ISAS - INTERNATIONAL SCHOOL
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THESIS FOR THE TITLE

OF

"MAGISTER PHILOSOPHIAE"

PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS

Section: Mathematics

Supervisor: Prof. Giovanni Mancini

Candidate: Maria Szlenk

Academic Year: 1983/1984

TRIESTE

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CONTENTS

INTRODUCTION

Let H $C^1(\mathbb{R}^{2N},\mathbb{R})$ and let J be a simplectic matrix $\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}$, where I- identity in $\mathbb{R}^N$, set H'= grad H, z= $(p,q)\in C^1(\mathbb{R},\mathbb{R}^{2N})$, $\dot{z}=\frac{dz}{dt}$ and consider the Hamiltonian system

(H-S) $\qquad -J\dot{z} = H'(z)$

with N degrees of freedom.

This equation creates a lot of problems connected with existence of periodic solutions. We can divide resalts in two groups: the first one states the existence of solutions on a prescribed energy surface, the second one the existence of solutions of a prescribed period.

As we can see in the following example, not all Hamiltonian systems have periodic solutions.

<u>Example 0.1.</u>

Let us consider the equation

(0.1) $\qquad \dot{x}= F(x)$, where F satisfies

(F1) $\qquad F\in C^2(\mathbb{R}^m, \mathbb{R}^m)$, $F(0)= 0$.

If we want to find small oscilations near the equilibrium $x= 0$, we should require, first of all, that $F'(0)$ has a pair of imaginary eigenvalues $\pm i\omega^*$. However this is not sufficent, in general, for (0.1) to have periodic soltions near 0. Let $m= 2$ and

(0.2)
$$\begin{aligned} \dot{x}_1 &= -x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= x_1 - x_2(x_1^2 + x_2^2) \end{aligned}$$

One has that $F'(0)$ has $\pm i$ as eigenvalues, while from

$$\frac{d}{dt}\ \frac{1}{2}(x_1^2+x_2^2) = x_1\dot{x}_1 + x_2\dot{x}_2 = -(x_1^2+x_2^2)^2$$

it follows that the solutions of (0.2) are not periodic because

$$x_1^2 + x_2^2 = -\frac{1}{2t+c}\ .$$

Liapunov has shown in a celebrated theorem [22] that (0.1) possesses periodic solutions near 0 provided it has an integral $\mathfrak{F}\in C^2(\mathbb{R}^m, \mathbb{R})$ with $\mathfrak{F}''(0)$ non-singular, that is:

Theorem 0.0.

Suppose F satisfies F1) and $F'(0)$ has $i\omega^*$ as eigenvalues with (algebraic) multiplicity 1. For all $k\in\mathbb{Z}$, $k\neq\pm1$, $ik\omega^*$ is not an eigenvalue of $F'(0)$. Moreover suppose (0.1) has an integral $\mathfrak{F}\in C^2(\mathbb{R}^m, \mathbb{R})$ veryfying $(F(x), \mathfrak{F}'(x))= 0$ and such that $\mathfrak{F}''(0)$ is non-singular. Then (0.1) has a family of $\frac{2\pi}{\omega}$ periodic solutions x_ω with $|x_\omega|_\infty \longrightarrow 0$ as $\omega\longrightarrow\omega^*$.

One can see that (H-S) has the prime integral i.e. exists such $\mathfrak{F}(x)$ that $(JH'(x), \mathfrak{F}'(x))= 0$ and it holds for $\mathfrak{F}(x)= H(x)$. There is a sense to look for solutions on prescribed energy level, because $H(z(t))=$ constant for every solution $z(t)$ of (H-S). Weinstein in [25] and Moser in [23] eliminated the non-resonance condition from the Liapunov Center Theorem in the following way:

Theorem 0.1.

Let H be C^2 in a neighbourhood of $z=0$, $H(0)=0$, $H'(0)=0$ and suppose the Hessian $H''(0)$ be positive definite. Then for every $\varepsilon>0$ small enough (H-S) has on $H(z)=\varepsilon$ at least N distinct periodic orbits.

These both results are local, the general one was obtained by Weinstein [26], who proved:

Theorem 0.2.

Let Ω be bounded, convex domain in R such that $\Sigma=\partial\Omega$, where $\Sigma=\{x\in\mathbb{R}^{2N}: H(x)=1\}$ and suppose $H'(z)\neq 0$, $\forall z\in\Sigma$. Then (H-S) has a periodic orbit on Σ.

This result was improved by Rabinowitz [24] who weakened the convexity assumption on Ω taking Ω star-shaped with respect to some point $x_0 \in \mathbb{R}^{2N}$. His result was as follows:

Theorem 0.3.

Suppose $H'(z) \neq 0$, $\forall z \in \Sigma$, $\Sigma = H^{-1}(1)$ and that Σ is radially diffeomorphic to $\partial B = \{x \in \mathbb{R}^{2N} : |x|=1\}$. Then (H-S) has a periodic orbit on Σ.

Then the result concerning multiple periodic solution of (H-S) on a prescribed energy level was stated by Ekeland and Lasry [17]. They proved:

Theorem 0.4.

Let $\Sigma = \partial\Omega$, Ω bounded, convex domain in $\mathbb{R}^{2N}$. Suppose there exist r, $R > 0$ with $R^2 < 2r^2$ such that $B_r \subseteq \Omega \subset B_R$. Then (H-S) has at least N different periodic orbits on $\Sigma = H^{-1}(1)$.

This result will be written later with more details as Theorem 1.3.2.

In this direction there are two results: by Berestycki, Lasry, Mancini, Ruf [12] (in this case Σ lies between two ellipsoids i.e. if $\mathcal{E}$ is given ellipsoid such that for suitable $\alpha, \beta > 0$, $\alpha\mathcal{E} \subset \Omega \subset \beta\mathcal{E}$, where Ω is star-shaped and compact then there are at least N distinct solutions of (H-S)), the another result is by Girardi and Matzeu [19] see Theorem 4.1.4.

In the investigation of closed orbits on an energy surface, Rabinowitz [24] introduced a device which translates the given problem into the problem of finding periodic solutions having prescribed period for a related Hamiltonian system, a problem which is in interest in itself. The basic result is the following:

Theorem 0.5.

Suppose $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$ and satisfies $H(z) = O(|z|^2)$ at $z=0$, $H \geq 0$ and $0 < H(z) \leqslant \theta(z, H_z)_{\mathbb{R}^{2N}}$ for $|z| \geqslant \bar{r}$, $\theta \in]0,\frac{1}{2}[$. Then $\forall T > 0$ there exists a non- constant T-periodic solution of (H-S).

A simple example 0.2 shows, nevertheless, that T in the above period , is not in general the real period of the solutions he finds: it could be as well T/2 or T/3 and so on.

Example 0.2.

Let $H(x) = \Phi(|x|^2)$, $\Phi \in C^\infty$, $\Phi'(z) > 0$ $\forall z$, and $\Phi'(0)=1$, $\Phi'(+\infty) = \infty$. The energy surfaces are spheres and the solution of (H-S) (at the energy level $H(z(t)) \equiv h$) are

$$\psi(h^{1/2})(\xi \cos\gamma(h)t + \eta \sin\gamma(h)t,\ \xi \sin\gamma(h)t - \eta\cos\gamma(h)t),$$

where $\xi, \eta \in R^N$, $|\xi|^2 = |\eta|^2 = 1$, $\psi = \Phi^{-1}$, $\gamma(h) = 2/\psi'(h)$, The corresponding periods are bounded from above. Thus, a periodic solution of "long" period T given by Theorem 0.5, could be a solution of minimal period $T/k \leqslant T_0$.

On the other hand in the search of geometrically distinct orbits, informations about the minimality of the period are crucial. A first result was obtained by Clarke and Ekeland [14] , see Theorem 1.2.0. , they proved that for every $T > 0$ (H-S) has a periodic solution having T as a minimal period, provided H is convex and "subquadratic". A partial result for H convex and "superquadratic" was given by Ambrosetti and Mancini [5] , they used a dual variational functional constrained to a suitable manifold, however they have to put some restriction on the G"- the second derivative of Fenchel conjugate of H, i.e. $(G'(y),y) \geqslant \lambda(G''(y),y)$ for some suitable constant $\lambda > 1$. A different method was used by Girardi and Matzeu in [19] to show the existence of N distinct periodic solutions with minimal period

$T > 0$ (see Theorem 3.1.3) when H is "superquadratic" and not necessarly convex. They developed the idea of finite dimensional approximation in the direction of "pseudo-iindex" theory.

The last result of " prescribed minimal period" is given by Ekeland and Hofer [16] . Their result is another progress of that obtained by Ambrosetti and Mancini and contain both "sub- and superquadratic" cases, i.e.

Theorem 0.6.

Let $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ and satisfies

a) $H''(x)$ positive definite for $x \neq 0$

b) $H(x)\|x\|^{-2} \longrightarrow 0$ when $\|x\| \longrightarrow 0$

c) $\exists r > 0$, $\exists \beta > 2$: $(H'(x), x) \geqslant \beta H(x)$ for $\|x\| \geqslant r$.

then for every $T > 0$ there exists a periodic solution $\bar{x}$ of (H-S) with minimal period T.

To prove this theorem they used a combination of: an index theory for periodic solutions of Hamiltonian system and a complete description of C^2 functionals near critical points of mountain-pass type (see Hofer [21]).

In the first part of this paper I give some ideas and examples of "dual variational method" and "Lusternik- Schnirelman critical points theory" (c.f. [7, 13; 17, 14, 5]) and so called "direct method"(c.f. [11, 8, 10]). In the second part I sketch the approach given by Ekeland and Hofer in the proof of Theorem 0.6., i.e. the cnnection between the Morse index and the order of an isotropy group which gives the possibility to find a solution with prescribed minimal period T. In the third part I discuss results obtained by Girardi and Matzeu in [18] and [19] .

I give also in that section some improvements of their results to obtain N distinct solutions with prescribed minimal period T under more general conditions on H , a kind of ellipsoid conditions. In the last section there will be presented some results concern the existence of periodic orbits on any given energy surface (c.f. [19, 6, 12, 24]).

I. METHODS USED TO STUDY HAMILTONIAN SYSTEMS IN THE PRESCRIBED PERIOD CASE

§1. Preliminars and notations.

Throughout the paper we shall use the following notations: if $x,y \in R^N$ then $x \cdot y$ denotes the Euclidean scalar product, $|x|^2 = x \cdot x$, $B_R := \{x \in R^{2N} : |x| < R\}$ We shall assume $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$, $\Sigma := \{x \in R^{2N} : H(x) = 1\}$. If E denotes a Banach space then $\langle \cdot , \cdot \rangle$, will denote a pairing between E and E', $\|\cdot\|$ denotes the norm in E. Let A be the operator defined on $\mathrm{dom}A = \{z \in H^{1,\alpha}(0,2\pi; \mathbb{R}^{2N}) : z(0)=z(2\pi)\}$, $\alpha^{-1}+\beta^{-1} = 1$, $\alpha, \beta > 0$, by the equation $Az=-J\dot{z}$. Let $E = \{u \in L^{\alpha}(0,2\pi; \mathbb{R}^{2N}) : \int_0^{2\pi} u=0\}$ and denote by $\|\cdot\|$ the norm in $L^{\alpha}(0,2\pi; \mathbb{R}^{2N})$. A, as an operator from domA E into E has a continuous inverse, which we'll call L. Note that $\ker A = \mathbb{R}^{2N}$ and L is a compact operator from E into $L^{\alpha}(0,2\pi; \mathbb{R}^{2N})$.

Remark 1.1.1.

Let z(t) be a T periodic solution of (H-S), then $v(t)= z(\lambda t)$ is a 2π periodic solution of (H-S)

$$(H\text{-}S)_\lambda \qquad -J\dot{v} = \lambda H'(v),$$

where $\lambda = T/2\pi$.

hence instead of looking for a T periodic solution of (H-S) it is sometimes more convenient to look for a 2π-periodic solution of $(H\text{-}S)_\lambda$.

Definition 1.1.1. Let f be a functional defined on a Banach space E, let M be a submanifold of E of class C^2. We say that the pair (f,M) satisfies the Palais-Smale condition (P-S) if :

(P-S) $\forall \{u_n\}_{n\in N} \in M$ such that $f(u_n)$ is bounded and $f'_{|M}(u_n) \longrightarrow 0$ as $n \longrightarrow \infty$

$\exists \{u_{n_k}\}_{k\in N}$ which is converging.

Remark 1.1.2.

It is easy to see that along the solution $z(t)$ of (H-S), the Hamiltonian function is constant, i.e., if $z(t)$ is any solution then $h_z := H(z(t)) = \text{const.}\ \forall t$.

