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QUASI-PARTICLES AT FINITE TEMPERATURE

Candidate:

Noureddine Chair

Supervisor:

Prof. Walter Thirring

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TRIESTE

SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

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" L'idée que les mathématiques
pouvaient en quelque sorte s'
adapter à des objets de notre
expérience me semblait remar-
quable et passionnante. "

(W. Heisenberg)

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PREFACE

In this review, we will be studying the behaviour of quasi-particles under the so-called Kubo-Martin-Schwinger (KMS) structure. The consequences of this condition can be summarized as follows:

(i) Assuming an infinite lifetime for quasi-particles, then their S-matrix is trivial: no interactions between the quasi-particles;

(ii) If the correlation functions decay sufficiently, then quasi-particles can be created by applying quasi-free field operators on the equilibrium state.

Due to the technicalities involved throughout this work, we attempted to render it self-consistent and tried to collect all the material needed for fulfilling this task. The outline of this review is as follows:
in Chapter 1, we shall be more concerned about the description of quasi-particles and the construction of the elementary excitations corresponding to quasi-particle states. In Chapter 2, we will give an account on the mathematical techniques needed for this review, including a detailed description of the KMS condition. Finally, in Chapter 3, we shall be concerned about the construction of creation and annihilation operators for quasi-paricles and collective excitations and a detailed proof of the results mentionned above.

CHAPTER 1

THE PHYSICAL PROBLEM

The Kubo-Martin-Schwinger (KMS) condition [1, 2, 3] is a convenient way of studying infinite quantum systems at equilibrium, unlike the grand canonical distribution which only makes sense for a finite system [4].

One of the advantages of the infinite system approach is that the surface effects have never to be considered; we deal purely with the properties of the bulk material. This approach however requires a more elaborated mathematical treatment than the finite system approach. A suitable mathematical tool for describing infinite systems is the so-called C^*-algebra, whose elements are operators representing local observables, so that averages make sense for those systems.

As this review is a very technical one and the concept of quasi-particles is discussed in most of it, we would like to give a clear picture of what quasi-particles are.

Let us start by considering a system which is contact with a heat reservoir. Actually, most of the physical systems are not isolated, but are immerged in a "thermal reservoir" [5].

Since the system communicates with the thermal reservoir, excitation and de-excitation processes are influenced by the exchange of energy between the system and the reservoir, whose presence maintains a certain number of excited quanta. Therefore, the absorption of energy by the system occurs in two ways: it is absorbed either by the excitation of additional quanta or by annihilation of holes of particles maintained by the reservoir.

The latter case corresponds to a quantum (hole) with negative energy, $-\hbar w$, and negative momentum, $-\hbar k$, being annihilated. This physical situation describes quasi-particle excitations of the system under consideration. The existence of quasi-particle spectrum was first proposed by Landau to explain the phenomenum of superfluidity of liquid Helium II.

Briefly, quasi-particles are elementary excitations which correspond to modified single-particle excitations. Examples of quasi-particles are [6]:

(i) Phonons – they correspond to excited states of a lattice vibration;

(ii) Cooper pairs – they appear in connection with the phenomenum of superconductivity. In the presence of an attractive interaction, the electrons in the neighborhood of the Fermi surface condensate into a state of lower energy, in which each electron is paired with one of opposite momentum and spin.

(iii) Magnons – in a ferromagnet, the low-lying excitations corresponding to oscillations in the electron-spin density fluctuations are known as spin-waves. The quantized spin-waves are called magnons, and they obey the Bose statistics.

(iv) Plasmons – they are elementary excitations of an interacting electron gas. In such a system, as a consequence of the Coulomb interaction, one can have a collective oscillation of the electron particle density, the plasma oscillation. In the limit of long wavelengths, the frequency of the oscillation is given by the plasma frequency

$$w_p = \left(\frac{4 \quad N e^2}{m} \right)^{\frac{1}{2}} ,$$

(iv) Plasmons (cont'd) -

N being the electron density.

The plasma frequency has no counterpart in the non-interacting electron system.

The quantum of the plasma oscillations is the plasmon.

In this review, we are interested in studying the consequences of the KMS structure on the behaviour of quasi-particles. The physical idea is that if we perturb a system which is at an equilibrium state, then such a perturbation will spread out to infinity just like a wave-packet in ordinary Quantum Mechanics. Mathematically, this is reflected by the energy-momentum spectrum of the system.

Generally speaking, the dispersion relation $E(k)$ for the elementary excitations will be different from the expression $E(k) = k^2 / 2m$ for the free particle [7]. It should be emphasized that the elementary excitations are the results of collective interactions of the particles of the system, and therefore belong to the system as a whole, but not to its individual particles.

In our review, we will make the following assumption: for a fixed k the energy spectrum is assumed to consist of a pure point spectrum, $E(k)$ plus a continuous background. The function $E(k)$ is not known a priori (+), but assumed to behave reasonably.

(+) For phonons, $E(k) \sim k$ as $k \longrightarrow 0$.

In RQFT however Poincaré invarinace implies that

$$E(k) = (k^2 + m^2)^{\frac{1}{2}} .$$

Due to the assumptions made about the energy-momentum spectrum, one finds that the correlation functions decay in the direction $x = vt$ as a power $t^{-3/2}$ in the limit t goes to infinity, just like the wave-packet of a non-relativistic particle. To understand better this statement, let us follow the reasoning below.

Let us work out the probability amplitude corresponding to the quasi-particle state $\psi \in \mathcal{H}$. The wave function corresponding to this state is given by $f(k)$, or equivalently, by the Fourier transform of $f(k)$ with respect to the measure $\mu(\underline{k}) = \delta (w - E(k))\, dk\, dw$:

$$f(x,t) = \int d^3K\, dw\, e^{i(\underline{K}\cdot\underline{x} - wt)} \{ f(\underline{K}) + \text{continuous background} \} .$$

Let us concentrate on the following integral

$$I = \left[\int d^3\underline{K}\, |f(K)|^2\, e^{i(\underline{K}\cdot\underline{x} - \varepsilon t)} + \text{continuous background} \right] .$$

If we assume that $f(k)$ is sufficiently smooth (by this we mean that $f(k)$ goes to zero faster than any power of k whenever k goes to infinity), then the dominant contribution to I arises, for t very large, from the region of integration where the phase of the oscillatory factor is stationary.

(k.x − Et = Δ (k) is stationary; the condition for it is $\frac{\partial \Delta(\underline{K})}{\partial K_x} \simeq 0$.
and suppose that this happens for $k = k_0$.)

Upon a Taylor expansion, I can be written as

$$I = |f(\underline{K}_0)|^2 \exp[(i\underline{K}_0 \cdot v - \varepsilon_{K_0})t] \cdot \int d^3K \exp(it \sum_{i,j} A_{ij} K_i K_j)$$

where

$$A_{ij} = \frac{1}{2} \frac{\partial^2 \Delta(K)}{\partial K_i \partial K_j}\bigg|_{K=K_0} = \frac{1}{2}\left(\frac{\delta_{ij}}{m} - \varepsilon''(K_0)\right)$$

In the literature this method is called the Method of Stationary Phase | 8 |.

By changing variable, the above integral can be brought into

$$I = \int d^3q \; e^{(i\varepsilon''(K_0)t)} \; e^{-iq^2 t}$$

so that the probability amplitude assumes the following form

$$\sim (\varepsilon''(K_0)t)^{-\frac{3}{2}} |f(K_0)| + \text{something decaying faster}$$

The next task is to construct the one-particle excitation corresponding to the quasi-particle state. Since the energy spectrum consists of a pure point spectrum plus a continuous background for each k (we follow here the arguments reported in ref. [9]).

As $|\Omega\rangle$ is cyclic, therefore one can always find an operator A belonging to the algebra of local operators ($\mathcal{A}$), such that

$$A^{*}|\Omega\rangle = |\psi\rangle + |\phi\rangle$$

where $|\psi\rangle$ is the quasi-particle state corresponding to $w = E(k)$ and $|\phi\rangle$ has an energy spectrum with no support in a neighborhood of $w = E(k)$.

Therefore to separate out this part, we translate A to get a new operator

$$A(x,t) = U(x,t) \; A \; U(x,t)^{-1}$$

and then smear it with a smooth function $g(x,t)$ (infinitely differentiable and vanishing outside a bounded set) ; to be precise, $g(x,t)$ has to be of the form of eq.( ) with the δ -contribution only. In this case,

$$\tilde{A}^{*} = \int dx \; U(x,t) \; A^{*} \; U(x,t)^{-1} \, g(x,t) \; ,$$

$$\tilde{A}^{*}|\Omega\rangle = |\psi\rangle \quad , \quad \tilde{A}|\Omega\rangle = 0.$$

To find the scattering matrix, it is required to define asymptotic states, which is done according to the expression below:

$$a_{\substack{in \\ out}}(f) = \lim_{t \longrightarrow \mp\infty} \int d^3x \;\; g(x,t) \, A(x,t) \quad .$$

When we apply $a_{in,\, out}(f)$ to the vector $|\Omega\rangle$ representing the equilibrium state, this operator will create one quasi-particle with a certain momentum distribution determined by f.

Now, the striking difference between a KMS Ω -state and the ground state at zero temperature appears:

$a_{in}(f_1)\, a_{in}(f_2)\, |0\rangle$ represents two quasi-particles with momentum distributions corresponding to f_1 and f_2 for $t \longrightarrow -\infty$, they differ from $a_{out}(f_1)\, a_{out}(f_2)\, |0\rangle$.

In contradiction, in the KMS representation $a_{out}(f) = a_{in}(f)$, in particular $a_{in}(f_1)\, a_{in}(f_2)\, |\Omega\rangle = a_{out}(f_1)\, a_{out}(f_2)\, |\Omega\rangle$ and asymptotically the same distributions reappears. That is, there is no collision between the quasi-particles,

$$S = 1.$$

This will be proved in Section 1 of Chapter 3.

By construction, the asymptotic fields $a_{in,\, out}(f)$ have a time dependence like free particles and to complete this picture we demonstrate the following properties:

(i) Their commutators or anti-commutators are c-numbers (the proof will be given in Section 3 of Chapter 3);

(ii)The truncated n-point functions taken with $|\Omega\rangle$ vanish for $n \geq 3$ (not for n = 2). This is due to the poor clustering properties of the correlation functions in the coordinate space of the non-relativistic many-body theory, which in turn is due to the long-lived collective

(ii) (cont'd)

excitations (Goldstone modes).

The state w has a clustering [10] with $\delta \geq 0$ if for every pair of observables A and B $| w(A \tau_x B) - w(A) w(\tau_x B) | = O(|x|^{-\delta})$ as $|x| \longrightarrow \infty$, where τ_x, $x \in R^{\nu}$, is the action of the translation group.

(iii) The algebra of fields generated by $a_{in,\ out}(f)$ is embedded in the whole algebra which is the bicommutant of the algebra, $\{\mathcal{A}''\}$. Mathematically, $\{\mathcal{A}''\}$ factors, i.e.

$$\{\mathcal{A}''\} = \mathcal{A}_0 \otimes \underline{\mathcal{B}}$$

where $\mathcal{A}_0$ is generated by $a_{in,\ out}(f)$ and (the commutant of $\mathcal{A}_0$); this will be shown in Section 5 of Chapter3.

The asymptotic fields $a_{in,\ out}(f)$ correspond to what is called in field theory the "dressed particles", and they were shown by Haag [11] to be <u>strong limits</u> of non-local polynomials in the operators referring to the bare particles.

Mathematically, this means that $a_{in,\ out}$ will not belong to $\mathcal{A}$ (the quasi-local algebra), but at least to $\mathcal{A}''$ (the weak closure of $\mathcal{A}$), example of this is the Hamiltonian of the system or the number of particles: they do not belong to $\mathcal{A}$, since they are global quantities.

Since at finite temperature ($T \neq 0$), unlike the ground state, a local perturbation may create a particle or annihilate a hole, therefore the energy of the infinite system at finite temperature is not bounded below. Speaking in physical terms, $a_{in,\ out}$ may create a particle or annihilate a hole.

There are operators a^{+} (or b^{+}) referring only to particles(or holes) which are constructed from $\mathcal{A}''$ and $\mathcal{A}'$,i.e. a linear combination of elements in $\mathcal{A}''$ and $\mathcal{A}$. The observables $a_{in,\ out}$ do not make distinction between creation of a particle and absorption of a hole. The Hamiltonian referring to a quasi-particle has the form

$$H = \int \frac{d^3k}{(2\pi)^3} E(k) \left[a^{+}(k)\, a^{+\,*}(k) - b^{+}(k)\, b^{+\,*}(k) \right]$$

with

$$a^{+\,*}(k)\ |\Omega\rangle = 0,$$

$$b^{+\,*}(k)\ |\Omega\rangle = 0.$$

Writting the Hamiltonian in this way, the infinite energy of the equilibrium state w is adjusted to zero.

The symmetry between particles and holes, $a^{+} \longrightarrow b^{+}$, $H \longrightarrow -H$ reflects the symmetry between $\{\mathcal{A}'\} \longleftrightarrow \{\mathcal{A}''\}$, which is the characteristic of the KMS representation (Ω is cyclic for both of them).

Finally, we come to end up this chapter by describing how the quasi-particles travel to infinity. To do this, we use the theory developped by Prigogine and his collaborators [12 , 13] (the Brussels group) on the time-monotonic operators and K-systems.

Their work can be summarized as follows:

It is possible to give a dynamical meaning to the second law of thermodynamics in quantum mechanics, in terms of the existence of the so-called Lyapounov variables,i.e. dynamical variables varying monotonically in time without becoming contradictory.

It turns out that the Lyapounov operator exists in quantum mechanics only if the system is infinite,and for a suitable Hamiltonian which is not bounded below (this is our case) .

Under this condition, there exists a self-adjoint operator T (internal time or age operator) conjugate to H on the orthogonal complement of $\mathcal{H}_{-\infty}$,i.e. $\mathcal{H}_{-\infty}^{\perp}$ where $\mathcal{H}_{-\infty}$ is the one-dimensional space spanned by constant functions, i.e.

$$[T , H] = iP_{\perp} ,$$

where

$$P_{\perp} = 1 - |\Omega\rangle\langle\Omega| ,$$

with spectrum of H $= [- \infty , + \infty]$.

Further, T satisfies

$$e^{iH\alpha} e^{iwT} e^{-iH\alpha} = e^{iw (T + \alpha)} \quad \text{(Weyl relation)},$$

i.e. the group generated by one shifts the other.

The Hilbert space $\mathcal{H}$ can be written in the spectral decomposition as

$$\mathcal{H} = \int_{-\infty}^{\infty} d\lambda \, \mathcal{H}_\lambda \quad , \quad \mathcal{H}_\lambda \cong L^2(R)$$

With some identification of these spaces, $\mathcal{H}$ can be written as $L^2(R) \times L^2(R)$ such that $H\psi(\lambda,\mu) = \lambda\,\psi(\lambda,\mu)$.

By using the Weyl relation, we have

$$e^{Tw}\,\psi(\lambda,\mu) = \psi(\lambda + w, \mu) .$$

The operator T is not however unique, since any $T' = T . 1 \otimes B$ with $B \in \mathcal{B}\,(L^2(R))$ (the set of bounded operators acting on $L^2(R)$) satisfies the same Weyl relation.

