ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Thesis submitted for the degree of

"Magister Philosophiae"

CONSISTENT DIMENSIONS OF STRING THEORIES

Candidate:

Ömer Faruk DAYI

Supervisor:

Prof.R.RAMACHANDRAN

Academic Year: 1984/85

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

CONSISTENT DIMENSIONS OF STRING THEORIES

I am grateful to Prof.R.RAMACHANDRAN for suggesting the subject,for fruitful discussions and for reading the manuscript.

CONTENTS:

INTRODUCTION 1

I. COVARIANT FORMULATION OFBOSONIC STRING

1.Equations of motion 2

2.Solution of equations of motion 4

3.Hamiltonian formalism and quantization 6

4.No-Ghost theorem 10

II. NEVEU-SCHWARZ AND RAMOND MODELS OF SPINNING STRING

1.Equations of motion and their solutions 19

2.Quantization 22

3.No-Ghost theorem 28

III. FACTORIZATION OF CLOSED STRING 33

IV. LIGHT-CONE GAUGE FORMULATION

1.Bosonic String 46

2.Spinning string 54

V. CONFORMAL INVARIANCE OF BOSONIC STRING 61

CONCLUSIONS 71

REFERENCES 72

INTRODUCTION:

Original dual theory introduced by Venetziano[1] was in four dimensions. After the calculation of one loop amplitudes[2] the problem of unitarity of S-matrix was raised and Lovelace[3] showed that unitarity is regained if dimension of space-time is 26, ground state is a tachyon and two complete sets of oscillators are eliminated from the physical Fock space. In the Neveu-Schwarz and Ramond models (abriviated as N.S.M and R.M. respectively) of spinning string consistent dimension (or critical dimension) showed up as 10. In ref.(4) the N.S.M. and R.M. are combined by taking the even G-parity part of N.S.M. and imposing both Majorana and Weyl conditions on the fields of R.M. (the latter is possible only in dimension=2 mod 8). This construction led to the superstring formulation[5] which is introduced in 10 dimensions. The superstring theories (including heterotic string) because of being finite and anomaly free theories for SO(32) and $E_8 \times E_8$ gauge groups and includes all the interactions , became a candidate for the theory of everything[6].

In this work we want to probe the reasons of consistent dimensions. Bosonic string theory gives all the formalism which one uses in superstrings and also in ref.(7) it is shown that one can obtain 10 dimensinal superstring theories from 26 dimensional bosonic string. Thus understanding of bosonic string is crucial. Because of these reasons we will give all the methods of obtaining the consistent dimension for bosonic string in detail. All of these methods can be applied to spinning string case also. Usually generalizing to the latter case is quite easy after having a good understanding of bosonic one. Thus we will neither give all the methods for spinning string theory nor be very precise in the methods which we give.

In refs.(5,8) it was shown that combination of N.S.M. and R.M. with the conditions of ref.(4) ,which we told above ,is equivalent to superstring. Hence the consistent dimension of N.S.M.-R.M. is directly the consistent dimension of superstrings also (it is also possible to show , only by using the superstring formulation, that superstring is consistent only in 10 dimensions).

I.COVARIANT FORMULATION OF BOSONIC STRING[9]:

I.1.Equations of motion :

The simplest step beyond apoint like object is a one -dimensional object.i.e. a string.During its evolution it spans a two-dimensional space which is called 'world sheet' and parametrized with σ , τ internal coordinates:$x^{\mu} = x^{\mu}(\sigma , \tau)$. μ is D-dimensional space-time index.We always take τ as an evolution, σ as a kinematical parameter and impose

$$\left(\frac{\partial x^{\mu}}{\partial \tau} \right)^2 \leqslant 0 \qquad\qquad \left(\frac{\partial x^{\mu}}{\partial \sigma} \right)^2 \geqslant 0$$

(we use a metric $\eta^{oo} = - \eta^{ii} = -1$, i=1,2,. . . ,D-1 and $\hbar = c = 1$ units).The area element for a two-dimensional surface embedded in space time is

$$d^2 A = \left\{ \left(\frac{\partial x^{\mu}}{\partial \tau} \frac{\partial x_{\mu}}{\partial \sigma} \right)^2 - \left(\frac{\partial x^{\mu}}{\partial \sigma} \right)^2 \left(\frac{\partial x^{\nu}}{\partial \tau} \right)^2 \right\}^{1/2} d\sigma \, d\tau \geqslant 0$$

Hence we postulate the following action for string[10]

$$S = - \frac{1}{2\pi\alpha'} \int_{-\infty}^{\infty} d\tau \int_{0}^{\pi} d\sigma \left[(\dot{x}\cdot x')^2 - x'^2 \dot{x}^2 \right]^{1/2} \tag{1.1}$$

where dot and prime indicate,respectively,derivatives with respect to τ and σ ,the space-time index is suppressed. α' is called Regge slope parameter and has $(\text{mass})^{-2}$ dimension(string tension T is related to α' as $T = 1/2\pi\alpha'$).

From the principle of least action :

$$x_{\mu} \longrightarrow x_{\mu} + \delta x_{\mu} (\sigma , \tau)$$

$$\delta S = \int d\tau \, d\sigma \left[\frac{\partial L}{\partial \dot{x}_{\mu}} \frac{d}{d\tau} \delta x_{\mu} + \frac{\partial L}{\partial x'_{\mu}} \frac{d}{d\sigma} \delta x_{\mu} \right]$$

$$= \int_{0}^{\pi} d\sigma \frac{\partial L}{\partial \dot{x}_{\mu}} \delta x_{\mu} \Big|_{\tau = -\infty}^{\infty} + \int_{-\infty}^{\infty} d\tau \frac{\partial L}{\partial x'_{\mu}} \delta x_{\mu} \Big|_{\sigma = 0}^{\pi} - \int d\sigma \, d\tau \left(\frac{\partial}{\partial \tau} \frac{\partial L}{\partial \dot{x}_{\mu}} + \frac{\partial}{\partial \sigma} \frac{\partial L}{\partial x'_{\mu}} \right) \delta x_{\mu} = 0 \tag{1.2}$$

During the variation of trajectory we keep the initial and final points of the string fixed :

$$\delta x_\mu(\tau=-\infty) = \delta x_\mu(\tau=\infty) = 0$$

Hence (1.2) leads to:

(i) edge condition : (a) $\dfrac{\partial L}{\partial x'_\mu} = 0$ at $\sigma = 0\,,\ \pi$ (1.3)

or (b)
$$\delta x_\mu(\sigma=0) = \delta x_\mu(\sigma=\pi)$$
$$\frac{\partial L}{\partial x'_\mu}(\sigma=0) = \frac{\partial L}{\partial x'_\mu}(\sigma=\pi) \qquad (1.4)$$

(ii) eqs. of motion :

$$\frac{\partial}{\partial \tau}\frac{\partial L}{\partial \dot{x}_\mu} + \frac{\partial}{\partial \sigma}\frac{\partial L}{\partial x'_\mu} = 0 \qquad (1.5)$$

The string which is defined with the edge condition (a) is called open string and the one with (b) is called closed string.

Instead of keeping the initial and final positions fixed,we may perform the variation by allowing only the motions which satisfy (1.5).For instance by performing a translation one can see that total momentum of string,defined as:

$$P^\mu = \int_{(c)} (d\sigma\, P^\mu_\tau + d\tau\, P^\mu_\sigma) = \int_0^\pi d\sigma\, P^\mu_\tau \qquad (1.6)$$

is conserved.In (1.6) (c) is any curve going from one boundary to the other and

$$P^\mu_\tau = \frac{\partial L}{\partial \dot{x}_\mu} \qquad P^\mu_\sigma = \frac{\partial L}{\partial x'_\mu} \qquad (1.7)$$

One the same lines one can show that angular momentum current is conserved,by performing an infitisimal Lorentz transformation.

In the following work we will use the open string edge condition (1.3) and when necessary give the results of closed string directly.

(1.3) and (1.5) written explicitly are

(i) edge condition: $$\frac{(\dot{x}\cdot x')\,\dot{x}_\mu - \dot{x}^2 x'_\mu}{\left[(\dot{x}\cdot x')^2 - x'^2\dot{x}^2\right]^{1/2}} = 0 \qquad \text{at } \sigma = 0\,,\pi \tag{1.8}$$

(ii) eq. of motion:

$$\frac{\partial}{\partial\tau}\left\{\frac{(\dot{x}\cdot x')\,x'_\mu - x'^2\dot{x}_\mu}{\left[(\dot{x}\cdot x')^2 - x'^2\dot{x}^2\right]^{1/2}}\right\} + \frac{\partial}{\partial\sigma}\left\{\frac{(\dot{x}\cdot x')\,\dot{x}'_\mu - \dot{x}^2 x'_\mu}{\left[(\dot{x}\cdot x')^2 - x'^2\dot{x}^2\right]^{1/2}}\right\} = 0\,. \tag{1.9}$$

Under this form they are unsolvable.However we have also following identities,coming from the definition of momentum densities :

$$P^\mu_\tau\cdot x'_\mu = 0 \qquad P^2_\tau + x'^2/4\pi^2\alpha'^2 = 0$$
$$P^\mu_\sigma\cdot \dot{x}_\mu = 0 \qquad P^2_\sigma + \dot{x}^2\;/4\pi^2\alpha'^2 = 0 \tag{1.10}$$

At $\sigma = 0\,,\pi$ $P^\mu_\sigma = 0$ so $\left.\dot{x}^2\right|_{\sigma=0,\pi} = 0$ .i.e. the end points of the string move at the speed of light.

I.2. Solution of the equations of motion :

One can explicitly show that action (1.1) is invariant under reparametrization of world sheet,however that is a direct cosequence of the definition:two-dimensional area embedded in space-time.Hence we choose a parametrization which simplifys the eqs. of motion. The simplest choice is an orthonormal system of coordinates on the surface.i.e.

$$x'\cdot\dot{x} = 0 \qquad x'^2 + \dot{x}^2 = 0\;. \tag{1.11}$$

In this parametrization

$$P^\mu_\tau = \frac{1}{2\pi\alpha'}\,\dot{x}^\mu \qquad P^\mu_\sigma = -\frac{1}{2\pi\alpha'}\,x'^\mu$$

so the eqs. of motion and edge conditions are

$$\ddot{x}_\mu - x''_\mu = 0$$
$$x'_\mu = 0 \quad \text{at } \sigma = 0\,,\pi\;. \tag{1.12}$$

General solution with these edge conditions is :

$$x_\mu = \sum_{n=0}^{\infty} X_{n\mu}(\tau)\cos n\sigma$$

Thus eqs. of motion read

$$\ddot{x}_n^\mu + n^2 x_n^\mu = 0 \qquad n = 0,1,2,\ldots \tag{1.13}$$

Let us introduce a_n^μ modes defined as :

$$a_n^\mu(\tau) = \frac{1}{2(2\alpha' n)^{1/2}}\left(\dot{x}_n^\mu(\tau) - in\,x_n^\mu(\tau)\right) \tag{1.14}$$

By making use of this definition and (1.13) we get for $n \neq 0$:

$$\begin{aligned} a_n^\mu(\tau) &= a_n^\mu\, e^{-in\tau} \\ a_n^{*\mu}(\tau) &= a_n^{*\mu}\, e^{in\tau} \end{aligned} \tag{1.15}$$

where $a_n^\mu \equiv a_n^\mu(0)$ and $a_n^{*\mu} \equiv a_n^{*\mu}(0)$. In $n = 0$ case

$$\ddot{x}_0^\mu = 0$$

has for solution

$$x_0^\mu = q^\mu + 2\alpha' p^\mu \tau \tag{1.16}$$

where p^μ is center of mass momentum defined as :

$$p^\mu = \frac{1}{2\pi\alpha'}\frac{d}{d\tau}\int_0^\pi d\sigma\; x^\mu(\sigma,\tau) \tag{1.17}$$

Finally, (1.14) – (1.16) lead to :

$$x^\mu(\sigma,\tau) = q^\mu + 2\alpha' p^\mu \tau - i(2\alpha')^{1/2}\sum_{n=1}^{\infty}\frac{1}{n^{1/2}}\left(a_n^{*\mu} e^{in\tau} - a_n^\mu e^{-in\tau}\right)\cos n\sigma \tag{1.18}$$

as solution of (1.12).

The eqs. of motion are solved but one must still take the constraints (1.11) into account which can be rewritten as :

$$(\dot{x} - x')^2 = 0 \qquad (\dot{x} + x')^2 = 0 \qquad \text{for} \quad 0 \leq \sigma \leq \pi$$

However if we extend analytically $x_\mu(\sigma,\tau)$ from $0 \leq \sigma \leq \tau$ to $-\pi \leq \sigma \leq \pi$ by

$$\dot{x}_\mu(-\sigma) = \dot{x}_\mu(\sigma) \qquad , \qquad x'_\mu(-\sigma) = -x'_\mu(\sigma) \tag{1.19}$$

we can unify the constraints as :

$$(\dot{x} + x')^2 = 0 \qquad \text{for} \qquad -\pi \leq \sigma \leq \pi \tag{1.20}$$

By defining

$$\alpha_0^\mu = 2i\alpha' p^\mu \qquad \alpha_n^\mu = (2\alpha')^{1/2}\, n^{1/2}\, a_n \qquad \alpha_{-n}^\mu = a_n^{*\mu} \tag{1.21}$$

as new modes,(1.20) reads

$$(\dot{x} + x')^2 = 4\alpha' \sum_{n=-\infty}^{\infty} e^{-in(\tau+\sigma)}\, L_n$$

where

$$L_n = \frac{1}{4\alpha'} \sum_{m=-\infty}^{\infty} \alpha^\mu_{n-m}\, \alpha_{m\mu} \tag{1.22}$$

Hence the constraints expressed as an infinite set of initial conditions

$$L_N = \frac{1}{4\alpha'} \sum_{n=-\infty}^{\infty} \alpha^\mu_{N-n}\, \alpha_{n\mu} = 0 \tag{1.23}$$

In particular $L_0 = 0$ gives the mass shell condition

$$m^2 = p_\mu p^\mu = \frac{1}{2\alpha'} \sum_{n=1}^{\infty} n\, a_n^{*\mu}\, a_{n\mu} \tag{1.24}$$

I.3.Hamiltonian formalisim and quantization:

Hamiltonian of a system which has constraints is given as[11]

$$H = H_0 + \sum_\alpha v_\alpha \phi_\alpha(p_i, q_j)$$

H_o is the canonical hamiltonian of the system. ϕ_α are primary constraints expressed as $\phi_\alpha = 0$. ϑ_α are constants in coordinates q_j and momentum p_i and their choice is equivalent to a gauge choice.The "time" evolution of an arbitrary function is given by

$$\dot{f}\,(p,q) = \left\{ f , H \right\} + \frac{\partial f}{\partial \tau}$$

An observable of the system is a function such that at given time

$$\left\{ f , \phi_\alpha \right\} = \sum_\beta d_{\alpha\beta}\, \phi_\beta$$

For applying this to string we assume equal time canonical Possion brackets :

$$\left\{ x^\mu(\sigma) , x^\nu(\sigma') \right\} = \left\{ P^\mu(\sigma) , P^\nu(\sigma') \right\} = 0$$

$$\left\{ x^\mu(\sigma) , P^\nu_\tau(\sigma') \right\} = \eta^{\mu\nu}\, \delta(\sigma - \sigma')$$

By extending σ interval from $[0 , \pi]$ to $[-\pi , \pi]$ by

$$P^\mu_\tau(-\sigma) = P^\mu_\tau(\sigma) \qquad x'_\mu(-\sigma) = -x'_\mu(\sigma) \quad \text{for} \quad [-\pi , 0]$$

the constraints can be expressed as:

$$L_n = \frac{1}{4} \int_{-\pi}^{\pi} d\sigma \; e^{in\sigma} \left[\pi\,(2\alpha')^{1/2}\, P^\mu_\tau + \frac{x'^\mu}{(2\alpha')^{1/2}} \right]^2 = 0 \qquad (1.25)$$

Possion bracket algebra of L_n closes as it should :

$$\left\{ L_n , L_m \right\} = i\,(m-n)\, L_{m+n}$$

Canonical hamiltonian reads to zero,thus:

$$H = \sum_{n=-\infty}^{\infty} \vartheta_n L_n$$

One can show that the orthonormal parametrization, seen before, is equivalent to the choice:

$$H = L_0 \tag{1.26}$$

Expanding as before

$$x^\mu = q^\mu - i\alpha_0^\mu \tau + i \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu(\tau) \cos n\sigma \tag{1.27}$$

$$P_\tau^\mu = \frac{1}{2\pi\alpha'} \left\{ -i\alpha_0^\mu + \sum_{n\neq 0} \alpha_n^\mu(\tau) \cos n\sigma \right\} \tag{1.28}$$

we can check that

$$\{ \alpha_n^\mu, \alpha_m^\nu \} = -2in\alpha' \eta^{\mu\nu} \delta_{m+n,0}$$

$$\{ q^\mu, \alpha_0^\nu \} = 2i\alpha' \eta^{\mu\nu}$$

or equivalently

$$\{ a_n^\mu, a_m^{\nu +} \} = -i \eta^{\mu\nu} \delta_{n,m}$$

$$\{ q^\mu, p^\nu \} = \eta^{\mu\nu}$$

Quantization is achieved by using the replacement

$$i \{ \text{Poisson bracket} \} \longrightarrow [\text{commutator}]$$

and taking the normal products. Thus we postulate

$$[a_n^\mu, a_m^{\nu +}] = \eta^{\mu\nu} \delta_{n,m} \tag{1.29}$$

$$[q^\mu, p^\nu] = i \eta^{\mu\nu} \qquad [x^\mu(\sigma,\tau), P_\tau^\nu(\sigma',\tau)] = \eta^{\mu\nu} \delta(\sigma - \sigma') \tag{1.30}$$

Introduce a $|0,p\rangle \equiv |0\rangle$ vector such that

$$a_n^\mu |0,p\rangle = 0 \quad ; \quad P_{op}^\mu |0,p\rangle \doteq p^\mu |0,p\rangle$$

Fock space is now spanned with the vectors of the type

$$\prod_n a^{+\lambda_n}_{n,\mu_n} |0\rangle \tag{1.31}$$

This space has an indefinite metric because of $\eta^{\mu\nu}$ which appers in (1.29),(1.30).

