TOPOLOGICAL METHODS
IN SUPERSYMMETRIC THEORIES

Camillo IMBIMBO

International School for Advanced Studies, Trieste, Italy

and

Ecole Normale Supérieure, Paris, France

INDEX

CHAPTER I

1. The Witten index

Let $\mathcal{H} = \mathcal{B} \oplus \mathcal{F}$ be the Hilbert space of the states of a quantum mechanical system, with $\mathcal{B}$ ($\mathcal{F}$) being the subspace of the bosonic (fermionic) states. Let F be a hermitian operator, called fermionic number, commuting with the Hamiltonian of the system, which defines the bosonic and fermionic states in the following way:

$$
\begin{aligned}
F|b\rangle &= 0 \qquad \text{if} \quad |b\rangle \in \mathcal{B} \\
F|f\rangle &= |f\rangle \qquad \text{if} \quad |f\rangle \in \mathcal{F}
\end{aligned}
\tag{1}
$$

A supersymmetrical system is characterized by the existence of one or more supersymmetry operators

$$Q^i : \mathcal{H} \longrightarrow \mathcal{H} \; ; \; (i = 1, \ldots, N)$$

such that

$$Q^i(\mathcal{B}) \subset \mathcal{F} \quad \text{and} \quad Q^i(\mathcal{F}) \subset \mathcal{B} \tag{2}$$

and satisfying the anticommutation relations:

$$2H\delta^{ij} = \{Q^i, Q^j\} \tag{3}$$

$$\{Q^i, (-1)^F\} = 0 \tag{4}$$

where H is the Hamiltonian of the system.

Such a theory is defined to have a N-extended supersymmetry. In everything which will follow the existence of more than one supersymmetry charge will not play any role; therefore we will restrict our analysis, without any loss of generality, to the case of a single supersymmetry generator (N=1).

Let us state some elementary remarks which follow directly from the previous definitions.

From the fundamental anticommutation relation (3), it follows that the energy of a supersymmetrical system is positive definite.

Eq.(3) implies also that Q commutes with the Hamiltonian:. Thus if $|b\rangle$ is a bosonic eigenstate of the Hamiltonian with eigenvalue $E \neq 0$ and with unit norm

$$H|b\rangle = E|b\rangle$$

$$F|b\rangle = 0$$

$$\langle b|b\rangle = 1$$

the state

$$|f\rangle \equiv Q/\sqrt{E}\,|b\rangle$$

will be a normalized, fermionic eigenstate of H with the same eigenvalue E

$$H|f\rangle = H\,Q/\sqrt{E}\,|b\rangle = \frac{Q}{\sqrt{E}}H|b\rangle = E|f\rangle$$

$$\langle f|f\rangle = \langle b|\frac{Q}{\sqrt{E}}\frac{Q}{\sqrt{E}}|b\rangle = \langle b|\frac{H}{E}|b\rangle = 1$$

Applying once again the operator $Q/\sqrt{E}$ on $|f\rangle$ we will get the original bosonic state $|b\rangle$:

$$\frac{Q}{\sqrt{E}}|f\rangle = \frac{Q}{\sqrt{E}}\frac{Q}{\sqrt{E}}|b\rangle = \frac{H}{E}|b\rangle = |b\rangle$$

In other words the operator $\tilde{Q} = Q/\sqrt{E}$ provides an isomorphism between the subspaces of the bosonic and of the fermionic eigenstates of the Hamiltonian with eigenvalue $E \neq 0$. Q has itself as inverse:

$$\tilde{Q} : |b\rangle \longrightarrow |f\rangle \quad ; \quad \tilde{Q}^2 = \mathbb{1} \tag{5}$$

Consider now the eigenstates of H with zero eigenvalue. (3) implies:

$$H|0\rangle = 0 \quad \Longrightarrow \quad Q^2|0\rangle = 0 \quad \Longrightarrow \quad \langle 0|Q^2|0\rangle = 0$$

thus

$$\| Q|0\rangle \| = 0$$

Assuming that the metric of the Hilbert space is positive definite,one concludes:

$$Q|0\rangle = 0$$

Vice-versa, if $|0\rangle$ is a state annihilated by the supersymmetry generator Q, $|0\rangle$ is an eigenstate of the Hamiltonian with zero energy:

$$Q|0\rangle = 0 \quad \Longrightarrow \quad Q^2|0\rangle = 0 \quad \Longrightarrow \quad H|0\rangle = 0 \tag{6}$$

Summarizing, we have learned the following about the structure of the irreducible representations of the supersymmetry algebra (3):

i) the representations with non-zero energy are doublets containing a fermionic and a bosonic state;

ii) the representations with zero energy are singlets, which can be either bosonic or fermionic.

Therefore the spectrum of a supersymmetric theory, when discrete, can be schematically represented as in fig.1: the eigenstates with non-zero energy always appear in pairs boson-fermion,while the zero energy states are singles.

Let us come now to the issue of spontaneous supersymmetry breaking. By definition, we will say that a given supersymmetrical quantum mechanical system exhibits spontaneous breakdown of supersymmetry if the ground state (vacuum) is not invariant under a supersymmetry transformation:

$$Q|0\rangle \neq 0 \tag{7}$$

From the previous discussion(eq.(6)) it follows that supersymmetry is broken if and only if the energy of the vacuum is different from zero.

From this point of view supersymmetry is quite different from ordinary symmetry. For an ordinary symmetry to be preserved, the existence of an invariant state is not enough: other, non invariant states, can become the vacuum of the theory if they are energetically favorite. If this happens, spontaneous breakdown of the corresponding symmetry takes place. On the contrary, in a supersymmetrical theory no breaking will occur if there exists just one invariant state.

Since in general the vacuum energy cannot be exactly determined, it is difficult to establish if supersymmetry is broken or not.

A criterion to check that would be to count the zero energy states: supersymmetry is broken if and only if this number is zero.

This may be technically very difficult. Suppose we have determined the number of zero energy states to a certain approximation,e.g. to a given order in perturbation theory. The slightest correction to this computation, e.g. the next term in the perturbative expansion, could invalidate our conclusion about supersymmetric breaking, since it can make eigenstates with zero energy become eigenstates with non zero energy and vice-versa.
Also results valid to all orders in perturbation theory can be invalidated by, even tiny, non perturbative effects. For example non-renormalization theorems[1] state that supersymmetry if not broken at tree level stays unbroken to all orders in perturbation theory. But these theorems cannot exclude that supersymmetry gets broken by, let us say, weak instantons effect, giving a non-zero contribution to the vacuum energy.

These kind of considerations have motivated Witten to introduce the concept of <u>index</u> of a supersymmetrical theory[2]. The index is defined as the <u>difference</u> between the number of bosonic and fermionic eigenstates of the Hamiltonian with zero energy:

$$\Delta_W \equiv n_B^{E=0} - n_F^{E=0} \tag{9}$$

It is easily seen that the knowledge of the Witten index gives information about the spontaneous supersymmetry breaking.

If $\Delta_W \neq 0$ supersymmetry is unbroken since either $n_B^{E=0}$ or $n_F^{E=0}$ (or both) are different from zero.

If on the contrary $\Delta_W = 0$, we are still left with two possibilities:

a) $n_B^{E=0} = n_F^{E=0} \neq 0$ implying that supersymmetry is unbroken;

b) $n_B^{E=0} = n_F^{E=0} = 0$ implying that supersymmetry is spontaneously broken.

Thus, the non-vanishing of the index Δ_W is a sufficient (but not necessary) condition for supersymmetry to be unbroken.

What makes Δ_W an useful object is the fact that it is what we will call a "topological" invariant: By topological invariant we mean a quantity which remains invariant under continuous changes of the parameters of the theory (masses, coupling constants, size of the quantization box, etc.) so long as these changes do not modify the asymptotic behavior of the potential.