To exclude all sorts of pathological construction, we shall require that T commutes the number of quasi-particles.

Now, if we can write

$$\int d^3k = \int_0^{\infty} dE \int d\Omega$$

where Ω is an angular variable, then our Hamiltonian for the quasi-particles assumes the following form

$$H = \int_{-\infty}^{\infty} d\varepsilon \, \varepsilon \int d\Omega \, C^*(\varepsilon,\Omega)\, C(\varepsilon,\Omega)$$

where

$$C(\varepsilon,\Omega) = \begin{cases} a^{+*}(\varepsilon,\Omega) & \text{for} \quad \varepsilon > 0 \\ b^{+*}(\varepsilon,\Omega) & \text{for} \quad \varepsilon < 0 \end{cases}$$

Thus, the canonical transformation

$$e^{iw\mathcal{T}} C(\varepsilon,\Omega)\, e^{-iw\mathcal{T}} = C(\varepsilon - w,\Omega)$$

generates $e^{iw\mathcal{T}}\, H\, e^{-iw\mathcal{T}} = H + w N$, where

$$N = \int d\varepsilon \int d\Omega\, C^{*}(\varepsilon,\Omega)\, C(\varepsilon,\Omega)$$

is the number of quasi-particles and holes. The age operator we are looking for therefore is

$$T = \mathcal{T} / N .$$

Formally

$$\mathcal{T} = \int_{-\infty}^{\infty} d\varepsilon\, d\Omega\, C^{*}(\varepsilon,\Omega)\, i \frac{\partial}{\partial \varepsilon} C(\varepsilon,\Omega)$$

An example of Lyapounov operator is $D = (\underset{\sim}{x} \cdot \underset{\sim}{P})$ (free time evolution), with $\frac{1}{P}\, D\, \frac{1}{P}$ would formally be conjugate to $H = \frac{1}{2} P^2$.

Therefore, we see that $\mathcal{T}$ is a generalization of $\frac{1}{|P|}\, D\, \frac{1}{|P|}$ and its increase with time can be illustrated by the monotonicity of D.

It says that in the direction of P, x increases with time.

Similarly, T <u>increases</u> because quasi-particles (holes) diffuse to infinity in the direction of k ($-k$) if E has a suitable form. (If T exists, then one can define an entropy operator M to be monotonic operator function of T , $M = f(T)$).

CHAPTER 2

MATHEMATICAL PRELIMINARIES

1. Unbounded Operators.

Let $\mathcal{H}$ be a vector space with a scalar product $\langle \cdot , \cdot \rangle$ and normed by the Euclidean norm (i.e.)

$$\varphi \in \mathcal{H} \longrightarrow \|\varphi\|^2 = \langle \varphi , \varphi \rangle$$

The sequence $\{\varphi_n\}$ is a Cauchy sequence if

$$\|\varphi_n - \varphi_m\| \to 0 \qquad \text{when} \qquad n, m \to \infty$$

We say that $\mathcal{H}$ is complete if every Cauchy sequence is convergent in $\mathcal{H}$.

We say that $\varphi_n \underset{n\to\infty}{\longrightarrow} \varphi$ if $\|\varphi_n - \varphi\| \to 0$, $n \to \infty$. With the above properties, the vector space $\mathcal{H}$ is called a Hilbert space.

A linear operator A is a linear transformation form $\mathcal{H} \to \mathcal{H}$. Its domain $\mathfrak{D}$(A) is the set of vectors $\psi \in \mathcal{H}$ such that $A\psi \in \mathcal{H}$ $(\|A\psi\| < \infty)$

A set of vectors $\mathfrak{D} \subset \mathcal{H}$ is dense in $\mathcal{H}$ if for all $\psi \in \mathcal{H}$; ψ is a limit of a sequence of vectors in $\mathfrak{D}$. The range of A, R(A) is subspace of $\mathcal{H}$. If A is a transformation from $\mathcal{H}_1$ onto $\mathcal{H}_2$ then R(A) is a subspace of $\mathcal{H}_2$. $\mathfrak{D}$(A) is not necessary with $\mathcal{H}$, if $\mathfrak{D}$(A) is dense in $\mathcal{H}$, we say that A is densely defined. An operator A is bounded if there exists a constant C such that

$$\|A\psi\| \leq c\|\psi\| \qquad \forall \psi \in \mathfrak{D}(A)$$

We denote by $\mathcal{B}(\mathcal{H})$ the set of bounded operators defined in the whole of $\mathcal{H}$.

$\mathcal{B}(\mathcal{H})$ is a normed space, if we define the norm of $A \in \mathcal{B}(\mathcal{H})$ by:

$$\|A\| = \sup_{\|\psi\| \leq 1} \|A\psi\| = \sup_{\psi \neq 0} \frac{\|A\psi\|}{\|\psi\|}$$

The least upper bound exists by definition (the smallest value of k). If the operators are unbounded, care is needed. The sum operator A + B in this case is defined by

$$\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$$

$$(A+B)\psi = A\psi + B\psi$$

Similarly for the product A·B

$$\mathcal{D}(A \cdot B) = \{ \psi \in \mathcal{D}(B) \quad , B\psi \in \mathcal{D}(A) \}$$

$$(A \cdot B)\psi = A(B\psi)$$

We denote by $A \subset B$ if $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $A\psi = B\psi \quad \forall \psi \in \mathcal{D}(A)$

(B is the extension of A). An operator A is closed if:

$$\psi_n \in \mathcal{D}(A) \qquad \psi_n \xrightarrow[n \to \infty]{} \psi \qquad A\psi_n \xrightarrow[n \to \infty]{} \varphi$$

implies: $\psi \in \mathcal{D}(A)$ and $\lim_{n \to \infty} A\psi_n = \varphi = A\psi$

If the operator A is not closed, we can in some case extend to a closed operator we say that an operator A is closable if:

$$\forall \ \psi_n \in \mathcal{D}(A) \qquad \psi_n \xrightarrow[n \to \infty]{} 0 \qquad A\psi_n \xrightarrow[n \to \infty]{} \varphi$$

implies that $\varphi = 0$. In this case we can define the extension of A to a closed operator $\tilde{A}$ such that

$$\mathfrak{D}(\tilde{A}) = \{ \varphi \quad \text{such that there exists} \quad \varphi_n \in \mathfrak{D}(A), \quad \varphi_n \to \varphi \quad A\varphi_n \to \psi \}, \text{ then}$$

$$\tilde{A}\varphi = \lim_{n\to\infty} A\varphi_n$$

The adjoint operator A^* of an operator A is defined only when $\mathfrak{D}(A)$ is dense, in this case

$$\mathfrak{D}(A^*) = \{ \psi \quad \text{such that there exists} \quad \psi' \quad \text{with} \quad \langle \psi, A\varphi \rangle = \langle \psi', \varphi \rangle, \ \forall \varphi \in \mathfrak{D}(A) \}$$

In the case of unbounded operators, we have to differentiate between Hermitian, symmetric and self-adjoint operators. An operator is called hermitian if its domain $\mathfrak{D}(A)$ is not restricted and if

$$\langle \varphi, A\psi \rangle = \langle A\varphi, \psi \rangle \qquad \forall \varphi, \psi \in \mathfrak{D}(A)$$

If $\mathfrak{D}(A)$ is dense we say that A is symmetric and we have $A \subset A^*$, A is self-adjoint if $A = A^*$, (A^* is always a closed operator). We say that A is essentially self-adjoint, if it admits a self-adjoint closure.

Convergence of operators

Definition: Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{B}(\mathcal{H})$ and let $A \in \mathcal{B}(\mathcal{H})$

a) $\{A_n\}$ converges weakly to A, written $\underset{n\to\infty}{w\lim} A_n = A$

If $\{A_n\psi\}$ converges weakly to $A\psi$ for each $\psi \in \mathcal{H}$

(i.e) $\langle (A_n - A)\psi, \varphi \rangle \to \infty \quad \forall \varphi \in \mathcal{H}$

b) $\{A_n\}$ converges strongly to A, written $\underset{n\to\infty}{s\lim} A_n = A$

if $\| A_n\psi - A\psi \| \to 0 \quad \forall \psi \in \mathcal{H}$

c) $\{A_n\}$ converges uniformly (or in norm) to A written

$\underset{n\to\infty}{u\text{-}\lim} A_n = A$ if $\| A_n - A \| \to 0$

uniform convergence $\Rightarrow$ strong convergence $\Rightarrow$ weak convergence

Vector-valued function

$\varphi(z)$ is a vector-valued function if $\varphi(z)$ ia a vector defined for every $z \in \mathfrak{D}$ where $\mathfrak{D}$ is a domain of a complex plane.

$\varphi(z)$ is a holomorphic (or analytic) in $\mathfrak{D}$ if $\varphi(z)$ is strongly differentiable for every z, this equivalent to say if $\langle \varphi(z), \psi \rangle$ is holomorphic in z for every $\psi \in \mathcal{H}$.

Operator-valued functions

Let A(z) be a function of z defined in the domain of a complex plane and is a bounded operator valued; $A(z) \in \mathcal{B}(\mathcal{H})$ for every $z \in \mathcal{D}$

We say that A(z) is an analytic family in $\mathcal{D}$ if A(z) is differentiable in the operator norm for every $z \in \mathcal{D}$. This is equivalent to say that $\langle A(z), \varphi, \psi \rangle$ is holomorphic for every $\varphi, \psi \in \mathcal{H}$. If A(z) is a function defined on D with unbounded operator-valued, closed, and such that $\mathcal{D}(A(z)) = \mathcal{D}$ is independent of z, and $A(z)\varphi$ is holomorphic in $\mathcal{D}$ for every $\varphi \in \mathcal{D}$ then we say that this is a family of type A. We say that a family A(z) is a holomorphic self-adjoint it is holomorphic in a domain $\mathcal{D}$ symmetric w.r.t. the real axis and if $A(z)^* = A(\bar{z})$. Finally, we come to the definition of the resolvant of an operator and its spectrum.

The resolvant os an operator A, denoted by $\rho(A)$ is the set of z such that (A-z) is invertible and its inversive $(A-z)^{-1}$ is bounded over the whole $\mathcal{H}$ .The spectrum of A is complement of ρ denoted by $\sigma(A)$. The spectrum of a self-adjoint operator is the subset the real line. The important property of a self-adjoint operator is the spectral decompositeness.

If A is a self-adjoint operator, then there exists a familty of orthogonal projections, $E(\lambda)$ $\lambda \in \mathbb{R}$ such that

$$E(\lambda_1)E(\lambda_2) = E(\lambda_1) \qquad \text{if} \quad \lambda_1 \leq \lambda_2$$

$$\underset{\varepsilon \to 0}{s\ lim}\ E(\lambda+\varepsilon) = E(\lambda) \qquad \varepsilon > 0$$

$$\underset{\lambda \to -\infty}{s\ lim}\ E(\lambda) = 0 \quad , \quad \underset{\lambda \to +\infty}{s \cdot lim}\ E(\lambda) = \mathbb{1}$$

In this case $\psi \in \mathcal{D}(A)$ if $\int \lambda^2 d < E(\lambda)\psi, \psi > < \infty$

For $\psi \in \mathcal{D}(A)$ we have:

$$A\psi = \int_{-\infty}^{+\infty} \lambda\, dE(\lambda)\psi$$

λ is an eigenvalue is A $E(\lambda) - E(\lambda-0) \neq 0$ where

$$E(\lambda-0) = \underset{\varepsilon \to 0}{s \cdot lim}\ E(\lambda-\varepsilon)$$

The set of eigenvalues corresponds to the point spectrum of A, denoted by $\sigma_p(A)$.

If we define $\mathcal{H}_p$ to be the closure of the set of vector generated by $\rho(\lambda)\mathcal{H}$ then we can decompose A into a direct sum of two operators

$$A = A_p + A_c \qquad \text{with} \quad \mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_c$$

where $\mathcal{H}_c$ is $\mathcal{H}_p^{\perp}$ (orthogonal of $\mathcal{H}_p$).

A_c has no point spectum in $\mathcal{H}_c$. Finally, let S ne an interval in $\mathbb{R}$, we define E(s) as a projection by

$$]a,b] \to E(]a,b]) = E(b) - E(a)$$

For every $\psi \in \mathcal{H}$ we define the measure on $\mathbb{R}$ by $\mu_\psi(s) = < E(s)\psi, \psi >$ (with $E(s)^2 = E(s)$)

Let $\mu_\alpha(s)$ be a Lebesgue measure of S, we say that the vector ψ is absolutely continuous w.r.t A if

$$\mu_\alpha(s) = 0 \quad \Rightarrow \quad \mu_\psi(s) = 0 \quad \Rightarrow \quad E(s)\psi = 0$$

The set of the absolutely continuous vectors is denoted by $\mathcal{H}_{ac}$.

ψ is singular continuous w.r.t. A if $\langle E(\lambda)\psi, \psi\rangle$ is continuous but it is singular w.r.t. Lebesgue measure (i.e) $\exists\, s_0$ such that $\mu_\alpha(s_0) = 0$ and $E(s_0)\psi = \psi$. The set of singular continuous vectors is denoted by $\mathcal{H}_{sc}$. The subspaces $\mathcal{H}_{ac}$, $\mathcal{H}_{sc}$ and $\mathcal{H}_p$ are closed and mutually orthogonal

$$\mathcal{H} = \mathcal{H}_{ac} \oplus \mathcal{H}_{sc} \oplus \mathcal{H}_p$$

$A = A_{ac} \oplus A_{sc} \oplus A_p$ the corresponding spectra are denoted σ_{ac}, σ_{sc} and σ_p respectively. The discrete spectrum of A is the set of isolated eigenvalues with a finite multiplicity.

The complement of the discrete spectrum in $\sigma(A)$ is the essential spectrum.

<u>Spectral projections</u> *

Definition: Let A be a bounded self-adjoint operator and Ω a Borel set of $\mathbb{R}$

$P_\Omega \equiv \chi_\Omega(A)$ is called a spectral projection of A.

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*) Spectral projection can be used to investigate the spectrum of A.

P_Ω is an orthogonal projection since $\chi_\Omega^2 = \chi_\Omega$

Definitions: A family of projections $\{P_\Omega\}$ obeying the following properties is called a projection-valued measure (p.v.m.)

a) each P_Ω is an orthogonal projection

b) $P_\emptyset = 0$ $\quad P_{(-a,a)} = \mathbb{1}$ for some a

c) If $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \emptyset$

for $n \neq m$, Then $P_\Omega = s\lim_{N\to\infty} \left(\sum_{n=1}^{N} P_{\Omega_n} \right)$

d) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$

If P_Ω is p.v.m., then $d(\varphi, P_\Omega \varphi)$ is an ordinary measure for any φ .

The symbol $d(\varphi, P_\lambda \varphi)$ is used when integrating w.r.t. this measure.

2. Distributions (Generalized Functions)

The theory of distributions is a generalization of classical analysis, which makes it possible to deal in a systematic manner with difficulties that previously had been overcome by an ad hoc construction. For example, if $(x,t) \quad R^2$, then $u = f(x+t) + g(x-t)$ satisfies the d'Alembert's equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 \tag{1}$$

provided that the functions f and g be twice differentiable. This restriction is both tedious and unnatural in many instances. It can be overcome by introducing the so-called weak solutions.