We now want to find the quantized forms of the constraints.For this aim we introduce new fields defined as [12] (in $\alpha' = 1/2$ units) :

$$Q^{\mu}(z) = q^{\mu} - ip^{\mu} \ln z - i \sum_{n=1}^{\infty} \frac{1}{n^{1/2}} \left(a^{\mu+}_{n} z^{n} - a^{\mu}_{n} z^{-n} \right) \tag{1.32}$$

$$P^{\mu}(z) = iz \frac{dQ^{\mu}(z)}{dz} = p^{\mu} + \sum_{n=1}^{\infty} n^{1/2} \left(a^{\mu+}_{n} z^{n} + a^{\mu}_{n} z^{-n} \right) \tag{1.33}$$

These have the following relations with the fields introduced before :

$$Q^{\mu}(e^{i\tau}) = x^{\mu} (\sigma = 0 , \tau)$$

$$P_{\mu}(z) = (\dot{x}_{\mu} + x'_{\mu})(\sigma , \tau) \qquad \text{for } z = \exp i(\tau + \sigma)$$

From (1.25) one deduces that

$$L_n = \frac{1}{4\pi} \int_{-\pi}^{\pi} d\sigma \, e^{in\sigma} : P^2 \left(\exp i(\tau + \sigma) \right) : \tag{1.34}$$

where the : : indicates normal ordering.It can be easly checked that L_n is τ indipendent,hence can be expressed as :

$$L_n = \frac{1}{2} \oint \frac{dy}{2 \; iy} y^{n} : P^2(y) : \tag{1.35}$$

where the integration is performed around the origin.Now by using (1.35) form of the constraints one can show that they form the Virasoro algebra :

$$\left[L_n , L_m \right] = (n-m) L_{n+m} + \frac{D}{12} n(n^2 -1) \delta_{n+m,o} \tag{1.36}$$

where D is space-time dimension.There is an ambigutiy in the definition of L_0 after normal ordering because of zero mode.For fixing this ambiguity we define the constraints on Fock space vectors as :

$$\left[L_n - \alpha(0)\,\delta_{n,0} \right] |\phi\rangle = 0 \qquad \text{for all } n \tag{1.37}$$

where $\alpha(0)$ is an arbitrary c-number.

I.4.No-Ghost theorem :

For the Fock space which is not positive definite and spanned by the states

$$|r\rangle = \prod_{n=1}^{\infty} \prod_{\mu_n=0}^{D-1} \left(a_n^{\dagger\,\mu_n} \right)^{\lambda_{n,\mu_n}} |0\rangle \tag{1.38}$$

the number operator is given as :

$$R = \sum_{n=1}^{\infty} n\, a_n^{\dagger\,\mu_n}\, a_n^{\mu_n} \tag{1.39}$$

Its effect on a vector $|r\rangle$ is :

$$R\,|r\rangle = M\,|r\rangle \tag{1.40}$$

where

$$M = \sum_{n=1}^{\infty} \sum_{\mu_n=0}^{D-1} n\, \lambda_{n,\mu_n} \quad , \quad M = 1,2,3,\ldots \tag{1.41}$$

is the level number of $|r\rangle$ .Now define R^M as the space of states at the level M. We know that physical states must satisfy (1.37),from the $n=0$ part (mass-shell condition) see that an on-shell state belonging to R^M has mass :

$$p^2 = 2(\,M - \alpha(0)) \tag{1.42}$$

Define "spurious states " as those states belong to R^M of the form :

$$| s \rangle = \sum_{n=1}^{\infty} c_n L_{-n} | s_n \rangle \tag{1.43}$$

where c_n's are c-numbers, $| s_n \rangle$ is an element of Fock space which does not have L_{-n} operator.They are orthogonal to any physical state :

$$\langle s | \phi \rangle = 0 \tag{1.44}$$

We now want to build a subspace of R^M which has obviously a positive norm: First define transverse states as follows(for the moment we take $\alpha(0) = 1$ without giving any reason but we will justify it later) : The momentum of them is constrained by $p^2 = 2(N - 1)$,thus we may choose a frame where

$$p = \left(\sqrt{2} (1 - N/2), 0 , \ldots , 0 , N/\sqrt{2} \right)$$

We then pick up the lightlike vector

$$k = (- 1/\sqrt{2}, 0, \ldots , 0, 1/\sqrt{2}) \tag{1.45}$$

such that $k\,p = 1$ and define

$$K_n = k \cdot \alpha_n = k \cdot \sqrt{n}\, a_n \qquad\qquad K_n^{\dagger} = K_{-n} \tag{1.46}$$

We now define transverse states through the conditions :

$$L_n | t \rangle = K_n | t \rangle = 0 \qquad\qquad n \geqslant 1 \tag{1.47}$$

Transverse states have positive semi-definite norm.

Before giving the theorem for the absance of ghosts we need to prove three lemmas [14]:

Lemma 1 : Let us consider a transverse state $|t\rangle \in R^M$ such that $\langle t|t\rangle \neq 0$.For such a $|t\rangle$ the states

$$|\{\lambda,\mu\},t\rangle = L_{-1}^{\lambda_1} \ldots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \ldots K_{-m}^{\mu_m} |t\rangle \tag{1.48}$$

span some subspace of R^N and are linearly independent,where

$$N = M + \sum_{r=1}^{n} r\lambda_r + \sum_{s=1}^{m} s\mu_s > M \tag{1.49}$$

and this subspace contains no solutions of $L_n|\phi\rangle = K_n \, |\phi\rangle = 0$.

Proof : (a) Suppose that we have linear relation betweenthe vectors $|\{\lambda,\mu\},t\rangle$

$$\left[\sum_{[\lambda,\mu]} c\,[\lambda,\mu]\; L_{-1}^{\lambda_1} \ldots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \ldots K_{-m}^{\mu_m}\right]|t\rangle = 0$$

$|t\rangle$ is obtained from vacuum by applying creation operators,hence for linear dependence the terms which have only creation operators,after expanding L's and K's in terms of a and $a^\dagger$, must cancel with each other.If we pick the $a_1^\dagger$ oscillators :

$L_{-1}^{\lambda_1} \ldots L_{-n}^{\lambda_n} K_{-1}^{\mu_1}$ contributes a term of the form

$$(p\cdot a_1^\dagger)^{\lambda_1}(a_1^\dagger\cdot a_1^\dagger)^{\lambda_2}(a_1^\dagger\cdot a_2^\dagger)^{\lambda_3}\ldots(a_1^\dagger\cdot a_{n-1}^\dagger)^{\lambda_n}(k\cdot a_1^\dagger)^{\mu_1}$$

and another set of $\{\lambda,\mu\}$ with same N which is called $\{\lambda',\mu'\}$ will give the following contribution in the same case :

$$(p\cdot a_1^\dagger)^{\lambda_1'}(a_1^\dagger\cdot a_1^\dagger)^{\lambda_2'}(a_1^\dagger\cdot a_2^\dagger)^{\lambda_3'}\ldots(a_1^\dagger\cdot a_{n-1}^\dagger)^{\lambda_n'}(k\cdot a_1^\dagger)^{\mu_1'}$$

The terms which maximize the sum $\lambda_1 + 2\lambda_2 + \ldots + \lambda_n + \mu_1$ must cancel each other, but the bases of each λ are different so this can be achieved if they have same λ_n (i.e.

if $\lambda_1' + 2\lambda_2' + \dots + \lambda_n' + \mu_1'$ is also maximized $\lambda_1 = \lambda_1', \dots, \lambda_n = \lambda_n', \mu_1 = \mu_1'$)
But this means that there is only one set $\{\lambda\}$ so there can not be any cancelation unless $\lambda_1 = \lambda_2 = \dots = \lambda_n = \mu_1 = 0$. If one uses the same reasoning for the other oscillators also, he can see that $c[\lambda, \mu] = 0$ unless $\lambda_1 = \dots = \lambda_n = \mu_1 = \dots = \mu_m = 0$ but $N - M > 0$ thus the latter can not be satisfyed.

(b) Let us now prove the second part of the lemma. It is easy to see that

$$[L_m, K_n] = -n K_{m+n} \tag{1.50}$$

$$[K_m, K_n] = 0 \tag{1.51}$$

Apply K_n to $|\{\lambda,\mu\}, t\rangle$ and commute it till yields $K_n|t\rangle = 0$. But during this process some K_r $r < 0$ will be created (see 1.50) so we end up with

$$K_n \,|\{\lambda,\mu\}, t\rangle = \sum_{\lambda',\mu'} c[\lambda', \mu'] \,|\{\lambda',\mu'\}, t\rangle$$

where $\mu' = \sum s\mu_s' \geq \mu = \sum s\mu_s$. If $n = 1$ the only commutator which does not produce K_r $r < 0$ is $[K_1, L_{-1}] = 1$. In this case μ is not increased.

Let us consider a superposition of $|\{\lambda,\mu\}, t\rangle$ vectors with fixed $|t\rangle$ and N, then ask this superposition be annihilated with all K_n. The terms with maximal μ will be mapped to greater μ so they must cancel each other, but as we have seen in (a), this is impossible. Hence we conclude that $\lambda_2 = \lambda_3 = \dots = \lambda_n = 0$. Applying K_1 on a term which contains $L_{-1}^{\lambda_1}$ gives $\lambda_1 L_{-1}^{\lambda_1 - 1} K_{-1}^{\mu_1} \dots K_{-m}^{\mu_m} |t\rangle$ so λ_1 must also zero. Hence $|\phi\rangle$ must be as follows :

$$|\phi\rangle = \sum_{\mu} c[\mu] K_{-1}^{\mu_1} \dots K_{-m}^{\mu_m} |t\rangle$$

when L_n is applyed on this state, it gives a superposition of $|\{0,\mu'\}, t\rangle$ vectors where μ' is increased. Thus the only solution of $L_n|\phi\rangle = 0$ is $\mu_1 = \dots = \mu_m = 0$, which is forbidden (see 1.49). This concludes the proof of the first lemma.

In the first lemma we started from a transverse state of R^M and constructed an independent set of vectors belonging to R^N where $N = M + \sum_r r\lambda_r + \sum_s s\mu_s > M$. These vectors are orthogonal to transverse vectors of R^N. Further this subspace does not contain any transverse vectors itself.

In order to obtain a complete basis for R^N we still have to vary $|t\rangle$ and this is done in the following lemma:

Lemma 2 : If $|t,M,v\rangle$ is an orthonormal basis for T^M the $|\{\lambda,\mu\},t,M,v\rangle$ states, defined like (1.48), give a basis for R^N where $N = M + \sum r\lambda_r + \sum s\mu_s$ and as N varies for the whole Fock space. Further T^M is positive definite.

Proof: $|t,M,v\rangle$'s are transverse states which belong to R^N and they form a subspace of it called T^N. We also define G^N as set of all $|\{\lambda,\mu\},t,M,v\rangle \in R^N$ such that $N > M$. We construct T^N and G^N as follows :

Start from $N = 0$ .i.e. the vacuum. It satisfies $L_n|0\rangle = K_n|0\rangle = 0$ for $n \geq 1$. Hence

$$T^0 = \{|0\rangle\} \quad , \quad G^0 = \{\emptyset\} \quad \text{(=null set).}$$

From these construct G^1 as:

$$G^1 = \{L_{-1}|0\rangle \ , \ K_{-1}|0\rangle\}$$

Orthogonal complement of it is made of states annihilated by $L_1 = L_{-1}^\dagger$, $K_1 = K_{-1}^\dagger$:

$$T^1 = \{a_{1,i}^\dagger|0\rangle \ ; \ i = 1, \ldots, D-2\}$$

Now build G^2 as:

$$G^2 = \{L_{-1}a_{1,i}^\dagger|0\rangle, L_{-2}|0\rangle, K_{-1}a_{1,i}^\dagger|0\rangle, K_{-2}|0\rangle, L_{-1}^2|0\rangle, K_{-1}^2|0\rangle, L_{-1}K_{-1}|0\rangle\}$$

Then construct its orthogonal complement and so on. Assume that we have been able to construct $T^0, T^1, \ldots, T^{N-1}$ and $G^0, G^1, \ldots, G^{N-1}$ which is true for $N = 1,2$. We now attempt to construct G^N and T^N.

By applying L_{-n} and K_{-n} to $T^0, T^1, \ldots, T^{N-1}$ we construct G^N. All the states thus obtained are $|\{\lambda,\mu\}, t, M, v\rangle$ states. For fixed $|t,M,v\rangle$, in Lemma 1, we showed that they are linearly independent. For the states with different $|t,M,v\rangle$ we have

$$\langle \{\lambda',\mu'\}, t', M', v' \mid \{\lambda,\mu\}, t; M, v\rangle = \delta_{MM'}\, \delta_{vv'}\; f(\lambda,\lambda',\mu,\mu',M,v)$$

To see this commute all L_{-n}, K_{-m} to left, L_n, K_m to right untill they annihilate. The only surviving contributions will come from c-number or L_0 operators which are diagonal in $|M,v\rangle$. Thus we get $\delta_{MM'}$, $\delta_{vv'}$. $\delta_{MM'}$ is obvious and $\delta_{vv'}$ is the result of selecting an orthonormal basis for $T^0, T^1, \ldots, T^{N-1}$ by hypothesis. Thus we conclude that $|\{\lambda,\mu\}, t, M, v\rangle$ states are linearly independent basis for G^N $(N > M)$.

Now we construct T^N as orthogonal complement of G^N and show that it is made of transverse states. $|t,N,v\rangle$ belongs to T^N so it should be orthogonal to all vectors $|\{\lambda,\mu\}, t, M, v\rangle$ because of their independence just proved. Hence

$$\begin{aligned}\langle t,N,v \mid L_{-1}^{\lambda_1} \ldots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \ldots K_{-m}^{\mu_m} \mid t, M, v'\rangle &= \\ = \langle t,N,v \mid L_{-1} \left[L_{-1}^{\lambda_1 - 1} \ldots \right] \mid t, M, v'\rangle &= 0\end{aligned} \tag{1.52}$$

for all $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m,$ M, v. The vectors

$$L_{-1}^{\lambda_1 - 1} \ldots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \ldots K_{-m}^{\mu_m} \mid t, M, v'\rangle$$

form a complete basis for R^{N-1} so they are linearly independent. Thus :

$$\langle t,N,v \mid L_{-1} = 0$$

identically.Another independent basis of G^N is obtained by commuting L_{-n}'s with each other and ordering them as $L_{-2}^{\lambda_2} L_{-1}^{\lambda_1} \ldots L_{-n}^{\lambda_n} \cdot L_{-2}^{\lambda_2 -1} L_{-1}^{\lambda_1} \ldots L_{-n}^{\lambda_n} |t,M,v'\rangle$ form a complete basis for R^{N-2} .Thus,like before,follows from this that

$$\langle t,N,v \mid L_{-2} = 0$$

identically.We coclude that

$$\langle t,N,v \mid L_{-n} = 0$$

If we put all the λ_n's equal to zero ,(1.52) reads :

$$\langle t,N,v \mid K_{-1}^{\mu_1} \ldots K_{-m}^{\mu_m} |t,M,v'\rangle = 0$$

On the same lines as before we conclude that

$$\langle t,N,v \mid K_{-m} = 0$$

We have used the fact that we can choose an orthonormal basis for T^N .But T^N is positive semi-definite so there may be some zero norm states which do not permit this choice.If $|t\rangle$ is a zero norm state belonging to a semi-positive Hilbert space it must be a null state(i.e.orthogonal to all states including itself).If there is such a state in T^N it would be also in G^N .i.e. In G^N there would be transverse state.But lemma 1 prohibts this,thus T^N contains no zero norm state that we may always apply orthonormalization procedure.

Hence we have been able to construct a positive definite subspace T^N for each value of R.Also we have constructed a complete basis for $R^N = T^N \oplus G^N$.

<u>Lemma 3</u> :Let $\mathcal{S}^{\ell_0}$ be the space of spurious states with $L_0 = \ell_0$ .Then L_1 defines a map $L_1 : \mathcal{S}^1 \longrightarrow \mathcal{S}^0$ and $\tilde{L}_2 = L_2 + (3/2)\, L_1^2$ defines a map $\tilde{L}_2 : \mathcal{S}^1 \longrightarrow \mathcal{S}^{-1}$ provided that D = 26 (Observe that $\mathcal{S}^1 \to \mathcal{S}^0$, $\mathcal{S}^1 \to \mathcal{S}^{-1}$ are enough to produce

all the other maps : $\mathcal{S}^{1} \longrightarrow \mathcal{S}^{1-m}$).