Because of this property, Δ_W is expected not to be sensitive to the specifc approximation one uses. In many cases one can exploit some convenient limit (great masses, small coupling constants, small size of the quantization box,etc.) to evaluate Δ_W easily and reliably.

From the index, stringent constraints on the possibility of dynamical supersymmetry breaking have been obtained in many class of theories(

To understand why the Witten index is a topological invariant let us look at fig.2.

Suppose that for given values $(g_0, m_0, \beta_0, \ldots)$ of the parameters (coupling constants, masses, size of the quantization box, ...) of the theory, the spectrum of the Hamiltonian appears as in fig.2a, with

$$\Delta_W (g_0, m_0, \beta_0, \ldots) = n_B^{E=0}(g_0, m_0, \beta_0, \ldots) - n_F^{E=0}(g_0, m_0, \beta_0, \ldots)$$

being the Witten index. (In the example in fig.2a, $n_B^{E=0}(g_0, m_0, \beta_0, \ldots)=3$ and $n_F^{E=0}(g_0, m_0, \beta_0, \ldots)=1$, so that $\Delta_W = 2$).

Now, imagine that we perturb the system adiabatically, changing continuously the parameters up to the values $(g_1, m_1, \beta_1, \ldots)$. Accordingly the energy spectrum will become as shown in fig.2b: the energy levels have moved and some states which initially had zero energy have become eigenstates with non-zero energy. But since ,as we explained before, the non zero energy states always come in pairs boson-fermion, these states will be even in number, half bosons and half fermions. It follows that the difference

$$\Delta_W (g_1, m_1, \beta_1, \ldots) = n_B^{E=0}(g_1, m_1, \beta_1, \ldots) - n_F^{E=0}(g_1, m_1, \beta_1, \ldots)$$

will not change. (In the example in fig.2b two states which had zero energy in fig.2a, have acquired a non-vanishing energy: $n_B^{E=0}(g_1, m_1, \beta_1, \ldots)=2$, $n_F^{E=0}(g_1, m_1, \beta_1, \ldots)=0$ and the index does not change $\Delta_W = 2-0=2$).

Analogously, suppose that for some other adiabatic trasformation which brings the values of the paramaters to $(g_2, m_2, \beta_2, \ldots)$ some eiggnstates that had non-zero energy, will go down to the zero energy level. Both the number of the bosonic and the fermionic zero energy states will increase the same amount, leaving the difference Δ_W the same.(In fig.2c $n_B^{E=0}(g_2, m_2, \beta_2, \ldots) = 4$, $n_F^{E=0}(g_2, m_2, \beta_2, \ldots) = 1$, with $\Delta_W = 4-2=2$ as in a and b).

In the former reasoning the assumption is made that the changes in the parameters can move the energy levels but are "gentle" enough not to make new eigenstates appear. It turns out that the allowed deformations of the parameters are those which do not change the behavior of the potential asymptotically (for large fields configurations).

The topological invariance of the Witten index can be traced to the fact that it can be thought of as the analytical index (in the usual mathematical sense) of a certain linear operator related to the supersymmetry charge.

Let us clarify this point: let us choose in the Hilbert space $\mathcal{H} = \mathcal{B} \oplus \mathcal{F}$ ($\mathcal{B}$ ($\mathcal{F}$) is the subspace of the bosonic (fermionic) states) a boson-fermion basis

$$\begin{pmatrix} b \\ f \end{pmatrix} \in \mathcal{H} \ ; \quad \begin{pmatrix} b \\ 0 \end{pmatrix} \in \mathcal{B} \ , \quad \begin{pmatrix} 0 \\ f \end{pmatrix} \in \mathcal{F}$$

The operator $(-1)^F$ will be represented by the (operator valued) matrix:

$$(-1)^F = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \tag{10}$$

The supersymmetry charge Q sends bosonic states into fermionic ones and vice-versa, that is it anticommutes with $(-1)^F$. Thus it will have the following matrix representation:

$$Q = \begin{pmatrix} 0 & L^{\dagger} \\ L & 0 \end{pmatrix} \tag{11}$$

where L^+ is the operator hermitian conjugate of L.

The zero energy states are annihilated by Q and therefore they have to satisfy the equations:

$$Q\begin{pmatrix} b \\ 0 \end{pmatrix} = 0 \qquad \Longleftrightarrow \qquad L\, b = 0$$
$$Q\begin{pmatrix} 0 \\ f \end{pmatrix} = 0 \qquad \Longleftrightarrow \qquad L^{+} f = 0 \tag{12}$$

One concludes that the Witten index is:

$$\Delta_W \equiv n_B^{E=0} - n_F^{E=0} = \text{dim ker}(L) - \text{dim ker}(L^{+}) \tag{13}$$

where ker(L) (ker(L^+)) is the subspace of the states annihilated by L (L^+), dim is the dimension of the subspace. The expression on the right-hand-side of eq.(13) is the definition of the index of a linear operator, which is known to be a topological quantity:

$$\Delta_W = \text{index}(L) \tag{14}$$

If we are dealing with a quantum mechanical system with a finite number of degrees of freedom, the corresponding L and L^+ will be linear differential operators defined on a finite dimensional manifold. In the case of an infinite number of degrees of freedom (field theory) L and L^+ will be functional differential operators on infinite dimensional manifolds.

Such a connection between the Witten index of supersymmetrical quantum mechanics and the analytical index of linear differential operators has lead to new technics to derive in a straightforward manner the classical "index theorems" of Atiyah-Singer and Callias-Bott-Seeley. We will come back to that in more details in the third chapter.

2. The superpartition function

The definition (9) we gave for the Witten index of a supersymmetrical theory may not be, as it stands, the most convènient for actual computations. We will discuss in this section alternative expressions for the index which not only will turn out to be convenient for explicit evaluations, but will also allow, in some cases, for interesting generalizations.

Let us first distinguish between the case when the energy spectrum is discrete and the case when it is continuous (in a box, if we are speaking of a field theory).

In the former case it is easily seen that the Witten index can be rewritten, formally, as a trace over the whole Hilbert space of the theory:

$$\Delta_W \doteq \mathrm{tr}\,(-1)^F \qquad \text{for theories with discrete spectrum} \tag{14}$$

In fact, in the trace (which is the sum over all the eigenstates of the Hamiltonian) the contributions coming from the non-zero energy states cancel in pairs, since for each bosonic state there is a fermionic one with the same energy but with opposite value of $(-1)^F$:

$$\mathrm{tr}\,(-1)^F = \sum_{\substack{\text{eigenstates}\\ \text{of } H}} (-1)^F = n_B^{E=0} - n_F^{E=0} + \sum_{\substack{\text{eigenstates}\\ \text{with } E>0}} (-1)^F =$$

$$= \Delta_W + \sum_{\substack{\text{bosonic}\\ \text{eigenstates}\\ \text{with } E>0}} (+1) + \sum_{\substack{\text{fermionic}\\ \text{eigenstates}\\ \text{with } E>0}} (-1) =$$

$$= \Delta_W + \sum_{E>0}{}' (1-1) = \Delta_W$$

The expression in eq.(14) is mathematically ill defined since it gives rise to a non convergent series: 1-1+1-1+... .
It is more correct, and more useful in practice, to introduce some kind of regularization:

$$\mathrm{tr}\,(-1)^F f(H) \tag{15}$$

where f(H) is a function of the Hamiltonian with the following properties:

(i) $f(0) = 1$

(ii) $f(H) \to 0$ for $H \to \infty$ sufficiently rapidly that the series defined by (15) is convergent.

