SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

ATTESTATO DI RICERCA

"MAGISTER PHILOSOPHIAE"

POLARIZATION OF THE MICROWAVE BACKGROUND RADIATION

CANDIDATO:

Elisa Milaneschi

RELATORE:

Prof. Gianfranco DE ZOTTI

Anno Accademico 1984/85

TRIESTE

INDEX.

INTRODUCTION

Observations of the microwave background radiation yields crucial information about the large-scale gravitational and scattering processes which have occured during the evolution of the universe. The isotropy of its intensity has been extensively studied and remains of great current interest, particularly with regard to theories of galaxy formation.

Up to now the only anisotropy detected is a dipole variation (Boughn et al.,1981; Gorenstein and Smoot, 1981), with a fractional amplitude of $(1.40 \pm 0.11) \cdot 10^{-3}$. For the quadrupole component we have an upper limit fractional amplitude of $2 \cdot 10^{-4}$ (Lubin, Epstein and Smoot, 1983; Fixsen, Cheng and Wilkinson,1983).

The dipole anisotropy is most straightforwardly interpreted as a Doppler effect due to the motion of the observer relative to the last scattering surface of the microwave radiation.Still a contribution to the dipole due to cosmological effects cannot be ruled out; however the quadrupole amplitude typically exceeds the dipole one in homogeneous anisotropic models (Fabbri, 1980).

Should subsequent measurements detect a quadrupole anisotropy, it could be originated both by a long density wave or by an universal expansion anisotropy. It is therefore an important problem to discriminate real cosmological effects in the properties of the cosmic background radiation. In this connection, it is very important to observe that in anisotropic cosmologies, as first pointed out by Rees (1968), the radiation anisotropy is coupled to the linear polarization by means of the Thomson scattering. It is well known that when a beam of natural light undergoes Thomson scattering, the radiation becomes partially polarized in the direction orthogonal to the scattering plane. The overall effect does not vanish after averaging over the incidence directions of the beams, if the beam intensity depends on the direction. This is just what happens for the background radiation in anisotropic cosmologies.

Since polarization cannot be produced by the peculiar motion of our reference frame, a comparative analysis of anisotropy and polarization would be a powerful test for cosmological models.

There is also a good deal to be learned from the study of polarization on small angular scales. In fact Thomson scattering effectively converts temperature fluctuations into partial polarization if the photon mean free path is smaller than the characteristic length of the density perturbations before the last scattering time. Then density waves, turbulent motions and gravitational waves can generate small scale polarization during the recombination.

No positive detections of cosmic polarization, but only upper limits, often referring to the Rayleigh-Jeans region, have been reported so far (Nanos,1979; Smoot and Lubin,1979; Lubin and Smoot,1981; Lubin et al., 1983). The most stringent limits, due to Lubin et al. (1983) are a few times 10^{-5} for the linear polarization and $7 \cdot 10^{-3}$ for the circular polarization, at large angular scales. In the millimetric region Caderni et al.(1978b) set limits of order 10^{-3} on the linear polarization at angular scales between 0.5° and 40°.

For comparison, the anisotropic models investigated by Rees(1968), Basko and Polnarev (1980) ,Negroponte and Silk (1980) predict polarization degrees of $10^{-4} \div 10^{-6}$ for radiation anisotropies of order 10^{-4}.

A crucial problem, however, concerns the survival of the primordial polarization during the propagation of photons over cosmological distances and within our galaxy. As a matter of facts, the radiation might be depolarized by random magnetic fields through Faraday rotation. However the polarization degree of the background radiation can affected also by a coherent magnetic field because of the finite thickness of the last scattering surface. As the background photons travelled different path lengths since their last scatterings and were subject to different Faraday rotations, the primeval polarization tends to be damped, and can conceivably be canceled out al least at certain wavelengths.

On the other hand, a peculiar general relativistic effect occurs even in vacuo (Brans,1975); the depolarizing effect is intrinsic to the radiation decoupling process in any anisotropic cosmologies. However, this was shown to be not very important (Caderni et al.,1978a; Fabbri and Breuer,1980).

The plan of this thesis is the following. The first section deals with large scale polarization. In chapter I we discuss the transfer equation for homogeneous cosmologies. The solutions for Bianchi type I models have been investigated by Rees(1968),Anile(1974), Basko and Polnarev (1980),Negroponte and Silk (1980) and

Stark (1981). Tolman and Matzner (Tolman and Matzner,1984; Tolman,1985) extended the investigation to Bianchi models of type I,V,IX and VII,namely, to the classes of homogeneous models which are direct generalization of the standard (closed, flat and open) Friedmann cosmologies. Their results are described in chapter II. In chapter III we consider the effects of a homogeneous magnetic field on the anisotropy and polarization of the background radiation in Bianchi type I models (Milaneschi and Fabbri,1985).

In the second section we deal with the small scale polarization in the microwave background.
A small scale polarization degree can be produced by local sources such as the scattering of the background radiation by clusters of galaxies (Sunyaev and Zeldovich,1980). However a more important source of small scale polarization are density waves. Kaiser(1983) and Bond and Efstathiou (1985) have estimated the polarization produced during the recombination in initially adiabatic perturbations, which according to the present theory of galaxy formation, grew via gravitational instability into the large inhomogeneities we observe today. They found a polarized component of the temperature anisotropy of about 10%.

In the last chapter we present the results of our own work (Milaneschi and Valdarnini,1985).

We have numerically integrated the transfer equation for the evolution of adiabatic perturbations in universe models with dark matter and calculated the ratio between polarization and radiation anisotropy as a function of the angular scale for different values of the density parameter Ω and of the spectral index of primordial density perturbations.

SECTION I

LARGE SCALE POLARIZATION OF THE MICROWAVE BACKGROUND RADIATION.

CHAPTER I

TRANSFER EQUATION FOR POLARIZED RADIATION IN HOMOGENEOUS COSMOLOGIES.

In this chapter we will follow the procedure used by Dautcourt and Rose(1978) to derive a transfer equation for polarized radiation in an orthonormal reference frame and in a general homogeneous curved spacetime.

First we will consider the transfer equation for non polarized radiation. To obtain a manifestly covariant description, a relativistically invariant photon distribution function N related to the radiation brightness I by $N = I/(c\, h^{-1} \nu^{3})$ must be used.

The Liouville theorem ensures that N does not change if one follows the motion of a particular photon in the absence of collisions. With an affine parameter s along the photon path one calculates the total change of N as:

$$\frac{dN}{ds} = 0 \qquad (1)$$

In an orthonormal frame $e_{(i)}$ (i=0,1,2,3) the four momentum k^μ of a photon satisfies the relations:

$$k^\alpha k^\beta \eta_{\alpha\beta} = 0 \tag{2}$$

$$\frac{dk^\alpha}{ds} = -\Gamma^\alpha_{\beta\gamma} k^\beta k^\gamma , \tag{3}$$

where $\Gamma^\alpha_{\beta\gamma}$ are the affine connection coefficients. Introducing a coordinate system x^μ of the space-time the eq.s (1) and (3) give:

$$k^\alpha e^\mu_{(\alpha)} \frac{\partial N}{\partial x^\mu} + \frac{\partial N}{\partial k^\alpha} \frac{dk^\alpha}{ds} = 0$$

$$k^\alpha e^\mu_{(\alpha)} \frac{\partial N}{\partial x^\mu} - \Gamma^\alpha_{\mu\beta} k^\gamma k^\beta \frac{\partial N}{\partial k^\alpha} = 0 . \tag{4}$$

The components of the four momentum k^α are not inde_ pendent; a standard representation is

$$k^0 = h\nu \quad , \qquad\qquad k^i = h\nu\, n^i$$

where n^i is the threee-dimensional ray direction. It is convenient to introduce polar coordinates (ϑ, ϕ) and two further directions a^i and b^i , forming with n^i an orthonormal triad:

$$n^i = (\sin\vartheta \cos\phi , \ \sin\vartheta \sin\phi , \ \cos\vartheta)$$

$$a^i = \frac{\partial n^i}{\partial \vartheta}$$

$$b^i = \frac{1}{\sin\vartheta} \frac{\partial n^i}{\partial \phi} .$$

We obtain:

$$e^{\rho}_{(o)}\frac{\partial N}{\partial x^{\rho}}+n^{i}e^{\rho}_{(i)}\frac{\partial N}{\partial x^{\rho}}-\nu\gamma^{o}\frac{\partial N}{\partial \nu}-\gamma^{i}\left(a^{i}\frac{\partial N}{\partial\theta}+\frac{b_{i}}{\sin\theta}\frac{\partial N}{\partial\phi}\right)=0, \tag{5}$$

where $\gamma^{\mu}=\Gamma^{\mu}{}_{\alpha\beta}n^{\alpha}n^{\beta}$.

In a homogeneous universe we choose to work in an orthonormal frame of reference which is itself homogeneous, in the sense that the spatial frame vectors are homogeneous vector fields. Following standard work (cf. Ellis,1967; Ryan and Shepley,1975) the metric is defined by:

$$ds^{2}=-dt^{2}+R^{2}(t)\,(e^{2\beta})_{ij}\,E^{i}E^{j}, \tag{6}$$

the time coordinate t labels successive homogeneous spatial surfaces, and R(t) is the usual cosmological length scale. The anisotropy is represented by the traceless 3X3 matrix $\beta_{ij}(t)$;the shear tensor is then defined by:

$$\sigma_{ij}=\frac{1}{2}\left[(e^{\beta})^{\cdot}{}_{ik}\,/(e^{\beta})_{kj}+(e^{\beta})^{\cdot}{}_{jk}\,/(e^{\beta})_{ki}\right].$$

The three vector fields E_i (dual to 1-forms E^i) have commutators(Lie derivatives) $[E_i,E_j]=C^{k}{}_{ij}E_k$, where the $C^{k}{}_{ij}$ are the structure constants of a transitive Lie group associated with the Bianchi cosmologies. This means that the E_i are homogeneous vector fields(Mac Callum,1979).

The orthonormal frame is defined by equating the

ds^2 above to $\eta_{\mu\nu}e^{\mu}e^{\nu}$ where $\eta_{\mu\nu}$ is the Minkowski metric of signature +2, and e_{α} (dual to 1-forms e^{α}) are the four frame vector fields. (The indexes μ, ν, α are frame indexes).

We choose:

$$e^0 = dt$$
$$e^i = R(t)\,(e^{\beta})_{ij}\,E^j,$$

and suppose that our frame is dynamically non-rotating, i.e. a vanishing "Fermi" rotation velocity. In such orthonormal frame, N is independent of the spatial coordinates x^i and the following relations hold:

$$\gamma^0 \nu = -\frac{d\nu}{dt}$$
$$\gamma^i a_i = -\frac{d\theta}{dt}$$
$$\gamma^i b_i \frac{1}{\sin\theta} = -\frac{d\phi}{dt} \qquad (7)$$

So the transfer equation in a homogeneous universe becomes:

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial \nu}\frac{d\nu}{dt} + \frac{\partial N}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial N}{\partial \phi}\frac{d\phi}{dt} = 0 \qquad (8)$$

To take into account the polarization properties of the radiation we may generalize the distribution function N to the two complex functions

$${}_0N = \frac{I + iV}{ch^4\nu^3}$$

$${}_2N = \frac{Q - iU}{ch^4\nu^3} \qquad (9)$$

The quantities I,Q,U and V are the usual Stokes parameters (Chandrasekhar,1960). The quantities ${}_0N$ and ${}_2N$ are space-time invariants like N; they change,however, if polarization directions are rotated. The polarization directions are two space-like directions orthogonal to the propagation direction k^μ, which can be rapresented by a complex null vector m^α, with

$$m_\alpha \bar{m}^\alpha = 1$$
$$m_\alpha m^\alpha = 0$$
$$m_\alpha k^\alpha = 0 . \qquad (10)$$

From the Maxwell equations one knows that m^α is parallel transported along the ray k^β(this preserves the orthogonality relation $m_\alpha k^\alpha = 0$ along the whole ray). If polarization directions are rotated by an angle α :

$$m^\mu \rightarrow m'^\mu = e^{i\alpha} m^\mu ,$$

the distribution functions ${}_0N$ and ${}_2N$ transform in the following way:

$${}_0N \rightarrow {}_0N' = {}_0N$$
$${}_2N \rightarrow {}_2N' = e^{2i\alpha} \, {}_2N .$$

Thus the quantities ${}_0N$ and ${}_2N$ have spin weight zero and two respectively(Appendix).

Now we want to write ${}_2N$ with respect to the directions a^i and b^i, which are not parallel propagated directions. Thus we must add to the transport equation for ${}_2N$ a term accounting for the relative twisting of the directions (a^i, b^i) with respect to a parallel propagated direction. I.e. ${}_2N$ must be replaced by ${}_2N\, e^{-i2\alpha}$ and the Liouville equation becomes:

$$\frac{d}{ds}\left({}_2N e^{-i2\alpha}\right) = e^{-i2\alpha}\left(\frac{d}{ds}\,{}_2N + (-i2)\frac{d\alpha}{ds}\,{}_2N\right) = 0. \tag{11}$$

Now we must find $\frac{d\alpha}{ds}$. To this purpose let us define the reference vector $t^i = 1/\sqrt{2}\,(a^i + i b^i)$, and differentiate t^i with respect to time:

$$\frac{dt^i}{dt} = -\frac{1}{\sqrt{2}}\left(a^i \gamma_i + i\, b^i \gamma_i\right) n^i + i\, t^i \operatorname{cotg}\theta\, b^j \gamma_j , \tag{12}$$

from the parallel transport law we have for the polarization vector:

$$\frac{dm^i}{dt} = -\Gamma^i_{\alpha\beta}\frac{k^\alpha}{k^0} m^\beta = -\left(\Gamma^i_{0e} + \Gamma^i_{ec} n^e\right) m^c . \tag{13}$$

For a rotation of an angle α around $\vec{n}$, the vector t^i transforms into $e^{i\alpha} t^i$; if at the source $m^i = t^i$ subsequently is $m^i = t^i e^{i\alpha}$, that is $m^i \bar{t}_i = e^{i\alpha}$. Differentiating this relation with respect to time we get:

$$\frac{d\alpha}{dt}\, i e^{i\alpha} = \frac{dm^i}{dt}\bar{t}_i + m^i \frac{d\bar{t}_i}{dt} , \tag{14}$$

and substituing eq.s (12) and (13) we find

$$\frac{d\alpha}{dt} = -\operatorname{cotg}\theta\, b^i \gamma_i - n_i \Omega^i - \frac{1}{2}\varepsilon^{jei} n_i n^k \Gamma_{jek} \tag{15}$$

where we have used the relations:

$$t^{[e} \bar{t}^{j]} = -i\varepsilon^{eji} n^i \frac{1}{2} ,$$

$$\Gamma_{ij0} = \varepsilon_{ijk}\Omega^k .$$

For completeness we have included the frame's Fermi rotation velocity Ω_i in this result, but of course we have chosen $\Omega_i = 0$.

The first term in (15) is due to the choice of a polar system of coordinate. In fact a variation of $\vec{n}$ in the direction of $\vec{b}$ does not produce simply a parallel transport of t^i along the unitary sphere, but also produces an extra rotation of $\vec{a}$ and $\vec{b}$ around $\vec{n}$, because, by definition, it must be tangent to the circle at θ = const.

The third term is due to the spatial curvature of the universe.

Substituting eq.(15) in (10) we obtain:

$$\frac{\partial_0 N}{\partial t} + \frac{\partial_0 N}{\partial \gamma}\frac{d\gamma}{dt} + \frac{\partial_0 N}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial_0 N}{\partial \phi}\frac{d\phi}{dt} = 0$$

$$\frac{\partial_2 N}{\partial t} + \frac{\partial_2 N}{\partial \gamma}\frac{d\gamma}{dt} + \frac{\partial_2 N}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial_2 N}{\partial \phi}\frac{d\phi}{dt} + 2i\left(\mathrm{ctg}\,\theta\, b^i \gamma_i + \frac{1}{2}\varepsilon^{jli} n_i n^k \Gamma_{jlk}\right) {}_2N = 0. \qquad (16)$$

Now we consider Thomson scattering by free electrons and generalize the equation (16) to a Boltzmann equation by adding a source term on the right side, $\left(\frac{\partial N}{\partial t}\right)_{coll.}$. The Thomson scattering source term for intensity I alone is $l^{-1}(J - I)$, where J is the emission term; the mean free path l is $(n_e \sigma_T)^{-1}$, where n_e is the free electron number density and σ_T is the Thomson scattering cross section. From the corresponding emission term for the Stokes parameters, we obtain (Chandrasekhar,1960) for the vector $\underset{\sim}{N} = \begin{pmatrix} {}_0N \\ {}_2N \end{pmatrix}$:

$$\left(\frac{\partial \underset{\sim}{N}}{\partial t}\right)_{coll.} = \ell^{-1}\int\frac{d\Omega'}{4\pi}\left[p_+(\theta,\phi;\theta',\phi')\,\underset{\sim}{N}(\theta',\phi') + p_-(\theta,\phi;\theta',\phi')\,\bar{\underset{\sim}{N}}(\theta',\phi')\right] - \ell^{-1}\underset{\sim}{N}. \qquad (17)$$

where p_+ and p_- are 2X2 matrices; their elements can be expanded in terms of spin weighted polynomials n^i, n^{ij} and m^{ij}(see Appendix) for both primed and unprimed angles; the result is:

$$p_- = \begin{pmatrix} \frac{3}{8}n^{ij}n'_{ij} - \frac{3}{4}n^i n'_i + \frac{1}{2}, & -\frac{3}{4}n^{ij}m'_{ij} \\ -\frac{3}{4}m^{ij}n'_{ij}, & \frac{3}{2}m^{ij}m'_{ij} \end{pmatrix}$$

$$p_+ = \begin{pmatrix} \frac{3}{8}n^{ij}n'_{ij} + \frac{3}{4}n^i n'_i + \frac{1}{2}, & -\frac{3}{4}n^{ij}\bar{m}'_{ij} \\ -\frac{3}{4}m^{ij}n'_{ij}, & \frac{3}{2}\bar{m}^{ij}\bar{m}'_{ij} \end{pmatrix}$$

Then the final expression for the transfer equation in the presence of Thomson scattering is:

$$\frac{\partial \underset{\sim}{N}}{\partial t} + \frac{\partial \underset{\sim}{N}}{\partial \nu}\frac{d\nu}{dt} + \frac{\partial \underset{\sim}{N}}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \underset{\sim}{N}}{\partial \phi}\frac{d\phi}{dt} - i\,2\frac{d\alpha}{dt}\,\delta_{\alpha 2}\underset{\sim}{N}$$

$$= -\ell^{-1}\underset{\sim}{N} + \ell^{-1}\int\frac{d\Omega'}{4\pi}\left(p_+\underset{\sim}{N} + p_-\bar{\underset{\sim}{N}}\right), \qquad (18)$$

where

$$\delta_{\alpha 2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

CHAPTER II

SOLUTION OF THE TRANSFER EQUATION TO THE FIRST ORDER IN SHEAR.

2.1 THE POLARIZATION OF THE MICROWAVE BACKGROUND RADIATION IN FLAT AND CLOSED MODELS.

In this paragraph we will describe the results relative to the large scale polarizazion in Bianchi models, in the absence of a uniform magnetic field, following in particular the paper by Tolman and Matzner(1984).

They consider models which are a generalization of the standard closed,flat and open Friedmann Robertson-Walker cosmolodies,that is Bianchi types IX,I,V and VII.

The distribution functions ${}_0N$ and ${}_2N$ are expanded in terms of the spin weighted polynomials and substituted into the transfer equation (18). By using the orthogonality relations the emission term integral may be evaluated yielding the exact result:

$$
{}_0J = \frac{1}{10}\,(\mathrm{Re}\,{}_0N_{ij})\,n^{ij} - \frac{3}{10}\,(\mathrm{Re}\,{}_2N_{ij})\,m^{ij} + \frac{i}{2}\,(\mathrm{I_m}\,{}_0N_i)\,n^i + \mathrm{Re}\,{}_0N_0
$$

$$
{}_2J = \frac{3}{5}\,\mathrm{Re}\,{}_2N_{ij}\,m^{ij} - \frac{1}{5}\,(\mathrm{Re}\,{}_0N_{ij})\,m^{ij}, \tag{19}
$$

only multipoles up to the quadrupole (${}_0N_{ij}$ and ${}_2N_{ij}$) appear, meaning that while higher order modes are damped by Thomson scattering through the absorption term I they are not regenerated in the scattering; but quadru_ pole and lower order modes undergo both a damping via I and an enhancement via J in the scattering.
Tolman and Matzner(1984) truncate the multipole expansion to the quadrupole order; this is a good approximation for flat and closed models, but for very open models significant higher order modes are generated by the spatial curvature, so these last models will be treated in a different way (Tolman,1985) and we will discuss them in the next paragraph.

Let us use as independent variable in the transfer equation the optical depth τ defined by

$$
\tau(t_1,t_2) = \int_{t_1}^{t_2} \ell^{-1}\,dt
$$

and define

$$
\xi = \frac{\partial(\ln {}_0N)}{\partial(\ln \nu)} .
$$

Invoking the orthogonality of the spin weighted polynomials in each of the expanded transfer equation, we obtain four coupled,complex, first order ordinary differential equations for the monopole, dipole and quadrupole moments of the Stokes parameters distribution functions:

$$\frac{\partial}{\partial\tau}\,{}_0N_0 + \ell\left(-\frac{1}{3}\Gamma^0{}_{rr}\,\xi\,{}_0N_0 + \hat{J}^k_0\,N_k + \hat{M}^{ke}\,{}_0N_{ke}\right) = -i\,\mathrm{Im}\,{}_0N_0$$

$$\frac{\partial}{\partial\tau}\,{}_0N_i + \ell\left(-\Gamma^0{}_{i0}\,\xi\,{}_0N_0 + \left\{\hat{A}_i{}^k\,\xi + \hat{B}^k_i\right\}{}_0N_k + \hat{C}^{ke}_0\,N_{ke}\right) =$$

$$-\mathrm{Re}\,{}_0N_i - \frac{i}{2}\,\mathrm{Im}\,{}_0N_i$$

$$\frac{\partial}{\partial\tau}\,{}_0N_{ij} + \ell\left(E_{ij}\,\xi\,{}_0N_0 + \left\{\hat{D}^k{}_{ij}\,\xi + \hat{H}^k_{ij}\right\}{}_0N_k + \left\{\hat{F}^{ke}_{ij}\,\xi + \right.\right.$$

$$\left.\left.\hat{G}^{ke}_{ij}\right\}{}_0N_{ke}\right) = -\frac{9}{10}\,\mathrm{Re}\,{}_0N_{ij} - \frac{3}{10}\,\mathrm{Re}\,{}_2N_{ij} - i\,\mathrm{Im}\,{}_0N_{ij}$$

$$\frac{\partial}{\partial\tau}\,{}_2N_{ij} + \ell\left(\hat{K}^{ke}_{ij} + i\,\hat{L}^{ke}_{ij}\right){}_2N_{ke} = -\frac{1}{5}\,\mathrm{Re}\,{}_0N_{ij} - \frac{2}{5}\,\mathrm{Re}\,{}_2N_{ij}$$

$$-i\,\mathrm{Im}\,{}_0N_{ij},$$

(20)

the real coefficients used here are defined as follows:

$$\hat{A}^k_i = -\frac{1}{5}\left(\Gamma^0{}_{ik} + \Gamma^0{}_{ki} + \Gamma^0{}_{rr}\,\delta_{ik}\right)$$

$$\hat{B}^k_i = -\Gamma^k{}_{i0} + \frac{1}{5}\left(\Gamma^i{}_{0k} - 4\,\Gamma^k{}_{0i} + \Gamma^0{}_{rr}\,\delta_{ik}\right)$$

$$\hat{C}^{ke}_i = -\frac{2}{5}\left(\Gamma^k{}_{ie} + \Gamma^e{}_{rr}\,\delta_{ik} + 3\,\Gamma^e{}_{00}\,\delta_{ik} + \Gamma^0{}_{e0}\,\delta_{ik}\,\xi\right)$$

$$\hat{D}_{ij} = -\Gamma^0{}_{j0}\,\delta_{ik}$$

$$\hat{E}_{ij} = -\Gamma^0{}_{ij}$$

$$\hat{F}^{kl}_{ij} = -\frac{1}{7}\left(\{\Gamma^{o}{}_{ik} + \Gamma^{o}{}_{ki}\}\delta_{jl} + \{\Gamma^{o}{}_{jl} + \Gamma^{o}{}_{lj}\}\delta_{ik} + \Gamma^{o}{}_{rr}\delta_{ik}\delta_{jl}\right)$$

$$\hat{G}^{kl}_{ij} = \left(\frac{4}{7}\Gamma^{i}{}_{ok} - \frac{10}{7}\Gamma^{k}{}_{oi} - 2\Gamma^{k}{}_{io} + \frac{2}{7}\Gamma^{r}{}_{or}\delta_{ik}\right)\delta_{jl}$$

$$\hat{H}^{k}_{ij} = -\Gamma^{k}{}_{ij} + \Gamma^{j}{}_{oo}\delta_{ik}$$

$$\hat{J}^{k} = -\frac{1}{3}\left(\xi\Gamma^{o}{}_{ko} + 2\Gamma^{k}{}_{oo}\right)$$

$$\hat{K}^{kl}_{ij} = \frac{6}{7}\Gamma^{o}{}_{ik}\delta_{jl} + \frac{2}{3}\Gamma^{i}{}_{ko}\delta_{jl} - \left(\frac{2}{7} + \frac{1}{3}\xi\right)\Gamma^{o}{}_{rr}\delta_{ik}\delta_{jl}$$

$$\hat{L}^{kl}_{ij} = \frac{1}{7}\left(\frac{11}{3}\Gamma^{t}{}_{rn}\varepsilon^{rtn}\delta_{ik}\delta_{lj} + \frac{4}{3}\Gamma^{t}{}_{r(i}\varepsilon^{|rt|j)}\right.$$
$$\left. - 2\Gamma^{t}{}_{r(l}\varepsilon^{|rt|j)} - 2\Gamma^{t}{}_{r(i}\varepsilon^{|rt|k)}\delta_{lj}\right).$$

(21)

Because n^{ij} and m^{ij} are symmetric and traceless, ${}_oN_{ij}$ and ${}_2N_{ij}$ are also, and therefore the coefficients listed above must be symmetric and traceless on the index pairs ij and kl. However for brevity in this listing we have not always done so.

These coefficients are functions of the affine connection coefficients $\Gamma^{\alpha}{}_{\beta\gamma}$, then they represent the effect of the gravitational field on the radiation.

It is a well known classical result that circular polarization (Stokes parameter V) obeys a separate transfer equation (Chandrasekhar, 1960); here we may generalize this to curved spacetime by noting that none of the modes of V is coupled to any other Stokes parameter, and vice-versa. We henceforth ignore the V term.

The equations are now simplified by converting to normalized functions, as follow. In our reference frame $\hat{J}^{\kappa}$ is zero and the term ${}_0N_{k\ell}\hat{M}^{k\ell}$ is a second order term in the shear; so at the lowest order in the shear we are left with the equation:

$$\frac{\partial}{\partial t}\,{}_0N_0 - \frac{\dot{R}}{R}\frac{\partial_0 N}{\partial \gamma}\,\gamma = 0.$$

Thus we have only the term due to isotropic cosmological expansion; the redshifted monopole is conserved, to first order in shear. We divide all the Stokes parameter functions by the monopole function, so for the vector $\underline{\underline{N}}$

$$\underline{\underline{N}} = \begin{pmatrix} 1 \\ {}_0N_i/{}_0N_0 \\ {}_0N_{ij}/{}_0N_0 \\ \mathrm{Re}\;{}_2N_{ij}/{}_0N_0 \\ \mathrm{Im}\;{}_2N_{ij}/{}_0N_0 \end{pmatrix} = \begin{pmatrix} 1 \\ D_i \\ I_{ij} \\ Q_{ij} \\ U_{ij} \end{pmatrix}$$

we obtain the equations:

$$\begin{aligned}
&\frac{\partial}{\partial \tau} D_i - \ell\,\hat{C}^{\kappa\ell}_{ij}\, I_{\kappa\ell} = \mathrm{Re}\, D_i \\
&\frac{\partial}{\partial \tau} I_{ij} - \ell\,(\hat{E}_{ij}\,\xi + \hat{A}^{k}{}_{ij}\, D_k) = \frac{9}{10} I_{ij} + \frac{3}{10} Q_{ij} \\
&\frac{\partial}{\partial \tau} Q_{ij} + \ell\,\hat{L}^{\kappa\ell}_{ij}\, U_{\kappa\ell} = \frac{1}{5} I_{ij} + \frac{2}{5} Q_{ij} \\
&\frac{\partial}{\partial \tau} U_{ij} - \ell\,\hat{L}^{\kappa\ell}_{ij}\, Q_{\kappa\ell} = U_{ij}
\end{aligned} \qquad (22)$$

Note that U_{ij} and Q_{ij} contribute to both the Stokes parameters U and Q because the polynomials m^{ij} which compare in the expansion of ${}_2N$ are complex. From equations (22) we see that if there is shear ($\hat{E}_{ij} = -\sigma_{ij}$) a quadrupole component is always generated from the monopole term and then by means of the Thomson scattering a quadrupole component in the polarization is created.

From the first of eq.s(22) we see that also a dipole term is generated by the quadrupole if $\hat{C}^{kl}_{i} \neq 0$; in Bianchi type I models this term is zero, so we do not have dipole anisotropy. The $\hat{L}^{kl}_{ij}$ coefficient, containing the spatial curvature, couples the parameters Q and U giving rise to the twist of the polarization vector.