In the next we will use also the following notations: let f be a functional $f: \mathbb{R}^{2N} \longrightarrow R$, $c \in R$,

(1.1.1.) $f^c := \{ z \in \mathbb{R}^{2N} : f(z) \leq c \}$

(1.1.2.) $K_c := \{ z \in R^{2N} : f(z) = c \text{ and } f'(z) = 0 \}$

A sphere of radius $r > 0$ will be denote as S_r i.e. $S_r := \{ z \in \mathbb{R}^{2N} : |z| = 1 \}$.

In the next sections we shall use the space $H^{1/2}(S_T, R^{2N})$.

Remark 1.1.3.

$H^{1/2}(S_T, \mathbb{R}^{2N})$ is a space of 2N tuples of T periodic functions which possess a derivative of order ½. In other words let $z \in C^{\infty}(\mathbb{R}, \mathbb{R}^{2N})$ be T-periodic, then z has a Fourier expansion $z = \sum_{k\in\mathbb{Z}} z_k \exp\{ikt/T\}$ with $z_k \in C^N$ and $\bar{z}_k = z_{-k}$. $E = (S_T, \mathbb{R}^{2N})$ is a closure of the set of such functions, under the (Hilbert) norm:

$$\left(\sum_{k\in\mathbb{Z}} (1+|k|)|z_k| \right)^{1/2} .$$

§.2 Dual variational principle.

If H is convex and C^1 it is possible to define the Legendre transform G of H in the following way:

Definition 1.2.1.

(1.2.1) $$G(u)= \sup \{u \cdot z - H(z) \ ; \ z \in \mathbb{R}^{2N}\}.$$

G inherits from H some properties (see for example 14).It is easy to see that

$G'(u) = z \Longleftrightarrow H'(z) = u$.

Remark 1.2.1.

It is well known that the solution of $(H-S)_\lambda$ coincide with the critical points of the functional K_λ defined on E as

(1.2.2) $$K_\lambda(z) = ½ \int_0^{2\pi} \langle z, J\dot{z} \rangle \, dt - \lambda \int_0^{2\pi} H(z) dt \ .$$

The dual variational method is used in the case when there are difficulties to find critical points of $K_\lambda(z)$. For instance when H is superquadratic $K_\lambda(z)$ is neither bounded from below nor from above. In dual variational method $K_\lambda(z)$ is replaced by its"dual functional" $f_\lambda(z)$ and thanks to "better" properties of $f_\lambda(z)$ it is possible to find critical points of $f_\lambda(z)$. A clear illustration of that method is the proof of theorem 2.2. from [5] (given by Ambrosetti and Mancini) which will be sketched later, another example can be find in [4] - the proof of the Weinstein's theorem.

Dual variational principle: Let $\lambda \in \mathbb{R}^+$ be given, if $u \in E \setminus \{0\}$ is a critical point of

(1.2.3) $$f_\lambda(u) = -\frac{\lambda}{2} \int_0^{2\pi} \langle u, Lu \rangle + \int_0^{2\pi} G(u) \quad , \ u \in E$$

then $z = G'(u)$ is a 2π-periodic solution of $(H-S)_\lambda$.

In fact: from $-\lambda\int_0^{2\pi}\langle v,Lu\rangle + \int_0^{2\pi}\langle G'(u),v\rangle = 0 \ \forall v\in E$, it follows that there exists $\xi\in\mathbb{R}^{2N}$ such that

$$-\lambda Lu + G'(u) = \xi. \tag{1.2.4}$$

Let $z = \lambda Lu + \xi$, then z is a solution of $(H\text{-}S)_\lambda$, because $-J\dot z = \lambda u$, on the other hand $z = G'(u)$, and hence we deduce that $-J\dot z = \lambda u = \lambda H'(z)$.

First of all I would like to present a result proved by Clarke and Ekeland [14] . They found a solution with prescribed minimal period, provided H subquadratic and convex.

Theorem 1.2.0.

Let $H\in C'(\mathbb{R}^{2N},\mathbb{R})$ be convex and $H(0) = 0$, $H'(0) = 0$. Moreover holds

$$\frac{H(\xi)}{|\xi|^2} \longrightarrow 0 \quad \text{as} \quad |\xi|\longrightarrow\infty, \qquad \frac{H(\xi)}{|\xi|^2}\longrightarrow\infty \quad \text{as} \quad |\xi|\longrightarrow 0.$$

Then $\forall T>0$, (H-S) has a periodic solution having T as minimal period.

Sketch of the proof:

Using duality method they have $f_\lambda(u) = -\frac{\lambda}{2}\int_0^{2\pi}\langle u,Lu\rangle + \int_0^{2\pi} G(u)$ is bounded from below, because here $\beta\in\,]1,2[$, so $\int_0^{2\pi}G(u)$ dominates the quadratic part of the functional for large u. From compactness and convexity they obtained that $f_\lambda(u)$ achives its negative minimum at some $\bar u$. Moreover holds $\int_0^{2\pi}\langle\bar u,L\bar u\rangle>0$. To prove that $\bar u$ has a minimal period equals 2π , let by contrary it will be $2\pi/k$ for some $k\geq 2$ $k\in\mathbb{Z}$ and let $v(t):=\bar u(t/k)$. They obtained a contradiction showing that $v(t)$ gives $f_\lambda(u)$ a strictly lower value than does $\bar u$ thanks to the following facts:

$\int_0^{2\pi}\langle v,Lv\rangle = k\int_0^{2\pi}\langle\bar u,L\bar u\rangle > 0$ and $\int_0^{2\pi}G(\bar u) = \int_0^{2\pi}G(v)$, hence

$$f_\lambda(v) = -\frac{\lambda}{2}\int_0^{2\pi}\langle v,Lv\rangle + \int_0^{2\pi}G(v) < -\frac{\lambda}{2}\int_0^{2\pi}\langle\bar u,L\bar u\rangle + \int_0^{2\pi}G(\bar u) = f_\lambda(\bar u),$$

which contradicts that $f_\lambda(\bar u) = \min_{u\in E} f_\lambda(u)$. So $G'(\bar u(t/\lambda))$ is a solution of (H-S) with minimal period $T = 2\pi/\lambda$.

■

Theorem 1.2.1. [5]

Let $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ and there exist $a_1, a_2, a_3, > 0$ and $\beta > 2$ such that

i) $a_1|z|^\beta \leq H(z) \leq a_2|z|^\beta \qquad \forall z \in \mathbb{R}^{2N}$

$H''(z)\xi \cdot \xi \geq a_3|z|^{\beta-2} \qquad \forall z \in \mathbb{R}^{2N}$ and $\forall \xi \in \mathbb{R}^{2N}, \ |\xi| \equiv 1;$

ii) $\beta H(z) \leq H'(z) \cdot z \qquad z \in \mathbb{R}^{2N}$;

iii) $\exists \mu \in]0,1[$ such that $\forall u \in \mathbb{R}^{2N}$, $u \neq 0$ it results $G''(u)u \cdot u \leq \mu G'(u) \cdot u$.

Then $\forall T > 0$ (H-S) has a periodic solution having T as minimal period.

Remark 1.2.2.

Due to the assumptions i) and ii) we have that H is strictly convex and superquadratic at zero and infinity.

Sketch of the proof:

The proof is based on "dual action principle", unfortunately since H is superquadratic f is not bounded from below nor from above. We are looking for stationary points of f as a saddle points. So, here is introduced a submanifold M on which f has a minimum, which appears to be a solution having T as minimal period. Due to Remark 1.1.1. and using the fact that λ plays no rôle in the arguments we will take $\lambda=1$, so $f = -\frac{1}{2}\int_0^{2\pi} u \cdot Lu + \int_0^{2\pi} G(u)$, where G is a Legendrê transform of H. As a manifold M we take $M = \{u \in E, \ u \neq 0; \ \langle f'(u), u \rangle = 0\}$, where E is as in preliminars. The pair (f,M) satisfies (P-S) condition (thanks to compactness of L and continuity of H': $L^\beta \longrightarrow L^\alpha$, $\beta^{-1}+\alpha^{-1}= 1$), moreover $\exists \bar{u} \in E$ such that $\int_0^{2\pi} \bar{u} \cdot L\bar{u} > 0$ ($\bar{u} = (\bar{\xi} \sin t, \bar{\xi} \cos t)$; $\bar{\xi} \in \mathbb{R}^N$, $\bar{\xi} \neq 0$) and assumptions on H and G" give us that $f_{|M}$ achieves its minimum in a point $v \in M$. Such v is a stationary point of f on E, i.e. $f'(v) = 0$ and holds

iv) $$\int_0^{2\pi} v \cdot Lv = \max \left\{ \int_0^{2\pi} u \cdot Lu \ : u \in E, \ u \neq 0, \ \int_0^{2\pi} G(su) \leq \int_0^{2\pi} G(sv) \ \forall s > 0, \ \int_0^{2\pi} \langle G'(u), u \rangle \leq \leq \int_0^{2\pi} \langle G'(v), v \rangle \right\}.$$

Using the last characterization of v it is possible to show that 2π is a minimal period of v. If not let $2\pi/m$ be a minimal period of v, $m>1$, $m\in\mathbb{Z}$. Set $v^*(t) = v(t/m)$, such $v^*\in E$, and $\int_0^{2\pi} G(sv^*) = \int_0^{2\pi} G(sv)$ $\forall s>0$, and $\int_0^{2\pi}\langle G'(v^*),v^*\rangle = \int_0^{2\pi}\langle G'(v),v\rangle$, but $\int_0^{2\pi} v^*\cdot Lv^* = m\int_0^{2\pi} v\cdot Lv$; since $\int_0^{2\pi} v\cdot Lv>0$ we have $\int_0^{2\pi} v^*\cdot Lv^* > \int_0^{2\pi} v\cdot Lv$ which contradicts the iv). For details see [5]. ∎

Ekeland, in his paper [15], under a bit different assumptions relatively to the Rabinowitz's theorem 0.5 (i.e. H is convex, grows slowly at infinity and quicly near the origin), obtained a similar result. To prove it, he looks for critical points of mountain-pass type(see Definition 2.1.), using the theorem of Ambrosetti and Rabinowitz [7]. His result is following:

<u>Theorem 1.2.2.</u>

Let $H:\mathbb{R}^{2N}\longrightarrow\mathbb{R}$ be convex, H has minimum at (0,0), $H(0,0) = 0$ and let there is a constant $\theta\in[0,\frac{1}{2}[$ such that $\forall\lambda>1$ and $\forall z = (p,q)\neq(0,0)$ it holds $H(\lambda z)\geqslant\lambda^{1/\theta}H(z)>0$. Then $\forall T>0$ $\exists$ a non-constant T-periodic solution of Hamiltonian equation $J\dot{z}\in\partial H(z(t))$ a.e., where ∂H is a subgradient of H, such that

(1.2.5) $$0<H(z(t)) = h \quad\text{and}\quad h\leqslant C/T^{\frac{4}{1-2\theta}}$$

where C is a constant depending only on θ and the minimum value of H at the S_1.

<u>Sketch of the proof.</u> First Ekeland proved the thesis in the case H is strictly convex and differentiable, then to obtain a general result he approximated H by a sequence of suitable strictly convex, differentiable functions. Since H is convex he used dual variational principle, studying the following functional: $f(\dot{p},\dot{q}) = \int_0^1 G(J(\dot{p},\dot{q})) - T\dot{p}\cdot q)dt$. He showed that f satisfies: (P-S) condition, $f(0,0) = 0$, f is bounded away from zero on the boundary of some open ball B and is zero again at some point (e,f) outside B. Under these conditions the Ambrosetti-Rabinowitz's theorem holds [7] which guarantees the existence of a non-trivial critical point

of f. Then using: the inequality $H(p,q) \leq \theta(p H'_p(p,q) + q H'_q(p,q))$, the equivalence $(p,q) \in \partial G(r,s) \Longleftrightarrow pr + qs = H(p,q) + G(r,s)$ and the constancy of H along the trajectory he got $c \geq h(\frac{1}{2\theta} - 1)$. From the last inequality and definition of critical level c ($c := \inf_{c \in C} \max_{0 \leq t \leq 1} f(c(t))$ where C is the set of continuous path such that $c(0) = (0,0)$, $c(1) = (e,f)$, see (2.5)), he obtained (1.2.5).

■

§3. Lusternik- Schnirelman critical points theory in the presence of S^1 symmetry.

When we want to calculate the number of distinct solutions on a fixed energy level a "normal"way is to use the idea of "index" (or "genus"); in more complicated cases even "pseudoindex" (for instance for indefinite functionals) c.f. [17, 9], which permits the classification of invariant sets.