Proof: L_{-n} $n \geq 3$ are generated by L_{-1}, L_{-2} that $|s\rangle \in \mathcal{S}^{1}$ is of the form

$$|s\rangle = L_{-1}|\lambda, 0\rangle + \tilde{L}_{-2}|\lambda, -1\rangle$$

where $L_0(|\lambda, n\rangle \equiv |n\rangle) = n|\lambda, n\rangle$. Then readily seen, by using the Virasoro algebra, that

$$L_1|s\rangle = L_{-1}L_1|\lambda, 0\rangle + \tilde{L}_{-2}L_1|\lambda, -1\rangle$$

$$L_0 L_1|s\rangle = 0$$

thus

$$L_1|s\rangle \in \mathcal{S}^{0}$$

Again by making the use of Virasoro algebra one can show that

$$\tilde{L}_2|s\rangle = L_{-1}\tilde{L}_2|\lambda, 0\rangle + \tilde{L}_{-2}L_2|\lambda, -1\rangle + (D/2 - 13)|\lambda, -1\rangle$$

where the first two terms on the right hand side form again a supirious state called $|s'\rangle$ and satisfies

$$L_0|s'\rangle = -|s'\rangle$$

therefore $|s'\rangle \in \mathcal{S}^{-1}$ But the third term is not a spurious state that it shoul be eliminated, follows from this that <u>D must be 26.</u>

This completes the proof of lemma 3.

Theorem : When the dimension of space-time is 26 any solution of (1.37), with $\alpha(0) = 1$, is of the form

$$|\phi\rangle = |t, N, v\rangle + |n\, s\rangle$$

where $|t,N,v\rangle \in T^N$ and $|n\, s\rangle$ is null spurious state.

Proof: Let $|\phi\rangle$ satisfies (1.37) with $\alpha(0) = 1$ and $\mathcal{H}^{l_0}$ be the space of the states with basis $K_{-1}^{\lambda_1} \ldots K_{-m}^{\lambda_m}|t, M, v\rangle$ satisfying $L_0 = l_0$. By lemma 2 we can write

$$|\phi\rangle = |\psi\rangle + |s\rangle \,, \quad |\psi\rangle \in \mathcal{H}^1 \quad |s\rangle \in \mathcal{S}^1$$

Further by lemma 3 $L_1 : \mathcal{S}^1 \longrightarrow \mathcal{S}^0$, $\tilde{L}_2 : \mathcal{S}^1 \longrightarrow \mathcal{S}^{-1}$ and also one can easly show that $L_1 : \mathcal{H}^1 \longrightarrow \mathcal{H}^0$ $\tilde{L}_2 : \mathcal{H}^1 \longrightarrow \mathcal{H}^{-1}$. $|\psi\rangle$ and $|s\rangle$ are independent states that

$$L_1 |\phi\rangle = 0 \quad \text{implies} \quad L_1 |s\rangle = 0 \quad L_1 |\psi\rangle = 0$$
$$\tilde{L}_2 |\phi\rangle = 0 \quad \text{implies} \quad \tilde{L}_2 |s\rangle = 0 \quad \tilde{L}_2 |\psi\rangle = 0$$

L_1 and $\tilde{L}_2$ generates all of the L_n $n > 0$ so we conclude that

$$L_n |s\rangle = L_n |\psi\rangle = 0$$

$|s\rangle$ is both physical and spurious therefore a null vector. By the argument of lemma 1 one can also show that

$$K_n |\psi\rangle = 0$$

so $|\psi\rangle \in T^N$. This concludes the proof of no-ghost theorem. Therefore we have shown that when the dimension of space-time is 26 snd the ground state is a tachyon the physical states decouple as positive definite norm states plus zero norm states.

II.NEVEU-SCHWARZ-RAMOND MODEL OF SPINNING STRING :

II.1.Equations of motion and their solutions :

The action for spinning string is given as [15,16] ($\alpha' = 1/2$) :

$$S = -\frac{1}{\pi}\int d\sigma\, d\tau\ \eta_{\mu\nu}\, e \Big[\frac{1}{2}\, \partial_\alpha x^\mu \partial_\beta x^\nu g^{\alpha\beta} + \frac{i}{2}\, \bar{S}^\mu \rho^\alpha \partial_\alpha S^\nu + i\, \bar{\Psi}_\alpha \rho^\beta \rho^\alpha S^\mu \partial_\beta x^\nu + \frac{1}{4}\, \bar{S}^\mu S^\nu\, \bar{\Psi}_\alpha \rho^\beta \rho^\alpha \psi_\beta \Big] \qquad (2.1)$$

where μ, ν are D-valued space-time , α, β two-dimensional vector indicies ($\alpha, \beta = 0 , 1$).A,B two-dimensional spinor ; a,b two-dimensional tangent space indicies.i.e. $g^{\alpha\beta} dx^\alpha \wedge dx^\beta = \eta^{\alpha\beta} \varepsilon_{ab} de^\alpha_a\, de^\beta_b$ (A,B = 1,2 ; a,b = 0,1).The latter indicies are suppressed in the action .

$x^\mu(\sigma, \tau)$ = D - dim. vector,

$S^\mu_A(\sigma, \tau)$ = D - dim. vector, 2 - dim. Majorana spinor,

$\bar{S}(\sigma, \tau) = S^T \rho^0$

$\psi^A_\alpha(\sigma, \tau)$ = 2 - dim. Rarita - Schiwinger field,

$e^\alpha_a(\sigma, \tau)$ = 2 - dim. vierbein ='zweibein',

$e = \det e^\alpha_a$

$g^{\alpha\beta}(\sigma, \tau)$ = 2 - dim. auxilary metric,

$\eta_{\mu\nu} = \mathrm{diag}\,(-1, 1, \ldots, 1)$

$$\rho^\alpha = e^\alpha_a\, \rho^a \qquad \rho^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \rho^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$\eta^{\alpha\beta} = \mathrm{diag}\,(-1, 1)$.

This action is invariant under the following two-dimensional transformations :

(i) General coordinate transformations (= reparametrization of world sheet),

(ii) Local Lorentz rotations,

(iii) Supersymmetry transformation which are given in refs. (15,16),

(iv) Weyl transformations which are given as :

$$x^{\mu} \longrightarrow x^{\mu}, \quad S^{\mu}_{A} \longrightarrow \Lambda^{-1/2} S^{\mu}_{A}, \quad e^{a}_{\alpha} \longrightarrow \Lambda e^{a}_{\alpha}, \quad \psi^{A}_{\alpha} \longrightarrow \Lambda^{1/2} \psi^{A}_{\alpha}$$

By using the principle of least action, variations with respect to x^{μ}, S^{μ}_{A} and e^{a}_{α} , ψ^{A}_{α} give, respectively, equations of motion and constraints which take the form given below, after the choice of orthogonal gauge.i.e .

$$e^{a}_{\alpha} = f\, \delta^{a}_{\alpha} \qquad \psi^{\alpha} = \rho^{\alpha} \lambda \tag{2.2}$$

Equations of motion :

$$\Box x^{\mu} = 0 \qquad \rho^{\alpha} \partial_{\alpha} S^{\mu} = 0 \tag{2.3}$$

Constraints :

$$T_{\alpha\beta} = \partial_{\alpha} x^{\mu} \partial_{\beta} x_{\mu} + \frac{i}{4} \bar{S}^{\mu} (\rho_{\alpha} \partial_{\beta} + \rho_{\beta} \partial_{\alpha}) S_{\mu} - \frac{1}{2} \eta_{\alpha\beta} (\partial_{\gamma} x^{\mu} \partial^{\gamma} x_{\mu}) = 0 \tag{2.4}$$

$$(\partial_{\alpha} x^{\mu}) \rho^{\alpha} \rho_{\beta} S_{\mu} = 0$$

with the following boundary conditions :

$$\begin{aligned} x'^{\mu} &= 0 && \text{at} \quad \sigma = 0, \pi , \\ S_{1} &= S_{2} && \text{at} \quad \sigma = 0 , \\ S_{1} &= \varepsilon S_{2} && \text{at} \quad \sigma = \pi , \quad \varepsilon = \pm 1 . \end{aligned} \tag{2.5}$$

For $S = 0$ (2.3) - (2.5) reduce to the eqs. of bosonic string given in (1.12). In (2.5) $\mathcal{E} = 1$ is called Ramond model [17] and $\mathcal{E} = -1$ is Neveu - Schwarz model [18].

After the gauge choice (2.2) we still have some symmetries in the theory which leave (2.3) - (2.5) invariant (more about this in section IV).

Let us solve the eqs. of motion and express the constraints in a manageable form. Solution of bosonic part of (2.3) - (2.5) is, as before :

$$x_\mu(\sigma,\tau) = q_\mu - i\alpha_{0\mu}\tau + i\sum_{n\neq 0}\frac{1}{n}\alpha_{n,\mu}\, e^{-in\tau}\cos n\sigma \tag{2.6}$$

and for the S field they are solved by setting

$$\begin{aligned} S_1^\mu &= \sum_n c_n^\mu\, e^{-in(\tau+\sigma)} \\ S_2^\mu &= \sum_n c_n^\mu\, e^{-in(\tau-\sigma)} \\ c_{-n} &= c_n^* \end{aligned} \tag{2.7}$$

if $\mathcal{E} = 1$ n is summed over all integers, positive and negative ,

if $\mathcal{E} = -1$ n is summed over all half-integers , positive and negative.

In order to express the constraints in a manageable form we extend the definition of σ from $[0,\pi]$ to $[-\pi,\pi]$ by defining the one component D-dimensional vector ψ as :

$$\begin{aligned} \psi(\tau,\sigma) &= S_1(\tau,\sigma) \qquad \text{if} \quad 0 \leq \sigma \leq \pi \\ \psi(\tau,\sigma) &= S_2(\tau,-\sigma) \qquad \text{if} \quad -\pi \leq \sigma \leq 0 \end{aligned} \tag{2.8}$$

Definition of x^μ is also analytically extended to $[-\pi,\pi]$ as given in (1.19). Because of the boundary condition at $\sigma = 0$, $\psi(\sigma,\tau)$ is analytic at that point. Thus ψ

is defined on $[-\pi, \pi]$ by

$$\psi = \sum_n c_n e^{-in(\tau+\sigma)} \tag{2.9}$$

Now the second constraint in (2.4) yields

$$(\partial_0 + \partial_1)\, x \cdot \psi = 0 \qquad \text{for} \qquad -\pi \leq \sigma \leq \pi \tag{2.10}$$

where ∂_0 and ∂_1 are derivatives with respect to τ and σ, respectively. This is nothing other than

$$P_\mu \cdot \psi^\mu = 0 \tag{2.11}$$

where as before

$$P_\mu = \sum_{n\neq 0} \alpha_{n\mu}\, e^{in(\tau+\sigma)} - i\,\alpha_{0\mu} \tag{2.12}$$

Because of having same $(\sigma+\tau)$ dependence, (2.11) is a constraint on the time independent variables $\alpha_{n\mu}$ and $c_{n\mu}$. The other condition in (2.5) is also expressible as

$$P^2 + i\,\psi(\partial_0+\partial_1)\psi = 0 \tag{2.13}$$

This again is a time independent constraint. Thus the basic eqs. of these models are definition of x^μ (2.6), ψ^μ (2.9) and the gauge conditions (2.11), (2.13).

II.2.Quantization:

In the first section, we have given the procudure for quantizing a system which has constraints, in Hamiltonian formalism.Hence we will not do all the steps but directly give equal τ commutation relations,which are somehow obvious from the constraint eqs.

$$\left[x^{\mu}(\sigma . \tau) \; , \dot{x}^{\nu}(\sigma', \tau) \right] = \pi \, \eta^{\mu\nu} \, \delta(\sigma - \sigma') \qquad (2.14)$$

$$\left\{ \psi^{\mu}(\sigma, \tau), \; \psi^{\nu}(\sigma', \tau) \right\} = \pi \; \eta^{\mu\nu} \; \delta(\sigma - \sigma') \qquad (2.15)$$

where $\{ , \}$ indicates the anticommutator.Quatization of x^{μ} proceeds as before.The modes of ψ^{μ} which are expressed in (2.9), now are operators which obey the folowing anticommutators :

$$\left\{ c^{\mu}_{n}, c^{\nu}_{m} \right\} = \frac{1}{2} \, \eta^{\mu\nu} \, \delta_{m+n,0} \qquad \begin{array}{l} m,\text{n=integer for } = 1 \\ m,\text{n=half-integer for } =-1 \end{array} \qquad (2.16)$$

Let us consider the two cases indipendently:

(i) $\varepsilon = 1$ case : n is integer, c^{μ}_{n}'s are regarded as annihilation or creation operators, respectively, when $n > 0$ or $n < 0$. According to (2.16) zero mode satisfys an algebra which is proportional to the algebra of D-dimensional γ^{μ} matrices.i.e.

$$\left\{ \gamma^{\mu}, \gamma^{\nu} \right\} = 2 \, \eta^{\mu\nu} \qquad (2.17)$$

A realizatioṇof this algebra exists for all even D and the rank of γ^{μ} matrices is $2^{D/2}$.

Thus we set

$$c^{\mu}_{0} = \frac{1}{2} \gamma^{\mu} \qquad (2.18)$$

SInce c^{μ}_{0} is a matrix and the operators c^{μ}_{n} $(n \neq 0)$ must anticommute with it, we define:

$$c^{\mu}_{n} = \frac{\gamma_5}{\sqrt{2}} \, d^{\mu}_{n} \qquad c^{\mu}_{-n} = \frac{\gamma_5}{\sqrt{2}} \, d^{\mu\,\dagger}_{n} \qquad n > 0 \qquad (2.19)$$

where

$$\left\{ d^{\mu}_{n} \, , d^{\nu\,\dagger}_{m} \right\} = \eta^{\mu\nu} \, \delta_{m,n} \qquad (2.20)$$

Here γ_5 is a matrix such that $\gamma_5^2 = 1$ and $\{\gamma_5, \gamma_\mu\} = 0$. If D/2 is even, as in four dimensions:

$$\gamma_5 = i \prod_{\mu=0}^{D-1} \gamma^\mu$$

if D/2 is odd:

$$\gamma_5 = \prod_{\mu=0}^{D-1} \gamma^\mu$$

Setting $z = \exp\left[i(\tau + \sigma)\right]$ we obtain the following expression for ψ

$$\psi^\mu(z) = \frac{1}{2} \Gamma^\mu(z) \tag{2.21}$$

where

$$\Gamma^\mu(z) = \gamma^\mu + \sqrt{2}\, \gamma_5 \sum_{n=1}^{\infty} \left(d_n^{\mu +} z^n + d_n^\mu z^{-n} \right) \tag{2.22}$$

is the field originally introduced by Ramond.

(ii) $\varepsilon = -1$ case: n is half-integer.We define the creation and annihilation operators $b_n^{\mu +}$, b_n^μ , respectively, as:

$$c_{-n}^\mu = \frac{1}{\sqrt{2}} b_n^{\mu +} \qquad c_n^\mu = \frac{1}{\sqrt{2}} b_n^\mu \qquad n > 0 \tag{2.23}$$

They obey the anticommutation relations which follow:

$$\{ b_n^\mu , b_m^{+\nu} \} = \eta^{\mu\nu} \delta_{m,n} \tag{2.24}$$

Now the $\psi^\mu(z)$ is:

$$\psi^\mu(z) = \frac{1}{\sqrt{2}} H^\mu(z) \tag{2.25}$$

where $H^{\mu}(z)$ is the field originally introduced by Neveu and Schwarz and expressed as:

$$H^{\mu}(z) = \sum_{n=1/2}^{\infty} \left(b_n^{\mu +} z^n + b_n^{\mu} z^{-n} \right) \tag{2.26}$$

n is half-integer moded that at z = 0 there is a branch point of the square root type. Hence two models have the same basic algebra, but the R.M. will describe half-integer spin while N.S.M. will desribe integer spin states.