By definition, these are functions u such that

$$\int u\left(\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2}\right) dx\, dt = 0 \tag{2}$$

for all sufficiently "good" functions ϕ , for example, for $\phi \quad C_c^2(R^2)$, the class of twice differentiable functions that vanish on the exterior of a bounded set.

In the theory of distributions, functions are replaced by linear forms on an auxiliary vector space, i.e. if V is a vector space over the field of complex numbers C, then a linear form on V is a homomorphism $V \longrightarrow C$. The linear forms on V are made into a vector space Hom(V,C) by defining: $\langle cu, \phi\rangle =$ $= c\langle u,\phi\rangle$ if $c \in C$ and $u \in$ Hom(V,C), and $\langle u+v, \phi\rangle = \langle u,\phi\rangle + \langle v,\phi\rangle$ If $u, v \in$ Hom(V,C), where $\phi \in V$ in each case.

The basic space in distribution theory is $C_c^{\infty}(R^n)$; its members are complex-valued functions on R^n which possess continuous derivatives of all orders, and vanish outside some bounded set. The notation $\mathcal{D}(R^n)$ is also used.

A continuous function $f : R^n \longrightarrow C$ determines a linear form on $\mathcal{D}(R^n)$ by the rule

$$\langle f, g \rangle = \int f \phi \, dx \quad , \quad \phi \in \mathcal{D}(R^n) \tag{3}$$

Conversely, it can be shown that this linear form determines f uniquely so that the space of continuous functions on R^n can be identified with a subspace of $\mathrm{Hom}(\mathcal{D}(R^n), C)$.

Now, if f is also continuously differentiable then the linear forms on $\mathcal{D}(R^n)$ determined by its derivatives are, by (3) and an integration by parts

$$\langle \frac{\partial f}{\partial x_i}, \phi \rangle = \int \phi \left(\frac{\partial f}{\partial x_i} \right) dx = - \int f \left(\frac{\partial \phi}{\partial x_i} \right) dx$$

$$i = 1, 2, \dots m \quad , \quad \phi \in \mathcal{D}(R^m)$$

i.e.

$$\langle \frac{\partial f}{\partial x_i}, \phi \rangle = - \langle f, \frac{\partial \phi}{\partial x_i} \rangle \tag{4}$$

Equation (4) makes sense for any linear form on $\mathcal{D}(R^n)$ and so provides a definition of the derivatives of such a form. Iterating eq.(4), one can obtain (generalized) derivatives of all orders.

Multiplication by infinitely differentiable functions can also define in analogy with eq.(4) :

$$\langle f u, \phi \rangle = \langle u, f \phi \rangle \quad , \qquad \phi \in \mathcal{D}(R^n) \tag{5}$$

The class of distributions is not the whole $\mathrm{Hom}(\mathcal{D}(R^n), C)$; it is the subspace consisting of continuous linear forms. This means that $\mathcal{D}(R^n)$ has been equipped with an appropriate topology.

Notation and Definitions.

The Euclidean norm $(x \,.\, x)^{\frac{1}{2}}$ will be written as $|x|$. $A \subset R^n$ is compact iff it is both closed and bounded.

The support of a function $f : X \longrightarrow C$ is the closure of the set $\{x \in X : f(x) \neq 0\}$ i.e. outside this closure $f(x) = 0$. We shall write supp f for this.

Dereivatives of higher orders are written as

$$\partial^{\alpha} f = \partial^{|\alpha|} / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$$

where $\alpha = (\alpha_1, \ldots \alpha_n)$, n multi-index α_i are non-negative integers; $|\alpha| = \alpha_1 + \cdots + \alpha_n$ order or length.

Definition: Let $X \subset R^n$ be an open set. A linear form T,

$$T : \mathcal{D}(R^n) \longrightarrow C$$

is called distribution if for every compact set $K \subset X$, there is a real number $c \geqslant 0$ and a non-negative integer N such that

$$|\langle T, \phi \rangle| \leqslant C \sum_{|\alpha| \leqslant N} \sup |\partial^\alpha \phi| \qquad (6)$$

for all $\phi \in \mathcal{D}(X^n)$ with supp $\phi \subset K$. The vector space of distributions on X is called $\mathcal{D}'(X)$.

THEOREM: Let $X \subset R^n$ be an open set, and let $f \in C^0(X)$ (the set of continuous functions). Then,

$$\langle f, \phi \rangle = \int f \phi \, dx \quad , \quad \phi \in \mathcal{D}(X) \qquad (7)$$

is a distribution.

Proof:

$$\left| \int f \phi \, dx \right| \leqslant \sup{}^{(+)} |\phi| \int f \, dx \quad , \quad \phi \in \mathcal{D}(K) \quad , \quad \partial^0 = 1$$

where $K \subset X$ is a compact set and

$$\mathcal{D}(K) = \left\{ \phi : \phi \in \mathcal{D}(R^n) \quad , \quad \sup \phi \subset K \right\}$$

Definition: Let $X \subset R^n$ be an open set. A sequence $(\phi_j)_{1 \leqslant j < \infty} \in \mathcal{D}(X)$ is said to converge (or tend) to zero in $\mathcal{D}(X)$ if:

(i) the support of the ϕ_j 's are contained in fixed compact subset of X and

(ii) for each multi-index α, the derivatives $\partial^\alpha \phi_j$ converge to zero uniformly.

THEOREM: A linear form T on $\mathcal{D}(X)$ is a distribution iff $\lim_{j \to \infty} \langle T, \phi_j \rangle = 0$ for every sequence (ϕ_j) which converges to zero in $\mathcal{D}(X)$ as $j \to \infty$.

Convergence of Distributions.

Definition: Let $X \subset R^n$ be an open set, and let $(T_j)_{1 \leq j < \infty}$ be a sequence of distributions on X. The sequence is said to converge in $\mathcal{D}'(X)$ to $T \in \mathcal{D}'(X)$ if

$$\lim \langle T_j , \phi \rangle = \langle T , \phi \rangle \qquad \text{for all } \phi \in \mathcal{D}(X).$$

For example, let f $C^0 \in (R)$ have its support in $[0,1]$ and suppose that $\int f(x)\, dx = 1$. Put $f_k(x) = k\, f(kx)$, $k = 1,2, \ldots.$

Then, $f_k(x) \to 0$ for any fixed $x \in R$ as $k \to \infty$. But, from $\int f(x)\, dx = 1$ and $\langle \delta, \phi \rangle = \phi(0)$ with $\phi \in \mathcal{D}(R)$, then

$$|\langle f_k, \phi\rangle - \phi(0)| \leq \left(\int |f(x)|\, dx\right) \sup\{|\phi(x) - \phi(0)| : 0 \leq x \leq 1\}$$

Hence, $f_k \to \delta$ in $\mathcal{D}'(R)$ as $k \to \infty$.

Operator-Valued Distributions.

In Quantum Field Theory one defines an operator-valued distribution $T(x)$ by saying that for every test function $\phi(x)$, $\langle T, \phi \rangle$ is a bounded linear operator on some Hilbert space. The physical motivation for this definition is that due to the incertainty principle the measurement of a field quantity at a point must be impossible.

Tempered Distributions.

Tempered distributions, as functionals, are continuous in a slightly weaker mode of convergence than the distributions defined over $\mathcal{D}$. The space of test functions over which tempered distributions are defined is denoted by $\mathcal{S}$. The functions in $\mathcal{S}$ are of class C^{∞} but do not necessarily have bounded support. They are howver required to go to zero as $|x| \longrightarrow \infty$ faster than any inverse power of x, and so all their derivatives.

For tempered distributions on R, or R^n, the space $\mathcal{S}$ (R^n) of test functions is defined by

$$\sup_{(x)} |x^P Q^{(k)}(x)| < \infty , \quad \text{respectively}$$

$$\sup_{(x)} \|\underline{x}\|^P \left| \partial^{|\alpha|}/\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n} \varphi(\underline{x}) \right| < \infty \tag{8}$$

for all p

Convergence in $\mathcal{S} = \mathcal{S}$ (R).

If Ψ , ϕ_j (j = 1,2,...) are test functions in $\mathcal{S}$, then $\phi_j \xrightarrow{\mathcal{S}} \Psi$ if for all P and all K

$$\sup |x^P \{ \phi_j^{(k)}(x) - \Psi^{(k)}(x)\} | \longrightarrow 0 \qquad \text{as } j \to \infty .$$

Definition: A tempered distribution f on R is a linear function on $\mathcal{S} = \mathcal{S}(R)$ which is continuous with respect to the convergence mode defined above, i.e.

$$\langle f, \phi_j \rangle \longrightarrow \langle f, \Psi \rangle \qquad \text{whenever } \phi_j \xrightarrow{\mathcal{S}} \Psi .$$

Fourier Transforms of Tempered Distributions.

Let $f(x)$ be a continuous function whose Fourier transform $\hat{f}(k)$ exists and is continuous. Regarded as distributions, f and $\hat{f}$ are the functionals $\langle f, \phi \rangle$ and $\langle \hat{f}, \phi \rangle$, respectively. The relation between them is given by the Parseval's identity

$$\langle \hat{f}, \phi \rangle = \int_{-\infty}^{+\infty} \hat{f}(x)\, \phi(x)\, dx = \int_{-\infty}^{+\infty} \hat{f}(x) \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} \hat{\phi}(k)\, e^{ikx}\, dk\, dx =$$

$$= \int_{-\infty}^{+\infty} \frac{1}{(2\pi)} \int_{-\infty}^{+\infty} \hat{f}(x)\, e^{ikx}\, dx\; \hat{\phi}(k)\, dk = \int_{-\infty}^{+\infty} f(k)\, \hat{\phi}(k)\, dk =$$

$$= \langle f, \hat{\phi} \rangle$$

Definition: If T is any tempered distribution, its Fourier transform $\hat{T}$ is defined as the distributions (functionals) given by

$$\langle \hat{T}, \phi \rangle = \langle T, \hat{\phi} \rangle \qquad \forall\ \phi \in \mathcal{S}$$

THEOREM: Let $T \in \mathcal{S}'$. Then $\hat{T}$ is a distribution in $\mathcal{S}'$.

Proof: If $\phi \in \mathcal{S}$, then $\hat{\phi} \in \mathcal{S}$. Also $\widehat{(\alpha \phi_1 + \beta \phi_2)} = \alpha \hat{\phi}_1 + \beta \hat{\phi}_2$

i.e. $\hat{T}$ is a linear functional.

Example: $\hat{\delta}(x) = \langle \delta x, e^{itx} \rangle = 1$

$\widetilde{D^\alpha \delta x} = (-i)^{|\alpha|} K^\alpha$

3. Algebraic Techniques.

The occurrence of many inequivalent irreducible representations of operators in the Hilbert space, $\mathcal{H}$, of an infinite system makes the algebraic approach more suitable as in the latter the algebra can be realized concretely as the algebra of operators acting in $\mathcal{H}$ in different manners which are non-isomorphic.

3.1. C^* - algebra.

We call C^* - algebra a set $\mathfrak{a}$ possessing the following structure elements.

(i) A complex linear structure ;

(ii) A bilinear associative product, $A.B$, $A,B \in \mathfrak{a}$.

(iii) A conjugate linear adjunction (involution):

$$A \in \mathfrak{a} \longrightarrow A^* \in \mathfrak{a} \text{ such that } A^{**} = A, \quad (AB)^* = B^* A^*,$$

for $A,B \in \mathfrak{a}$;

(iv) A norm $||A||$ with the following algebraic properties:

1. $|| A B || \leq || A || \; || B ||$

2. $|| A^* || = || A ||$, $A \in \mathfrak{a}$

3. $|| A^* A || = || A ||^2$, $A \in \mathfrak{a}$

for which $\mathfrak{a}$ is required to be complete (i.e. convergence in $\mathfrak{a}$ of all Cauchy sequences of $\mathfrak{a}$ with respect to the distance $|| A - B ||$).

For example, let us first consider the set C of all complex functions $f(x)$ on the real which vanish for $|x| \longrightarrow \infty$ with conjugation as involution and a norm $|| f || = \sup |f(x)|$. The set C is an Abelian C^*-algebra.

Next, let us consider the set of all bounded operators $\mathcal{B}(\mathcal{H})$ in a Hilbert space $\mathcal{H}$. The norm $\| A \|$ of a bounded operator is defined by

$$\| A \| = \sup \frac{\| A\psi \|}{\|\psi\|} \quad \text{with } \psi \neq 0 .$$

Equipped with this norm and with the usual definition of the adjoint, $\mathcal{B}(\mathcal{H})$ is a C^*-algebra.

The importance of a C^*-algebra in physics follows from the fact that it can be shown that any C^*-algebra is isomorphic to a subalgebra of $\mathcal{B}(\mathcal{H})$.

3.2. States.

Definition: A linear functional $w(A)$ on a C^*-algebra $\mathfrak{a}$ is called a state if $w(A^* A) \geqslant 0$ and if the norm $\| w \| = w(1) = 1$.

A state w has the following properties:

1. $| w(A) | = \| A \|$,
2. $| w(A^* B) |^2 \leq w(A^*A) \cdot w(B^*B)$.

The states form a convex set, i.e. if w_1 and w_2 are states each linear combination $\lambda w_1 + (1 - \lambda) w_2$ with $0 \leq \lambda \leq 1$ is again a state.

Definition: A state w is pure if there does not exist a number λ, $0 < \lambda < 1$ and two different states w_1 and w_2 such that $w = \lambda w_1 + (1 - \lambda) w_2$. A state which is not pure is called a mixed state.

Definition: An automorphism of $\mathfrak{A}$ is a linear mapping

$$A \in \mathfrak{A} \longrightarrow A_\tau \in \mathfrak{A}$$

such that
$$(AB)_\tau = A_\tau B_\tau ,$$
$$(A^*_\tau) = A^*_\tau ,$$
$$\| A_\tau \| = \| A \| .$$

An automorphism of $\mathfrak{A}$ gives rise to a transformation $w \mapsto w_\tau$ of the states defined by $w_\tau (A) = w(A_\tau)$ for all A .

We say that w is invariant if $w_\tau = w$.

3. 3. Representations.

Definition: Given an abstract C^*-algebra $\mathfrak{A}$, a representation π of $\mathfrak{A}$ on the Hilbert space $\mathcal{H}$ is a map $A \in \mathfrak{A} \longrightarrow \pi(A)$ from $\mathfrak{A}$ to bounded operators on $\mathcal{H}$ with the following properties:

1. Linearity;
2. Multiplicativity, $\pi(AB) = \pi(A) . \pi(B)$;
3. * - representation property, $\pi(A^*) = \pi(A)^*$, $A,B \in \mathfrak{A}$.

The representation π is called faithful whenever

$$\pi(A) = 0 \Longrightarrow A = 0.$$

In other words, faithful representations yield exact (or concrete) realizations of $\mathfrak{A}$.

THEOREM: Let $\mathfrak{A}$ be a simple C^*-algebra. Then either $\mathfrak{A}$ is isomorphic to the algebra of compact operators on some Hilbert space in which case it has only one irreducible representation up to unitary equivalence or $\mathfrak{A}$ has a family of mutually inequivalent representations with the power of the continuum. This shows the difference between ordinary quantum mechanics and quantum field theory.