Let E be a complex Hilbert space. The generic element of S^1 will be denote by s ($s \in R/2\pi\mathbb{Z}$) or by e^{is}. Let T be an unitary representation of S^1 in E, that is $T_s \in \mathrm{Isom}(E)$ is defined for all s, $\|T_s u\| = \|u\|$, $\forall u \in E$, $T_{s+s'} = T_s T_{s'}$ (hence $T_0 = id$, $T_{-s} = T_s^{-1}$) and $s \longrightarrow T_s$ is continuous.

Definition 1.3.1.

a) A representation R of S^1 in $\mathbb{C}^k$(or an S^1-action on $\mathbb{C}^k$) will be called regular (or is said to acts freely on E) if it only has a trivial fixed point space i.e. $\mathrm{Fix}R = \{0\}$, in other words $R_s u = u \ \forall s \Longrightarrow u = 0$.

b) A set $X \subset E$ is said to be invariant (under T) if $T_s X = X$, $\forall s$.

c) A functional $f: E \longrightarrow R$ is invariant if $f(T_s u) = f(u)$, $\forall u \in E$, $\forall s$.

d) A mapping $\Phi: E \longrightarrow \mathbb{C}^k$ is said to be equivariant with respect to (T,R) if

$$\Phi \circ T_s = R_s \circ \Phi \quad \forall s.$$

Example 1.3.1.

Given $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{Z}^k$, let for $\xi = (\xi_1, \dots, \xi_k)$, $R_s^\alpha \xi := (e^{i\alpha_1 s}\xi_1, \dots, e^{i\alpha_k s}\xi_k)$. R^α is an S^1-action, it is regular if and only if $\alpha_1, \dots, \alpha_k$ are all nonzero. Moreover, due to the Peter-Weyle representation theorem, any S^1-action on $\mathbb{C}^k$ is of the form R^α for some k-tuple in some orthonormal basis.

Let $M_k(A;R) = C_{eq}(A, \mathbb{C}^k_R \setminus \{0\})$ be the space of all continuous maps $\Phi: A \longrightarrow \mathbb{C}^k \setminus \{0\}$ which is equivariant with respect to (T,R). We denote by E° the space of fixed points of T : $E^\circ = \{u \in E: T_s u = u, \ \forall s \in R/2\pi\mathbb{Z}\}$. We assume that E° is finite dimensional. Let us define two classes of subsets of E:

(1.3.1) $\Gamma = \{X \subset E: \ X \cap E^\circ = \emptyset$, X is closed and invariant under T $\}$

$\Gamma_c = \{X \in \Gamma: \ X$ is compact $\}$

Definition 1.3.2.

For $X \in \Gamma$ we define:

(1.3.2) $i(X) := \inf\{k \in \mathbb{N};$ there exists a regular S^1-action R on $\mathbb{C}^k$ with $M_k(X;R) \neq \emptyset\}$

$i(\emptyset) := 0$

$i(X) := +\infty$ if $X \neq \emptyset$ and no such $k \in \mathbb{N}$ can be found.

Properties of index i are included in the following

Lemma 1.3.1.

i) $X \subset Y \Longrightarrow i(X) \leqslant i(Y)$

ii) $i(X \cup Y) \leqslant i(X) + i(Y)$

iii) if $\psi \in C(X,Y)$ is equivariant with respect to (T,Id) then $i(Y) \geqslant i(X)$

iv) if X is compact, then $i(X) < \infty$ and $\exists \delta > 0$ such that $i(X) = i(N_\delta(X))$, where $N_\delta(X)$ denotes the δ neighbourhood of X.

Remark 1.3.1.

Let S^n denote the following T-invariant set in E : $S^n = \{ u=(x\cos t + y\sin t,\ x\sin t - y\cos t):\ x,y \in \mathbb{R}^N,\ |x|^2 + |y|^2 = 1 \}$, then $i(S^n) = n$.

Remark 1.3.2.

Let h be T-invariant, $h \in C^1(E, \mathbb{R})$, $M = \{ u \in E,\ u \neq 0,\ h(u) = 0 \}$. Suppose M is a smooth manifold in E and look for critical points of f constrained to M, i.e. the points $u \in E$ such that $f'(u) = \omega h'(u)$ for some $\omega \in \mathbb{R}$.

In Lusternik-Schnirelman critical points theory we proceed in the following way : for every $j \leq i(M)$ (notice that $M \in \Gamma$) we define M_j to be the class of all compact $X \in \Gamma$, $X \subset M$, such that $i(X) \geq j$; suppose $M_j \neq \emptyset$, $\forall j \leq i(M)$ and set

(1.3.3) $$c_j := \inf_{X \in M_j} \sup_{u \in X} f(u)$$

We have $c_1 \leq c_2 \leq c_3 \ \ldots$; $c_j < \infty$ and $c_1 > -\infty$ if f is bounded from below on M, suppose also the pair (f,M) satisfies (P-S) condition and M is a smooth manifold then it is possible to show that every c_j is a critical level for $f_{|M}$ i.e. there exists at least one $u \in M$ such that $f(u) = c_j$ and $f'_{|M}(u) = 0$. Moreover if $c_l = \ldots = c_{l+r} = c$ then $i(K_c) \geq r+1$, where K_c is given by (1.1.2).

Remark 1.3.3.

If u is a critical point of f on M then every $v \in \mathcal{O}(u) := \{ T_s u:\ s \in [0, 2\pi[\}$ is a critical point, too. From the above discussion follows that

i) $c_j \neq c_i$ then the coresponding critical points u_j, u_i are such that $\mathcal{O}(u_j) \neq \mathcal{O}(u_i)$.

ii) if $c_l = \ldots = c_{l+r} = c$ then at level c there are at least r+1 critical points u_{l+k}, $k = 0,\ldots,r$ such that $\mathcal{O}(u_{l+k})$ are mutualy distinct.

As an example of using dual action principle and Lusternik-Schnirelman critical points theory we present a proof of the theorem stated by Ekeland and

Lasry [17] , here Theorem 0.4. Now we give sketch of the proof of that theorem.

The idea of the proof is following: the existence of periodic Hamiltonian trajectories on Σ follows from existence of solutions of related Hamiltonian system, having a prescribed minimal period (as in [4]). Then they look for critical points of a functional on a suitable manifold M. It is possible with precise estimates to pick out a part of M where the S^1-action is free, then they applicate the index theory and this gets the result.

Step1. Instead of studying (H-S) with any given H , using convexity of Ω we can replace H by $\mathcal{H}$ defined as $\mathcal{H}(z) = a^{\beta}$ for $\beta > 2$ arbitrary, where $a \in \mathbb{R}^+$ is found in the following way: $\forall z \in \mathbb{R}^{2N}$, $z \neq 0$ $\exists a \in \mathbb{R}^+$ and a unique $\bar{z} \in \Sigma$ such that $z = a\bar{z}$. $\mathcal{H}$ has properties: $\mathcal{H} \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ is strictly convex, $\forall z \in \Sigma$ $\mathcal{H}(z) = 1$ and $\mathcal{H}'(z) \neq 0$, the Hamiltonian system

$$(1.3.4) \qquad -J\dot{z} = \mathcal{H}'(z)$$

has on Σ the same periodic orbits as (H-S); for details see [4] .Let $z_1,\dots,z_n$ be n distinct periodic solutions of (1.3.4) with minimal period 2π . Set $h_i := H(z_i(t)) = \text{const.}$ (Remark 1.1.2) and

$$w_i(t) = h_i^{-1/\beta} z_i(h_i^{(\beta-2)/\beta} t)$$

From the β-homogenity of $\mathcal{H}$ follows that w_i are solutions of (1.3.4) satisfying $\mathcal{H}(w_i(t)) = 1$. Since $\mathcal{H}'(z) \neq 0$ $\forall z \in \Sigma$, then w_i are periodic trajectories on Σ. Such trajectories are distinct: in fact if $h_i \neq h_j$ for some i,j then $w_i \neq w_j$ (because $z_i \neq z_j$); otherwise the claim follows from the fact that minimal period of w_i is $2\pi h_i^{(2-\beta)/\beta}$.

Step 2. Let G be a Legendrė transform of $\mathcal{H}$. G is strictly convex and α-homogeneus ($\alpha^{-1} + \beta^{-1} = 1$). We shall look for N- distinct critical points of f on E having

2π as a minimal period , where

(1.3.5.) $$f(u) = -\tfrac{1}{2}\int_0^{2\pi} u\ Lu + \int_0^{2\pi} G(u)$$

The minimal period of a stationary point v of f is 2π when it is so of z,

$z \equiv Lv + \xi$, $\xi \in \mathbb{R}^{2N}$.

Step 3. Now we define a manifold M in E. Let $h(u) = \langle f'(u), u\rangle = -\int_0^{2\pi} u\cdot Lu +$ $+\alpha\int_0^{2\pi} G'(u)\cdot u = -\int_0^{2\pi} u\cdot Lu + \alpha\int_0^{2\pi} G(u)$. Set $M := \{ u \in E;\ u \neq 0 : h(u) = 0\}$. For M it follows that $\langle h'(u), u\rangle = -2\int_0^{2\pi} u\cdot Lu + \alpha^2\int_0^{2\pi} G(u) = \alpha(\alpha - 2)\int_0^{2\pi} G(u)$. Hence

$\langle h'(u), u\rangle \neq 0 \quad \forall u \in M$. M is a smooth manifold in E. Moreover if $v \neq 0$ is a critical point of f constrained on M it results

(1.3.6.) $$f'(v) = \omega h'(v),$$

for some $\omega \in \mathbb{R}^+$. From (1.3.6) it follows:

(1.3.7) $$\langle f'(v), v\rangle = \lambda\langle h'(v), v\rangle .$$

The left-hand side of (1.3.7) is equal to $h(v) = 0$, the right hand-side gives $\langle h'(v), v\rangle \neq 0$, thus $\omega = 0$. Namely $v \neq 0$ is a critical point of f on E if and only if v is a critical point of f on M. Note that

(1.3.8) $$f_{|M}(u) = \frac{2-\alpha}{2}\int_0^{2\pi} G(u) = \frac{2-\alpha}{2\alpha}\int_0^{2\pi} u\cdot Lu$$

and also

(1.3.9) $$\lambda w \in M,\ w \neq 0 \quad \text{iff} \quad \lambda^{2-\alpha}\int_0^{2\pi} w\cdot Lw = \alpha\int_0^{2\pi} G(w).$$

In particular there is a T-equivariant map $\psi : S^N \longrightarrow M$ defined by $\psi(w) = \lambda w$ with λ given by (1.3.9). Here it suffieces to note that M_N (defined in Remark 1.3.2) is not empty, because $\psi(S^N) \subset M_N$. In fact $\psi(S^N)$ is compact and $i(\psi(S^N)) \geqslant i(S^N) =$ in view of Lemma 1.3.1 iii) and Remark 1.3.1.

Step 4. The pair (f,M) satisfies (P-S), moreover from (1.3.8)it follows that f is bounded from below on M. So we are in the position to apply arguments from Remark 1.3.2. i.e. Lusternik-Schnirelman theory.

Since (f,M) satisfies (P-S) then f attains its minimum on M, let denote it by m and let $m := f(\bar{v})$. Let M* be a set of points $u \in M$ such that $u = \sum_{k\in\mathbb{Z}} u_{hk}\exp(ihkt)$ for some integer $h \geqslant 2$. The following lemma holds:

Lemma 1.3.2.

$\min\{f(u) : u \in M^*\} = f(2^{\frac{1}{2-\alpha}}.\bar{v}) = 2^{\frac{\alpha}{2-\alpha}}\, m.$

The proof of that lemma is carried out by using (1.3.8) and (1.3.9), can be found in [6] .

Step 5. Let $m^* = \min\{f(u) : u \in M^*\}$ and $\mathbb{M} = \{u \in M : f(u) < m^*\}$, it is easy to see that every critical point of $f_{|M}$ such that $v \in \mathbb{M}$ will have 2π as minimal period. In order to find N- distinct critical points of $f_{|M}$ lying in $\mathbb{M}$ we first state

Lemma 1.3.3.

Under the assumption of Theorem 0.4 it follows that $\psi(S^N) \subset \mathbb{M}$.

Using the result from Lemma 1.3.3. the proof of Theorem 0.4 can be easily completed. In fact $\psi(S^N) \subset M_N$, it follows that $c_N = \inf_{X\in M_N} \sup_{u\in X} f(u) < \sup_{\psi(S^N)} f(u) < m^*$. Hence the result follows because of Remark 1.3.2 and Remark 1.3.3.

