}\right] \\ \qquad +dD\frac{\dot a\dot A}{aA}-\Lambda=\kappa^2(V+\rho_{rad}) \qquad (V=V_{BG}) \\ \frac{\ddot a}{a}+(d-1)\left[\left(\frac{\dot a}{a}\right)^2+\frac{k_d}{a^2}\right]+D\frac{\dot a\dot A}{aA}=\frac{2\Lambda}{d+D-1}+\kappa^2\left(\frac{2V+AV'}{d+D-1}-p_{rad}\right) \\ \frac{\ddot A}{A}+(D-1)\left[\left(\frac{\dot A}{A}\right)^2+\frac{k_D}{A^2}\right]+d\frac{\dot a\dot A}{aA}=\frac{2\Lambda}{d+D-1}+\kappa^2\left(\frac{2V+AV'}{d+D-1}\right. \\ \qquad\qquad \left.-\frac{AV'}{D}-p_{RAD}\right). \end{cases}$$

In order to find the conditions that in a late time, low temperature ($T \ll 1/A$) limit, the universe is described by the Friedmann equations, the following conditions are imposed

$$(3.19)\quad \begin{cases} \dot A = 0 \\ A = A_0 \\ p_{RAD} = 0 . \end{cases}$$

The set of equations (3.18) now reduces to

$$(3.20)\quad \begin{cases} \frac{d(d-1)}{2}\left[\left(\frac{\dot a}{a}\right)^2+\frac{k_d}{a^2}\right]-\bar{\varkappa}^2\bar{\rho}_r=\Lambda-\frac{D(D-1)}{2}\frac{k_D}{A_0^2}+\varkappa^2 V(A_0) \\ \frac{\ddot a}{a}+(d-1)\left[\left(\frac{\dot a}{a}\right)+\frac{k_d}{a^2}\right]-\bar{\varkappa}^2\bar{p}_r=\frac{2\Lambda}{d+D-1}+\varkappa^2\left(\frac{2V(A_0)+A_0V'}{d+D-1}\right) \\ \frac{(D-1)k_D}{A_0^2}=\frac{2\Lambda}{d+D-1}+\varkappa^2\left[\frac{2DV(A_0)+A_0V'(A_0)(1-d)}{D(d+D-1)}\right] \end{cases}$$

where $\bar{\rho}_r$ and $\bar{p}_r$ are the four-dimensional energy density and pressure of radiation defined by $\bar{\rho}_r=\rho_{rad}/\Omega_D$ and $\bar{p}_r=p_{rad}/\Omega_D$. Comparing the first two equations in (3.20) to the Friedmann equations, we can obtain the conditions

$$(3.21)\quad \begin{cases} \Lambda=\frac{D(D-1)}{2}\frac{k_D}{A_0^2}-\varkappa^2 V(A_0) \\ \Lambda=-\varkappa^2\left[\frac{1}{2}A_0V'(A_0)+V(A_0)\right], \end{cases}$$

when the third equation can be implied by these two.

When the potential shape is determined from the underlying Kaluza-Klein theory, these two equations serve to find the parameter A_0 of the internal space and the cosmological constant, Λ_F, that is usually necessary for a Friedmann universe to exist at a late time.

Following Maeda (1985), by introducing a 'potential' for the curvature term of the internal space

$$(3.22)\quad V_{CUR}=-\frac{D(D-1)}{2\varkappa^2}\frac{k_D}{A^2},$$

and defining the total effective potential by

$$(3.23)\quad V_{TOT}=V_{CUR}+V_{BG}+\frac{\Lambda}{\varkappa^2},$$

the conditions (3.21) then become

$$(3.24) \quad \begin{cases} V_{TOT}(A_0) = 0 \\ V'_{TOT}(A_0) = 0 . \end{cases}$$

In other words, this total potential has to have an extremum at a point A_0 where the potential vanishes as well. When all these conditions are satisfied, we are assured that this particular Kaluza-Klein theory and model of compactification admits the late time behaviour of a [four-dimensional Friedmann universe] X [constant compact internal space] $(F^4 \times K^D)$.

3.4 Local stability analysis

Now that we have a set of equilibrium configurations for the late time behaviour of the universe, we must also investigate its stability against small perturbations (Bailin, Love and Vayonakis 1984).
In the cases of open ($k_d = -1$) or flat ($k_d = 0$) universe, consider the following expansion (Maeda 1985)

$$(3.25) \quad \begin{cases} \log a(t) = \alpha(t) = \alpha_0(t) + \delta\alpha(t) \\ \log A(t) = \beta(t) = \beta_0 + \delta\beta(t) \end{cases}$$

where $a_0(t) = \exp(\alpha_0(t))$ is the Friedmann solution in the radiation-dominated era and the constant $\beta_0 = \log A_0$ is the solution of equations (3.21),

the dynamical equation for $\delta\beta$ is given by, for $\delta\alpha$, $\delta\beta$ small,

$$(3.26) \quad \delta\ddot{\beta} + d\,\dot{\alpha}_0\,\delta\dot{\beta} + B_0\,\delta\beta = 0$$

where $B_0 = \dfrac{\chi^2 (d-1)}{D(d+D-1)} V''_{TOT}(\beta_0)$.

The general solution (or asymptotic solution for $k_d = -1$) is

(3.27) $$\delta\beta = t^{-k} [C_1 H_k^{(1)} (\sqrt{B_0}\, t) + C_2 H_k^{(2)} (\sqrt{B_0}\, t)]$$

where C_1 and C_2 are constants,

$H_k^{(1)}$ and $H_k^{(2)}$ are the Hankel functions, and

$$k = \begin{cases} \dfrac{d-1}{2(d+1)} & \text{for } k_d = 0 \\ \dfrac{d-1}{2} & \text{for } k_d = -1 . \end{cases}$$

If $B_0 > 0$, the solution (3.27) is a damped oscillation. The asymptotic behaviour is

$$\delta\beta \sim t^{-k-\frac{1}{2}} \quad \text{for } t \to \infty .$$

But if $B_0 < 0$, the solution (3.27) grows exponentially and thus A_0 is an unstable point.

Anyalysis of the constraint equation also show that, if $B_0 > 0$,

$$\delta\alpha \sim t^{-k-\frac{1}{2}} \quad \text{for } t \to \infty .$$

The condition $B_0 > 0$ is the same as

(3.28) $$V''_{TOT}(\beta) \big|_{\beta=\beta_0} > 0 .$$

In summary, the conditions for the existence of a $F^4 \times K^D$ solution which is stable against small perturbation is the existence of a local minimum of the 'total' effective potential V_{TOT} defined in equation (3.23) which also vanishes at this point.

3.5 An attractor and a graceful return

A cosmological model based on a Kaluza-Klein theory that can explain all the coupling constants and masses of the fundamental particles still cannot explain the existence of the present universe if the solutions of the dynamical equations are of a run-away type instead of settling down to a Friedmann universe with a constant internal space, an attractor universe. It is not easy to say anything definite, especially when we do not have a complete quantum theory of gravity to investigate the universe before or near Planck time t_p. As a consequence we do not know how the universe looks like when it emerges from a quantum gravitational era[1]. Perhaps what we can best do is to investigate the range of initial parameters in which the universe is an attractor. If this range is big enough that a natural set of initial conditions will guarantee a graceful return to the present universe, the model can be considered as natural. (e.g. see Maeda and Nishino 1985.)

To this end, we consider the equation for β ($= \log A$) again. If β approaches β_0 asymptotically, we will find the $F^4 \times K^D$ solution anyway. The last equation in equations (3.18) can be rewritten as, for $p_{RAD}=0$,

$$\ddot{\beta} + (d\dot{\alpha} + D\dot{\beta})\dot{\beta} + \frac{(D-1)k_D}{A^2} = \kappa^2 (\tilde{p}_D - \frac{\tilde{T}}{d+D-1}) + \frac{2\Lambda}{d+D-1} . \tag{3.29}$$

Consider a dynamical system of β by introducing the 'energy' E and the 'potential' U as

$$\begin{cases} E = \frac{1}{2}\dot{\beta}^2 + U(\beta) \\ U(\beta) = -\int_{\beta_0}^{\beta} d\beta' f(\beta') \end{cases} \tag{3.30}$$

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1 Quantum Cosmology is a way to find our initial conditions.

where $f(\beta) = -(D-1)k_D e^{-2\beta} + \varkappa^2(\tilde{p}_D - \frac{\tilde{T}}{d+D-1}) + \frac{2\Lambda}{d+D-1}$

$$= \frac{\varkappa^2}{d+D-1}(2V_{TOT} - \frac{d-1}{D}V'_{TOT}) .$$

Then the equation (3.29) can be rewritten as

$$\dot{E} = -(d+D)\dot{y}\dot{\beta}^2 \tag{3.31}$$

where y is defined in (3.8).