For taking the constraints (2.11) into account we introduce supergauge operators as:

$$\mathcal{L}_n = \oint \frac{dz}{2\pi i z} z^n \sqrt{2} \; : \psi(z) \cdot P(z) : \tag{2.27}$$

where $\psi(z)$ is given by (2.21) or (2.25) and we call $\mathcal{L}_n$,respectively, F_n or G_n gauge. In each case contour encircles the origin. The other constaints (2.13) can be written as:

$$P^2 - 2 \psi(z) z \frac{d}{dz} \psi(z) = 0 \tag{2.28}$$

so they lead to L_n gauge operators defined as:

$$L_n = \frac{1}{2} \oint \frac{dz}{2\pi i z} z^n \left[: P^2(z) : - 2 \psi(z) z \frac{d}{dz} \psi(z) : \right] \tag{2.29}$$

where again it depends on model.Now, the following algebra of supergauge and gauge operators are direct:

$\varepsilon = 1$ (Ramond Model) :

$$\left\{ F_m, F_n \right\} = 2 L_{m+n} + (D/2) m^2 \delta_{m+n,0}$$

$$\left[L_m , F_n \right] = \left((m/2) - n \right) F_{m+n} \tag{2.30}$$

$$\left[L_m , L_n \right] = (m - n) L_{m+n} + (D/8)\, m^3 \, \delta_{m+n,o}$$

<u>$\mathcal{E} = -1$ (Neveu-Schwarz Model) :</u>

$$\left\{ G_m , G_n \right\} = 2 L_{m+n} + \frac{D}{2} \left(m^2 - \frac{1}{4} \right) \delta_{m+n,o}$$

$$\left[L_m , G_n \right] = \left(\frac{m}{2} - n \right) G_{m+n} \tag{2.31}$$

$$\left[L_m , L_n \right] = (m - n) L_{m+n} + \frac{D}{8}\, m(m^2 - 1)\, \delta_{m+n,o}$$

Difference between the two algebras can be removed if one redifines L_o in R.M. as follows:

$$L_o^{new} = \bar{L}_o = L_o + D/16 \tag{2.32}$$

Then the two algebras are identical. Let us now define the spectrum of these models:

<u>$\mathcal{E} = 1$ (R.M) :</u>

The physical state vactors are defined as the states satisfying the following conditions:

$$\begin{aligned} F_n \,|\psi\rangle &= 0 \qquad && n > 0 \\ L_n \,|\psi\rangle &= 0 \qquad && n > 0 \end{aligned} \tag{2.33}$$

Since there is an ambiguity in the quantum value of L_o we may impose

$$\left(L_o - \alpha_F(0) \right) |\psi\rangle = 0 \tag{2.34}$$

mass-shell condition on physical states but F_0 is related to L_0 as $F_0^2 = L_0$ so we also impose the following condition on those states:

$$\left(F_0 + i\, \alpha_F'^{1/2}(0) \right) |\psi\rangle = 0 \tag{2.35}$$

Ground state is

$$|\psi\rangle_0 = |0, p\rangle\, u(p)$$

where u(p) is a spinor satisfying Dirac eq.i.e. $(\gamma \cdot p - m)\, u(p) = 0$ (m is ground state mass). Mass-shell condition reads:

$$(L_0 + m^2/2)\, |\psi\rangle = \left(p^2/2 - m^2/2 + \sum_{n=1}^{\infty} n\, a_n^{\dagger} \cdot a_n + \sum_{n=1}^{\infty} n\, d_n^{\dagger} \cdot d_n \right) |\psi\rangle = 0 \tag{2.36}$$

$\underline{\varepsilon = -1 \text{ (N.S.M)}}$:

For G gauges there is no ambiguity that we define physical states as:

$$\begin{aligned} G_n |\psi\rangle &= 0 \qquad n = 1/2, 3/2, \ldots \\ L_n |\psi\rangle &= 0 \qquad n \geq 1 \end{aligned} \tag{2.37}$$

But for L_0 still there is an ambiguity while quantizing that mass-shell condition is given as:

$$\begin{aligned} &(L_0 - \alpha(0))\, |\psi\rangle = 0 \\ &\left(p^2/2 - \alpha(0) + \sum_{n=1}^{\infty} n\, a_n^{\dagger} \cdot a_n + \sum_{m=1/2}^{\infty} m\, b_m^{\dagger} \cdot b_m \right) |\psi\rangle = 0 \end{aligned} \tag{2.38}$$

In this model one defines G-parity, which will be preserved by the interactions,as:

$$G = (-1)^{\left(\sum_m b_m^{\dagger} b_m - 1 \right)}$$

II.3. No – Ghost theorem :

Proof of this theorem follows a similar line of argument as the one used in bosonic string. Thus we will not give the proofs of the lemmas which we need for constructing a basis for all Fock space. In the proof of theorem we will make use of projection operator on transverse states instead of giving a lemma like the third one of bosonic case. Because of the symmetry between $H_{\mu}(z)$ and $i/\sqrt{2}\ \Gamma^{\mu}(z)$, we will present the results side by side for N.S.M and R.M.

The number operator R takes half-integer values for N.S.M. and integer values for R.M. and the physical states are defined by (2.33) – (2.35) and (2.37) – (2.38). Let us define the transverse subspace as the space of states which obey (1.47) constraints supplemented by

$$H_n \,|\, t\rangle = 0 \qquad\qquad n > 0$$

where

$$H_n = \frac{i\,\gamma_5}{\sqrt{2}}\ k\cdot d_n \qquad\qquad \text{for} \quad \text{R.M.}$$

$$H_n = k\cdot b_n \qquad\qquad \text{for N.S.M.}$$

<u>Lemma 1</u> : For fixed $|\, t\rangle$ belonging to R^{M}, such that $\langle t \,|\, t\rangle \neq 0$, the states constructed as:

$$F_{-1}^{\varepsilon_1} \ldots F_{-a}^{\varepsilon_a} H_{-1}^{\delta_1} \ldots H_{-b}^{\delta_b} L_{-1}^{\lambda_1} \ldots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \ldots K_{-m}^{\mu_m} |\, t\rangle \qquad \text{for R.M.}$$

$$G_{-1/2}^{\varepsilon_1} \ldots G_{1/2-a}^{\varepsilon_a} H_{-1/2}^{\delta_1} \ldots H_{1/2-b}^{\delta_b} L_{-1}^{\lambda_1} \ldots L_{-n}^{\lambda_n} K_{-1}^{\mu_1} \ldots K_{-m}^{\mu_m} |\, t\rangle \qquad \text{for N.S.M.} \tag{2.39}$$

where $\varepsilon_i , \delta_i = 0$ or 1, and

$$N = M + \sum_i i\,\varepsilon_i + \sum_j j\,\delta_j + \sum_r r\,\lambda_r + \sum_s s\,\mu_s \geq M \qquad \text{for R.M.}$$

$$N = M + \sum_i (i - \tfrac{1}{2})\,\varepsilon_i + \sum_j (j - \tfrac{1}{2})\,\delta_j + \sum_r r\,\lambda_r + \sum_s s\,\mu_s \geq M \qquad \text{for N.S.M.} \tag{2.40}$$

span some subspace of R^N and they are linearly indipendent. This subspace does not contain any transverse state.

Lemma 2: If $|\,t,M,v\rangle$ is an orthonormal basis for T^M, the states defined as in (2.39) give a basis for the space of states with the value of number operator given as (2.40), but now $N - M \geq 0$, and as N varies for the whole Fock space. Further T^M is positive definite (The proof is again by iteration).

We may now proceed as before to give no-ghost theorem (14,19) but instead of it we will look for a projection operator onto the space of transverse states (20,21).

By using the k vector (1.45) define ligth-cone coordinates P_+, P_-, Ψ_+, Ψ_- (see also 4.8).Then using the gauge conditions written classically as:

$$P \cdot \Psi = 0 \quad , \quad P^2 - 2z\,\Psi\cdot\Psi' = 0 \qquad \Psi' = \frac{d}{dz}\Psi(z)$$

we solve $P_{+T}(z)$ as a function of P_-, P^i, Ψ_-, Ψ^i $(i = 1,2,\ldots,D-2)$.Let us now introduce the Laurent expansion of an arbitrary operator x(z)

$$x(z) = \sum_{n=-\infty}^{\infty} \langle x(z)\rangle_n \, z^{-n}$$

$$\langle x(z)\rangle_n = \oint \frac{dz}{2\pi i z}\, z^n\, x(z)$$

In this way we define L_n, F_n, G_n, H_n as before and also

$$D_n = \left\langle \frac{1}{P_-}\left(1 - 2z\,\frac{\Psi_-\cdot\Psi_-'}{P_-^2}\right)\right\rangle_n \qquad n = \text{integer}$$

$$B_n = \left\langle \frac{\sqrt{2}}{(P_-)^{3/2}}\,\frac{d}{dz}\left\{\frac{\Psi_-}{(P_-)^{1/2}}\right\}\right\rangle_n \qquad \begin{array}{l} n = \text{integer for R.M.} \\ n = \text{half-integer for N.S.M.} \end{array}$$

The difference between $P_{+,T} = \langle P_{+,T}(z) \rangle_0$ and $P_+ = \langle P_+(z) \rangle_0$ is obtained as:

$$P_{+,T} - P_+ = \left\langle \frac{1}{P_-} \left(1 - 2z \frac{\psi_- \cdot \psi_-'}{P_-} \right) \left(-\frac{1}{2} P^2 + z\, \psi \cdot \psi' \right) + \right.$$

$$\left. + \frac{2z}{(P_-)^{3/2}} \frac{d}{dz} \left\{ \frac{\psi^-}{(P_-)^{1/2}} \right\} P \cdot \psi \right\rangle_0$$

By using the operators which we introduced, one obtains for $E = P_{+,T} - P_+$ the following expression:

$$E = (\bar{L}_0 - \lambda)(D_0 - 1) + F_0 B_0 + \sum_{n=1}^{\infty} \left[F_{-n} B_n - B_{-n} F_n + L_{-n} D_n + D_{-n} L_n \right] \quad \text{for R.M.}$$

$$E = (L_0 - \alpha(0))(D_0 - 1) + \sum_{r=1/2}^{\infty} (B_{-r} G_r + G_{-r} B_r) + \sum_{n=1}^{\infty} (L_{-n} D_n + D_{-n} L_n) \quad \text{for N.S.M.}$$

where $\bar{L}_0$ is defined in (2.32) and λ is taking care of normal ordering ambiguities. The algebra of the new operators B_n, D_m is as follows for R.M. :

$$[F_n , D_m] = 2\, B_{n+m}$$

$$\{F_n , B_m\} = -\frac{1}{2} (3n + m)\, D_{n+m}$$

$$[L_n , D_m] = -(2n + m)\, D_{n+m}$$

$$[L_n , B_m] = -\left(\frac{3}{2} n + m\right) B_{n+m}$$

$$[D_n , D_m] = \{B_n , B_m\} = [D_n , B_m] = 0$$

The same algabra is obtained for N.S.M. by replacing F_n with G_n. By making use of this algebra one can show that for fermions (R.M.)

$$[L_n , E] = -n\, L_n + D_n \left\{ n^3 \left(\frac{D}{8} - \frac{5}{4}\right) + n \left(3\lambda - \frac{D}{8}\right) \right\}$$

We want to use E to define a projection operator that its commutator with L_n must give again L_n ,follows from this that:

$$D = 10 \qquad \lambda = D/16 \Longrightarrow \alpha_F(0) = 0$$

For N.S.M. the calculation of the same commutator yields to following conditions:

$$D = 10 \qquad \alpha(0) = 1/2$$

In each model, with these conditions, one can show that

$$[X_n, E] = -n X_n \qquad X_n = L_n, K_n, D_n, F_n \text{ or } G_n, B_n, H_n$$

Now using the basis of states previously derived and these commutation relations, one shows that E has negative or zero eiqenvalues(integer for R.M., half-integer and integer for N.S.M.) and the zero eqienvalue is reached only in the transverse subspace. The projectors onto the transverse subspace are then defined as

$$P = \oint \frac{dy}{2\pi i y} \, y^{E} \qquad \text{in R.M.}$$

$$P = \oint \frac{dy}{2\pi i y} \, y^{2E} \qquad \text{in N.S.M.}$$

If $|\phi_1\rangle$ and $|\phi_2\rangle$ are two on-shell physical states, one can show that

$$\langle \phi_1 | P | \phi_2 \rangle = \langle \phi_1 | \phi_2 \rangle$$

Therefore if $|\phi\rangle$ is an element of physical subspace one can write it as:

$$|\phi\rangle = P\,|\phi\rangle + (1-P)\,|\phi\rangle$$

where the former is an element of T^N and the latter is null spurious state. This completes the proof of not having any ghost state for

$D = 10$

and

ground state has $\quad m^2 = -1/2 \quad$ for N.S.M. ,

$\quad\quad\quad\quad\quad\quad\quad m^2 = 0 \quad$ for R.M.

III. FACTORIZATION OF CLOSED STRING :

Interactions of strings can be introduced as splitting of one string into two and vice versa. Now we want to calculate non-planar loop by using the vertex function for ground state emission of bosonic string.i.e.

$$V(k,z) = \; : e^{ik\cdot Q(z)} : \exp(-\frac{1}{2} \gamma_0^2 k^2) \tag{3.1}$$

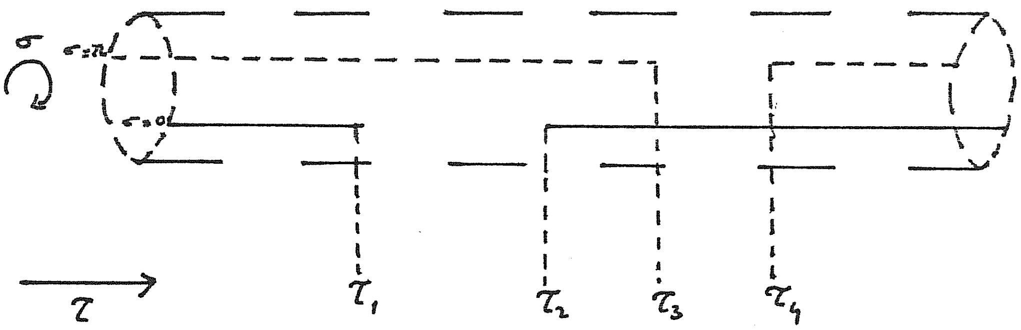
where

$$Q_\mu(z) = \sum_{m=0}^{\infty} \gamma_m (a_{m\mu} z^{-m-\varepsilon} + a^\dagger_{m\mu} z^{m+\varepsilon}) \qquad \text{in } \varepsilon \longrightarrow 0 \text{ limit}$$

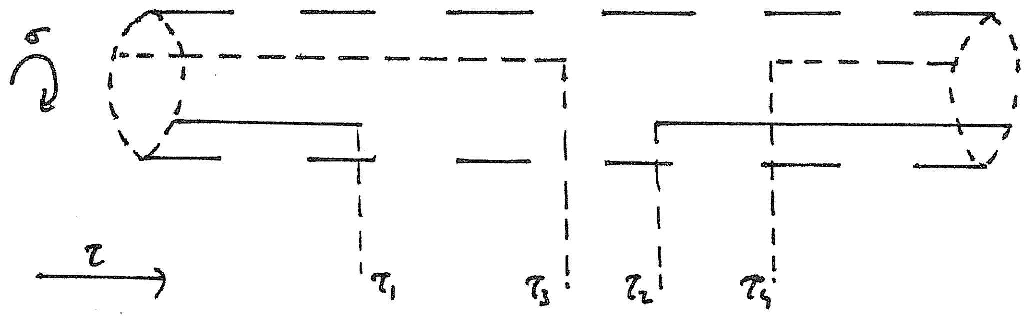
$$\gamma_m = \left[\frac{\Gamma(m+2\varepsilon)}{m!} \right]^{1/2} \tag{3.2}$$

Definition of Q_μ is equivalent to (1.32), but now a_0 is treated as the other non-zero mode operators.i.e. $a_0 |0\rangle = 0$.

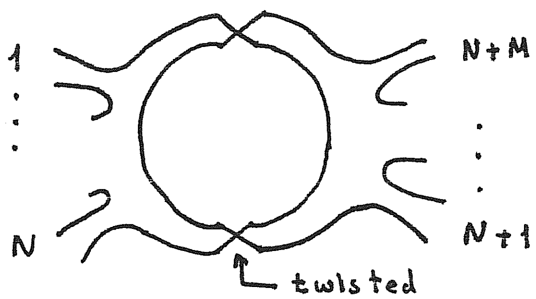
If we represent the strings by arcs of circles[22] we have the following diagram for 4-string amplitude:



Two strings join at $\tau_1, \sigma=0$ and split into two again at $\tau_2, \sigma=0$. The other end ($\sigma=\pi$) joins at τ_3 and splits at τ_4. For getting a Lorentz covariant amplitude we must add to this one the diagram which differs only in the order of ineraction times.i.e.



Two strings join at $\tau_1, \sigma = 0$ and at τ_3 the other ends also join to form a closed string. The string then splits into two at times τ_2 and τ_4. This diagram when, generalized to N+M strings, is equivalent to non-planar loop,given as:



(in the planar case all the interactions are at $\sigma = 0$ or $\sigma = \pi$). We will now calculate this loop amplitude and show that closed string factorizes if and only if D = 26 and $\alpha(0) = 1$ (which is first shown in ref.3).

The amplitude of non-planar loop , with the vertex operator (3.1) is [23]:

$$\mathcal{L} = \mathrm{Tr} \left\{ \Omega \, P \prod_{i=1}^{N} \left(V(k_i,1) \, \Delta \right) \prod_{i=N+1}^{N+M} \left(V(k_i,1) \, \Delta \right) \right\} \tag{3.3}$$

where P is the projection operator onto physical states, Ω is twist operator and Δ is the propagator which is given in integral representation as:

$$\Delta = \int_0^1 dx \; x^{L_0 - 2} \tag{3.4}$$

where

$$L_0 = \sum_{n=0}^{\infty} (n+\varepsilon)\, a_n^{+} \cdot a_n \qquad \text{and} \qquad [a_n^{\mu}, a_m^{+\nu}] = \delta_{mn}\, \eta^{\mu\nu} \tag{3.5}$$

The trace is taken over all the oscillator modes.