3.4. Gel'fand - Naimark - Segal (GNS) construction.

The link between states and representations is given as follows: each state w of a C^*-algebra $\mathfrak{A}$ determine a representation π_w of $\mathfrak{A}$ on a Hilbert space $\mathcal{H}_w$, with a vector $\xi_w \in \mathcal{H}_w$, in such a way that

$$w(A) = (\xi_w) | \pi_w (A) | (\xi_w) , \quad A \in \mathfrak{A}$$

with the vector ξ_w cyclic for π_w (i.e. the set $\pi_w(A)\xi_w$, $A \in \mathfrak{A}$ is norm dense in $\mathcal{H}_w$. The triplet $(\pi_w, \mathcal{H}_w, \xi_w)$ is called the GNS construction of $\mathfrak{A}$ (one says that w generates the representation π_w).

3.5. Von Neumann Algebras.

Norm Topology.

A bounded operator A has a norm $||A||$. This norm defines a topology; this is the simplest topology one can introduce for the notion of convergence.

We say that A is the norm limit of the sequence A_n if $\| A - A_n \| \longrightarrow 0$. The realization of a C^*-algebra is closed in this topology.

Weak Topology.

We say that A is the weak limit of the sequence A_n if $(\Psi, A\Phi) = \lim_{n \to \infty} (\Psi, A_n \Phi)$ for all Ψ and Φ.

The weak topology is weaker than the norm topology in the sense that

$A_n \xrightarrow{\text{norm}} A$ implies $A_n \xrightarrow{\text{weak}} A$ but not conversely.

Definition: A weakly closed algebra[(+)] $\mathcal{R}$ of bounded operators containing the identity is called a Von Neumann algebra or W^*-algebra.

Definition: The commutant $\mathcal{R}'$ of the Von Neumann algebra is the set of all bounded operators commuting with all operators of $\mathcal{R}$. Von Neumann's theorem says that $\mathcal{R} = \mathcal{R}''$, i.e. $\mathcal{R}$ coincides with its bicommutant.

Definition: A Von Neumann algebra is a factor if the center $C = \mathcal{R} \cap \mathcal{R}'$ does not contain other operators than multiples of the identity.

The representation $\pi(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathcal{H}$ is norm closed.

We can increase the algebra $\pi(\mathfrak{a})$ by adding all weak limit points.

This leads to the Von Neumann algebra $\overline{\pi(\mathfrak{a})}^{w} = \mathcal{R}$

(+) i.e. contains all the weak points of $\mathcal{R}$; it ca be shown that the weakly closed algebra coincides with the strongly closed one.

3.6. Primary State.

Given a state w over a C^*-algebra $\mathcal{A}$. By the GNS construction, we obtain a representation h_w of $\mathcal{A}$ by operators acting on a Hilbert space . $h_w(a)''$ is a Von Neumann algebra. If this is a factor, i.e. if $h_w(a)'' \cap h_w(a)' = \lambda \mathbb{1}$, then w is called a primary state.

The link between pure state and irreducibility is expressed by the following

THEOREM: The cyclic representation determined by a pure state is irreducible. Conversely, every vector of an irreducible representation gives rise to a pure state.

Remarks: An automorphism $A \longmapsto A_\tau$ is inner if there exists a unitary element $Q_\tau \in \mathcal{A}$ such that $A_\tau = Q \; A \; Q^{-1}$, if Q such an automorphism is an outer one.

Let $A \longrightarrow A$ be an outer automorphism, and let $\pi(A)$ be a representation. Then the mapping $\pi(A) \longrightarrow \pi(A_\tau)$ is an automorphism of $\pi(A)$. It will be implementable if there exists a unitary operator U_τ such that $\pi(A_\tau) = U_\tau \; (A) \; U_\tau^{-1}$.

THEOREM: If ω is invariant in the cyclic representation defined by w, then the unitary operator U_τ is uniquely determined by $U_\tau \Omega = \Omega$.

Cyclic and Separating Vectors.

Let M be a self-adjoint subset of $\mathcal{B}(\mathcal{H})$ and $\phi \in \mathcal{H}$. We say that ϕ is cyclic for M if $M\phi$ is dense in $\mathcal{H}$; that ϕ is separating for M if $A \in M$ and $A\phi = 0 \Longrightarrow A = 0$.

Quasi-Local Algebra.

Because our apparatus has finite dimensions, the observations one can make on infinite systems in bound to be local, and hence the following definitions:

The C^*-algebra that describes the infinite system is obtained as a union of local algebras $\mathcal{A}(\Lambda)$ completed(+) in norm where Λ is a bounded region in space, i.e. $\mathcal{A} = \overline{\bigcup_{\Lambda} \mathcal{A}(\Lambda)}^{\|\;\|}$. The operators $A \subset \mathcal{A}$ are quasi-local operators.

The Algebra of Observables at Infinity.

Let $\mathcal{A}$ be a quasi-local algebra, containing a set $\{\mathcal{A}_\Lambda\}$ of collections of C^*-subalgebras corresponding to the open bounded region Λ in space. Locality means that, given $A \in A_\Lambda$, $B \in \Lambda_M$ and $\Lambda \cap M = \phi$, then $[A, M] = 0$. Now, define $\tilde{\mathcal{A}}_\Lambda$ to be the C* -subalgebra of $\mathcal{A}$ generated by $\{\mathcal{A}_M : M \cap \Lambda = \phi\}$. Therefore, if $\mathcal{A}_\Lambda$ is interpreted as the algebra of observables inside Λ, then $\tilde{\mathcal{A}}_\Lambda$ corresponds to the algebra of observables measurable outside Λ, i.e. $\mathcal{A}_\Lambda$ and $\tilde{\mathcal{A}}_\Lambda$ commute.

If π is the * - representation of $\mathcal{A}$ on $\mathcal{H}_\pi$ and $\mathcal{B}_\pi$ is the intersection of all $\pi(\tilde{\mathcal{A}}_\Lambda)$ (weakly closed), i.e. $\mathcal{B}_\pi = \bigcap \overline{\pi(\tilde{\mathcal{A}}_\Lambda)}$. Then $\mathcal{B}_\pi$ may be interpreted as the algebra of observables outside any bounded open set and is referred to as the algebra at infinity.

(+) The reason of completing the algebra has a physical origin, the time evolution of $A(\Lambda)$ will be contained in the quasi-local algebra, but not in the local algebra. Think of a localized wave-packet in QM Hilbert space. Once the time evolution acts on $A(\Lambda)$, it becomes non-localized, though its norm is preserved.

4. The Kubo-Martin-Schwinger Condition (KMS).

The Kubo-Martin-Schwinger was proposed initially as boundary conditions determining the solutions of an infinite set of differential equations fulfilled by the Green functions describing equilibrium states. Meanwhile it was realized that the KMS condition which has a simple expression in the algebraic framework of field theory can be substituted to the "Gibbs ansatz" as the basis of equilibrium statistical mechanics.

The advantage of the KMS condition, however, is that it is valid for an infinite system, whilst the Gibbs ansatz can only hold for a finite system (as the partition function for an infinite system is divergent).

Before passing to the KMS condition, let us look at the traditional way of constructing equilibrium statistical mechanics based on the Gibbs ansatz.

I. Finite Systems.

Let H' be the Hamiltonian for a system in a box V, with a particle number operator N and chemical potential μ. The expectation value of an observable represented by an operator A is given by

$$\omega(A) = \frac{Tr\{[\exp -\beta(H'-\mu N)]\, A\}}{Tr[\exp -\beta(H'-\mu N)]}$$

with $H = H' - \mu N$

$$\omega(A) = \frac{Tr\{[\exp -\beta H]\, A\}}{Tr[\exp -\beta H]} \qquad (1)$$

H is a self-adjoint operator and gives therefore rise to a continuous group of unitary operators in $\mathcal{H}$, i.e. $U_t = \exp(iHt)$ (Stone's theorem).

ω defines a normalized state on all bounded operators on $\mathcal{H}$ (i.e. $\mathcal{B}(\mathcal{H})$), $\omega(1) = 1$.

Under the (*) automorphism $A \longrightarrow A_t = U_t A U_t^{-1}$ of $\mathcal{B}(\mathcal{H})$, eq.(1) gives

$$\omega(A_t) = \frac{\mathrm{Tr}\{[\exp -\beta H] \exp iHt \exp -iHt\}}{\mathrm{Tr}[\exp -\beta H]}$$

Due to the invariance of the trace for cyclic permutations, we have

$$\omega(A_t) = \omega(A)$$

Let $z = t + i\gamma$ be a complex number and $A \in \mathfrak{a}$, then

$$A_z = e^{iHz} A e^{-iHz}$$

will, <u>in general</u>, not be a bounded operator. But

$$A_z e^{-i\beta H} = e^{iHt} e^{-H\gamma} A e^{-iHt} e^{-H(\beta-\gamma)}$$

will be bounded and of trace class(+) for $0 \leq \gamma \leq \beta$; without this restriction the trace of the above expression may not exist.

Similarly, $e^{-\beta H} A_z$ will be of trace class for $-\beta \leq \gamma \leq 0$. So, for any $A,B \in \mathfrak{a}$, the function

$$F_{AB}(z) = \mathrm{Tr}\, A_z\, e^{-\beta H}$$

(+) An operator $A \in \mathcal{L}(\mathcal{H})$ is called trace class iff $\mathrm{tr}\,|A| < \infty$.

is well-defined in the strip $0 \leq \gamma \leq \beta$ and analytic in the open strip $0 < \gamma < \beta$ and conditions at the boundaries. This statement follows from the fact that $H\, e^{-\alpha H}$ is a bounded operator for any $\alpha > 0$; for $\alpha = 0$ this is not true as H is unbounded.

Similarly, the function

$$G_{AB}(z) = \mathrm{Tr}\; e^{-\beta H} A_z B$$

is analytic in the strip $-\beta < \gamma < 0$ and continuous at the boundaries.

We have, for $\gamma = 0$,

$$F_{AB}(t) = \omega(B A_t) \quad ; \quad G_{AB}(t) = \omega(A_t B)$$

The trace property under cyclic permutations yields

$$F_{AB}(t + i\beta) = G_{AB}(t)$$

Proof: Due to the invariance of the trace for cyclic permutations, we have $\omega(A_t) = \omega(A)$. Let A and $B \in \mathcal{B}(\mathcal{H})$, then omitting the denominator in (1), with $\{e_\alpha\}$ set of eigenstates of H and corresponding eigenvalues ε_α.

$$\begin{aligned}
\omega(B A_{t+i\beta}) &= \sum_\alpha e^{-\beta \varepsilon_\alpha} \langle e_\alpha | B\, U_{t+i\beta} A\, U^{-1}_{t+i\beta} | e_\alpha \rangle \\
&= \sum_\alpha e^{-\beta \varepsilon_\alpha} \langle e_\alpha | B\, U_t\, e^{-\beta H} A | e_\alpha \rangle e^{-i\varepsilon_\alpha t} e^{\beta \varepsilon_\alpha} \\
&= \sum_\alpha \langle e_\alpha | U_t^{-1} A\, U_t\, e^{-\beta H} B | e_\alpha \rangle \\
&= \mathrm{Tr}\left(e^{-\beta H} A_t B \right) = \omega(A_t B_t)
\end{aligned}$$

(by cyclic permutations)

So,

$$\omega(B A_{t+i\beta}) = \omega(A_t B)$$

Eq. (6) is called the KMS boundary condition. For the passage to the thermodynamic limit, it is convenient to use the functions F and G only for real times.

Define, with $z = t + i\gamma$, $f(z) = \int \hat{f}(\varepsilon)\, e^{iz\varepsilon}\, d\varepsilon$, where $\hat{f}(\varepsilon) \in \mathcal{D}$. By Paley-Wiener theorem (B. Simon), $f(z)$ is analytic over the entire z-plane.

Multiplying (6) by $f(t)$ and integrating, we can shift the integration path on the left-hand side within the analyticity domain of F_{AB} and obtain

$$\int f(t)\, F_{AB}(t+i\beta)\, dt = \int f(t-i\beta)\, F_{AB}(t)\, dt$$

$$= \int f(t-i\beta)\, \omega(BA_t)\, dt$$

$$= \int f(t)\, \omega(A_t B)\, dt$$

The validity of eq.(7) for any test function $f \in \mathcal{D}$ and any pair $A,B \in \mathfrak{a}$ is equivalent to the KMS boundary conditions. To see this, let $\hat{F}(\varepsilon)$ and $\hat{G}(\varepsilon)$ denote the Fourier transforms of $F(t) = \omega(BA_t)$ and $G(t) = \omega(A_t B)$. Then, eq.(7) gives

$$e^{-\beta\varepsilon} \int \hat{f}(\varepsilon)\, e^{it\varepsilon}\, F(t)\, dt\, d\varepsilon = \int \hat{f}(\varepsilon)\, e^{it\varepsilon}\, G(t)\, dt\, d\varepsilon$$

Therefore, for any $f \in \mathcal{D}$, we have

$$\hat{F}(\varepsilon) = e^{\beta\varepsilon}\, \hat{G}(\varepsilon)$$

F and G are considered as distributions over $\mathcal{D}$.

Since $F(t)$ and $G(t)$ are bounded, continuous functions of t and hence $\hat{F}$, $\hat{G}$ are distributions over $\mathcal{S}$.

$\mathcal{D} \subset \mathcal{S}$, the relation (8) holds for tempered distributions $\hat{F}$, $\hat{G}$. This implies the analyticity of the distributions $F(z)$, $G(z)$ in the open strips

$$-\beta \leq \gamma \lesssim 0 \quad , \quad -\beta < \gamma < 0$$

$F(t)$, $G(t)$ are bounded, continuous functions, so by eq.(3) they are boundary values of $F(t+i\gamma)$ for $\gamma=0$ and $\gamma=\beta$ respectively.

Summarizing, analyticity of $F(z)$, $G(z)$ with KMS boundary conditions implies that eq.(6) is equivalent to eq.(7) .

II. Infinite Systems.

Equation (7) written above holds for a finite system, i.e. we should write it as

$$\int f(t-i\beta)\, \omega_V(B A_t^V)\, f(t)\, dt = \int f(t)\, \omega_V(A_t^V B)\, dt$$

where ω_V is the Gibbs state for volume V and $A_t^V = U_V(t)\, A\, U_V(t)^{-1}$.

In order to prove that the KMS boundary condition holds for an infinite system, we have actually to prove that

$$\lim_{V\to\infty} \int \omega_V(B A^V_t)\, f(t)\, dt = \int \omega(B A_t)\, f(t)\, dt$$

with $f(t) \in \mathcal{J}$.

The integrands are uniformly bounded functions of t.

It turns out that the limit exists under the following assumptions:

1. Our observables are quasi-local operators, the algebra to which they belong is denoted by $\mathcal{A}$ (operators which can be approximated in the norm topology by operators belonging to a finite region), for which the Hamiltonian is additive, i.e. if we define the system into two parts then the difference between the Hamiltonian of the system and the sum of the Hamiltonians corresponding to the parts is just a surface term.

2. Let $V_{n+1} \supset V_n$ and $\bigcup V_n = R^3$ (the whole space).

   With (1) holding and $A \subset \mathfrak{a}(V)$ (algebra of bounded operators), then

$$\lim_{V_n \to \infty} \omega_{V_n}(A) = \omega(A)$$

with $V \subset V_n \subset V_{n+1}$.