At $\beta = \beta_0$, we have the following properties

$$\begin{cases} U(\beta_0) = 0 \\ U'(\beta_0) = -f(\beta_0) = 0 \\ U''(\beta_0) = -f'(\beta_0) = \frac{\varkappa^2(d-1)}{D(d+D-1)} V''_{TOT}(\beta_0) . \end{cases}$$

So, if $\beta = \beta_0$ is a local minimum of V_{TOT}, as is required in the last section, so is $U(\beta)$. There is a finite region near $\beta = \beta_0$ where $V_{TOT}(\beta) \geqslant 0$ because β_0 is a local minimum. Let

$$(\beta'_1, \beta'_2) = \{ \beta \mid \beta_0 \in (\beta'_1, \beta'_2) \text{ and } V_{TOT} \geqslant 0 \ \ \forall \beta \in (\beta'_1, \beta'_2) \} .$$

For $k_d = 0$ or -1, since the constraint equation gives

$$\dot{y} = \pm\sqrt{[\frac{dD}{d+D}(\dot{\beta}-\dot{\alpha})^2 - \frac{d(d-1)}{a^2}k_d + 2\kappa^2\rho_m + 2V_{TOT}]/[(d+D)(d+D-1)]} , \tag{3.32}$$

so $\dot{y}$ will have a definite sign in the evolution of the universe if β dose not go out of this region. Let β_1 and β_2 be the points where $U(\beta)$ first becomes maximal on the two sides of β_0, setting

$$U_0 = \min [U(\max(\beta_1, \beta'_1)), U(\min(\beta_2, \beta'_2))]$$

there exists a finite trapped region

$$T = \{ (\beta, \dot{\beta}) \mid E(\beta, \dot{\beta}) < U_0 \} ,$$

in which the system is always dissipative ($\dot{E} < 0$) if $\dot{y} > 0$ when the universe is trapped into this region. The universe will then approach the solution $\beta = \beta_0$ asymptotically. This situation is explained diagrammatically in Figures 3-1 and 3-2. In Chapter IV, a toy model is discussed and the details of this dynamical system is further explained.

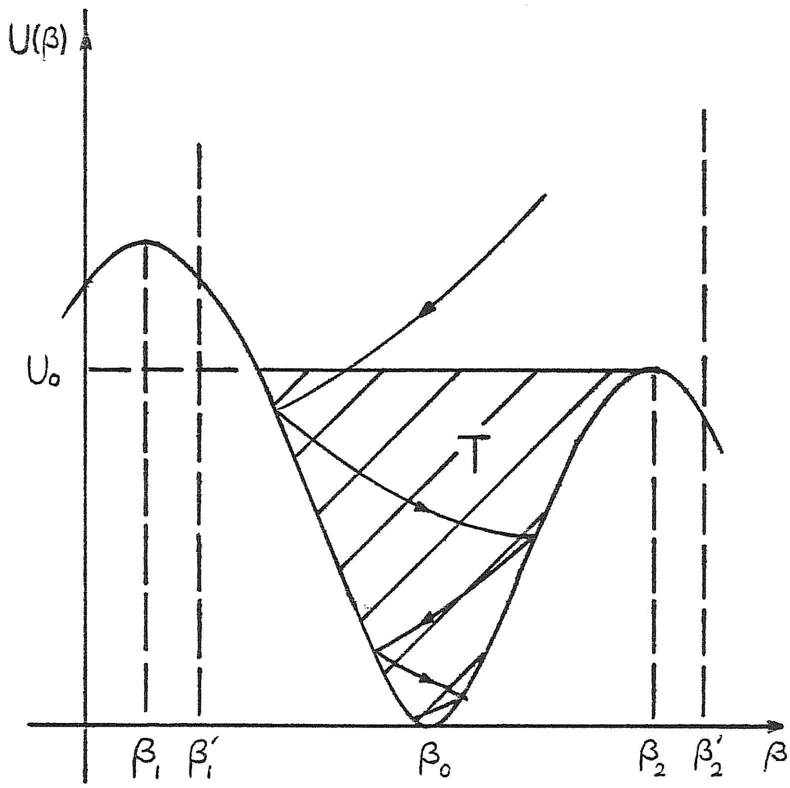


Figure 3-1

The schematic diagram of a potential or the dynamical system for β (equation (3.28)). The shaded region is the trapped region T. If $\dot{y} > 0$ when the universe enters this region, this system is dissipative and the universe approaches $\beta = \beta_0$ asymptotically as shown by the path.

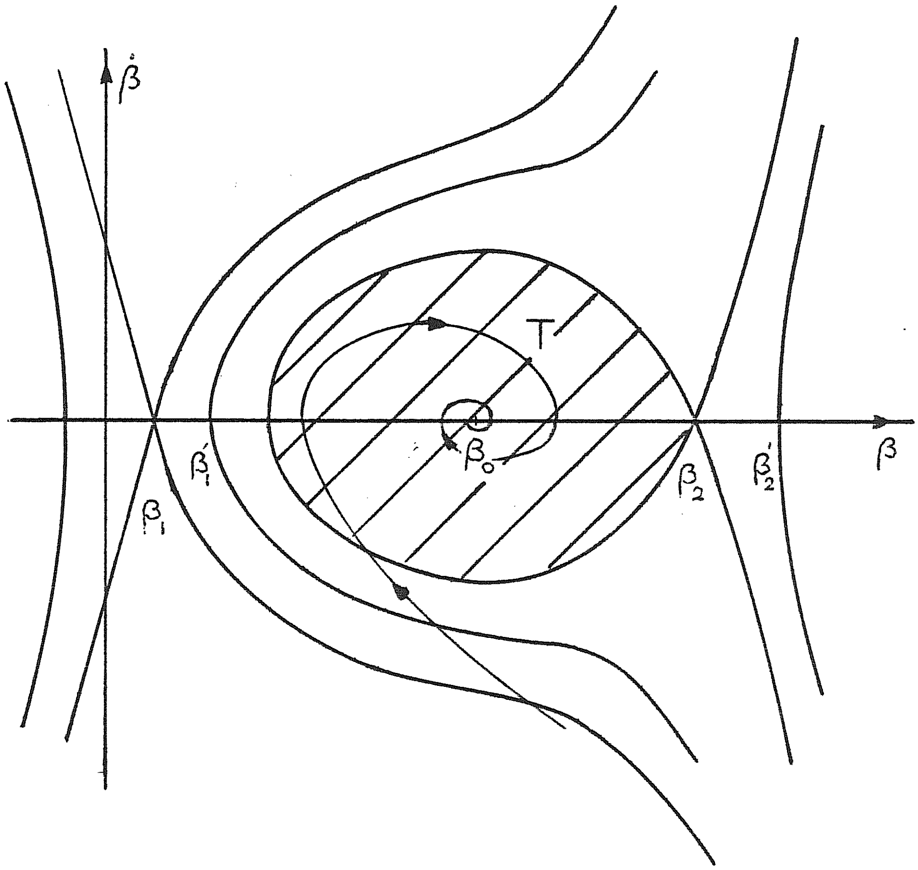


Figure 3-2

The contours of equipotentials of the potential mentioned in Figure 3-1 in the phase space of $(\beta,\dot{\beta})$. Once the universe is trapped in the region T, it will spiral in to the $F^4 \times K^D$ solution.

One last point of caution is that the scale factor of the internal space should not become smaller than Planck length during its evolution. If it does, we hope that the universe can emerge again with such parameters that the scenario considered is valid. Otherwise, we are not sure that the generalization of the classical Einstein equations into higher dimensions is justified.

3.6 Further considerations

So far we have developed a tool to gauge the 'graceful return' of a higher-dimensional universe to the present universe. However, there are some more side issues of Kaluza-Klein cosmologies that might, to the best, solve some cosmological problems or provide alternative solutions to them; to the worst, undermine the nice picture or impose constraints on the models. As is mentioned in section 3.2, it is the effective matter Lagrangian that 'controls' the dynamics of the universe through equation (3.11). This matter Lagrangian may have a lot of contributions in different orders of importance. There might be particles created in a dynamical space-time due to anisotropy (Maeda 1984). This could has some effects on the dynamics of the universe, but will not be discussed in this section. Another important contribution is that of thermal effect. If we consider the universe to start at very high temperature ($T \gg 1/a$, $1/A$), the particles present may be considered as thermal radiation in the higher-dimensional space-time. When the temperature of the universe drops and the internal space shrinks ($1/a \gg T \gg 1/A$), these higher-dimensional radiation can be considered to have 'decayed' into four-dimensional radiation and an infinite tower of massive particles. Important consequences of this 'decaying' process are discussed in the following subsections.

3.6.1 Temperature, entropy and inflation

Temperature is a very important concept in cosmology and in field theories. A beautiful example is the phase transitions in gauge theories and the very interesting consequences that it might have in cosmology considered by Linde (1979) and Guth (1981). In a higher-dimensional space-time, when the scale factors of the two subspaces change during the dynamical evolution and finally settle down on a Friedmann universe with a highly compactifed internal space, we have to change our perspective from higher-dimension to four-dimension. However, it is incorrect to consider an effective four-dimensional temperature as different from the higher-dimensional temperature (Barr and Brown 1984). This idea of true four-dimensional temperature being the same as the true (4+D)-dimensional temperature can easily be demonstrated when we consider the periodicity of the imaginary time of the particle fields in thermal equilibrium. Because of dimensional reduction, we can always express the field ψ in terms of the eigin-functions $\phi_l(y)$

$$\psi(it, x, y) = \sum_l \psi_l(it, x)\, \phi_l(y)$$

which of course shows that the periodicity $\beta(=1/T)$ of the imaginary time t remains unchanged. Thus, the temperature is the same for both four- and (4+D)-dimensional spaces.