Sketch of the proof of Lemma 1.3.3. From ß-homogenity of $\mathcal{H}(z)$ and definition of Σ one has

$$R^{-\beta}|x|^{\beta} \leqslant \mathcal{H}(z) \leqslant r^{-\beta}|x|^{\beta}, \tag{1.3.10}$$ hence one has an estimate:

$$a_2 r^{\alpha} \leqslant \int_0^{2\pi} G(u) \leqslant a_2 R^{\alpha} \text{ for } \forall\, u \in \Sigma, \quad a_2 = 2\pi a_1, \text{ where } a_1 = 1/\alpha \cdot \beta^{-1/\beta - 1}.$$

Now let $\bar{v} = \lambda \bar{w}$, $\bar{w} \in \Sigma$. Set $b = \max\{\frac{1}{2}\int_0^{2\pi} u\, Lu\}$, one has $b = \frac{1}{2}\int_0^{2\pi} \bar{u}\, L\bar{u}$, where $\bar{u} \in S^n$. Using (1.3.9) and (1.3.10) , and definition of b one gets

(1.3.11) $$m = f(\lambda \bar{w}) \geqslant a_3 r^{2\frac{\alpha}{2-\alpha}} \quad , \text{ where } \quad a_3 = \frac{2-\alpha}{2}\left(\frac{\alpha}{2b}\right)^{\alpha/(2-\alpha)} \cdot a_2^2 \cdot \frac{1}{2-\alpha} \ .$$

Let $u = \psi(w)$ with $w \in S^N$, from $u = \lambda w$, $b = \frac{1}{2}\int_0^{2\pi} w \cdot Lw$ $\forall w \in S^N$, (1.3.8) and (1.3.10) it follows $f(\psi(w)) \leqslant a_3 R^{2\frac{\alpha}{2-\alpha}}$, hence

(1.3.12) $$\hat{m} = \max_{w \in S^n} f(\psi(w)) \leqslant a_3 R^{2\frac{\alpha}{2-\alpha}} \ .$$

The condition $R^2 < 2r^2$ implies $\hat{m} < a_3 2^{\frac{\alpha}{2-\alpha}} r^{2\frac{\alpha}{2-\alpha}}$ and by (1.3.11) $\hat{m} < 2^{\frac{\alpha}{2-\alpha}} m = m^*$.

Therefore $\hat{m} < m^*$ as we wanted to show.

§4. Direct method.

The aim of this paragraph is to introduce the definition of pseudo-index and to give theorems which will be usefull to study directly a functional in the case when H is "superquadratic" and to find the critical points of this functional.

Example 1.4.1.

The following functional $f(z) = -\frac{1}{2}\int J\dot{z}\, z - \lambda \int H(z)$ is neither bounded from below nor from above, moreover there doesn't exist a weakly continuous function Φ such that $f + \Phi$ would be bounded or from below, or from above. The critical points of $f(z)$ (if exist) are of saddle type with both stable and unstable manifolds infini - te dimensional.

If we assume something more about H: $H''(z)$ is bounded $\forall z \in \mathbb{R}^{2N}$ or H is convex, then we can apply respectively methods of Amann and Zehnder ([1, 2, 3] , a kind of Liapunow-Schmidt finite dimensional reduction method), or the one introduced by Clarke and Ekeland [14] . To avoid these two restrictions it is necessary to stu- dy a suitable functional $f(z)$ directly.

Let H be "superquadratic" and moreover let $|H'(z)| \leqslant k_1 + k_2|z|^\alpha$ for some k_1, $k_2 > 0$, then the functional f is Fréchet differentiable in the Hilbert space $E = W^{1/2}(S^1, \mathbb{R}^{2N})$ (see Remark 1.1.3). We shall study a functional f(z)

(1.4.1) $$f(z) = \tfrac{1}{2}\langle z, Lz\rangle - \int H(z) \quad \forall z \in E,$$

where $\langle v, Lw\rangle = I'(w)(v) \quad \forall\, v, w \in E$ and I is the extension of the functional $\int_0^T v\cdot J\dot{v}$ from $C^{\infty}(S_T; \mathbb{R}^{2N})$ into E.

Let now $E = E^+ \oplus E^- \oplus E^0$ be the orthogonal decomposition of E with respect to the action $\tfrac{1}{2}\langle v, Lv\rangle$, $E^0 = \ker L \simeq \mathbb{R}^{2N}$. To introduce the pseudo-index in this particular case let us define the group of linear homeomorphisms $\mathcal{H}$, from E into itself as

(1.4.2) $$\mathcal{H} = \{\, U = e^{tL},\ t \in \mathbb{R} \,\}.$$

Note that E^+ is L-invariant under $\mathcal{H}$ i.e. $UE^+ = E^+ \quad \forall U \in \mathcal{H}$.

Let $\mathcal{H}^*$ be a class of mappings from E into itself such that

- $h \in \mathcal{H}$, h is S^1-equivariant
- h is a homeomorphism from E into itself of the kind $h = e^{tL} + \Phi$ with Φ compact

Let Γ be a class of closed, S^1- invariant subsets of E. Following Benci [9] we shall define a pseudo-index i*:

<u>Definition 1.4.1.</u>

(1.4.3) $$i^*(A) := \min_{h \in \mathcal{H}^*} i(h(A) \cap E^+ \cap S_\varrho), \quad \forall A \in \Gamma;$$

where i is the index (1.3. 2). We call a couple $(\mathcal{H}^*, i^*)$ a pseudo-index theory.

There are some advantages which gives us a pseudo-index. We remark that $i(X) = +\infty$ if $X \cap \mathrm{Fix}(T_s) \neq \emptyset$ (where T_s is a unitary representation of S^1), and this implies $c_k = \inf_{i(X) \geq k} \sup_{u \in X} f(u) = -\infty$, provided f is not bounded from below. Moreover the pseudoindex is invariant under the deformation flow η, if only $\eta \in \mathcal{H}^*$

The pseudo-index is a map $i^* : \Gamma \longrightarrow \mathbb{N} \cup \{+\infty\}$ with the following <u>properties:</u>

i*) $i^*(A) \leq i(A) \quad \forall A \in \Gamma$

ii*) if $A \subset B$ then $i^*(A) \leqslant i^*(B)$ $\forall A, B \in \Gamma$

iii*) $i^*(\overline{A \setminus B}) \geqslant i^*(A) - i(B)$ $\forall A, B \in \Gamma$

iv*) $i^*(h(A)) = i^*(A)$ $\forall h \in \mathcal{H}^*$ $\forall A \in \Gamma$.

Theorem 1.4.1. [9]

Let $H^+ = E^+$, $H^- \in \Gamma$ be linear subspaces of E such that $E = H^+ \oplus H^-$.
Suppose that $Fix(T_s) \cap H^+ = \{0\}$, $Fix(T_s) \subset H^-$, $\dim(H^- \cap H^+) < +\infty$ and that H^- is invariant for every $U \in \mathcal{H}$. Then

$$i^*(H^-) = \tfrac{1}{2}(\dim(H^+ \cap H^-)). \tag{1.4.4}$$

Definition 1.4.2.

If $f \in C'(E, \mathbb{R})$ and $c_0, c_\infty \in \mathbb{R}$ (with $c_0 < c_\infty$), we say that f, c_0, c_∞, satisfies the property (P) with respect to $\{\Gamma, \mathcal{H}^*\}$ if

a) f^c, $K_c \in \Gamma$ and K_c is compact for every $c \in [c_0, c_\infty]$,

b) $\forall c \in [c_0, c_\infty]$, $\forall N = N_\delta(K_c)$ $\exists \varepsilon > 0$ and $\eta \in \mathcal{H}^*$ such that $\eta(f^{c+\varepsilon} \setminus N) \subset f^{c-\varepsilon}$.

The property (P) is strictly related to (P-S) condition and in our case it is possible to construct such $\eta \in \mathcal{H}^*$ as a flow which is related to a vector field being a suitable approximation of the vector field f'.

Lemma 1.4.1. (Deformation lemma , [9])

Let $f \in C'(E, \mathbb{R})$ be a functional which satisfies following assumptions:

f1) $f(u) = \tfrac{1}{2}\langle Lu, u\rangle + \Phi(u)$, where L is bounded, selfadjoint operator and $\varphi = \Phi'$ is compact.

f2) f satisfies (P-S) condition with $f(u_n) \longrightarrow c \in [c_0, c_\infty]$.

Then $\forall c \in [c_0, c_\infty]$, $\forall N = N_\delta(K_c)$ $\exists \varepsilon > 0$ and $\exists\, \eta = e^{-\delta L} + \psi$ (where $\delta > 0$ is a constant and ψ a compact operator) such that

$$(f^{c+\varepsilon} \setminus N) \subset f^{c-\varepsilon}. \tag{1.4.5}$$

Moreover if

f3) f is T_s- invariant

then η is T_s-equivariant, i.e. $\eta\in\mathcal{H}^*$.

Remark 1.4.1.

The functional (1.4.1) related to H superquadratic satisfies assumptions f1)-f3)

Now having a deformation lemma we are able to state some multiplicity results.

Proposition 1.4.1. [12] .

Let f satisfies f1)-f3). Define

(1.4.6) $$c_k := \inf_{i^*(X)\geqslant k} \sup_{u\in X} f(u).$$

Assume futhermore that for some $k,m\in\mathbb{N}$, exists such $X\in\Gamma$: $i^*(X) \geqslant k+m-1$ and that $0 < c = c_k = c_{k+1} = \dots = c_{k+m-1} < +\infty$.

If $K_c \cap Fix(T_s) = \emptyset$ then c is a critical value of f and $i(K_c) \geqslant m$.

Proof of this proposition use standard arguments connected with deformation lemma, Since $K_c \cap Fix(T_s) = \emptyset$ and K_c is invariant and compact by (P-S), $K_c\in\Gamma$ and $i(K_c)$ is well defined and finite. Futhermore $\exists\,\delta>0$ such that $N_\delta(K_c)$ verifies $i(N_\delta(K_c)) = i(K_c)$ due to (1.3.1 iv)). Moreover, by deformation lemma we are able to constru ct for any $\varepsilon>0$, $\varepsilon<\bar\varepsilon$ small enough a flow $\eta : E \longrightarrow E$. To prove the proposition we argue by contradiction and suppose that $i(K_c) \leqslant m-1$. By definition of $c_{k+m-1} = c$ there exists A with $i^*(A) \geqslant k+m-1$ such that $\max_{x\in A} f(x) \leqslant c+\varepsilon$, i.e. $A \subset f^{c+\varepsilon}$. Now let $B = \eta(\overline{A\setminus N_\delta(K_c)})$. By the following property of η : $\eta(x) = x\ \forall x$ such that $|f(x) - c| \geqslant \bar\varepsilon$ we know $\eta(0) = 0$, since η is an equivariant homeomorphism, it is clear that $B\in\Gamma\setminus\{0\}$. By property iii*) we know

$i^*(B) \geqslant i^*(A) - i(N_\delta(K_c))$. Hence, since $i(N_\delta(K_c)) = i(K_c) \leqslant m-1$ and $i^*(A) \geqslant$ $\geqslant k+m-1$ yields $i^*(B) \geqslant k$.

On the other hand from a property (1.4.5) we infer that $B \subset f^{c-\varepsilon}$ which is the contradiction to the fact $c = c_k$ as $i^*(B) \geqslant k$. ∎

From the viewpoint of applications it is crucial to have conditions which will a priori guarantee that the c_k constructed by (1.4.6), k= 1, 2;... satisfy $0 < c_k < +\infty$. We want to prove this in our concrete case; thus let $H_{\bar{k}}$ be a $2\bar{k}$ dimensional subspace of E^+, S^1-invariant, and put $H^+ = E^+$, $H^- = H_{\bar{k}} \oplus E^- \oplus E^0$. Then, due to Theorem 1.4.1 one has $i^*(H^-) = \bar{k}$.

<u>Proposition 1.4.2.</u>

Let $f \in C^1(E, R)$ satisfies f1)-f3). Let H^+ and H^- be as above, $E = H^+ \oplus H^-$, and let f be bounded on H^- from above. Then $\forall k \leqslant \bar{k}$ c_k is finite.

<u>Proof</u>. It follows from the definition of c_k. We have

$$c_1 \leqslant c_2 \leqslant \ldots \leqslant c_{\bar{k}} = \inf_{i^*(X) \geqslant \bar{k}} \sup_{u \in X} f(u) \leqslant \sup_{H^-} f(u) .$$

∎

<u>Proposition1.4.3.</u>

Let f be as in Proposition 1.4.2 and be strictly positive on $E^+ \cap S_\varrho$ for some $\varrho > 0$. Then $c_k > 0$ for every $k \in \mathbb{N}$.