If the energy spectrum is discrete the non-zero energy eigenstates also cancel in the trace (15) because of the fermion-boson degeneration. Thus the trace (15) reduces to the Witten index:

$$\Delta_W = \mathrm{tr}\,(-1)^F f(H) \qquad \text{for theories with discrete spectrum} \tag{16}$$

A common choice for f(H) is:

$$f(H) = e^{-\beta H} \qquad \beta \in \mathbb{R}^+ \tag{16'}$$

or:

$$f(H) \equiv \frac{z}{z + H} \qquad z \in \mathbb{C} \tag{16"}$$

(in the last chapter we will study an example where this choice turns out to be useful).

The expression (16') defines the object:

$$Z_S(\beta) \equiv \mathrm{tr}(-1)^F e^{-\beta H} \tag{17}$$

which is called the superpartition function of the theory.

The superpartition function admits a functional integral representation[3]. Let $\mathcal{L}_E(\varphi, \psi)$ be the euclidean Lagrangian of a d-dimensional supersymmetric theory, quantized in a box of size β, and let us schematically indicate with φ (ψ) the bosonic (fermionic) fields. It can be shown that:

$$Z_s(\beta) = \int_{\substack{\text{Periodic}\\ \text{Boundary}\\ \text{Conditions}}} [d\varphi\, d\psi]\, e^{-\int_{|x|\leq\beta} d^dx\, \mathcal{L}_E(\varphi,\psi)} \tag{18}$$

where in the path integral one takes periodic boundary conditions for both bosons and fermions . (To get the ordinary partition function one has to choose periodic boundary conditions for the bosonic variables and anti-periodic ones for the fermions).

As we said, the functional integral (18) does not actually depend on β when the energy spectrum is discrete:

$$\Delta_W = Z_s(\beta) \qquad \text{for theories with discrete spectrum} \tag{19}$$

When the theory admits a continuous portion in the energy spectrum eq.(19) is, in general, false[(4),(5),(10),(32)]. What can happen in such a case, is that the contributions in the trace $\mathrm{tr}(-1)^F e^{-\beta H}$ of the bosonic and of the fermionic non-zero energy states belonging to the continuous part of the spectrum do not cancel. Even if each level is still doubly degenerate due to supersymmetry, the densities of the bosonic and of the fermionic energy eigenstates are not guaranteed to be equal. If non-equality is the case, the superpartition function acquires a non-trivial β-dependence, and thus cannot concide with the Witten index which is a constant number.

Nevertheless, in some cases there still exists the following connection between the two concepts:

$$\Delta_W = \lim_{\beta\to\infty} Z_S(\beta) \tag{20}$$

It is easy to see that this holds when zero is an isolated point in the energy spectrum, that is when the continuous part of the spectrum is separated from zero by a finite gap. In a mathematical language one says that a sufficient condition for eq.(20) to hold is that the Hamiltonian be a Fredholm operator.[6] One should stress that this condition is not necessary: there exist Hamiltonians having a continuous spectrum starting from zero which satisfy eq.(20), an important example being the supersymmetrical Hamiltonians around a classical monopole background[7],[11]

When the Fredholm condition is valid, we can write for the superpartition function:

$$\begin{aligned} Z_S(\beta) &= \int_0^\infty [n_B(E)-n_F(E)]\,e^{-\beta E}\,dE = \\ &= \Delta_W + \int_{\varepsilon>0}^\infty [n_B(E)-n_F(E)]\,e^{-\beta E}\,dE \end{aligned} \tag{21}$$

where $n_B(E)$ ($n_F(E)$) is the bosonic (fermionic) density of the energy eigenstates with eigenvalue E. $\varepsilon > 0$ is the energy gap separating the continuous portion of the spectrum from E=0. (The non zero-energy states belonging to the discrete part of the spectrum cancel as usual).

The integrand in the last term in eq(21) now converges uniformly in the interval (ε , $+\infty$), so that one can bring the limit inside the intergral:

$$\lim_{\beta\to\infty} Z_s(\beta) = \Delta_W + \lim_{\beta\to\infty} \int_\varepsilon^\infty [n_B(E)-n_F(E)]e^{-\beta E}\,dE =$$
$$= \Delta_W + \int_0^\infty [n_B(E)-n_F(E)] \lim_{\beta\to\infty} e^{-\beta E}\,dE = \Delta_W$$

It is quite clear the the same result stays true even if the difference $n_B(E) - n_F(E)$ does not vanish as $E \to 0$ so long as it is not too singular.

For theories with a continuous spectrum the superpartion function $Z_s(\beta)$ is a more general object than the index. In particular, for a certain class of theories, one can recover the index taking the limit of $Z_s(\beta)$ for $\beta \to \infty$. For finite β , $Z_s(\beta)$ gives information about the entire spectrum of the theory in contrast with $\Delta_W = n_B^{E=0} - n_F^{E=0}$ which only gives information about the zero energy states.

CHAPTER II

1. Supersymmetric quantum mechanics

We will begin to analyse the simplest example of supersymmetrical system: supersymmetric quantum mechanics with one bosonic degree of freedom[13]. Supersymmetry, contrary to ordinary symmetries, can display spontaneous breakdown even in systems with a finite number of degrees freedom. This fact explains why supersymmetrical quantum mechanics has been extensively [13],[14] studied as a simple toy-model to understand the kinds of concepts we discussed in the previous chapter : index, dynamical supersymmetry breaking,etc..

The Lagrangian of the system is:

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - \frac{1}{2}\left[W'(\varphi)\right]^2 + \frac{i}{2}\bar{\psi}\dot{\psi} + W''(\varphi)\bar{\psi}\psi \qquad (22)$$

where φ is a bosonic quantum variable and $\psi = \begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix}$ is a two-real spinor, $\bar{\psi} \equiv \psi^T\sigma_1$ is its conjugate. $W(\varphi)$ is called the superpotential and the prime indicates derivation respect to the argument.

It can be checked that the above Lagrangian is invariant (up to surface terms) under the supersymmetry trasformation:

$$\delta\varphi = \bar{\epsilon}\psi$$
$$\delta\psi = \left[-i\partial_t + W'(\varphi)\right]\varphi\,\epsilon$$

The Hamiltonian corresponding to the Lagrangian (22)is:

$$2H = p^2 + [W'(\varphi)]^2 - i\chi_1\chi_2 W''(\varphi) \tag{23}$$

p is the momentum conjugate to φ . χ_A, A=1,2, are hermitian fermionic variables satisfying the anticommutation relations:

$$\{\chi_A , \chi_B\} = 2\,\delta_{AB} \tag{24}$$

The hermitian supersymmetry generators are easily seen to be:

$$Q_1 = \sqrt{\tfrac{1}{2}}\,(p\chi_1 + W'\chi_2)$$
$$Q_2 = \sqrt{\tfrac{1}{2}}\,(p\chi_2 - W'\chi_1) \tag{25}$$

A concrete representation of the fermionic variables satisfying the anticommutation relations (24) is provided by the 2x2 Pauli matrices:

$$\chi_1 = \sigma_1 \qquad \chi_2 = \sigma_2 \tag{26}$$

The operator $(-1)^F$ must anticommute with Q_A :therefore it is diagonal in the representation (26):

$$(-1)^F = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{27}$$

In this basis the supersymmetry charge Q_1 looks like:

$$Q_1 = \begin{pmatrix} 0 & L^+ \\ L & 0 \end{pmatrix} \tag{28}$$

with:

$$L \equiv p + i\, W'(\varphi) \tag{29}$$

and similarly for Q_2 .