The transfer equations may be considered as a single matrix differential equation:

$$\frac{\partial}{\partial \tau} \underline{\underline{N}} + \ell \hat{R} \underline{\underline{N}} = - S \underline{\underline{N}} , \qquad (23)$$

where $\hat{R}(t)$ is the matrix of coefficients involving space-time curvature:

$$R(t) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{C}^{kl}_{i} & 0 & 0 \\ \xi \hat{E}_{ij} & \hat{H}^{k}_{ij} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\hat{L}^{kl}_{ij} \\ 0 & 0 & 0 & \hat{L}^{kl}_{ij} & 0 \end{pmatrix}$$

and S is the constant matrix due to the scattering:

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 9/10 & 3/10 & 0 \\ 0 & 0 & 1/5 & 2/5 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

The eq.(23) has general solution:

$$\underline{\underline{N}}(\tau) = e^{S\tau}\Big[1 + \int_{\tau_o}^{\tau} \ell M(\tau')\,d\tau' + \int_{\tau_o}^{\tau} M(\tau')\,\ell \int_{\tau_o}^{\tau'} M(\tau'')\,\ell\, d\tau''\, d\tau' + \ldots\Big]\, e^{-S\tau_o}\, \underline{\underline{N}}(\tau_o), \qquad (24)$$

where

$$e^{S\tau} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & e^{\tau} & 0 & 0 & 0 \\ 0 & 0 & e_1(\tau) & 3e_3(\tau) & 0 \\ 0 & 0 & 2e_3(\tau) & e_2(\tau) & 0 \\ 0 & 0 & 0 & 0 & e^{\tau} \end{pmatrix}$$

with

$$e_1(\tau) = \frac{1}{7}\left(6e^{\tau} + e^{3/10\tau}\right)$$

$$e_2(\tau) = \frac{1}{7}\left(e^{\tau} + 6e^{3/10\tau}\right)$$

$$e_3(\tau) = \frac{1}{7}\left(e^{\tau} - e^{3/10\tau}\right)$$

and $M(\tau)$ is:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \hat{C}^{ke}_{i}\, e^{-\tau} e_1(\tau) & 3\hat{C}^{ke}_{i}\, e^{-\tau} e_3(\tau) & 0 \\ -\xi\, e_1(-\tau)\sigma_{ij} & e^{\tau} e_1(-\tau)\, \hat{H}_{ij}{}^{k} & 0 & 0 & 3e(-\tau)e^{\tau} L^{k}_{ij} \\ -\xi\, 2e_3(-\tau)\sigma_{ij} & 2e^{\tau} e_3(-\tau)\, \hat{H}^{k}_{ij} & 0 & 0 & e_2(-\tau)e^{\tau} L^{k}_{ij} \\ 0 & 0 & 2e^{-\tau} e_3(\tau)\, \hat{L}^{ke}_{ij} & -e_2(\tau)e^{-\tau} L^{ke}_{ij} & 0 \end{pmatrix}$$

Let us take as initial conditions:

$$\underline{\underline{N}}(\tau_0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

The first term of the series is:

$$\underline{\underline{N}}(\tau) = \underline{\underline{N}}(\tau_0) + e^{S\tau} \int_{\tau_0}^{\tau} e M(\tau')\, d\tau'\, N(\tau_0)$$

$$= \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ -e_1(\tau) \int_{\tau_0}^{\tau} e_1(-\tau')\, \xi\sigma_{ij} e\, d\tau' - 3e_3(\tau)\, 2\int_{\tau_0}^{\tau} e(-\tau')\, \xi\sigma_{ij} e\, d\tau' \\ 2e_3(\tau) \int_{\tau_0}^{\tau} 2e_1(-\tau')\, \xi\sigma_{ij} e\, d\tau' - e_2(\tau)\, 2\int_{\tau_0}^{\tau} e_3(-\tau')\, \xi\sigma_{ij} e\, d\tau' \\ 0 \end{pmatrix}.$$

At the present time $\tau = 0$, so $e_1(0)=1$, $e_2(0)=1$, $e_3(0)=0$ and

$$I_{ij} = \int_{\tau_0}^{0} - \xi \ell e_1(-\tau')\sigma_{ij}\, d\tau' = \xi \int_0^{\tau_0} \frac{1}{7}\left(6e^{-\tau'} + e^{-\frac{3}{10}\tau'}\right)\ell\sigma_{ij}\, d\tau'$$

$$Q_{ij} = \int_{\tau_0}^{0} - \xi \ell 2 e_3(-\tau')\sigma_{ij}\, d\tau' = \xi \int_0^{\tau_0} \frac{2}{7}\left(e^{-\tau'} - e^{-\frac{3}{10}\tau'}\right)\ell\sigma_{ij}\, d\tau'$$

(26)

In Bianchi type I models all the other terms of the series are zero because $M(\tau')\cdot M(\tau'') = 0$,so eq.s (26) represent the complete solution to this order of approximation. If $\hat{C}^{ke}_{i}$, $\hat{H}^{ke}_{i}$, $\hat{L}^{ke}_{i}$ are different from zero, the other terms of the series are present; however we can consider eq.s (26) as the solution before the recombination between matter and radiation, for $l \to 0$, so that the other terms of the series become negligible, since they are multiplied by higher powers of l. So if $\sigma_{ij}\, l$ is a constant we have at the beggining of the decoupling (for $\tau_0 \to \infty$):

$$I_{ij} = \xi \frac{4}{3}\sigma_{ij}\ell$$

$$Q_{ij} = -\xi \frac{2}{3}\sigma_{ij}\ell \, .$$

(27)

The same result is obtained if we use an expansion in powers of l for optically thick stages(see Dautcourt and Rose,1978). Let us write the system (22) in the following way:

$$\ell\left(\frac{\partial}{\partial t} D_i + \hat{C}^{ke}_{i} I_{ke}\right) = - D_i$$

$$\ell\left(\frac{\partial}{\partial t} I_{ij} - \xi\sigma_{ij}\right) = -\frac{9}{10} I_{ij} - \frac{3}{10} Q_{ij}$$

$$\ell\left(\frac{\partial}{\partial t} Q_{ij} + \hat{L}^{ke}_{ij} U_{ij}\right) = -\frac{1}{5} I_{ij} - \frac{2}{5} Q_{ij}$$

$$l\left(\frac{\partial}{\partial t}U_{ij}+\hat{L}^{kl}_{ij}\,Q_{ij}\right)=-U_{ij}$$

(28)

If 1 = 0 the rhs constitute a system of algebraic equations for the expansion coefficients, from which we obtain:

$$D_i=0,\quad I_{ij}=0,\quad Q_{ij}=0,\quad U_{ij}=0,$$

only unpolarized isotropic radiation will be present in this case. Thus considering ${}_0N_0$ as of zero order in 1 and the remaining coefficients of first order, we have to first order in 1:

$$D_i=0$$
$$\frac{9}{10}I_{ij}+\frac{3}{10}Q_{ij}=\xi l\sigma_{ij}$$
$$\frac{1}{5}I_{ij}+\frac{2}{5}Q_{ij}=0$$
$$U_{ij}=0\ ,$$

from which:

$$Q_{ij}=-\frac{2}{3}\sigma_{ij}\,l\xi$$
$$I_{ij}=\frac{4}{3}\sigma_{ij}\,l\xi\ .$$

In the opposite case (no scatterings), we find from eq.(26), $Q_{ij}=0$ and $I_{ij}=-\xi\int\sigma_{ij}\,dt$. Let us now consider the case of axisymmetric Bianchi type I models. In this case we can take the shear tensor of diagonal form and choosing the symmetry axis along the i=3 direction, we have $\sigma_{11}=\sigma_{22}$, then $I_{11}=I_{22}$ and $Q_{11}=Q_{22}$.
As a consequence of the axial symmetry the Stokes para_

meter $U=0$. In fact by definition $U = I_p \sin 2\chi$, where I_p is the intensity of the polarized component and χ is the angle by which the reference direction (along the meridians) is rotated with respect to the polarization axis; because of the symmetry is $\chi = 0$ or $\chi = \pi/2$ (the sign of Q discriminates between the 2 possibilities). Then

$$_0N = {}_0N_0 + (\mu^2 - \frac{1}{3})({}_0N_{33} - {}_0N_{11})$$

$$_2N = \frac{Q}{ch^4\nu^3} = \frac{1}{2}(1-\mu^2)({}_2N_{33} - {}_2N_{11}) .$$

Since

$$Q_{ii} = 2\xi \int \sigma_{ii} \ell \, e_{3}(-\tau) d\tau$$

the optically thick solution is:

$$Q = -\xi ch^4\nu^3 \frac{1}{3} \ell (\sigma_{33} - \sigma_{11})(1-\mu^2) {}_0N_0 \qquad (29)$$

So the polarization is proportional to $\sin^2\theta$ (is null along the symmetry axis), as is expected because of the axial symmetry. Moreover, by definition $Q = I_p \cos 2\chi$, so the polarization is along the meridians if the greatest expansion is along the simmetry axis (being $\xi = -1$ in the Railegh-Jeans region).

Then in axisymmetric Bianchi type I models the two parameters I and Q are sufficient to caracterize the radiation field.

Tolman and Matzner have numerically integrated the system (22) to first order in shear in Bianchi universes of type IX,I and V.

For the calculation the evolution of shear and of the ionization fraction are needed; for the shear evolution the results of Collins and Hawking (1973) have been used:

$$\sigma_{ij} = \sigma_{ij}|_0 \,(1+z)^3, \qquad \text{type I and V} \qquad (30)$$

and

$$\sigma_{ij} = A_{ij}\sqrt{1+\Omega_0 z} + B_{ij}(1+z)^{3/2}\sqrt{1+\Omega_0 z}\,(1+3z\Omega_0)(4\Omega_0-1)^{-1}, \qquad \text{type IX} \qquad (31)$$

with A_{ij} and B_{ij} constant.

The first term of eq. (31), which is generated by the coupling between spatial curvature and anisotropic expansion, is neglected in the numerical integration of the transfer equation; however, we note, in some cases, in particular in the case of a second ionization, the anisotropy mode driven by the curvature could be more important than the other one. For the fractional ionization x(z) of the matter during decoupling they have used the numerical results of Peebles(1968). Generally, x begins to decrease from unity at about $z \simeq 1900$, drops most rapidly around $z \simeq 1400$ and from $z \simeq 800$ on, it is down to a constant residual fraction $\simeq 3 \cdot 10^{-5}\,\Omega_0^{-1/2}$.

Tolman and Matzner find, in agreement with Negroponte and Silk(1980), that quite a significant error is made by assuming a step function behaviour for x(z), either at decoupling or at reheating. In fact in the step model of Negroponte and Silk, in which x(z) = 1 for $z > 1500$ and $x(z) = 10^{-5}$ for $x \leq 1500$, they find a polarization degree $P \simeq 10^2 \left.\frac{\Delta H}{H}\right|_0$ (where $\left.\frac{\Delta H}{H}\right|_0$ is the present anisotropy of the Hubble constant), while with an accurate model for the fractional ionization history they obtain $P \simeq 10^3 \cdot \left.\frac{\Delta H}{H}\right|_0$.

This is due to the different number of scattering n_s between the recombination and now; in the step model is $n_s = 0.03$ and in the other case is $n_s = 10^3$. Since an higher number of scatterings may increase the polarization but always isotropizes the radiation field, a radiation anisotropy about two times greater is found in the step model. However, we note that an anisotropy calculation following the Sachs-Wolfe method (1967), in which a step-model with a last scattering surface shifted to $z=1000$ is used, gives rise to an error on the anisotropy degree only of some per cent .

Tolman and Matzner have also considered the case of a second ionization of the intergalactic medium in which the time of reheating is extended from z_1 to z_2 , with z_1-z_2 equal 2 or 3 and z_1 less than or equal 10.

Let us now consider the results of the numerical integration.

In Bianchi type I and V the coefficient $\hat{L}^{\kappa e}_{ij}$,which is a function of the spatial curvature, vanishes, so only in type IX models a twisting of the direction of polarization is expected.The rotation angle with respect to the meridians is independent of the observational direction (see fig. 1); the twist of polarization is large (of the order of 60° for $\Omega_0 = 1.5$), so it should be easily observable if the polarization could be. This result suggests an important observational test of the spatial curvature of the standard models.

The anisotropy and polarization results are presented in fig. 2 , 3 , 4 where the present amplitudes of the quadrupole I_{ij} and dipole D_i intensity anisotropies are given in units of $10^4 \left.\frac{\sigma_{ij}}{H}\right|_0$; the degree of polarization

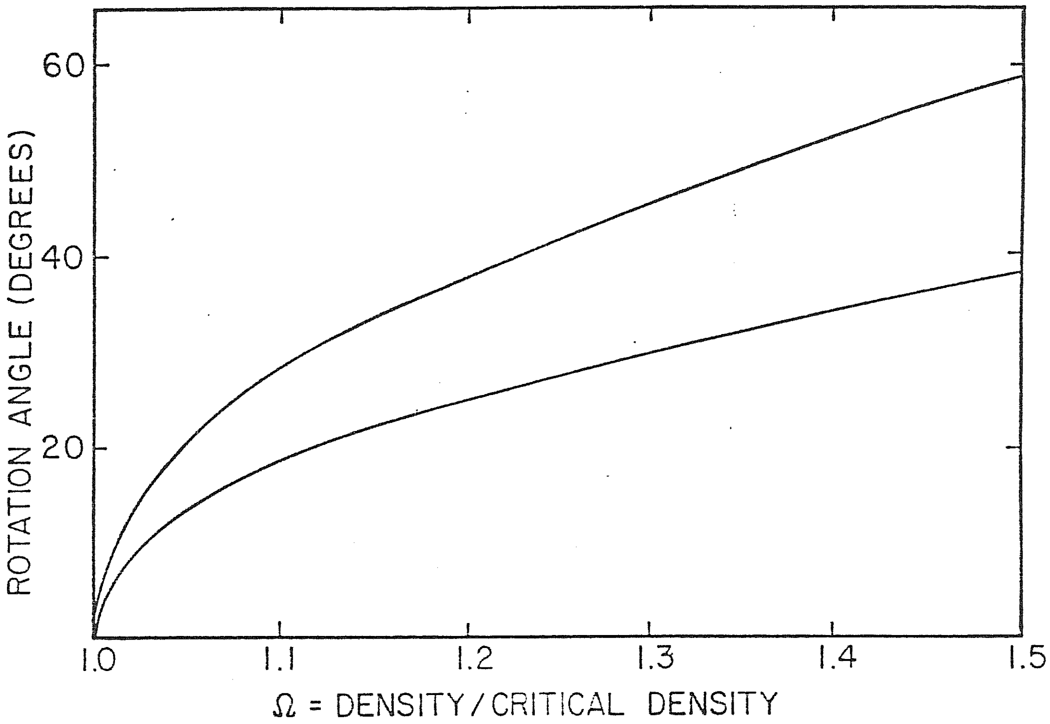

ROTATION ANGLE (DEGREES)
20
40
60
1.0
1.1
1.2
1.3
1.4
1.5
Ω = DENSITY/CRITICAL DENSITY

Fig. 1

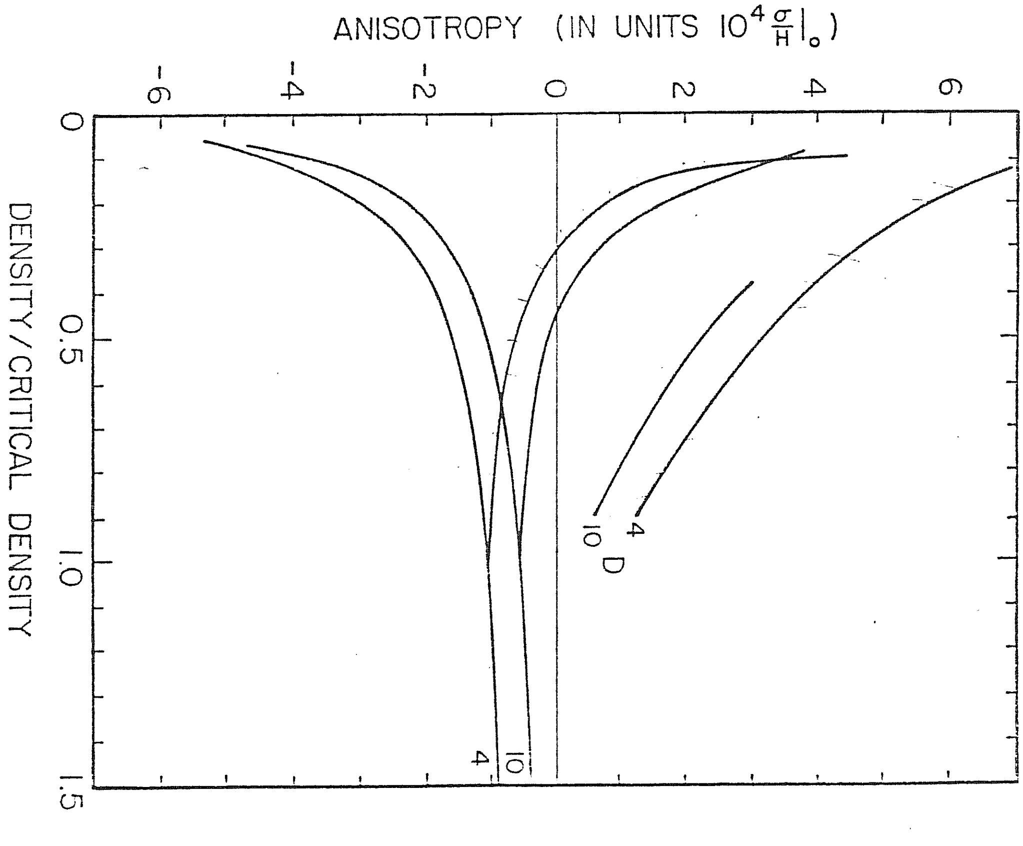

ANISOTROPY (IN UNITS $10^4 \frac{\sigma}{H}|_o$)
-6
-4
-2
0
2
4
6
DENSITY/CRITICAL DENSITY
0
0.5
1.0
1.5
4
10
D

Fig. 2

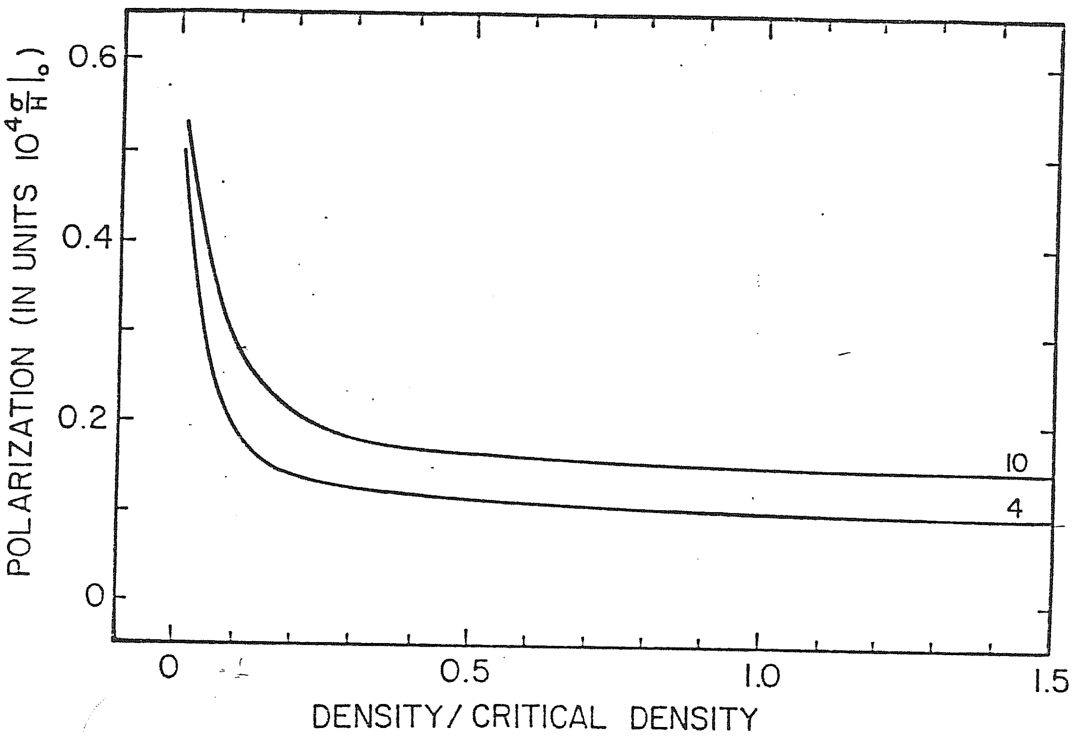

POLARIZATION (IN UNITS $10^4 \frac{\sigma}{H}|_0$)
0.6
0.4
0.2
0
10
4
0
0.5
1.0
1.5
DENSITY / CRITICAL DENSITY

Fig. 3

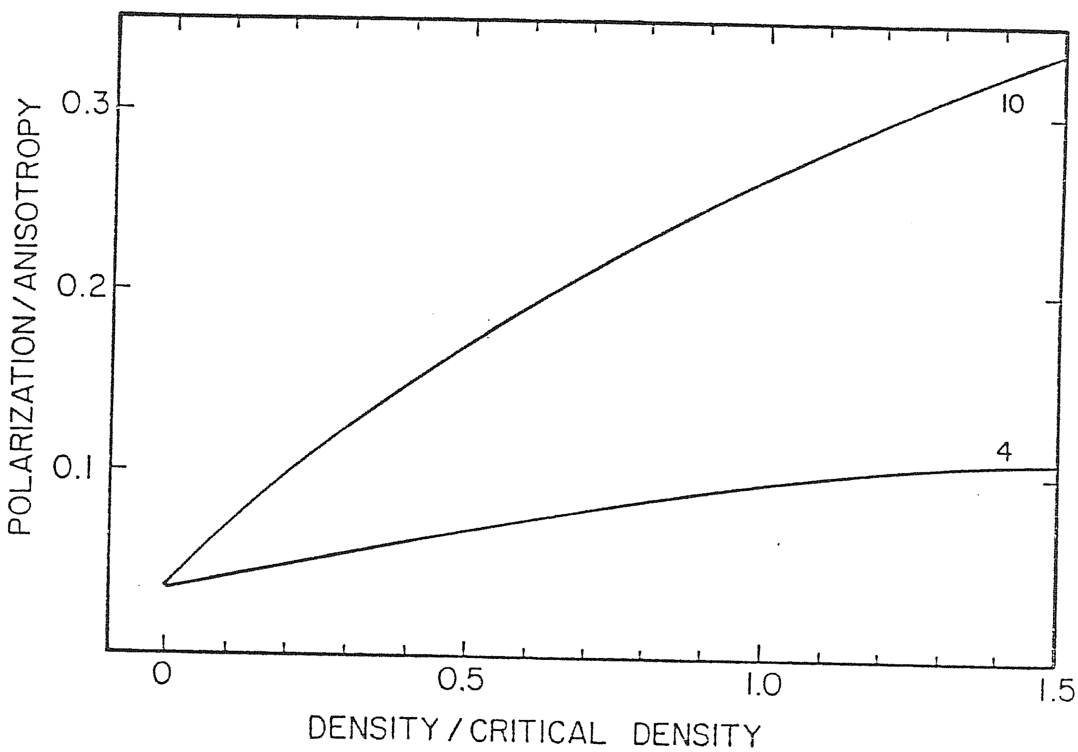

POLARIZATION / ANISOTROPY
0.3
0.2
0.1
10
4
0
0.5
1.0
1.5
DENSITY / CRITICAL DENSITY

Fig. 4

in the quadrupole mode P_{ij}, also in units of $10^{-4}\,\frac{\sigma_{ij}}{H}/_0$.
The Hubble constant was 100 km s^{-1} Mpc^{-1} throughout.
The quadrupole intensity amplitude is normally negative, simply because the maximum of the shear corresponds to the greatest redshift and therefore a minimum of the intensity. In closed models, where a dipole is never generated, and in open models in which the shear is in a plane perpendicular to the preferred direction so that no dipole is formed, $|I_{ij}|$ increases uniformly in models with decreasing Ω_0. This is expected, since a lower density results in less numerous scatterings, allowing the anisotropy to build up by differential redshifting. In open models in which the formation of a dipole anisotropy is possible, they find that the dipole increases while the quadrupole decreases with decreasing Ω_0. However for very open models ($\Omega_0 \lesssim .1$) there are integration problems due to having neglect higher order multipoles, so this procedure fails.
The polarization and anisotropy degree are sensitive to a second ionization of the diffuse gas. Their values are a function of the optical depth or number of scatterings n_s between the reheating time t_h and the present time t_0. If $n_s \ll 1$ ($z_h \ll 10$), the reionization has no effect; if $n \gtrsim 1$ the two polarization states decrease at a different rate. The maximum polarization is attained for $n_s = 1.72$. More scatterings tend to damp the reionization anisotropy of both the polarization states; in the limit $n_s \gtrsim 30$ polarization and anisotropy drop to the asymptotic values:

$$I = \xi \frac{4}{3} \Delta H_0 \ell$$

$$Q = -\xi \frac{2}{3} \Delta H_0 \ell$$

Consequently the ratio P/I increases with τ_h reaching a maximum equal to 2.

The behaviour of the polarization degree depends on the function $e_3(-\tau)$, which couples polarization and anisotropy. For Bianchi type I models with a second ionization, we have:

$$\underline{\underline{N}}(\tau=0) = e^{-S\tau_h}\,\underline{\underline{N}}(\tau_h) + \int_{\tau_h}^{0} \ell M(\tau')\,d\tau' \; e^{-S\tau_h}\,\underline{\underline{N}}(\tau_h)$$

Setting

$$\underline{\underline{N}}(\tau_h) = \begin{pmatrix} 1 \\ 0 \\ I_{ij}(\tau_h) \\ Q_{ij}(\tau_h) \\ 0 \end{pmatrix}$$

we obtain:

$$\underline{\underline{N}}(\tau=0) = \begin{pmatrix} 1 \\ 0 \\ I_{ij}(\tau_h)\,e_1(-\tau_h) + Q_{ij}(\tau_h)\,3e_3(-\tau_h) \\ I_{ij}(\tau_h)\,2e_3(-\tau_h) + Q_{ij}(\tau_h)\,e_2(-\tau_h) \\ 0 \end{pmatrix} +$$

$$\begin{pmatrix} 0 \\ 0 \\ -\xi\int_{\tau_h}^{0} e_1(-\tau')\,\ell\,\sigma_{ij}\,d\tau' \\ -\xi\int_{\tau_h}^{0} 2e_3(-\tau')\ell\,\sigma_{ij}\,d\tau' \\ 0 \end{pmatrix}$$

Then, starting with a large radiation anisotropy $I_{ij}(\tau_h)$ the polarization reaches its maximum value in correspondence of the maximum value of the function $e_3(-\tau_h)$, that is for $\tau_h = 1.72$.

As noted by Basko and Polnarev (1980) an interesting property of the quadrupole anisotropy and polarization is they drop as $e^{-\frac{3}{10}\tau}$ and not as $e^{-\tau}$. The reason is that in the course of scattering an anisotropic, non polarized component of the radiation field does not simply becomes isotropic, but partly transforms into a polarized component. The scattering of the latter results in its partial depolarization, with a certain fraction being transformed into the anisotropic, non polarized component. Thus, the anisotropic and the polarized components of the radiation field make a single complex which decays more slowly than $e^{-\tau}$.

Tolman and Matzner have also considered some type IX models with $\Omega_0 > 1$ and $\Omega_B = 0.12$, because of the upper limit set on Ω_B by studies of nucleosynthesis (Yang et al. 1984). The results for the dark matter dominated closed models did not significantly vary with Ω_0 for a fixed $\Omega_B = .12$. In every case the anisotropy in these models was about two times larger (due to less free electrons) than the corresponding model in fig.2, while the polarization was smaller (again due to less scatterings) so that the ratio P/I is about 0.045 for all dark matter models.

Moreover measuraments of the quadrupole polarization may provide an important test on wether the dominant source of the quadrupole is shear or are matter inhomo_ geneities. In fact, while the essentially random or statistical origin of the anisotropy in the case of inhomogeneities works against the development of large scale polarization, the opposite is true in the case of shear(see Sec.II).

Finally, if a quadrupole anisotropy in the microwave background will be observed, the ratio of polarization to anisotropy may provide a good test on the reheating time and on the hydrogen mass fraction of the universe.

2.2 THE POLARIZATION OF THE MICROWAVE BACKGROUND RADIATION IN OPEN UNIVERSES.

This problem has been extensively treated by Tolman(1985). The cosmological models he considers are homogeneous models with negative spatial curvature ($\Omega_0 < 1$) and a small expansion anisotropy. It is well known that the spatial curvature in these models has the effect of distorting the radiation anisotropy by focussing it onto a relatively small angular scale (Novikov,1968). In a model with shear, the radiation anisotropy has a quadrupolar angular dependence when it is generated, but by the present time it has been squeezed into a single small intense "hot spot", the magnitude of the effect depends on Ω_0. In the case of very open universes the standard approach of expanding in multipole moments becomes extremely difficult, because the effect of interest occurs on all angular scales from 90° to very few degrees.

Then a direct or Monte Carlo method is used to compute numerically the entire histories of a number of photons, and therefore to determine their distribution on the sky. The method relies heavily on the symmetries of the problem, particularly spatial homogeneity. This symmetry implies that any physical quantity cannot have a dependence on spatial position. Cylindrical symmetry is also assumed for the shear, the direction of the photon being thus specified by a single parameter, the angle from the symmetry axis.

The evolution of photons is the outcome of three processes: shear which redshifts the photons at a rate dependent on their direction, thus introducing intensity anisotropy; spatial curvature which deflects the photons into one preferred direction, thus distorting the aniso_ tropy; and Thomson scattering which changes the direction and polarization of the photon istantaneously and without changing the total intensity.

The study of the first two processes makes use of the equations of motion for the free propagation of photons. Taking the symmetry axis along the polar axis, they are:

$$\dot{k} = -\sigma\left(\tfrac{3}{2}\cos^2\theta - \tfrac{1}{2}\right) \qquad a)$$

$$\dot{\phi} = 0 \qquad b)$$

$$(\cos\theta)^{\cdot} = -\sin^2\theta\left(R^{-1}e^{-\beta} + \tfrac{3}{2}\cos\theta\right) \qquad c)$$

$$(32)$$

where k = k^{o} R (is the renormalized energy), and $\sigma = \sigma_{33}$, $\beta = \left(\frac{2}{3}\beta_{ij}\beta^{ij}\right)^{1/2}$. The factor R^{-1} identifies the spatial curvature term which gives rise to the distorsion; note that it is never positive, so the spatial curvature must always bends photon paths toward cos ϑ = -1, i.e. the symmetry axis. The second term does not have just one sign, but will always be very small due to the factor σ . In the case of isotropy ($\sigma = \beta = 0$), the second term of eq.(32 c) apparently implies that spatial curvature is still producing a distorsion because photon paths are redshifted alike (eq. 32 a) and so only perfect isotropy can be observed.

The equations of motion have been integrated in Bianchi type V models, where the shear evolve as $\sigma(z) = \sigma_0 (1+z)^3$ (Collins and Hawking,1973). The results of the simulation are shown in the fig.5,6,7,8,9,which are graphs of the intensity anisotropy $\frac{\Delta I}{I}$ and the ratio of polarized intensity to anisotropic intensity P/ΔI against the dire_ ction ϑ on the sky in units of the present shear to expansion ratio $\frac{\sigma}{H}\Big|_0$.
Statistical fluctuations on the scale of the bin size (1°) are seen in all the results; they should be ignored since only structure which is larger than the average fluctuation

can be considered physical. The models considered have $\Omega_0 = 0.1$ and no reheating (fig. 5), reheating at z = 4 (fig. 6) 10 (fig. 7), 100 (fig. 8), as well as $\Omega_0 = 0.3$ with $z_r = 10$ (fig. 9).