Assume the reaction time scale of a gas of bosons and fermions to be short compared to cosmic time scale so that a thermal equilibrium is maintained. The system can then be characterized by a single temperature T and the entropy in any comoving volume remains constant. Suppose that the particles are massless in the higher-dimensional space-time, since the D-dimensional subspace is compact, as shown in equation (2.18), the energy of a particle with three-momentum $\vec{p}$ is given by

$$E^2 = \vec{p}^{\,2} + A^{-2} f \qquad (3.33)$$

where A is the scale factor of the internal space, and

f is an eigen-value of a Laplacian built on $\dot{\gamma}_{MN}$.

The energy is thus the distance from the origin in figure 3-3. The

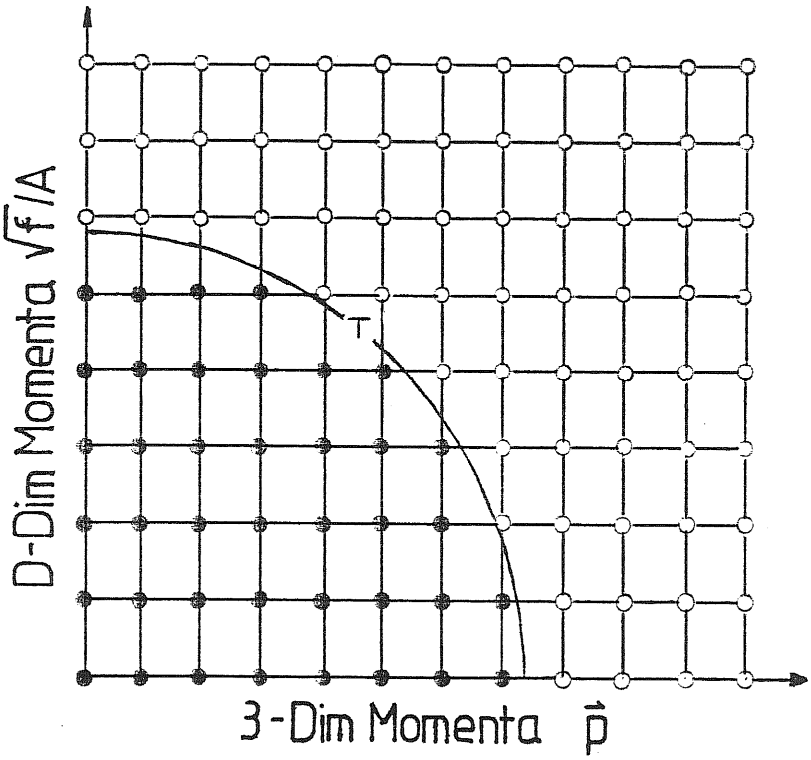


Figure 3-3

A schematic representation of the energy level of the (3+D) dimensional field $E=[p^2+f/A^2]^{1/2}$. The energy is the distance to the origin. The circular arc is the temperature. Solid and empty circles denote filled and unfilled energy levels respectively.

circular arc is the temperature within which the energy levels are filled ($E \leqslant T$). As the size A(t) of the compact dimension shrinks, the level spacing in the vertical direction gets larger. Conservation of entropy makes sure that the number of filled levels remains the same. If a(t) remains roughly constant, the temperature T will increase as is shown in figure 3-4. Since

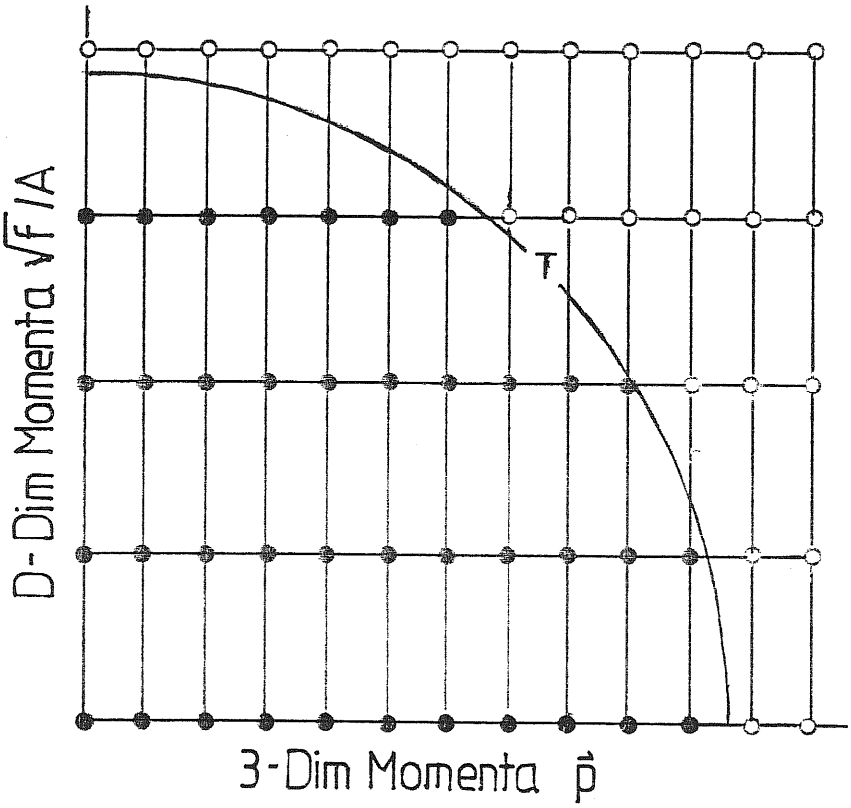


Figure 3-4

Follows figure 3-3, as the scale factor A shrinks, the level spacings in the vertical direction become larger. Conservation of entropy implies that T has increased, if a is roughly unchanged. Note that the higher modes of the D-dimensional kinetic energy are being frozen out.

$$\frac{\dot{T}}{T} = -\frac{\dot{R}}{R} = -\dot{y} = -\frac{1}{d+D}\left(d\frac{\dot{a}}{a} + D\frac{\dot{A}}{A}\right) \simeq -\frac{D}{d+D}\frac{\dot{A}}{A},$$

T will not increase as fast as 1/A. As a result, the massive particles in the four-dimensional world will be progressively frozen out. Eventually when $T \ll 1/A$, very few massive particle will remain, and we are left with a hot gas of massless particles in the four-dimensional space-time. (Alternatively, a increases very fast, but the qualitative picture of the freeze-out and a boost in entropy is the same.)

This increase in temperature can be viewed as a large increase in entropy in the four-dimensional space-time. Of course, the entropy in a higer-dimensional co-moving volume is unchanged. Sahdev (1984), Abbott, Barr and Ellis (1984,1985) and Kolb, Lindley and Secker (1984) have discussed the inflationary behaviour due to this large influx of entropy. By a simple thermodynamical argument, Abbott, Barr and Ellis showed that a large number of internal dimensions ($\sim$ 40) is needed if the inflation is substantial enough. However, as is pointed out by Maeda (1985), if we use Casimir effect to avoid the second singularity during this 'pole-inflation' and a fine-tuned cosmological constant to create an attractor, a graceful return is not always achieved after this inflation. This point will be discussed, with some computer results, in Chapter IV.

3.6.2 The massive particles from dimensional reduction

The picture of freeze-out of the massive particles is analogous to that of heavy particles in standard cosmology. However, it has been assumed that there are no quantum numbers that forbid the decay of the higher modes. In fact, the massive particles due to dimensional reduction (called pyrgons) always carry conserved quantum numbers, k, whereas the zero mode does not (equation (2.12)). It is obvious that when a particle ϕ of quantum number k decays into two particles ϕ_1 and ϕ_2 with quantum numbers k_1 and k_2 respectively, the conservation law requires $k_1 + k_2 = k$. So no pyrgons can decay to zero modes only. The only way to get rid of the pyrgons is by pyrgon-antipyrgon annihilation (Kolb and Slansky 1984).