As explained in the previous section, the Witten index of this model is the analytical index of the operator L :

$$\Delta_W = \text{index}(L) \tag{30}$$

The model (22) is simple enough that the question of the spontaneous breaking of supersymmetry can be directly addressed. We have just to evaluate the number of independent, normalizable solutions of the linear differential equations:

$$\begin{aligned} L\,\psi_+(\varphi) = 0 \quad &\Longleftrightarrow \quad \frac{1}{i}\frac{d}{d\varphi}\psi_+(\varphi) + i\,W'(\varphi)\,\psi_+(\varphi) = 0 \\ L^+\,\psi_-(\varphi) = 0 \quad &\Longleftrightarrow \quad \frac{1}{i}\frac{d}{d\varphi}\psi_-(\varphi) - i\,W'(\varphi)\,\psi_-(\varphi) = 0 \end{aligned} \tag{31}$$

The first one has solution:

$$\psi_+(\varphi) = e^{\int_0^{\varphi} W'(x)\,dx} \tag{32}$$

while the second gives

$$\psi_-(\varphi) = e^{-\int_0^{\varphi} W'(x)\,dx} \tag{33}$$

For ψ_+ and ψ_- to be accetable as quantum states, they have to be normalizable. This depends on the superpotential $W(\varphi)$.

Let us choose, for concreteness, $W(\varphi)$ to be a polynomial in φ . One has to distinguish between the case when

$$W'(\varphi) \sim \varphi^{(even)} \quad as \quad \varphi \to \pm\infty \tag{34}$$

and the case when

$$W'(\varphi) \sim \varphi^{(odd)} \quad as \quad \varphi \to \pm\infty \tag{35}$$

From (34) it follows that:

$$\int^{\varphi} W'(x)\,dx \sim \varphi^{(odd)}$$

so that nor ψ_+ neither ψ_- are normalizable. Thus, when (34) holds supersymmetry is spontaneously broken since there exist no normalizable zero-energy state. Accordingly:

$$\Delta_W = 0 \tag{36}$$

In the case (35) we have

$$\int^{\varphi} W'(x)\,dx \sim \varphi^{(even)}$$

so that either ψ_+ or ψ_- (according to sign of $W'(\varphi)$) is normalizable. In either cases there exists one (bosonic or fermionic) zero energy eigenstate and supersymmetry is unbroken. The Witten index is:

$$\Delta_W = \pm 1 \tag{37}$$

In this example we have been able to compute separately the number $n_B^{E=0}$ and $n_F^{E=0}$ of the bosonic and fermionic zero energy states; thus the knowledge of the Witten index does not add any further information. But, suppose we were given only with the values (36) or (37) of Δ_W . (In a subsequent section we will expose a method to get Δ_W without solving the Schrodinger equations (31)). Imagine that in the case (34) the potential is as in fig.3a (even number of vanishing minima), while in the case (35) it is as in fig.3b (odd number of vanishing minima). At least in the case of fig.3b we would have concluded from the non-vanishing of the index that supersymmetry is not broken. For the case of fig. 3a no conclusion could have been drawn.

Let us stress that perturbation theory could not have distinguished between the cases 3a and 3b and would have told us that supersymmetry is unbroken in both cases[(13),(14)]. The non-renormalization theorems[(1)] apply also to this situation and state that the quantum effective potential, obtained from the expansion around one of the minima in fig.3 vanishes at the same minimum. Because of the shape of the potential in fig.3 one could imagine that istantons, for example, may be responsible for non-perturbative, dynamical breaking of supersymmetry.

The knowledge of the index guarantees that dynamical supersymmetry breaking does not occur in the case of fig.3b nor because of instantons neither because of whatever non-perturbative effect. This is somewhat typical of the incomplete but exact statements about the quantum behavior of the theory that the index allows to do.

Explicit computations[(13),(14)] show that istantons do induce spontaneous breaking of supersymmetry only in the case of fig.3a. One can arrive to the same conclusion by means of the Callias trace formula for operator on open manifold[(4)] which will be discussed in the next chapter[(15),(16)].

2. The constant configuration prescription: theories with discrete spectrum

The aim of this section is to illustrate a technique, called ultra-local prescription[(9)], to evaluate the superpartition function of a supersymmetrical theory by means of its functional integral representation.

For a wide class of theories, not containing gauge fields, this seems to be the simplest and most direct way to calculate this quantity.

In this section we will consider theories which display, when quantized in a box, a discrete energy spectrum. The case of a continuous energy spectrum will be studied in a following section.

Let us start from the functional integral expression[3]:

$$\Delta_W = Z_S(\beta) = \int_{P.B.C.} [d\varphi \, d\bar{\psi} \, d\psi] \exp\left[- \int d^d x \, \mathcal{L}_E(\varphi, \psi, \bar{\psi})\right] \tag{38}$$

In order to compute (38), we want to exploit its remarkable property: topological invariance. Since $Z_S(\beta)$ does not depend on β, we can take the limit $\beta \to 0$ in the functional integral (38).
Let first imagine to shrink the spatial size of the quantization box to zero. Exapanding the fields in Fourier components along the spatial coordinates, one realizes that the non-zero modes have large energies (of $O(1/\beta)$): thus they do not contribute to the index.
One is left only with the zero modes, that is with the configurations independent on the spatial coordinates. The theory has been dimensionally reduced to 0+1 dimensions, without changing the value of $Z_S(\beta)$. At this point, one can still send to zero the length of the time interval where the "paths" are taken. This corresponds to the high temperature limit of a statistical mechanical ensamble. Standard arguments[17] say that in this limit constant loops in field space give the dominant contribution

to (38) as far as the bosonic degrees of freedom are concerned. One may invoke supersymmetry to state the the same is true for the fermions. More formally, given a 0+1 dimensional system:

$$\mathcal{L}_E(\varphi,\psi,\bar{\psi}) = \frac{1}{2}\dot{\varphi}^2 + \bar{\psi}\dot{\psi} + V(\varphi,\psi,\bar{\psi}) \tag{39}$$

with a potential not depending on the derivatives of the fields, we decompose the fields in Fourier components:

$$\varphi(t) = \frac{1}{\sqrt{\beta}} \sum_n \exp\left[-i(2\pi n/\beta)t\right]\varphi_n\,,\quad (\varphi_n^* = \varphi_{-n}) \tag{40}$$

and the same for the fermions because they also obey periodic boundary conditions. Let us rewrite (38) as:

$$Z_S = \Big\{\exp\Big(-\int_0^\beta dt\, V(\delta/\delta J, \delta/\delta\bar{\xi}, \delta/\delta\xi)\Big) \times \tag{41}$$
$$\times \Big[\int [d\varphi\, d\psi\, d\bar{\psi}] \exp\Big(-\int_0^\beta dt\Big[\frac{1}{2}\dot{\varphi}^2 + \bar{\psi}\dot{\psi} + J\varphi + \bar{\xi}\psi + \bar{\psi}\xi\Big]\Big)\Big]\Big\}_{J,\xi,\bar{\xi}=0}$$

Using the decomposition (40) one gets:

$$Z_{1s} = \left[\exp\left(-\int_0^\beta dt\, V(\delta/\delta J, \cdots)\right)\right) \left(\int \frac{d\varphi_0\, d\psi_0\, d\bar{\psi}_0}{(2\pi)^{1/2}} \exp(-J_0\varphi_0 - \bar{\xi}_0\psi_0 - \bar{\psi}_0\xi_0) \cdot$$

$$\times \prod_{n\neq 0} \frac{\beta}{2\pi n} \exp[-J_n(\beta/2\pi n)J_n] \prod_{n\neq 0}\left(\frac{2\pi n}{\beta}\right) \exp[-\bar{\xi}_n(\beta/2\pi n)\xi_n]\Big)\Big]_{J,\xi,\bar{\xi}=0} =$$

(42)

$$= \left[\exp\left(-\int_0^\beta dt\, V(\delta/\delta J, \cdots)\right)\right)\left(\int \frac{d\varphi_0\, d\psi_0\, d\bar{\psi}_0}{(2\pi)^{1/2}} \exp[-(J_0\varphi_0 + \bar{\xi}_0\psi_0 + \bar{\psi}_0\xi_0)]\,[1 + O(\beta)]\Big)\Big]_{J,\xi}$$