The cylindrical symmetry of the model implies that only two directions for the net polarization are realizable: parallel to the meridians, from pole to pole, or perpendicular to this; thus U = 0 is the expected result.
The hot spot in $\Delta I/I$ is evident at the pole on the right with width (that is, semidiameter of the spot maximum) in agreement with Novikov (1968): 3.5° for Ω_0=0.1, 11.8° for Ω_0=0.3; however, also evident is how quickly scattering reduces it. The sky is otherwise featurless in $\Delta I/I$.
The polarization of the radiation in the hemisphere opposite the spot is roughly constant and positive if we take the expansion greatest along the symmetry axis. The hemisphere which contains the direction of the spot shows three different features in the polarization. First, roughly centered between pole and equator there is a large band of strong polarization which has the direction of negative P/I (opposite to that of the rest of the sky). these regions cover 15% (fig.6,b), 20% (fig.7,b) and 30% (fig.9,b) of the area of the sky, just below the observed

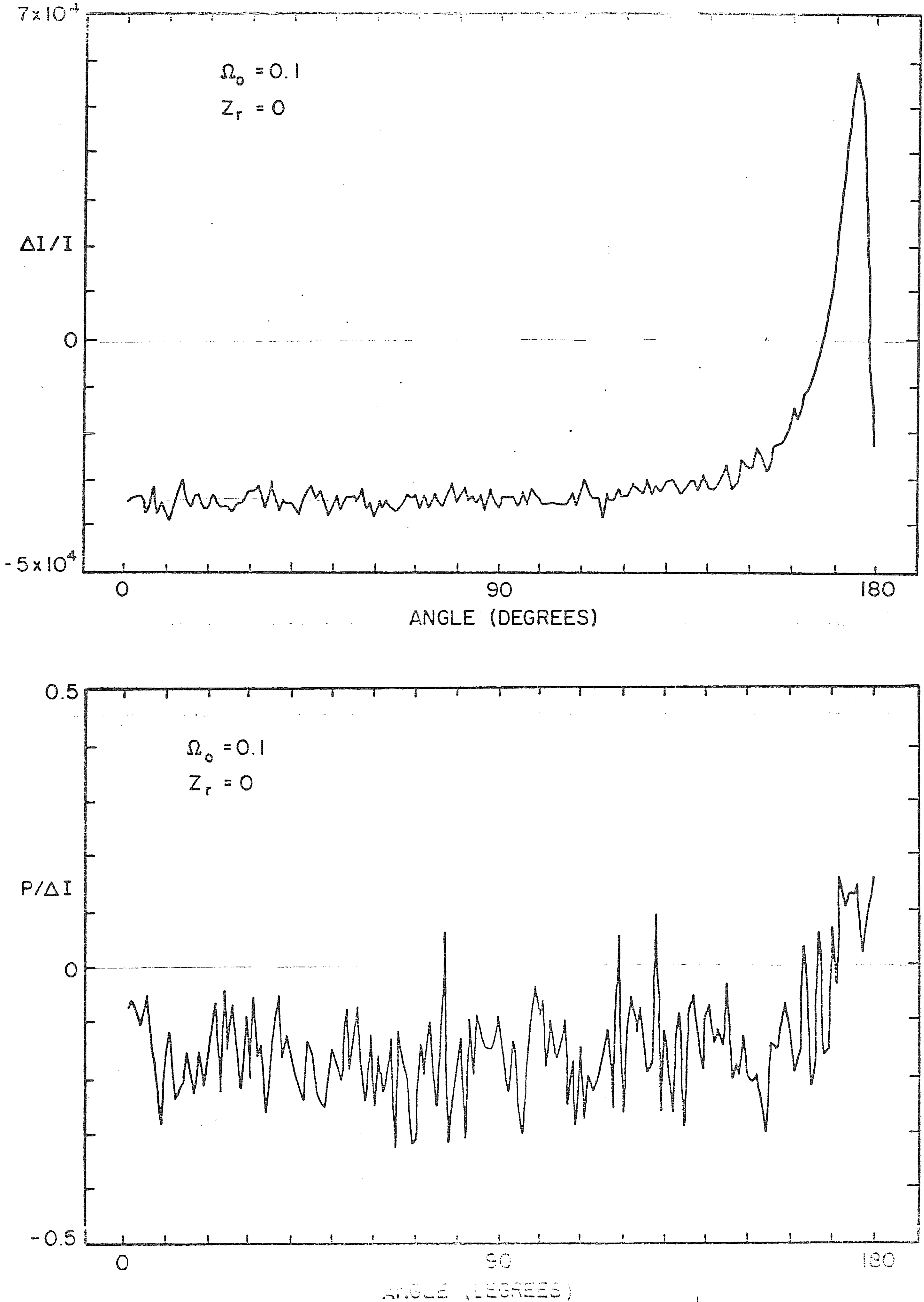
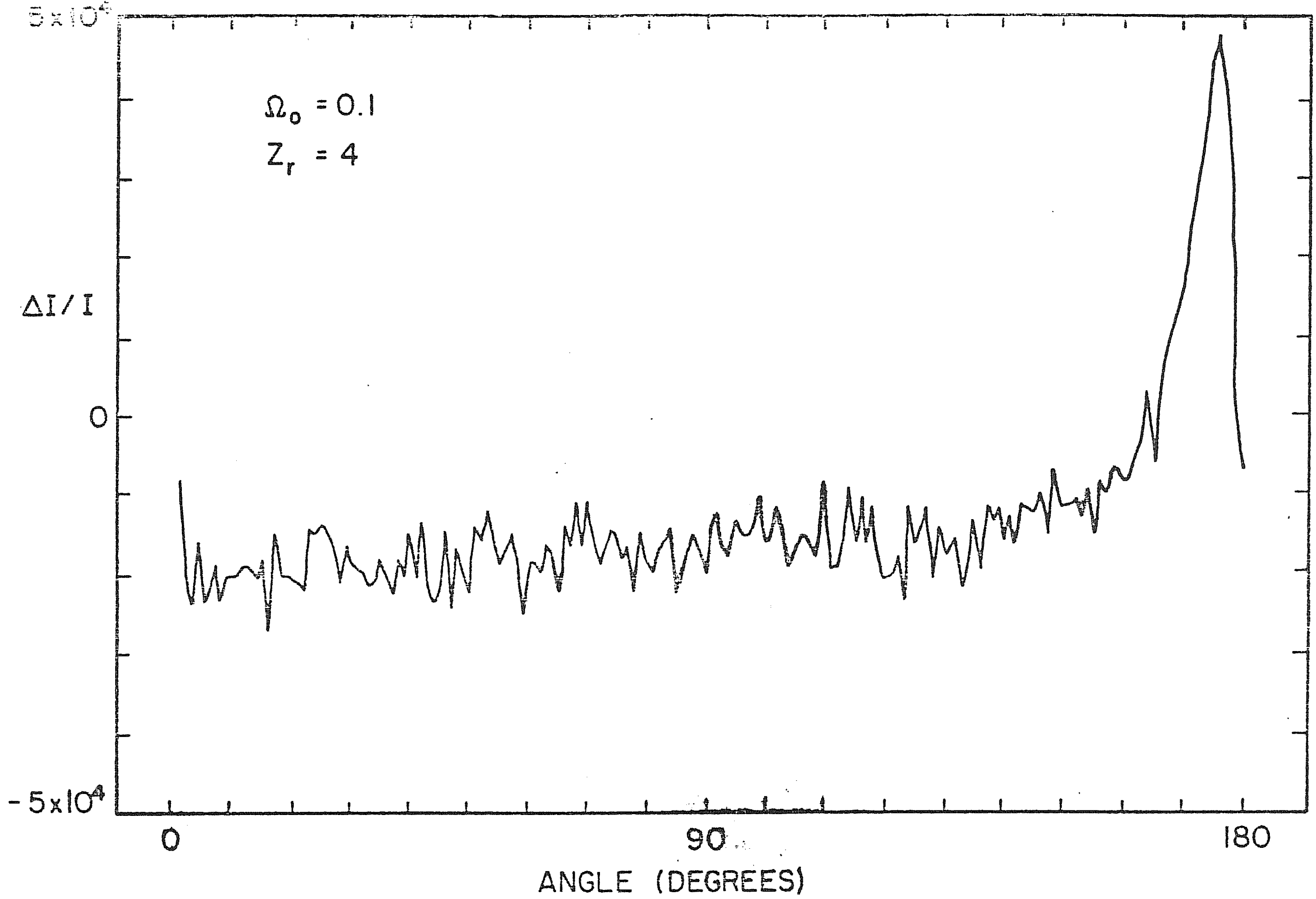
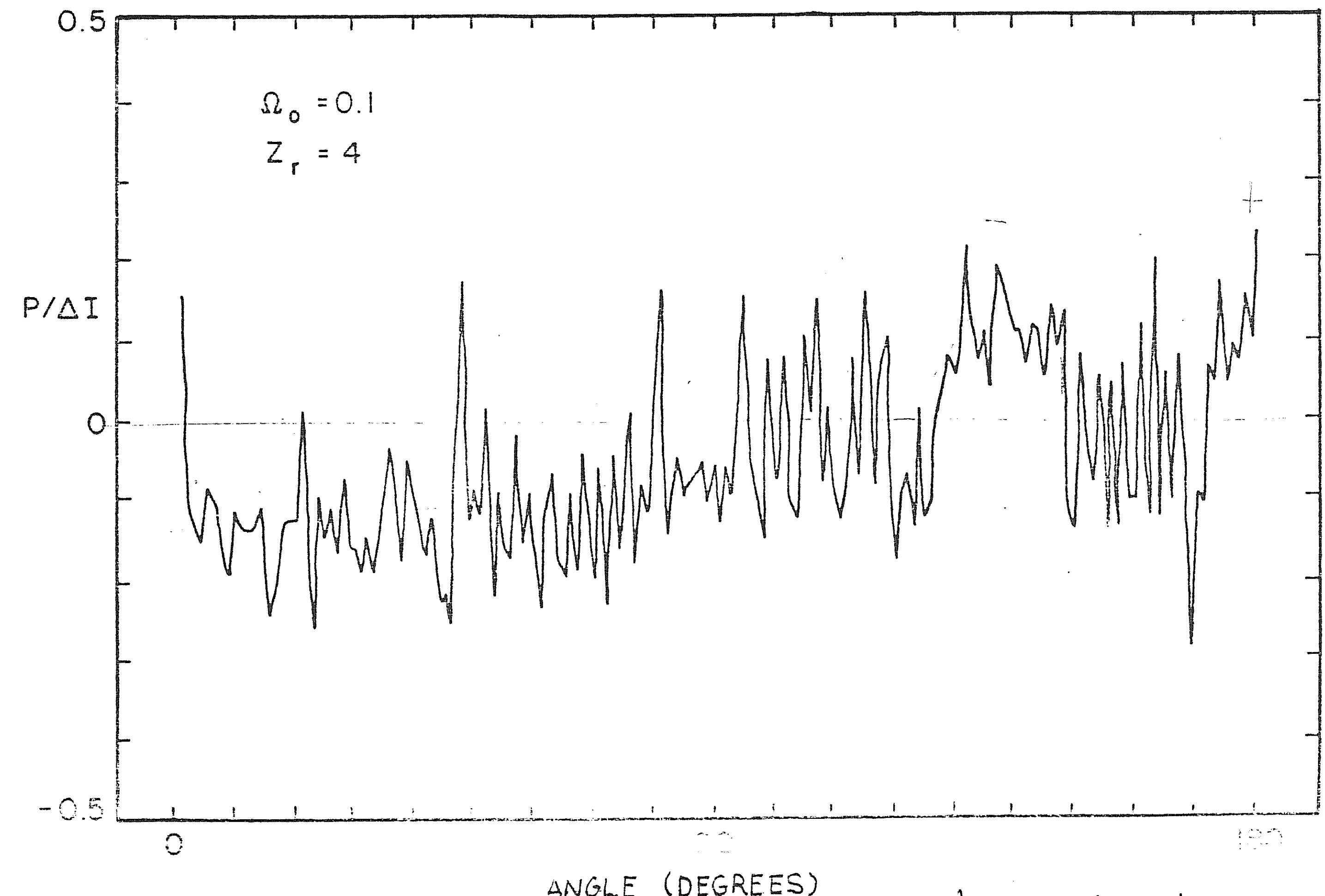

7x10^4
Ω_0 = 0.1
Z_r = 0
ΔI/I
0
-5x10^4
0
90
180
ANGLE (DEGREES)
0.5
Ω_0 = 0.1
Z_r = 0
P/ΔI
0
-0.5
0
90
180
ANGLE (DEGREES)

Fig 5 a, b

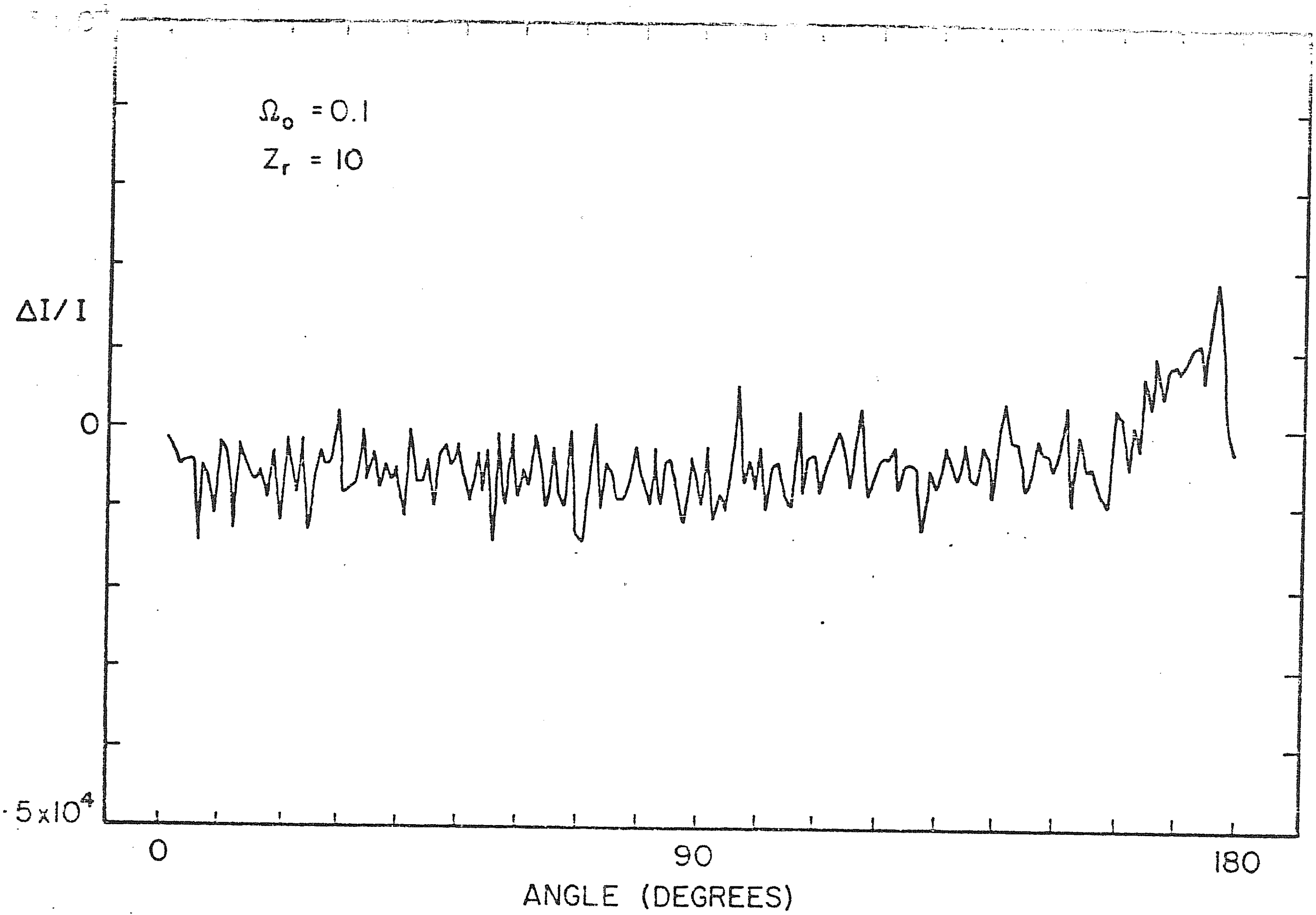


5x10^4
Ω_o = 0.1
Z_r = 4
ΔI/I
0
-5x10^4
0
90
180
ANGLE (DEGREES)

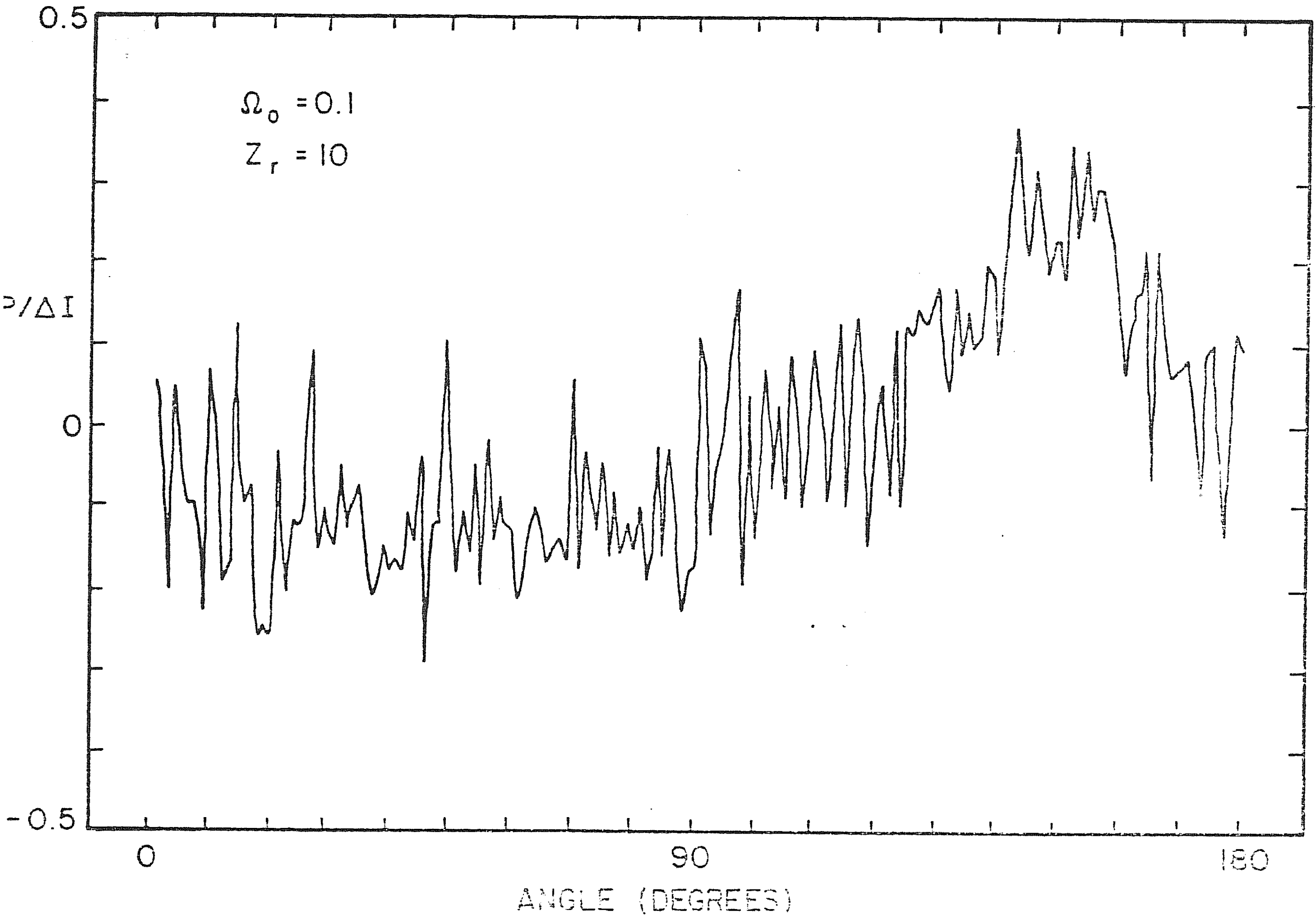


0.5
Ω_o = 0.1
Z_r = 4
P/ΔI
0
-0.5
0
ANGLE (DEGREES)

fig 6 a, b

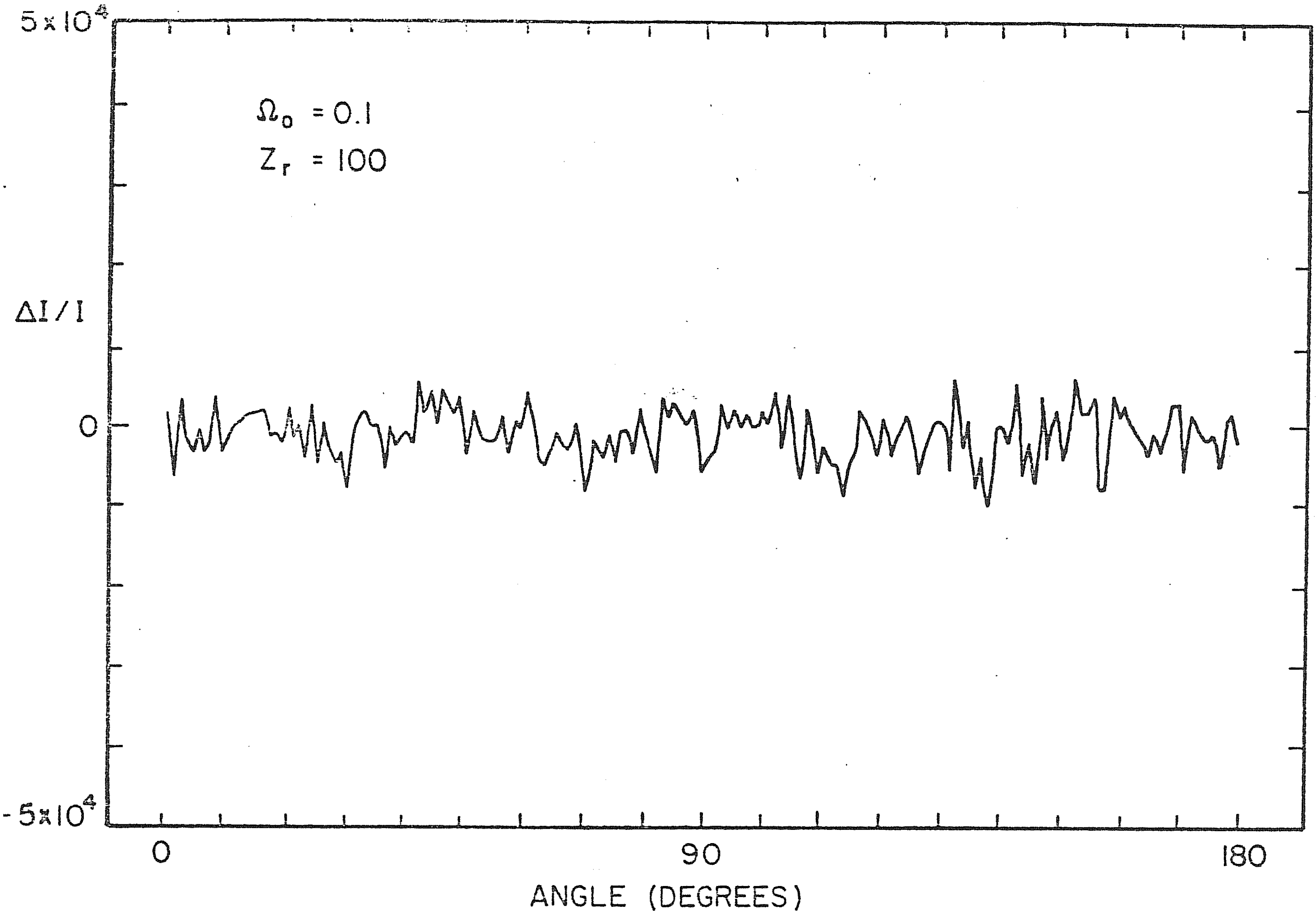


Ω_o = 0.1
Z_r = 10
ΔI/I
0
-5x10⁴
0
90
180
ANGLE (DEGREES)



0.5
Ω_o = 0.1
Z_r = 10
P/ΔI
0
-0.5
0
90
180
ANGLE (DEGREES)

fig 7a, b


5x10^4
Ω_0 = 0.1
Z_r = 100
ΔI/I
0
-5x10^4
0
90
180
ANGLE (DEGREES)

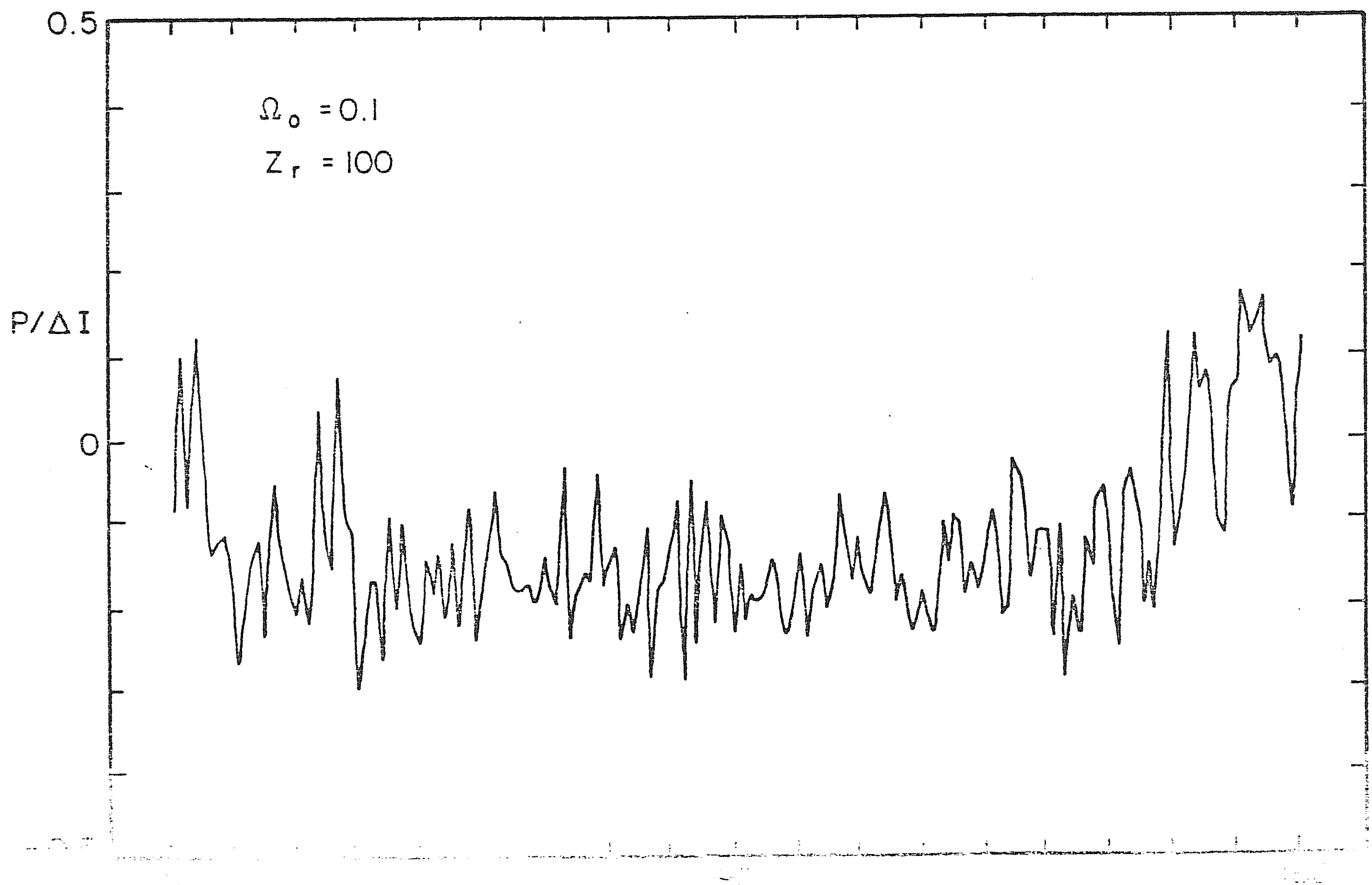

0.5
Ω_0 = 0.1
Z_r = 100
P/ΔI
0
ANGLE (DEGREES)

fig 8a,b

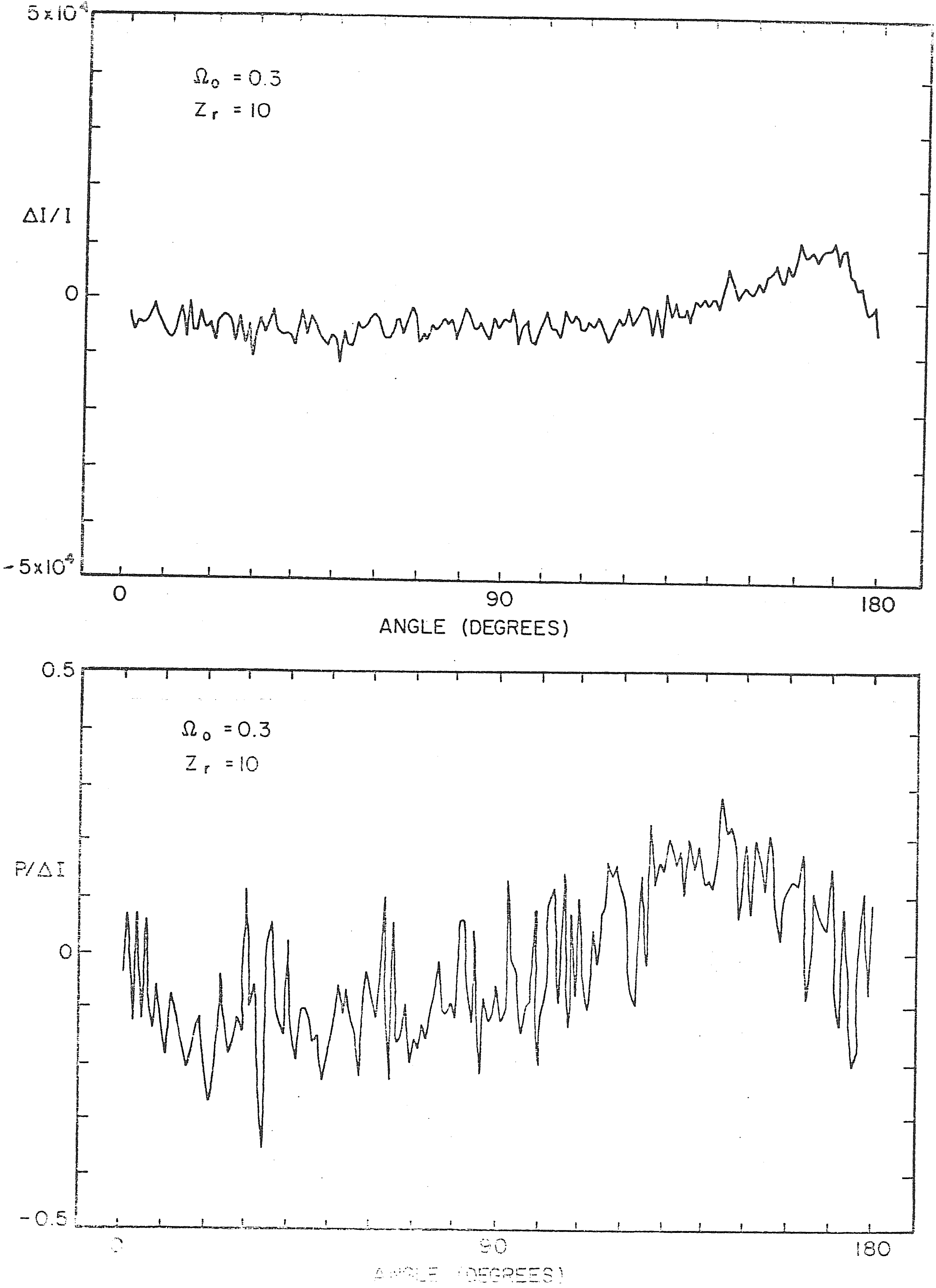

5x10⁻⁴
Ω₀ = 0.3
Zr = 10
ΔI/I
0
-5x10⁻⁴
0
90
180
ANGLE (DEGREES)
0.5
Ω₀ = 0.3
Zr = 10
P/ΔI
0
-0.5
0
90
180
ANGLE (DEGREES)

Fig 9 a, b

limits for $\frac{\sigma}{H}\Big|_0 \simeq 3\cdot 10^{-8}$. Second, the spot itself is polarized in the positive P/I direction, but at a significant level only if there is an unpolarized region near the equator and in fig. 6,b also surrounding the spot.

Reheating introduces scattering, so that the very narrow spot is converted into polarization, but with opposite direction and distributed broadly over the sky. Where this polarization is comparable in magnitude to the already present polarization, cancellation occurs and an unpolarized region results; in other direction the new polarization dominates and the unusual direction of polarization would be observed.

In general the shear could be triaxial, and have an additional cos 2ϕ variation. The hot spot in that case would be split into four parts, with alternately positive and negative peaks. But then even a little scattering would quickly mix the positive and negative regions and so damp the spot amplitude. The polarization would also develop a cos 2ϕ variation, but in contrast would be less sensitive to scattering because it has features on larger angular scales. The magnitude of P/Δ I is similar to that expected in a flat universe (20%), except for the unpolarized region.