If the universe were ever at a temperature comparable to A^{-1}, the pyrgons would have been present. The stable pyrgons that have not annihilated contribute to the present energy density of the universe. Since the total energy density of the universe is less than $2\rho_c$ where ρ_c is the critical density, this serves as a very severe constraint on the models. If annihilation were negligible, then today the number density of pyrgons would be comparable to that of neutrinos and the pyrgon mass m_ϕ has to be less than 100eV. The number density n_ϕ satisfies the equation

$$\dot{n}_\phi = [(n_\phi^{eq})^2 - n_\phi^2]\,\sigma_A\,|v| - \Gamma_E n_\phi \tag{3.34}$$

where n_ϕ^{eq} is the equilibrium number density,

σ_A is the annihilation cross-section ($\simeq \frac{\alpha^2}{m_\phi^2}$), and

Γ_E is the expansion rate of the universe, given by

$$\Gamma_E = (\rho_\phi + \rho_\gamma)^{1/2}/\bar{m}_p\ ,\quad \rho_\gamma \sim T^4\ ,\quad \rho_\phi = n_\phi m_\phi\ .$$

In a typical Kaluza-Klein theory, α is given by

$$\alpha = e^2 = \frac{\bar{I}_p^2}{A^2} \simeq \frac{m_\phi^2}{\bar{m}_p^2}\ .$$

It is expected that the pyrgons fall out of equilibrium when $T \lesssim m_\phi$. At this time, if we compare the annihilation rate Γ_A and the expansion rate Γ_E, we have

$$\frac{\Gamma_A}{\Gamma_E} \simeq \frac{m_\phi^3}{\bar{m}_p^3} \ll 1\ .$$

Thus, the primordial pyrgons would be as abundant as primordial neutrinos since annihilation is not effective.

To enhance the annihilation, Kolb and Slansky (1984) assumed that the pyrgons carry an additional charge that is similar to α and set this value to 1. They plot the ratio $r = n_\phi / n_\gamma$ as a function of T / m_ϕ as in figure 3-5.

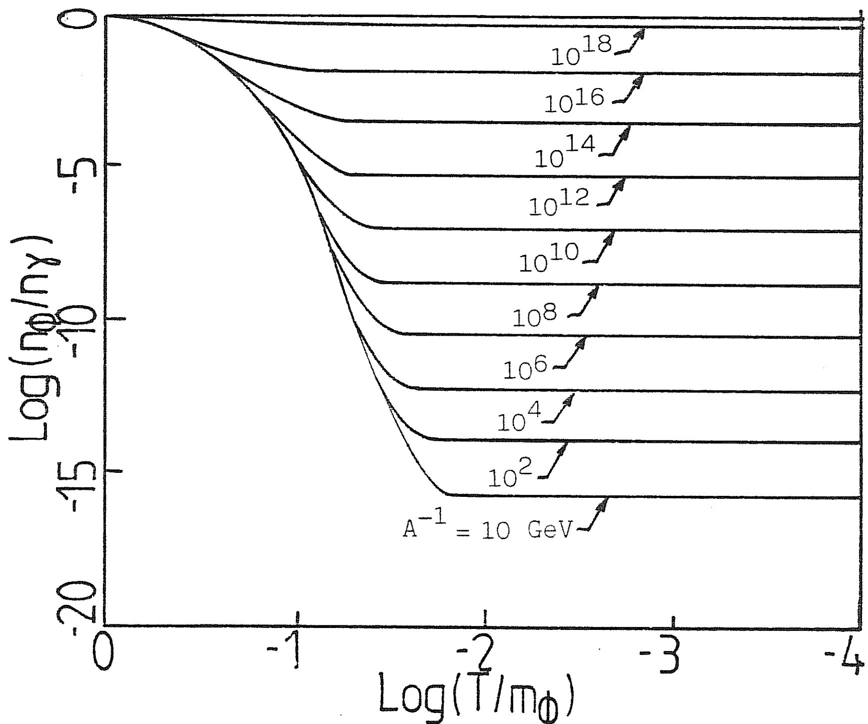


Figure 3-5

The ratio r of the number density of the pyrgons n_ϕ to the photon number density as a function of T/m_ϕ for various values of the compactification scale A.

Today $T \sim 10^{-13}$ GeV and the energy density contributed by the photons is $\rho_\gamma \sim n_\gamma T_\gamma \sim 10^{-4} \rho_c$. The requirement that $\rho_\phi \leqslant \rho_c$ becomes

$$r = \frac{n_\phi}{n_\gamma} = \frac{\rho_\phi / m_\phi}{\rho_\gamma / T_\gamma} \leqslant 10^{-9} A .$$

Using the results from figure 3-5, this is satisfied only if

$$A > (10^6 \text{ GeV})^{-1} \qquad (3.35)$$

at decoupling time of the pyrgons.

However, this limit can be relaxed if a large amount of energy is created after the decoupling of the primordial pyrgons (e.g. an inflation). Perhaps this is not too bad as we already have the primordial monopoles that have to be suppressed by an inflation.

IV DETAIL STUDIES OF A SIMPLE MODEL

As a toy model, consider a universe of (4+D) dimensions. The metric is given in the form (3.2). The subspaces of dimensionalities three and D are taken to be maximally symmetric. A compactified maximally symmetric space in D dimensions is in the form of a sphere, S^D, and $k_D = 1$ in equations (3.5). The matter Lagrangian consists of a background part, which is given by the one-loop effect as discussed by Candelas and Weinberg (1984), and a radiation part. When D is odd, the one-loop effective potential is finite and is given in the form C_D / A^4. Furthermore, in the cases $D = 3 \pmod 4$, the contributions from bosons and fermions are of the same sign. The system of equations (3.9) are solved using computer techniques. κ is taken to be 1. In order that the gauge coupling constants have reasonable values (Weinberg 1983), the number of scalars is taken to be 1 000 and 10 000. D is set to be 3 and 7 in the calculations. The calculations start at some time $t_1 > t_p$, the Planck time. The scale factors of the two subspaces are a_1 and A_1 (a_1, $A_1 > l_p$) and the temperature is T_1 ($T_1 \gg 1/a$, $1/A$). From this time onwards, we may consider the generalized Einstein equations (3.1) to be valid.

4.1 The Candelas-Weinberg model

Although it has been shown that contributions due to the quantum fluctuations of graviton fields are much more important (1 graviton $\sim$ 1 000 scalars) than that of the scalars (Orbóñez and Rubin 1984), we consider a lot of scalars as an approximation. The derivation of the effective potential due to a number of massless particles in the space-time specified above is outlined below. (a is assumed to be much larger than A. For the details see Candelas and Weinberg 1984.)

The action for a minimally coupled massless scalar field ϕ is given by

(4.1) $$S[\phi] = -\frac{1}{2}\int d^4x \int d^Dx \sqrt{-g}\, \phi_{;\mu}\phi^{;\mu}$$

The effective potential is defined through the relation

(4.2) $$\exp\left(-i\int d^4x\, V\right) = \int \mathcal{D}\phi\, e^{i S[\phi]}$$

and is given by the infinite sum of the one-loop graphs due to the infinite tower of massive particles in four-dimensions.

(4.3) $$V = -\frac{i}{2}\int \frac{d^4k}{(2\pi)^4} \sum_{l=0}^{\infty} d_l \log\left(k^2 + \frac{f_l}{A^2}\right)$$

where f_l is the eigen-value of the D-dimensional Laplacian on a unit-sphere and d_l is the degeneracy, given by

$$f_l = l\,(l + D - 1)$$
$$d_l = (2l + D - 1)\,\frac{\Gamma(l + D - 1)}{\Gamma(D)\, l!}\,.$$

In the case of fermions, the eigen-value and degeneracies are those of the Dirac operator and are denoted by $f_l^{(\frac{1}{2})}$ and $d_l^{(\frac{1}{2})}$. The total potential is then given by

$$V = \lim_{n\to 4}\left[-\frac{1}{2}(4\pi)^{-\frac{n}{2}}\Gamma\left(-\frac{n}{2}\right) A^{-n}\left\{ b\,\zeta_D^{(0)}(-n) - 4f\,\zeta_D^{(\frac{1}{2})}(-n)\right\}\right]$$

where b and f are the number of bosons and fermions respectively, and

$$\zeta_D^{(0)}(z) = \sum_l d_l\, f_l^{-z} \quad \text{for bosons}$$
$$\zeta_D^{(\frac{1}{2})}(z) = \sum_l d_l^{(\frac{1}{2})}\, f_l^{(\frac{1}{2})\,-z} \quad \text{for fermions.}$$

In the case of D being odd, the effective potential is given by

(4.4) $$V = \frac{C_D}{A^4}$$

with $C_D = b\,C_D^{(0)} + f\,C_D^{(\frac{1}{2})}$.

The values of $C_D^{(0)}$ and $C_D^{(\frac{1}{2})}$ are given in Table 4-1.

Table 4-1

Values of $C_D^{(0)}$ and $C_D^{(\frac{1}{2})}$, the one-loop potential for a single massless real spin-0 and spin-½ fields on an D-dimensional sphere of unit radius

D	$C_D^{(0)}$	$C_D^{(\frac{1}{2})}$
1 (a)	-5.05576×10^{-5}	2.022304×10^{-4}
(b)	4.73982×10^{-5}	-1.895928×10^{-4}
3	7.56870×10^{-5}	1.945058×10^{-4}
5	4.28304×10^{-4}	-1.140405×10^{-4}
7	8.15883×10^{-4}	5.958744×10^{-5}
9	1.13389×10^{-3}	-2.99172×10^{-5}
11	1.32932×10^{-3}	1.477709×10^{-5}
13	1.37403×10^{-3}	-7.242740×10^{-6}
15	1.25249×10^{-3}	3.537614×10^{-6}
17	9.55916×10^{-4}	-1.725405×10^{-6}
19	4.79352×10^{-4}	8.412070×10^{-7}
21	-1.79909×10^{-4}	-4.101970×10^{-7}

Since the circle S^1 is multiply connected, for D=1 we have the choice of whether or not to allow the field to change sign around the circle. These two cases of untwisted and twisted fields are labelled here as (a) and (b), respectively.