Noting that

$$\frac{\delta}{\delta J(t)} = {\sum_n}' \frac{1}{\sqrt{\beta}} \exp(i2\pi n t/\beta)\, \frac{\delta}{\delta J_n}$$

one obtains, taking the limit $\beta \to 0$:

$$Z_{1s} = \int \frac{d\varphi_0\, d\psi_0\, d\bar{\psi}_0}{(2\pi)^{1/2}} \exp[-\beta V(\varphi_0/\sqrt{\beta}, \psi_0/\sqrt{\beta}, \bar{\psi}_0/\sqrt{\beta})]$$

After a rescaling of the fields:

$$Z_s = \Delta_W = \int \frac{d\varphi_0\, d\psi_0\, d\bar{\psi}_0}{(2\pi\beta)^{1/2}} \exp[-\beta V(\varphi_0, \psi_0, \bar{\psi}_0)] \qquad (43)$$

In general, the normalization is $1/(2\pi\, \beta^d)^n$ where d is the space-time dimension and n is the number of scalar supermultiplets. The normalization comes from the integral over the Kinetic terms, but the easiest way to find it is to observe that the index is always 1 for a free massive theory.

Armed with the ultra-local prescription (43), we can derive a number of results concerning scalar supersymmetric theories.

(i) Supersymmetrical quantum nechanics revisited

Let us re-derive the results (36)-(37) for the index of supersymmetrical quantum mechanics, making use of the path integral approach. The superpotential is chosen in such a way that the Hamiltonian has a discrete spectrum, for example as in (34) or in (35). The formula (43) gives for :

$$\Delta_W = \int [d\varphi\, d\bar\psi\, d\psi]\, \exp\left[-\int_0^\beta dt \left(\frac{1}{2}\dot\varphi^2 + \bar\psi\dot\psi + \frac{1}{2}W'(\varphi)^2 + W''(\varphi)\bar\psi\psi\right)\right] =$$

$$= \int \frac{d\varphi\, d\psi\, d\bar\psi}{(2\pi\beta)^{1/2}} \exp\left(-\beta\left[\frac{1}{2}(W'(\varphi))^2 + W''(\varphi)\bar\psi\psi\right]\right) =$$

$$= \int \frac{d\varphi}{(2\pi/\beta)^{1/2}} \exp\left[-\frac{\beta}{2}(W'(\varphi))^2\right] W''(\varphi)$$

Performing the change of variables:

$$y \equiv W'(\varphi)$$

one gets:

$$\Delta_W = \int_{W'(-\infty)}^{W'(+\infty)} \frac{dy}{(2\pi/\beta)^{1/2}}\, e^{-\frac{1}{2}\beta y^2} = \begin{cases} 1 & \text{in the case (35)} \\ 0 & \text{in the case (34)} \end{cases} \tag{44}$$

Exactly the same result follows for the N=1 Wess-Zumino model in two dimensions, since when reduced to constant configurations it becomes identical with the above model.

(ii) The complex Wess-Zumino models (N=2 in d=2 or N=1 in d=4)

The Lagrangian is :

$$\mathcal{L} = \frac{1}{2}\partial_\mu \bar{\varphi}\, \partial^\mu \varphi - \frac{1}{2} W'(\varphi)\overline{W'(\varphi)} + \frac{1}{2}\bar{\psi} \not\partial \psi + W''(\varphi)\bar{\psi}\psi + \overline{W''(\varphi)}\bar{\psi}$$

and the ultra-local prescription gives:

$$\Delta_W = \int \frac{d\varphi\, d\bar{\varphi}}{2\pi\beta^2}\, \beta^4\, W''(\varphi)\, \overline{W''(\varphi)}\, \exp\left[-\beta^2 W'(\varphi)\, \overline{W'(\varphi)}\right]$$

After a change of variables

$$\eta \equiv W'(\varphi)$$

which is always invertible if we choose $W'(\varphi)$ to be a polynomial, one obtains:

$$\Delta_W = \int \frac{d\eta \, d\bar{\eta}}{2\pi/\beta^2} \exp\left(-\beta^2 \bar{\eta}\eta\right) = \text{the degree of the polynomial } W'(\varphi) \tag{45}$$

(iii) The non linear supersymmetric σ-model

We will choose to work with the real models (N=1 in d=2)[18] but the same reasoning works for N=2 in d=2, the Kähler models[19].

The superpartition function is:

$$Z_S = \int [d\varphi \, d\bar{\psi} \, d\psi] \exp\left(-\int d^2x \left[\frac{1}{2} g_{ij}\partial_r\varphi^i\partial^r\varphi^j + \frac{1}{2} g_{ij}(\varphi)\bar{\psi}^i \not{D}\psi^j + \right.\right.$$
$$\left.\left. + \frac{1}{12} R_{ijkl}(\varphi)\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l\right]\right) \tag{46}$$

where the scalar fields φ^i are coordinates of the compact, n-dimensional, Riemannian manifold M, and ψ^i are 2-dimensional Majorana spinors (i,j = 1,...,n). Using again formula (43) for the index, one has:

$$\Delta_W = \int \frac{d\varphi \, d\bar{\psi} \, d\psi}{(2\pi\beta^2)^{n/2}} \exp\left[-\frac{1}{12}\beta^2 R_{ijkl}(\varphi)\bar{\psi}^i\psi^j\bar{\psi}^k\psi^l\right]$$

We can perform the Grassman integration using Berezin's formula. Expanding the exponential, the only term which can contribute must have n ψ's and n $\bar{\psi}$'s, to match the number of objects in the integration measure. Since the expansion of the exponential contains only even powers of ψ 's and $\bar{\psi}$'s, it follows that the integral vanishes for odd n. For even n we get:

$$\Delta_W = \int \frac{\prod_i d\varphi^i d\psi^i d\bar{\psi}^i}{(2\pi\beta^2)^{n/2}} \frac{(-1)^{n/2}}{(n/2)!} \left(\frac{1}{12}\beta^2 R_{ijk\ell}\bar{\psi}^i\psi^j\bar{\psi}^k\psi^\ell\right)^{n/2}$$

Changing from the ψ , $\bar{\psi}$ basis to the component basis $\psi = \begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$ gives a factor of three in the Lagrangian after appropriate use of the cyclic identities for $R_{ijk\ell}$. The resulting integral over Grassman numbers is trivial and gives a product of ϵ -symbols. Thus, we have:

$$\Delta_W = \int_M \frac{\prod_i d\varphi^i}{(2\pi)^{n/2}} \frac{1}{(n/2)!} \frac{1}{4^{n/2}} \epsilon^{i_1\cdots i_n} \epsilon^{j_1\cdots j_n} R_{i_1 i_2 j_1 j_2} \cdots R_{i_{n-1} i_n j_{n-1} j_n} \tag{47}$$

This is nothing but the Chern-Weil expression[20] for the Euler-Poincaré characteristic of a riemannian manifold. We have thus recovered the Witten's result[2]

$$\Delta_W(\text{sigma model}) = \text{Euler characteristic of the manifold} \tag{48}$$

Let us remark that we assumed that the manifold M is compact. In the non-compact case complications may arise.First of all the ultra-local prescritpion may turn out not to be correct because of the continuous spectrum of the Hamiltonian. Secondly, it has been shown that in a class of non-compact, non-linear σ-models there exist infinitely many states with zero energy which make the index infinite[21].