Thus in open universes it has been shown that features in the polarization arise on three different scales, while the intensity anisotropy is featureless except for a single spot confined to less than one percent of the sky; so polarization could be more accessible to observation than intensity anisotropy.

2.3 EFFECT OF COMPTONIZATION ON THE MICROWAVE BACKGROUND RADIATION IN BIANCHI TYPE I UNIVERSES.

Stark (1981,b) calculated the dependence of the polarization and quadrupole component of the cosmic background radiation on frequency, optical depth and temperature of a reheated intergalactic gas in an axisymmetric Bianchi type I universe. He shows that Comptonization effects on the polarization and quadrupole component can become important for gas temperature $T_c > 10^{7\div 8}$ K.

The observations do not esclude the presence of an intergalactic hydrogen with density as high as the critical density; however the gas temperature T_e must be $> 10^5$ K in order to evoid detecting too much neutral hydrogen, while it must also be $T_e < 3 \cdot 10^8$ K in order to evoid detecting too much thermal emission in the X-ray region.

First Stark obtains the zero temperature ('$T_e = 0$') solution for the radiation properties of a cold intergalac_ tic gas; this solution is in agreement with that found by Basko and Polnarev (1980), Negroponte and Silk (1980). Then using this as a zero order solution, he finds the general solution for the intensity and polarization as functions of frequency, angle and optical depth of reheated gas for the finite temperature gas correct to $O(kT_e/m_e c^2)$.

The solution is given in terms of the vector

$$\tilde{J} = \begin{pmatrix} \tilde{I} \\ \tilde{Q} \end{pmatrix} = \frac{1}{x^3} \begin{pmatrix} I \\ Q \end{pmatrix}$$

where $x = \dfrac{h\nu}{kT_{rec}} \left(\dfrac{t}{t_{rec}} \right)^{2/3}$, with T_{rec} temperature of the radiation

at the recombination time t_{rec}, and can be written in the following way:

$$\tilde{J} = n_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \frac{\Delta H_0}{n_e \sigma_T c} (1-\mu^2) \begin{pmatrix} e \\ f \end{pmatrix} + \frac{\langle \beta^2 \rangle}{3} \begin{pmatrix} r \\ 0 \end{pmatrix} +$$

$$\frac{\langle \beta^2 \rangle}{3} (1-\mu^2) \frac{\Delta H_0}{n_e \sigma_T c} \begin{pmatrix} p \\ q \end{pmatrix} + \frac{\Delta H_0}{n_e \sigma_T c} \frac{\langle \beta^2 \rangle}{3} \begin{pmatrix} t \\ 0 \end{pmatrix} ,$$

with $$\frac{\langle \beta^2 \rangle}{3} \equiv \frac{K T_e}{m_e c^2} .$$

Substituing this form of solution into the radiative transfer equation containing the temperature corrections (see Stark 1981,a), he finds, after identifying the isotropic and quadrupole contributions for the various orders:

$$\frac{\Delta I}{I} = \frac{K T_e}{m_e c^2} \tau \frac{e^x x}{e^x - 1} \left(x \operatorname{cotgh} \frac{x}{2} - 4 \right)$$

This result agrees with that given by Sunyaev and Zeldovich (1969); the Comptonization effects change the photon (and therefore intensity) spectrum for an isotropic universe.

Let us now consider the Comptonization corrections to the polarization properties. The ratio of the polarization due to Comptonization to the zero temperature polarization $\frac{\delta^T}{\delta^0}$ is found to be of order:

$$\frac{\delta^T}{\delta^0} \simeq 10 \max(1, \tau) \frac{K T_e}{m_e c^2} \simeq \max(1, \tau) \frac{1}{5} \left(\frac{T_e}{10^8 K} \right)$$

Then appreciable changes to the zero temperature predic_ tion occur for temperatures $T_e > 10^{7 \div 8}$ K (depending on the optical depth of the reheated gas).

The ratio $\frac{\delta^T}{\delta^0}$ depends strongly on the frequency (see fig.10) The change in sign around $x \simeq 6$ means that for $x < 6$ the comptonized radiation is polarized at right angles with respect to the zero temperature result (i.e. it is perpendicular to the meridian plane). We see a large increase at high x due to the sharp fall off (and con_ sequent large frequency derivatives) of the Planck spectrum in this region. The Comptonization contribution to the quadrupole component $(\Delta T)^T$ divided by its zero temperature value is found to be:

$$\frac{(\Delta T)^T}{(\Delta T)^0} \simeq 5\tau \frac{kT_e}{m_e c^2} \simeq \frac{\tau}{10} \left(\frac{T_e}{10^8 K}\right) .$$

Fig. 11 shows clearly that (as for polarization) the Comptonization effects become rapidly more important at high frequencies. The general frequency dependence of the Comptonization distortions to the spectrum, polari_ zation and quadrupole component are all very similar.

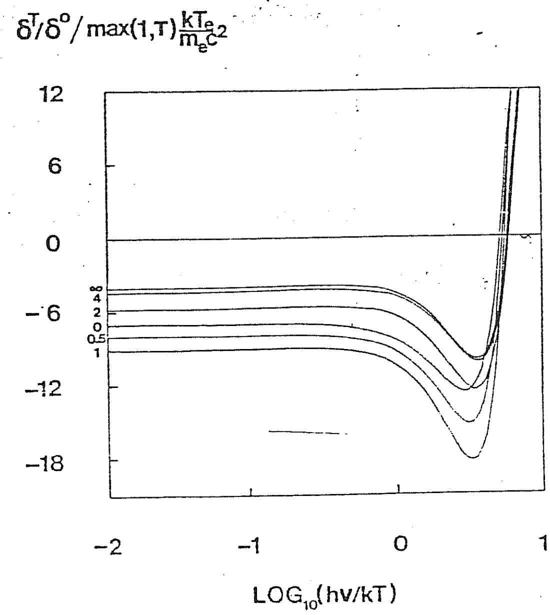


Figure The frequency dependence of the first order fractional Comptonization contribution to the degree of polarization (divided by $\max(1,\tau)\ kT_e/m_e c^2$) of the microwave background for the optical depths of reheated gas indicated. The actual fractional Comptonization changes (i.e. the contribution due to Comptonization divided by the zero temperature result) in per cent are obtained by multiplying these results by $\sim 2\max(1,\tau)\ (T_e/10^8\,\mathrm{K})$.

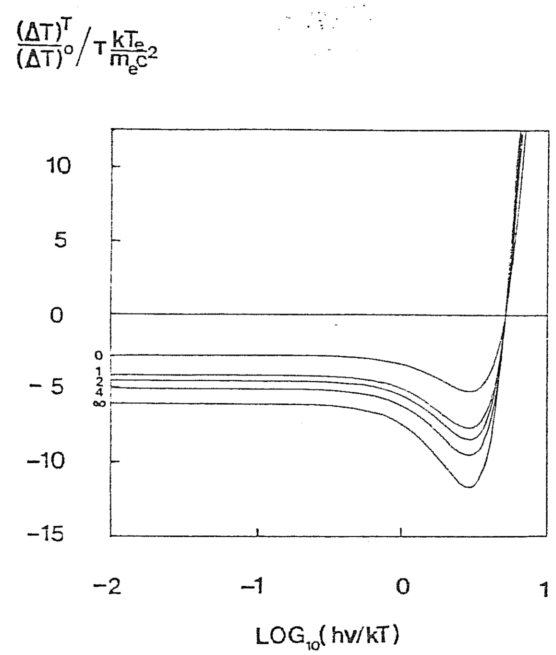


Figure The frequency dependence of the first-order fractional Comptonization contribution to th quadrupole temperature component (divided by $\tau\, kT_e/m_e c^2$) of the microwave background for the optic depths of reheated gas indicated. The actual fractional Comptonization changes (i.e. the contribution d to Comptonization divided by the zero temperature result) in per cent are obtained by multiplying the results by $\sim 2\tau\,(T_e/10^8\,\mathrm{K})$.

CHAPTER III

POLARIZATION OF THE MICROWAVE BACKGROUND RADIATION IN BIANCHI TYPE I MODELS WITH A HOMOGENEOUS MAGNETIC FIELD.

Astron. Astrophys. 59, ▮▮▮ ▮▮▮ (1985) 02(03.12.1; 03.09.1)

ASTRONOMY AND ASTROPHYSICS

Polarization of the microwave background radiation in Bianchi type-I cosmological models with a homogeneous magnetic field

E. Milaneschi* and R. Fabbri

Istituto di Fisica Superiore, Università di Firenze Via S. Marta 3, I-50139 Firenze, Italy

Received March 29, 1984, accepted February 25, 1985

Summary. In this paper we study the anisotropy and polarization properties of the 3 K cosmic background radiation in homogeneous anisotropic cosmological models filled with a uniform magnetic field. We generalize the polarization transfer equation to magnetic Bianchi models by adding a Faraday rotation term. By means of a multipole expansion, we solve the transfer equation for the quadrupole and octuple terms to the first order in the collision time for Bianchi type-I models. The solution is valid for the cosmological scenario with a secondary ionization of the intergalactic plasma and also for the time preceding thermal decoupling. The polarization degree is found to decrease for decreasing frequency according to a simple law, and the polarization pattern over the sky is affected by Faraday rotation in a characteristic way, quite different from space curvature effects. The polarization properties of the background radiation could therefore provide a powerful test for the existence of a homogeneous magnetic field, to be added to current astronomical tests.

Key words: cosmology – cosmic background radiation

1. Introduction

The angular distribution of the cosmic background radiation exhibits an anisotropy of about 10^{-3} on the dipole scale, while contrasting results have been reported on the quadrupole scale (Fabbri et al., 1980, 1982; Ceccarelli et al., 1982a; Fixsen et al., 1983; Lubin et al., 1983a). Although the dipole anisotropy may be entirely due to the peculiar motion of the Earth with respect to the radiation frame of reference, anisotropies at somewhat smaller scales, if genuinely extragalactic, would be linked to structures on a cosmological scale. It is therefore an important problem to discriminate real cosmological effects in the properties of the cosmic background. In this connection, it is very useful to observe that in anisotropic cosmologies, as first pointed out by Rees (1968), the radiation anisotropy is coupled to the linear polarization by Thomson scattering. Since large scale polarization cannot be produced by the peculiar motion of our frame of reference, a comparative analysis of anisotropy and polarization would be a powerful test for cosmological models of the universe.

In fact, no positive detections of cosmic polarization have been reported up to now. Only upper limits on the polarization degree are available, often referring to the Rayleigh-Jeans region (Nanos, 1979; Smoot and Lubin, 1979; Lubin and Smoot, 1981; Lubin et al., 1983b). The most stringent limits, due to Lubin et al. (1983b), are a few times 10^{-5} for the linear polarization and 7.10^{-3} for the circular polarization, at large angular scales and at a wavelength of 0.9 cm. In the millimetric region Caderni et al. (1978b) set limits of order 10^{-3} on the linear polarization at angular scales between 0.5° and 40°. For a comparison, the models investigated by Rees (1968), Basko and Polnarev (1980) and Negroponte and Silk (1980) predict polarization degrees of $10^{-4} \div 10^{-6}$ for anisotropies of order 10^{-4}.

However, in order to correctly interpret the implications of the experiments we should carefully examine the possible role of depolarizing mechanisms. In fact, a depolarizing effect is intrinsic to the radiation decoupling process in any anisotropic cosmologies (Brans, 1967). However, this was shown to be not very important (Caderni et al., 1978a; Fabbri and Breuer, 1980). More relevant is the problem of the possible influence of large scale magnetic fields. The depolarizing property of random magnetic fields is familiar in astrophysics. In our cosmological context, however, the polarization degree of the 3 K background can be affected also by a coherent magnetic field because of the finite thickness of the last scattering hypersurface. As the background photons traveled different path-lengths since their last scatterings and were subject to different Faraday rotations, the primeval polarization tends to be damped, and can conceivably be canceled out at least at certain wavelengths. Ceccarelli et al. (1982b) first attempted to evaluate this effect averaging the polarization degree over the last scattering interval by means of the familiar weighting factor $e^{-\tau}$, with τ the optical thickness of the cosmological medium. However, the work of Basko and Polnarev (1980) on nonmagnetic models shows that a brute force weighting procedure is not applicable because of a peculiar anisotropy-polarization coupling. As also noted by Negroponte and Silk (1980), simplified approaches are not successful in calculating the polarization, while they work much better for the radiation anisotropy.

The only reliable method to simultaneously evaluate both the anisotropy and the polarization of the cosmic background is to write down and solve a general relativistic transfer equation for the Stokes parameters of the radiation. Several authors have studied the transfer equation for homogeneous anisotropic cosmologies in the absence of magnetic fields. The solutions for Bianchi type-I models have been investigated by Rees (1968),

Send offprint requests to: R. Fabbri

* Present address: International School for Advanced Studies (SISSA), Trieste, Italy

14

Anile (1974), Basko and Polnarev (1980), Negroponte and Silk (1980), and Stark (1981). Tolman and Matzner (1981) extended the investigation to Bianchi models of type I, V, IX and VII, namely, to the classes of homogeneous models which are direct generalizations of the standard (closed, flat and open) Friedmann cosmologies.

In this paper we analyse the effects of a homogeneous magnetic field on the anisotropy and the polarization of the background radiation in Bianchi type I models.

To this purpose, in Sect. 2 we generalize the transfer equation for polarized radiation to the case of magnetic Bianchi models filled with a cosmic medium at rest in the homogeneity frame. No restriction is posed on the Bianchi type in this general formulation. However, we assume a non-relativistic intergalactic medium, so that the transfer equation is obtained by simply adding a Faraday rotation term. In the following sections we restrict ourselves to type I models, and solve the transfer equation (to first order in the expansion anisotropy) by expanding the photon distribution functions into multipoles up to the octupole term. The solution is to the first order in the mean free time for Thomson scattering. Therefore it reasonably describes the state of the radiation field down to low redshifts only in the case of a substantial reheating of the intergalactic plasma.

Our solution shows that, while the anisotropy pattern is weakly affected by a cosmological magnetic field, substantial modifications are produced in the linear polarization state. The polarization degree monotonically decreases for increasing radiation wavelength, while the polarization pattern correspondingly tends to change from quadrupolar to octupolar. The octupole term would dominate in the polarization pattern at observation wavelengths $\lambda_0 > 10\,\text{cm}$ for a magnetic strength (at the present epoch) $B_0 \simeq 10^{-9}$ Gauss.

Admittedly, the existence of a so high magnetic field is quite controversial. Evidence for a field larger than 10^{-9} Gauss was claimed to arise from measurements of the Faraday rotation for the radiowaves emitted by quasars (Sofue et al., 1968) and of the rotation measure field within spiral galaxies (Tosa and Fujimoto, 1978; Sofue et al., 1980), Removal of local (i.e., Galactic) contributions from the Faraday rotation of distant sources, however, leads to upper limits on B_0 (Ruzmaikin and Sokoloff, 1977); a recent limit claimed by Vallée (1983), is as low as $B_0 \lesssim 3 \times 10^{-11}$ Gauss, and contradicts the argument based on the galactic structure. Further, the analysis of Welter et al. (1984) shows that the extragalactic components of the rotation measures of about 100 QSO's can be explained by magnetized discrete clouds along the line of sight. In this connection, we also recall that Lawler and Dennison (1982) provide evidence for strongly inhomogeneous fields in the intracluster medium. In both cases, the inhomogeneity scale is some tens of kiloparsecs.

Although the presence of extragalactic magnetic fields may render the detection of cosmological effects more difficult, we believe that the cosmic background may offer a better test than discrete sources. Since a universal magnetic field would produce a large-scale polarization pattern, experimentalists can search for low-order harmonics with large-beamwidth detectors, so that small-scale effects are smeared out. (We also point out that the cosmic background polarization probes the magnetic field at redshifts larger than those of QSO's, where magnetized discrete clouds are probably absent).

A serious problem which cannot be removed by large-beamwidth detectors is, of course, the large-scale distribution of the magnetic field in our Galaxy. However, as we shall see below, even field strengths as low as $\sim 10^{-11}$ Gauss would produce measurable effects in the cosmic background polarization, and a spatial analysis could separate them from local contributions. -11

The magnetic field effects turn out to be approximately proportional to $B_0\lambda_0^2$. (This is only a low-field approximation in our cosmological context!). Such a frequency dependence, as well as the spatial distribution, make them quite different from the cosmological curvature effect discovered by Matzner and Tolman (1982).

Therefore, the 3K background radiation might be usefully used as a cosmic probe for an intergalactic magnetic field, effective over a redshift range much larger than the one of astronomical sources. On the other hand, any reasonable value of B_0 implies negligible effects at $\lambda_0 \approx 1\,\text{mm}$, so that millimetric measurements of the polarization pattern would provide clean information on the geometric structure of the universe.

2. The transfer equation

Homogeneous cosmologies can be described by the metric

$$ds^2 = -dt^2 + e^{2\alpha}(e^{2\beta})_{ij}E^iE^j,$$

with E^i the three homogeneous one-form fields which are invariant under a transitive group of motions on the spacelike hypersurfaces (Ellis, 1967). One can also introduce the orthonormal frame $\omega^i = e^{\alpha}(e^{\beta})_{ij}E^j$ and, from this, the usual polar coordinates $\{\theta, \phi\}$ on the celestial sphere. For the homogeneity of space, all physically relevent quantities depend only on t, θ and ϕ. The transfer equation for polarized radiation propagating in a medium of Thomson scatterers at rest in the homogeneous frame can be written as

$$\frac{\partial\hat{n}}{\partial t} + \frac{\partial\hat{n}}{\partial\nu}\frac{d\nu}{dt} + \frac{\partial\hat{n}}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat{n}}{\partial\phi}\frac{d\phi}{dt} + \hat{R}(t,\theta,\phi)\cdot\hat{n} = \frac{1}{l}\left[-\hat{n} + \int\frac{d\Omega'}{4\pi}\,\hat{P}(\Omega,\Omega')\cdot\hat{n}\right]. \tag{1}$$

where the symbolic vector $n^{\wedge}$ can be expressed in terms of the standard Stokes parameters and the corresponding photon occupation numbers

$$\hat{n} \equiv \frac{c}{h\nu^3}\begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} = \begin{pmatrix} n_I \\ n_Q \\ n_U \\ n_V \end{pmatrix}. \tag{2}$$

Also, l is the photon mean free time. $\hat{P}$ is a standard transfer matrix (Chandrasekhar, 1950) describing Thomson scattering, and the matrix

$$\hat{R} = \frac{d\psi}{dt}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{3}$$

describes the variation of the Stokes parameters due to the transport of the orthonormal reference vectors for polarization along the photon geodesics. The angle ψ measures the corresponding rotation of the polarization vector, and its time derivative was given by Dautcourt and Rose (1978). The transfer equation can also be written in a more compact way introducing the complex photon distribution

$$\hat{N} \equiv \begin{pmatrix} {}_1N \\ {}_2N \end{pmatrix} = \frac{1}{ch^4\nu^3} \begin{pmatrix} I + iV \\ Q - iU \end{pmatrix}. \tag{4}$$

Then we can write

$$\frac{\partial \hat{N}}{\partial t} + \frac{\partial \hat{N}}{\partial \nu}\frac{d\nu}{dt} + \frac{\partial \hat{N}}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial \hat{N}}{\partial \phi}\frac{d\phi}{dt} - 2i\frac{d\psi}{dt}\hat{M}\cdot\hat{N}$$
$$= \frac{1}{l}\left[-\hat{N} + \int d\Omega'(\hat{p}^+ \cdot \hat{N} + \hat{p}^- \cdot \hat{N}^*)\right], \tag{5}$$

where the matrix $\hat{M} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, and the star denotes complex conjugation. The matrices $\hat{p}^+$ and $\hat{p}^-$, which give the emission term for Thomson scattering, are:

$$\hat{p}^+ = \begin{pmatrix} \frac{3}{8} n^{ij}(\theta,\phi) n_{ij}(\theta',\phi') + \frac{3}{4} n^i(\theta,\phi) n_i(\theta',\phi') + \frac{1}{2}, & -\frac{3}{4} n^{ij}(\theta,\phi) \overset{*}{m}_{ij}(\theta',\phi') \\ -\frac{3}{4} m^{ij}(\theta,\phi) n_{ij}(\theta',\phi'), & \frac{3}{2} m^{ij}(\theta,\phi) \overset{*}{m}_{ij}(\theta',\phi') \end{pmatrix},$$

$$\hat{p}^- = \begin{pmatrix} \frac{3}{8} n^{ij}(\theta,\phi) n_{ij}(\theta',\phi') - \frac{3}{4} n^i(\theta,\phi) n_i(\theta',\phi') + \frac{1}{2}, & -\frac{3}{4} n^{ij}(\theta,\phi) m_{ij}(\theta',\phi') \\ -\frac{3}{4} m^{ij}(\theta,\phi) n_{ij}(\theta',\phi'), & \frac{3}{2} m^{ij}(\theta,\phi) m_{ij}(\theta',\phi') \end{pmatrix}. \tag{6}$$

where n and m are the polynomials formed from spin-weighted spherical harmonics (see Dautcourt and Rosé, 1978). The properties of a specific cosmological model affect Eq. (5) through the explicit expressions of the functions $\frac{d\theta}{dt}, \frac{d\phi}{dt}, \frac{d\psi}{dt}, \frac{d\nu}{dt}$ and l, which in turn depend on the time evolution of e^α and $(e^\beta)_{ij}$.

The inclusion of a source-free homogeneous magnetic field affects both the evolution of the cosmological metric and the structure of the transfer equation.

In practice, we shall restrict ourselves to cosmologies where both the anisotropy of the expansion rate and the magnetic field are small. This means that the averaged expansion factor e^α evolves as in the isotropic limit, i.e. $\propto (1 + z)^{-1}$, with z the usual cosmological redshift. Also, the magnetic field does not contribute to an appreciable extent to the cosmic matter density: for B_0 as large as 10^{-9} Gauss at $z = 0$ we should have $\rho_{field}/\rho_{matter} \lesssim 10^{-8}$ for $z \lesssim 10^3$. The presence of a magnetic field, however, affects the behavior of the anisotropy of the expansion. For instance, let us consider the dust-magnetic solution given by Thorne (1967) for axisymmetric Bianchi type-I (i.e., euclidean) models, whose geometry is described by the metric

$$ds^2 = -dt^2 + A^2(dx^2 + dy^2) + W^2 dz^2.$$

Here A and W denote the cosmic scale factors

$$A = e^\alpha(e^\beta)_{11} = e^\alpha(e^\beta)_{22},$$
$$W = e^\alpha(e^\beta)_{33}.$$

Thorne's (1967) solution is

$$t = \frac{2}{(3\mu)^{1/2}}(A + 6\delta)(A - 3\delta)^{1/2},$$
$$W = A + 12\delta + \varepsilon_D A^{-1}(A - 3\delta)^{1/2} - 72\delta^2 A^{-1}, \tag{7}$$

where ε_D is an anisotropy parameter, $\delta = \beta/\mu$, and the constants β and μ are defined by

Then the fractional anisotropy of the expansion rate is given by

$$\frac{1}{W}\frac{dW}{dt}\left(\frac{1}{A}\frac{dA}{dt}\right)^{-1} - 1 \simeq \frac{3}{4}\frac{\varepsilon_D(1 + z)^{3/2}}{R_0^{3/2}} + 6\frac{\delta(1 + z)}{R_0}, \tag{8}$$

where R_0 denotes the value of $(A^2W)^{1/3}$ at the present epoch. Therefore, the expansion anisotropy is the superposition of two modes, ε_D and δ being mutually independent. The first mode, which decay as $(1 + z)^{3/2}$, is present also in the absence of the magnetic field; the second mode is excited by the magnetic field (δ being proportional to the field strength) and decay more slowly, as $(1 + z)$. Since according to observation the anisotropy of the expansion rate was at most of the order of 10^{-4} at $z \simeq 1000$, it can be shown that the mode driven by the magnetic field would dominate after the thermal decoupling for $B_0 \simeq 10^{-8}$ Gauss; also, for $B_0 = 10^{-9} \div 10^{-10}$ Gauss, this anisotropy mode will become more important than the first one at later epochs. Other solutions for type-I magnetic models have been provided by Jacobs (1969). Their qualitative behaviors for large times and small anisotropies are similar. In other Bianchi types, one also finds curvature driven modes, whose time evolution is not relevant to our purposes.

In the following, the temporal evolution of the (small) anisotropy of the expansion rate does not need to be specified. In order to generalize the transfer equation to the magnetic case, we only need to make some assumptions on the thermal history of the universe.

Here we assume that the cosmological medium can be treated (1) as a "cold" plasma, so that we can neglect kinetic energy terms and relativistic effects, and (2) as a tenuous gas, endowed with characteristic frequencies much smaller than the radiation frequency. Synchrotron emission and absorption can thereby be ignored, and collisional absorption is negligible in comparison with Thomson scattering.

Let us discuss briefly the above assumptions. The cosmological plasma can be treated as cold for electron temperatures $T_e \lesssim 10^7$ K (cf. Stark, 1981). This condition is well satisfied for a long period before the recombination of the primordial hydrogen, and in particular close to the thermal decoupling ($z \simeq 10^3$). For the secondary ionization of the cosmological plasma, models have been constructed where T_e can be as large as $\sim 10^8$ K at $z \simeq 3$ (Sunyaev and Zeldovich, 1970; Field and Perrenod, 1977). However, so high temperatures are not demanded by observation, and for the sake of simplicity we shall maintain the low-temperature approximation as in Eqs. (1) and (5).

As to assumption (2), we observe that the mean free path for Thomson scattering is $cl \simeq 10^{29}(1 + z)^{-3}$ cm for a present electron density $n_{e0} \simeq 10^{-5}$ cm^{-3}. It is then much greater than the Larmor radius, which is of order $10^{10}(1 + z)^{-2}$ cm or $10^{12}(1 + z)^{-2}$ cm for $B_0 \simeq 10^{-9}$ Gauss or 10^{-11} Gauss, respectively. However, the electric force acting on electrons during collisions is still much larger than the magnetic force; therefore the Thomson scattering terms in the transfer equation keep the same form as in the absence of the magnetic field. Next, let us estimate the plasma frequency ω_p and the gyrofrequency ω_H, which are given by

$$\omega_p = \left(\frac{4\pi n_e e^2}{m_e}\right)^{1/2},$$
$$\omega_H = \frac{eB}{cm_e}, \tag{9}$$

with obvious meanings of the symbols. The ratios of ω_p and ω_H to the radiation frequency are roughly given by

$$\frac{\omega_p}{\omega} \simeq 10^{-9}(1+z)^{1/2},$$

$$\frac{\omega_H}{\omega} \simeq 10^{-13}(1+z), \tag{10}$$

taking the values $n_{e0} = 10^{-5}\,\mathrm{cm}^{-3}$, $B_0 = 10^{-9}$ Gauss, and $\omega = 10^{11}(1+z)\mathrm{s}^{-1}$. Since just before the thermal decoupling both ratios are as small as $\sim 10^{-8}$, for the entire period under investigation we can treat the radiation transfer in the limit of "weak anisotropy" of the plasma, in which the two natural magneto-ionics modes are regarded (a) as transverse and (b) as having identical ray paths (see Melrose, 1980). In this regime the only important effect produced by a cosmic magnetic field is the Faraday rotation. It is well known that in the quasicircular approximation the axial ratio of the polarization ellipse of the two natural modes and the difference between the two refraction indexes are:

$$T = -\frac{\cos\theta^*}{|\cos\theta^*|} - \frac{\omega_H}{2\omega}\frac{\sin^2\theta^*}{\cos\theta^*},$$

$$\Delta n = \frac{\omega_p^2\omega_H}{\omega^3}|\cos\theta^*|, \tag{11}$$

where θ^* is the angle between the magnetic field vector and the direction of propagation of the radiation. Moreover, since the refractive index is very close to unity, we can assume that the photon paths are space-time null geodesics.

In conclusion, our transfer equation will be obtained by simply adding a Faraday rotation term to Eq. (1); and since (1) is written in a polar system associated to a local orthonormal frame, such a term has exactly the same form as in the ordinary plasma physics. Therefore, taking the magnetic field in the $\theta = 0$ direction, we have

$$\frac{\partial\hat n}{\partial t} + \frac{\partial\hat n}{\partial v}\frac{dv}{dt} + \frac{\partial\hat n}{\partial\theta}\frac{d\theta}{dt} + \frac{\partial\hat n}{\partial\phi}\frac{d\phi}{dt} + \hat R(t,\theta,\phi)\cdot\hat n$$
$$= \frac{1}{l}\left[-\hat n + \int\frac{d\Omega'}{4\pi}\,(\Omega,\Omega')\cdot\hat n\right] + \hat r_{FR}\cdot\hat n, \tag{12}$$

where matrix $\hat r_{FR}$ has the form

$$\hat r_{FR} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -r_v & 0 \\ 0 & r_v & 0 & -r_Q \\ 0 & 0 & r_Q & 0 \end{pmatrix} \tag{13}$$

with

$$r_v = -\frac{2T}{T^2+1}\,\omega n\,\Delta n, \quad r_Q = -\frac{T^2-1}{T^2+1}\,\omega n\,\Delta n.$$

Using (11) we obtain also

$$r_v = \frac{\omega_p^2\omega_H}{\omega^2}\cos\theta - \frac{\omega_p^2\omega_H^2}{\omega^3}\frac{\sin^2\theta\cos\theta}{|\cos\theta|}$$

$$r_Q = -\frac{\omega_p^2\omega_H^2}{2\omega^3}\sin^2\theta. \tag{14}$$

In our cosmological context, to the zero-th order in the expansion anisotropy we can write

$$\frac{\omega_p^2\omega_H}{\omega^2} = \frac{C}{l}, \tag{15a}$$

where

$$C = \frac{4\pi e^3 B_0}{c^2 m_e^2 \sigma_T \omega_0^2} \tag{15b}$$

is a time-independent adimensional quantity, which may be of order unity for realistic values of B_0 and ω_0. Thus, the leading term of r_v is important in Eqs. (12) and (14), while the terms proportional to $\omega_p^2\omega_H^2\omega^{-3} = (C/l)(\omega_H/\omega)$ can be dropped.

Equation (12) holds for homogeneous cosmologies of any Bianchi type (with the cosmic matter at rest in the homogeneous frame of reference). The only limitation on the model is due to the fact that a homogeneous magnetic field is only compatible with Bianchi types I, II, VI$_0$, VII$_0$ and VII$_h$ (Tsoubelis, 1979).

In the following we shall restrict ourselves to Bianchi type I spaces. In such spaces the terms $(\partial\hat n/\partial\theta)(d\theta/dt)$, $(\partial\hat n/\partial\phi)(d\phi/dt)$ and $\hat R\cdot\hat n$ give contributions of the second order in the shear tensor and can be neglected in our first order treatment. Thus we have simply

$$\frac{\partial\hat n}{\partial t} + \frac{\partial\hat n}{\partial v}\frac{dv}{dt} = -\frac{1}{l}\left[\hat n - \int\frac{d\Omega'}{4\pi}\hat P(\Omega,\Omega')\cdot\hat n\right] + \hat r_{FR}\cdot\hat n. \tag{16}$$

Taking also advantage of the simplified form of $\hat r_{FR}$ (where in particular $r_Q = 0$), it turns out that it is very convenient to rewrite Eq. (16) in terms of the distribution functions ${}_0N$ and ${}_2N$, which are eigenfunctions of the operator $\hat r_{FR}$, corresponding to the eigenvalues $r_1 = 0$ and $r_2 = i\omega\,\Delta n$, respectively. (In this approximation $\hat r_{FR}$ truly represents the Faraday rotation, in the strict sense of rotation of the polarization plane). Then the transfer equation takes the form:

$$\frac{\partial_0 N}{\partial t} + \frac{\partial_0 N}{\partial v}\frac{dv}{dt} = -\frac{1}{l}\left[{}_0N - \int\frac{d\Omega'}{4\pi}(p^+_{0A\,A}N + p^-_{0A\,A}N^*)\right],$$
$$\frac{\partial_2 N}{\partial t} + \frac{\partial_2 N}{\partial v}\frac{dv}{dt} = -\frac{1}{l}\left[{}_2N - \int\frac{d\Omega'}{4\pi}(p^+_{2A\,A}N + p^-_{2A\,A}N^*)\right]$$
$$- 2i\frac{d\chi}{dt}\,{}_2N, \tag{17}$$

where

$$2\frac{d\chi}{dt} = \frac{C}{l}\cos\theta, \tag{18}$$

and χ is the angle between the major axis of the polarization ellipse and the reference vector for the polarization.