<u>Proof.</u> It follows from the definitions of i^* and c_k. We have

$$0 < \inf_{S_\varrho \cap E^+} f(u) \leqslant \inf_{i^*(X) \geqslant 1} \sup_{u \in X} f(u) = c_1 \leqslant \ldots .$$

∎

II. RESULT BY EKELAND AND HOFER

The aim of this section is to present a recent result obtained by Ekeland and Hofer [16] .They approached in a original way to the problem of finding solutions with prescribed minimal period using Morse index and a description of mountain-pass type critical points. Their result for Hamiltonian system in superquadratic case is the following:

Theorem 2.1.

Suppose $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$ satisfies

(2.1) $H''(x)$ is positive definite for $x \neq 0$

(2.2) $H(x)\|x\|^{-2} \longrightarrow 0$ when $\|x\| \longrightarrow 0$

(2.3) $\exists\ r > 0,\ \exists\, \beta > 2 : H'(x)\cdot x \geqslant \beta\, H(x)$ for $\|x\| \geqslant r$.

Then, for any $T > 0$, there exists a periodic solution $\bar{x}$ of (H-S) with minimal period T.

The assumptions of this theorem are as in the Rabinowitz's one ([22] , here Theorem 0.5), but the assumption (2.1) which states in particular that H is strictly convex. Note that Rabinowitz's result is weaker than this one namely he proved the existence of solution for any period T which need not be minimal (see example 0.1). First Ekeland and Hofer state some abstract results, then they make an application to super- and subquadratic Hamiltonian systems.

Let us put the same notations as in [16] :

$$\text{(A0)} \qquad \begin{cases} H \in C^2(\mathbb{R}^{2N}, \mathbb{R}) \ \text{ is strictly convex} \\ H(x) > H(0) = 0 \quad \text{for all } x \neq 0 \\ H(x)\|x\|^{-1} \longrightarrow +\infty \quad \text{as } \|x\| \longrightarrow +\infty \end{cases}$$

Denote by $G: \mathbb{R}^{2N} \longrightarrow \mathbb{R}$ the Fenchel conjugate of H,

$L^{\alpha}_{o} := \{ u \in L^{\alpha}(0,T; \mathbb{R}^{2N}) : \int_0^T u dt = 0 \}$ with $1 < \alpha < +\infty$

(2.4) $\quad f(u) := \int_0^T (\frac{1}{2}(Ju, \Pi u) + G(-Ju))dt$ where Πu is the primitive of u in L^{α}_{o}.

Suppose that α and H stisfy:

(A1) $$\begin{cases} \|G'(y)\| \leqslant c_1 \|y\|^{\alpha-1} + c_2 \\ \|H'(x)\| \leqslant c_3 \|x\|^{\frac{1}{\alpha-1}} + c_4 \end{cases}$$, where c_i; $i = 1,\dots 4$, are any constants.

Remark2.1.

Under the assumptions (A0) and (A1) to find T-periodic solutions of (H-S) is equivalent to find critical points of the C^1 functional f (2.4) on L^{α}_{o} (due to "dual variational principle"). To do the latter one uses Ambrosetti-Rabinowitz theorem [7] (see the proof of Ekeland's theorem, here Theorem 1.2.2.). From this theorem it is possible to obtain the existence of at least one non-constant T-periodic solution. The question is when T is a minimal period, so we have to know something more about the critical points which are found in this way. This was achieved by Hofer [20] who gave the following definition and theorem.

Definition 2.1.

Let $\bar{u} \in L^{\alpha}_{o}$ be a critical point of f. We say $\bar{u}$ is of mountain-pass type (m-p), if for all open neighbourhoods U of $\bar{u}$ the set $\{ u \in U : f(u) < f(\bar{u}) \}$ is neither empty nor path connected.

Proposition 2.1.

Let f satisfies (A0), (A1), (P-S). Let u_o , $u_1, \in L^{\alpha}_{o}$, $u_o \neq u_1$ and define

(2.5) $\quad C = \{ c \in C^{0}([0,1] , L^{\alpha}_{o}) : c(0) = u_o, c(1) = u_1$

(2.6) $\quad d = \inf_{c \in C} \sup_{0 \leqslant t \leqslant 1} f(c(t))$

(2.7) $\quad e = \max \{ f(u_o), f(u_1) \}$

If, moreover $d > e$ then K_d contains either a local minimum or a critical point of (m-p) type.

The following theorem is the crucial one:

Theorem 2.2.

Assume (A0) and (A1). Let $\hat{x}$ be a T-periodic, non-constant solution of (H-S), and assume there exists a neighbourhood U of $\hat{x}([0,T])$ in $\mathbb{R}^{2N}$ and a constant $k > 0$ such that

(2.8) $$H''(x)y \cdot y \geqslant k\|y\|^2 \quad \forall x \in U, \quad \forall y \in \mathbb{R}^{2N}.$$

Set $\hat{u} = \frac{d\hat{x}}{dt}$, a critical point of f in L^α_0. If $\hat{u}$ is a local minimum or has (m-p) type then $\hat{x}$ has minimal period T;

Idea of the proof. Ekeland and Hofer define index $m(\hat{x})$ of the critical point $\hat{u}$ and they prove that if $\hat{u}$ has (m-p) type then $m(\hat{x}) \leqslant 1$. If f were C^2 and $\hat{u}$ "nondegenerate", $m(\hat{x})$ would be simple Morse index and the inequality $m(\hat{x}) \leqslant 1$ would follow from the Morse lemma. Unfortunately f is C^1 and $\hat{u}$ may be degenerate, to deal with this difficulty Ekeland and Hofer introduce a finite dimensional reduction and achieve the result using a lot of technicalities.

Then they prove that $m(\hat{x}) + 1 \geqslant O(\hat{x})$, where $O(\hat{x})$ is the order of the isotropy group of $\hat{x}$. If $\hat{u}$ is a local minimum then $O(\hat{x}) = 1$, so $\hat{x}$ has minimal period T or T/2. The second possibility is then eliminated due to the definition of (m-p) type point and S^1-invariancy of f.

Now we are going to give some more details of the proof. Let us consider the following problem:

(I) $$\begin{cases} \dot{x} = JH'(x) \\ x(0) = x(T) \end{cases}$$

Definition. 2.2.

A solution $\hat{x}$ of problem (I) will be called admissible if it is non-constant, and if $H''(\hat{x}(t))$ is invertible for all t. It follows that the condition (2.8) is satisfied.

Lemma 2.1.

Let $\hat{x}$ be an admissible solution, and set $\hat{u} = \frac{d\hat{x}}{dt}$, Then there exists a unique symmetric bilinear map $Q : L^2_o \times L^2_o \longrightarrow \mathbb{R}$ such that:

$$Q(u_1,u_2) = \frac{d}{ds_1}\frac{d}{ds_2} f(\hat{u} + s_1 u_1 + s_2 u_2)_{s_1 = s_2 = 0} \tag{2.9}$$

for all u and u in L_o .

We obtain that Q has a form: Q = K + L , where K is compact , $L^{1/2}$ is equivalent to the standard norm on L^2_o .

Definition 2.3.

a) Q is called the Hessian of f at $\hat{u}$.

b) The Morse index $m(\hat{x})$ of $\hat{x}$, where $\hat{x}$ is a critical point of f is equal to the dimension of a maximal negative eigenspace of $Q(\hat{u})$ (i.e.the dimension of maximal subspace $V \subset E$ such that $Q(\hat{u})v.v < 0$ $\forall v \in V,\ v \neq 0$).

c) We say that $O(\hat{x})$ is the order of the isotropy subgroup of $\hat{x}$ for the S^1 action a_s on T-periodic functions: $a_s x(t) = x(t + sT)$ if $O(\hat{x})$ is the greates integer k such that $\hat{x}$ is T/k -periodic.

The relation between $m(\hat{x})$ and the behaviour f(u) near critical point $\hat{u}$ is given by the following theorem:

Theorem 2.3.

Assume (A0) and (A1). Let $\hat{x}$ be an admissible solution of problem (I), and $\hat{u} = \frac{d\hat{x}}{dt}$.

Then $\hat{u}$ is a critical point of f and:

a) if $\hat{u}$ is a local minimum then $m(\hat{x}) = 0$

b) if $\hat{u}$ is of (m-p) type then $m(\hat{x}) = 1$

c) if $m(\hat{x}) = 1$ then there exists an open neighbourhood W of $\hat{u}$ such that the set $\{u \in W : f(u) < f(\hat{u})\}$ has at most two path components. If they are exactly two, say $\mathfrak{F}_1$ and $\mathfrak{F}_2$, and if we denote by λ_1 the negative eigenvalue of Q and v_1 the associated eigenvector, we have for all $\eta > 0$ sufficently small

(2.10) $$\hat{u} + \eta v_1 \in \mathfrak{F}_1 , \quad \hat{u} - \eta v_2 \in \mathfrak{F}_2 .$$

Ekeland and Hofer proved this theorem using the extension of the Morse lemma to the case when critical points $\hat{u}$ occur in manifolds (and hence are degenerate), following Gromoll and Meyer [20] and a finite dimensional reduction to obtain the result for $f \in C^1$ (not C^3).

Proposition 2.2.

The following relation holds:

(2.11) $$O(\hat{x}) \leq m(\hat{x}) + 1.$$

In the proof of this relation is used the idea of conjugate points:

Definition 2.4.

Let $\hat{x}$ be an admissible T -periodic solution. We say that $\hat{x}(t_2)$ is conjugate to $\hat{x}(t_1)$ with $t_1 < t_2$, if the linear problem :

(2.12) $$\begin{cases} \dot{y} = JH''(\hat{x}(t))y \\ y(t_1) = y(t_2) \end{cases}$$

has a non-zero solution. The multiplicity of the conjugate point $\hat{x}(t_2)$ is the number of lineary independent solutions problem (2.12) possesses.

These points satisfy the following properties :

Proposition 2.3.

a) If $\hat{x}$ is a T-periodic solution of (I) then $\hat{x}(kT)$ is conjugate to $\hat{x}(0)$ for $\forall k \in \mathbb{Z}$.

b) If $\hat{x}$ is an admissible T-periodic solution then its index $m(\hat{x})$ is equal to the number of times $s \in [0,T]$ such that $\hat{x}(s)$ is conjugate to $\hat{x}(0)$, each counted with its multiplicity.

Having the last property it is possible to complete the proof of Theorem 2.2. We can easily see that for an admissible T-periodic solution $\hat{x}$ Proposition 2.2. holds (i.e. $O(\hat{x}) \leqslant m(\hat{x}) + 1$). In fact, say $\hat{x}$ is T/k periodic, then there are at least $(k-1)$ points conjugate to $\hat{x}(0)$ on the arc $[0,T]$, namely $T/k, \ldots, (k-1)T/k$, by Proposition 2.3 $m(\hat{x}) \geqslant k-1$. ∎

Corollary 2.1.

a) If $m(\hat{x}) = 0$ then $\hat{x}$ has minimal period T.

b) If $m(\hat{x}) = k$ then the minimal period of $\hat{x}$ cannot be smaller than $T/(k+1)$

Now we return to the proof of Theorem 2.2. If $\hat{u}$ is a local minimum then $m(\hat{x}) = 0$ (by Theorem 2.3.a)), so $\hat{x}$ has as a minimal period T (Corollary 2.1). If $\hat{u}$ is of (m-p) type either $m(\hat{x}) = 0$ (then a minimal period is T), or $m(\hat{x}) = 1$ and $\hat{x}$ has a minimal period T or T/2. Fortunately the second possibility doesn't hold under the assumptions (A0) and (A1). Using the characterization of (m-p) type point, from Theorem 2.3.c) and Definition 2.1 we can find an open neighbourhood $\mathfrak{P}$ of $\hat{u}$ which is not path connected and has exactly two components $\mathfrak{P}_1$ and $\mathfrak{P}_2$. Since f is S^1-invariant we have that also $\mathfrak{P}$ is S^1-invariant. As $\hat{u}$ is T/2 -periodic then it is possible to construct a path which connects $a_s(\mathfrak{P}_1)$ with $a_s(\mathfrak{P}_2)$, $s \in S^1$, which means that $\mathfrak{P}$ is path connected and gives a contradiction. ∎

III. PRESCRIBED MINIMAL PERIOD IN "NONCONVEX CASE".

§ 1. Results obtained by Girardi and Matzeu.

For H nonconvex there is a result given by Girardi and Matzeu [18] and then a more general result in [19] . There is a possibility to slightly improve both these results. First we write down their theorems.

Theorem 3.1.1.