The previous computation is an example of a general method[8],[22],[23] derive the Atiyah-Singer theorem[24] for all classical complexes and its G-index generalization. It can be shown[2] that there exists the following correspondece between invariant quantities of the non-linear, supersymmetrical σ-models and the topological invariants of the underlying manifold M:

$$
\begin{aligned}
\mathrm{tr}(-1)^F &= \chi(M) \\
\mathrm{tr}(-1)^F K &= \mathrm{lef}\ (k) \\
\mathrm{tr}\ (Q\ \ ) &= \mathrm{sign}\ (M) \\
\mathrm{tr}\ (Q\ \ K) &= \mathrm{sign}\ (k)
\end{aligned}
\tag{49}
$$

where $k : M \longrightarrow M$ is an isometry of the manifold M and K is the corresponding quantum operator commuting with the Hamiltonian. Q_5 is the generator of the discrete chiral trasformation $\psi \longrightarrow \gamma_5 \psi$. $\chi(M)$ is the Euler charateristic of M, lef the Lefschetz number, and sign refers to the signature of M or k. The idea of the method consists to express the supertraces in the left-hand-sides of eqs.(49) as supersymmetrical path-integrals. One evaluates these path-integrals in a manner similar to that one we explained in eqs.(46),(47); thus, one obtains expressions for the index densities of the topological invariants in the right-hand sides of eqs.(49). An example will be given in the third chapter.

Let us remark that it is quite surprising that the non-linear supersymmetrical σ-model contains so many informations about the topology of the underlying manifold.

3. Theories with a continuous spectrum

(i) The super-Liouville example

There are cases when the ultra-local prescription apparently does not seem to give the right answer for the Witten index. Consider the N=1 real super-Liouville model in d=2:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi + \frac{m^2}{g^2}\exp g\varphi + \frac{1}{2}\bar{\psi}\not{\partial}\psi + \frac{m}{\sqrt{2}}\exp\left(\frac{g\varphi}{2}\right)\bar{\psi}\psi \tag{50}$$

which is of the Wess-Zumino type, with:

$$W'(\varphi) = \sqrt{2}\,\frac{m}{g}\exp\left(\frac{g\varphi}{2}\right)$$

Using the formula (43),one gets[(9),(28),(32)]:

$$Z_S = \int_0^\infty \frac{dy}{(2\pi/\beta^2)^{1/2}}\exp\left(-\frac{\beta^2}{2}y^2\right) = \frac{1}{2} \tag{51}$$

Clearly this cannot be equal to the Witten index which is defined to be an integer. One is forced to conclude that the Hamiltonian of the model (50) has a continuous spectrum: the constant configurations prescription might fail to reproduce the correct superpartition function and $Z_S(\beta)$ is not guaranteed to concide with the index.

Since the validity of (43) is doubtful, one has to resort to some other method to evaluate $Z_S(\beta)$. Because of the existence of a local Nicolai mapping (25),(26),(27) for this model, $Z_S(\beta)$ can be exactly computed(29). A bit surprisingly, the rigorous computation based on the Nicolai mapping gives the same answer of the ultra-local prescription:

$$Z_S(\beta) = 1/2$$

It follows that even the weak equality (20) does not hold:

$$\Delta_W \neq \lim_{\beta\to\infty} Z_S(\beta) = 1/2$$

It is therefore proven that the Hamiltonian of the super-Liouville model has a continuous spectrum starting from zero and the difference between the bosonic and the fermionic densities of energy eigenstates is rather singular as the energy tends to zero.

(ii) Theories with Fredholm Hamiltonians

In this and in the following section we will restrict our analysis to Hamiltonians that have a spectrum in which the continuous part is separated from zero by a finite gap (Fredholm Hamiltonians).

For such theories, we expect, in general, a non trivially β-dependent superpartition function. Nevertherless equality (19) holds.

In 1979 Callias, Bott and Seeley(5) gave an explicit formula for the trace(§)

$$\mathrm{tr}\left(e^{-\beta L^{+}L/2} - e^{-\beta LL^{+}/2}\right) \tag{52}$$

where L is a linear, differential operator defined on the non-compact manifol R^n

$$L \equiv \gamma^i \partial_i \otimes 1\!\!1_m + i\, 1\!\!1_p \otimes \Phi(x) \tag{53}$$

γ^i, i=1,...,n are the gamma matrices in n dimensions. $\Phi(x)$ is a nxn hermitian, (asymptotically) unitary matrix chosen in such a way that L is Fredholm. The trace (52) can be thought of as the superpartition function

$$Z_s(\beta) = \mathrm{tr}(-1)^F e^{-\quad H} \tag{54}$$

of the supersymmetric Hamiltonian:

$$2H = L^{+}L\,(1-F) + LL^{+}F \tag{55}$$

(F is the fermion operator).

We are not interested for the moment in the specific form of the Callias formula for the trace (52):later on we will derive it by means of "supersymmetrical" methods. The important fact is that, for each value of β , the trace (52) is a topological invariant since it is expressed by a surface integral which depends only on the asymptotic behavior of the potential.

(§) Actually they computed the trace

$$\mathrm{tr}\left(\frac{z}{z+L^{+}L} - \frac{z}{z+LL^{+}}\right) \quad , \quad z \in \mathbb{C}$$

which is easily seen to be related to (52) by a Laplace trasform.

Accordingly in the following we will assume the porperty of topological invariace for $Z_S(\beta)$ from the very beginning. Using the functional integral representation of the superpartition function we will show that, also in this case, $Z_S(\beta)$ can be explicitly calculated as an ordinary integral over constant configurations. Let us stress that this fact is not obvious since $Z_S(\beta)$ turns out to be non trivially dependent on β .

The previous assumption is justified by the Callias-Bott-Seeley theorem for quantum mechanics of the type (55),(53) (n odd), but it can consistently applied to quantum mechanical system not of this form (in particular with n even) and also to field theory. In the latter case such a generalization is mathematically non trivial because the relevant differential operators are defined on infinite dimensional manifolds.

(iii) <u>More on the constant configuration prescription</u>

Consider supersymmetric quantum mechanics with an <u>arbitrary</u> number of degrees of freedom having as euclidean Lagrangian:

$$\mathcal{L}_E = \frac{1}{2}(\dot{\varphi}^i)^2 + \frac{1}{4}\chi^i_A\dot{\chi}^i_A + \frac{1}{2}(W_{,i}(\varphi))^2 - \frac{i}{4}\epsilon^{AB}W_{,ij}\chi^i_A\chi^j_B \quad (56)$$

where i,j =1,...,n; A=1,2 . φ^i are the bosonic variables, χ^i_A are the fermions. $_{,i}$ denotes differentiation with respect to φ^i and $W(\varphi)$ is the superpotential.

The superpartition function is given by the functional integral:

$$Z_s(\beta) = \int_{P.B.C.} [d\varphi \, d\chi^i_A] \exp\left(-\int_0^\beta dt \, \mathcal{L}_E\right) \tag{57}$$

In ordere to assure the Fredholm character of the energy spectrum $W_{,i}(\varphi)$ is chosen to satisfy the asymptotic condition:

$$|W_{,i}(\varphi)|^2 \xrightarrow{|\varphi| \to \infty}$$ a homogeneous function of order zero bounded below by a positive constant

According to our ipothesis, we can deform the potential in the following way:

$$W_{,i}(\varphi) \longrightarrow W_{,i}(\lambda\varphi), \quad \lambda \text{ an arbitrary parameter} > 0 \tag{58}$$

without affecting the superpartition function, since (58) corresponds to a "stretching" of the potential leaving the value at infinity fixed. Thus, invariance under the deformation (58) implies:

$$Z_s(\beta) = \int_{P.B.C.} [d\varphi^i d\chi^i_A] \exp\left(-\int_0^\beta dt \left[\frac{1}{2}\dot{\varphi}^{i2} + \frac{1}{4}\chi^i_A \dot{\chi}^i_A + \frac{1}{2}(W_{,i}(\lambda\varphi))^2 - \frac{i}{4}\epsilon^{AB}\chi^i_A \chi^j_B W_{,ij}(\lambda\varphi)\right]\right) \tag{59}$$

Performing a change of integration variables with unit jacobian

$$\lambda \varphi^i \rightarrow \varphi^i$$
$$\sqrt{\lambda}\, \chi^i_A \rightarrow \chi^i_A$$

this reduces to:

$$Z_{,S}(\beta) = \int_{P.B.C.} [d\varphi^i d\chi^i_A] \exp\Big(-\int_0^\beta dt \Big[\frac{1}{2}\frac{\dot{\varphi}_i^2}{\lambda^2} + \frac{1}{4}\frac{\chi^i_A \dot{\chi}^i_A}{\lambda} + \frac{1}{2}(W_{,i}(\varphi))^2 - \frac{i}{4}\epsilon^{AB}\chi^i_A \chi^j_B W_{,ij}(\varphi)\Big] \tag{60}$$

At this point one may carry out a decomposition in Fourier modes of $\varphi^i(t)$ and $\chi^i_A(t)$, exactly as we did in sec. II.2 .