By using the definition of twist operator:

one can show that

$$\Omega \left\{ \prod_{i=1}^{N} V(k_i, \rho_i) \right\} \Omega \; P = \left\{ \prod_{i=1}^{N} V'(k_i, \rho_i) \right\} P$$

where V' is the same vertex except $\rho \longrightarrow \rho e^{-i\pi}$ for zero mode. By observing that L_0 generates dilatations:

$$x^{L_0}\, V(z)\, x^{-L_0} = V(xz) \tag{3.6}$$

we may take the propagators to right and by using the physical state projection operator, which we do not give the explict form, we write (3.3) as:

$$\mathcal{L} = \int_0^1 \frac{dw}{w^2} \int \prod_{i=2}^{N+M} \frac{d\rho_i}{\rho_i} \; (-1)^{N-1}\, S(w)\, \mathrm{Tr} \left\{ w^{L_0} \prod_{i=1}^{N} V'(k_i, \rho_i) \prod_{i=N+1}^{N+M} V(k_i, \rho_i) \right\} \tag{3.7}$$

where $\rho_i = x_1 \cdot \cdot \cdot x_{i-1}$, $w = x_1 x_2 \cdot \cdot \cdot x_{N+M}$, V' is the same vertex as V except the zero modes of the a_n oscillators have the sign of ρ reversed. $S(w)$ is coming from the elimination of spurious states and given as:

$$S(w) = f^{E}(w)\, (1-w)^{F} \tag{3.8}$$

where E is the number of complete set of oscillators which are eliminated, F is the number of additinonal first mode oscillators which are eliminated and f(w) is partition function (because it multiplys the amplitudes as $f(w)^{-D}$):

$$f(w) = \prod_{n=1}^{\infty} (1 - w^{n}) \tag{3.9}$$

The range of the variables ρ_i is [25]:

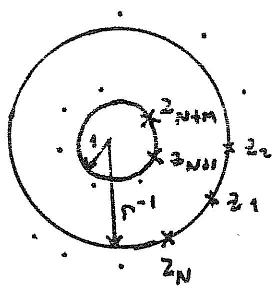
$$-1 = \rho_1 < \rho_2 < \cdot \cdot \cdot < \rho_N < -w < 0 < w\, \rho_{N+1} < \rho_{N+M} < \cdot \cdot \cdot < \rho_{N+1}$$

$$w < \rho_{N+M} < 1$$

At $w \longrightarrow 1$ limit partition function goes to 1 that amplitude may become divergent. In fact we are interested in this limit, so it is useful to perform Jacobi transformation[26]:

$$\begin{aligned} &\tau = (2\pi i)^{-1} \ln w && \tau' = -1/\tau && r = \exp i \pi \tau' \\ &\nu_i = (2\pi i)^{-1} \ln \rho_i && \nu_i' = \nu_i/\tau && z_i = \exp 2\pi i \nu_i' \end{aligned} \tag{3.10}$$

In z-plane non-planar loop becomes:



Measure of the integral , simply, transforms as:

$$\int_0^1 \frac{dw}{w^2} \int \prod_{i=2}^{N+M} \frac{d\rho_i}{\rho_i} (-1)^{N+1} =$$

$$= 2\int_0^1 \frac{dr}{r} \left(\frac{-\pi}{\ln r}\right)^{N+M+1} w^{-1} \oint_{r^{-1}} \prod_2^N \frac{dz_i}{z_i} \oint \prod_{N+1}^{N+M} \frac{dz_i}{i\, z_i} \tag{3.11}$$

The integrations are around the circles of radii indicated by the subscripts on $\oint$, with the restrictions:

$$0 = \theta_1 < \theta_2 < \cdots < \theta_N \quad \text{and} \quad \theta_{N+1} < \theta_{N+2} < \cdots < \theta_{N+M} < \theta_{N+1} + 2\pi$$

Let us now discuss Jacobi transformation of the trace term of (3.7). The traces which we are interesed in are of the form:

$$\mathrm{Tr}\, w^{L_0} \prod_{i=1}^{M} V(k_i, \rho_i, a, a^\dagger) \tag{3.12}$$

The most convenient way to calculate the trace is to use coherent states,which are defined as [24]:

$$| z\rangle = e^{z a^\dagger} |0\rangle \tag{3.13}$$

Some useful properties of it are [27]:

$$\begin{aligned} \langle z' | z\rangle &= \exp(z'^* z) \\ a | z\rangle &= z | z\rangle \\ \exp(z' a^\dagger) | z\rangle &= | z+z'\rangle \\ x^{a^\dagger a} | z\rangle &= | xz\rangle \end{aligned} \tag{3.14}$$

Unit element can be written as:

$$I = \frac{1}{\pi} \int dRe\,z \; dIm\,z \quad | z \rangle \langle z | \quad e^{-|z|^2} \tag{3.15}$$

Thus if we introduce for each mode an operator like (3.15),(3.12) reads:

$$\prod_n \frac{1}{\pi} \int d^2 z_n \langle z_n | \, w^{(n+\varepsilon) a_n^\dagger a_n} \prod_{i=1}^{M} \exp\left[i k_i \gamma_n a_n \, \rho_i^{-n+\varepsilon} \right] \qquad x$$

$$x \exp\left[i k_i \gamma_n a_n^\dagger \rho_i^{n-\varepsilon} \right] \exp(-\frac{1}{2} \gamma_0^2 k_i^2 \, | z_n \rangle \quad e^{-|z_n|^2}$$

where $d^2 z_n = dRe\,z_n \; dIm\,z_n$ and vector indicies are suppressed. Hence we may calculate the non-zero and zero modes separately:

(i)Non-zero modes:

We need to calculate terms like:

$$\frac{1}{\pi} \int d^2 z \; e^{-|z|^2} \quad \langle z | \, W^{a^\dagger a} \prod_{i=1}^{M} \exp(q_i^+ a^\dagger) \quad \exp(q_i^- a) \quad | z \rangle \tag{3.16}$$

where

$$q_i^+ = \frac{i}{\sqrt{n}} \, k_i \rho_i^n \quad ; \quad q_i^- = \frac{i}{\sqrt{n}} \, k_i \rho_i^{-n} \quad ; \quad W = w^n \tag{3.17}$$

By using the properties of coherent states (3.14) it is easy to show that

$$\langle z | W^{a^\dagger a} \prod_{i=1}^{M} \exp(q_i^+ a^\dagger) \quad \exp(q_i^- a) \quad | z \rangle = \exp(\beta) \langle z | z' \rangle \tag{3.18}$$

where

$$\beta = A + Bz \qquad\qquad z' = Wz + C \tag{3.19}$$

$$A = \sum_{j=2}^{M} q_j^{+} \sum_{i=1}^{j-1} q_i^{-} \quad ; \quad B = \sum_{i=1}^{M} q_i^{-} \quad ; \quad C = W \sum_{i=1}^{M} q_i^{+}$$

When we evaluate the integral over z,(3.16) yields to:

$$\frac{1}{1-W} \exp\left[\left(\frac{-n^{-1} W}{1-W} \right) \sum_{i,j=1}^{M} k_i \cdot k_j \left(\frac{\rho_i}{\rho_j} \right)^n - n^{-1} \sum_{j>i}^{M} k_j \cdot k_i \left(\frac{\rho_j}{\rho_i} \right)^n \right.$$

We now multiply over n and use

$$\frac{1}{1-w^n} = \sum_{m=0}^{\infty} (w^n)^m$$

in the exponent to get the following expression in the $w = x_1 x_2 \cdot \ . \ . \ x_M$ variable:

$$\left(\prod_n \frac{1}{1-w^n} \right)^D \exp\left\{ \sum_{n=1}^{\infty} (-1/n) \sum_{m=0}^{\infty} \sum_{i,j} k_i \cdot k_j \, (c_{ij} w^{m+1})^n \right\} \times$$

$$\times \exp\left\{ \sum_{n=1}^{\infty} (-1/n) \sum_{j>i}^{M} k_i \cdot k_j \, c_{ji}^n \right. \qquad (3.20)$$

where $c_{ij} = \rho_i / \rho_j$ and D is space-time dimension.We use (3.9) and the expansion:

$$\ln (1-x) = \sum_{n=1}^{\infty} (-1/n) \, x^n$$

in (3.20) to get:

$$f^{-D}(w) \left(\exp \sum_{m=0}^{\infty} \sum_{i,j} \ln (1 - c_{ij} w^{m+1})^{k_i \cdot k_j} \right) \exp \sum_{j>i} \ln (1 - c_{ji})^{k_i \cdot k_j} =$$

$$= f^{-D}(w) \prod_{j>i} (1 - c_{ji})^{k_i \cdot k_j} \prod_{m=1}^{\infty} \prod_{i,j} (1 - c_{ij} w^m)^{k_i \cdot k_j} \qquad (3.21)$$

In (3.21) there is not any constraint on k_i's. If we put the condition:

$$k_1 + k_2 + \cdot \cdot \cdot + k_M = 0$$

(3.20) reads:

$$f^{-D}(w) \prod_{m=1}^{\infty} \prod_{j>i} \frac{(1 - c_{ji}\, w^{m-1})(1 - c_{ij}\, w^{m})}{(1 - w^{m})^2} \tag{3.22}$$

(ii) Zero modes:

The contribution from the zero modes is:

$$\int d^2 z \quad e^{-|z|^2} \langle z \mid w^{\varepsilon a^\dagger a} \quad \exp(q_i^+ a^\dagger) \exp(q_i^- a) \exp(-\tfrac{1}{2} \gamma_0^2 k_i^2) \mid z\rangle \tag{3.23}$$

where

$$q_i^+ = i k_i \gamma_0 \rho_i^{\varepsilon} \qquad ; \qquad q_i^- = i k_i \gamma_0 \rho_i^{-\varepsilon}$$

Like non-zero mode case we use (3.14) and then perform the integral. Thus (3.23) reads:

$$[1/(1-w)]^D \exp\left(\frac{B\,C}{1 - w^{\varepsilon}} + A\right)$$

where

$$A = \sum_{j=2}^{M} q_j^+ \sum_{i=1}^{j-1} q_i^- - \frac{1}{2} \gamma_0^2 \sum_{i=1}^{M} (k_i)^2$$

$$B = \sum_{i=1}^{M} q_i^-$$

$$C = w^{\varepsilon} \sum_{i=1}^{M} q_i^+$$

By using

$$\delta(x) = \lim_{\varepsilon \to 0^+} (\pi \varepsilon)^{-1/2} \, e^{-x^2/\varepsilon}$$

form of the Dirac delta function and observing that in the same limit

$$\gamma_0^2 \sim \frac{1}{2\varepsilon}$$

it is easy to find that the contribution from the zero modes as:

$$(2\pi/-\ln w)^{D/2}\ \delta\Big(\sum_i k_i\Big) \prod_{j>i}\left[c_{ij}^{1/2}\ \exp\frac{\ln^2 c_{ji}}{2\ln w}\right]^{k_i \cdot k_j} \tag{3.24}$$

Let us now write (3.22) and (3.24) in more compact form by using the theta functions, defined as [28]:

$$\Theta_1(v|\tau) = 2q_0\, q^{1/4}\ \sin\pi v \prod_{n=1}^{\infty}(1-2q^{2n}\cos 2\pi v + q^{4n}) \tag{3.25}$$

$$\Theta_2(v|\tau) = 2q_0\, q^{1/4}\ \cos\pi v \prod_{n=1}^{\infty}(1-2q^{2n}\cos 2\pi v + q^{4n}) \tag{3.26}$$

where

$q = \exp i\pi\tau$ and $q_0 = \prod_{n=1}^{\infty}(1-q^{2n})$

Thus, if we use these definitions, (3.22) can be written as:

$$f^{-D}(w) \prod_{i<j} \tilde{\psi}(c_{ji}, w)^{k_i \cdot k_j} \tag{3.27}$$

where we introduced a new function $\tilde{\psi}(x, w)$, which is defined as follows:

$$\tilde{\psi}(x,w) = -2\pi i x^{1/2}\ \Theta_1\left(\frac{\ln x}{2\pi i}\,\Big|\,\frac{\ln w}{2\pi i}\right) \Big/ \Theta_1'\left(0\,\Big|\,\frac{\ln w}{2\pi i}\right) \tag{3.28}$$

$\Theta_a'(v\,|\,\tau)$ is the derivative of $\Theta_a(v\,|\,\tau)$ with respect to v.

Zero mode contribution (3.24) can also be written in more compact form in terms of theta functions as:

$$(2\pi/-\ln w)^{D/2}\ \delta^D\Big(\sum_i k_i\Big)\prod_{j>i}\left[\frac{\psi(c_{ji},w)}{\tilde{\psi}(c_{ji},w)}\right]^{k_i\cdot k_j} \tag{3.29}$$

where we used a new function defined as:

$$\psi(x,w) = -2\pi i\left[\exp\left(\ln^2 x/2\ln w\right)\right]\Theta_1\left(\frac{\ln x}{2\, i}\,\Big|\,\frac{\ln w}{2\, i}\right)\Big/\Theta_1'\left(0\,\Big|\,\frac{\ln w}{2\, i}\right) \tag{3.30}$$

Till now we did not use the properties of the non-planar loop.In the non-planar loop when i and j strings are on the same boundary therewill not be any change but when they are on the opposite boundaries there will be $-c_{ji}$ instead of c_{ji} in the zero mode contribution. Thus the trace term in (3.12) is found as follows:

$$\left(\frac{2\pi}{-\ln w}\right)^{D/2}\delta^D\Big(\sum_i k_i\Big)f(w)^{-D}\prod_{i<j}\left\{\psi(|c_{ji}|,w)\ \text{or}\ \psi_T(|c_{ji}|,w)\right\}^{k_i\cdot k_j} \tag{3.31}$$

one takes ψ when i and j are on the same boundary and ψ_T in the opposite case.The latter is defined as:

$$\psi_T(x,w) = 2\pi\left[\exp\left(\frac{\ln^2 x}{2\ln w}\right)\right]\Theta_2\left(\frac{\ln x}{2\, i}\,\Big|\,\frac{\ln w}{2\, i}\right)\Big/\Theta_1'\left(0\,\Big|\,\frac{\ln w}{2\, i}\right) \tag{3.32}$$

Under the (3.10) Jacobi transformations ψ and ψ_T transform as (29):

$$\psi(\rho_j/\rho_i, w) = i(-\pi/\ln r)(z_j/z_i)^{-1/2}\ \tilde{\psi}(z_j/z_i, r^2) \tag{3.33}$$

$$\psi_T(|\rho_j/\rho_i|, w) = (-\pi/\ln r)\ r^{-1/2}\ \tilde{\psi}(z_j/z_i, r^2) \tag{3.34}$$

Thus ,after Jacobi transformation,the trace term in (3.7) reads:

$$(2\pi/-\ln w)^{D/2}\ f(w)^{-D}\ \delta^D(\sum_i k_i) \prod_{2=j>i}^{N} \psi(\rho_j/\rho_i, w)^{k_i\cdot k_j} \times$$

$$\times \prod_{N+1\leq i<j=N+2}^{N+M} \psi(\rho_j/\rho_i, w)^{k_i\cdot k_j} \prod_{N\geq i<j=N+1}^{N+M} \psi_T(|\rho_j/\rho_i|, w)^{k_i\cdot k_j}$$

$$\equiv (2\pi/-\ln w)^{D/2}\ f(w)^{-D}\ \delta^D(\sum_i k_i)\ A \qquad (3.35)$$

Let us define

$$q = \sum_{i=N+1}^{N+M} k_i \qquad \text{and} \qquad \alpha(0) = \frac{1}{2}\ k_i^2$$

and assume

$$\sum_{i=1}^{N+M} k_i = 0$$

Therefore we have the following relations:

$$\left(\sum_{i=1}^{N+M} k_i\right)^2 = 2q^2 + 2 \sum_{N\geq i<j=N+1}^{N+M} k_i k_j = 0$$

$$\sum_{2=j>i}^{N+M} k_i\cdot k_j = \sum_{2=j>i}^{N} k_i k_j + \sum_{N+2=j>i\geq N+1}^{N+M} k_i k_j - q^2 = -\alpha(0)\,(M+N)$$

When we use these in A, which is defined by (3.35), we get

$$A = i^{q^2-(M+N)\alpha(0)} \left(\frac{-\pi}{\ln r}\right)^{-(M+N)\alpha(0)} r^{q^2/4} \prod_{j>i}^{N} (z_i/z_j)^{k_i\cdot k_j/2} \quad \times$$

$$\times \prod_{j>i} \tilde{\psi}(z_j/z_i, r^2)^{k_i\cdot k_j}$$

Partition function also transforms in a simple way

$$f(w) = (-\pi/\ln r)^{-1/2}\, w^{-1/24}\, r^{1/12}\, f(r^2) \tag{3.36}$$

We would like to express the trace in terms of vertices which depend on z's: Between vertices on the same boundary ,there is a correlation

$$\prod_{j>i} (z_i/z_j)^{k_i\cdot k_j/2}$$

which is just one gets from zero modes. This correlation does not exist between the strings on different boundaries.Thus we project on the vacuum in the zero modes at the position of twists(without $\delta(\sum k_i)$).

Thus after (3.10) transformations (3.35) yields to

$$\delta^D(\sum_i k_i)\; i^{q^2} \left[\frac{-\pi}{\ln r}\right]^{-(M+N)\,d(0)} r^{q^2/4-D/12}\, w^{D/24} \times$$

$$\times\ \mathrm{Tr}\left\{ r^{2L_0}\, |0\rangle_0\langle 0|\prod_{i=1}^{N} V(k_i,z_i)\, |0\rangle_0\langle 0| \prod_{i=N+1}^{N+M} V(k_i,z_i)\right. \tag{3.37}$$

where $|0\rangle_0\langle 0|$ indicates the projection of the zero modes onto vacuum.