3. Solution of the transfer equation to the first order in the mean free time.

The photon distribution functions ${}_0N$ and ${}_2N$ have spin weights zero and two respectively; therefore they can be expanded by means of the spin-weighted polynomials, which form complete orthogonal sets of functions in the spaces of square-integrable functions of appropriate spin weight (Dautcourt and Rose, 1978).

Thus we have

$$
\begin{aligned}
{}_0N &= {}_0N_0 + {}_0N_i n^i + {}_0N_{ij} n^{ij} + {}_0N_{ijk} n^{ijk} + \cdots, \\
{}_2N &= {}_2N_{ij} m^{ij} + {}_2N_{ijk} m^{ijk} + \cdots,
\end{aligned} \tag{19}
$$

In the ${}_0N$ expansion, $n^i = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ is the unit vector and

$$
\begin{aligned}
n^{ik} &= n^i n^k - \frac{1}{3}\delta^{ik}, \\
n^{ikl} &= n^i n^k n^l - \frac{1}{5}(\delta^{ik} n^l + \delta^{il} n^k + \delta^{kl} n^i).
\end{aligned} \tag{20a}
$$

In the ${}_2N$ expansion we have

$$
\begin{aligned}
m^{ik} &= m^i m^k, \\
m^{ikl} &= m^{ik} n^l + m^{il} n^k + m^{lk} n^i,
\end{aligned} \tag{20b}
$$

where

$$
m^i = \frac{1}{\sqrt{2}}\left(\frac{\partial n^i}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial n^i}{\partial\phi}\right) \tag{20c}
$$

is a complex quantity of spin weight 1.

The number of indices of the n-polynomials and m-polynomials charaterizes the multipole order of the corresponding contributions to anisotropy and polarization. Thus for instance, n^{ij} and m^{ij} are quadrupole terms, and n^{ijk} and m^{ijk} are octupole terms.

3.1. Solution to the quadrupole order

For the moment, let us truncate the expansions (19) to the quadrupole order, as done by previous authors dealing with nonmagnetic models. Using the orthogonality relations between spin-weighted polynomials and keeping only the terms to first order in the shear, we obtain for the expansion coefficients ${}_0N_0$, ${}_0N_i$, ${}_0N_{ij}$ and ${}_2N_{ij}$:

$$
\begin{aligned}
&\frac{\partial {}_0N_0}{\partial t} - \xi\, {}_0N_0\dot{\alpha} = -\frac{1}{l}\, i \,\mathrm{Im}\, {}_0N_i, \\
&\frac{\partial {}_0N_i}{\partial t} - \xi\, {}_0N_i\dot{\alpha} = -\frac{1}{l}\left(\mathrm{Re}\, {}_0N_i + \frac{i}{2}\mathrm{Im}\, {}_0N_i\right), \\
&\frac{\partial {}_0N_{ij}}{\partial t} - \xi({}_0N_{ij}\dot{\alpha} + \sigma_{ij}\, {}_0N_0) = -\frac{1}{l}\left(\frac{9}{10}\mathrm{Re}\, {}_0N_{ij} + \frac{3}{10}\mathrm{Re}\, {}_2N_{ij} + i\,\mathrm{Im}\, {}_0N_{ij}\right), \\
&\frac{\partial {}_2N_{ij}}{\partial t} - \xi\, {}_2N_{ij}\dot{\alpha} = -\frac{C}{l} C_{ij}^{kl}\, {}_2N_{kl} - \frac{1}{l}\left(\frac{1}{5}\mathrm{Re}\, {}_0N_{ij} + \frac{2}{5}\mathrm{Re}\, {}_2N_{ij} + i\,\mathrm{Im}\, {}_2N_{ij}\right),
\end{aligned} \tag{21}
$$

where

$$
\xi = \frac{\nu}{{}_0N}\frac{\partial {}_0N}{\partial \nu},
$$

σ_{ij} denotes the shear tensor given by

$$
2\sigma_{ij} = \frac{d}{dt}(e^{\beta})_{ki}(e^{-\beta})_{kj} + \frac{d}{dt}(e^{-\beta})_{kj}(e^{\beta})_{ki},
$$

and (identifying the $i = 3$ direction with the magnetic field vector, i.e. the $\theta = 0$ direction)

$$
C_{ij}^{kl} = -\frac{1}{6}(\delta^{kl}\varepsilon^{jl3} + \delta^{li}\varepsilon^{jk3} + \delta^{kj}\varepsilon^{li3}).
$$

From the first of the equations (21) we see that the isotropically redshifted monopole intensity is conserved to the first order in the shear. Moreover, the circular polarization modes, namely $\mathrm{Im}\, {}_0N_0$, $\mathrm{Im}\, {}_0N_i$ and $\mathrm{Im}\, {}_0N_{ij}$, are not coupled to any other Stokes parameter; so, since cosmologically significant circular polarization is not expected to be present primordially and would be strongly damped at any rate according to (21), we set $V = 0$. Introducing further the adimensional quantities I_{ik} and Q_{ik} defined by

$$
\begin{aligned}
\mathrm{Re}\,\frac{{}_0N_{ik}}{{}_0N_0} &= I_{ik}, \\
\mathrm{Re}\,\frac{{}_2N_{ik}}{{}_0N_0} &= Q_{ik},
\end{aligned} \tag{22}
$$

we get the following system of equations

$$
\begin{aligned}
\frac{\partial}{\partial t} I_{ij} - \xi\sigma_{ij} &= -\frac{1}{l}\left(\frac{9}{10} I_{ij} + \frac{3}{10} Q_{ij}\right), \\
\frac{\partial}{\partial t} Q_{ij} &= \frac{1}{l}\left[\frac{C}{3}(\varepsilon^{jk3} Q_{ik} + \varepsilon^{ik3} Q_{kj}) - \frac{1}{5} I_{ij} - \frac{2}{5} Q_{ij}\right].
\end{aligned} \tag{23}
$$

In the very early stages of the universe, although there can be a large expansion anisotropy and an intense primordial magnetic field, the interactions between matter and radiation are so frequent, that we may consider the radiation field as being uniform, isotropic and unpolarized; thus only the monopole differs from zero. Subsequently, when we get close to the recombination of the primordial hydrogen, the polarization degree of the radiation becomes appreciable. The mean free path for Thomson scattering being still small in comparison with the Hubble radius, we can use the approximation appropriate to optically thick stages, i.e., solve the equations to first order in l (Dautcourt and Rose, 1978). By this approximation, we get

$$
\begin{aligned}
&\frac{9}{10} I_{ij} + \frac{3}{10} Q_{ij} = \xi l \sigma_{ij}, \\
&\frac{1}{5} I_{ij} + \frac{2}{5} Q_{ij} - \frac{1}{3} C(Q_{ik}\varepsilon^{jk3} + Q_{kj}\varepsilon^{ik3}) = 0
\end{aligned} \tag{24}
$$

We thereby obtain an algebraic system of equations. We will give its explicit solution in the case of (Bianchi type-I) spaces with diagonal metric. In this connection, we recall that the diagonalization of the metric tensor in magnetic models, as shown by Ryan et al. (1982), is possible if and only if the magnetic field is parallel to one of the eigenvectors of dg_{ij}/dt on a nonsingular $t = \mathrm{const}$ hypersurface. Setting $\sigma_{ij} = \sigma_{ii}\delta_{ij}$ in (24), we obtain the solution

$$
\begin{aligned}
I_{ii} &= \frac{4}{3}\xi l\left(\sigma_{ii} + \frac{\delta_{i1} - \delta_{i2}}{3}\frac{C^2\,\overline{\Delta H}}{1 + 4C^2}\right), \\
Q_{ii} &= -\frac{2}{3}\xi l\left[\sigma_{ii} + 2(\delta_{i1} - \delta_{i2})\frac{C^2\,\overline{\Delta H}}{1 + 4C^2}\right], \\
I_{12} &= \frac{2}{9}\frac{\xi l C\,\overline{\Delta H}}{1 + 4C^2}, \\
Q_{12} &= -\frac{2}{3}\frac{\xi l C\,\overline{\Delta H}}{1 + 4C^2}, \\
I_{13} &= I_{32} = Q_{13} = Q_{32} = 0,
\end{aligned} \tag{25}
$$

where $\overline{\Delta H} = \sigma_{22} - \sigma_{11}$ is the azimuthal anisotropy of the cosmological model. Since, as we have remarked, this solution describes the state of the radiation field only before recombination and after a substantial reheating of the cosmological plasma, in general we should integrate Eq. (23) numerically in order to derive the anisotropy and polarization of the background radiation at the present epoch. In this case, Eq. (25) provides the "initial conditions" at some redshift $z \gtrsim 1500$ to be used for the integration. However, the memory of the early events is lost if the optical depth between the secondary ionization and us is large: the period of free propagation between the recombination and the onset of reheating does not affect the subsequent state of the radiation, and (25) is an approximate description of its properties at low redshift. Numerical calculations performed by various authors in non-magnetic models show that the first-order analytic solutions are reasonable approximations down to $z = 0$ (the observation epoch) if the reheated gas is able to scatter each background photon at least ~ 10 times.

Therefore, although Eq. (25) cannot cover all of the usual cosmological scenarios, still it is worth discussing its physical implications.

First we observe that in axisymmetric Bianchi type-I models the expansion coefficients have the same form as in the absence of the magnetic field: because of $\Delta H = 0$, the off-diagonal coefficients vanish; also $I_{11} = I_{22}$ and $Q_{11} = Q_{22}$. If, on the other hand, the universe is not axisymmetric, the Faraday rotation gives rise to non-vanishing off-diagonal terms (besides modifying the diagonal ones). The temperature and polarization pattern may be drastically changed. For example, consider the Stokes parameters Q and U:

$$Q = \frac{1}{3} c h^4 \nu^3 \xi I_0 N_0 \left[\frac{3}{2} (\sigma_{11} + \sigma_{22}) \sin^2 \theta + \frac{\overline{\Delta H}}{2(1+4C^2)} (1 + \cos^2 \theta)(\sin 2\phi - 2C \cos 2\phi) \right],$$

$$U = -\frac{1}{3} c h^4 \nu^3 \xi I_0 N_0 \frac{\overline{\Delta H}}{1+4C^2} (\sin 2\phi + 2C \cos 2\phi). \tag{26}$$

In non-axisymmetric Bianchi I spaces we have, in the absence of the magnetic field, $U = 0$ when ϕ is an integer multiple of 90°. (The polarization is there parallel or perpendicular to the principal planes of the shear, according to whether Q is positive or negative). In the presence of a magnetic field, U vanishes on the "cosmic meridians" corresponding to

$$\operatorname{tg} 2\phi = \frac{2Q_{12}}{Q_{11} - Q_{22}} = -2C. \tag{27}$$

For $B_0 = 10^{-9}$ Gauss and $\lambda_0 = 1$ cm, the zeros of U are displaced by 6°; for $B_0 = 3.10^{-11}$ Gauss we have the same displacement for $\lambda_0 = 6$ cm.

A more detailed discussion of the linear polarization pattern will be given after calculating the octupole approximation. Before closing this subsection, we notice that a non-vanishing circular polarization arises when we consider the full expressions of r_v and r_Q (see Eq. (14)). Let us introduce the quantity

$$\varepsilon = \frac{\omega_H^2 \omega_p^2 l}{\omega^3 C}, \tag{28}$$

which is $\lesssim 10^{-13}(1+z)$ for the above values of B_0 and λ_0. Equation (25) gives the solution to the zero-th order in ε, and can be used to generate the first-order solution by iteration. We find

$$\frac{\operatorname{Im} {}_0N_0}{\operatorname{Re} {}_0N_0} \sim \frac{\operatorname{Im} {}_0N_{ij}}{\operatorname{Re} {}_0N_{ij}} \sim \varepsilon C Q_{ij},$$

so that Bianchi type-I models predict a circular polarization enormously less than the linear polarization.

3.2. Solution to the octupole approximation

Let us now consider also the octupole terms in the expansion of the distribution functions ${}_0N$ and ${}_2N$. Inserting (19) in Eq. (17) keeping the ${}_0N_{ijk}$ and ${}_2N_{ijk}$ terms, we find that the radiation intensity in the octupole term is not coupled to the monopole and quadrupole terms. If we suppose ${}_0N_{ijk}$ to vanish initially, they will remain identically equal to zero. Otherwise, they are strongly damped and result unmeasurable. The equations for the linear polarization coefficients ${}_2N_{ij}$ and ${}_2N_{ijk}$ are:

$$\frac{\partial}{\partial t} {}_2N_{ij} - \xi \dot{\alpha} \, {}_2N_{ij} = \frac{1}{l} \left[C(C_{ij}^{kl} \, {}_2N_{kl} + 5iD_{ij}^{klr} \, {}_2N_{klr}) + \frac{1}{5} \operatorname{Re} {}_0N_{ij} + \frac{2}{5} \operatorname{Re} {}_2N_{ij} + i \operatorname{Im} {}_2N_{ij} \right],$$

$$\frac{\partial}{\partial t} {}_2N_{ijk} - \xi \dot{\alpha} \, {}_2N_{ijk} = -\frac{1}{l} \left[\frac{7iC}{3} (E_{ijk}^{rl} \, {}_2N_{rl} + F_{ijk}^{rlm} \, {}_2N_{rlm}) + {}_2N_{ijk} \right], \tag{29}$$

where

$$D_{ij}^{klr} = \int m^{klr} (m^{ij})^* n^3 \frac{d\Omega}{4\pi},$$

$$E_{ijk}^{rl} = \int m^{rl} (m^{ijk})^* n^3 \frac{d\Omega}{4\pi},$$

$$F_{ijk}^{rlm} = \int m^{rlm} (m^{ijk})^* n^3 \frac{d\Omega}{4\pi}.$$

If we introduce the adimensional Stokes parameters:

$$\operatorname{Re} \frac{{}_2N_{ij}}{{}_0N_0} = Q_{ij},$$

$$\operatorname{Re} \frac{{}_2N_{ijk}}{{}_0N_0} = Q_{ijk},$$

$$\operatorname{Im} \frac{{}_2N_{ij}}{{}_0N_0} = U_{ij},$$

$$\operatorname{Im} \frac{{}_2N_{ijk}}{{}_0N_0} = U_{ijk}, \tag{30}$$

and separate real and imaginary parts, Eqs. (29) take the following form:

$$\frac{\partial}{\partial t} Q_{ij} = -\frac{1}{l} \left\{ \frac{1}{5} I_{ij} + \frac{2}{5} Q_{ij} - C \left[\frac{1}{6} (\varepsilon^{jl3} Q_{il} \cdot 2 + \varepsilon^{il3} Q_{jl}) + \frac{5}{7} U_{ij3} \right] \right\},$$

$$\frac{\partial}{\partial t} U_{ijk} = -\frac{1}{l} \left\{ U_{ijk} + \frac{C}{9} \left[\delta^{3k} Q_{ij} + \delta^{i3} Q_{jk} + \delta^{j3} Q_{ik} - \frac{2}{5} (\delta^{jk} Q_{3i} + \delta^{ik} Q_{3j} + \delta^{ij} Q_{3k}) \right] \right\}. \tag{31}$$

$$\frac{\partial}{\partial t}U_{ij} = -\frac{1}{l}\left\{U_{ij} + C\left[-\frac{1}{6}(\epsilon^{jl3}U_{il} \cdot 2 \quad + \epsilon^{il3}U_{jl}) + \frac{5}{7}Q_{ij3}\right]\right\},$$

$$\frac{\partial}{\partial t}Q_{ijk} = -\frac{1}{l}\left\{Q_{ijk} + \frac{C}{9}\left[\delta^{k3}U_{ij} + \delta^{i3}U_{jk} + \epsilon^{j3}U_{ik} - \frac{2}{5}(\delta^{jk}U_{3i} + \delta^{ik}U_{3j} + \delta^{ij}U_{3k})\right]\right\}. \qquad (32)$$

From Eqs. (32) we see that the coefficients U_{ij} and Q_{ijk} are coupled to each other, but not to other Stokes parameters. Since Eqs. (32) again contain damping terms on the right-hand sides, we shall henceforth limit ourselves to Eqs. (21) for I_{ij} and Eqs. (31) for Q_{ij} and U_{ijk}. This means that no octupole term appears in the anisotropy pattern.

From the second of Eqs. (31) we see that, if the quadrupole term Q_{ij} differs from zero, an octupole term is produced in the polarization, which in turn affects Q_{ij}. This coupling arises from the presence of the magnetic field: for $C = 0$ all multipoles of order higher than the quadrupole would vanish (to the first order in σ_{ij}).

Again we solve the equations to the first order in the photon mean free time l.

The quadrupole coefficients for the anisotropy of intensity are

$$I_{11} = \frac{2}{9}\frac{\xi l}{1 + \frac{5}{21}C^2}\left[\frac{126 + 25C^2}{21}\sigma_{11} + \frac{2C^2\overline{\Delta H}}{4C^2 + \left(1 + \frac{5}{21}C^2\right)^2} + \frac{\frac{2}{3}C^2\sigma_{33}}{7 + 3C^2}\right],$$

$$I_{22} = \frac{2}{9}\frac{\xi l}{1 + \frac{5}{21}C^2}\left[\frac{126 + 25C^2}{21}\sigma_{22} - \frac{2C^2\overline{\Delta H}}{4C^2 + \left(1 + \frac{5}{21}C^2\right)^2} + \frac{\frac{2}{3}C^2\sigma_{33}}{7 + 3C^2}\right],$$

$$I_{12} = -\frac{2}{9}\frac{\xi lC\overline{\Delta H}}{4C^2 + \left(1 + \frac{5}{21}C^2\right)^2},$$

$$I_{33} = \frac{2}{3}\xi l\frac{2 + \frac{5}{7}C^2}{1 + \frac{3}{7}C^2}\sigma_{33}, \qquad (33)$$

and the quadrupole coefficients for the linear polarization:

$$Q_{11} = -\frac{\xi l}{1 + \frac{5}{21}C^2}\left[\frac{2}{3}\sigma_{11} + \frac{\frac{4}{3}C^2\overline{\Delta H}}{4C^2 + \left(1 + \frac{5}{21}C^2\right)^2} + \frac{\frac{4}{9}C^2\sigma_{33}}{7 + 3C^2}\right],$$

$$Q_{22} = -\frac{\xi l}{1 + \frac{5}{21}C^2}\left[\frac{2}{3}\sigma_{22} - \frac{\frac{4}{3}C^2\overline{\Delta H}}{4C^2 + \left(1 + \frac{5}{21}C^2\right)^2} + \frac{\frac{4}{9}C^2\sigma_{33}}{7 + 3C^2}\right],$$

$$Q_{33} = -\frac{2}{3}\frac{\xi l\sigma_{33}}{1 + \frac{3}{7}C^2},$$

$$Q_{12} = -\frac{2}{3}\frac{\xi lC\overline{\Delta H}}{4C^2 + \left(1 + \frac{5}{21}C^2\right)^2}. \qquad (34)$$

As already remarked, no octupole terms are present for the intensity. The octupole coefficients for the linear polarization are given by

$$U_{113} = \frac{2}{27}\frac{\xi lC}{1 + \frac{5}{21}C^2}\left[\sigma_{11} + \frac{2C^2\overline{\Delta H}}{4C^2 + \left(1 + \frac{5}{21}C^2\right)^2} - \frac{2}{5}\frac{\sigma_{33}}{1 + \frac{3}{7}C^2}\right],$$

$$U_{223} = \frac{2}{27}\frac{\xi lC}{1 + \frac{5}{21}C^2}\left[\sigma_{22} - \frac{2C^2\overline{\Delta H}}{4C^2 + \left(1 + \frac{5}{21}C^2\right)^2} - \frac{2}{5}\frac{\sigma_{33}}{1 + \frac{3}{7}C^2}\right],$$

$$U_{123} = \frac{\frac{2}{27}\xi lC^2\overline{\Delta H}}{4C^2 + \left(1 + \frac{5}{21}C^2\right)^2},$$

$$U_{333} = \frac{\frac{2}{45}\xi lC\sigma_{33}}{1 + \frac{3}{7}C^2}. \qquad (35)$$

These solutions tend to solutions previously given by Tolman and Matzner (1984) in the limit $C \to 0$. The coefficients I_{ij} and Q_{ij} in Eqs. (33) and (34) differ from the corresponding expressions in (25) because of the coupling with the third order multipole. However, the corrective terms are proportional to C^2. Therefore, if we truncate the multipole expansion to the quadrupole terms ab initio, the resulting quadrupole coefficients are correct up to the first order in C. The corresponding approximation is good only for low fields (e.g., $B_0 \lesssim 10^{-9}$ Gauss at $\lambda_0 = 1$ cm).

The octupole coefficients are of the order of C times the quadrupole coefficients. In fact, C may be larger than unity at sufficiently large wavelengths. For instance, $C \geq 1$ at $\lambda_0 \gtrsim 10$ cm for $B_0 = 10^{-9}$ Gauss. Therefore the polarization octupole coefficients may become more important than the quadrupole ones; however, since $Q_{ij} \simeq \xi l\sigma_{ij}/C^2$ for $C \to \infty$, also U_{ijk} tends to zero as C^{-1}. Therefore the large-scale polarization is destroyed, whereas the quadrupole intensity approaches the value $I_{ij} = \frac{10}{9}\xi l\sigma_{ij}$. For axisymmetric, Bianchi type-I models we confirm the result that the quadrupole off-diagonal terms vanish, but the octupole coefficients U_{113}, U_{223}, U_{333} appear, and the quadrupole diagonal coefficients are affected accordingly.

4. Polarization and anisotropy patterns of the background radiation

The degree of linear polarization is defined by the relation:

$$P = \frac{(Q^2 + U^2)^{1/2}}{I}. \tag{36}$$

The direction of polarization can be described by the angle χ defined by

$$\chi = \frac{1}{2} \operatorname{arctg} \frac{U}{Q}. \tag{37}$$

To the octupole terms we get

$$P^2 = [\mathrm{Re}(Q_{ij}m^{ij} + iU_{ij3}m^{ij3})]^2 + [\mathrm{Im}(Q_{ij}m^{ij} + iU_{ij3}m^{ij3})]^2 \tag{38}$$

and

$$\chi = \frac{1}{2} \operatorname{arctg}\left[\frac{\mathrm{Im}(Q_{ij}m^{ij} + iU_{ij3}m^{ij3})}{\mathrm{Re}(Q_{ij}m^{ij} + iU_{ij3}m^{ij3})}\right]. \tag{39}$$

Using the results of the previous section, we shall first give explicit expressions for P and χ in the case of axisymmetric models. From Eqs. (34) and (35) we get very simple expressions:

$$P = \frac{\xi I \sigma_{11} \sin^2 \theta}{1 + \frac{3}{7} C^2}\left(1 + \frac{9}{45} C^2 \cos^2 \theta\right)^{1/2} \tag{40}$$

and

$$\chi = \frac{1}{2} \operatorname{arctg}\left(\frac{3}{5} C \cos \theta\right). \tag{41}$$

Clearly, the polarization degree vanishes along the symmetry axis. The maximum polarization is for $\theta = \theta_m$, where

$$\theta_m = \frac{\pi}{2} \qquad \text{for } C^2 < \frac{50}{9},$$
$$\sin^2 \theta_m = \frac{2}{3}\,\frac{25 + 9C^2}{9C^2} \qquad \text{for } C^2 > \frac{50}{9}. \tag{42}$$

As to the polarization direction, we observe that for $C = 0$, the angle χ vanishes and the polarization direction is parallel or perpendicular to the cosmic meridians, according to whether the universe is expanding along the symmetry axis more or less rapidly than the average. (This holds true also for the exact solution, not only to the first order in θ). For $C \neq 0$ the polarization direction is tilted by an amount depending on the photon propagation direction. On the cosmic equatorial plane, $\theta = \pi/2$, we have $\chi = 0$. Near the poles, we have the maximum rotation

$$\chi_m = \pm\frac{1}{2} \operatorname{arctg} \frac{3C}{5}, \tag{43}$$

which increases with the ratio B_0/ω_0^2. We have $\chi_m \approx 2^0$ at $\lambda_0 = 1\,\mathrm{cm}$ for $B_0 = 10^{-9}$ Gauss, and at $\lambda_0 = 6\,\mathrm{cm}$ for $B_0 = 3.10^{-11}$ Gauss.

The effects of a cosmic magnetic field on the polarization may thereby be quite large. On the other hand, the anisotropy pattern for axisymmetric models is described by

$$\Delta I \equiv (I_{11} - I_{33})\left(\cos^2 \theta - \frac{1}{3}\right)$$
$$= \frac{2 + \frac{5}{7} C^2}{1 + \frac{3}{7} C^2}\, \xi I \sigma_{11}\left(\cos^2 \theta - \frac{1}{3}\right). \tag{44}$$

For any values of C, the anisotropy amplitude is little different from its value in the absence of a magnetic field, and the angular pattern remains quadrupolar. The ratio $P/\Delta I$, which is of order unity for $C \ll 1$, decreases for increasing magnetic field and goes like C^{-1} for large C.

Therefore, a powerful test for the existence of a cosmic magnetic field would be, rather than the anisotropy, the frequency-dependent polarization pattern. For $B_0 = 0$, P is practically constant in the Rayleigh-Jeans region, although it changes like λ_0^{-1} in the Wien region. In our axisymmetric case, in the Rayleigh-Jeans limit

$$\frac{P(B_0)}{P(0)} = \frac{\left(1 + \frac{9}{25} \tilde{\alpha}^2 B_0^2 \lambda_0^4 \cos^2 \theta\right)^{1/2}}{1 + \frac{3}{7} \tilde{\alpha}^2 B_0^2 \lambda_0^4}, \tag{45}$$

where $\tilde{\alpha} = e^3/(\pi c^4 m_e \sigma_T)$. For B_0 as large as 10^{-9} Gauss, fairly strong variations would occur in the centimetric range: the ratio (45), which is close to unity for $\lambda_0 = 1\,\mathrm{cm}$, would be reduced to about $\frac{1}{2}$ for $\lambda_0 = 5\,\mathrm{cm}$. For $B_0 = 10^{-11}$ Gauss the same considerations apply to the range $\lambda_0 = 10 \div 50\,\mathrm{cm}$.

We note also that the variations of P are of the second order in B_0 and fourth order in λ_0 for the hypothesis of axial symmetry around the magnetic field vector. For Bianchi type-I non-axisymmetric models, the magnetic corrections are of first order in C and therefore in B_0. We find that, to this order of approximation, in the Rayleigh-Jeans limit, the polarization degree is

$$\begin{aligned}\frac{P^2}{I^2} = {} & (\sigma_{11} + \sigma_{22})^2 \sin^4 \theta + \frac{(\sigma_{22} - \sigma_{11})^2}{9}\left[\cos^2 \theta \sin^2 2\phi\right. \\ & \left. + \frac{\cos^2 \theta + 1}{4} \cos^2 2\phi\right] - \frac{\sigma_{22}^2 - \sigma_{11}^2}{6} (\cos^2 \theta + 1) \\ & \sin^2 \theta \cos 2\phi - \tilde{\alpha} B_0 \lambda_0^2 [(\sigma_{22} - \sigma_{11})^2 \sin 2\phi \\ & \cos 2\phi \frac{11 \cos^4 \theta - 16 \cos^2 \theta + 7}{162} + (\sigma_{22}^2 - \sigma_{11}^2) \sin 2\phi \\ & \sin^2 \theta \frac{53 \cos^2 \theta + 105}{810} + \frac{\sigma_{22} - \sigma_{11}}{81} \sin^2 2\theta \\ & \sin 2\phi(\sigma_{11} \cos^2 \phi + \sigma_{22} \sin^2 \phi)].\end{aligned} \tag{46}$$

Thus even in the absence of axial symmetry the ratio of the polarization degree to the anisotropy of intensity is of order unity for low values of $\tilde{\alpha} B_0 \lambda_0^2$, and decreases substantially for $\tilde{\alpha} B_0 \lambda_0^2 \gtrsim 1$.

5. Conclusions

In this paper we evaluated the effects of a cosmological magnetic field on the background radiation in Bianchi type-I spaces, to the first order in the expansion anisotropy and the Thomson mean free time.

We have seen that the polarization degree is a decreasing function of the parameter $C = 4\pi e^3 B_0/c^2 m_e^2 \sigma_T \omega_0^2$, which is proportional to the rotation of the polarization plane between two consecutive scatterings. Expanding the radiation field up to the octupole terms, we have found that, aside from a frequency-dependent multiplicative factor, the anisotropy pattern of the radiation is the same as in the absence of the magnetic field, being quadrupolar. However, the polarization pattern is substantially modified, because the Faraday rotation produces a coupling between the quadrupole and the higher-order multipoles of the polarization. An obviously important test for the existence of a cosmic magnetic field is the decreasing of the polarization degree P itself with the observational frequency: while for the special case of axisymmetric Bianchi type-I models the variations of P are of second order in B_0 and fourth order in λ_0, in general for Bianchi type-I non-axisymmetric models, the magnetic corrections are of first order in B_0 and second order in λ_0. However, since we should be able to discriminate the effects of the Galactic magnetic field, also the observation of the polarization pattern constitutes an important test. As a matter of facts, in the presence of a cosmic magnetic field, the polarization pattern would be tilted across the meridians of the celestial sphere by an angle χ, whose dependence on the sky direction should be consistent with anisotropy measurements. In this connection, we also notice that the contribution of the galactic field B_q to the rotation measure is estimated to be about $14|\mathrm{cotg}\, b|\sin l\,\mathrm{m}^{-2}$, where b and l are the usual galactic coordinates (cfr. Ruzmaikin and Sokoloff, 1977). For $B_0 = 3 \times 10^{-11}$ Gauss, Eq. (43) implies an effective cosmological rotation measure of about $20\,\mathrm{m}^{-2}$, so that the cosmic polarization effects would be comparable in magnitude to the galactic contribution, but with a different spatial distribution. Therefore, we could still detect the existence of such a small magnetic field. Also, inhomogeneous extragalactic fields with random rotation measures of about $50\,\mathrm{m}^{-2}$ (Welter et al., 1984) should not be a serious problem for the detection of cosmic effects of the above order of magnitude.

As Tolman and Matzner (1982) have shown, also a positive spatial curvature gives place to a distortion of the polarization pattern, but in a quite different way: in the absence of a magnetic field, the contribution of curvature to the rotation angle is independent of the direction of propagation of the radiation, and moreover independent of frequency. So we could easily distinguish between the two cases by means of a spatial and/or spectral analysis.

In conclusion, even for magnetic fields much lower than claimed by Sofue et al. (1968, 1980) we can have measurable effects on the cosmic background polarization in the Rayleigh-Jeans wavelength region. On the other hand, for reasonable strengths of B_0, the intergalactic Faraday rotation cannot dopolarize the cosmic radiation at least around $\lambda_0 \approx 1$ mm.

Acknowledgments. We wish to thank an anonymous referee, whose suggestions have led us to a substantial improvement of the manuscript.