The effective potential of the 'background' field is thus given by

$$V_{BG} = \frac{1}{\Omega_D(A)}\frac{C_D}{A^4} \tag{4.5}$$

where $\Omega_D(A)$ is the volume of the internal space given by

$$\Omega_D(A) = \frac{2\pi^{\frac{D+1}{2}}}{\Gamma(\frac{D+1}{2})}A^D$$

The 'total' potential as defined in equation (3.20) is given by

$$V_{TOT} = V_{CUR} + V_{BG} + \Lambda_F . \tag{4.6}$$

Solving the equations for an $F^4 \times S^D$ late time solution, we obtain

$$A_0 = \left(\frac{(D+4)}{D(D-1)} \frac{C_D}{\Omega_D(1)} \right)^{\frac{1}{D+2}}$$

(4.7)

$$\Lambda_F = \frac{(D+2)}{2} \frac{C_D}{\Omega_D(1)} A_0^{-\frac{D+4}{D+2}}$$

Accepting this fine-tuning condition on the cosmological constant Λ, it is easy to show that the stability condition (3.28) is satisfied.

$$V''_{TOT} = \frac{(D+2)D(D-1)}{A_0^4} > 0 .$$

The total potential at A_0 is actually a global minimum as is depicted in figure 4-1.

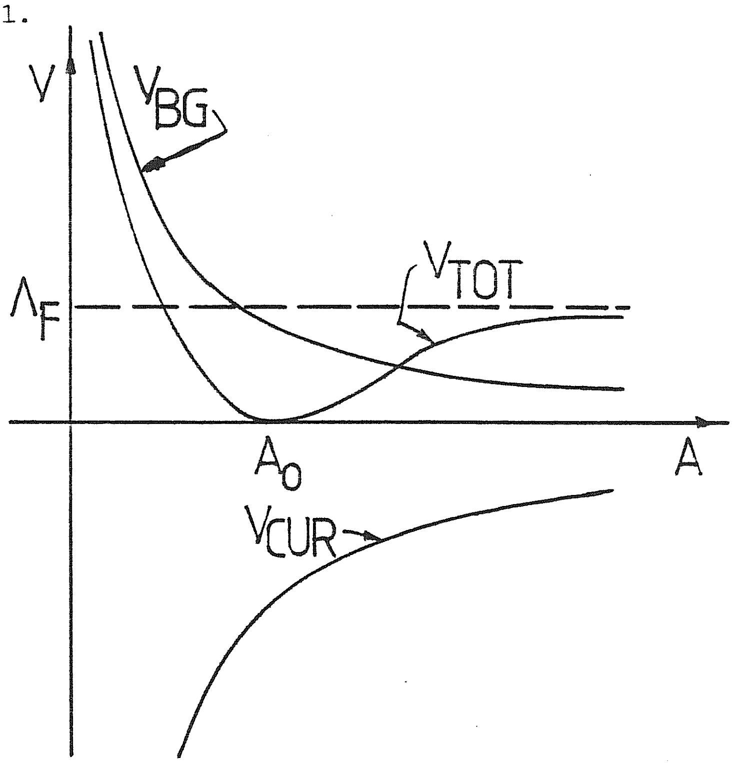


Figure 4-1

The 'total' potential as a sum of three terms

4.2 The attractor property

In order to assess the attractor property of this model, the dynamical system defined in equations (3.30) and (3.31) is used.

$$(4.8) \quad \begin{cases} E = \frac{1}{2}\dot{\beta}^2 + U(\beta) \\ U(\beta) = -\int_{\beta_0}^{\beta} d\beta' \, f(\beta') \\ \quad = -\frac{(D-1)}{2} e^{-2\beta} + \frac{4 b C_D^{(0)}}{D(D+4)\Omega_D(1)} e^{-(D+4)\beta} \\ \quad -\frac{2\Lambda}{D+2}\beta - (\beta \rightarrow \beta_0) \end{cases}$$

and the dynamical equation is

$$(4.9) \quad \dot{E} = -(d+D)\dot{y}\dot{\beta}^2$$

where y is defined as log (R), $R = a^d A^D$.

In the cases D=3 and D=7, by putting in b=1 000 and 10 000, and assuming that at some time the volume of the universe is increasing ($\dot{y} > 0$) and the temperature is low ($1/A \gg T \gg 1/a$), the system is integrated by numerical techniques. The energy density and principal pressures are those of a four-dimensional radiation plus Casimir energy

$$(4.10) \quad \begin{cases} \rho = \frac{\pi^2 b}{30\Omega_D(A)} T^4 + \frac{b C_D^{(0)}}{\Omega_D(A) A^4} \\ p_d = \frac{\rho}{d} - \frac{b C_D^{(0)}}{\Omega_D(A) A^4} \\ p_D = \frac{4}{D} \frac{b C_D^{(0)}}{\Omega_D(A) A^4} \end{cases}$$

In both cases, the behaviour of the systems are very similar. Figure 4-2 shows the 'potential' U (ß) and 'energy' $E(\beta,\dot{\beta})$ of the system (4.8) as the universe evolves in 'conformal' time η. The phase-space diagram of the spiralling in is plotted in figures 4-3 and 4-4. It is interesting to note that the $\dot{\beta}$ dependence of $\dot{E}$ makes the system highly dissipative when $\dot{\beta}$ is large but spirals in very slowly at the end.

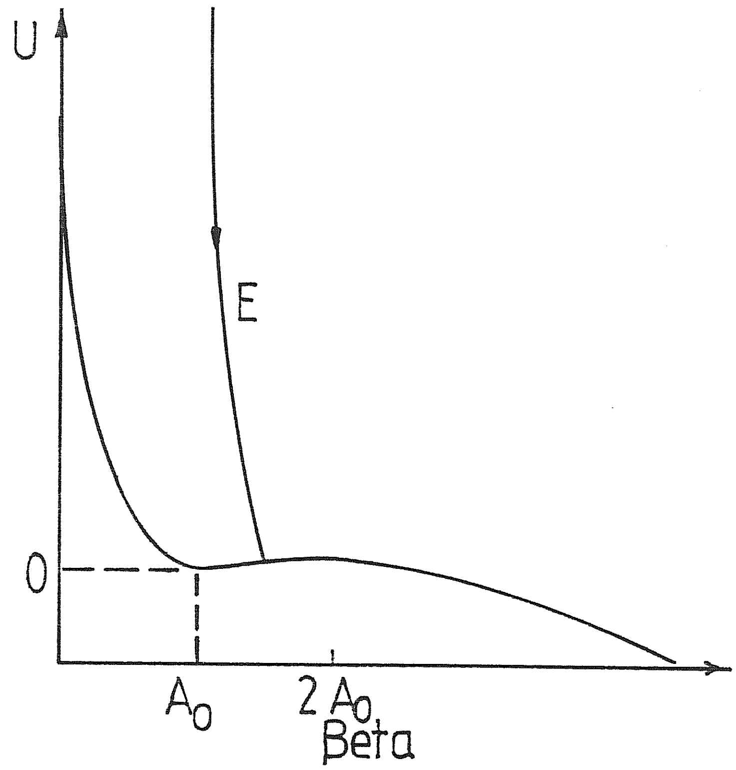
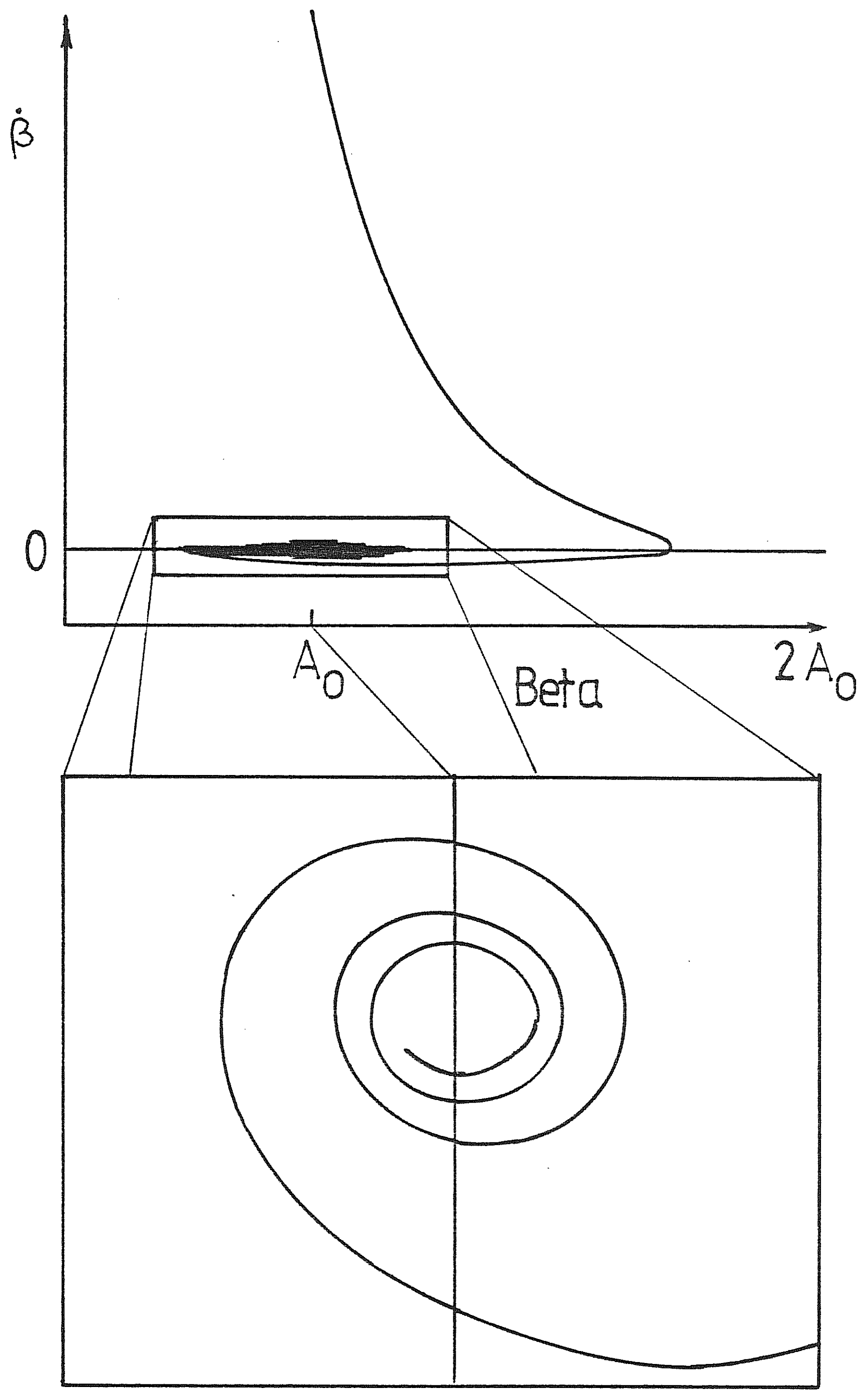