H1) $H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$

H2) $H(z) \geqslant 0$, $\forall z \in \mathbb{R}^{2N}$, $H(z) = 0$ iff $z = 0$

H3) $\exists$ $\beta > 2$, $\beta H(z) \leqslant H'(z)\cdot z$

H4) $\exists$ $a_2 \geqslant 0$: $|H'(z)| \leqslant a_2|z|^{\beta-1}$

H5) $H(z) \geqslant a_1|z|^{\beta}$ for some $a_1 \geqslant 0$.

Let H1)-H5) be verified, then there exists an integer number $\bar{n} \geqslant 2$ depending on a_1, a_2, β such that for every $T > 0$ there exists a solution z of (H-S) and an integer $m \in \{1,\dots, \bar{n}-1\}$ such that z has minimal period T/m.

As a consequence of this theorem Girardi and Matzeu proved the following

Theorem 3.1.2.

Let H be convex and H1) - H5) be verified. One can determine a function $g:R^+ \longrightarrow R^+$ with $g(\beta) > \beta$, $\beta > 2$ such that z has minimal period T. Further one has

(3.1.1) $$\lim_{\beta\to\infty} (g(\beta) - \beta) = +\infty .$$

They improved their result [19] stating:

Theorem 3.1.3.

Let H satisfies H1), H3), H4), H5): and let the further hypothesis be verified:

H6) $a_2/a_1 < \sqrt{2}\,\beta$.

Then for every $T > 0$, there exist at least N distinct periodic solutions of (H-S) having minimal period T.

Theorem 3.1.3. is more general than 3.1.1, but its proof is partly based on the proof of Theorem 3.1.1. We shall only point out the steps of proofs of these theorems. Details we'll give in the improvements of Girardi's and Matzeu's results stated in III.§2.

Sketch of the proof of Theorem 3.1.1. Let

(3.1.2) $$E = \{ z \in H^1([0,T]; \mathbb{R}^{2N}) : z(0) = z(T) \}.$$

The solution of (H-S) coincide with the critical points of the functional f defined on E as

(3.1.3) $$f(v) = \tfrac{1}{2}\int_0^T v(t)\cdot J\dot{v}(t)\,dt - \int_0^T H(v(t))dt \qquad \forall v \in E.$$

Lemma 3.1.1. (Wirtinger inequality)

Let $z \in E$ has minimal period T/m, $m \in \mathbb{N}$. If $\dot{z} \in L^2([0,T], \mathbb{R}^{2N})$ then

(W-I) $$\|z\|_{L^2} \leq \frac{T}{2\pi m} \|\dot{z}\|_{L^2}.$$

Proof. The expansion of z in its Fourier series is given by

$$z(t) = \sum_{k\in\mathbb{Z}} z_k \exp(i2\pi mkt/T) \qquad t \in [0,T],$$

hence $$\dot{z}(t) = 2\pi m/T \sum_{k\in\mathbb{Z}} k\, z_k \exp(i2\pi mkt/T) \qquad t \in [0,T], \text{ so}$$

$$\|z\|_{L^2} = \Big(\sum_{k\in\mathbb{Z}} (z_k)^2\Big)^{1/2} \leq \Big(\sum_{k\in\mathbb{Z}} k(z_k)^2\Big)^{1/2} = \frac{T}{2\pi m}\|\dot{z}\|_{L^2}.$$

■

Lemma 3.1.2.

If H1) - H5) are satisfied then zero is an isolated critical value of f.

Lemma 3.1.3.

There exists a minimum positive critical value of f, say c_T.

Lemma 3.1.4.

Let $z \in E$ be a solution of (H-S) with $h_z = H(z(t))$ $\forall t \in [0,T]$. Then one has:

(3.1.4.) $$h_z \geqslant \frac{K_1}{T^{\beta/(\beta-2)}}$$, where

(3.1.5) $$K_1 = a_1(2\pi\beta a_1/a_2^2)^{\beta/(\beta-2)}.$$

Lemma 3.1.5.

Let $z \in E$ be a solution of (H-S) with h_z as above, such that $f(z) = c_T$. Then one has

(3.1.6) $$h_z \leqslant K_2 \cdot \left(\frac{1}{T}\right)^{\beta/(\beta-2)}$$, where

(3.1.7) $$K_2 = \frac{2}{\beta-2}\pi\left(\frac{\pi}{a_1}\right)^{2/(\beta-2)}.$$

To obtain this estimate Girardi and Matzeu used the result obtained by Rabinowitz in [24] .Having all these lemmas we are able to complete the proof of Theorem 3.1.1.

(3.1.8) Let $\bar{n} := \min\left\{ n \in \mathbb{N}:\ n \geqslant 2\ ,\ \ n > \left(\frac{K_2}{K_1}\right)^{(\beta-2)/\beta} \right\}$, where K_1 and K_2 as above.

Let z be a critical point of f such that $c_T = f(z)$ is a minimum positive critical value of f (which exists due to Lemma 3.1.3;). If T is not the minimum period of z , then let $m \in \mathbb{Z}$, $m \geqslant 2$ be such that T/m is a period of z : we want to show that $m < \bar{n}$. By contradiction,if $m \geqslant \bar{n}$ then one would have from (3.1.8.) and (3.1.6.)

(3.1.9) $$\frac{T}{m} \leqslant \frac{T}{\bar{m}} < T\cdot\left(\frac{K_1}{K_2}\right)^{(\beta-2)/\beta} \leqslant \left(\frac{K_2}{h_z}\right)^{(\beta-2)/\beta}\cdot\left(\frac{K_1}{K_2}\right)^{(\beta-2)/\beta} = \left(\frac{K_1}{h_z}\right)^{(\beta-2)/\beta} ,$$

but now applying lemma 3.1.4 to z as a solution of period T/m one has

(3.1.10) $$\left(\frac{K_1}{h_z}\right)^{(\beta-2)/\beta} \leqslant \frac{T}{m}$$, which contradicts (3.1.9) .

∎

Remark 3.1.1.

In the proof of Theorem 3.1.2 Girardi and Matzeu showed that $\left(\frac{K_2}{K_1}\right)^{\beta-2/\beta} < 2$, where K_2 is an estimate due to Ekeland [15] for H convex. The theorem is still true also in nonconvex case, with a different $g(\beta)$ and $a_2/a_1 \neq 4$.

Remark 3.1.2.

We can assume without loss of generality that $a_1 = 1$ in H5) (Remark 1.1;1) .

Sketch of the proof of Theorem 3.1.3. Let E and f be as in I.§4. Under the hypothesis of this theorem Lemma 3.1.2 still holds and Lemma 3.1.3 is a consequence of Proposition 4.3. Moreover the following inequality is true:

$$c_{min} \geq \frac{\beta-2}{2} \cdot \left(\frac{2\pi\beta}{a_2^2}\right)^{\beta/(\beta-2)} \cdot \left(\frac{1}{T}\right)^{2/(\beta-2)} \quad , \text{ where} \tag{3.1.11}$$

$$c_{min} := \min \{ f(v) : \ v \in E, \ f'(v) = 0 \ , \ f(v) > 0 \} = c_T .$$

To obtain the minimality of a period T Girardi and Matzeu proved the crucial

Lemma 3.1.6.

Let $v \in E$ be such that $f(v) > 0$, $f'(v) = 0$. If

$$f(v) < \frac{\beta-2}{2} \cdot \left(\frac{4\pi\beta}{a_2^2}\right)^{\beta/(\beta-2)} \cdot \left(\frac{1}{T}\right)^{2/(\beta-2)} \tag{3.1.12}$$

then v has minimal period T . We define $g(\beta) := \frac{\beta-2}{2} \left(\frac{4\pi\beta}{a_2^2}\right)^{\beta/(\beta-2)} \cdot \left(\frac{1}{T}\right)^{2/(\beta-2)}$.

To obtain N different solutions Girardi and Matzeu use direct method (the idea of pseudoindex). Let E be decomposed like in I§4. We choose H_k^- and H^- in the same way as in I§4. We can easily see that for $H(z) = |z|^\beta$ (due to Remark 1.4.1.), Propositions : 1.4.1, 1.4.2 and 1.4.3 hold . If F is a functional related to $H(z) = |z|^\beta$ and c_k^F are defined like in (1.4.6) then we have

$$0 < c_1^F \leq c_2^F \leq \ \ldots \leq c_N^F \leq c_{min}^F = h(\beta) := \pi^{\beta/(\beta-2)} \left(\frac{2}{\beta T}\right)^{2/(\beta-2)} \cdot \left(\frac{\beta-2}{\beta}\right)$$

Moreover for critical points related to f we have $c_i \leq c_i^F$, so

$$c_{min} \leq c_k \leq c_k^F = h(\beta), \qquad k = 1 ,\ldots,N . \tag{3.1.13}$$

Now let $z_i \in E$, $i = 1,\ldots,N$ be such that

(3.1.14) $\qquad f(z_i) = c_i, \quad f'(z_i) = 0 \quad ,$

where c_i is defined as in (1.4.6). If $c_i = c_j$, for some $i,j \in \mathbb{N}$, $i \neq j$ then there exist infinitely many points satisfying (3.1.14).

We must only to show that each such z_i from (3.1.14) has as minimal period T.

By contrary, if not, then one would have , by (3.1.12)

(3.1.15) $$c_i \geqslant h(\beta) \cdot \left(\frac{\beta}{a_2}\right)^{2\beta/(\beta-2)} \cdot 2^{\beta/(\beta-2)}$$

but taking into account (3.1.13) the constant a_2 should satisfy $a_2 > \sqrt{2}\ \beta$, which contradicts H6) .

■

§ 2. Some improvements of results given by Girardi and Matzeu.

Combining the results obtained by Girardi and Matzeu i.e. the Theorems 3.1.1 and 3.1.3. it is possible to state the following

Theorem 3.2.1.

Let $\xi \in \mathbb{R}^{2N}$, $\xi = (\xi_1, \dots, \xi_N, \xi_{N+1}, \dots, \xi_{2N})$, $\omega_i \in \mathbb{R}$, $i = 1,\dots,N$, are such that $0 < \omega_1 \leq \dots \leq \omega_N$.

K1) $\quad H \in C^1(\mathbb{R}^{2N}, \mathbb{R})$

K2) $\quad \beta H(\xi) \leqslant H'(\xi)\cdot\xi \qquad \forall \xi \in \mathbb{R}^{2N}, \ \beta > 2$

K3) $\quad H(\xi) \geqslant a_1 \left(\sum_{i=1}^{N} \frac{\omega_i}{2} \left(\xi_i^2 + \xi_{i+N}^2 \right) \right)^{\beta/2}$

K4) $\quad |H'(\xi)| \leqslant a_2 \left(\sum_{i=1}^{N} \frac{\omega_i}{2} \left(\xi_i^2 + \xi_{i+N}^2 \right) \right)^{(\beta-1)/2}$.

K5) $\quad a_2 < \sqrt{\omega_N} \cdot \beta$

If K1) - K5) is verified then $\forall T > 0$ (H-S) has at least one solution having T as minimal period.

Remark 3.2.1.

From K2) and K4) follows that

K6) $$H(\xi) \leq \frac{a_2}{\beta} \cdot \sqrt{\frac{2}{\omega_1}} \left(\sum_{i=1}^{N} \frac{\omega_i}{2} (\xi_i^2 + \xi_{i+N}^2) \right)^{\beta/2}$$

In fact $H(\xi) \leq \frac{1}{\beta} H'(\xi)\cdot\xi \leq \frac{1}{\beta} |H'(\xi)|\ |\xi| \leq \frac{a_2}{\beta} \left(\sum_{i=1}^{N} \frac{\omega_i}{2} (\xi_i^2 + \xi_{i+N}^2) \right)^{(\beta-1)/2} \cdot |\xi|$

$$\leq \frac{a_2}{\beta} \sqrt{\frac{2}{\omega_1}} \left(\sum_{i=1}^{N} \frac{\omega_i}{2} (\xi_i^2 + \xi_{i+N}^2) \right)^{\beta/2} .$$

Remark 3.2.2.

The assumptions K1) - K5) are more general than these given in Theorems 3.1.1. and 3.1.3. Notice that for $\omega_i = 2$ for every $i = 1,\dots, N$ they reduce to the assumptions H1) -H5).

Remark 3.2.3.

As it was ealier mentioned we can put $a_1 = 1$ (see Remark 3.1.2)

Proof is given in two steps, due to Proposition 1.4.3 which guarantees the existence of a strictly positive, minimal critical value of f, since 0 is an isolated critical value of f (this we are going to prove).

Let $E = H^{1/2}(S_T, \mathbb{R}^{2N})$; critical points of the functional f are T-periodic solutions of (H-S), where $f(v) = \frac{1}{2}\langle v,Kv\rangle - \int_0^T H(v)$, K is defined as L in (1.4.1). We are going to show:

Lemma 3.2.1.