References

Anile A.M.: 1974, *Astrophys. Space Sci.* **29**, 415
Basko M.M., Polnarev A.G.: 1980, *Monthly Notices Roy. Astron. Soc.* **191**, 207
Brans C.H.: 1967, *Astrophys. J.* **147**, 76
Caderni N., Fabbri R., Melchiorri B., Melchiorri F., Natale V.: 1978a, *Phys. Rev.* **D17**, 1901
Caderni N., Fabbri R., Melchiorri B., Melchiorri F., Natale V.: 1978b, *Phys. Rev.* **D17**, 1908
Ceccarelli C., Dall'Oglio G., Bernardis P. Masi S., Melchiorri B., Melchiorri F., Moreno G., Pietranera L., Pucacco G.: 1982b, Proc. XVII Rencontre de Moriond, eds. J. Audouze and J. Tran Thanh Van, p. 191
Ceccarelli C., Dall'Oglio G., Melchiorri B., Melchiorri F., Pietranera L.: 1982a, *Astrophys. J.* **260**, 484
Chandrasekhar S.: 1950, Radiative Transfer, Clarendon Press, Oxford
Dautcourt G., Rose K.: 1978, *Astron. Nachr.* **299**, 13
Ellis G.F.R.: 1967, *J. Math. Phys.* **8**, 1171
Fabbri R., Breuer R.A.: 1980, *N. Cim.* B **58**, 113
Fabbri R., Guidi I., Melchiorri F., Natale V.: 1980, *Phys. Rev. Letters* **44**, 1563
Fabbri R., Guidi I., Natale V.: 1982, *Phil. Trans. Roy. Soc. London* A **307**, 67
Fennelly A.J.: 1982, *Astrophys. J.* **252**, 410
Field G.B., Perrenod S.C.: 1977, *Astrophys. J.* **215**, 717
Fixsen D.J., Cheng E.S., Wilkinson D.T.: 1983, *Phys. Rev. Letters* **50**, 620
Jacobs K.C.: 1969, *Astrophys. J.* **155**, 379
Lawler J.M., Dennison, B.: 1982, *Astrophys. J.* **252**, 81
Lubin P.M., Smoot G.F.: 1981, *Astrophys. J.* **245**, 1
Lubin P.M., Epstein G.L., Smoot G.F.: 1983, *Phys. Rev. Letters* **50**, 616
Lubin, P., Melese P., Smoot G.F.: 1983, *Astrophys. J. Letters* **273**, L51
Matzner R.A., Tolman B.W.: 1982, *Phys. Rev.* **D 26**, 2951
Melrose D.B.: 1980, Plasma Astrophysics, Gordon and Breach Science publishers
Nanos G.P.: 1979, *Astrophys. J.* **232**, 241
Negroponte J., Silk J.: 1980, *Phys. Rev. Letters* **44**, 1433
Rees M.J.: 1968, *Astrophys. J. Letters* **153**, L1
Ruzmaikin A.A., Sokoloff D.D.: 1977, *Astron. Astrophys.* **58**, 247
Ryan M.P.Jr., Waller S.M., Shepley L.C.: 1982, *Astrophys. J.* **254**, 425
Smoot G.F., Lubin P.M.: 1979, *Astrophys. J. Letters* **234**, L117
Sofue Y., Fujimoto M., Kawabata K.: 1968, *Publ. Astron. Soc. Japan* **20**, 388
Sofue Y., Takamo T., Fujimoto M.: 1980, *Astron. Astrophys.* **91**, 335
Stark R.F.: 1981, *Monthly Notices Roy. Astron. Soc.* **195**, 127
Sunyaev R.A., Zel'dovich Ya. B.: 1970, *Astrophys. Space Sci.* **7**, 3
Thorne K.S.: 1967, *Astrophys. J.* **148**, 51
Tolman B.W., Matzner R. A.: 1984, *Proc. Roy. Soc.* (in press)
Tosa M., Fujimoto M.: 1978, *Publ. Astron. Soc. Japan* **30**, 315
Tsoubelis D.: 1979, *Lett. N. Cim.* **26**, 274
Vallée J.P.: 1983, *Astrophys. Letters* **23**, 85
Welter G.L., Perry J.J., Kronberg P.P.: 1984, *Astrophys. J.* **279**, 19

E. Milaneschi and R. Fabbri: Polarization of the microwave background radiation
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SECTION II

SMALL SCALE POLARIZATION OF THE MICROWAVE BACKGROUND RADIATION.

CHAPTER IV

MICROWAVE BACKGROUND POLARIZATION IN THE DIRECTION OF CLUSTERS OF GALAXIES.

Sunyaev and Zeldovich (1980) have pointed out that small scale polarization of the microwave background radiation is produced by scattering in clusters of galaxies. X-ray observations have shown that rich clusters of galaxies contain a large amount of hot intergalactic gas (Forman et al., 1978; Cooke et al., 1978). Hence clusters of galaxies may be considered as fully ionized gas clouds with high temperature and finite optical depth with respect to Thomson scattering.

The scattering of the cosmic background radiation on these clouds of intergalactic gas opens the possibility of measuring the velocity of each cloud relative to the coordinate frame determined by the background radiation.

The radial motion of the cloud produces a variation of the background temperature in the direction of the cluster:

$$\frac{\Delta T}{T} \simeq -\frac{v_r}{c}\tau .$$

Here τ is the optical depth of the cloud with respect to the Thomson scattering, v_r is the radial component of the peculiar velocity of the cloud (positive v_r corresponds to a recession velocity exceeding that corresponding to Hubble's law). Superpoded to this effect there is another temperature variation due to scattering on free hot electrons (Sunyaev and Zeldovich, 1970); however this latter effect is frequency dependent , so measurements at two different wavelengths (one in the Rayleigh-Jeans region and the other in the Wien region) could give an estimate of the radial velocity of the cloud.

Moreover, we have the possibility, at least in principle, to measure the tangential component of the peculiar velocity of the cloud v_T using the observation of the microwave background polarization of the scattered radiation.In the clusters of galaxies there are two different polarizing effects.

a) Scattering on a single electron.

A single moving electron sees an anisotropic background radiation field. In each direction the Planckian form of the spectrum is conserved; however, the radiation temperature depends on the angle ϑ between the line of sight and the direction of motion according to the well known formula:

$$T_0 = T_r \frac{\sqrt{1-\beta^2}}{1+\beta\mu_0} \tag{38}$$

where $\beta = \frac{v}{c}$; T_0, $\mu_0 = \cos\vartheta_0$ and the angle ϑ_0 are measured in the rest frame of the electron. The expansion of eq. (38) in powers of β has a quadrupole component of order β^2:

$$T_0 = T_r\left[1-\beta\mu_0+\beta^2\left(\mu_0^2-\frac{1}{3}\right)+\ldots\right] \tag{39}$$

The angular distribution of the radiation intensity has a form:

$$J = J_r\left(1+a\mu_0+C\left(\mu_0^2-\frac{1}{3}\right)+\ldots\right) \tag{40}$$

Then after scattering on a free electron linear polarization must arise in the direction perpendicular to the direction of its motion.

Let us calculate the polarization degree using the scattering matrix (Chandrasekhar,1960). For a single scattering we have:

$$\begin{pmatrix} I_{\parallel}(\mu) \\ I_{\perp}(\mu) \end{pmatrix} = \frac{3}{8}\int_{-1}^{1} d\mu' \begin{pmatrix} 2(1-\mu'^2)(1-\mu^2)+\mu^2\mu'^2 & \mu^2 \\ \mu'^2 & 1 \end{pmatrix} \begin{pmatrix} I_{\parallel}(\mu') \\ I_{\perp}(\mu') \end{pmatrix}$$

(41)

where $I_{\parallel}$ and $I_{\perp}$ are radiation components with electric vector along two orthogonal directions. If the radiation field is not polarized

$$I_{\parallel} = I_{\perp} = I_0/2$$

In our case I_0 is given by (40), so from (41) we find:

$$\begin{pmatrix} I_{\parallel}(\mu) \\ I_{\perp}(\mu) \end{pmatrix} = \frac{1}{2} I_0 \begin{pmatrix} 1 + \frac{1}{5} C(1+\mu^2) - \frac{1}{3} C \\ 1 + \frac{2}{5} C - \frac{1}{3} C \end{pmatrix},$$

then

$$p = \frac{I_{\parallel} - I_{\perp}}{I_{\parallel} + I_{\perp}} = \frac{1}{10} C(\mu^2 - 1), \qquad (42)$$

to the first order in C.

In the case of small τ there is unscattered light in the direction of the cloud. The scattered light contributes only a small fraction τ to the total intensity. Therefore

$$p = 0.1 \, \tau \beta_t^2 .$$

The transformation to the observer's reference frame does not change this value.

To give an estimation of p, we take $\tau = \frac{1}{20}$ (this value comes from the observed "Sunyaev and Zeldovich effect"

in the direction of the cluster of galaxies Abell 2218);
on the other hand from the observrd radiation anisotropy, $\frac{\Delta T}{T} \simeq 10^{-3}$, using the formula $\frac{\Delta T}{T} = -\frac{v_r}{c}\tau$, we can give an upper limit to $v_r \lesssim 6000$ km/s). Then we find:

$$\rho \lesssim 2 \cdot 10^{-6},$$

which is more than one order of magnitude below the actual experimental sensitivity.

b) Effect of finite optical depth.

There is another effect connected with two consecutive scatterings, i.e. with finite optical depth of the cloud. Although τ is small, it is possible that effects of the order of τ^2 may become observable. These effects are absent when we consider scattering on a single electron.

In the rest frame of the cloud the unperturbed radiation field is, to the first order in β,

$$T_0 = T_r (1 - \beta \mu_0)$$
$$J_0 = J_r (1 - k \beta \mu_0),$$

where the coefficient k depends on ν.
Due to the symmetry of the angular dependence of Thomson scattering relative to the change $\vartheta_0 \to \pi - \vartheta_0$, $\mu_0 \to -\mu_0$, the

scattered radiation becomes isotropic and the dipole term $\kappa\beta\mu$ vanishes in the scattered radiation field. It is obvious that the isotropic radiation field does not change under the action of Thomson scattering. Therefore we must take into account only the uncompensated decrease, due to scattering, of the dipole component of the radiation measured in a frame moving with the cloud.

For example, for a plasma sphere with constant electron density within its boundary, we obtain to the first approximation:

$$J_0 = J_r(1 - \kappa\beta\mu_0) \quad , \qquad -1 < \mu_0 < 0$$

and

$$J_0 = J_r[1 - \kappa\beta\mu_0 e^{-\tau\mu_0}], \qquad 0 < \mu_0 < 1,$$

where $\tau = 2\sigma_T n_e R$.

Such a distribution contains the second harmonics with coefficient $\sim \kappa\beta\tau$. The scattered light becomes polarized, the polarization having opposite sign on different sides of the cloud. Taking into account that only a small fraction τ of the detected radiation is scattered in the cloud, we obtain:

$$p = \pm \frac{\kappa\beta\tau^2}{20} \sin^2\theta_0 .$$

In the Rayleigh-Jeans region $\kappa = 1$ and with $\tau = 0.1$, $v_t = 3000$ km/s $\rightarrow \beta_t = 0.01$ we find:

$$p_{max} = \pm 2.5 \cdot 10^{-6}.$$

The thermal effect can also produce a polarization of the microwave background radiation, infact it decreases the intensity in the Rayleigh-Jeans region, producing an anisotropy in the radiation field. Therefore even in the case $\sigma=0$, it can lead to the polarization of the cosmic background in the direction to the cloud. In this case the coefficient of the quadrupole anisotropy is $\simeq \frac{KT_e}{m_e c^2}\tau$. So after one single scattering the polarization degree is:

$$P \simeq 0.1 \frac{KT_e}{m_e c^2} \tau \sin^2 \theta_o ,$$

and for a small finite optical depth is:

$$P \simeq 0.1 \frac{KT_e}{m_e c^2} \tau^2 \sin^2 \theta_o ,$$

which is always of the order of 10^{-6}.

CHAPTER V

SMALL SCALE POLARIZATION OF THE MICROWAVE BACKGROUND RADIATION IN THE ADIABATIC THEORY.

Another source of small scale polarization are the primordial perturbations which are supposed to have grown up into the observed structures of the universe.
In this case the expected polarization degree is larger than that predicted by Sunyaev and Zeldovich (1980) to accompany the microwave temperature decrease in the direction of clusters and is also larger than the expected polarization from discrete sources.

Kaiser (1983) calculated the evolution of initially adiabatic perturbations through the epoch of recombination with particular regard for the polarizing properties of Thomson scattering. He makes the following simplifying assumptions: (i) the self gravity of the perturbations may be neglected, since he considers perturbations below the Jeans mass, (ii) the universal expansion may be neglected, since the time scale for recombination is small compared to the expansion time scale at decoupling

A spectrum of initially adiabatic sound waves is propagated through the recombination epoch, $1500 \lesssim z \lesssim 900$, then a linear approximation is used. The general perturbation is decomposed into Fourier components in which the perturbed quantities all vary as $X(\vec{r},t) = Re[X(t)e^{i\vec{k}\cdot\vec{r}}]$ and which evolve independently.

The fractional perturbation of the radiation field $J(\mu,\vec{r},t)$ is also a function of the angle between the observation direction $\vec{l}$ and the wave vector $\vec{k}$, being $\mu = \vec{k}\cdot\vec{l}$.

Because through the Thomson scattering a linear polarization arises whenever radiation with a non-zero quadrupole moment of intensity is scattered, a multipole expansion of the radiation field at least to the second order is required. In fact Kaiser expands the intensity of the radiation up to the quadrupole moment, neglecting the coupling with higher multipoles. This approximation is good for the optically thick regime of the perturbation that is $k\,t_c \ll 1$, where t_c is the mean free time for Thomson scattering, but becomes less accurate at later times.

The procedure he uses is the following. First he generalize the transfer equation given by Peebles and Yu

(1970) to allow for Thomson scattering, introducing the scattering matrix. Then he solves analitically this equation for the optically thick regime taking the moments up to the second and expanding the transfer equation to the first two non-vanishing orders in the mean free path of the photons. In this regime all the relevant quantities are supposed to vary proportionally to exp $[Y\ t]$, where Y is found to be:

$$Y = \pm \frac{iK}{\sqrt{3a}} - \frac{1}{6} K^2 t_c \left(1 - \frac{14}{15}\frac{1}{a} + \frac{1}{a^2}\right),$$

where

$$a = \left(1 + \frac{3}{4}\frac{\rho_m}{\rho_r}\right).$$

Then the damping rate is:

$$\Gamma_{Thom} = \frac{1}{6} K^2 t_c \left(1 - \frac{14}{15}\frac{1}{a} + \frac{1}{a^2}\right).$$

If we compare this quantity with the result for isotropic scattering given by Peebles (1980), $\Gamma_{iso} = \frac{1}{6} K^2 t_c \left(1 - \frac{6}{5}\frac{1}{a} + \frac{1}{a^2}\right)$ we see that Γ_{Thom} exceeds Γ_{iso} by a factor 4/3 in the radiation dominated limit, showing that the effective viscosity is increased.

The solution valid in the optically thick regime are then used as an initial sequence to integrate numerically the transfer equation when the optical depth of the perturbation is intermediate. The integration stops at a redshift z_f = 850. The final results are the total rms fluctuations obtained summing the contribution from diffe_ rent plane waves in quadrature.

Fig. 12 (a,b,c) shows the spectrum of fluctuations e emerging from decoupling for an initial spectrum $\delta^2 \propto k^4$. δ_g, $\mathcal{J}_o$ and P_o represent the rms density fluctuations, the rms radiation fluctuation and rms polarization fluctuation respectively.

From these figures we see that the polarized component is expected to be about the 20% of the rms intensity fluctua_ tion.

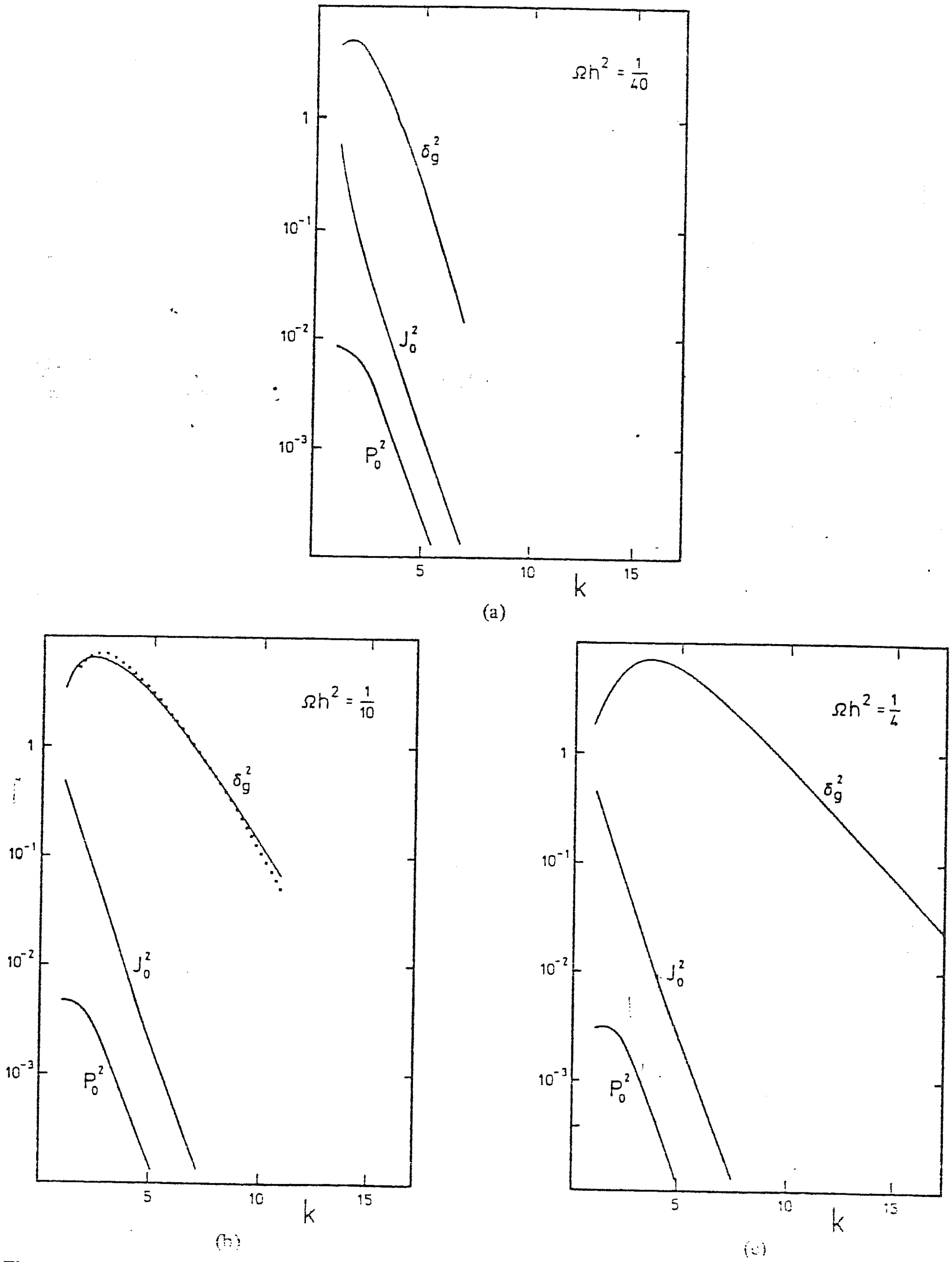

Figure 1. The spectra of density, temperature and polarization fluctuations emerging from decoupling for a spectrum with $n = 4$ initially, and for three values of the parameter Ωh^2. The dotted line in Fig. 1(b) is Peebles' (1981) result for the same parameters.

57/85/A

SMALL SCALE POLARIZATION OF THE COSMIC BACKGROUND RADIATION

E. Milaneschi and R. Valdarnini

International School for Advanced Studies (SISSA), Trieste, Italy

September 1985

SUMMARY

We study the polarization of the microwave background radiation on small angular scales, induced by adiabatic density perturbations in universes dominated either by cold, warm and hot dark matter or by baryons.

We have computed the ratio between polarization and radiation anisotropy $\equiv (P/(\Delta I/I))$ as a function of angular scale. This ratio is found to be typically $\simeq 10\%$; it turns out to be larger on very small angular scales and approaches a constant value on scales larger than $\simeq 40' \; \Omega^{2/3} h^{1/3}$.

It depends also on the value of the spectral index n of primeval density fluctuations, being larger for larger n. For fixed values of the angular scale and of n the $P/(\Delta I/I)$ ratio is the same, irrespective of the nature of dark matter.

Finally, this ratio decreases with Ω; for $\Omega = 0.2$ it is a factor $\simeq 2$-4 smaller than for $\Omega = 1$.

Thus a positive detection of polarization in the cosmic background radiation at small angular scales would provide useful constraints both on Ω and on the value of n.

Key words: polarization - cosmic background radiation - dark matter.

1. INTRODUCTION

The small scale polarization of the cosmic background radiation has been studied by Kaiser (1983) and Bond and Efstathiou (1984). Kaiser (1983) pointed out that a sizeable degree of polarization is associated to small scale anisotropies of the cosmic microwave background induced by adiabatic density perturbations. This is a very interesting result since the predicted ratio between polarization and anisotropy (0.2) is much larger than that associated to other competing sources of small scale anisotropies (such as Poisson fluctuations in the direction of discrete sources, Sunyaev-Zeldovich effect in rich cluster of galaxies). Thus measuring this ratio would greatly ease the singling out of the truly primordial temperature fluctuations.

Kaiser's (1983) result, however, is preliminary in that it is based on a number of simplifying assumptions. In fact he neglects the self gravity of the perturbations, since he considers perturbations below the Jeans mass, and the expansion of the Universe through the recombination.

A more refined calculation was later carried out by Bond and Efstathiou (1984) who, however, only give the ratio between the correlation function of the polarization and the correlation function of the radiation anisotropy.

Here we present a full analysis of the problem. We compute the ratio between polarization and radiation anisotropy as a function of angular scale for a variety of cosmological scenarios. We consider model universes in which dark matter is made of massive neutrinos (N case), warm particles with a rest mass in the KeV range (tentatively gravitinos, G case) and cold particles (C case). In these cases we take $\Omega = 1$ and $\Omega_b = 0.03$ for the baryonic contribution to Ω. For comparison we also consider a baryonic model for the dark matter with $\Omega = \Omega_b = 0.2$ (B case) and a cold dark matter case with $\Omega = 0.2$ and $\Omega_b = 0.03$ (C2 case). In addition we analyse the dependence of the results on the spectrum of density perturbations.

We have numerically integrated the perturbed Liouville equation for the collisionless particles and the collisional Boltzmann equation for the perturbations to the radiation Stokes parameters.

In sect. 2 we describe the analytical framework of our model and in sect. 3 the numerical integration scheme is discussed. In sect. 4 we present the results and draw some conclusions.

2. THEORY

2.1 Fundamental assumptions

The metric element, in the standard synchronous gauge, reads

$$ds^2 = dt^2 - a^2(t)\left[\delta_{\alpha\beta} - h_{\alpha\beta}\right] dx^\alpha dx^\beta , \tag{2.1}$$

where t is the cosmic time, a(t) is the scale factor and $h_{\alpha\beta}$ are the metric perturbations (hereafter we take $c = k_B = 1$). The total density is

$$\rho = \rho_r + \rho_X + \rho_b , \tag{2.2}$$

where ρ_r, ρ_X and ρ_b are the energy densities of radiation, collisionless particles and baryons, respectively. We parametrize the Hubble constant as $H = 100h\ \mathrm{Km\ sec^{-1}\ Mpc^{-1}}$ and take the present value of the background radiation temperature to be 2.7°K (Smoot et al. 1985). The time unit τ is chosen to be

$$\tau = \sqrt{3 / 8\pi G \rho_r a^4}$$

and $l = a_o c\tau$ is the length unit; the index o refers to the present epoch.

In these units the equation for a(t) is

$$\frac{da}{dt} = \frac{1}{a}\sqrt{1 + (\rho_X + \rho_b)/\rho_r} . \tag{2.3}$$

At high red-shifts the collisionless particles (hereafter X particles) obey the Fermi-Dirac distribution function with zero chemical potential. The ratio ρ_X/ρ_r is

$$\frac{\rho_X}{\rho_r} = c_X e_{1X} , \quad e_{1X} = \int_0^\infty p^2 q \frac{dp}{e^p + 1} , \quad c_X = \frac{15 g_X \alpha_X^4}{2\pi^4} , \quad q^2 = p^2 + (m_X / T_X)^2 \tag{2.4}$$

where β_X is the number of spin states for the X particles, $\alpha_X = T_X/T_{rad}$ and p is the particle momentum in T_X units. The mass of the particle is fixed by the above parameters: $m_X = 70.5\Omega_X h^2/\beta_X\alpha_X^3$ eV. The mass associated to the length scale $\lambda = 2\pi a(t)\, k^{-1}$ (k = wavenumber) is $M = \frac{\pi}{6}\rho\lambda^3$.

In the linear approximation all the variables of interest can be decomposed into plane waves. We choose the x^3 axis along the wave propagation direction and call θ the angle between the x^3 axis and $\vec{p}$.

2.2 Liouville equation for collisionless particles

We assume that the distribution function of the X particles, f, is given by

$$f = \left\{ e^{p/(1+g)} + 1 \right\}^{-1}, \tag{2.5}$$

where g is a small correction to the unperturbed distribution. In our co-ordinate system the Liouville equation for a plane wave reads (Peebles 1982)

$$\frac{\partial g}{\partial t} + i\,\frac{K\mu}{a}\,\frac{p}{q}\,g = \frac{1}{4}\left[(1-\mu^2)\dot{h} + (3\mu^2-1)\dot{h}_{33}\right] \equiv \frac{Y}{4}, \tag{2.6}$$

where $g = g(t,k,\mu,p)$, $h = h^{\alpha}_{\alpha}$, $\mu = \cos\theta$. The dependence of g on μ can be decomposed into Legendre polynomials (Valdarnini 1985):

$$g = \sum_{\ell=0}^{\infty} \left\{ \sigma_{2\ell}\,\frac{4\ell+1}{2}\,P_{2\ell}(\mu) + i\,\sigma_{2\ell+1}\,P_{2\ell+1}(\mu) \right\}. \tag{2.7}$$

Then eq. (2.6) becomes

$$\begin{cases} \dot{\sigma}_0 = \dfrac{K}{a}\,\dfrac{p}{q}\,\sigma_1 + \dfrac{\dot{h}}{3} \\ \dot{\sigma}_\ell = (-)^{\ell}\dfrac{K}{a}\,\dfrac{p}{q}\left\{\sigma_{\ell+1}(\ell+1) + \sigma_{\ell-1}\,\ell\right\}\dfrac{1}{2\ell+1} + \left(\dot{h}_{33} - \dfrac{\dot{h}}{3}\right)\dfrac{\delta_{\ell 2}}{5}, \quad \ell \geq 1 . \end{cases} \tag{2.8}$$

2.3 Matter-radiation and field equations

(a) Transfer equation for polarized radiation and matter equations

We assume that at high red-shifts ($z > 10^7$) the photon distribution is isotropic and unpolarized. Linear polarization will be produced by Thomson scattering of radiation anisotropies due to density perturbations. Polarized radiation is described by the four Stokes parameters I, Q, U and V (Chandrasekhar 1960). Our decomposition into plane waves guarantees that for each plane wave the radiation field possesses axial symmetry with respect to the wave propagation direction, so that U = 0 in our reference frame. We further assume that there is not circular polarization (V=0).

Let us define J_l, J_r as the fractional perturbations of the radiation distribution relative to two orthogonal directions of linear polarization, which are in turn orthogonal to the k direction. Then $I = ½(J_l + J_r)$ and $Q = ½(J_l - J_r)$. The transfer equation is (Kaiser 1983):

$$\frac{\partial}{\partial t}\begin{bmatrix} I \\ Q \end{bmatrix} + i\mu\frac{k}{a}\frac{p}{q}\begin{bmatrix} I \\ Q \end{bmatrix} - \gamma\begin{bmatrix} 1 \\ 0 \end{bmatrix} =$$

$$t_c^{-1}\left\{\frac{3}{16}\int_{-1}^{+1}\begin{pmatrix} 3-\mu'^2-\mu^2+3\mu^2\mu'^2 & 1-\mu'^2-3\mu^2(1-\mu'^2) \\ 1-3\mu'^2-\mu^2+3\mu^2\mu'^2 & 3-3\mu'^2-3\mu^2(1-\mu'^2) \end{pmatrix}\begin{pmatrix} I' \\ Q' \end{pmatrix} d\mu' \right.$$

$$\left. + 4\mu i v_b\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} I \\ Q \end{bmatrix}\right\} . \qquad (2.9)$$

Here $I = I(\mu,k,t)$, $Q = Q(\mu,k,t)$, iv_b is the baryonic matter velocity and

$$t_c = \left(n_e(z)\,\sigma_T\, c\, x(z)\right)^{-1},$$

where n_e is the electron number density, σ_T the Thomson scattering cross-section and x(z) the ionization degree. The polarization parameter Q-iU is a spin-two quantity for a rotation of the l and r axes around the $\vec{k}$ direction (Dautcourt and Rose 1978). Then Q-iU can be expanded in a series of spin-two polynomials. Owing to the axial symmetry these polynomials reduce to the associated Legendre polynomials $P_{l,2} = (1-\mu^2)\frac{d^2}{d\mu^2} P_l(\mu)$. Then we have decomposed the Q parameter in the series

$$Q = \sum_{\ell=2}^{\infty} q_\ell (-i)^\ell P_{\ell,2}(\mu). \tag{2.10}$$

For the fractional radiation brightness I the following expansion has been used:

$$I = \sum_{\ell=0}^{\infty} \delta_\ell (-i)^\ell P_\ell(\mu) . \tag{2.11}$$

The radiation density contrast is

$$\delta_r = \frac{1}{2}\int_{-1}^{+1} I \, d\mu = \delta_0 . \tag{2.12}$$

The transfer equation (2.9) becomes

$$\begin{aligned}
\dot\delta_0 &= \frac{2}{3}\dot h - \frac{k}{a}\frac{\delta_1}{3} ,\\
\dot\delta_1 &= -t_c^{-1}\left[4v_b + \delta_1\right] - \frac{k}{a}\left[\frac{2}{5}\delta_2 - \delta_0\right] ,\\
\dot\delta_2 &= -t_c^{-1}\left[\frac{9}{10}\delta_2 + \frac{6}{5}q_2\right] - 2\left(\dot h_{33} - \frac{\dot h}{3}\right) - \frac{k}{a}\left[\frac{3}{7}\delta_3 - \frac{2}{3}\delta_1\right] ,\\
\dot\delta_\ell &= -\frac{k}{a}\left[\frac{\ell+1}{2\ell+3}\delta_{\ell+1} - \frac{\ell}{2\ell-1}\delta_{\ell-1}\right] - t_c^{-1}\delta_\ell ,\\
\dot q_2 &= -\frac{k}{a}\frac{5}{7}q_3 - t_c^{-1}\left[\frac{\delta_2}{20} + \frac{2}{5}q_2\right] ,\\
\dot q_\ell &= -\frac{k}{a}\left[\frac{\ell+3}{2\ell+3}q_{\ell+1} - \frac{\ell-2}{2\ell-1}q_{\ell-1}\right] - t_c^{-1}q_\ell , \quad \ell \geq 3 .
\end{aligned} \tag{2.13}$$

It must be noted that in eq. (2.13) the polarization is produced only through the scattering of the quadrupole moment of the radiation field.