The KMS boundary condition for the infinite system can be written as

$$\lim_{V \to \infty} \omega_V (B A_t^V) = \omega(B A_t)$$

Another form which can be shown to be equivalent to the KMS boundary condition, namely,

$$\omega(AB) = \omega\left(B_{-\frac{1}{2}i\beta} A_{\frac{1}{2}i\beta}\right)$$

where $A, B \in \tilde{\mathcal{A}} \subset \mathcal{A}$ (a subset of all elements $A \in \mathcal{A}$, of which the Fourier transform $\hat{A}(\varepsilon)$ of A_t has compact support).

Properties of a KMS State ω .

1. A KMS state ω is invariant in time, i.e. $U_t \Omega = \Omega$, Ω is the cyclic vector associated to ω .

2. The generator of the unitary operator is H : $U_t = e^{iHt}$. The spectrum of H is continuous, running from $-\infty$ to $+\infty$ with a single discrete point at 0.

3. Existence of a conjugation operator J , i.e. an anti-unitary operator J with $J^2 = 1$, which transforms the von Neumann algebra $\mathcal{R}$ generated by $\pi(\mathfrak{a})$ into its commutant:

$$J \mathcal{R} J = \mathcal{R}'$$

4. with the additional property $J C J = C^*$ for all center elements and $J \Omega = - \Omega$.

5. Ω is cyclic for $\mathcal{R}'$, in particular, if $S(A) = J \pi(A) J$, the set of vectors $S(A) \Omega$ is dense in $\mathcal{H}$.

6. The unbounded operator T, defined by

$$T \pi(A) \Omega = S(A)^* \Omega$$

has an inverse, and $T \pi(A) T^{-1} = \pi(A_{\frac{1}{2} i\beta})$ for all $A \in \mathfrak{a}$.

It can be shown that

$$T = e^{-\beta H/2}$$

CHAPTER 3

IMPLICATIONS OF THE KMS CONDITION ON THE QUASI-PARTICLES

3.1. CONSTRUCTION OF THE CREATION AND ANNIHILATION OPERATORS FOR QUASI-PARTICLE. COLLECTIVE EXCITATIONS AND RELATED TOPICS.

Notations and assumptions :

Consider an infinite quantum system of interacting particles, respectively spins which is at equilibrium and at inverse temperature $\beta = \frac{1}{KT}$ (k is the Boltzmann constant).

The state is assumed to be translationally invariant and pure phase ie a factor phase.

The K.M.S. state is denoted by ω and its G.N.S. vector by Ω.

The algebra of observables (the quasi - local algebra) is denoted by $\mathcal{A}$ with elements A,B,..., its commutant by $\mathcal{A}'$ with elements A,B,... and the weak closure by $\mathcal{A}''$ (the bicommutant of $\mathcal{A}$).

Time and space translations are implemented by a strongly continuous unitary group $U(t,x)$ with self-adjoint generators H , P with

$$H\Omega = P\Omega = 0 .$$

Because we are in the K.M.S. condition, Ω is cyclic and separating, this allows us to use tools developped in QFT in connection with scattering.

The appropriate Hamiltonian, which describes the physical behaviour of the system should become simpler, when expressed by means of new objects

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In the field theory the vacuum is a cyclic vector for smeared fields (corresponding to undressed particles) which would represent collective excitations respectively quasi-particles. The stable quasi-particle and continuous background (due to the decay of some quasi-particles) are assumed to span the whole Hilbert space $\mathcal{H}$.

If no stable quasi-particle is present, our results will be empty (there will be some more remarks at the end of the chapter).

The point spectrum of (H , P) is assumed to contain a contribution concentrated on the hypersurface $\{\omega = \varepsilon(K)\}$ with $\varepsilon(K) > 0$ and continuous This sharp excitation corresponds to quasi-particles of infinite lifetime ie the width corresponding to the dispersion relation $\omega = \varepsilon(K)$ is zero.

Because we are in a K.M.S. state so there must be a sharp excitation located on the hypersurface $\{\omega = -\varepsilon(K)\}$ ie a mirror excitation [14] so that the change in energy is zero.

Physically this correspond to a hole excitation or saying the same thing this is an anihilation of an excitation with $\omega = \varepsilon(K)$ in the state Ω.

In the construction of creation and annihilation operators corresponding to quasi-particles and holes, we will denote by $P_{\pm}$, the part of the projection-valued measure $E_H(d\omega)\,E_P(dK)$ with support on the manifolds (the hypersurface $\omega = \pm\,\varepsilon(K)$).

We assume that $\mu := \mu_+ \cup \mu_-$ (:= means by definition) exhausts the singular continuous spectrum of the resolution of the identity $\int E_H(d\omega)\,E_P(dK)$.

The remaining parts of the projection-valued measures will be assumed to consist of those projections on Ω and on the absolutely continuous part.

Remarks:

(i) The energy of a KMS state extends from $-\infty$ to $+\infty$, so that in a temperature state the whole R^4 is covered with points of the joint spectrum $(H, \underline{P})$. Therefore the sharp excitation branches will be embedded in the continuum of the spectrum.

(ii) Apart from the BCS model of superconductivity, there will be no mass gap, i.e. we have $\mathcal{E}(0) = 0$; in other words, the situation is in some sense similar to Relativistic Quantum Field Theory when zero mass particles are present. This allows one to use the Buchholz scattering theory of massless particles. [15]

Our next task is to construct the creation and annihilation operators for collective excitations, the "one-particle" and "one-hole" excitation branches of which are respectively $\mu_{\pm}$.

By definition of cyclicity of, Ω is dense in $\mathcal{H}$ and by definition of the spectrum there is an $A \in \mathcal{A}$ such that $P_+ A \Omega \neq 0$ and this corresponds to a one-particle excitation. The KMS conditions imply that $P_- A^* \Omega \neq 0$, which corresponds to a one-hole excitation.

To put it in a more elegant form, $(w,k) \in$ spectral support of A. As the spectral support of A^* (adjoint of A) = - spectral support of A, one has as a consequence that $(-w,-k) \in$ spectral support A^*, which is defined by the Arveson spectrum operator and state respectively (16).

$(w,k) \in$ spectral support of A iff $\int f(x,t)\, A(x,t)\, dx\, dt \neq 0$ for all test functions f with $\tilde{f}(w,k) \neq 0$, $\tilde{f}$ being the Fourier transform of f. Taking the complex conjugate of both sides give $\int \bar{f}(x,t)\, A^*(x,t)\, dx\, dt \neq 0$ provided $\tilde{f}(-w,-k) \neq 0$. Using now the spectral decomposition of the strongly continuous unitary group $U(t,x)$, we obtain

$$A(t,x)\Omega = U(t,x)A\Omega = \frac{1}{(2\pi)^2}\int e^{-iwt}\, e^{iK.x}\, E(dw,dK)A\Omega \quad (1.1)$$

where $A\Omega$ is a vector in $\mathcal{H}$, $E(dw\, dk)$ is the projection-valued measure, therefore $A(t,x)\Omega$ is the Fourier transform of a vector measure.

The operator-valued distributional[(+)] Fourier transform of $A(x,t)$ is defined by

$$\int f(t,x)A(t,x)\,dx\,dt =: \int \tilde{f}(w,K)\tilde{A}(w,K)\,dK\,dw \qquad (1.2)$$

For a suitable test function f, for which $\tilde{f}(w,k) \neq 0$, we see that using the Arveson spectrum of operators, the support of the Fourier transform of $A(x,t)$ is in the Arveson spectrum of A with respect to the automorphism $U(x,t)$ generated by $(H, \underline{P})$. Eq.(1.1) shows that $\tilde{A}(w,k)\Omega$ is a vector-valued measure which by assumption contains a δ -type singular contribution concentrating on $\mu = \mu_+ \cup \mu_-$.

(+)Parseval's equality: $f,g \in L^2 \Longrightarrow (2\pi)^4 \int dx\, f(x)g(x) = \int dp\, \tilde{f}(-p)\tilde{g}(p)$.

In order to obtain an operator with energy-momentum support located on $\mu_{\pm}$, we define the following sequence of operators:

$$A_{T,\pm}(f) = \int \tilde{h}(T.(\omega \mp \varepsilon(K)))\, \tilde{f}(K)\, \tilde{A}(\omega,K)\, d\omega\, d\underline{K} \qquad (1.3)$$

where $\tilde{h} \in \mathcal{D}(R)$, with $\tilde{h}(0) = 1$, $\int ds\, h(s) = 1$, $f \in \mathcal{D}(R^3)$.

As $T \longrightarrow \infty$, one hopes that all contribution coming from $A_T(f)$ that does not belong to quasi-particles, collective excitations located on $\mu_{\pm}$, will be eliminated. To illustrate this fact, we can take $h_T(s) = h(Ts)$ as a δ-Dirac sequence, since we have $\int h_T(s)\, ds = 1$ for every T.

The support of $h_T(s)$ is contained in supp h(s)/T, therefore as $T \longrightarrow \infty$, the support of $h_T(s)$ will be concentrated at the origin with $\int ds\, h(s) = 1$, i.e. $h(s) = \delta(w \pm \varepsilon(k))$, which means that the energy-momentum spectrum is located on $\mu_{\pm}$.

<u>LEMMA</u>: The norm $\lim_{T \longrightarrow \infty} A_{T,\pm}(f)\, \Omega$ exists. (1.4)

<u>Proof</u>: $\tilde{A}(w,k)\, \Omega$ is a vector-valued measure which is continuous in the vector norm(*) $[\, \tilde{A}(w,k)\, \Omega\, dw\, d\underline{k} = E(dw\, d\underline{k})\, A\Omega$, i.e.

$|| \int \tilde{g}_n(w,k)\, A(w,k)\, \Omega\, || \longrightarrow 0$ with $n \longrightarrow \infty$, for all sequence $\{ g_n \}$, $\check{g} \in \mathcal{D}$, converging to zero in the sup norm(++), in an arbitrary but fixed compact set $\subset R^4$.

(*) The projection-valued measure is continuous in norm.

(++) The set of bounded operators B(s,F) of a set S into a normed vector space F is a normed vector space with the norm $|| A ||_S = || A || = \sup_{x \in S} | A(x) |$.

This is the so-called sup norm.

Now, we split the vector-valued measure into a pure point contribution (pp), a singular continuous (sc) an an absolutely continuous (ac) which are mutually orthogonal, i.e. $E(dw\, d\underline{k}) = E_{pp}(dw\, d\underline{k}) + E_{ac}(dw\, d\underline{k}) + E_{sc}(dw\, dk)$, and $E_{pp}(dw\, d\underline{k}) = 0$, by assumption.

For the absolutely continuous part , eq.(   ) gives

$$\lim_{T\to\infty} \| \tilde{h}(T(w \mp \varepsilon(k)))\, \tilde{f}(\underline{k})\, E_{ac}(dw.d\underline{k})\, A\Omega \|^2 =$$

$$= \lim_{T\to\infty} \int (A\Omega, E_{ac}(dw.d\underline{k})\, A\Omega) \cdot |\tilde{h}(w,k)|^2\, |\tilde{f}(k)|^2$$

where $E^2_{ac}(dw\, dk) = E_{ac}(dw\, dk)$, $\| \ .\ \|^2 = (\ .\ ,\ .\ )$ were used.

Because both $\tilde{h}$ and $\tilde{f}$ have compact support, the least upper bound of $|\tilde{h}|^2 . |\tilde{f}|^2$ is attained(*) which in turn implies the following inequality

$$\lim_{T\to\infty} \left\| \int \tilde{h}(T(w \mp \varepsilon(k)))\, \tilde{f}(\underline{k})\, E_{ac}(dw.d\underline{k})\, A\Omega \right\|^2 \le$$

$$\lim_{T\to\infty} \left(\int_{K_T} (A\Omega, E_{ac}(dw.d\underline{k})\, A\Omega) \right) \sup |\tilde{h}(w,\underline{k})|^2\, |\tilde{f}(k)|^2$$

K_T is the support of $h_T . f$. As $T \longrightarrow \infty$, the support will be concentrated at the origin, so its Lebesgue measure is zero, but $E_{ac}(dw\, d\underline{k})$ is absolutely continuous with respect to the Lebesgue measure, i.e.

$E_{ac}(dw\, d\underline{k}) = 0$ as $T \longrightarrow \infty$. So,

$$\lim_{T\to\infty} \left\| \int \tilde{h}(T\cdot(w \mp \varepsilon(k)))\, \tilde{f}(k)\, E_{ac}(dw.d\underline{k})\, A\Omega \right\|^2 = 0 \quad (1.5)$$

(*) Product of two continuous functions of compact support is a continuous function and with compact support, which is the intersection of the supports of the two functions.

On the other hand:

$$\Big\| \int \tilde{h}(T(\omega \mp \varepsilon(k)))\tilde{f}(k)\, E_{sc}(d\omega . d\underline{k})\, A\Omega \; -$$
$$\int \tilde{h}(T'(\omega \mp \varepsilon(k)))\, E_{sc}(d\omega . d\underline{k})\, A\Omega \Big\|^2 = \Big(A\Omega, \int |(\tilde{h}_T - \tilde{h}_{T'})|^2 |\tilde{f}|^2 E_{sc}(d\omega . d\underline{k}) A.$$

$E_{sc}(dw\, dk)$ has its support on $\mu_{\pm}$, as $T, T' \longrightarrow \infty$, $\tilde{h}_T - \tilde{h}_{T'} = 0$ as $\tilde{h}(0) = 1$ (supported at the origin).

This means:

$$\Big\| \int \tilde{h}(T(\omega \mp \varepsilon(k)))\tilde{f}(k)\, E_{sc}(d\omega\, d\underline{k})\, A\Omega - \int \tilde{h}(T'(\omega \mp \varepsilon(k)))\, E_{sc}(d\omega\, d\underline{k})\, A\Omega \Big\|^2 \equiv \qquad (1.6)$$

or, in other words, $\| A_T \Omega \|$ is a Cauchy sequence.

This in turn implies that the norm $\lim A_{T,\pm}(f)\,\Omega$ exists, since

$$\big|\, \| A_T \Omega \| - \| A_{T'} \Omega \| \,\big| \leq \| (A_T - A_{T'})\Omega \| \longrightarrow 0.$$

To obtain a true operator, we have to show that this limit exists on a dense set of vectors.

<u>THEOREM</u>: i) There exists densily defined operators $a_{\pm}(f)$ with

$A_{T,\pm}(f) \longrightarrow a_{\pm}(f)$ strongly on a dense domain of definition on which $a_{\pm}$ are closable.

ii) On this dense domain, we have the following time evolution of $a_{\pm}(f)$:

$$U_t\, a_{\pm}(f)\, U_{-t} = a_{\pm}\big(e^{i\varepsilon(-i\underline{\nabla}).t} f \big), \qquad (1.7)$$

i.e. quasi-free evolution.

Proof: With B' in the commutant of $\mathcal{A}$, i.e. $B' \in \mathcal{A}'$, we have that

$$\lim_{T \to \infty} A_{T,\pm} B'\Omega = \lim_{T \to \infty} B' A_{T,\pm} \Omega = B' P_{\pm} A \Omega \quad (1.8)$$

(since the support of $A_{T,\pm}$ is concentrated on $\mu_{\pm}$ for $T \to \infty$).

So, we can define $a_{\pm} B'\Omega =: B' P_{\pm} A \Omega$.