Keeping track of all the factors,namely (3.8),53.11),(3.36),(3.37),we find (3.7) as :

$$\mathcal{L} = 2\delta^D(\sum k_i)\,(-1)^{N+M}\, i^{q^2+\frac{E}{2}} \int_0^1 dr \left[\frac{-i\pi}{\ln r}\right]^{(N+M)(1-d(0))+1-E/2} \times$$

$$\times\ w^{(D-E)/24-1}\,(1-w)^F\, r^{-1-(D-E)/12+q^2/4}\left[f(r^2)\right]^E \oint_{r^{-1}} \prod_{i=2}^{N}\frac{dz_i}{z_i} \oint_1 \prod_{N+1}^{N+M}\frac{dz_i}{z_i} \times$$

$$\times\ \mathrm{Tr}\left\{ r^{2L_0}\, |0\rangle_0\langle 0|\prod_{i=1}^{N} V(k_i,z_i)\, |0\rangle_0\langle 0| \prod_{i=N+1}^{N+M} V(k_i,z_i)\right\} \tag{3.38}$$

The trace and $f(r^2)$ are power series around $r=0$, so the nature of the singularity at $w \rightarrow 1$ $(r \rightarrow 0)$ will depend only explicitly displayed dependence on r and $\ln w$. The nature of the singularity must not depend on N and M. Because of $(-i\pi/\ln r)$ term this requires $\alpha(0)=1$. But still the singularity is not acceptable because there is, in general, a cut in the q^2 plane which is not due to open string states and is therefore forbidden by unitarity. If we arrange to be left with no factors of $\ln r$ or w the cut will reduce to poles. This gives the conditions

$D = 26$ $\qquad E = 2$ $\qquad F = 0$

which supplement

$\alpha(0) = 1$

In these conditions one can show that the poles emerging are the closed string states .

In Neveu-Schwarz model of spinning string non-planar loop also gets some new contributions from the b_n operators that the conditions for having poles instead of cuts will be as follows:

$D = 10$ $\qquad E = 2$ $\qquad F = 0$ $\qquad \alpha(0) = 1|2$.

IV. LIGHT - CONE GAUGE FORMULATION :

IV.1.Bosonic String :

IV.1.(a)Solutions of equations of motion:

The orthonormal gauge conditions (1.11),namely

$$\dot{x}\cdot x' = 0 \qquad\qquad \dot{x}^2 + x'^2 = 0 \tag{4.1}$$

do not specify the choice of the coordinates along the string,because there exist infinitly many orthonormal systems on a surface.If we perform a change of coordinates

$$\tilde{\sigma} = \tilde{\sigma}(\sigma,\tau) \qquad\qquad \tilde{\tau} = \tilde{\tau}(\sigma,\tau)$$

and impose that orthonormality conditions (4.1) are preserved,we end up with the following conditions:

$$\frac{\partial\tilde{\sigma}}{\partial\tau} = \frac{\partial\tilde{\tau}}{\partial\sigma} \qquad\qquad \frac{\partial\tilde{\sigma}}{\partial\sigma} = \frac{\partial\tilde{\tau}}{\partial\tau} \tag{4.2}$$

or equivalently

$$\Box\,\tilde{\tau} = 0 \qquad\qquad \Box\,\tilde{\sigma} = 0$$

$$\Box = \frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\sigma^2} \tag{4.3}$$

Let us pick up an arbitrary space-like vector n^μ .i.e. $n^2 \geqslant 0$. We can choose

$$n_\mu x^\mu(\sigma,\tau) = \lambda\tau \tag{4.4}$$

as our new variable because x^μ satisfies d'Alembert equation(see 1.12). Let us now find λ.

In orthonormal coordinates

$$P^{\mu}_{\tau} = \frac{1}{2\pi\alpha'} \dot{x}^{\mu}$$

and it is related to total momentum as given in (1.6). Hence

$$n_{\mu} P^{\mu} = \lambda / 2\alpha'$$

and the parametrization is fixed as:

$$n_{\mu} x^{\mu} = 2\alpha' (n_{\mu} P^{\mu}) \tau \tag{4.5}$$

The meaning of this equation is simple: We intersect the surface $x^{\mu}(\sigma, \tau)$ with the plane $n_{\mu} x^{\mu} = 2\alpha' n_{\mu} P^{\mu} \tau$ so the τ variable is uniquely defined.i.e. we have chosen one of the orthonormal coordinate systems.

We define the light-cone gauge as the gauge where

$$n^{\mu} = (-1, 0, \ldots ,0, 1) \tag{4.6}$$

that

$$x^{+} = 2\alpha' P^{+} \tau \qquad P^{+}_{\tau} = \frac{1}{\pi} P^{+} \tag{4.7}$$

where we have introduced light-cone coordinates

$$x^{\pm} = \frac{1}{\sqrt{2}} (x^{0} \pm x^{D-1}) \tag{4.8}$$

Let us now see that dynamical variables are the transverse ones. It follows from (1.10) that in any gauge

$$P^{\mu}_{\tau} \cdot x'_{\mu} = 0 \qquad P^2_{\tau} + \frac{(x')^2}{4\pi^2 \alpha'^2} = 0 \tag{4.9}$$

In the light-cone gauge, (4.7), the former of (4.9) reads

$$x^{-\prime} = \frac{\pi}{p^+} P^i_{\tau} x^{i\prime} \qquad i = 1,2, \ldots ,D-2 \tag{4.10}$$

and the latter yields to:

$$P^-_{\tau} = (1/2\pi\ p^+) \left[\pi^2 (P^i_{\tau})^2 + (x'^i)^2 / 4\alpha'^2 \right] \tag{4.11}$$

p_{μ} is center of mass momentum which is defined in (1.17). As before the eqs. of motion for transverse coordinates are

$$\ddot{x}^i - x''^i = 0 \tag{4.12}$$

When the x_i's are known, x^+ and x^- are also known up to an integral constant q^-. Hence the dynamical variables are $x_i(\sigma, \tau)$, q^-, p^+.

We expand x_{μ} in oscillator modes as before (see (1.18) and (1.21)) :

$$x_{\mu}(\sigma, \tau) = q_{\mu} - i\alpha^0_{\mu}\tau - i\sum_{n=1}^{\infty} \frac{1}{n} \left[\alpha_{-n\mu} e^{in\tau} - \alpha_{n\mu} e^{-in\tau} \right] \cos n\sigma \tag{4.13}$$

but now ligth-cone variables have

$$\alpha^+_n = 0 \quad \text{for } n \neq 0$$

$$\alpha^-_n = \frac{1}{2} \int_0^{\pi} d\sigma \left[4\alpha' P^-_{\tau} \cos n\sigma + (2i/\pi)\, x^{-\prime} \sin n\sigma \right] \tag{4.14}$$

(4.10) ,(4.11) when used in (4.14) yields to:

$$\alpha_n^- = \frac{1}{2\alpha' p^+} L_n^\perp \tag{4.15}$$

$$L_n^\perp = (1/4\alpha') \sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i \tag{4.16}$$

Zero mode (n=0) gives:

$$\alpha_0^- = \int_0^\pi d\sigma \, (2\alpha' P_\tau^-) = 2\alpha' P^- = \frac{1}{2\alpha' p^+} L_0^\perp$$

threfore we find the following mass formula:

$$\alpha' (\text{mass})^2 = \alpha' p^\mu p_\mu = \alpha' (-2 p^+ p^- + p^i p^i) = (1/2\alpha') \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i \tag{4.17}$$

(b) Quantization in Hamiltonian formalism:

In this gauge we found the primary constraints as (4.10) and (4.11) ,also showed that the independent variables are q^- ,p^+ ,$x^i(\sigma , \tau)$ at classical level. In Hamiltonian formalism we take P_τ^i also as an independent variable,thus we formulate the following Possion brackets:

$$\{ x^i , x^j \} = \{ P_\tau^i , P_\tau^j \} = 0$$

$$\{ x^i(\sigma, \tau) , P_\tau^j(\sigma', \tau) \} = \delta^{ij} \delta(\sigma - \sigma')$$

$$\{ q^- , p^+ \} = - 1 \tag{4.18}$$

$$\{ P_{\tau+} , x_i \} = \{ P_{\tau+} , P_{\tau i} \} = \{ q_- , x_i \} = \{ q_- , P_{\tau i} \} = 0$$

Hamiltonian is (cf.(1.26))

$$H = \frac{L_0^{\perp}}{2\alpha'} = 2\alpha' P_+ P_- = \pi\alpha' \int_0^{\pi} d\sigma \left[(P_\tau^i)^2 + (x'^i)^2/(2\alpha'\pi)^2 \right] \tag{4.19}$$

One can check this by verifing that the eqs. of motion following from the canonical Poisson brackets and the Hamiltonian formalism :

$$\dot{f} = \frac{\partial f}{\partial \tau} + \{f, H\}$$

are same.

It is now easy to quantize the system by using the relation between Poisson brackets and commutators as defined in section one and impose the normal products on classically defined quantities:

$$[a_n^i, a_m^{+j}] = \delta^{ij} \delta_{m,n} \text{ etc. . . .} \tag{4.20}$$

In this gauge the space of the vectors

$$\prod_i \prod_{n=1}^{\infty} (a_{n,i}^{+})^{\lambda_n} |0\rangle \tag{4.21}$$

has positive definite metric. The states of string are hence desrcibed by purely transverse oscillators. The non-transverse oscillators are expressed in terms of the transverse ones through the equations:

$$\alpha_n^- = \frac{1}{p^+} \left[L_n - \delta_{n,0}\, \alpha(0) \right] \tag{4.22}$$

where the c-number $\alpha(0)$ appers due to normal ordering ambiguities in L_0. Here L_0 is the transverse one ($L_0^{\perp}$) in $2\alpha' = 1$ units:

$$L_n = \frac{1}{2} \sum_{m=-\infty}^{\infty} \alpha_{n-m}^i \alpha_m^i \tag{4.23}$$

Covariance of this gauge is not manifest but before giving some arguments about this let us calculate mass of the ground state from the zero point fluctuations of string [31]. It follows from (4.17) that at classical level

$$\alpha' (\text{mass})^2 = \sum_{n=1}^{\infty} n\, a_0^{+i} a_n^i = \frac{1}{2} \sum_{n=1}^{\infty} n\, (a_n^{+i} a_n^i + a_n^i a_n^{+i})$$

After quantizing procudure we use the commutators of (4.20) that mass formula reads

$$\alpha' (\text{mass})^2 = \sum_{n=1}^{\infty} n\, (a_n^{+i} a_n^i + \frac{\delta^{ii}}{2}) \qquad (4.24)$$

Therefore at the lowest level(i.E. all $\lambda_n = 0$ in (4.21))

$$\alpha' (\text{mass})_0^2 = \frac{d}{2} \sum_{n=1}^{\infty} n$$

where d = D − 2 = number of transverse components. This is a divergent quantity but we may regularize it by using Riemann zeta function which is defined as [32]:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad \text{Re}(s) > 1 \qquad (4.25)$$

This function can be analytically continued to negative values of s as:

$$\zeta(-m) = -\frac{B_{m+1}}{m+1} \qquad m = 1,2,3, \ldots \qquad (4.26)$$

where B_r is Bernoulli's number and defined as:

$$z\,(e^z - 1)^{-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$$

and

$B_2 = 1/6$

These lead to

$$\zeta(-1) = -\frac{B_2}{2} = -1/12 \tag{4.27}$$

therefore the ground state mass reads

$$\alpha'(\text{mass})_o^2 = -d/24$$

First exited level $a_1^{\dagger i}|0\rangle$ has

$$\alpha'(\text{mass})_1^2 = \alpha'(\text{mass})_o^2 + 1 = -(d-24)/24$$

but this state is constructed from transverse oscillators only, so for giving a Lorentz invariant meaning to $(\text{mass})^2$ it must be zero, this leads to:

$$\alpha'(\text{mass})_1^2 = 0 = -(d-24)/24 \Longrightarrow \quad d = 24 \quad \text{or} \quad D = d+2 = 26$$

and the ground state has $\alpha'(\text{mass})_o^2 = -1$ .i.e. a tachyon.

(c) Covariance of the gauge:

In light-cone gauge only the transverse operators are dynamical that the Poincàre algebra, given below, is not manifestly satisfied:

$$[P^\mu, P^\nu] = 0$$

$$[P^\mu, M^{\alpha\beta}] = i(\eta^{\mu\alpha}P^\beta - \eta^{\mu\beta}P^\alpha) \tag{4.28}$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(\eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} + \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\sigma}M^{\mu\rho})$$

where $M^{\mu\nu}$ is a hermitian operator given as [33]:

$$M^{\mu\nu} = (\tfrac{1}{2}) \int_0^{\pi} d\sigma \;\; (x^{\mu} P_{\tau}^{\nu} + P_{\tau}^{\nu} x^{\mu} - x^{\nu} P_{\tau}^{\mu} - P_{\tau}^{\mu} x^{\nu}) \tag{4.29}$$

Working out in normal modes separately for the various choices of μ and ν one finds ($\alpha' = \tfrac{1}{2}$):

$$\begin{aligned}
M^{ij} &= (q^i p^j - q^j p^i) - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^i_{-n} \alpha^j_n - \alpha^j_{-n} \alpha^i_n\right) \\
M^{i+} &= q^i p^+ \\
M^{+-} &= -\frac{1}{2} \left(q^- p^+ + p^+ q^- \right) \\
M^{i-} &= \frac{1}{2} (q^i \alpha_0^- + \alpha_0^- q^i) - q^- p^i - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^i_{-n} \alpha^-_n - \alpha^-_{-n} \alpha^i_n\right)
\end{aligned} \tag{4.30}$$

where

$$\alpha_0^- = p^- = \frac{1}{p^+} \left(L_0 - \alpha(0)\right) \quad ; \quad \alpha_n^- = \frac{1}{p^+} L_n \quad \text{for} \quad n \neq 0$$

Classically (i.e. with Poisson bracket algebra)(4.28) is satisfied in any space-time dimension because the algebra of gauge operators is:

$$\{ L_n , L_m \} = (n - m) L_{n+m}$$

After quantization except $[M^{i-} , M^{j-}]$ commutator,which should be zero, is not very hard to calculate and observe that Poincàre algebra is satisfied independently from space-time dimension. But the commutator of M^{i-} will depend on dimension of space-time,as one can calculate from the following form of M^{i-}:

$$M^{i-} = -q^- p^i + (1/2p^+)\,(q^i L_0 + L_0 q^i - 2\,\alpha(0)\, q^i) - (i/p^+) \sum_{n\geq 1}^{\infty} \frac{1}{n} (\alpha^i_{-n} L_n - L_{-n} \alpha^i_n) \qquad (4.31)$$

and using Virasoro algebra and some other useful commutators given below:

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{d}{12} (n^3 - n)\, \delta_{n+m,0} \qquad (4.32)$$

$$[L_n, \alpha^j_m] = -m\, \alpha^j_{m+n}$$

$$[L_n, q_i] = -i\, \alpha^i_n$$

Now the calculation is cumbersome but direct and gives the following result (33) :

$$[M^{i-}, M^{j-}] = -\frac{1}{2(p^+)^2} \sum_{m=1}^{\infty} \left[m(1 - d/24) + \frac{1}{m} (d/24 - \alpha(0)) \right] \times$$

$$\times \left(\alpha^i_{-m} \alpha^j_m - \alpha^j_{-m} \alpha^i_m \right) \qquad (4.33)$$

For arbitrary values of $D = d + 2$ and the theory is not covariant. If and only if

$D = d + 2 = 26$ $\qquad \alpha(0) = 1$

the theory is covariant in the light-cone gauge.

IV.2. Spinning String:

If we introduce new variables

$$\sigma^{\pm} = \frac{1}{2} (\tau \pm \sigma)$$
$$\partial_{\pm} = \frac{\partial}{\partial \sigma^{\pm}} \qquad (4.34)$$

the eqs. of motion (2.3) and the constraints (2.4) now read

$$\partial_- S_1^\mu = 0 \qquad \partial_+ S_2^\mu = 0 \tag{4.35}$$

$$\partial_+ \partial_- x^\mu = 0 \tag{4.36}$$

$$i S_1 \overleftrightarrow{\partial}_+ S_1 + \frac{1}{2} (\partial_+ x)^2 = 0 \tag{4.37}$$

$$i S_2 \overleftrightarrow{\partial}_- S_2 + \frac{1}{2} (\partial_- x)^2 = 0 \tag{4.38}$$

$$S_1 \partial_+ x + \partial_+ x \cdot S_1 = 0 \tag{4.39}$$

$$S_2 \partial_- x + \partial_- x \cdot S_2 = 0 \tag{4.40}$$

where we used $a \overleftrightarrow{\partial} b = a(\partial b) - (\partial a)b$. These are in orthonormal gauge (2.2). We have still residual invariance under the following transformations

$$S_1 \longrightarrow S_1 + f(\sigma_+)\, \partial_+ x$$

$$S_2 \longrightarrow S_2 + g(\sigma_-)\, \partial_- x$$

$$x \longrightarrow x - 4i \left(\int^{\sigma_+} f \frac{\partial S_1}{\partial \sigma} d\sigma + \int^{\sigma_-} g \frac{\partial S_2}{\partial \sigma} d\sigma \right)$$

where f and g are small but arbitrary functions which anticommute with S_i's. Invariance of (4.35) and (4.36) are obvious, so let us see that also the constraints are invariant:

$$\delta\,(\text{eq. } 4.37) = -2i\, \partial_+ \left[f(\partial_+ x \cdot S_1 + S_1 \cdot \partial_+ x) \right] = 0 \tag{4.41}$$

$$\delta\,(\text{eq. } 4.39) = 4f \left[i S_1 \overleftrightarrow{\partial}_+ S_1 + (\partial_+ x)^2 / 2 \right] = 0 \tag{4.42}$$

and similar relations hold for (4.38),(4.40) with S_2 and ∂_-. Right hand sides of (4.41) and (4.42) are proportional to the constraints (4.39) and (4.37),respectively. Thus Hamiltonian (built up from constraints) and the constraints form a closed algebra at cllasical level.