It is easily seen that the constant modes give a contribution of O(1) in λ, while the non-constant modes are at least O(λ). From the overall λ-independence, one concludes that, in the present case also, the superpartition function is given by the functional integral reduced to constant configurations:

$$Z_{,S}(\beta) = \int \frac{\prod_i d\varphi^i\, 2\, d\chi^i_1 d\chi^i_2}{(2\pi\beta)^{k/2}} \exp\Big[-\beta\Big(\frac{1}{2}W_{,i}(\varphi)^2 - \frac{i}{4}\epsilon^{AB}\chi^i_A \chi^j_B W_{,ij}(\varphi)\Big)\Big] \tag{61}$$

Let us remark that while in the case of discrete energy spectrum the ultra-local prescription (61) was "physically" intuitive (since $Z_s(\beta)$ is actually β-independent, one can imagine to "decouple" the non-constant modes sending the size of the quantization box to zero and giving them large masses), for theories with continuous spectrum it is not so. Rather, it is the specific structure of the Lagrangian (61) which requires that the non-constant modes cancel each other, somehow, in order that $Z_s(\beta)$ be a topological invariant. There exist supersymmetrical Hamiltonians, still Fredholm, for which the constant configurations prescription cannot be justified.

Integrating the fermions in (61), one obtains:

$$Z_s(\beta) = \int \left(\prod_i dq^i\right)\left(\frac{\beta}{2\pi}\right)^{n/2} \det \| W_{,ij}(q) \| \exp\left[-\frac{\beta}{2}\left(W_{,i}(q)\right)^2\right]$$

Making the change of variable

$$y^i = W_{,i}(q) \tag{62}$$

the final result for the superpartition function is:

$$Z_s(\beta) = N \int_{B^n} \left(\prod_i dy^i\right)\left(\frac{\beta}{2\pi}\right)^{n} e^{-\beta\frac{(y^i)^2}{2}} \tag{63}$$

where B^n is the image space of the mapping (62). N is the winding number of the map from S^{n-1} to S^{n-1} given by the restriction of (62) to the sphere at infinity. An integral representation of N is given by:

$$N = \frac{n}{\Omega^{(n)}} \int d\hat{W}_{,1} \wedge \dots \wedge d\hat{W}_{,n} \tag{64}$$

where $\hat{W}_{,i} \equiv \frac{W_{,i}}{|W_{,i}|}$ and $\Omega^{(n)}$ is the solid angle in R^n:

$$\Omega^{(n)} = \frac{2\pi^{n/2}}{\Gamma(\frac{1}{2}n)}$$

To get the Witten index one has to take the limit of (63) as $\beta \to \infty$. Clearly in this limit the gaussian equals 1 (recall the asymptotic condition on W (φ)) and hence

$$\Delta_W = \lim_{\beta \to \infty} Z_s(\beta) = N \tag{65}$$

One can find a more explicit formula for the integral in (63), when $|W_{,i}(\varphi)| \xrightarrow{|\varphi^i| \to \infty} \nu$, a constant independent on the angles. The image space B^n becomes a sphere of radius ν, and thus:

$$Z_s(\beta) = N \int_0^{\nu} \left(\frac{\beta}{2\pi}\right)^{n/2} d\rho\, \rho^{n-1}\, \Omega^{(n)} e^{-\beta \rho^2/2} =$$

$$= N\, \Omega^{(n)} \int_0^{\sqrt{\frac{\beta}{2}}\,\nu} \frac{dt}{(\pi)^{n/2}}\, t^{n-1} e^{-t^2} \tag{66}$$

Formula (66) agrees with the Callias-Bott-Seeley result (see next chapter) for n odd. For n even it gives a new trace formula. As pointed out at the beginning of this section, if one takes for granted that $Z_S(\beta)$ is a topological invariant even for field theories, the very same formulae (63)-(66) obtained hold also there.

(iv) Gauge theories

As Witten pointed out[2] it is in general difficult to compute the index for theories which contain particles that stay massless for all the values of the parameters. Of course this is just the case for gauge theories when there not occurs complete breakdown of the gauge symmetry.

The difficulty lays in the fact that the energy spectrum becomes continuous, starting from zero, when there are massless particles around, due to their zero momentum modes. Under these circumstances the concept of index becomes tricky to define.

To overcome this obstacle it is useful to introduce some sort of generalization of the concept of index.[2]

Let X be any operator commuting with the supersymmetry charges and, hence, with the Hamiltonian:

$$[X, Q] = 0 \quad \Longrightarrow \quad [X, H] = 0$$

Let P_λ be the projector into the subspace of the eigenstates of H which are eigenstates of X with eigenvalue λ. The quantities

$$\mathrm{tr}(-1)^F e^{-\beta H} P_\lambda \qquad ; \qquad \mathrm{tr}\, f(X) e^{-\beta H}(-1)^F \qquad (67)$$

similarly to the superpartition function $\mathrm{tr}(-1)^F e^{-\beta H}$, are topologically invariant. Like the index, if one of them is non-zero, supersymmetry is unbroken.

For gauge theories, even if $\mathrm{tr}(-1)^F e^{-\beta H}$ can be ill defined and/or tricky to evaluate for the reasons explained above, it is possible to choose P_λ and X so that the related traces (67) are well defined and easy to compute.

We will consider the pure N=1 super Yang-Mills theory. Its field content is: a non abelian gauge field A_μ and a Majorana spinor ψ^a in the adjoint representation of the gauge group. The lagrangian is:

$$\mathcal{L}_{SYM} = -\frac{1}{4}(F^a_{\mu\nu})^2 + i\,\bar{\psi}^a(\not{D}\psi)^a \qquad (68)$$

For sake of simplicity we will take SU(2) as gauge group. We will work in the gauge $A_0 = 0$.

As usual to compute the index we are interested in the zero-momentum modes. The strategy will be that one of quantizing only the zero -momentum modes which have zero energy classically: in the weak coupling they are expected to have energy much lower than the others modes. The zero-momentum modes that have classicaly zero energy satisfy:

$$F_{ij} = 0 \implies [A_i, A_j] = 0 \qquad A_i = A_i^a\, T^a \text{ constant matrices}$$

Hence, A_i must belong to the Cartan subalgebra of the gauge group SU(2). This means that they are proportional to the same generator, which we can choose, conventionally, to be T^3 :

$$A_i = h_i T^3 \tag{69}$$

Analogously, the fermionic zero-momentum modes carrying classicaly zero energy are:

$$\psi = \psi^a T^a = \epsilon T^3 \tag{70}$$

where ϵ is a constant Majorana spinor.