Therefore, $\lim_{T \to \infty} A_{T,\pm}(f) B'\Omega = a_{\pm}(f) B'\Omega$.

We should remark that $B'\Omega$ is dense in $\mathcal{H}$ (one of the properties of the KMS condition is that Ω is cyclic for $\mathcal{A}'$) and, since the left-hand side of eq.(1.8) is well-defined and it exists in norm.

From properties of the KMS conditions (Chapter II), we have

$$\begin{aligned} B'\Omega &= JBJ\Omega = JB\Omega = TB^*\Omega \\ &= e^{-\beta H/2} B^*\Omega = e^{-\beta H/2} B^* e^{\beta H/2}\Omega, \quad H\Omega = 0 \\ &= \hat{B}^*\Omega \quad (1.9) \end{aligned}$$

with $\hat{B} := e^{\beta H/2} B e^{-\beta H/2}$

Now, if we assume that B has a compact (w,k) support, so does $\hat{B}$. This means that $a_{\pm}$ are also defined on the dense set $\tilde{\mathcal{A}}\Omega$. $\tilde{\mathcal{A}}$ is the set of elements with (w,k) support.

The closability(+) of $a_{\pm}$ follows from the fact that $a_{\pm}$ are densely defined.

(+) If T is a densely defined operator on a Hilbert space $\mathcal{H}$, then T is closable iff $D(T^*)$ is dense (see ref. B. Simon), since as $T \to \infty$, $w = \pm \varepsilon(k)$, $\underset{\sim}{k} = -i \underset{\sim}{\nabla}$ (operator form for k).

Proof (cont'n): The domain of a^* is defined by

$$D_{a^*} = \{\psi \mid \exists \psi' \text{ s.t. } (\psi \mid a\phi) = (\psi' \mid \phi)\}$$

For $A', B' \in \mathcal{A}'$, we have

$$(A'\Omega, a_\pm B'\Omega) = \lim_{T\to\infty} (A'\Omega, A_{T,\pm} B'\Omega)$$

$$= \lim_{T\to\infty} (A' A^*_{T,\pm} \Omega, B'\Omega) = (A' P_\mp \Omega, B'\Omega) \quad (1.10)$$

By comparing with D_{a^*}, we deduce that

$$\psi = A'\Omega \quad , \quad \psi' = A' P_\mp A\Omega$$

This shows that $a^*_\pm A'\Omega = \lim A^*_{T,\pm} A'\Omega = \psi'$,i.e.

$a^*_\pm$ is densely defined, hence the closability of $a_\pm$.

This finally proves (i).

For the proof of (ii), we see that with $B \in \tilde{\mathcal{A}}$, $e^{iHt} B e^{-iHt}$ is also in $\tilde{\mathcal{A}}$,i.e. has a (w,k) compact support. By using (i),

$$\lim \mathcal{U}_t A_{T,\pm}(f) \mathcal{U}_{-t} B\Omega =$$

$$= \mathcal{U}_t (\text{st.} \lim A_{T,\pm}(f) \mathcal{U}_{-t} B\Omega)$$

$$= \mathcal{U}_t a_\pm(f) \mathcal{U}_{-t} B\Omega \quad (1.11)$$

On the other hand

$$\mathcal{U}_t A_{T,\pm}(f) \mathcal{U}_{-t} = \frac{1}{(2\pi)^2} \int \tilde{h}(T(w \mp \varepsilon(K))) e^{-iwt} f(K) \tilde{A}(w,\underline{K}) \, dw \, d\underline{K} \quad (1.12)$$

(1.12) can be seen , by working out $\langle \Omega | \mathcal{U}_t A_{T,\pm}(f) \Omega \rangle$ and using $e^{iHt} |\Omega\rangle = |\Omega\rangle$, which converges on $\tilde{\mathcal{A}}$ towards

$$\lim_{T\to\infty} \frac{1}{(2\pi)^2} \int \tilde{h}(T(w \pm \varepsilon(K))) e^{-i\varepsilon(K)t} \tilde{f}(\underline{K}) \tilde{A}(w,\underline{K}) \, dw \, d\underline{K} = a_\pm (e^{\mp i\varepsilon(-i\nabla)t} f$$

$$(1.13)$$

Corollary: With B' and C' $\in \mathcal{A}'$, we have

$$a_{\pm} B' C' \Omega = B' a_{\pm} C' \Omega \qquad (1.14)$$

If $a_{\pm}$ is a bounded operator, then it is in $\mathcal{A}''$ (as $\mathcal{A}''$ (bicommutant of $\mathcal{A}$) consists of bounded operators that commute with $\mathcal{A}'$).

For $a_{\pm}$ unbounded, then $a_{\pm}$ are affiliated(+) with $\mathcal{A}''$ provided their domain of definition are sufficient nice.

The unboundedness of $a_{\pm}$, physically speaking, corresponds to bosonic character of $a_{\pm}$ (for example, Cooper pairs, phonons).

Notice in RQFT (at zero temperature) or in the ground state formalism, the $\lim_{T \to \infty} A_T \Omega$ will exist. The difference however is that the system being in temperature state, one obtains a well-defined operator $a_{\pm}$ without deformong the limit time $t \longrightarrow \pm \infty$. This shall be seen in detail in the sequel.

In the ground state formalism, one proceeds as in the Haag-Ruelle ,
that is, f(x,t) is a solution of the wave equation

$$-i \partial_t f = \varepsilon(-i \underline{\nabla}) f$$

Defining the operator according to (1.15)

$$B(x,t) := \int e^{-i\omega t} e^{i \underline{K} \cdot \underline{x}} \tilde{A}(\omega, K) \tilde{h}(t(\omega - \varepsilon(K))) \, d\omega \, d\underline{K}$$

we have then to perform the limit

(+) An operator a on a Hilbert space $\mathcal{H}$ is said to be affiliated with W^* algebra $\mathcal{A} = \mathcal{A}''$ on $\mathcal{H}$ in case $U a U^{-1} = a$ for all unitaries $U \in \mathcal{A}'$.

$$\lim_{t \to \pm\infty} \int f(x,t) B(x,t)\, dx =$$

$$= \lim_{t \to \pm\infty} \int e^{-iwt}\, e^{i\varepsilon(K)t} \int f(x,0)\, e^{i\underline{K}.\underline{x}}\, dx\, \tilde{h}(t(w \mp \varepsilon(K)))\, \tilde{A}(w,K)\, dw\, d\underline{K}$$

$$= \lim_{t \to \pm\infty} \int e^{-i(w-\varepsilon(K))t}\, \tilde{h}(t.(w \mp \varepsilon(\underline{K})))\, \tilde{f}(\underline{K})\, \tilde{A}(w,\underline{K})\, dw\, d\underline{K} \qquad (1.16)$$

provided it exists on a dense domain of definition.

For a KMS state, because of the oscillatory term $e^{-i(w - \mathcal{E}(k))t}$, the above equation would yield the same results as the one obtained from eq.(1.16).

Corollary: For a KMS state, we have $a_{in}(f) = a_{out}(f) = a(f)$, with $a(f)$ defined in (1.7), i.e. $S = 1$. (1.17)

Proof: For a non-trivial result, we have to show that $a_{in} = a_{out}$ on a dense domain of definition. Take eq.(1.16) and perform the limits $t \longrightarrow \pm\infty$ with both sides being applied to the dense set $\{A'\Omega\}$, i.e.

$$\lim_{t \to \pm\infty} \int f(x,t) B(x,t)\, dx\, B'\Omega =$$

$$= \lim_{t \to \pm\infty} \int e^{-i(w-\varepsilon(K))t}\, \tilde{h}(t(w \mp \varepsilon(\underline{K})))\, \tilde{f}(\underline{K})\, \tilde{A}(w,\underline{K})\, dw\, d\underline{K}\, B'\Omega$$

due to the concentration of $\tilde{h}_t$ along $\mu_\pm$, time dependence drops out completely for $t \longrightarrow \pm\infty$; in other words, the RHS is $B'\, a_\pm(f)\Omega$.

The LHS of (i) can be written as

$$\lim_{t\to\pm\infty} \iint e^{i\underline{k}\,\underline{x}}\, dx\, f(x,0) \int e^{i\varepsilon(k)t - i\omega t}\, d\underline{k}\, d\omega\, \tilde{h}(t\cdot(\omega \mp \varepsilon(k)))\, \tilde{A}(\omega,k)\, \bar{B}\Omega$$

$$= \lim_{t\to\pm\infty} B' \int \tilde{f}(k)\, \tilde{h}(0)\, \tilde{A}(\pm\varepsilon(\underline{k}), \underline{k})\, d\underline{k}\, \Omega =$$

$$= B' \int \tilde{f}(k)\, \tilde{A}(\underline{k}, \pm\varepsilon(k))\, d\underline{k}\, d\Omega = B' a(f)\Omega$$

Therefore,

$$a_{in}(f) = a_{out}(f) = \int \tilde{f}(k)\, \tilde{A}(\underline{k}, \pm\varepsilon(k))\, d\underline{k} = a(f)$$

Corollary: With Ω a KMS state, $A, B \in \mathcal{A}$, $B_2 \in \tilde{\mathcal{A}}$, we have $(B_1\Omega \mid A(x,t)\ B_2\Omega)$ is the Fourier transform of a measure.

Proof: Using eq.(1.9), we have $e^{\beta H/2} B\Omega = \hat{B}'\Omega$. Therefore,

$$(B_1\Omega, A(x,t) B_2\Omega) = (B_1\Omega, A(x,t)\, e^{-\beta\frac{H}{2}}\, \hat{B}_2\Omega)$$

Using again eq. (1.9), (i) can be written as

$$(B_1\Omega, A(x,t)\, J\, \hat{B}_2^*\Omega) = (B_1\Omega, A(x,t)\, \hat{B}_2'^*\Omega) =$$

$$= (\hat{B}_2' B_1\Omega, A(x,t)\Omega)$$

Proof (cont'd) : Introducing the spectral resolution of (H,P), i.e.

$$A(x,t) = \frac{1}{(2\pi)^2} \int e^{-i\omega t} e^{i\underline{K}.\underline{x}} E(d\omega . d\underline{K}) A$$

gives

$$(B_1 \Omega , A(x,t) B_2 \Omega) = \frac{1}{(2\pi)^2} \int e^{-i\omega t} e^{i\underline{K}.\underline{x}} (\hat{B}_2^{*\prime} B_1 , A\Omega) E(d\omega . d\underline{K})$$

i.e. the Fourier transform of a measure.

3.2. The Structure of the n-Point Functions.

In order to prove that the "multiparticle" states are built from excitations which asymptotically move freely (hinted by $U_t \quad (f)\, U_{-t} = a_{\pm}(e^{i\varepsilon(-i\nabla).t} f)$ eq.(1.7)), one has to look at the n-point functions, $\Omega\,(A^1_{T,\pm} \ldots\, A^n_{T,\pm}\Omega)$, and in this way one also could prove that if $A_{T,\pm} \longrightarrow a_{\pm}$, then $A^1_T \ldots A^n_T\, \Omega$ would converge towards $a^1 \ldots. a^n\, \Omega$.

To do so, one can work either in the coordinate space or in energy-momentum space. However, working in the latter case is more attractive as there is no dependence of $\tilde{h}_T$ on the shape of the dispersion law $\omega = \varepsilon\,(K)$ (since $\tilde{h}$ is concentrated along the manifolds $\mu_{\pm}$). The elements of the quasi-local algebra $\mathcal{A}$ are bounded, well-defined operators, therefore the n-point functions $W = (\Omega\;(A^1(x,t) \;\ldots\ldots\; A^n(x_n,t)\Omega)$ are continuous(+) functions, and by Bochner's theorem (18) their Fourier transforms are measures (in general complex).

The Fourier transform of $(\Omega\,(A^1(x_1,t_1) \ldots\ldots A^n(x_n,t_n)\Omega)$ is denoted by $\tilde{W}$, with $P_i := (\omega_i \,,\, k_i)$ and $X_i := (t_i \,,\, x_i)$. Translation invariance of Ω , $U\,\Omega = \Omega$, implies that W depends only the differences (18) $x_i - x_{i+1}$. Therefore,

$$\int dx_1\, dx_2 \ldots\ldots dx_n \exp\,(-i\,\Sigma\, P_i\, x_i)\; W(x_1, \ldots\ldots x_n)$$

(+) Boundedness of a function implies continuity.

$$= \int dx_1 \dots dx_n \exp\{[-i P_1(x_1-x_2) + (P_1+P_2)(x_2-x_3) + \dots$$

$$+ (P_1 + \dots P_{n-1})(x_{n-1}-x_n)]\} \exp[-i(\sum_{i=1}^{n} P_j)x_n] W(x_1-x_2, \dots, x_{n-1}-x_n)$$

$$= \int \exp[-i(\Sigma P_i)x_n] dx_n \int dx_1 \dots dx_{n-1} W(x_1-x_2, \dots x_{n-1}-x_n).$$

$$. \exp[-i P_1(x_1-x_2) + (P_1+P_2)(x_2-x_3) + \dots (P_1+\dots P_n)(x_{n-1}-x_n)]$$

$\tilde{W} := \hat{W}(P_1, P_1+P_2, \dots, P_1+P_2+\dots+P_{n-1})\ \delta(P_1+P_2+\dots+P_n)$, where a factor $(2\pi)^2$ is included in $\hat{W}$, the Fourier transform of $W(x_1-x_2, \dots, x_{n-1}-x_n)$.

The Dirac δ-measure restricts the set of variables to (4n-4)-dimensional submanifold (one condition from ω_i, and three from k_i), because of the projection of some variables $P_{i_1} + \dots + P_{i_k}$ on the vacuum Ω a further δ-function, e.g. $\delta(P_{i_1} + \dots + P_{i_k})$ will occur. So, we have the following assumption:

$\hat{\tilde{W}}(P_1, P_1+P_2, \dots, P_1+\dots+P_{n-1})$ is a (complex) measure in the variables $q_i := P_1 + \dots + P_i$.

The outcome of $\hat{\tilde{W}}$ being a measure is that its support properties are determined by the assumption made about the energy-momentum spectrum of $(H, \underline{P})$ and we have the following spectrum of $\hat{\tilde{W}}$:

i)The measure $\hat{\tilde{W}}$ contains singular, respectively singular continuous contributions with respect to each of the above variables

α) some of the $(P_1 + \dots + P_k)$ are projected on Ω ,

β) some of the $(P_1 + \dots + P_k)$ may vary over subsets with non-void interior of the submanifolds $\mu_\pm$, i.e. the subsets contain interior points.

Because of the assumption made about the spectrum of $(H, \underline{P})$ in Section (1), any subsets of $\mu := \mu_+ \cup \mu_-$ having another dimension than μ itself have zero measure (apart from $P_{i_1} + \dots\dots + P_{i_k}$ projected on Ω). (2.1)

ii)The rest of the spectrum is absolutely continuous.

Now, we come to show that in the limit $T \longrightarrow \infty$, the n-point functions $(\Omega \,|\, A^1_T \dots\dots A^n_T \,\Omega)$ converge towards the n-point functions of a quasi-free system, i.e.

$$
(\Omega \,|\, A^1_T \dots\dots A^n_T \,\Omega) \longrightarrow
\begin{cases}
\sum_P \prod_{(i_1 i_2)} (\Omega \,|\, a_{i_1} a_{i_2}) & \text{(Bosons)} \\
\sum_P \delta_P \prod_{(i_1 i_2)} (\Omega \,|\, a_{i_1} a_{i_2} \Omega) & \text{(Fermions)}
\end{cases}
\tag{2.2}
$$

where the A's are normalized to $(\Omega \,|\, A\Omega) = 0$.