0 is an isolated critical value of f .

Step 1. Let v be such that $f(v) > 0$ and $f'(v) = 0$, if

(3.2.5) $$f(v) < \frac{\beta-2}{2} \left(\frac{4\pi\beta}{a_2^2}\right)^{\beta/(\beta-2)} \cdot \left(\frac{1}{T}\right)^{2/(\beta-2)} =: g(\beta)$$

then v has T as minimal period (compare with Lemma 3.1.6).

Step 2. A minimum positive critical value satisfies:

(3.2.6) $$c_{min} < \frac{\beta-2}{2} \cdot \left(\frac{4\pi}{\omega_N \beta}\right)^{\beta/(\beta-2)} \cdot \left(\frac{1}{T}\right)^{2/(\beta-2)}$$

(compare with Lemma 3.1.7)

Proof of Lemma 3.2.1. Let $z(t)$ be a solution, $h_z := H(z(t))$; from the definition of $f(z)$ one has

(3.2.7) $$f(z) \geq \frac{\beta-2}{2} h_z T .$$

If 0 is not an isolated critical value, then there exists a sequence $\{z_n\}_{n\in\mathbb{N}} \in E \setminus \{0\}$ such that : $J\dot{z}_n = H'(z_n)$ and $f(z_n) \longrightarrow 0$. Let $z_n = (\varphi_1^n, \dots \dots, \varphi_{2N}^n)$, $M_n := \max_{t\in[0,T]} |(\sum \frac{\omega_i}{2}((\varphi_i^n)^2 + (\varphi_{i+N}^n)^2))^{1/2}|^{\beta-2}$, hence we have

(3.2.8) $$M_n \leq (h_n)^{(\beta-2)/\beta} \leq \left(\frac{2f(z_n)}{(\beta-2)T}\right)^{(\beta-2)/\beta} ,$$

where $h_n = H(z_n(t))$. Now we estimate M_n using Hölder and Wirtinger inequalities (Lemma 3.1.1.) and K2), K3), K4). Let $v = \hat{v} + c$, where v is a solution and $\hat{v}$ is such that $\int_0^T \hat{v}dt = 0$, $c \in \mathbb{R}^{2N}$.

$$\beta \int_0^T H(v) \leq \int_0^T J\dot{v}\cdot v \leq \frac{T}{2\pi} \|\dot{v}\|_{L^2}^2 = \frac{T}{2\pi}\|J\dot{v}\|_{L^2}^2 = \frac{T}{2\pi}\|H'(v)\|_{L^2}^2 \leq$$
$$\leq \frac{T}{2\pi} a_2^2 \int_0^T (\sum_{i=1}^N \frac{\omega_i}{2}(\varphi_i^2 + \varphi_{i+N}^2))^{\beta-1} \leq \frac{T}{2\pi} a_2^2 M \int_0^T (\sum_{i=1}^N \frac{\omega_i}{2}(\varphi_i^2 + \varphi_{i+N}^2))^{\beta/2} \leq$$
$$\leq \frac{T}{2\pi} a_2^2 M \int_0^T H(v) \quad , \text{ where}$$

(3.2.9) $$M := \max_{t\in[0,T]} |(\sum_{i=1}^N \frac{\omega_i}{2}(\varphi_i^2 + \varphi_{i+N}^2))^{1/2}|^{\beta-2} .$$

Hence

(3.2.10) $$M \geq \frac{2\pi\beta}{T a_2^2}$$

which is always strictly positive. The same estimate as (3.2.10) holds for every M_n, so M_n doesn't go to zero, that means also z_n doesn't converge to zero. ∎

Proof of the step 1. If v has not T as minimal period then $\exists\, m \in \mathbb{Z}$, $m \geq 2$

such that T/m is a minimal period. In that case due to the (W-I) (Lemma 3.1.1)

(3.2.11) $$\|\hat{v}\|_{L^2} \leqslant \frac{T}{4\pi} \|\dot{v}\|_{L^2} .$$

Using the same notation as in the proof of Lemma 3.2.1, moreover: K2), (3.2.11), definition of M (3.2.9), inequality (3.2.7) we are able to write the following inequalities:

$$\beta \int_0^T H(v) \leqslant \int_0^T v \cdot H'(v) = \int_0^T v \cdot J\dot{v} = \int_0^T \hat{v} \cdot J\dot{\hat{v}} \leqslant \frac{T}{4\pi} \|J\dot{v}\|^2_{L^2} = \frac{T}{4\pi} \|H'(v)\|^2_{L^2} \leqslant$$

$$\leqslant \frac{T}{4\pi} a_2^2 \int_0^T \left| \sum_{i=1}^N \frac{\omega_i}{2} (\varphi_{\xi_i}^2 + \varphi_{\xi_{i+N}}^2) \right|^{\beta-1} \leqslant \frac{T}{4\pi} a_2^2 M \int_0^T \left| \sum_{i=1}^N \frac{\omega_i}{2} (\varphi_{\xi_i}^2 + \varphi_{\xi_{i+N}}^2) \right|^{\beta/2} \leqslant$$

$$\leqslant \frac{T}{4\pi} a_2^2 M \int_0^T H(v),$$

hence

(3.2.12) $$M \geqslant \frac{4\pi\beta}{a_2^2 T} .$$

Since

$$H(v) \geqslant M^{\beta/(\beta-2)} \geqslant \left(\frac{4\pi\beta}{a_2^2 T} \right)^{\beta/(\beta-2)} \quad \text{one has}$$

$$f(v) \geqslant \frac{\beta-2}{2} \cdot \left(\frac{4\pi\beta}{a_2^2} \right)^{\beta/(\beta-2)} \cdot \left(\frac{1}{T} \right)^{2/(\beta-2)}$$

∎

Proof of the step 2. Let us introduce the following notations:

$$H_0(\varphi) = \left(\sum_{i=1}^N \frac{\omega_i}{2} \left(\varphi_{\xi_i}^2 + \varphi_{\xi_{i+N}}^2 \right) \right)^{\beta/2}$$

$$f_0(v) = \tfrac{1}{2} \langle v, Lv \rangle - \int_0^T H_0(v)$$

$$c^0_{min} = \inf \left\{ c \in R^+ : \ f_0(z) = c \ \text{ and } \ f_0'(z) = 0, \ \ z \in E \setminus \{0\} \right\}$$

$$c^0_k = \inf_{i^*(X) \geqslant k} \ \sup_{z \in X} \ f_0(z)$$

Since, by K3) one has

(3.2.13) $$f_0(v) \geqslant f(v) \quad \forall v \in E$$

(3.2.14) $$c^0_k \geqslant c_k$$

Now we are going to give an estimate for c^0_{min} from above.

On the complex plane (H-S), with $H = H_o$, can be written as

$$\dot{\xi} = i\, H_o'(\xi) \quad , \text{ where } \xi \in C^N \simeq R^{2N}$$

Since $H_o'(\xi) = (\ldots, \omega_j \cdot \beta/2 \, (\sum_{i=1}^{N} \frac{\omega_i}{2}(\xi_i^2 + \xi_{i+N}^2))^{\beta/2-1} \xi_j, \ldots)$, we have

(3.2.15) $$\dot{\xi}_j = i\,\omega_j \cdot \beta/2 \, (H_o(\xi))^{(\beta-2)/\beta} \xi_j \, .$$

If ξ is a solution then $H_o(\xi(t)) = \text{const} = h_o$. Solving (3.2.15) we have

$$\xi_j = b_j \exp(i\,\omega_j \beta/2 \; h_o^{(\beta-2)/\beta} t) \quad , \text{ where } b_j \text{ and } h_o \text{ are constants },$$

which will be found in the following way:

i) as ξ_j is T periodic, then holds

$$\omega_j \cdot \beta/2 \cdot h_o^{(\beta-2)/\beta} = \frac{2\pi}{T} \quad , \text{ hence}$$

$$h_o = \left(\frac{4\pi}{\omega_j \beta T}\right)^{\beta/(\beta-2)}$$

ii) from β-homogenity of H_o, we have for $\xi_j = b_j \tilde{\xi}_j$

$$h_o = H_o(b_j \tilde{\xi}_j) = b_j^\beta H_o(\tilde{\xi}_j) = b_j^\beta \left(\frac{\omega_j}{2}\right)^{\beta/2} \quad \text{hence}$$

$$b_j = h_o^{1/\beta} \cdot \left(\frac{2}{\omega_j}\right)^{1/2}$$

Using these results (i), ii)) we can finaly write:

(3.2.16) $$\xi_j = h_o^{1/\beta} \left(\frac{2}{\omega_j}\right)^{1/2} \exp\,(i(2\pi/T)\;t)$$

<u>Claim.</u> $f_o(\xi_N) < g(\beta)$, where $g(\beta)$ is given by (3.2.5). Due to the previous step this claim finishes the proof. We have

$$f_o(\xi_j) = -\frac{i}{2}\int_0^T \xi_j \cdot \dot{\xi}_j - \int_0^T H_o(\xi_j), \quad \text{from (3.2.16) we obtain}$$

(3.2.17) $$f_o(\xi_j) = -\frac{i}{2}\int_0^T h_o^{1/\beta}\left(\frac{2}{\omega_j}\right)^{1/2} \cdot i \cdot \frac{2\pi}{T} \cdot \exp(-i\tfrac{2\pi}{T}t) \cdot h_o^{1/\beta}\left(\frac{2}{\omega_j}\right)^{1/2} \cdot \exp(i\tfrac{2\pi}{T}t) - T h_o =$$

$$= \frac{1}{2}\int_0^T h_o^{2/\beta} \cdot \frac{2}{\omega_j} \cdot \frac{2\pi}{T} - T h_o = h_o^{2/\beta} \cdot \frac{2\pi}{\omega_j} - T h_o =$$

$$= \left(\frac{1}{T}\right)^{2/(\beta-2)} \left(\frac{4\pi}{\omega_j \beta}\right)^{\beta/(\beta-2)} \left(\frac{\omega_j \beta}{4\pi} \cdot \frac{2\pi}{\omega_j} - 1\right) = \left(\frac{1}{T}\right)^{2/(\beta-2)} \left(\frac{4\pi}{\omega_j \beta}\right)^{\beta/(\beta-2)} \cdot \frac{\beta-2}{2} \; .$$

Obviously we have $f_o(\xi_j) \geq c^o_{min}$.

Let now $j = N$ (notice that ω_N is the hightest frequency) , then

$$f_o(\xi_N) = \left(\frac{1}{T}\right)^{2/(\beta-2)} \cdot \frac{\beta-2}{2} \cdot \left(\frac{4\pi}{\omega_N \beta}\right)^{\beta/(\beta-2)}$$

Since $a_2 < \sqrt{\omega_N} \cdot \beta$ we have $g(\beta) > \left(\frac{1}{T}\right)^{2/(\beta-2)} \cdot \frac{\beta-2}{2}\left(\frac{4\pi}{\omega_N \beta}\right)^{\frac{\beta}{\beta-2}}$ that means

$$f_o(\xi_N) < g(\beta).$$

■

Remark 3.2.4.

Thanks to inequalities $0 < \omega_1 \leq \ldots\ldots \leq \omega_N$ one has from (3.2.17)

$f_o(\xi_N) \leq f_o(\xi_{N-1}) \leq \ldots\ldots \leq f_o(\xi_1)$. To obtain $f_o(\xi_i) < g(\beta)$ $\forall i = 1,\ldots,N$ it is enough to change the condition K5) to the more restrictive:

K5') $\quad a_2 < \sqrt{\omega_1} \cdot \beta$

Now let us state the similar result to this one given by Girardi and Matzeu (here Theorem 3.1.3.)

Theorem 3.2.2.

Let all assumptions of Theorem 3.2.1. be verified but K5), and let K5') holds then $\forall T > 0$ (H-S) has at least N distinct solutions having T as minimal period.

In the proof we use the results obtained in the proof of previous theorem. We follow also the proof given by Girardi and Matzeu.

Proof. Let $E = E^+ \oplus E^- \oplus E^o$, where E^+, E^-, E^o are subspaces of E:

$$(3.2.18)\quad \begin{cases} E^+ = \mathrm{span}\{(\sin jt)e_k - (\cos jt)e_{k+N};\ (\cos jt)e_k + (\sin jt)e_{k+N}:\ j \in \mathbb{N},\ 1 \leq k \leq N\} \\ E^- = \mathrm{span}\{(\sin jt)e_k + (\cos jt)e_{k+N};\ (\cos jt)e_k - (\sin jt)e_{k+N}:\ j \in \mathbb{N},\ 1 \leq k \leq N\} \\ E^o = \ker L \simeq \mathbb{R}^{2N} \end{cases}$$

Let L be an operator defined in I.§4.