Finally the continuity and motion equations for matter are

$$\left\{\begin{aligned} \dot{\delta}_b &= \frac{\dot{h}}{2} + \frac{k v_b}{a} \\ \dot{v}_b &= t_c^{-1}\frac{\rho_r}{\rho_b}\left[-i f_r - \frac{4}{3}v_b\right] - \frac{\dot{a}}{a}v_b \end{aligned}\right. . \tag{2.14}$$

Here δ_b is the baryonic density contrast and

$$f_r = \frac{1}{2}\int_{-1}^{+1} I\mu\, d\mu = -i\,\frac{\delta_1}{3} \tag{2.15}$$

is the radiation energy flux.

(b) Field equations

To complete our system of equations we have still to specify the equations for the gravitational field. In the weak field approximation these equations read (LSS, §82)

$$\ddot{h} + 2\frac{\dot{a}}{a}\dot{h} = 6\left[\delta_r + c_X e_{1X}\,\delta_{X,\nu} + \frac{\rho_b}{2\rho_r}\delta_b\right]a^{-4}$$

$$\dot{h}_{33} - \dot{h} = \frac{6}{ka^3}\left[-i f_r + \frac{\rho_b}{\rho_r}v_b + c_X e_{1X} f_X\right] , \tag{2.16}$$

where

$$\delta_{X,\nu} = \frac{1}{2e_{1X}}\int_0^\infty \sigma_0\left[p^2 + \frac{1}{2}a^2\right]\frac{p^3}{4}\frac{e^p}{(e^p+1)^2}\,dp ,$$

$$f_X = \frac{1}{2e_{1X}}\int_0^\infty \sigma_1\, p^4\,\frac{e^p}{(e^p+1)^2}\,dp . \tag{2.17}$$

For the X particles the density contrast is

$$\delta_X = \frac{1}{2e_{1X}} \int_0^\infty \bar{\sigma}_0\, p^3 q\, \frac{e^p}{(e^p+1)^2}\, dp \ . \tag{2.18}$$

In the cold case $\delta_{X,g} \simeq \frac{\delta_X}{2}$ and eqs. (2.8) are replaced by

$$\dot{\delta}_X \simeq \frac{\dot{h}}{2} \ . \tag{2.19}$$

3. NUMERICAL APPROACH

3.1 Approximate solutions

(a) Matter and radiation

At high red-shifts $t_c \to 0$ and the numerical integration of eqs. (2.13) becomes difficult (Peebles and Yu 1970). To avoid this problem we have expanded the I and Q quantities in powers of t_c. Approximate solutions of eqs. (2.9) have then been used in the first stages of the integration (see below). This approach is similar to that used by Peebles and Yu (1970).

In the $t_c = 0$ limit the transfer equation (2.9) has the solution

$$I = \delta_r + i\,4\mu v_b \equiv I_0 \quad , \quad Q = 0 \equiv Q_0 \ . \tag{3.1}$$

Writing $I = I_0 + t_c I_1$, $Q = Q_0 + t_c Q_1$, we find the following iterative solutions of eq. (2.9)

$$I = I_0 + t_c \left\{ -\frac{\delta_2}{4}\eta_p P_2(\mu) - C + \frac{d}{dt} t_c \left(\frac{\delta_2}{4}\eta_p P_2(\mu) + C \right) + \right.$$
$$\left. + \frac{ik\mu}{a} t_c \left[\frac{\delta_2}{4} P_2(\mu)\,\eta_p + C \right] \right\} \tag{3.2}$$
$$Q = Q_0 - t_c \frac{\delta_2}{8}\eta_p$$

where

$$C = \frac{d}{dt} I_0 + \frac{ik\mu}{a} I_0 - Y$$
$$\delta_2 = 2\left[1 + \frac{\eta_p}{3}\right]\left[\frac{4}{3}\frac{k v_b}{a} + \frac{\dot{h}}{3} - h_{33}\right] , \tag{3.3}$$

and $\eta_p = 1$. If $\eta_p = 0$ we recover Peebles and Yu (1970) results for unpolarized radiation. The radiation energy flux is, to the second order in t_c,

$$-i f_r = \frac{4}{3} v_b + t_c \left\{ - \frac{k \delta_r}{3a} \frac{1}{1+R} + \frac{4}{3} \frac{\dot{a}}{a} \frac{v_b}{1+R} + t_c \frac{\Delta}{1+R} \right\} , \tag{3.4}$$

where

$$\Delta = \frac{k \delta_r}{3a} \frac{\dot{a}}{a} \frac{2(1+2R)}{(1+R)^2} - \frac{4}{3} \frac{\ddot{a}}{a} \frac{v_b}{(1+R)} - \frac{4}{3} \left(\frac{\dot{a}}{a}\right)^2 \frac{v_b}{(1+R)^2} (1+3R)$$

$$+ \frac{k}{3a} \left[\frac{2}{3} \dot{h} \left(\frac{1}{1+R} + \frac{2}{5} \right) - \frac{4}{15} \dot{h}_{33} + \frac{4}{3} \frac{k v_b}{a} \left[\frac{1}{1+R} - \frac{4}{5} \right] + \frac{\eta_0}{16} \delta_2 \right] , \tag{3.5}$$

and $R = \frac{4}{3} \rho_r / \rho_b$. Then, in the limit of small t_c, the system of equations for matter and radiation reduces to

$$\dot{\delta}_r = \frac{2}{3} \dot{h} + \frac{4}{3} \frac{k v_b}{a} + \frac{k t_c}{a} \left[\frac{4}{3} \frac{\dot{a}}{a} v_b - \frac{k \delta_r}{3a} \right] \frac{1}{(1+R)} ,$$

$$\dot{\delta}_b = \frac{\dot{h}}{2} + \frac{k v_b}{a} ,$$

$$\dot{v}_b = - \frac{k \delta_r}{4a} \frac{R}{1+R} - \frac{\dot{a}}{a} \frac{v_b}{(1+R)} + \frac{3}{4} t_c \frac{R}{1+R} \Delta . \tag{3.6}$$

(b) Collisionless particles

The numerical integration starts at an initial red-shift z_{in} such that $\nu t_{in} >> 1$ for all the wavelengths of interest. In this limit eq. (2.8) can be integrated analytically, giving (Valdarnini 1985):

$$\sigma_0 = \frac{\dot{h}}{3} t_{in}$$

$$\sigma_1 = - \frac{2}{45} k \frac{t_{in}^2}{a_{in}} (2\dot{h}_{33} + \dot{h})$$

$$\sigma_2 = (\dot{h}_{33} - \dot{h}/3) t_{in}/5$$

$$\sigma_3 = -\frac{2}{35} k \frac{t_{in}^2}{a_{in}} \left(\dot h_{33} - \frac{\dot h}{3} \right)$$

$$\sigma_\ell = 0 \quad , \ \ell \geq 4 . \tag{3.7}$$

If X particles are ultrarelativistic at $z = z_{in}$ $\delta_{X,g} \simeq \delta_X$ and with the assumption $\delta_r = \delta_X = D(t) \propto t$ we have (Peebles 1982)

$$\dot h = \frac{3}{2} D/t$$

$$\dot h_{33} = \frac{39\omega + 35}{38\omega + 30} \frac{D}{t}$$

$$v_b = -\frac{kD}{6a} t$$

$$\delta_b = \frac{3}{4} \delta_r \tag{3.8}$$

where $\omega = (\rho_X + \rho_b)/\rho_r$. In the cold DM case $\delta_X = \frac{3}{4} \delta_r$ and the second of eqs. (3.8) becomes

$$\dot h_{33} \simeq \frac{35 D}{30 t} \quad ,$$

3.2 Numerical integration

The integration is performed from the initial red-shift $z_{in} = 10^9$ down to $z_f = 800$. The timestep is chosen to be $\Delta t = 10^{-2} \mathrm{Min}\,(a/\dot a, \lambda)$. For the integration we have used a Merson routine of the CERN Library; the relative accuracy has been fixed to be 10^{-4}, at each step of the integration. The differential equation for the ionization degree $x(z)$ has been numerically computed from $z = 2000$ down to $z = 800$, taking into account the presence of X particles (Peebles 1968, Bonometto et al. 1983). The values of $x(z)$ obtained in this way have been stored and used in the main computation.

The expansion of the g, I and Q variables into orthogonal polynomials has been truncated at a suitable l_{MAX} ($\simeq 200$), such that the last term of the series is much smaller ($\simeq 10^{-2}$) than the $l = 3$ term at $z = z_f$ for each of the variables under consideration. The integration over p for the σ's has been done by a ten point Gauss-Laguerre quadrature. The accuracy of this integration has been checked in a previous work (Valdarnini 1985). The system of eqs. (3.6) switches to eqs. (2.13) when

$$\frac{3}{4} t_c \frac{R}{1+R} \Delta \simeq \varepsilon_c \left[-\frac{k\delta_r}{4a} \frac{R}{1+R} - \frac{\dot{a}}{a} \frac{\sigma_b}{1+R} \right],$$

$$\frac{k t_c}{a} \left[\frac{4}{3} \frac{\dot{a}}{a} \sigma_b - \frac{k\delta_r}{3a} \right] \frac{1}{1+R} \simeq \varepsilon_c \left[\frac{2}{3} \dot{h} + \frac{4}{3} \frac{k\sigma_b}{a} \right], \qquad (3.9)$$

with $\varepsilon_c = 10^{-2}$.

In our integration we spanned the mass range $M_i < M < 10^{20} M_\odot$, with M_i depending on the considered case ($M_i = 10^6 M_\odot$ in the C case, $M_i = 10^{10} M_\odot$ in the G case and $M_i = 10^{13} M_\odot$ for the N and B cases). We took four values of k for each decade in M. Initial density perturbation spectra of the form $|\delta_r|^2 = D^2 k^n$ have been assumed at $z = z_{in}$. In an actual calculation we have set n = 0, in the linear approximation the results appropriate to any other spectral index can be obtained just by multiplying the final δ's by $k^{n/2}$.

4. RESULTS

4.1 Observable quantities

After recombination we can neglect scattering terms in eq. (2.9). Then, at the present epoch t_o, the fractional brightness perturbation I and the polarization Q are (Peebles and Yu 1970)

$$I(t_o, k, \mu) = I(t_f, k, \mu)\, e^{-ik\mu d} - \frac{2 i a_o}{k} \mu h_o + \frac{2 a_o}{k^2} \frac{d}{dt} \left[a \dot{h} \right]_o ,$$

$$Q(t_o, k, \mu) = Q(t_f, k, \mu)\, e^{-ik\mu d}, \quad d = \int_{t_f}^{t_o} \frac{dt}{a(t)}, \qquad (4.1)$$

where t_f is the time corresponding to the red-shift $z_{fin} = 800$ and

$$\overline{I}(t_g, k, \mu) = I(t_g, k, \mu) + 2i\frac{\dot{a}_g}{k}\mu h_g - \frac{2a_g}{k^2}\frac{d}{dt}\left[\dot{a}h\right]_g$$

In the calculation of the small scale temperature fluctuation of the CBR the gravitational field terms in the first of eq.s (4.1) are unimportant.

The radiation correlation function is defined as

$$C_R(\theta,\sigma) = \frac{1}{16}\left\langle \delta_r(\vec{x},\hat{n}_1),\ \delta_r(\vec{x},\hat{n}_2)\right\rangle , \tag{4.2}$$

where n_1, n_2 are unit vectors along the direction of observation, $\cos\theta = n_1 \cdot n_2$ and σ is the beam width. In the limit $\theta \ll 1$ eq. (4.2) becomes

$$C_R(\theta,\sigma) = \frac{1}{16}\frac{V}{4\pi^2}\int_0^\infty k^{n+2}\,dk\int_{-1}^{+1} d\mu\,\left|I(t_0,k,\mu)\right|^2 J_0\left(k\theta r_0\sqrt{1-\mu^2}\right)$$

$$\cdot \exp\left\{-k^2\sigma^2 r_0^2(1-\mu^2)\right\}, \tag{4.3}$$

where $r_0 = 2c/\Omega H$, V is a normalizing volume and J_0 a Bessel function. The mean square temperature fluctuation is then (Doroshkevich et al. 1978)

$$\left(\frac{\Delta T}{T}\right)^2_{\theta,\sigma} = \frac{3}{2}\left[C_R(0,\sigma) - \frac{4}{3}C_R(\theta,\sigma) + \frac{1}{3}C_R(2\theta,\sigma)\right], \tag{4.4}$$

where ΔT is the difference between the temperature measured along a central beam and the average of the temperatures measured in two directions spaced by an angle θ with respect to the central beam.

The corresponding quantities for the polarization have been defined in the same way. The small scale mean square polarization is

$$P^2(\theta,\sigma) = 16\cdot\frac{3}{2}\left[C_P(0,\sigma) - \frac{4}{3}C_P(\theta,\sigma) + \frac{1}{3}C_P(2\theta,\sigma)\right]. \tag{4.5}$$

To normalize our radiation spectra we introduce the rms fluctuation in mass $\delta M/M$ within a randomly placed sphere of radius R. It is found (Peebles 1980)

$$\left(\frac{\delta M}{M}\right)^2 = \frac{V}{2\pi^2}\int_0^\infty k^{n+2}\left|\delta_k\right|^2 W(kR) , \tag{4.6}$$

where

$$W(y) = \frac{9}{y^6}\left[\sin y - y\cos y\right]^2 \quad .$$

We normalize $\frac{\delta M}{M} = 1$ at the present epoch on the length scale $R = 8h^{-1}$ Mpc. The same normalization has been used in all the considered cases. In a neutrino dominated universe other normalization procedures make use of the two-point correlation function $\xi(r)$ (Bond and Efstathiou 1984). Different choices of the normalization may result in differences in the calculated $\Delta T/T$ as large as a factor of two (Vittorio and Silk 1985). We stress however that we are primarily interested in the ratio polarization/anisotropy, which is obviously independent of normalization.

4.2 Results and discussion

Our $\Delta T/T$ always agrees with those previously computed by several authors (Bonometto et al. 1984, Bond and Efstathiou 1984, Vittorio and Silk 1984). In Figs. 1 we show the temperature fluctuations versus the angular scale for two different antenna beams ($\sigma = 1.5'$ and $\sigma = 3'$). Different curves are labelled by the corresponding values of the spectral index n. Figs. 1 can be compared with Figs. 1 (a and b) of Bonometto, Lucchin and Valdarnini (1984), where the quoted values for σ must be doubled.

In Table 1 we quote the C_p/C_r ratios for $\theta = 0 = \sigma$ in the C case. Other cases give similar results. The n=1 case agrees with the corresponding values reported by Bond and Efstathiou (1984). In Fig. 2a we show the ratio between the polarization degree and the fractional radiation anisotropy $P/(\Delta I/I)$, as a function of the angular scale for $\sigma = 1.5'$ and a cold dark matter dominated universe. This ratio turns out to be very similar irrespective of the nature of dark matter (cold, warm or hot). This happens because in these models the polarization is produced through the scattering of radiation by free electrons, whose number is the same at each red-shift in all the three considered cases.

From Fig. 2 it can be seen that $P/(\Delta I/I)$ is a decreasing function of θ. The contribution to the integrals in eq. (4.4) is negligible for perturbations with $k\theta r_o < 1$ or $M > M_c(\theta)$ where

$$M_c(\theta) \simeq 2.10^{14} \Omega^{-2} h^{-1} \left(\theta/1'\right)^3 M_\odot \, . \qquad (4.7)$$

In Figs. 4 we plot, as a function of the mass scale and for n=0, the averaged final spectra $T_r = \sqrt{\overline{I^2}}$, $T_p = \sqrt{\overline{Q^2}}$, where

$$\overline{I^2(k,t_o)} = \frac{1}{2}\int_{-1}^{+1} d\mu \, |I(\mu,k,t_o)|^2$$

$$\overline{Q^2(k,t_o)} = \frac{1}{2}\int_{-1}^{+1} d\mu \, |Q(\mu,k,t_o)|^2 \, . \qquad (4.8)$$

Fig. 4a refers to the C case and Fig. 4b to the B case. From Fig. 4a it is clear that on larger mass scales $T_r(k)$ increases faster than $T_p(k)$. Then as θ is increased $M_c(\theta)$ becomes higher and the $P/(\Delta I/I)$ ratio decreases.

For $k \to 0$ $|T_p/T_r| \simeq 10^{-2}$. In fact for the longest wavelengths the recomination can be treated as instantaneous: so, while temperature anisotropy is generated by potential fluctuations, there are not enough scatterings to convert it into polarization (this result is analogous to that found by Negroponte and Silk,1980), when a step-function is used for the ionization degree).

At a fixed θ, decreasing n results in a decrease of $P/(\Delta I/I)$, since the contribution of $T_r(k)$, $T_p(k)$ in the integrals in eq. (4.4) is weighted by the k^{n+2} factor, which enhances the high mass contribution to the integral if n is lowered.

In Fig. 3 the $P/(\Delta I/I)$ ratio is shown as a function of the angular scale in the baryonic case ($\Omega = \Omega_b = 0.2$, B case). This ratio, $[P/(\Delta I/I)]_B$, has the same behaviour, versus θ and n, of the corresponding $[P/(\Delta I/I)]_C$ ratio in the cold DM case; the value of $[P/(\Delta I/I)]_B$, however, is smaller by a factor increasing from $\simeq 2$ to $\simeq 4$, as θ increases. This difference is mainly due to the different values of Ω considered in the two cases. In the

B case $\Omega = 0.2$ and at a fixed θ, $M_c(\theta)$ is $(0.2)^{-2}$ higher than in the C case; then $[P/(\Delta I/I)]_B$ is smaller than $[P/(\Delta I/I)]_C$. The $P/(\Delta I/I)$ ratio in the n=0 case reaches the asymptotic value for $M_c(\theta) \simeq 10^{19} M_\odot$, that is for

$$\theta \simeq 40' \, \Omega^{2/3} h^{1/3} . \tag{4.9}$$

In fact $[P(\Delta I/I)]_B$ becomes constant for $\theta > 15'$ while $[P/(\Delta I/I)]_C$ approaches a constant value for $\theta > 40'$.

These conclusions are confirmed by a comparison between the $[P/(\Delta I/I)]_B$ ratio and the $[P/(\Delta I/I)]_{C2}$ ratio (Fig. 2b) in the C2 case ($\Omega = 0.2, \Omega_b = 0.03$). The bending of $[P/(\Delta I/I)]_{C2}$ to approach a constant value occurs roughly at the same angular scale as found for $\Omega_b = 0.2$ baryon dominated case. It should be noted that the values of $[P/(\Delta I/I)]_{C2}$ are somewhat higher than $[P/(\Delta I/I)]_B$. This is due to the different values of Ω_b considered in the two cases. The $T_p(k)/T_r(k)$ ratio is higher in the B case than in C2 (compare figs. 4b and 4c) for perturbations with $M < 10^{15} M_\odot$, owing to the larger number of scatterings which photons of these perturbations undergo in the first case. However, because of the Gaussian factor $e^{-k^2\sigma^2 r_o^2(1-\mu^2)}$, the contribution of the fluctuation spectra to the integrals in eq. (4.4) is negligible for perturbations with $M < M_c(\sigma) \simeq 7.10^{14} \Omega^{-2} h^{-1} (\sigma/1.5')^3 M_\odot$. The $P/(\Delta I/I)$ ratio is smaller in the B case than in C2 since the values of $T_r(k)$ are close to the k=0 limit down to $M \simeq 10^{17} M_\odot$ in the first case, while in C2 case $T_r(k)$ begins to decrease at $M < 10^{18} M_\odot$. The different behaviour of $T_r(k)$ is due to the different values of the baryonic Jeans mass M_J in the two cases. In both cases the universe becomes matter dominated at $z \simeq z_{eq} = 4.2\ 10^4 \Omega h^2 \simeq 5.10^3$. At $z < z_{eq}$ and prior to recombination M_J takes the constant value $M_J \simeq 5.10^{17} M_\odot$ in the B case, while $M_J \simeq 10^{17} (z/z_{eq})^{-3/2} M_\odot$ in C2. In the latter case the value of M_J at recombination is $\simeq 10^{18} M_\odot$ and the total mass of a perturbation on the corresponding scale is $M \simeq \frac{\Omega}{\Omega_b} 10^{18} M_\odot \simeq 7.10^{18} M_\odot$.

In Fig. 5, a,b the polarization degree $P(\theta,\sigma)$ is shown as a function of the angular scale, for $\sigma = 1.5'$ and the N and C cases, respectively. In both cases $P(\theta,\sigma)$ reaches the asymptotic value $P \simeq 10^{-5}$ for $\theta > 10'$ and $n > 0$. Negative values of the primordial spectral index n are not favoured by numerical simulations on large scale for the matter distribution in the N and C case (White et al. 1983, Davis et al. 1985).

Up to now there has been no positive detection of polarization of the cosmic background radiation. At large angular scales Lubin, Melese and Smoot (1983) set the upper limit of 7.10^{-5} on the polarization degree. On small and intermediate angular scales (between 30' and 40°) only one experiment has been performed (Caderni et al. 1978) which sets upper limits of the order of 10^{-3}. Then a positive detection of linear polarization in the CBR would require an improvement by a factor ten in the sensitivity of the present instrumentation.

On the other hand, we stress that a positive detection of linear polarization of the CBR at angular scales $\theta > 15'$ will provide information on the value of Ω and will provide useful constraints for the value of the spectral index n of primeval density perturbations.

In table 2 we list the rms quadrupole component of the radiation anisotropy and of polarization. The definition of Wilson and Silk (1981) has been used for the quadrupole moment of the fluctuations. On the quadrupole scale the present upper limit for the radiation anisotropy is $Q_R < 2.10^{-4}$ (Lubin, Epstein and Smoot, 1983; Fixsen, Cheng and Wilkinson, 1983). In all the three cases considered, $n = -1$ is ruled out, the $n=0$ case is still possible, if we allow for various uncertainties and in particular for the uncertainity in the normalization. The quadrupole moment of the polarization is $Q_p \simeq 10^{-8}$, i.e. about three orders of magnitude smaller than the quadrupole anisotropy, thus it is completely unobservable with the present instrumentation.

After the recombination a second ionization of the intergalactic medium will increase the ratio $P/(\Delta I/I)$ on small angular scales if the corresponding optical depth τ is of order unity (Basko and Polnarev 1980, Negroponte and Silk 1980, Tolman and Matzner 1984). In a non baryonic dark matter dominated Universe with $\Omega_b = 0.03$, reheating of the intergalactic medium must occur at $z_{reh} > 100$ in order to reach $\tau \simeq 1$. However such values for

z_{reh}, as discussed by Bond and Efstathiou (1984), are not plausible. In a baryonic Universe second ionization of the intergalactic medium associated with galaxy formation is most likely to occur at $z_{reh} \lesssim 4$–5 (Sherman 1979, Osmer 1982). In the $\Omega_b = 0.2$ case, $\tau \simeq 0.2$ for $z_{reh} = 5$ and the small scale polarization does not change in an appreciable way.

Then we conclude that, in the models considered here, a second ionization of the intergalactic medium is not likely to affect our results.

ACKNOWLEDGEMENTS

We thank G. De Zotti for helpful discussions and comments which improved the work.

REFERENCES

Basko, M.M., Polnarev, A.G., 1980, M.N.R.A.S., 191, 207.

Bond, J.R., Efstathiou, G., 1984, Ap. J. Letters, 285, L45.

Bonometto, S.A., Lucchin, F., Occhionero, F., Vittorio, N., 1983, Astr. and Ap., 123, 118.

Bonometto, S.A., Lucchin, F., Valdarnini, R., 1984, Astr. and Ap., 140, L27.

Caderni, N., Fabbri, R., Melchiorri, B., Melchiorri, F., Natale, V., 1978, Phys. Rev. D, 17, 1908.

Chandrasekhar, S., 1960, Radiative Transfer (New York: Dover).

Dautcourt, G., Rose, K., 1978, Astr. Nachr., 299, 13.

Davis, M., Efstathiou, G., Frenk, C.S., White, S.D.M., 1985, Ap. J., 292, 371.

Doroshkevich, A.G., Zel'dovich, Ya.B., Sunyaev, R.A., 1978, Sov. Astr. 5, 22.

Fixsen, D.J., Cheng, E.S. Wilkinson, D.T., 1983, Phys. Rev. Letters, 50, 620.

Kaiser, N., 1983, M.N.R.A.S., 202, 1169.

Lubin, P.M., Epstein, G.L. Smoot, G.F., 1983, Phys. Rev. Letters, 50, 616.

Lubin, P., Melese, P., Smoot, G., 1983, Ap. J. (Letters), 273, L51.

Negroponte, J., Silk, J., 1980, Phys. Rev. Letters, 44, 1433.

Osmer, P.S., 1982, Ap. J., 253, 28.

Peebles, P.J.E., 1968, Ap. J., 153, 1.

Peebles, P.J.E., Yu, J.T., 1970, Ap. J., 162, 815.

Peebles, P.J.E. 1980, The Large Scale Structure of the Universe (Princeton: Princeton University Press).

Peebles, P.J.E., 1982, Ap. J., 258, 415.

Scherman, R.D. 1979, Ap. J., 232, 1.

Smoot, G.F., De Amici, G., Friedman, S.D., Witebsky, G., Sironi, G., Bonelli, G., Mandolesi, N., Certigliani, S., Marigi, G., Partridge, R.B., Danese, L., De Zotti,G., 1985, Ap. J. (Letters), 291, L23.

Tolman, B.W., Matzner, R.A., 1984, Proc. Roy. Soc. London, A., 392, 391.

Uson, J.M., Wilkinson, D.T., 1984, Ap. J. (Letters), 277, L1.

Valdarnini, R., 1985, Astr. and Ap., 143, 94.

Vittorio, N., Silk, J., 1984, Ap. J. (Letters), 285, L39.

Vittorio, N.,Silk, J., 1985, Ap. J. (Letters), 293, L1.

White, S.D.M., Frenk, C.S., Davis, M., 1983, Ap. J. (Letters), 274, L1.

Wilson, M.L., Silk, J., 1981, Ap. J., 243, 14.

TABLE 1

n	C_p/c_r
-2	1.10^{-2}
-1	2.10^{-2}
0	3.10^{-2}
1	7.10^{-2}
2	11.10^{-2}

Table 1: Ratio between the polarization and the radiation anisotropy correlation functions for $\theta = \sigma = 0$ in a cold dark matter dominated universe with $\Omega = 1$ and $\Omega_b = 0.03$.

TABLE 2

C case

n	Q_R	Q_P
-1	$2.7\ 10^{-3}$	$1.6\ 10^{-7}$
0	$1.4\ 10^{-4}$	$2.9\ 10^{-8}$
1	$6.6\ 10^{-6}$	$7.5\ 10^{-9}$
2	$3.2\ 10^{-7}$	$3.3\ 10^{-9}$

N case

n	Q_R	Q_P
-1	$4.2\ 10^{-3}$	$3.4\ 10^{-7}$
0	$2.9\ 10^{-4}$	$8.3\ 10^{-8}$
1	$1.9\ 10^{-5}$	$3.1\ 10^{-8}$
2	$1.4\ 10^{-6}$	$1.9\ 10^{-8}$

G case

n	Q_R	Q_P
-1	$2.7\ 10^{-3}$	$1.6\ 10^{-7}$
0	$1.4\ 10^{-4}$	$2.9\ 10^{-8}$
1	$6.7\ 10^{-6}$	$8.4\ 10^{-9}$
2	$3.2\ 10^{-7}$	$3.8\ 10^{-9}$

Table 2: Radiation and polarization rms quadrupole anisotropy, Q_R and Q_P, for dark matter dominated universes with $\Omega = 1$ and $\Omega_b = 0.03$.

FIGURE CAPTIONS

Fig. 1a Residual temperature fluctuations as a function of angular scale for a neutrino dominated universe with $\Omega = 1$ and $\Omega_b = 0.03$, evaluated for an antenna beam $\sigma = 1.5'$. Different curves correspond to different values of the spectral index n. The observational upper limit of Uson and Wilkinson (1984) is also shown.

Fig. 1b The same as for Fig. 1a, except for $\sigma = 3'$.

Fig. 2a Ratio between the polarization degree and the fractional radiation anisotropy as a function of angular scale for a universe dominated by cold particles with $\Omega = 1$, $\Omega_b = 0.03$ and for antenna beam $\sigma = 1.5'$. Different curves refer to different values of the spectral index n.

Fig. 2b The same for Fig. 2a, except for $\Omega = 0.2$, $\Omega_b = 0.03$.

Fig. 3 As in Figs. 2 but for a purely baryonic model with $\Omega = 0.2 = \Omega_b$.

Fig. 4 Averaged spectra of the radiation anisotropy and polarization as a function of the perturbation mass scale for a) a cold particle dominated universe ($\Omega = 1$, $\Omega_b = 0.03$); b) a baryonic model with $\Omega = 0.2 = \Omega_b$; c) a cold particle dominated universe with $\Omega = 0.2$, $\Omega_b = 0.03$.

Fig. 5 Polarization degree as a function of the angular scale for a) a neutrino dominated universe with $\Omega = 1$ and $\Omega_b = 0.03$. b) a cold particle model with $\Omega = 1$ and $\Omega_b = 0.03$. In both cases $\sigma = 1.5'$.