Figure 4-2

The potential U and energy E of the system as defined by equations (4.8) and (4.9). D=7, b=10 000, ß starts at β_0=-.3. The phase-space diagrams are shown in figures 4-3 and 4-4.



Figures 4-3 and 4-4

The phase-space diagram of (ß,ß̇) during the spiralling in to the $F^4 \times S^D$ condiguration. Figure 4-3 shows the sharp drop in ß̇ when it is large. Figure 4-4 shows the slow end.

One tentative conclusion is that the point A_0 is an attractor if the following conditions are fulfilled when the temperature of the universe drops below 1/A:

(i) The volume of the universe is increasing;

(ii) the radius of the internal space is less than $2 A_0$.

In this calculation, we have started at a low temperature so that the energy density is not dominated by radiation. It is, thus, interesting to see how the universe looks like in the high temperature regime and what the conditions for the universe to enter the low temperature regime with the parameters required are.

4.3 The high temperature regime

We have seen in the last section that this model does admit an attractor behaviour, but only if the parameters satisfy certain conditions at the time it enters the low temperature regime $T < 1/A$. If we believe in a Hot Big Bang, we might like to ask about the behaviour of the universe at high temperature and see whether it will enter the low temperature regime satisfying the conditions lay out in the last section.

In this regime, the matter field contributes to the stress-energy tensor in the form of a (4+D)-dimensional radiation

$$(4.12) \qquad \begin{cases} \rho_{rad} = (3+D)\, b \dfrac{\zeta(4+D)\,\Gamma\left(\frac{4+D}{2}\right)}{\frac{4+D}{2}} T^{4+D} \\ p_{rad} = p_{RAD} = \dfrac{\rho_{rad}}{3+D} \end{cases}$$

The equations (3.7) are solved numerically with the stress-energy tensor given by the sum of (4+D)-dimensional radiation and Casimir energy. The initial conditions depend on how the universe emerges from the quantum era and is unclear to us. A natural guess is to take a and A to be of order l_p. For the cases D=3 and D=7, and with b=1 000 and 10 000, the system is evolved and the following points are noted.

(1) Since the constraint equation in (3.7) admits both positive and negative roots for y' if we fix y, z and z', the sign of y' can be chosen to be positive or negative, to start with. Once this sign is chosen, it does not change its sign in a later time.

(2) If we choose $y' < 0$. The proper volume $R^{(d+D)}$ of the universe is monotonically decreasing. The qualitative picture critically depends on:

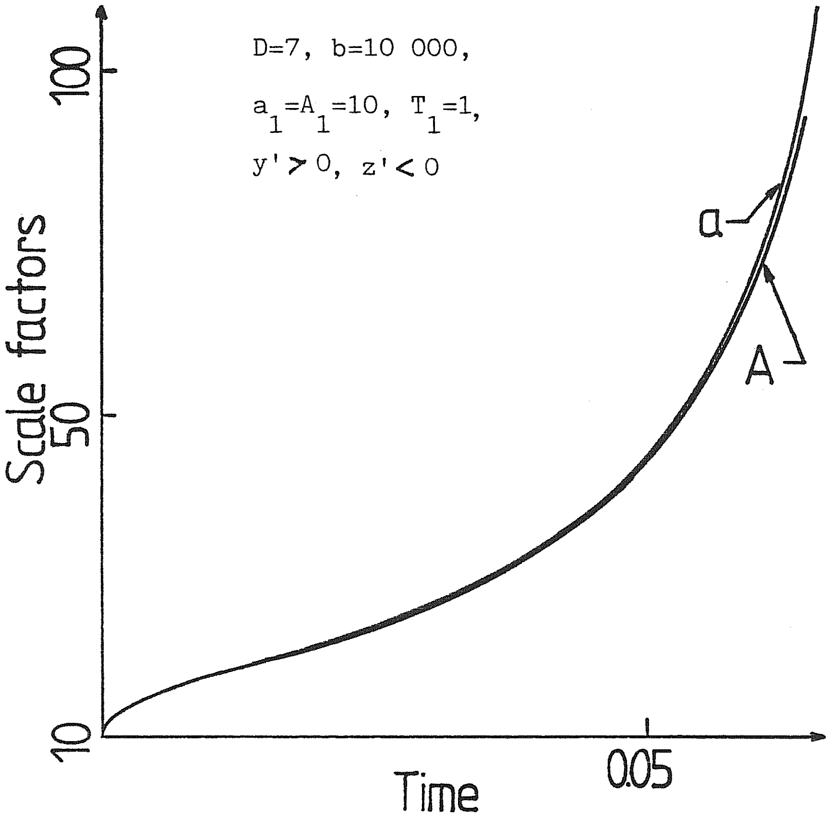
(i) If $z' < 0$, i.e., $\dot{A}/A < \dot{a}/a$, the internal space will compactify and we have a 'pole-inflation'. However, since in this case the system (4.10) is antidissipative, the decrease in A was not stopped before the time when the universe reenters an quantum era. So the present universe cannot be achieved in this simple form.

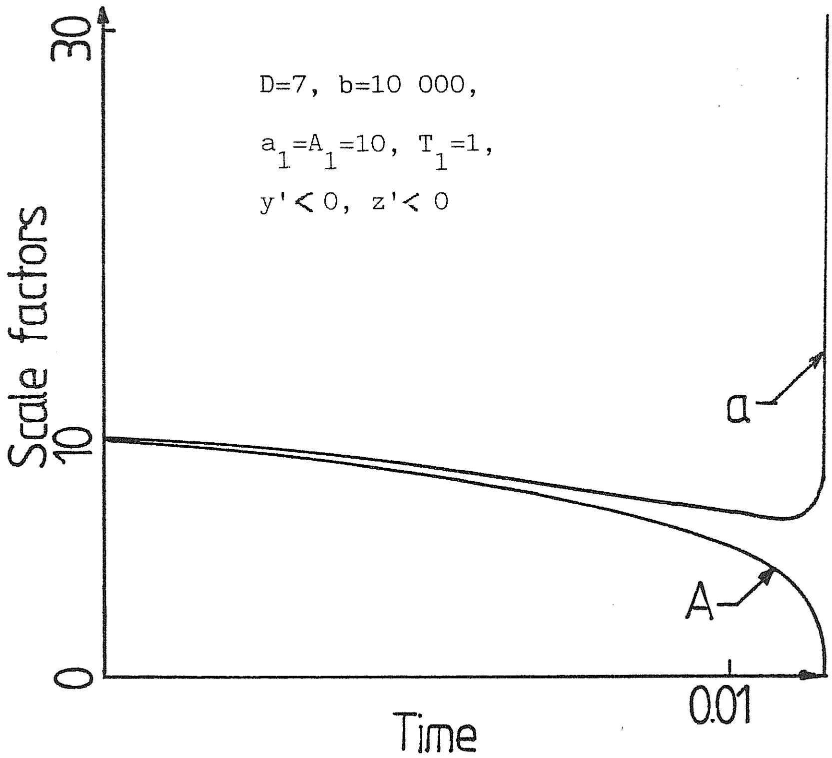
(ii) If we start with $z' > 0$, the behaviour is similar to 2(i) but with the scale factor a shrinking and A inflated. This clearly is not acceptable.