We want to show the existence of N distinct T-periodic solutions for a functional f_o which is related to Hamiltonian $H_o = (\sum_{i=1}^{N} \frac{\omega_i}{2}(\varphi_i^2 + \xi_{i+N}^2))^{\beta/2}$. As one can easily see such f_o satisfies conditions f1) - f3) from Lemma 1.4.1. Hence Proposition 1.4.1 holds. Since we can find a subspace $H_{\bar{k}}$ of E^+ :

(3.2.19) $$H_{\bar{k}} = H_N = \{ v_N(t) = (\varphi\cos(2\pi/T) + \eta\sin(2\pi/T);\ \xi\sin(2\pi/T) - \eta\cos(2\pi/T)) : (\varphi,\eta) \in \mathbb{R}^N \times \mathbb{R}^N \}$$

and due to Theorem 1.4.1 we have that $i^*(H^-) = N$, where $H^- = E^- \oplus E^o \oplus H_N$, so also the thesis of Proposition 1.4.1 holds.

We can also find such $\varrho > 0$ that $f_o(v) \geqslant c_o > 0$ for $\|v\| = \varrho$ and $v \in E^+$. In fact from definition of E^+ we have

$$f_o(v) \geqslant \tfrac{1}{2}\|v\|^2 - \int_0^T H_o(v) = \tfrac{1}{2}\|v\|^2 - \int_0^T (\sum_{i=1}^{N} \frac{\omega_i}{2} (\varphi_i^2 + \xi_{i+N}^2))^{\beta/2} \geqslant$$

$$\geqslant \tfrac{1}{2}\|v\|^2 - \left(\frac{\omega_N}{2}\right)^{\beta/2} \int_0^T |v|^\beta \geqslant \frac{1}{2}\|v\|^2 - \left(\frac{\omega_N}{2}\right)^{\beta/2} \cdot c_\beta \|v\|^\beta$$

since $H^{1/2}$ is continuously embedded into L^β, there exists $c_\beta > 0$ such that the last inequality holds. For $\|v\|$ sufficiently small the right-hand side is bigger than zero, so there exists $\varrho > 0$ and $c_o > 0$ such that

(3.2.20) $$f_o(v) \geqslant c_o > 0.$$

Hence thanks to Proposition 1.4.3 $c_k^o > 0$ for every $k \in \mathbb{N}$.

Moreover f_o is bounded from above on H^-. In fact, let $v \in H^-$, $v = v_N + v^- + v^o$, where $v^- \in E^-$, $v^o \in E^o$, v_N is like in (3.2.19) ;then one has

$$\tfrac{1}{2}\langle v, Lv\rangle \leqslant \tfrac{1}{2}\langle v_N, Lv_N\rangle = \frac{1}{2}\int_0^T v_N J\dot{v}_N = \frac{1}{2}\cdot\frac{2\pi}{T}\int_0^T |v_N|^2 \leqslant \frac{\pi}{T}\int_0^T |v|^2$$

Now from Hölder inequality

$$\tfrac{1}{2}\langle v, Lv\rangle \leqslant \frac{\pi}{T}\cdot T^{(\beta-2)/\beta}\left(\int_0^T |v|^\beta\right)^{2/\beta} \leqslant \frac{\pi}{T}\, T^{(\beta-2)/\beta}\cdot\frac{2}{\omega_1}\cdot\left(\int_0^T H_o(v)\right)^{2/\beta}$$

Hence

$$f_0(v) = \tfrac{1}{2}\langle v, Lv\rangle - \int_0^T H_0(v) \leq \frac{\pi}{T}\, T^{(\beta-2)/\beta} \cdot \frac{2}{\omega_1} \cdot \left(\int_0^T \Big(\sum_{i=1}^N \frac{\omega_i}{2}\,(\varphi_i^2 + \varphi_{i+N}^2)\Big)^{\beta/2}\right)^{2/\beta} - \tag{3.2.21}$$

$$- \int_0^T \Big(\sum_{i=1}^N \frac{\omega_i}{2}\,(\varphi_i^2 + \varphi_{i+N}^2)\Big)^{\beta/2} \leq \frac{\beta-2}{2}\cdot\left(\frac{1}{T}\right)^{2/(\beta-2)} \cdot \left(\frac{4\pi}{\omega_1 \beta}\right)^{\beta/(\beta-2)} = h(\beta)$$

The last inequality follows by noting that the maximum value on R^+ of the function $b: \mathbb{R} \longrightarrow R$ defined as $b(x) = \frac{2\pi}{\omega_1 T}\cdot T^{\frac{\beta-2}{\beta}} x^2 - x^\beta$ coincides with the right-hand side of (3.2.21). One obtains the maximum of $b(x)$ for $x = \left(\frac{4\pi}{\omega_1 T\beta}\right)^{1/(\beta-2)} \cdot T^{1/\beta}$. This fact gives us that for every $k \leq \bar{k}$, c_k is finite (see Proposition 1.4.2).

We have that $c_i < c_i^\circ$ $\forall i$, where c_i is a critical value related to the functional f (c_i as in (1.4.6.)), in particular $c_N \leq c_N^\circ$. Moreover $\sup\{f_0(v) : v \in H^-\} = h(\beta)$ (see (3.2.21.)). Due to Proposition 1.4.1 (notice that $i^*(H^-) = N$) we have at least N distinct solutions; the minimality period T we obtain following exactly the calculations which are made in the proof of Theo - rem 3.1.3.

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IV. PERIODIC SOLUTIONS OF HAMILTONIAN SYSTEMS ON PRESCRIBED ENERGY LEVEL

§1. Results.

The first existence result was established by Weinstein [26] (here Theorem 0.2). then Rabinowitz [18] weakened the convexity assumption on Ω, taking Ω star-shaped with respect to some point $x_0 \in \mathbb{R}^{2N}$ (see Theorem 0.3). The next problem concerns the number of periodic orbits on a given energy level. In this direction the first result was obtained by Ekeland and Lasry [17] , here Theorem 0.4, the next by Ambrosetti and Mancini [6] i.e.

Theorem 4.1.1.

Let Ω be a compact, strictly convex subset of $\mathbb{R}^{2N}$, with interior $\mathring{\Omega}$ containing 0 and C^2 boundary Σ. Assume further there exist $r, R \in \mathbb{R}^+$ and $k \in \mathbb{Z}$, $2 \leq k \leq N$ with $R < \sqrt{k}\cdot r$, such that $B_r \subset \mathring{\Omega} \subset B_R$. Then there exist at least $[n/(k-1)]$ distinct periodic Hamiltonian trajectories on Σ ($[a] = \min\{k \in \mathbb{Z}: a \leq k\}$)

The problem,which appeared next, was following:"Is it possible to obtain a result similar to Theorem 4.1.1 in the case of Σ star-shaped?" The answer is positive and given recently by Girardi and Matzeu [19] and by Berestycki, Lasry, Mancini and Ruf [12] . The last result is also a generalization of Theorem 4.1.1 in the sense that Σ is not only star-shaped but also "connected between" two ellipsoids. To state this theorem let us introduce the following notations: let $\mathcal{K}$ be a matrix defined by

$$\mathcal{K} = \begin{bmatrix} \omega_1 & & & & & 0 \\ & \ddots & & & & \\ & & \omega_N & & & \\ & & & \omega_1 & & \\ & & & & \ddots & \\ 0 & & & & & \omega_N \end{bmatrix}$$

where $\omega_i \in \mathbb{R}^+$, $i = 1,\dots,N$. The set $\mathcal{E} = \{ \frac{1}{2} \mathcal{X}x \cdot x = \sum_{i=1}^{N} \frac{\omega_i}{2}(x_i^2 + x_{i+N}^2) \leq 1 \}$ defines an ellipsoid in $\mathbb{R}^{2N}$.

(4.1.1.) Let $H \in C^2(\mathbb{R}^{2N}, \mathbb{R})$, $H'(x) \neq 0$ for all $x \in \Sigma$ and let Σ be a C^2 - manifold strictly star-shaped with respect to the origin and bounds the set $\Omega = \{ x \in \mathbb{R}^{2N} : H(x) \leq 1 \}$, which is compact.

(4.1.2) Let $\alpha \mathcal{E} \subset \Omega \subset \beta \mathcal{E}$ for some $0 < \alpha < \beta$.

From these conditions follows the existence of $\varrho > 0$ which is the largest positive real such that

(4.1.3) $$T_x \Sigma \cap B^o_\varrho = \emptyset \qquad \forall x \in \Sigma$$

where B^o_ϱ is the interior of a ball.

<u>Theorem 4.1.2.</u> 12

Given $\mathcal{E}$, there exists a constant $\delta = \delta(\varrho^2/\alpha^2, \omega_1, \dots, \omega_N) > 0$ such that (H-S) possesses at least N distinct periodic orbits on any surface Σ satisfying (4.1.1) - (4.1.3) with $\beta^2/\alpha^2 < 1 + \delta$.

Before presenting the result given by Girardi and Matzeu in [19] there are necessary some preliminaries.

Let Σ be a C^2 - regular manifold of $\mathbb{R}^{2N}$, strictly star-shaped, $0 \in \Omega$, $\Sigma = \partial\Omega$.
Let $R := \max\{ |z| : z \in \Sigma \}$, $r := \min\{ |z| : z \in \Sigma \}$, $d := \min\{ d(z) : z \in \Sigma \}$, where $d(z) = z \cdot n(z)$ is the distance between the tangent hyperplanes to Σ at the point z and the origin of $\mathbb{R}^{2N}$. Let $a(z)$ be a "gauge" function related to Σ, it is known that $a \in C^2(\mathbb{R}^{2N} \setminus \{0\}, \mathbb{R}) \cap C^1(\mathbb{R}^{2N}, \mathbb{R})$ and $\Sigma = \{ z \in \mathbb{R}^{2N} : a(z) = 1 \}$. Moreover one has:

Σ1) $$\frac{1}{R}|z| \leqslant a(z) \leqslant \frac{1}{r}|z| \qquad \forall z \in \mathbb{R}^{2N}$$

Σ2) $$z \cdot a'(z) = a(z) \qquad \forall z \in \mathbb{R}^{2N}$$

$\Sigma 3)$ $\quad \max \{ |a'(z)| : z \in \Sigma \} = \frac{1}{d}$

If the Hamiltonian function is defined as $\bar{H}(z) = (a(z))^{\beta}$, $\forall \beta > 2$, then the following conditions are verified:

$\Sigma' 0)$ $\quad \Sigma = \{ z \in \mathbb{R}^{2N} : \bar{H}(z) = 1 \}$

$\Sigma' 1)$ $\quad \frac{1}{R^{\beta}} |z|^{\beta} \leq \bar{H}(z) \leq \frac{1}{r^{\beta}} |z|^{\beta} \qquad \forall z \in \mathbb{R}^{2N}$

$\Sigma' 2)$ $\quad \beta H(z) = z \cdot H'(z) \qquad \forall z \in \mathbb{R}^{2N}$

$\Sigma' 3)$ $\quad |H'(z)| \leq \frac{\beta}{d\, r^{\beta-1}} |z|^{\beta-1}$

Now it is possible to state

Theorem 4.1.3. [19]

Let Σ be as above and let

(4.1.4) $\quad R^2 < \sqrt{2}\, rd$

Then Σ carries at least N distinct periodic Hamiltonian trajectories.

§2. Proof of Theorem 4.1.3.

From Theorem 3.1.3 follows

Corollary 4.2.1.

Let $\bar{H}$ be defined as in §1 and let

(4.2.1) $\quad R^{\beta} < \sqrt{2}\, d\, r^{\beta-1}$.

Then $\forall T > 0$ there exists at least N distinct periodic solutions of (H-S) with $H := \bar{H}$ having minimal period T .

Proof. It is easy to see that all the assumptions of Theorem 3.1.3.are satisfied with $a_1 = \frac{1}{R^{\beta}}$ and $a_2 = \frac{\beta}{d\, r^{\beta-1}}$ ∎

Proof of Theorem 4.1.4. From (4.1.4) it follows that for $\varepsilon > 0$ sufficiently small one has

$$\left(\frac{R^2}{r}\right)^{1+\varepsilon} < \sqrt{2}\, d$$

Hence, since one can always suppose (maybe after rescaling) that $r > 1$, one has

$$R^{2+2\varepsilon} < \sqrt{2}\, d\, r^{1+2\varepsilon}$$

Putting now $\beta = 2 + 2\varepsilon$ the thesis follows from Corollary 4.2.1 and Theorem 3.1.3.

∎

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