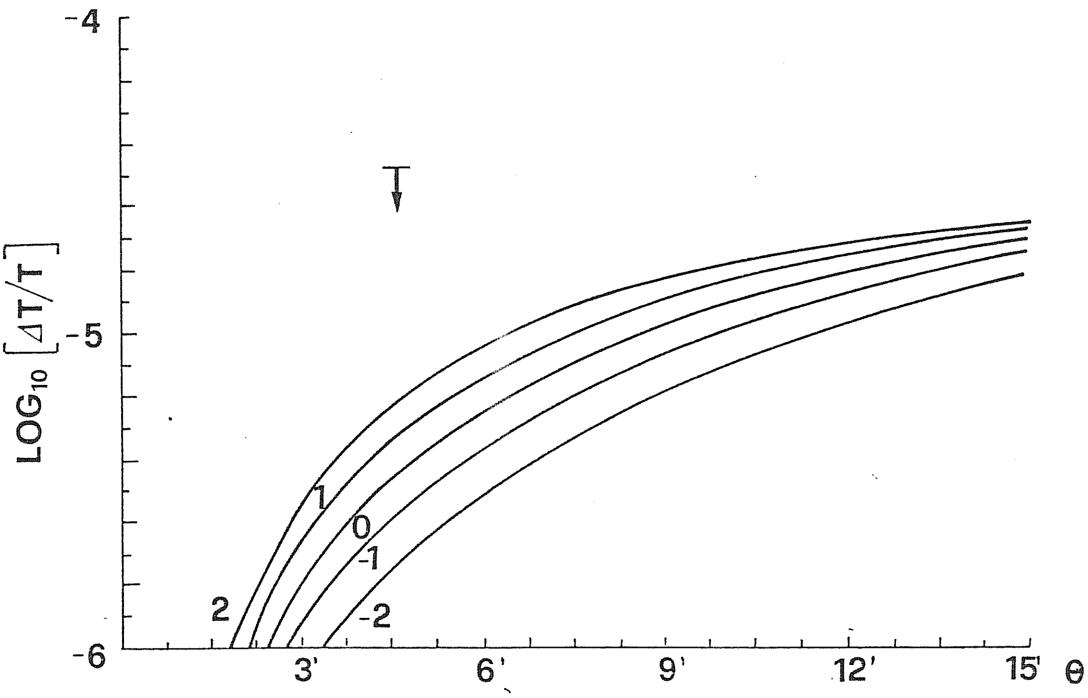

Fig. 1a

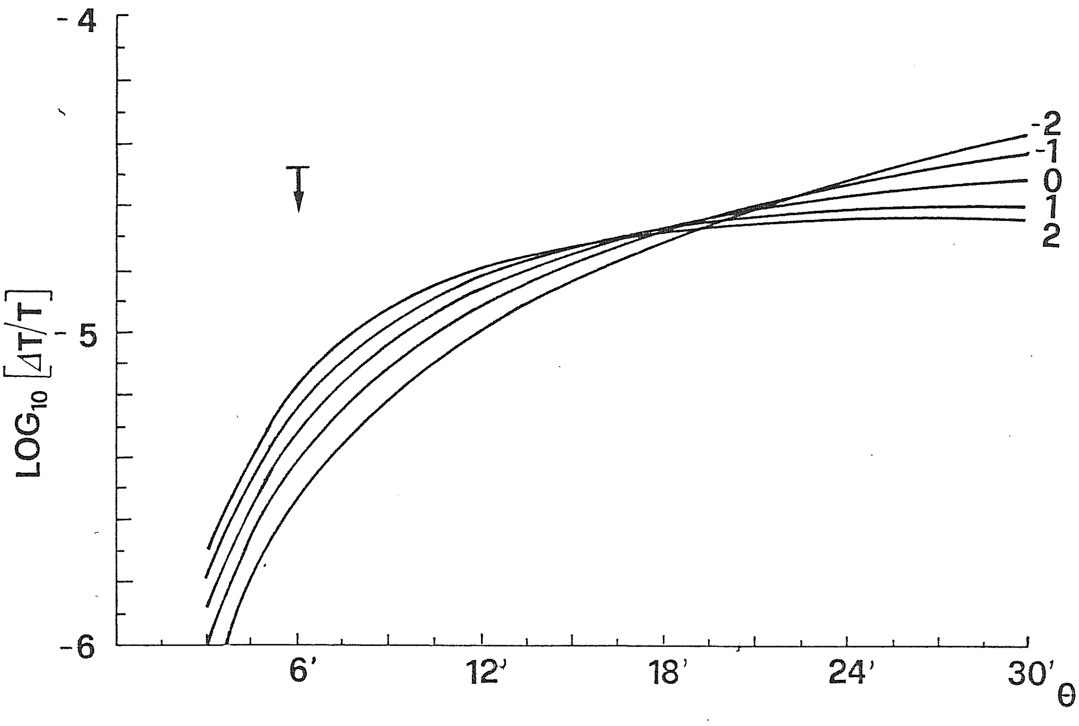

Fig. 1b

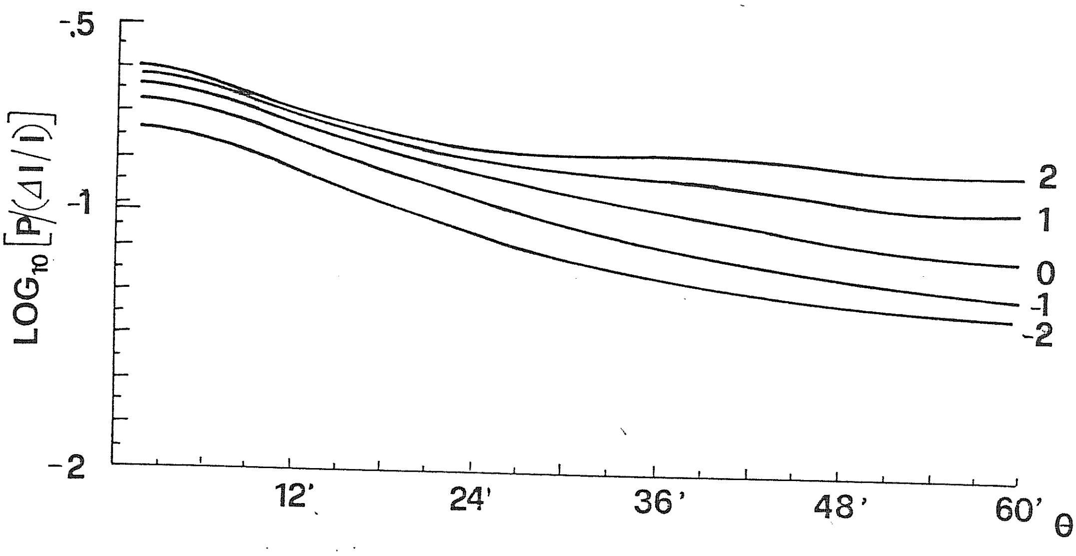

Fig. 2a

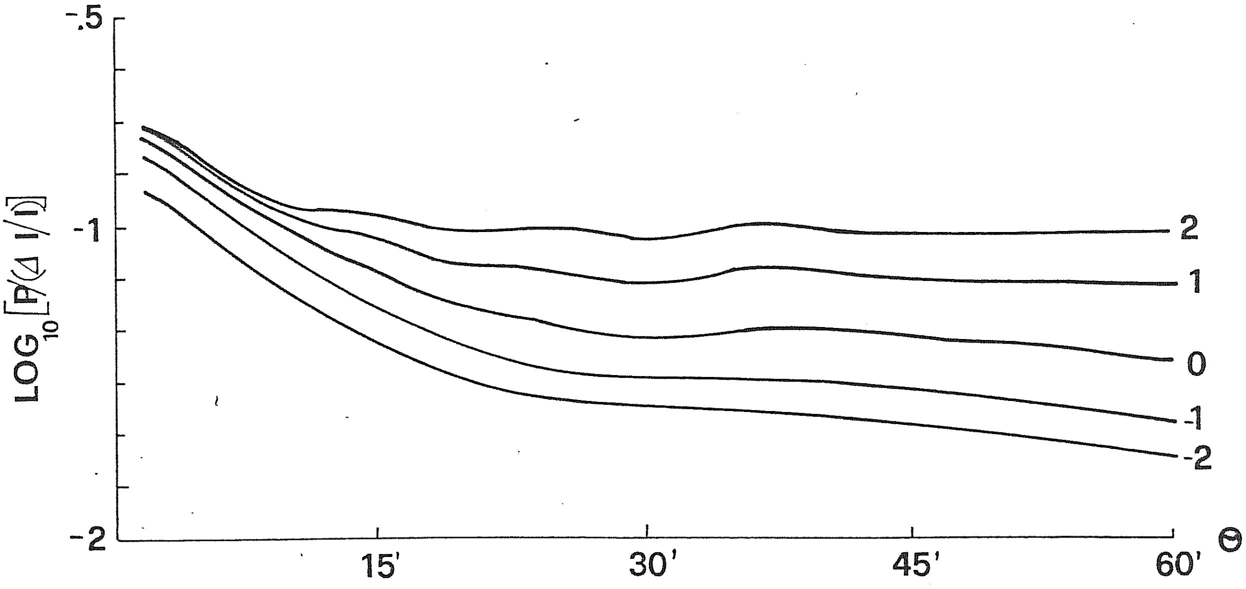

Fig. 2b

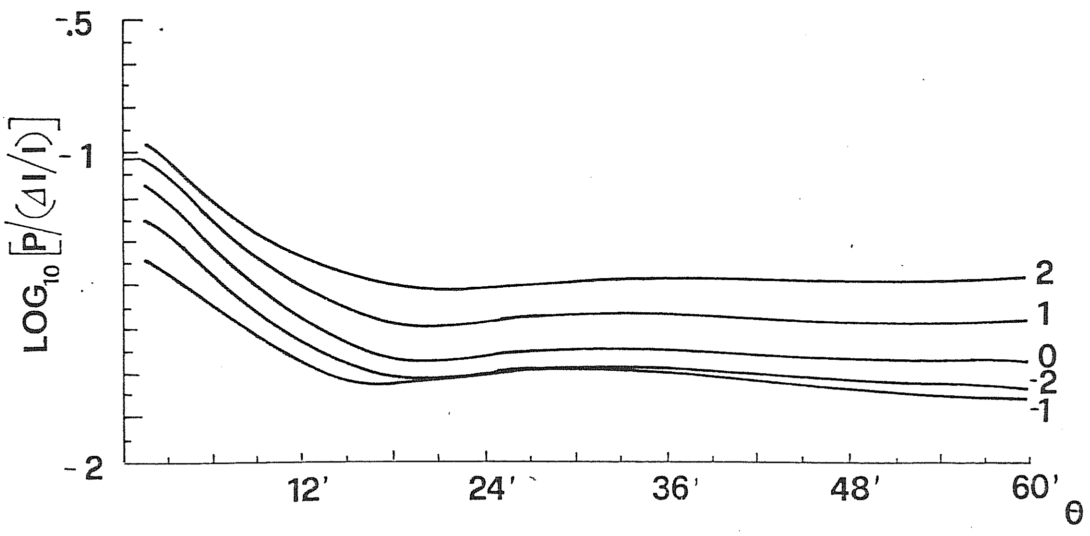

Fig. 3

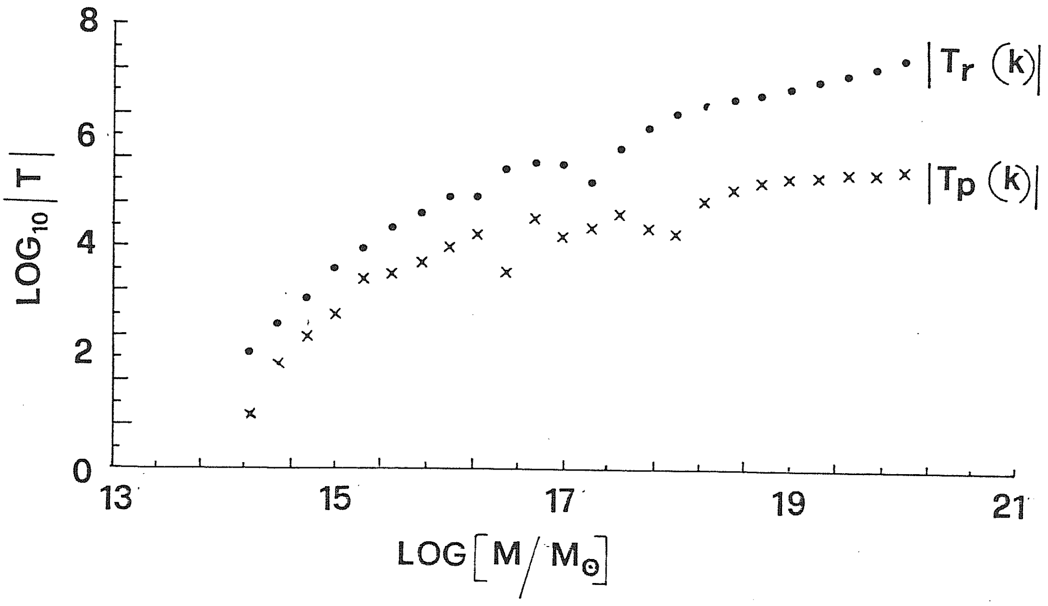

Fig. 4a

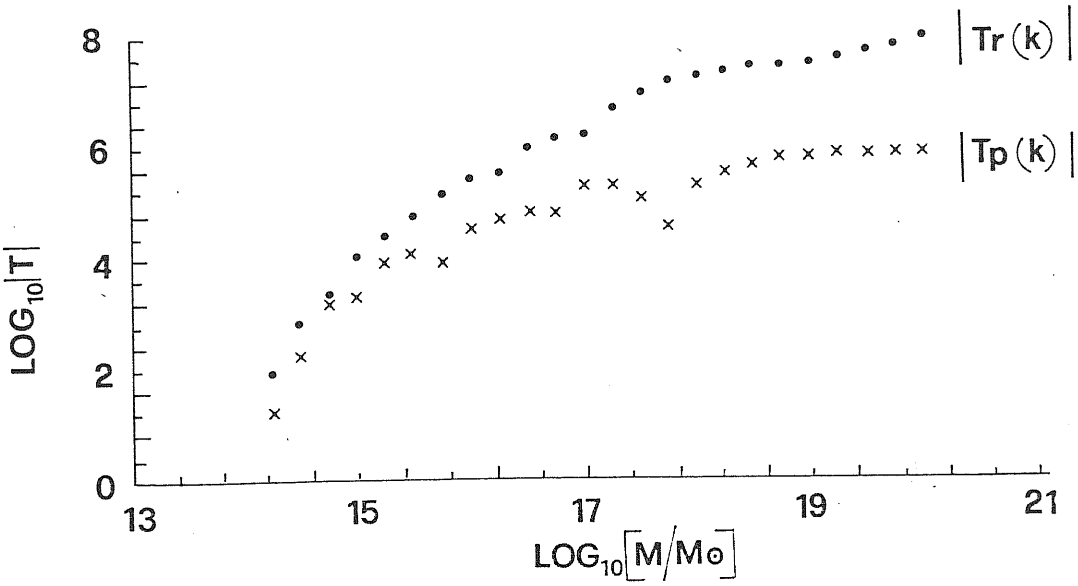

Fig. 4b

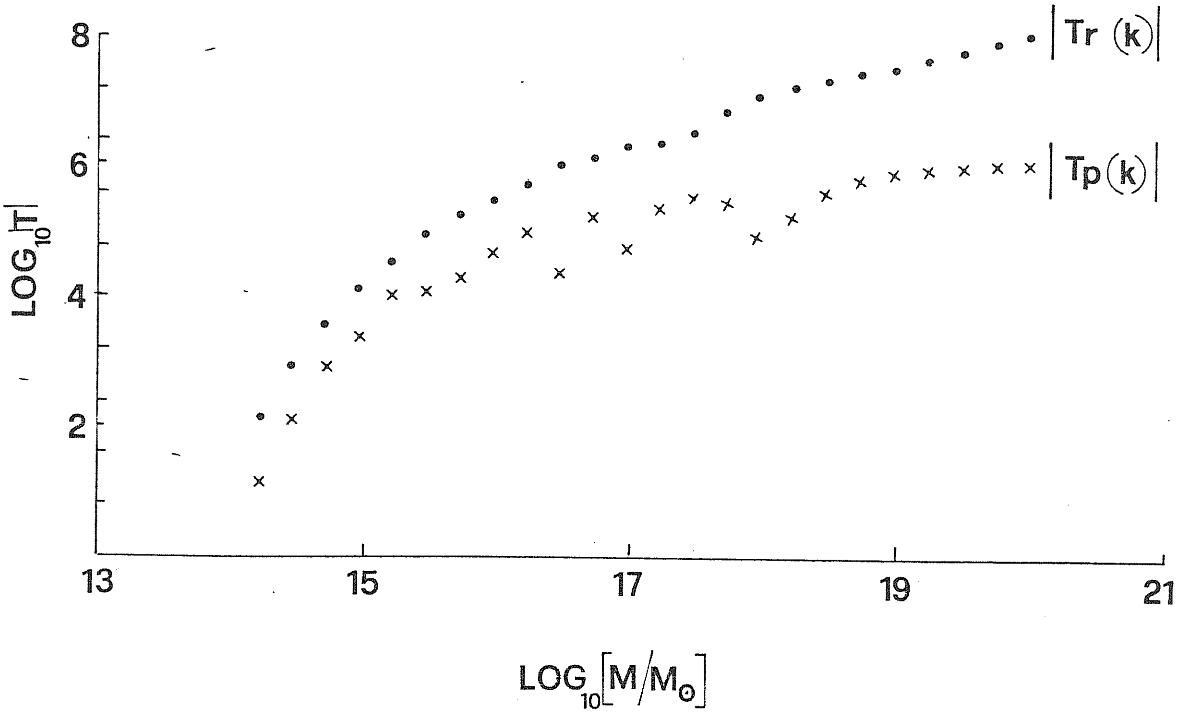

Fig. 4c

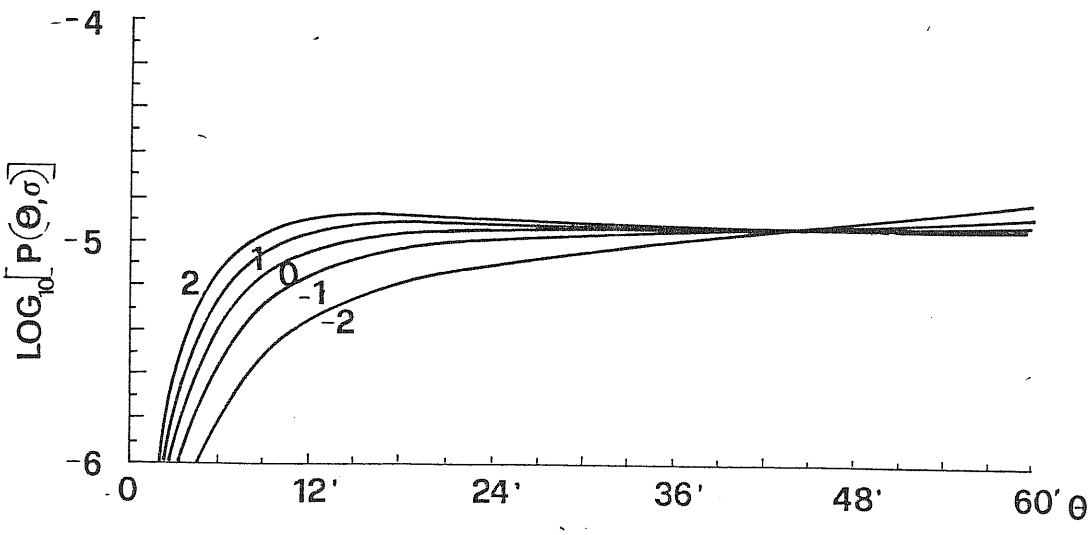

Fig. 5a

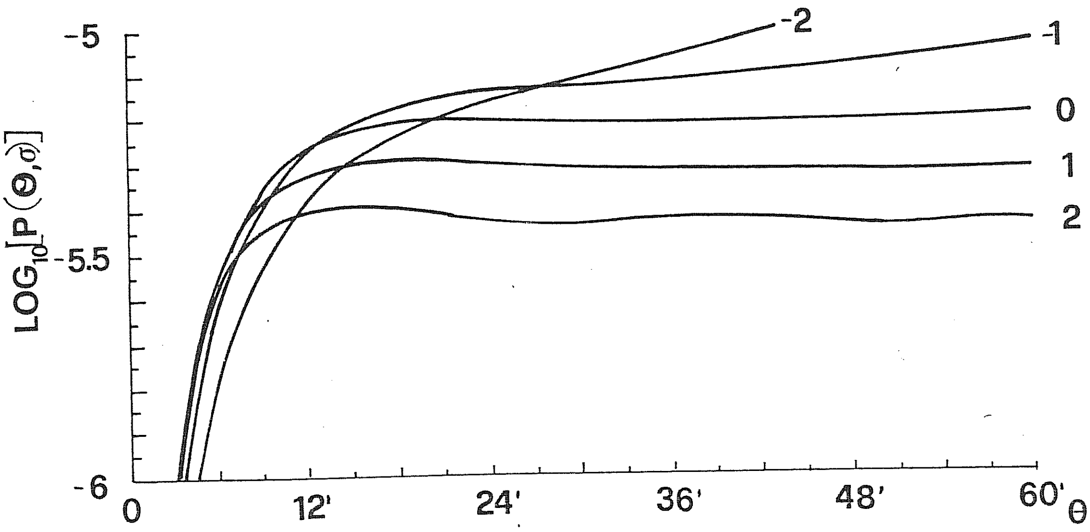

Fig. 5b

APPENDIX

1. SPIN S SPHERICAL HARMONICS

A quantity y is said to have spin weight s if it transforms under a rotation through an angle ψ about the radial direction $y \to y\, e^{is\psi}$.

Newmann and Penrose (1966) have introduced and developed techniques for spin weighted quantities on the unit sphere. Let us briefly introduce them here.

We employ the usual θ, ϕ coordinate system, then the radial vector $\vec{n}$ is: $n^i = (\sin\theta \cos\phi\ ,\ \sin\theta \sin\phi\ ,\ \cos\theta)$.

Associated with spin s quantities are the spin raising and lowering operators $\eth$ and $\bar{\eth}$, respectively.

The operator $\eth$ as applied to a quantity ${}_sY$ of spin s can be defined as:

$$\eth_s Y = -\sin^s\theta \left(\frac{\partial}{\partial\theta} + \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\sin^{-s}\theta\ {}_sY\right).$$

The resultant quantity has spin weight s+1. Similarly

$$\bar{\eth}_s Y = -\sin^{-s}\theta \left(\frac{\partial}{\partial\theta} - \frac{i}{\sin\theta}\frac{\partial}{\partial\phi}\right)\left(\sin^{s}\theta\ {}_sY\right)$$

has spin weight s−1.

Applying to spherical harmonics Y just s times gives us a set of orthogonal functions which when normalized to unit constitute a complete set of prthonormal functions in the space of square-integrable functions of spin weight s . The corresponding recurrence formulae are given by:

$$\eth_s Y_{\ell m} = [(\ell-s)(\ell+s+1)]^{1/2} \; {}_{s+1}Y_{\ell m}$$

$$\bar{\eth}_s Y_{\ell m} = -[(\ell+s)(\ell-s+1)]^{1/2} \; {}_{s-1}Y_{\ell m} .$$

The orthogonality relations are given by (note, no general orthogonality relations exist between spin s spherical harmonics with different spin values):

$$\int {}_sY_{\ell m} \; {}_s\bar{Y}_{\ell' m'} \, d\mu \, d\phi = \frac{4\pi}{3} \; {}_sN_{\ell' m'} \, \delta_{\ell \ell'} \, \delta_{m m'} .$$

2. POLYNOMIAL EXPANSION ON THE UNIT SPHERE.

The calculations involving an expansion in spherical harmonics are greatly simplified if polynomials related to spherical harmonics are introduced (Anderson and Spiegel, 1971).

We first consider spin zero quantities. They may be defined, separately for even and odd numbers of indices, as follow:

$$n^{i_1 \dots i_{2n}} = \sum_{r=0}^{n} A_{rn}\, n^{(i_1 \dots} \, n^{i_{2r}} \delta^{i_{2r+1} i_{2r+2}} \cdots \delta^{i_{2n-1} i_{2n})}$$

$$n^{i_1 \dots i_{2n+1}} = \sum_{r=0}^{n} B_{rn}\, n^{(i_1 \dots} \, n^{i_{2r+1}} \delta^{i_{2r+2} i_{2r+3}} \delta^{i_{2n} i_{2n+1})} ,$$

where the bracket around a number of indices indicates symmetrization. The numerical coefficients are:

$$A_{rn} = \frac{(-1)^{n-r}\,(2n+2r)!\,(2n!)^2}{(2r)!\,(n+r)!\,(n-r)!\,(4r)!}$$

$$B_{rn} = \frac{2n+1}{2r-1}\, A_{rn} .$$

The functions of lowest order are:

$$n=1$$

$$n^{ik} = n^i n^k - \frac{1}{3}\delta^{ik}$$

$$n^{ikl} = n^i n^k n^l - \frac{1}{5}(\delta^{ik} n^l + \delta^{il} n^k + \delta^{kl} n^i).$$

Polynomials of different order are orthogonal:

$$\int n^{i_1 \dots i_r}\, n^{k_1 \dots k_s} \frac{d\Omega}{4\pi} = 0 , \quad \text{if } r \neq s .$$

The "n-polynomials" are symmetric in all indices and trace_less for all pairs of indices:

$$n^{i_1 \dots i_n} = n^{(i_1 \dots i_n)}$$

$$n^{i_1 \dots i_n} \delta^{i_r i_s} = 0 , \qquad 1 \leq r \leq n,\ 1 \leq s \leq n,\ r \neq s.$$

They may be represented as linear combinations of ordinary (spin-zero) spherical harmonics.

A regular scalar function $f(\vartheta, \phi)$ defined on the unit sphere may be expanded in terms of these polynomials according to:

$$f = f_0 + f_i n^i + f_{ik} n^{ik} + f_{ikl} n^{ikl} + \dots$$

Only the traceless part of the coefficients $f_{ikl\dots}$ is required for a calculation of f, we therefore assume that the coefficients are traceless. They are given by:

$$f_0 = \int f \, d\Omega / 4\pi$$
$$f_i = \int f \, n^i \, 3 \, d\Omega / 4\pi$$
$$f_{ik} = \int f \, n^{ik} \, \tfrac{15}{2} \, d\Omega / 4\pi$$
$$f_{ikl} = \int f \, n^{ikl} \, \tfrac{7}{6} \, d\Omega / 4\pi .$$

The other polynomials we are interested in are those connected with spin 2 harmonics. Here

$$m^{ik} = m^i m^k \qquad \text{with} \qquad m^i = \tfrac{1}{\sqrt{2}} (a^i + i b^i)$$
$$m^{ikl} = m^{ik} n^l + m^{il} n^k + m^{kl} n^i$$

and in general:

$$m^{i_1 \dots i_{2k}} = \sum_{r=1}^{k} p_{rk} \, m^{(i_1 i_2} n^{i_3} \cdots n^{i_{2r}} \delta^{i_{2r+1}, i_{2r+2}} \delta^{i_{2k-1}, i_{2k})}$$

$$m^{i_1 \dots i_{2k+1}} = \sum_{r=1}^{k} q_{rk} \, m^{(i_1 i_2} n^{i_3} \cdots n^{i_{2r+1}} \delta^{i_{2r+2}, i_{2r+3}} \delta^{i_{2k}, i_{2k+1})}$$

are quantities symmetric in all indices and traceless for all pairs of indices. The m-polynomials are linear combinations of spin 2 harmonics. Like n-polynomials also m-polynomials constitute an orthonormal set on the unit sphere, since:

$$\int m^{i_1 \dots i_r} \, \bar{m}^{k_1 \dots k_s} \, \frac{d\Omega}{4\pi} = 0 \, , \qquad \text{for} \quad r \neq s \, .$$

A spin 2 quantity $g(\vartheta, \phi)$ may be expanded according to:

$$g = g_{ik} \, m^{ik} + g_{ikl} \, m^{ikl} + \dots$$

where the expansion coefficients must be symmetric in all indices and traceless in every pair of indices.

Explicitly one has for the lowest orders:

$$g_{ik} = 5 \int g(\theta, \phi) \, \bar{m}^{ik} \, \frac{d\Omega}{4\pi}$$

$$g_{ikl} = \frac{7}{3} \int g(\theta, \phi) \, \bar{m}^{ikl} \, \frac{d\Omega}{4\pi} \, .$$

REFERENCES.

Andersen,J.L.,Spiegel,E.A.,1972,Ap.J.,171,127.

Anile A.M.,1974, Ap. and Sp. Scie.,29,415.

Basko M.M.,PolnarevA.G.,1980,M.N.R.A.S.,191,207.

Bond J.R.,Efstathiou,G.,1984,Ap.J. (Letters),285,L45.

Bonometto S.A.,Lucchin F.,Occhionero F.,Vittorio N.,1983, Astr. and Ap.,123,118.

Bonometto S.A.,Lucchin F.,Valdarnini R.,1984,Astr. and Ap.,140,L27.

Boughn S.P.,Cheng E.S.,Wilkinson D.T.,1981,Ap.J.(Letters), 243,L113.

Brans C.H.,1975,Ap.J.,197,1.

Caderni N.,Fabbri R.,Melchiorri B.,Melchiorri F.,Natale V., 1978a,Phys.Rev.D,17,1901.

Caderni N.,Fabbri R.,Melchiorri B.,Melchiorri F.,Natale V., 1978b,Phys.Rev.D,17,1908.

Chandrasekhar S.,1960, Radiative Transfer,New York,Dover.

Collins C.B.,Hawking S.W.,1973,M.N.R.A.S.,162,307.

Cooke B.A.,Ricketts M.J.,Maccacaro T.,Pye J.P.,Elvis M., Watson M.G.,Pounds K.A.,McHardy I.,Maccagni D.,Seward F.D.,Page C.G.,Turner M.,1978,M.N.R.A.S.182,489.

Dautcourt G.,Rose K.,1978,Astron.Nachr.,299,13.

Davis M.,Efstathiou G.,Frenk C.S.,White S.D.M.,1985,
Ap.J.,292,371.

Doroshkevich A.G.,Zeldovich Ya.B.,Sunyaev R.A.,1978,Sov.
Astr.,5,22.

Ellis G.F.R.,1967,J.Math.Phys.,8,1171.

Fabbri R.,1980,Phys.Lett.,79A,21.

Fabbri R.,Breuer R.A.,1980,N.Cim.B 58,113.

Fixsen D.J.,Cheng E.S.,Wilkinson D.T.,1983,Phys.Rev.Lett.
50,620.

FormanF.,Jones C.,Cominský L.,Julian P.,Murray S.,Peters G.,
Tananbaum H.,Giacconi R.,1978,Ap.J.Suppl.,38,357.

Gorenstein M.V.,Smoot G.F.,1981,Ap.J.,244,361.

Kaiser N.,1983,M.N.R.A.S.,202,1169.

Lubin P.M.,Smoot G.F.,1981,Ap.J.,245,1.

Lubin P.M.,Epstein G., Smoot G.F.,1983,Bull.American Phys.
Soc.,28,30.

Lubin P.M.,Melese P.,Smoot G.F.,1983,Phys.Rev.Lett.273,L51.

Mac Callum M.A.H., in General Relativity: an Einstein
Centenary Survey, ed. Hawking,S.W.,and Israel,W.
1979,New York:Cambridge University.

Milaneschi E.,Fabbri R.,1985, Astr.&Ap. in press.

Nanos G.P.,1979,Ap.J.,232,241.

Negroponte J.,Silk J.,1980,Phys.Rev.Lett.,44,1433.

Newmann E.,Penrose R.,1966,J.Math.Phys.,7,863.

Novikov I.D.,1968,Astr.Zh.,45,538(Sov.Astr.AJ 12,427,1968).

Osmer P.S.,1982,Ap.J.,253,28.

Peebles P.J.E.,1968,Ap.J.,153,1.

Peebles P.J.E.,Yu J.T.,1970,Ap.J.,153,1.

Peebles P.J.E.,1980, The Large Scale Structure of the Universe.(Priceton:Princeton University Press).

Peebles P.J.E.,1982,Ap.J.,258,415.

Rees M.J.,1968,Ap.J.(Letters),153,L1.

Ryan M.P.Jr.,Shepley,L.C.,1975,Homogeneous Relativistic Cosmologies(Princeton N.J.:Princeton University).

Sachs R.K.,Wolfe A.M.,1967,Ap.J.,147,76.

Scherman R.D.,1979,Ap.J.,232,1.

Smoot G.F.,Lubin P.M.,1979,Ap.J.(Letters),234,L117.

Smoot G.F.,De Amici G.,Friedman S.D.,Witebsky G., Sironi G.,Bonelli G.,Mandolesi N.,Cortiglioni S., Morigi G.,Partridge R.B.,Danese L.,De Zotti G., 1985,Ap.J.(Letters),291,L23.

Stark R.F.,1981a,M.N.R.A.S.,195,115.

Stark R.F.,1981b,M.N.R.A.S.,195,127.

Sunyaev R.A.,Zeldovich Ya. B.,1969,Ap.Space Sci.,4,301.

Sunyaev R.A.,Zeldovich Ya.B.,1970,Ap.Space Sci.7,3.

Sunyaev R.A.,Zeldovich Ya.B.,1980,M.N.R.A.S.,190,413.

Tolman B.W.,Matzner R.A.,1984,Proc.Roy.Soc.Lond.A,392,391.

Tolman B.W.,1985,Ap.J.,290,1.

Uson J.M.,Wilkinson D.T.,1984,Ap.J.(Letters),277,L1.

Valdarnini R.,1985,Astr.& Ap.,143,94.

Vittorio N.,Silk J.,1984,Ap.J.(Letters),285,L39.

Vittorio N.,Silk J.,1985,Ap.J.(Letters),293,L1.

White S.D.M.,Frenk C.S.,Davis M.,1983,Ap.J.(Letters), 274,L1.

Wilson M.L.,Silk J.,1981,Ap.J.,243,14.

Yang S.,Turner M.S.,Steigman,G.,Schramm D.N.,Olive K.A., 1984,Ap.J.,281,493.