This algebraic property allows us to choose a component of S_i^μ ,say S_i^+,to be

$$S_i^+ = 0 \qquad (i = 1,2) \tag{4.43}$$

(4.35) – (4.40) are also invariant under conformal reparametrizations:

$$\begin{aligned} d\,\xi^{i\prime} &= \Lambda^i{}_j\, d\xi^j \\ x^\mu(\xi) &\longrightarrow x^\mu(\xi') \\ S^\mu(\xi) &\longrightarrow \Omega(\xi')\, S^\mu(\xi') \end{aligned} \tag{4.44}$$

where

$$\xi^0 = \tau \qquad \xi^1 = \sigma$$

$$\Lambda^i{}_j(\xi) = e^{\lambda(\xi)} \begin{pmatrix} \cosh \varphi(\xi) & \sinh \varphi(\xi) \\ \sinh \varphi(\xi) & \cosh \varphi(\xi) \end{pmatrix}_{ij} \tag{4.45}$$

$$\Omega(\xi) = \exp\left(-\tfrac{1}{2}\lambda(\xi)\right) \exp\left(-\tfrac{1}{2}\varphi(\xi)\,\sigma_3\right)$$

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From (4.44) and (4.45) one can show that if we call $\delta \xi^i = u^i$ it satisfies

$$\partial_i u_j + \partial_j u_i - \eta_{ji} \partial^l u_l = 0 \tag{4.46}$$

and this is equivalent to the condition (4.2), which is used to choose light-cone gauge. Therefore in the spinning string case we call ligth-cone gauge the gauge where the following conditions hold:

$$S_i^+ = 0 \qquad i = 1,2. \qquad \text{and} \qquad x^+ = p^+ \tau \tag{4.47}$$

Now quantization of transverse coordinates goes as before. If we use ψ^μ, defined in (2.8), instead of S_1^μ and S_2^μ we get for N.S.M.

$$\begin{aligned} x^i(\sigma, \tau) &= q^i + p^i \tau + i \sum_{n \neq 0} \frac{1}{n} \alpha^i_n e^{-in} \cos n\sigma \\ \psi^i(\sigma, \tau) &= (1/\sqrt{2}) \sum_r b^i_r e^{-ir(\tau+\sigma)} \qquad r = \text{half-integers} \\ & \qquad b^i_{-r} = b^{i\dagger}_r \end{aligned} \tag{4.48}$$

where

$$[\alpha^i_m, \alpha^j_n] = m \, \delta_{m+n,0} \delta^{ij} \; ; \qquad \{b^i_r, b^j_s\} = \delta_{r+s,0} \, \delta^{ij} \tag{4.49}$$

$$[p^i, q^j] = -i \, \delta^{ij} \qquad [q^i, q^j] = [p^i, p^j] = 0$$

After introducing gauge and supergauge operators L_n, G_r we can solve light-cone components from the constraint equations as follows:

$$x^- = q^- + \frac{1}{p^+} (L_0 - \alpha(0)) \tau + \frac{i}{p^+} \sum_{n \neq 0} \frac{1}{n} L_n e^{-in\tau} \cos n\sigma \tag{4.50a}$$

$$\psi^-(\sigma,\tau) = (1/2p^+) \sum_r G_r e^{-ir(\tau+\sigma)} \qquad r = \text{half integers} \tag{4.50)b}$$

where ($\alpha_0^i = p^i$)

$$\begin{aligned} L_n &= (½) \sum_m : \alpha^i_{n-m} \alpha^i_m : + (½) \sum_r (r - n/2) : b^i_{n-r} b^i_r : \\ G_r &= (½) \sum_s : \alpha^i_{r-s} b^i_s : \end{aligned} \tag{4.51}$$

and they satisfy the same algebra given in (2.31) except with d = D-2 instead of D.

Classically $(\text{mass})^2$ is (by solving the constraints for $\partial_+ x^-$ in terms of transverse components)

$$\alpha' m^2 = \sum_{n=1}^{\infty} \alpha^i_{-n} \alpha^i_n + \sum_{r=1/2}^{\infty} r\, b^i_{-r} b^i_r \tag{4.52}$$

We symmetrize the first part and antisymmetrize the second and then impose (4.49) commutators that we end up with the following zero point mass energy coming from quantum fluctuations of string:

$$\alpha' m_o^2 = \frac{d}{2} \left[\sum_{n=1}^{\infty} n - \sum_{r=1/2}^{\infty} r \right] \tag{4.53}$$

This is a divergent quantity,so we must regularize it. If one wants to use zeta function regularization he is constricted to use only $\zeta(-1) = \sum_{n=1}^{\infty} n$:the unique quantity which may have physical significance. Therefore we express $\sum_{r=1/2} r$ as:

$$\begin{aligned} \sum_{r=1/2}^{\infty} r = \sum_{n=1}^{\infty} (n - ½) &= \frac{1}{2} \sum_{n=1}^{\infty} (2n-1) + \frac{1}{2} \sum_{n=1}^{\infty} 2n - \sum_{n=1}^{\infty} n \\ &= \frac{1}{2} \left[\sum_{n=1}^{\infty} (2n-1) + \sum_{n=1}^{\infty} 2n \right] - \sum_{n=1}^{\infty} n \\ &= \frac{1}{2} \sum_{n=1}^{\infty} n - \sum_{n=1}^{\infty} n = -\frac{1}{2} \sum_{n=1}^{\infty} n \end{aligned} \tag{4.54}$$

Thus by using (4.27)

$$\alpha' m_o^2 = \frac{d}{2} \sum_{n=1}^{\infty} n (1 + 1/2) = (3d/4) \zeta (-1) = -\frac{d}{16}$$

First exited level is $b^i_{-1/2} | 0 \rangle$,so that the mass of it is

$$\alpha' m_1^2 = -\frac{d}{16} + \frac{1}{2} = -\frac{d-8}{16} \tag{4.55}$$

and again for giving a physical meaning to mass $m_1^2 = 0$. Follows from this that

$$d = D - 2 = 8 \Longrightarrow D = 10 \qquad \alpha' m_o^2 = \frac{-1}{2} \tag{4.56}$$

In this gauge dynamical variables are the transverse ones only that there is not manifest Poincàre invariance. (4.29) Poincàre generators are generalized to N.S.M. by [35] adding to its integrand a term like $S_1^\mu S_1^\nu + S_2^\mu S_2^\nu$. Therefore we find the following normal mode expansions of the generators in terms of transverse oscillators:

$$M^{ij} = q^i p^j - q^j p^i - i \sum_{n=1}^{\infty} \frac{1}{n} \left(\alpha^i_{-n} \alpha^j_n - \alpha^j_{-n} \alpha^i_n \right) - 2i \sum_{r=1/2}^{\infty} b^i_{-r} b^j_r$$

$$M^{i+} = q^i p^+$$

$$M^{+-} = -\frac{1}{2} (q^- p^+ - p^+ q^-) \tag{4.57}$$

$$M^{i-} = i q^- \alpha^i_0 + \frac{1}{\sqrt{2}\, 2 p^+} \left[q^i (L_0 - \alpha(0)) + (L_0 - \alpha(0)) q^i \right] -$$

$$- \frac{i}{2p^+} \sum_{n=1}^{\infty} \left(\alpha^i_{-n} L_n - L_n \alpha^i_n \right) + \frac{i}{p^+} \sum_{r=1/2}^{\infty} \left(G_{-r} b^i_r - b^i_{-r} G_r \right)$$

Like the bosonic string case all the algebra is satisfied with Poisson bracket algebra in any space-time dimensions,because of not having dimensional dependence in the algebras of gauge and supergauge operators. In the quantized version it is not hard to show that except

$[M^{i-}, M^{j-}]$ all the algebra is satisfied. The calculation of this one is achieved by using the following commutators:

$$[\alpha^i_m, L_n] = m\,\alpha^i_{m+n} \qquad [\alpha^i_m, G_r] = m\,b^i_{m+r}$$
$$[b^i_r, L_n] = (r + \tfrac{1}{2})\,b^i_{r+n} \qquad \{b^i_r, G_s\} = \alpha^i_{r+s} \tag{4.58}$$

and also (2.31) with $D \rightarrow d$ replacement and (4.49). Direct calculation gives the following result:[35]

$$[M^{i-}, M^{j-}] = -\frac{1}{(p^+)^2}\left\{ \sum_{n=1}^{\infty} \left[n(1 - d/8) - \frac{1}{n}(2\alpha(0) - d/8) \right] \times \right.$$
$$\times \left[\alpha^i_{-n}\alpha^j_n - \alpha^j_{-n}\alpha^i_n \right] \times \tag{4.59}$$
$$\left. \times \sum_{r=1/2}^{\infty} \left[4r^2 (1 - d/8) - (2\alpha(0) - d/8) \right] \left[b^i_{-r} b^j_r - b^j_{-r} b^i_r \right] \right\}$$

Therefore Lorentz covariance is achieved if and only if the following conditions are fulfilled:

$$d = D - 2 = 8 \Longrightarrow D = 10 \qquad \alpha(0) = \tfrac{1}{2} \quad . \tag{4.60}$$

V. CONFORMAL INVARIANCE OF BOSONIC STRING :

Instead of Nambu action (1.1), which should be minimized with respect to x for finding the eqs. of motion, Polyakov introduced [36]

$$S = \frac{1}{2} \int d^2\xi \sqrt{g}\, g^{ab}\, \eta^{\mu\nu} \partial_a x_\mu \partial_b x_\nu \tag{5.1}$$

where $\xi^1 = \tau$, $\xi^2 = \sigma$ which are the coordinates of two-dimensional Euclidean world-sheet. g^{ab} is two-dimensional auxilary metric,

$$g = \det \| g_{ab} \| \tag{5.2}$$

and $\eta^{\mu\nu}$ is Euclidean space-time metric. Now the action is supposed to be minimized with respect to $g^{ab}(\xi)$ and also $x_\mu(\xi)$. Variation with respect to g^{ab} forces the energy-momentum tensor to be zero:

$$T_{ab} = \partial_a x_\mu \partial_b x^\mu - \frac{1}{2} g_{ab} g^{cd} \partial_c x_\mu \partial_d x^\mu = 0 \tag{5.3}$$

It has the following solution for g_{ab} :

$$g_{ab} = \partial_a x_\mu \partial_b x^\mu \tag{5.4}$$

which again gives the Nambu action when used in (5.1). Therefore minimizing with respect to x_μ will give the same eqs. of motion which we found before.

Like the orthonormal gauge introduced in section I (see (1.11)) we may use the reparametrization invariance of the action in such a way that the metric becomes conformally Euclidean:

$$g_{ab} = \rho(\xi)\, \delta_{ab} \tag{5.5}$$

At classical level a change performed in conformal factor ρ will not effect the system but after quantization there may be some anomalies which prohibit this kind of transformation. If we use path integral formalism, integral depends on conformal factor only through the conformal anomaly. There are some different ways to find the anomaly term (37,38), but we will ilustrate here the method which dependes on BRS invariance (39, 40).

It is known that BRS invariance plays a centeral role in deriving the Ward - Takahashi identities and also BRS invariance specifies the physical Fock space (41). Hence we want to give path integral quantization of (5.1) in BRS invariant manner.

In superfield notation BRS symmetry is given with the following algebra(42):

$$\{Q, Q\} = 0$$

$$[D, Q] = -i\,Q \qquad (5.6)$$

$$[D, D] = 0$$

The superspace is defined with the coordinates (ξ_a, θ) ,where θ is a real element of Grassmann algebra and BRS charge Q generates the translation $\theta \longrightarrow \theta + \lambda$ where λ is also a real element of Grassmann algebra . Dilatation D generates the transformation $\theta \longrightarrow e^{\rho}\theta$ where ρ is a real number.

By using the general coordinate transformation on the world-sheet(or equivalently reparametrization) and replacing the parameter of the transformation ε^a according to:

$$\varepsilon^a \longrightarrow i\,\theta\,\eta^a(\xi) \qquad (5.7)$$

we find the definitions of the superfields relavant for our discussion as:

$$\tilde{X}^\mu(\xi, \theta) = \tilde{x}^\mu(\xi) + i\,\theta\left[\eta^a\partial_a + \frac{1}{2}(\partial_a\eta^a)\right]\tilde{x}^\mu(\xi) \qquad d = 0$$

$$\tilde{g}_{ab}(\xi,\theta)=\tilde{g}_{ab}(\xi)+i\,\theta\left\{\left[\eta^c\partial_c-\frac{1}{2}(\partial_c\eta^c)\right]\tilde{g}_{ab}(\xi)+\right.$$

$$\left.+\ \partial_a\eta^c\,\tilde{g}_{cb}(\xi)+\partial_b\eta^c\,\tilde{g}_{ca}(\xi)\right\}\qquad d=0 \tag{5.8}$$

$$[g(\xi,\theta)]^{1/n}=[g(\xi)]^{1/n}+i\,\theta\left[\eta^a\partial_a+\frac{2}{n}(\partial_a\eta^a)\right][g(\xi)]^{1/n}\qquad d=0$$

$$\eta^a(\xi,\theta)=\eta^a(\xi)+i\,\theta\,\eta^b\,\partial_b\,\eta^a(\xi)\qquad d=1$$

$$\zeta_a(\xi,\theta)=\zeta_a(\xi)+\theta\ B_a(\xi)\qquad d=-1$$

where ζ and η correspond to Faddeev-Popov ghosts and d is BRS dimension to be defined below. The BRS symmetry is realized by, for example

$$e^{\lambda Q}\,\eta^a(\xi,\theta)\,e^{-\lambda Q}=\eta^a(\xi,\theta+\lambda) \tag{5.9}$$

$$e^{i\rho D}\,\eta^a(\xi,\theta)\,e^{-i\rho D}=e^{d\rho}\,\eta^a(\xi,e^{\rho}\theta) \tag{5.10}$$

First component of a superfield with d = odd anticommutes with itself and also with θ . In (5.8)

$$\tilde{x}^\mu(\xi,\theta)=[g(\xi,\theta)]^{1/4}\,x^\mu(\xi,\theta)$$

$$\tilde{g}_{ab}(\xi,\theta)=g_{ab}(\xi,\theta)\,/\,[g(\xi,\theta)]^{1/4} \tag{5.11}$$

These variables are chosen such that path integral measure preserves (5.9) BRS symmetry. To treat conformally Euclidean gauge (5.5) we further define

$$A_1(\xi)=\tilde{g}_{12}(\xi)\quad;\ A_2(\xi)=\frac{1}{2}\left[\tilde{g}_{11}(\xi)-\tilde{g}_{22}(\xi)\right]$$

$$C(\xi)=\frac{1}{2}\left[\tilde{g}_{11}(\xi)+\tilde{g}_{22}(\xi)\right] \tag{5.12}$$

and the superfields of them as:

$$A_1(\xi,\theta) = A_1(\xi) + i\,\theta\Big\{\left[\eta^a\partial_a + \frac{1}{2}(\partial_a\eta^a)\right]A_1(\xi) + \\ + \left[(\partial_2\eta^1) - (\partial_1\eta^2)\right]A_2(\xi) + \left[(\partial_1\eta^2) + (\partial_2\eta^1)\right]C(\xi)\Big\}$$

$$A_2(\xi,\theta) = A_2(\xi) + i\,\theta\Big\{\left[\eta^a\partial_a + \frac{1}{2}(\partial_a\eta^a)\right]A_2(\xi) + \left[(\partial_1\eta^2) - \\ - (\partial_2\eta^1)\right]A_1(\xi) + \left[(\partial_1\eta^1 - \partial_2\eta^2)\right]C(\xi)\Big\} \tag{5.13}$$

$$C(\xi,\theta) = C(\xi) + i\,\theta\Big\{\left[\eta^a\partial_a + \frac{1}{2}(\partial_a\eta^a)\right]C(\xi) + \\ + \left[(\partial_1\eta^2) + (\partial_2\eta^1)\right]A_1(\xi) + \left[(\partial_1\eta^1) - (\partial_2\eta^2)\right]A_2(\xi)\Big\}$$

Gauge fixing Lagrangian for the conformally Euclidean gauge($A_1 = A_2 = 0$)is given by a BRS invariant projection:

$$\mathcal{L}_g(\xi) = \int d\theta\,\left[i\,\zeta_1(\xi,\theta)\,A_1(\xi,\theta) + i\,\zeta_2(\xi,\theta)\,A_2(\xi,\theta)\right] \tag{5.14}$$

Path integral for the action (5.1) thus becomes

$$Z = \int \prod_\xi \mathcal{D}C(\xi)\,\mathcal{D}A(\xi)\,\mathcal{D}\tilde{X}^\mu(\xi)\,\mathcal{D}B(\xi)\,\mathcal{D}\zeta(\xi)\,\mathcal{D}\tilde{\eta}(\xi)\ \times \\ \times \exp\left\{-\frac{1}{2}\int d^2\xi\,\sqrt{g}\,g^{ab}\,\partial_a X^\mu(\xi)\,\partial_b X_\mu(\xi) + \int d^2\xi\,\mathcal{L}_g(\xi)\right\} \tag{5.15}$$

where

$$\tilde{\eta}^a(\xi) = \sqrt{g}\,\eta^a(\xi)$$

The action is BRS invariant by construction: Superfields are costructed from the θ independent parts by reparametrization with parameter (5.7), but (5.1) is reparametrization invariant and gauge fixing part is a BRS invariant projection.