(iii) If we start with z'=0, the behaviour 2(i) or 2(ii) will happen depending on whether D is larger or smaller then 3.

(3) If we choose $y' > 0$, the volume of the universe increases with time. So an attractor behaviour might occur. However, the scale factors increase very fast and it is very unlikely that the scale factor A is of order A_0 when the universe enter the low temperature regime. This is a result of the large value of the fine-tuned cosmological constant.

The two cases with $y' < 0$ and $y' > 0$ are plotted in figures 4-5 and 4-6.





Figures 4-5 and 4-6

The time-evolution of the scale factors

4.4 Concluding remarks

As a result of this study, it may be concluded that for this simple model to be able to describe the present universe, the following set of initial conditions is required:

(i) the volume of the universe is increasing all the time ($y' > 0$);

(ii) the value of A is strictly less than $2\,A_0$;

(iii) the initial temperature is low compared to 1/A; and

(iv) the increase of α is faster than that of β.

After testing a large range of initial data, it is found that the first three conditions are crucial for this picture to work since

(a) the dynamical system (4.8) and (4.9) is not dissipative if $y < 0$;

(b) the first maximum of U occurs at $2A_0$, above which U is unbounded from below and A will increase forever regardless of the sign of y' once it gets there (run-away); and

(c) the temperature drops very slowly (compared to 1/A) until A 'explodes', so the condition (ii) cannot be achieved. (Fig. 4-7)

This picture, thus, cannot explain the present universe if we believe that the temperature of the universe is high when it emerges from the quantum era. However, we cannot say anything definite unless we have a complete theory to describe this regime. Nevertheless, quantum cosmology should be mentioned as a possible way to tell the initial conditions we want.

The above considerations might be improved if we take into account that the internal space may be a 'squashed' sphere instead of a round one. In this case, the absolute minimum of the potential may be at a different place. Since the squashing parameter enters the Einstein-Hilbert Lagrangian, it will behave like a 'scalar' field and the

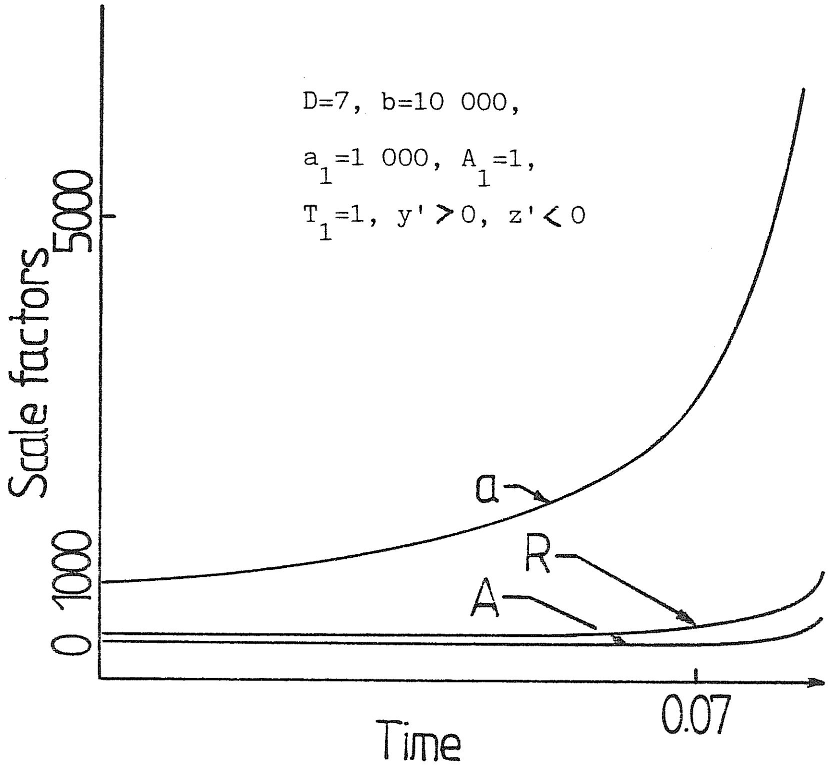


Figure 4-7

dynamical system will be different. This model is in the category of class II theories mentioned in section 3.2.

As is mentioned in section 4.1, the assumption that the gravitational Casimir effect can be negligible is not a good one. A full analysis including this effect is very important in understanding the overall influence of Casimir effect on Kaluza-Klein Cosmologies.

These improvements, however, are not expected to make the $F^4 \times K^D$ configuration an attractor for all initial conditions since the reason of the unboundedness of the potential U in the dynamic system of ß is the fine-tuned cosmological constant which is necessary in this kind of models. We should investigate more realistic models in order to make definite conclusions on the initial conditions.

APPENDIX A

In this appendix, the basic results in differential geometry needed to work with anholonomic basis are summarized. For convenience, the head (^) used to denote the tetrad indices is dropped.

Given a set of basis vectors $\{ e_i \}$, the structure coefficients $C_{ij}{}^k$ are defined by

(A1) $$[e_i , e_j] = C_{ij}{}^k e_k .$$

The connection coefficients $\Gamma^k{}_{ij}$ are defined through the covariant derivative of the basis vectors

(A2) $$\nabla_{e_j} e_i = \Gamma^k{}_{ij} e_k .$$

Comparing this with the basic property of two vector fields

(A3) $$[e_i , e_j] = \nabla_{e_i} e_j - \nabla_{e_j} e_i ,$$

we have

(A4) $$C_{ij}{}^k = \Gamma^k{}_{ji} - \Gamma^k{}_{ij} .$$

On the other hand, from the requirement that the metric is covariantly constant, i.e.,

(A5) $$\nabla_{e_k} g_{ij} = e_k (g_{ij}) - \Gamma_{jik} - \Gamma_{ikj} = 0$$

where Γ_{ijk} is defined as $\Gamma^l{}_{jk} g_{il}$,

we have

(A6) $$e_k (g_{ij}) = \Gamma_{jik} - \Gamma_{ikj} .$$

It follows from (A4) and (A6) that

$$\Gamma_{kij} = \frac{1}{2}\left[e_j(g_{ik}) + e_i(g_{jk}) - e_k(g_{ij}) + C_{ikj} + C_{kji} - C_{ijk}\right]. \tag{A7}$$

If the basis one-forms are denoted by $\{w^i\}$, from the properties of duality and exterior derivative of forms, we have

$$\begin{aligned} dw^k &= -\frac{1}{2} C_{ij}{}^k w^i \wedge w^j \\ &= -\Gamma^k{}_{ji} w^i \wedge w^j . \end{aligned} \tag{A8}$$

Define the connection one-form $w^i{}_j$ by

$$w^i{}_j = \Gamma^i{}_{jk} w^k , \tag{A9}$$

we arrive at the first Cartan equation

$$dw^i = - w^i{}_j \wedge w^k . \tag{A10}$$

The curvature two-form is given by the second Cartan equation

$$\mathcal{R}^i{}_j = d w^i{}_j + w^i{}_k \wedge w^k{}_j . \tag{A11}$$

This is obtained by considering the second exterior derivative of a vecotr field v: $d^2 v = \mathcal{R} v$, and is to be compared with

$$\mathcal{R}^i{}_j = \frac{1}{2} R^i{}_{jkl} w^k \wedge w^l , \tag{A12}$$

where $R^i{}_{jkl}$ are the components of the Riemann tensor.

APPENDIX B

In this appendix, the calculation of the Ricci tensor and the curvature scalar in a five-dimensional Kaluza-Klein theory is performed. The formalism developed in the last appendix is used.

In the five-dimension theory, a Killing vector is assumed in the extra dimension. The following one-forms are used:

(B1)
$$\begin{cases} w^m = d\,x^m \\ w^5 = d\,x^5 + A_m\, d\,x^m . \end{cases}$$

Note that a general metric is used to describe the curved space-time in four dimensions. The exterior derivatives of these one-forms are given by

$$\begin{cases} d\,w^m = 0 \\ d\,w^5 = A_{m,n}\, d\,x^n \wedge d\,x^m \\ \qquad\;\; = \frac{1}{2} F_{mn}\, w^m \wedge w^n \end{cases}$$

where $F_{mn} = \partial_m A_n - \partial_n A_m$.