Eliminating the "vacuum" from the intermediate states, we introduce the truncated n-point function defined by

$$
W(x_1, \dots, x_n) = W_c(x_1, \dots, x_n) + \sum_{I \neq 1} W_c(x_1 \dots x_{i_p}) W_c(x_{j_1} \dots x_{j_r}) \,.\, W_c(x_1 \dots x) \tag{2.3}
$$

each term in the sum corresponds to a particular way of dividing the n points $x_1, \dots\dots, x_n$ into several groups. The sum extends over all possible such groupings and the order in each grouping is the natural order.

For example,

$$
\langle \Omega \,|\, A_1(x_1) A_2(x_2) \Omega \rangle = \langle \Omega \,|\, A_1(x) A_2(x_2) \Omega \rangle_c \quad \text{with } \langle \Omega | A | \Omega \rangle = 0
$$

i.e. $$W(x_1, x_2) = W_c(x_1, x_2)$$

$$
W(x_1 x_2 x_3 x_4) = W_T(x_1 x_2 x_3 x_4)_c + W_T(x_1 x_2) W_T(x_3 x_4) + W_T(x_1 x_3) W_T(x_2 x_4) + W_c(x_1 x_4) W_c(x_2 x_3) \,. \tag{2.4}
$$

THEOREM: If the n-point functions $\tilde{W}_n$ are measures having a structure as in the above assumption, then the truncated correlation functions $(\Omega | A^1_T \ldots\ldots A^n_T \Omega)^c$ with $n \geq 3$ vanish in the limit $T \longrightarrow \infty$.

Proof: The process of truncation eliminates the contributions in $\tilde{W}_n$ located on submanifolds of "too low" dimension (i.e., Ω), so the truncated correlation functions $\tilde{W}^c_n$ will contain only singular contributions which are located on submanifolds of the type (i, β) of (3.2) and purely continuous part. Using the expression for A_T (Section 2), we have

$$(\Omega | A^1_T \ldots A^n_T \Omega)^c = \int \tilde{h}_{1,T}(p_1) \ldots \tilde{h}_{n,T}(p_n) \hat{\tilde{W}}^c(p_1, p_1+p_2, \ldots, p_1+\ldots p_{n-1}) .$$

$$. \delta(p_1 + \ldots + p_n)\, dp_1 \ldots dp_n \qquad (3.5)$$

where the test functions $\tilde{f}_1(k_1)$ ..... $\tilde{f}_n(k_n)$ are absorbed in $\tilde{h}_{1,T}$ ..... $\tilde{h}_{n,T}$. The energy-momentum support of $A_{i,T}$ being concentrated on an open set O_i of $\mu := \mu_+ \cup \mu_-$ in the limit $T \longrightarrow \infty$ (since $\tilde{h}_{i,T}(P)$ will not leave $\mu_\pm$ as $T \longrightarrow \infty$). This in turn implies that the support, say, of $(A_{2T} . A_{1T})$ will be contained in $O_1 + O_2$ (*) .

Integrating Eq.(3.5) with respect to P_n , we have

$$\int \tilde{h}_{1,T}(p_1) \ldots \tilde{h}_{n-1}(p_{n-1}) \tilde{h}_{n,T}(-(p_1 + \ldots + p_{n-1})) \hat{\tilde{W}}(p_1, \ldots, p_1+p_2+\ldots+p_{n-1}) .$$

$$. dp_1 \ldots dp_{n-1} \qquad (3.6)$$

The above limit as $T \longrightarrow \infty$ is zero on the absolutely continuous part of $\hat{\tilde{W}}^c$ (as the projection-valued measure there does not contribute as $T \longrightarrow \infty$, Section (1)).

(*) Think of the support of the convolution of 2 functions,
supp(f * g) $\subset$ supp(f) + supp(g).

In order that the above limit be different than zero, $(P_1 + \ldots\ldots + P_{n-1})$ must generate a set with non-void interior on μ for the P_i's varying over $O_i \subset \mu$.

The set has to be open, since only singular contribution of the type (i, β) can occur (singular continuous contribution in $\hat{\tilde{W}}$).

It will be shown that this is not possible for $n \geqslant 3$.

$P_1 + \ldots\ldots + P_{n-1} \in \mu$ means in particular that

$$\varepsilon(\underline{k}) = \pm\, \varepsilon(\underline{k}_1) \pm \varepsilon(\underline{k}_2) \pm \ldots \pm \varepsilon(\underline{k}_{n-1}) \,, \quad \underline{k} = \underline{k}_1 + \ldots + \underline{k}_{n-1} \tag{3.7}$$

The $\pm$ comes from $P_i \in \mu_\pm$.

By looking at eq.(3.6), we see that the number of independent variables are $4(n-1)$; but we have totake away the projectors onto the vacuum Ω which are $(n-1)$ projections (i.e., the number of independent variables are $3n-3$.

Therefore the support of the singular measure of eq.(3.6) is a submanifold of $3n-3$ dimensions in order that the integral be different than zero for $T \to \infty$.

Another constraint howeever has to be taken into account via eq.(3.7) which is $\tilde{h}_n(-(P_1 + \ldots + P_{n-1}))$, which implies $P_1 + \ldots\ldots + P_{n-1} = P$ on $\mu_\pm$.

Thus the dimension of the submanifold is $3n-4$ (the maximum number of variables).

Hence the variables are restricted to submanifolds of dimension $\leq 3n-4$.

This means that the singular support does not contribute in the limit $T \to \infty$ for $n \geqslant 3$. This proves the theorem.

The above results have some bearing with physics. The energy-momentum conservation forbids most of decays, respectively, mutual annihilation of quasi-particles respectively holes so that the corresponding phase space is restricted.

3.3 Algebraic considerations.

In this section we exploit the algebraic properties of the multiple commutators and the specific situation in the K.M.S. state.

This approach is not equivalent to that in Section (2), which was a geometric one. Here it is assumed that the joint spectrum of energy and momentum the concentration is on an hypersurface, in such a way that the strong limit

$$\text{st} \lim A_{T, I}(f)$$

exists on a dense set $\tilde{\mathcal{A}}\Omega$ with compact (ω, K)-support; and we assume that Ω is an analytic vector for $a^{\pm}(f)$.

Under these assumptions we show that $a^{\pm}(f)$ are respectively creation and annihilation operators. The followig lemma is needed to support that in the lim for $T \to \infty$, the commutator (respectively the anticommutator) tends to a C number:

Lemma : Assume that the dispersion relation $\omega = \varepsilon(K)$ is given by $\varepsilon(K) = K^2$. Assume further that for A, B fixed

$$D_{A,B}(x,t) \doteq \| [A(x,t), B]_{\pm} \|^{1/2} \qquad (3.1)$$

is integrable in x for fixed t and its Fourier transform has at most a δ - type singularity at $\varepsilon(K) = \omega$. Then

$$\lim [A_T, [B_T, C_T]_{\pm}] = 0 \qquad (3.2)$$

A, B, C all either even or odd,

$$\lim_{T \to \infty} [A, [B_T, C]_{\pm}] = 0 \qquad (3.3).$$

The proof of this lemma is based on certain commutators identities and on the Schwartz inequality for a norm $\| \ \|$. The rest are straight calculations.

This Lemma is thoroughly proved in ref. [19] .

Equation (3.3) says that $[A_T, B_T]_{\pm}$ converges to an element of the centre (which was assumed to be trivial), i.e.:

$$\lim_{T \to \infty} \left([A_T, B_T]_{,} - \lambda_T \mathbb{1} \right) = 0$$

The estimate for λ_T is missing; so the above is not sufficient to show that $a^{\pm}(f)$ are creators and annihilations operators. Therefore, to obtain the particle structure, we have to show in a modified version of the above Lemma that the truncated n point functions, $n > 2$, vanish.

To do so we can use the techniques already developed in the Buchholz theory of scattering [15] .

We are however in the K.M.S. state, where the spectrum goes from $-\infty$ to $+\infty$, so that the spectrum condition (*) does not hold.

Theorem : Let B be an operator of $\mathcal{A}''$ with stictly positive (negative) compact support in the energy, i.e.

$$B|\Omega\rangle = \int dE(\omega)\, \chi_{\Lambda}(\omega) B|\Omega\rangle \qquad \int dE(\omega) = \mathbb{1}$$

$\chi_{\Lambda}(\omega)$ is the characteristic function defined by

$$\chi_{\Lambda}(\omega) = \begin{cases} 1 & \text{if } \omega \in \Lambda \\ 0 & \text{if } \omega \notin \Lambda \end{cases}$$

with $\Lambda \subset R^{+}$ or $\Lambda \subset R^{-}$

(*) The hamiltonian H is positive

Then there exists an operator B^{+} affilated to $\mathcal{B}(\mathcal{H})$ (bounded operators on $\mathcal{H}$), s.t.

$$B_b^{+}|\Omega\rangle = B|\Omega\rangle$$
$$(B_b^{+})^{*}|\Omega\rangle = 0 \tag{3.4}$$

and

$$B_b^{+} = \begin{cases} \text{St}\lim \sum_{n=0}^{\infty} (\tau_{in} B - (\tau_{-i(n+\frac{1}{2})} B^{*})') & \text{, w positive} \\ \text{St}\lim \sum_{n=0}^{\infty} (-\tau_{-i(n+1)} B + (\tau_{i(n+\frac{1}{2})} B^{*})') & \text{, w negative} \end{cases}$$

where b stands for bosons (the constuction holds for both bosons and fermions).

The limit exists on $\mathcal{A}|\Omega\rangle$, and we have that $B' \in \mathcal{A}$ is defined by: $B'|\Omega\rangle = e^{-H/2} B^{*}|\Omega\rangle$

Remark : with $\mathcal{A}'' \cup \mathcal{A}'$ generating an irreducible C^{*} algebra the Kadison transitivity theorem, which is discussed in detail in ref. (20) , tells us that with $\Psi \perp \Omega$ we can find an hermitian $A \in \mathcal{A}'' \cup \mathcal{A}$ so that $A\Psi = \Psi$, $A\Omega = 0$. Hence with $\Psi := B\Omega$ normalized so that $(\Omega|B\Omega) = 0$, we have $(AB)\Omega = B\Omega$, $(AB)^{*}\Omega = B^{*}A\Omega = 0$ i.e. we can construct an operator B^{+} with $B^{+}\Omega = B\Omega$, $(B^{+})^{*}\Omega = 0$

Notice that the expressions for B_b^{+} say that we can split $B \in \mathcal{A}''$ into two components which look like in a certain sense creation and annihilation operators of certain excitation and respectively holes, similar to the splitting into positive and negative frequency part of ordinary field operators inFock space. (2.1)

In order to obtain the above decomposition, we write $B_b^{\dagger}(\omega)$ as a linear combination of $B(\omega)$, $(B^*(\omega))'$ i.e.

$$B_b^{\dagger}(\omega) := \alpha(\omega) B(\omega) + \beta(\omega)(B^*(\omega))' \tag{3.5}$$

where $\alpha(\omega), \beta(\omega)$ are coefficients to be determined, in such a way that

$$B_b^{\dagger}(\omega)\Omega = B(\omega)\Omega \qquad (B_b^{\dagger}(\omega))^*\Omega = 0 \tag{3.6}$$

are satisfied.

To determine α, β, we need the following relations

$$\begin{aligned} B^*(\omega) &= (B(-\omega))^* \\ B'(-\omega) &= (B(\omega))' \end{aligned} \tag{3.7}$$

(as B^* has support only for $\omega < 0$)

which follows from

$$JB\Omega = B'\Omega = e^{-H/2} B^*\Omega \tag{3.8}$$

For $\beta = 1$, $\omega \neq 0$, we have

$$\begin{aligned} (B^*(\omega))'\Omega &= (B^*(-\omega))'\Omega = JB^*(-\omega)\Omega = e^{-\omega/2} B(\omega)\Omega \\ (B_b^{\dagger}(\omega))\Omega &= (\alpha(\omega) + \beta(\omega) e^{-\omega/2}) B(\omega)\Omega \\ (B_b^{\dagger}(\omega))^*\Omega &= [\tilde{\alpha}(-\omega) + \tilde{\beta}(-\omega) e^{-\omega/2}) B^*(\omega)\Omega \end{aligned} \tag{3.9}$$

Therefore from (3.6) we arrive at the two equations

$$\begin{aligned} \alpha(\omega) + \beta(\omega) e^{-\omega/2} &= 1 \\ \tilde{\alpha}(-\omega) + \tilde{\beta}(-\omega) e^{-\omega/2} &= 0 \end{aligned} \tag{3.10}$$

with solutions

$$\alpha(w) = \frac{-1}{1-e^{w}} \quad , \quad \beta(w) = \frac{e^{w/2}}{1-e^{w}}$$

$$\tilde{\alpha}(-w) = \frac{-1}{e^{w}-1} \quad , \quad \tilde{\beta}(-w) = \frac{e^{w/2}}{e^{w}-1}$$

So, the final expression for $B_b^+(w)$ is given by

$$B_b^+(w) = \left(\frac{-1}{1-e^{w}}\right) B(w) + \left(\frac{e^{w/2}}{1-e^{w}}\right) (B^*(w))'$$

$$= \left[-\left(-\sum_{n=0}^{\infty} e^{-nw} e^{-w} B(w) + \sum_{n=0}^{\infty} e^{nw} e^{w/2} (B^*(w)) \right] $$

$$= \sum_{n=0}^{\infty} \left[-\tau_{-i(n+1)} B + \left(\tau_{i(n+\frac{1}{2})} B^* \right)' \right]$$

where $\tau_i(w) = e^{iw}$ (it looks like the time evolution).

$$\alpha(w) + \beta(w)\, e^{-w/2} = 1 \qquad (1)$$

$$\tilde{\alpha}(-w) + \tilde{\beta}(-w)\, e^{-w/2} = 0 \qquad (2)$$

Taking the complex conjugate of (2), we obtain $\alpha(w) + \beta(w)\, e^{w/2}$ (2')

Combining (1) and (2') gives

$$\beta(w)\left[e^{-w/2} - e^{w/2} \right] = 1 \qquad \therefore\ \beta(w) = \frac{e^{w/2}}{1-e^{w}}$$

$$\alpha(w) = \frac{-1}{1-e^{w}}$$

Now from

$$\alpha(w) + \beta(w)\, e^{-w/2} = 1$$

one gets

$$\tilde{\alpha}(-w) + \tilde{\beta}(-w)\, e^{w/2} = 1$$

$$\tilde{\alpha}(-w) + \tilde{\beta}(-w)\, e^{-w/2} = 0$$

Combining (1') and (2), one arrives at

$$\tilde{\beta}(-w) = \frac{e^{w/2}}{e^{w} - 1}$$

$$\tilde{\alpha}(-w) = \frac{-1}{e^{w} - 1}$$

Similarly, for $w > 0$, we obtain

$$B_b^{+}(w) = \sum_{n=0}^{\infty} \left(\tau_{in} B - \left(\tau_{-i(n+\frac{1}{2})} B^{*} \right)' \right)$$

THEOREM: If there exists a gauge transformation U such that

$U\,A|\Omega\rangle = A|\Omega\rangle$ for A even,

$U\,A|\Omega\rangle = -A|\Omega\rangle$ for A odd,

then we can construct Fermi-type operators B_f^+ for odd elements B affiliated to $\mathcal{B}(\mathcal{H})$ such that

$$B_f^+|\Omega\rangle = B|\Omega\rangle \quad , \quad (B_f^+)^*|\Omega\rangle = 0;$$

$$B_f = \text{st lim} \sum_{n=0}^{\infty} (-1)^n \left(\tau_{in} B - U(\tau_{-i(n+\frac{1}{2})} B^*)'\right) \qquad w \text{ positive}$$

$$B_f = \text{st lim} \sum_{n=0}^{\infty} (-1)^n \left(\tau_{-i(n+1)} B - U(\tau_{i(n+\frac{1}{2})} B^*)'\right) \qquad w \text{ negative.}$$

The use of such an U is reminiscent to the way Jordan and Wiegner managed to construct infinite dimensional representations of the Fermi-Dirac commutation relations (22).