If we write the measure in terms of superfields

$$d\mu = \prod_{\xi} \mathcal{D}C(\xi,\theta)\, \mathcal{D}A(\xi,\theta)\, \mathcal{D}\tilde{X}^{\mu}(\xi,\theta)\, \mathcal{D}[g(\xi,\theta)^{1/2}\eta(\xi,\theta)]\, \mathcal{D}\zeta(\xi,\theta) \times \mathcal{D}B(\xi) \tag{5.16}$$

we see that it is same with the measure of (5.15) up to a Jacobian factor of the form

$$\exp\left\{ \frac{i}{2}\theta \int d^2\xi\ \partial_a \left[\eta^a(\xi) A_w(\xi) \right] \right. \tag{5.17}$$

We pick η^a as a field which has vanishing normal component at the boundary:

$$n_a \eta^a(\xi) = 0 \qquad \text{at the boundary} \tag{5.18}$$

that (5.17) yields to one:

$$\ln (5.17) = \frac{i}{2}\theta \oint \eta^a(\xi) A_w(\xi)\, ds_a = 0 \tag{5.19}$$

We now want to illustrate the origin of the Jacobian factor arising from the following transformation

$$\tilde{x}^{\mu}(\xi) \longrightarrow \tilde{x}'^{\mu}(\xi) = \exp\left[\frac{1}{2}\alpha(\xi)\right] \tilde{x}^{\mu}(\xi) \tag{5.20}$$

To define the Euclidean path integral precisely, we first expand the variable $\tilde{x}^{\mu}(\xi)$ in terms of a complete set $\psi_n^{\mu}(\xi)$:

$$\tilde{x}^{\mu}(\xi) = \sum_n c_n \psi_n^{\mu}(\xi) \equiv \sum_n \langle \xi | n \rangle c_n \tag{5.21}$$

$$\int d^2\xi\ \psi_n^{\mu}(\xi)\, \psi_{\mu m}(\xi) = \delta_{nm} \tag{5.22}$$

We then write measure of $\tilde{x}^\mu$ as:

$$\prod_{\xi} \mathcal{D}\tilde{X}^\mu(\xi) = \det\left[\langle \xi | n \rangle\right] \prod_n dC_n = \prod_n dC_n \tag{5.23}$$

and take this expression as our primary definition of the integral measure(i.e. at the begining we must choose ψ_n such that $\det\left[\langle \xi | n \rangle\right] = 1$). After performing (5.20) transformation we again expand it in terms of $\psi_n^\mu(\xi)$:

$$\tilde{x}'^\mu(\xi) = \sum_n C_n' \psi_n^\mu(\xi)$$

where ($\alpha(\xi)$ is taken infinitisimal)

$$C_n' = \sum_m \int d^2\xi \, \psi_n{}^\mu \, e^{\alpha(\xi)/2} \, \psi_m^\mu \, C_m$$

$$= C_n + \int d^2\xi \sum_m \psi_n{}^\mu \, \frac{\alpha(\xi)}{2} \, \psi_m^\mu \, C_m$$

$$\equiv C_n + \frac{\alpha(\xi)}{2} \sum_m \langle n | m \rangle \, C_m$$

Therefore the new measure is

$$\prod_n dC_n' = \left[\det\left(\delta_{nm} + \frac{\alpha(\xi)}{2} \langle n | m \rangle\right)\right] \prod_n dC_n$$

$$= \left[\exp \mathrm{Tr} \ln\left(\delta_{nm} + \frac{\alpha(\xi)}{2} \langle n | m \rangle\right)\right] \prod_n dC_n$$

$$= \left[\exp \sum_n \frac{\alpha(\xi)}{2} \langle n | n \rangle\right] \prod_n dC_n$$

$$= \left[\exp \frac{\alpha(\xi)}{2} A_W(\xi)\right] \prod_n dC_n \tag{5.24}$$

where we defined $A_W(\xi)$ as :

$$A_W(\xi) = \sum_n \psi_n^{\mu}(\xi)\, \psi_n^{\mu}(\xi) \tag{5.25}$$

which will give the conformal anomaly when carefully evaluated.

(5.15) is BRS invariant but contains auxilary fields A and B. We thus integrate over A(ξ) and B(ξ) :

$$Z = \int \prod_{\xi} \mathcal{D}C(\xi)\, \mathcal{D}\tilde{X}^{\mu}(\xi)\, \mathcal{D}\zeta(\xi)\, \mathcal{D}\tilde{\eta}(\xi) \times$$

$$\times \exp\left\{ -\frac{1}{2}\int d^2\xi\, \partial_a\left(\frac{\tilde{X}^{\mu}}{\sqrt{\xi}}\right) \partial_a\left(\frac{\tilde{X}_{\mu}}{\sqrt{\xi}}\right) + \int d^2\xi\, \zeta \sqrt{\xi}\, \gamma\!\!\!/ \left[\frac{\tilde{\eta}}{\xi}\right]\right\} \tag{5.26}$$

where $\sqrt{g(\xi)} = \left[g(\xi)\right]^{1/4} = |\, C(\xi)\, |$ in this gauge and

$$\zeta(\xi) = \begin{pmatrix} \zeta_1(\xi) \\ \zeta_2(\xi) \end{pmatrix} \qquad \tilde{\eta}(\xi) = \begin{pmatrix} \tilde{\eta}_1(\xi) \\ \tilde{\eta}_2(\xi) \end{pmatrix} \qquad \gamma\!\!\!/ = \sigma^1 \partial_1 + \sigma^2 \partial_2$$

where σ^1 and σ^2 are the usual Pauli matrices.

We want to use the procudure described before that we expand $\tilde{\eta}$ and ζ as:

$$\tilde{\eta}(\xi) = \sum_n a_n \psi_n(\xi) \quad , \quad \zeta(\xi) = \sum_n b_n \varphi_n(\xi) \tag{5.27}$$

$$\int \psi_n^T(\xi)\, \psi_m(\xi)\, d^2\xi = \int \varphi_n^T(\xi)\, \varphi_m(\xi)\, d^2\xi = \delta_{nm} \tag{5.28}$$

Jacobian factor of (5.23) is equal to one when we take ψ_n and φ_n such that they diagonalize $\sqrt{\xi}\, \gamma\!\!\!/ \frac{1}{\xi}$ in (5.26):

$$\int d^2\xi\, \zeta(\xi) \sqrt{\xi}\, \gamma\!\!\!/ \left[\frac{1}{\xi}\tilde{\eta}(\xi)\right] = \sum_n \lambda_n b_n a_n \tag{5.29}$$

This is achieved by taking ψ_n and φ_n as:

$$H_{\tilde\eta}\, \Psi_n(\xi) = \lambda_n^2\, \Psi_n(\xi) \qquad H_{\zeta}\, \varphi_n(\xi) = \lambda_n^2\, \varphi_n(\xi) \tag{5.30}$$

where $H_{\tilde\eta}$ and H_{ζ} are, respectively, n = 1 and n = -2 case of hermitian operator

$$H = -\,\rho^{-(n+1)/2}\, \partial\!\!\!/\, \rho^{n}\, \partial\!\!\!/\, \rho^{-(n+1)/2} \tag{5.31}$$

Let us now evaluate Jacobian factor arising from the conformal transformation[40]:

$$\tilde{x}^{\mu}(\xi) \longrightarrow e^{\frac{1}{2}\alpha(\xi)}\tilde{x}^{\mu}(\xi) \qquad \tilde\eta(\xi) \rightarrow e^{\alpha(\xi)}\,\tilde\eta(\xi) \qquad \zeta(\xi) \rightarrow e^{-\frac{\alpha(\xi)}{2}}\,\zeta(\xi) \tag{5.32}$$

The method which is illustrated in (5.20) - (5.25) leads to the calculation of:

$$A_W(\xi) = \sum_n \Psi_n^T(\xi)\, \Psi_n(\xi)$$

where Ψ_n is the eigenfunction of H given by (5.31) :

$$H\, \Psi_n = \lambda_n \Psi_n$$

We regularize $A_W(\xi)$:

$$A_W(\xi) = \lim_{M\to\infty} \sum_n \Psi_n^T(\xi)\; e^{-\lambda_n^2/M^2}\; \Psi_n(\xi)$$

$$= \lim_{M\to\infty} \sum_n \Psi_n^T(\xi)\; e^{-H/M^2}\; \Psi_n(\xi)$$

We now change the basis vectors from $\Psi_n(\xi)$ to $e^{ik\cdot\xi}$:

$$A_W(\xi) = \lim_{M\to\infty} \mathrm{Tr} \int \frac{d^2k}{(2\pi)^2}\; e^{-ik\cdot\xi}\; e^{-H/M^2}\; e^{ik\cdot\xi}$$

where the trace is over two dimensional spinor index. Now the calculation is straightforward:

Commute $\exp(ik\cdot\xi)$ through $\exp(-H/M^2)$ and rescale $k \longrightarrow Mk$, then expand the resulting expression in powers of $1/M$. The result is as follows:

$$A_W(\xi) = \lim_{M\to\infty} 2\left\{\frac{3n+1}{24\pi}\left[-\partial^2 \ln\rho\right] + \frac{\rho}{4\pi}M^2\right\} \tag{5.33}$$

Conformal anomaly for $\tilde{x}^\mu(\xi)$ is obtained from (5.33) by setting $n = 0$ and dividing it to the trace factor 2. Therefore Jacobian of (5.32) conformal transformation reads

$$\exp\left\{\int d^2\xi \frac{\alpha(\xi)}{24\pi}\left[\frac{D}{2} - 8 - \left(-\frac{1}{2}\right)(-10)\right]\left[-\partial^2 \ln\rho + 2\mu^2\rho\right]\right\}$$

$$= \exp \int d^2\xi \, \frac{\alpha(\xi)}{48\pi}(D-26)\left[-\partial^2 \ln\rho + 2\mu^2\rho\right] \tag{5.34}$$

where μ^2 is regularization dependent divergent quantity. As it is obvious D = space-time dimension is arising from the summition over vector index of $\tilde{x}^\mu$. Since the action (5.26) is invariant under (5.32) transformation and $\rho(\xi) \to e^{\alpha(\xi)}\rho(\xi)$ we obtain Ward - Takahashi identity by the variational method as:

$$\left\langle \rho(\xi)\frac{\delta\tilde{S}}{\delta\rho(\xi)}\right\rangle = \left\langle \frac{D-26}{48\pi}\left(-\partial^2 \ln\rho(\xi) + 2\mu^2\rho(\xi)\right)\right. \tag{5.35}$$

where $\tilde{S}$ is the total action in (5.26). Integration of (5.35) gives a new action, so that Z reads

$$Z = Z(\rho = 1)\int \prod_{\xi} \mathcal{D}C(\xi) \exp\left\{\frac{D-26}{12\pi}\int d^2\xi\left[\frac{1}{2}\left(\partial_\mu \ln|C|\right)^2 + \frac{1}{2}\mu^2 C^2\right]\right\}$$

In our discussion we ignored the boundary conditions which are discussed in ref.42. If we want to get rid of the conformal anomaly we must take space-time dimension as follows:

D = 26

In other dimensions one must deal with the new Z.i.e. With Lioville modes also. In anomaly free dimension we may always remain in conformal gauge and even with a conformal reparametrization we may pass to light-cone gauge.

One can also show that conformal anomaly of spinning string cancels out when the space-time dimension is equal to ten (43).

CONCLUSIONS:

At the tree level one can regard D-dimensional space-time, where $D < D_{cr.}$ and $D_{cr.}$ is 26 for bosonic string and 10 for R.N.S. string, as a subspace of the $D_{cr.}$-dimensional one. But the string dynamics single out the critical dimension. Therefore either in bosonic or superstring case the consistent dimensions are larger than four.

The extra dimensions would be taken seriously on the basis of modern Kaluza-Klein idea [44]: $D_{cr.} - 4$ space-like dimensions can not be observed at present-day energies because they are compactified with a small length scale. Nevertheless one must derive an effective theory, which is valid at present-day energies and also in four-dimensions, for getting some experimental evidence.

REFERENCES:

(1) G. Veneziano, Nuovo Cim. 57 A (1968) 190.

(2) D.J. Gross, A. Neveu, J. Scherk and J.H. Schwarz, Phys. Rev. D 2 (1970) 692.

(3) C. Lovelace, Phys. Lett. 34 B (1971) 500.

(4) F. Gliozzi, J. Scherk and D. Olive, Nucl. Phys. B 122 (1977) 253.

(5) M.B. Green and J.H. Schwarz, Nucl. Phys. B 181 (1981) 502.

(6) A recent review on the subject is: J.H. Schwarz, Lectures on Superstring Theory, Caltech preprint CALT - 68 - 1247 (1985).

(7) A. Cosher, F. Englert, H. Nicolai and A. Taormina, Consistent Superstrings as Solutions of the D=26 Bosonic String Theory, CERN preprint CERN - TH.4220|85 (July, 1985)

(8) M.B. Green and J.H. Schwarz, Nucl. Phys. B 198 (1982) 252.

(9) J. Scherk, Rev. Mod. Phys. 47 (1975) 123.

(10) Y. Nambu, Proc. Int. Conf. on Symmetries and Quark Models (Gordon and Breach, 1970) 269.

(11) P.A.M. Dirac, Can. J. Phys. 2 (1950) 129 and Proc. Roy. Soc. (London) A 246 (1958) 326.

(12) S. Fubini and G. Veneziano, Nuovo Cim. 64 A (1969) 811 and Nuovo Cim. 67 A (1970) 29.

(13) There are some different ways of showing this algebra:

R.C. Brower and C.B. Thorn, Nucl. Phys. B 31 (1971) 163.

L. Brink, D. Olive and J. Scherk, Nucl. Phys. B 61 (1973) 173.

D. Friedan, Recent Advances in Field Theory and Statistical Mechanics, eds. J.B. Zuber and R. Stora, Les Houches Session (1982) 34.

(14) P. Goddard and C.B. Thorn, Phys. Lett. 40 B (1972) 235.

(15) S. Deser and B. Zumino, Phys. Lett. 65 B (1976) 369.

(16) L. Brink, P. Di Vecchia and P. Howe, Phys. Lett. 65 B (1976) 471.

(17) P. Ramond, Phys. Rev. D 3 (1971) 2415.

(18) A. Neveu and J.H. Schwarz, Nucl. Phys. B 31 (1971) 86.

(19) E.F. Corrigan and P. Goddard, Nucl. Phys. B 68 (1974) 189.

(20)L. Brink and D. Olive, Nucl. Phys. B 56 (1973) 253.

(21) P. Ramond, Phys. Rev. D 9 (1973) 3427.

(22) S. Mandelstam, Phys. Rep. 13 C (1974) 259.

(23) L. Clavelli and J.A. Shapiro, Nucl. Phys. B 57 (1973) 490.

(24) D. Amati, C. Bouchiat, J.L. Gervais, Lett. Nuovo Cim. II (1969) 399.

(25) M. Kaku and C.B. Thorn, Phys. Rev. D 1 (1970) 2860.

(26) A. Neveu and J. Scherk, Phys. Rev. D 1 (1970) 2355.

(27) V. Alessandrini? D. Amati, M. Le Bellac and D.I. Olive, Phs. Rep. 1 C (1971) 269.

(28) Bateman Mauscript Project, Higher Transcendental Functions, ed. A. Erdélyi, vol.II. (McGraw Hill, New York, 1953) page 355.

Such references will be abbreviated as B.II.355.

(29) B.II.370.

(30) B.II.357, B.II.370.

(31) L. Brink, Superstrings, CERN preprint CERN-TH.4006/84 (September, 1984).

(32) B.I.32 - 39.

(33) P. Goddard, J. Goldstone, C. Rebbi and C.B. Thorn, Nucl. Phys. B 56 (1973) 109.

(34) A. Neveu, J.H. Schwarz and C.B. Thorn, Phys. Lett. 35 B (1971) 529;

J.L. Gervais and B. Sakita, Nucl. Phys. B 34 (1971) 632.

(35) Y. Iwasaki and K. Kikkawa, Phys. Rev. D 8 (1973) 440.

(36) A.M. Polyakov, Phys. Lett. 103 B (1981) 207.

(37) E. Onofri and M.A. Virasoro, Nucl. Phys. B 201 (1982) 159.

(38) D. Friedan, Recent Advances in Field Theory and Statistical Mechanics, eds. J.B. Zuber and R. Stora, Les Houches Session (1982) 34.

(39) K. Fujikawa, Phys. Rev. D 25 (1982) 2584.

(40) K. Fujikawa, Path Integral Quantization of Gravitational Interactions-Local Symmetry Properties-, Lectures given at Kyoto Summer Institute RRK 85 - 21 (July 1985).

(41) T. Kugo and I. Ojima, Nucl. Phys. B 144 (1978) 234.

K. Fujikawa, Prog. Theor. Phys. 63 (1980) 1364.

(42) B. Durhuus, P. Olesen and J.L. Peterson, Nucl. Phys. B 198 (1982) 157.

(43) A.M. Polyakov, Phys. Lett. 103 B (1981) 211.

(44) J. Scherk and J.H. Schwarz, Phys. Lett. 57 B (1975) 463.