Comparing equations (B2) and equations (A8), we have the non-vanishing structure constants given by

(B3)
$$C_{mn}^{\;\;5} = - F_{mn} .$$

Using equations (A7) and (A9), we can construct the connection one-forms

(B4)
$$\begin{aligned} w_{mn} &= \bar{w}_{mn} - \frac{1}{2} g_{55} F_{mn}\, w^5 \\ w_{m5} &= - \frac{1}{2} g_{55} F_{mn}\, w^n \end{aligned}$$

where $\bar{w}$ is the connection one-form of the four-dimensional space time

$$\bar{w}^m{}_n = \Gamma^m{}_{nk}\, w^k .$$

Using equation (A11), we have the components of the curvature two-form given by

(B5)
$$\begin{cases} \mathcal{R}^m{}_n = d\bar{w}^m{}_n - \frac{1}{2} g_{55,k} F^m{}_n\, w^k \wedge w^5 - \frac{1}{2} g_{55} F^m{}_{n,k}\, w^k \wedge w^5 \\ \qquad - \frac{1}{2} g_{55} F^m{}_n F_{pq}\, w^p \wedge w^q + \bar{w}^m{}_p \wedge \bar{w}^p{}_n \\ \qquad - \frac{1}{2} g_{55} (F^n{}_p\, w^5 \wedge \bar{w}^p{}_n + \bar{w}^m{}_p \wedge F^p{}_n\, w^5) \\ \qquad - \frac{1}{4} g_{55} F^m{}_p F_{nq}\, w^p \wedge w^q \\ \mathcal{R}^m{}_5 = - \frac{1}{2} g_{55,p} F^m{}_n\, w^p \wedge w^n - \frac{1}{2} g_{55} F^m{}_{n,p}\, w^p \wedge w^n \\ \qquad - \frac{1}{2} g_{55} F^n{}_p\, \bar{w}^m{}_n \wedge w^p - \frac{1}{4} g_{55}^2 F^m{}_n F^n{}_p\, w^5 \wedge w^p . \end{cases}$$

By comparing this with equation (A12), we can get the components of the Riemann tensor

(B6)
$$\begin{cases} R^m{}_{npq} = \bar{R}^m{}_{npq} - \frac{1}{2} g_{55} F^m{}_n F_{pq} - \frac{1}{4} g_{55} (F^m{}_p F_{nq} - F^m{}_q F_{np}) \\ R^m{}_{5n5} = \frac{1}{4} g_{55}^2 F^{mp} F_{np} . \end{cases}$$

From these, we can calculate the Ricci tensor and the curvature scalar

(B7)
$$\begin{cases} R_{mn} = \bar{R}_{mn} - \frac{1}{2} g_{55} F_{mp} F_n{}^p \\ R_{55} = \frac{1}{4} g_{55}^2 F^{mn} F_{mn} \\ R = R^m{}_m + R^5{}_5 \\ \quad = \bar{R} - \frac{1}{4} g_{55} F^{mn} F_{mn} . \end{cases}$$

BIBLIOGRAPHY

Abbott R B, Barr S M and Ellis S D 1984, 1985 'Kaluza-Klein Cosmologies and inflation I, II' Phys Rev D30, 720; D31, 673.

Albrecht A and Steinhardt P J 1982 'Cosmology for Grand Unified Theories with radiatively induced symmetry breaking' Phys Rev Lett 48, 1220.

Bailin D, Love A and Vayonakis C E 1984 'Kaluza-Klein Cosmologies at late time' Phys Lett 142B, 344.

Bardeen J, Steinhardt P J and Turner M S 1982 'Spotaneous creation of almost scale-free density perturbations in an inflationary universe' Phys Rev D28, 679.

Barr S M and Brown L S 1984 'Entropy from extra dimensions' Phys Rev D29, 2779.

Candelas P and Weinberg S 1984 'Calculation of gauge couplings and compact circumferences from self-consistent dimensional reduction' Nucl Phys B237, 397.

Casimir H B G 1948 Proc Kon Ned Akad Wetenschap, ser B, 51, 793.

Chodos A and Myers E 1984 'Gravitational constribution to the Casimir energy in Kaluza-Klein theories' Ann Phys 156, 412.

Chodos A and Myers E 1985 'Gravitaional Casimir energy in non-Abelian Kaluza-Klein theories' Phys Rev D31, 3064.

Coleman S and Weinberg E 1973 'Radiative corrections as the origin of spontaneous symmetry breaking' Phys Rev D7, 1888.

DeWitt B S 1965 'Dynamical theory of groups and fields' Gordon and Breach Science Publishers, N Y.

Einstein A and Bergmann P 1938 'On a generalization of Kaluza's theory of electricity' Ann Math 39, 683.

Fritzsch H and Gell-Mann M 1972 ' Current algebra—quark and what else?' in 'Proceedings of the 16th international conference on high energy physics vol 2' edited by Jackson J D and Roberts A (National Accelerator Laboratory, Batavia, Illinois).

Fritzsch H, Gell-Mann M and Leutwyler H 1973 'Advantages of the color octet gluon picture' Phys Lett B47, 365.

Glashow S L 1961 'Partial symmetry of weak interactions' Nucl Phys 22, 579.

Gross D J and Wilczek F 1973 'Asymptotically free gauge theories I' Phys Rev D8, 3633.

Guth A H 1981 'Inflationary universe: a possible solution to the horizon and flatness problems' Phys Rev D23, 347.

Guth A H and Pi S-Y 1982 'Fluctuations in the new inflationary universe' Phys Rev Lett 49, 1110.

Hawking S W 1982 'The development of irregularities in a single bubble inflationary universe' Phys Lett 115B, 295.

Kaluza Th 1921[1] 'Zum unitätsproblem der physik' Sitzungsber Preuss Akad Ziss Berlin Math Phys K1, 966.

Klein O 1926[1] 'Quantentheorie und finfdimensional' Z Phys 37, 895.

Kolb E W Lindley D and Seckel D 1984 'More dimensions-less entropy' Phys Rev D30, 1205.

Kolb E W and Slansky R 1984 'Dimensional reduction in the early universe: where have the massive particles gone?' Phys Lett 135B, 378.

Lee B W and Weinberg S 1977 'Cosmological lower bound on heavy-neutrino mass' Phys Rev Lett 39, 165.

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1 English translations of these articles can be found in the book edited by Lee H C and 'Unified theories of more than four dimensions including exact solution' edited by De Sabbata V and Schmutzer E (World Scientific 1983).

Lee H C 1984 edited 'An introduction to Kaluza-Klein theories' (World Scientific).

Lim C S 1984 'Spontaneous gauge symmetry breaking in higher-dimensional theories' preprint BNL-35738.

Linde A D 1979 'Phase transitions in gauge theories and cosmology Rep Prog Phys 42, 389.

Linde A D 1982 'A new inflationary universe scenario: a possible solution of the horizon, flatness, homogeneity, isotropy and primodial monopole problems' Phys Lett 108B, 389.

Maeda K 1984 'Effect of particle creation on Kaluza-Klein cosmologies' Phys Rev D30, 2482.

Madea K 1985 'Stability and attractor in Kaluza-Klein cosmology I' SISSA preprint 34/85/A.

Maeda K and Nishino H 1985 'Cosmological solutions in D=6, N=2 Kaluza-Klein supergravity-Friedmann universe without fine-tuning' Phys Lett 154B, 358; 'Attractor universe in six-dimensional N=2 supergravity Kaluza-Klein theory' Phys Lett 158B, 381.

Misner C W Thorne K S and Wheeler J A 1973 'Gravitaion' (Freeman, San Francisco).

Nambu Y 1966 'A systematics of hadrons in subnuclear physics' in 'Preludes in Theoretical Physics' edited by de Shalit A, Feshbach H and van Hove L (North Holland).

Okada J 1985 'Symmetry breakings in the Kaluza-Klein theory' preprint HUPD-8503.

Ordóñez C R and Rubin M A 'Graviton diminance in quantum Kaluza-Klein theory' preprint UTTG 18-84.

Sahdev D 1984 'Towards a realistic Kaluza-Klein cosmology' Phys Lett 137B, 155.

Sakharov A D 1967 'Violation of CP invariance, C asymmetry and baryon asymmetry of the universe' Zh Ek Teor Fiz 5, 32. (English translation JETP Lett 5, 24.)

Salam A 1968 'Weak and Electromagnetic interactions' in 'Proceedings Nobel symposium on elementary particle physics' edited by Svartholm N (Stockholm: Almquist, Forlag A B).

Salam A and Strathdee J 1982 'On Kaluza-Klein theory' Ann Phys 141, 316.

Sarmadi M H 1984 'Spontaneous compactification in quantum Kaluza-Klein theories' ICTP preprint IC/84/3.

Starobinsky A A 1982 'Dynamics of phase transitions in the new inflationary universe scenario and generation of perturbations' Phys Lett 117B, 175.

Steigman G Schramm D N and Gunn J e 1977 'Cosmological limits to the number of massive leptons' Phys Lett B66, 202.

Weinberg S 1967 'A model of leptons' Phys Rev Lett 19, 1264.

Weinberg S 1972 'Gravitation and cosmology' (John Wiley and son).

Weinberg S 1973 'Non-Abelian gauge theories of the strong interactions' Phys Rev Lett 31, 494.

Weinberg S 1983 'Charges from extra dimensions' Phys Lett 125B, 265.