Remark: The above constructions do not work for w=0; it is not clear whether there exists a satisfying extrapolation for $w \longrightarrow 0$.

The problem may be a purely technical one or be of a physical origin.

A splitting of the element B can be seen by inverting the transformations (3.12) and (3.13).

$$B = \sum_{n=0}^{\infty} \tau_{in} (1-\tau_i) B = B_b^+ + (B_b^*)^{+*}$$

$$B = \sum_{n=0}^{\infty} (-1)^n \tau_{in} (1+\tau_i) B = B_f^+ + (B_f^*)^{+*}$$

This means that we have constructed <u>densely defined</u> operators B^+ so that $B = B^+ + B^{*+*}$ holds with the B^+'s, B^{+*}'s having certain properties which are shared by creation and annihilation operators.

Next, observe that because of

$$B_i^+ \Omega = B_i \Omega \quad \text{and} \quad (B_i^+)^* \Omega = 0,$$

we replace $(\Omega, B_1 \,\ldots\ldots\, B_n \Omega)$ the operator B_n by B_n^+ and then commute it to the left, i.e. we will have an expression like

$(\Omega, \;\ldots\ldots\; B_{n-1} B_{n-2} (\Omega [B_{n-1}, B_n^*] + B_n^* B_{n-1}) \Omega)$. So,

iterating in this way, we get a sum of expressions each of them contains a commutator of the type $[B, B_n^+]$. The remaining term with B_n^+ on the extreme left vanishes because of $(B_n^+)^* \Omega = 0$.

At the end, we have only a sum of commutators taken with Ω.

For a 4-point function in the case of bosons and fermions, we have

$$W(B_1 B_2 B_3 B_4) = \left\{ \begin{array}{l} W([B_1^{*+*} B_2]_\pm [B_3, B_4^+]_\pm) \pm \\ W(B_2 [B_1^{*+*}, [B_3, B_4^+]_\pm]) \mp \\ W([B_1, B_3^+]_\pm [B_2, B_4^+]_\pm) \pm \\ W(B_1 [[B_2, B_4^+]_\pm, B_3^+]) + \\ W([B_1, B_4^+]_\pm [B_2, B_3]_\pm) \pm \\ W([[B_1, B_4^+]_\pm, B_3^+] B_2) \quad ; \end{array} \right.$$

$$\lim W_4^t (B_{1T} B_{2T} B_{3T} B_{4T}) = 0$$

If $\lim [B_{iT}^{+} , [B_{jT} , B_{kT}^{+}]] = 0$.

To show that $\lim [B_{1\,T,N}^{+} , [B_{2T} , B_{3\,T,N}^{+}]_{\pm}] = 0$, see ref. (19). This implies that $w_n^{+} (B_{1T} \ldots\ldots B_{nT}) = 0$, i.e. w is a quasi-free state on $a_1 \ldots\ldots a_n$.

From $w(A A^{*}) = | w(A) |^2$ or $w(A^{*}) = w(A)^{*}$, we have

$$\lim w([B_{1T}, B_{2T}])_{\pm} [B_{1T}, B_{2T}]^{*}) = \lim |w([B_{1T}, B_{2T}])|^2$$

which implies

$$\lim \langle \Omega | w([B_{1T}, B_{2T}]_{\pm} [B_{1T}, B_{2T}]^{*}) | \Omega \rangle =$$

$$= \lim \| w([B_{1T}, B_{2T}]) \Omega \|$$

i.e.

$$\mathrm{st} \lim [B_{1T}, B_{2T}] | \Omega \rangle =$$

$$= \lim ([B_{1T}, B_{2T}]_{\pm}) | \Omega \rangle$$

or

$$\mathrm{st} \lim w([B_{1T}, B_{2T}]) = \lim w([B_{1T}, B_{2T}])$$

i.e. a c-number (since w is a mapping from the algebra of operators into $\mathbb{R}$).

3.4. The Structure of the Multi(Quasi) - Particle States.

It was shown in the previous sections that the strong limit of A_T when T tends to infinity exists on the dense set $\tilde{\mathcal{A}}\Omega$, with $\tilde{\mathcal{A}}$ (set of operators with compact (w,k) support) and that the n-point functions desintegrate into products of two-point functions for $T \rightarrow \infty$. It was also shown that the commutators (anticommutators) of the A_T's become c-numbers. With the above results, we would like to have

$$\lim_{T \rightarrow \infty} A_T^1 \ldots\ldots A_T^n \Omega = a^1 \ldots\ldots a^n \Omega$$

<u>Corollary</u>: The weak limit of $A_T^1 \ldots\ldots A_T^n$ for $T \rightarrow \infty$ <u>exists</u>. Let it be denoted by $\varphi(A^1, \ldots\ldots, A^n)$.

<u>Proof</u>: Let $B' \in \mathcal{A}'$ (the commutant of $\mathcal{A}$).

We prove the corollary by induction :

for n=1 the statement is true, as $B' A_T^{1\,*} \Omega$ converges strongly (strong convergence implies weak convergence).

Now, suppose the statement is true for n-1. Therefore,

$$\lim_{T \rightarrow \infty} (B'\Omega , A_T^1 \ldots\ldots A_T^n \Omega) = \lim_{T \rightarrow \infty} (B' A_T^{*} , A_T^2 , \ldots . A_T^n \Omega)$$

$$= \lim_{T \rightarrow \infty} (B' P_- \Omega , A_T^2 \ldots\ldots A_T^n \Omega) =$$

$$= (B' P_- \Omega , \psi(A_2 , \ldots . A^n) \Omega) .$$

Then, the sequence $A_T^1 A_T^2 \ldots . A_T^n \Omega$ converges on the dense set of vectors $\mathcal{A}'\Omega$. But, the sequence above is uniformly bounded in T (consequence of the results described in Section 2), i.e. we have $\| A_T^1 \ldots . A_T^n \Omega \| \leq C$, and using the fact that B' is bounded, one can conclude that the weak limit exists. This follows basically from the Schwarz inequality.

Corollary: Denoting the weak limit $A_T^1 \ldots . A_T^n \Omega$ by $\psi(A^1 \ldots . A^n)$, we have

i) $\psi \in D_a$ (i.e., ψ is in the domain of a), $\bar{a}^{(+)}$;

ii) $\bar{a}\psi(A^1 \ldots . A^n) = \psi(A, A^1, \ldots, A^n)$.

Proof: To prove that ψ is in the domain of a is equivalent to proving

$$|B'\Omega \quad \psi(A', \ldots . A^n)| \leq C \|B'\|$$

which is so by uniform boundedness. So, we write similarly as before

$$\lim_{T \to \infty} (B'\Omega | A_T \cdot A_T^1 \ldots . A_T^n \Omega) = (B'\Omega | \psi(A, A' \ldots A^n) =$$

$$= (a^* B'\Omega | \psi(A' \ldots . A^n)) = (B'\Omega | a^{**}\psi(A', \ldots . A^n))$$

$$= (B'\Omega | \bar{a}\psi(A' \ldots . A^n)) \quad .$$

Then, in this way we can define the states $a^1 \ldots . a^n \Omega$ recursively, i.e.

$$\psi(A') = \bar{a}_1 \Omega$$

$$\psi(A', A^2) = \bar{a}_2 \psi(A') \qquad , \quad [A_1, A_2] = 0$$

(+) $\bar{a}$ denotes the closure of a.

Corollary: $A_T^1 A_T^2 \ldots\ldots A_T^n \Omega \xrightarrow[T]{} \psi(A^1 \ldots\ldots A^n)$ in the norm.

Proof: Using the fact that in the limit $T \longrightarrow \infty$ the n-point functions decompose into sums of products of two-point functions, i.e.

$$(\Omega | A_T^1 \ldots\ldots A_T^n \Omega) \longrightarrow$$

$$\longrightarrow \begin{cases} \sum (\Omega, a_1, a_2 \Omega) \ldots\ldots (\Omega, a_{n-1} a_n \Omega) & \text{for even } n \\ \\ 0 & \text{for odd } n \end{cases}$$

we define the scalar product between the states $\psi(A_m, \ldots\ldots, A_1)$ and $\psi(A_{m+1}, \ldots\ldots, A_n)$. This scalar product will be defined exactly in the same way as the above expression. Therefore, one can introduce the norm for $\psi(A^1 \ldots\ldots A^n)$ which is the weak limit of $A_T^1 \ldots\ldots A_T^n \Omega$,i.e.

$$\lim_{T \to \infty} \| A_T^1 \ldots\ldots A_T^n \Omega \| = \| \psi(A^1 \ldots\ldots A^n) \| \quad .$$

3.5 The Algebraic Structure.

In this section, we would like to show that the algebra of quasi-free fermions and bosons $\mathcal{A}^0$ are embedded into the whole algebra $\mathcal{A}''$.

THEOREM: If $\mathcal{A}_0$ is a quasi-free algebra of fermions, then

$$\mathcal{A}'' = \mathcal{A}_0 \otimes \mathcal{B} \quad , \quad \mathcal{B} \subset \mathcal{A}_0'$$

i.e. $\mathcal{A}''$ factors.

Proof: $\mathcal{A}_0$ has a quasi-free time evolution with the state $\omega|_{\mathcal{A}^0}$ (restricted to $\mathcal{A}^0$); its KMS state is unique (there is a unique equilibrium state for each β for a fermion gas)(23). This in turn means that there is only one phase, hence the state is a factor, mathematically this is equivalent to

$$\mathcal{A}_0'' \cap \mathcal{A}' = \lambda \mathbb{1}$$

Defining

$$\mathcal{B} = \mathcal{A}_0' \cap \mathcal{A}_0''$$

then

i.e.

$$\mathcal{A}_0'' \cap \mathcal{B} = \lambda \mathbb{1}$$

so confirming

$$\mathcal{B} \subset \mathcal{A}_0'$$

Applying the proposition 4.20 of ref.(20), which says: If $(\mathcal{M}, \mathcal{H})$ is a factor ($\mathcal{M}$ is a Von Neumann algebra), then the map defined by

$$\pi : \sum_{i=1}^{n} x_i \otimes x_i' \in \mathcal{M} \otimes \mathcal{M}' \longrightarrow$$

$$\sum_{i=1}^{n} x_i x_i' \in \mathcal{L}(\mathcal{H})$$

is an isomorphism of $\mathcal{M} \otimes \mathcal{M}'$ onto the algebra generated by $\mathcal{M}$ and $\mathcal{M}'$ with $\mathcal{M}$ being a factor, then $(\mathcal{M} \cup \mathcal{M}')'' = \mathcal{L}(\mathcal{H})$, the whole algebra of bounded operators.

The situation in our case is

$$\pi(\mathcal{A}_0 \otimes \mathcal{B}) = (\mathcal{A}_0 \cup \mathcal{B})'' = \mathcal{A}$$

where

$$\mathcal{A}_0 \equiv \mathcal{M} \quad , \quad \mathcal{B} = \mathcal{M}' \ (\textit{more-less})$$

$$\mathcal{B} \subset \mathcal{A}_0'$$

is a <u>homomorphism</u> as $\mathcal{B} \subset \mathcal{A}_0'$.

To show that π is an isomorphism we should see that $\mathcal{A}_0$ is simple (the state corresponding to it is unique). Therefore, two- sided non-trivial ideals of $\mathcal{A}_0 \otimes \mathcal{B}$ are of the form $1 \otimes I$ or $\mathcal{A}_0 \otimes \mathbb{1}$, where I is an ideal of $\mathcal{B}$.

$\pi(\mathcal{A}_0 \otimes 1)$ and $\pi(1 \otimes \mathcal{B})$ are faithful,

Since $\pi(\mathcal{A}_0)$ and $\pi(\mathcal{B})$ cannot be mapped into 0, then the kernel is trivial.

<u>THEOREM</u>: If $\mathcal{A}_0$ is a quasi-free algebra of bosons and $\mathcal{A}''$ is a factor, then .

$$\mathcal{A}'' = \mathcal{A}_0 \otimes \mathcal{B}$$

<u>Proof</u>: In the case of bosons (there are several thermodynamical phases) , hence the non-uniqueness for each temperature, therefore we have to argue differently . To show that $\mathcal{A}_0'' \cap \mathcal{A}_0' = \lambda \mathbb{1}$, which was the key in proving the previous theorem. The <u>quasi-free algebra $\mathcal{A}_0$</u> is a quasi-local algebra. Using the definition given by Lanford and Ruelle (24) for the algebra at infinity, we see that $\mathcal{A}_0'' \cap \mathcal{A}_0'$ is a subalgebra of the algebra at infinity. The proposition (1) given in (2.3) shows that the algebra at infinity is trivial, therefore

$$\mathcal{A}_0'' \cap \mathcal{A}_0' = \lambda \mathbb{1}$$

which shows that

$$\mathcal{A}'' = \mathcal{A} = \mathcal{A}_0 \otimes \mathcal{B} \quad .$$

3.6. Resonances and Quasi-Particles.

From our assumption made in Section 1 on the joint spectrum of energy and momentum where the lifetime of quasi-particles is infinite, we deduced that the S-matrix was trivial, therefore there is no freedom for perturbations and small interactions between quasi-particles.

If we have quasi-particles with a finite lifetime (e.g., resonances), the assumptions would be changed and there would be no singularity at $w = \varepsilon(k)$. The finite lifetime of the quasi-particles is expressed by an excitation branch with a certain width, but which is still peaked along the idealized energy-mpmentum curve.

Spontaneous symmetry breaking shows that the branch has to shrink to an exact $\delta(w)$ for $k=0$, i.e. the lifetime goes to infinity as k approaches 0. So, the Fourier transform of the correlation function, which is a measure, has the following form:

$$\tilde{W}(k,w) = W(k) \frac{1}{\lambda(k)} \tilde{h}\left(\frac{w-\varepsilon(k)}{\lambda(k)}\right)$$

where $\lambda(k) \longrightarrow 0$ for $k \longrightarrow 0$.

$\tilde{w}(k,w)$ without $w(k)$ is a sequence in w with $\{k\}$ as the index set.

$\lambda(k)$ is the measure of the width of the excitation branch.

It turns out that the spectral concentration of energy and momentum implies that the resonance can be approximated by a creation or annihilation operator applied to the equilibrium state, and the real time evolution by a quasi-free one.

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