We, therefore, will quantize the Lagrangian:

$$\mathcal{L}_{eff} = V \left[\frac{1}{g^2} \sum_{i=1}^{3} (\dot{h}_i)^2 + \bar{\epsilon}\, i\gamma^0 \dot{\epsilon} \right]$$

whose Hamiltonian is:

$$H_{eff} = \frac{g^2}{V} \sum_{i=1}^{3} \pi_i^2 \tag{71}$$

(V is the volume of the quantization box). π_i are the momenta conjugate to h_i : $\pi_i = \frac{1}{i}\frac{\partial}{\partial h_i}$.

To quantize (71), one has to know the domain of the variables h_i .

Consider the gauge trasformations:

$$U_i \equiv e^{i \frac{2\pi}{\beta} x_i T^3} \tag{72}$$

Under (72) the gauge fields trasform like

$$A_i \xrightarrow{U_j} A_i + \frac{2\pi}{\beta} T_3 \delta_{ij}$$

that is, the zero-momentum modes are shifted by a constant:

$$h_i \xrightarrow{U_j} h_i + \frac{2\pi}{\beta}\delta_{ij} \qquad (73)$$

The gauge trasformations (72) are topologically trivial(they are called "small" gauge trasformations). By virtue of the Gauss law, the physical states must be annihilated by them. Therefore, the wave functionals of the physicals states must be periodic in h_i ,with period $2\pi/\beta$.

There exists another gauge trasformation:

$$G \equiv e^{i\pi T^2} \qquad (74)$$

which is also topologically trivial, since it is constant. Under (74):

$$h_i \xrightarrow{G} -h_i \quad ; \quad \epsilon \xrightarrow{G} -\epsilon$$

Invariance under the G operator (74) implies that the wave functionals of the physical states are even in h_i and contain an even number of fermionic, zero-momentum, creation operators.

The evaluation of the index is now trivial. The Hamiltonian (71) has four zero energy eigenstates

$$|0\rangle \;;\; \epsilon^{+}_{\alpha}|0\rangle \;;\; \epsilon^{+}_{1}\epsilon^{+}_{2}|0\rangle , \quad \alpha = 1,2 \qquad (75)$$

where $|0\rangle$ is the bosonic "vacuum" with constant wavefunction $\psi_0 = 1$.

ϵ^{+}_{α} are the zero-momentum, fermionic, creation operators. Between the states (75), only $|0\rangle$ and $\epsilon^{+}_{1}\epsilon^{+}_{2}|0\rangle$ are invariant under the "small" gauge trasformations (74). They are both bosonic. Therefore:

$$\Delta_W = \mathrm{tr}\,(-1)^F = 2 \quad \text{for } SU(2)$$

A direct generalization of this argument to the general case of a simple gauge group of rank r, gives[2]:

$$\Delta_W = r+1 \tag{76}$$

We conclude that supersymmetry is not spontaneously broken in pure super-Yang-Mills theory.

Let us end this paragraph with some remarks about the limitations of the previous result.

A major question is if the addition of matter to pure super-Yang-Mills can trigger spontaneous supersymmetry breaking. As long as one adds massiv supermultiplets, the index will not change and supersymmetry remains unbroken. The same is true if one couples supermatter in real representations of the gauge group, since it is always possible to give a gauge invariant mass to real matter. The difficult case is when the supermatter lays in <u>complex</u> representations of the gauge group: it cannot get mass preserving gauge invariance. As usual the zero-momentum bosonic modes make difficult to compute Δ_W. Besides that, in supersymmetric theories there appear frequently scalar potentials which are vanishing in certain directions in field space. Therefore the zero momentum modes of the scalar fields can become arbitrarly large at no cost in energy, so that they can develop a continuous energy spectrum starting from zero.We already pointed out that under these circumstances the concept of index becomes difficult to define. The relation between the index and the superpartition function may break down (see the super-Liouville example), and even its topological character becomes doubtful. In some cases such difficulties can be overcome using particular boundary conditions which eliminate the zero modes[(2),(30)].

One might ask if it is possible to derive the previous result about the index of super-Yang-Mills via the functional integral representation of the superpartion function, as we did for scalar theories..At present this program seems to present difficulties. The method we used for scalar theories is justified when the energy spectrum is discrete, while we expect this is not the case for gauge theories. Only restricting the supertrace to a subspace we have been able, in the above derivation, to discretize the spectrum.

An alternative and more rigorous computation of the index for gauge theories would,actually, be welcome: the correctness of (76) has been recently questioned in ref.(31), where it is pointed out that ambiguities may arise in the definition of the index for gauge theories.

CHAPTER III

1. Supersymmetrical quantum mechanics and index theorems

The index theorems demonstrates the equality of analytical indices (related to the solutions of partial differential equations) to topological invariants.

We already remarked in chapter I that the Witten index of a supersymmetrical quantum mechanical system is the analytical index of a linear differential operator related to the supersymmetry charge. In a basis in which the operator $(-1)^F$ is diagonal

$$(-1)^F = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

the hermitian supersymmetry charge looks like:

$$Q = \begin{pmatrix} 0 & L^+ \\ L & 0 \end{pmatrix} \quad ; 2H \equiv Q^2 \tag{77}$$

The Witten index is the analytical index of the operator L:

$$\Delta_W = \text{index } (L)$$

Therefore to compute the analytical index of a given differential operator L, we will choose the following strategy:

(a) we will build a supersymmetrical quantum mechanical model having the Q defined in (77) as supersymmetry generator.

(b) We will compute the superpartition function of the model built in (a) via the functional integral representation.

Step (b) includes two subcases:

(b_1) The operator L is defined on a compact manifold and has a discrete spectrum. Thus the superpartition function $Z_S(\beta)$ is β-independent and equal to the index of L. We can take the limit $\beta \to 0$ in the path integral representation of $Z_S(\beta)$. The path integral becomes straightforward to evaluate since in the limit $\beta \to 0$ the constant configurations will dominate(8),(22),(23),(9).

(b_2) The operator L is defined on a non-compact space and has a continuous spectrum. $Z_S(\beta)$ is, in general, β-dependent. If L is a Fredholm operator, $Z_S(\beta)$ is a topological invariant for each value of β. Because of that, in some cases, the path integral representing $Z_S(\beta)$ can be evaluated by means of the ultra-local prescription(10). One obtains the index of L taking the limit:

$$\text{index } (L) = \lim_{\beta \to \infty} Z_S(\beta)$$

We will see that the step (b_1) provides a derivation of the Atiyah-Singer index theorem, while step (b_2) gives the Callias-Bott-Seeley trace formula for operators on open spaces.

2. The Atiyah-Singer index theorem

Consider the Dirac operator on a d-dimensional, riemannian, compact manifold M in presence of a non-abelian gauge field:

$$\not{D} \equiv \gamma^a e_a^\mu \left(\partial_\mu + \frac{1}{2} \omega_\mu^{ab} \sigma_{ab} + i A_\mu^\alpha T^\alpha \right) \tag{78}$$

γ^a, a=1,...,d, are the gamma matrices in d dimensions; $\sigma_{ab} \equiv \frac{1}{4} [\gamma_a, \gamma_b]$
ω_μ^{ab} is the spin connection:

$$\omega_\mu^{ab} = - e^{\nu b} (\partial_\mu e_\nu^a - \Gamma^\lambda_{\mu\nu} e_\lambda^a)$$

e_μ^a are the vierbein:

$$e_\mu^a e_\nu^a = g_{\mu\nu} \quad ; \quad e_\mu^a e^\nu_a = \delta_\mu^\nu$$

The matrices $(T^\alpha)_{AB}$ provides a representation of the gauge group G.
α=1,...,dim(G); A,B = 1,...,m and m is the dimension of the representation.

The index of the Dirac operator $\not{D}$ is defined to be:

$$\text{index} (\not{D}) = \text{dim ker}(\not{D}_R) \quad - \quad \text{dim ker}(\not{D}_L)$$

where

$$\not{D}_{\substack{L \\ R}} = \not{D} \left(\frac{1 \pm \gamma_5}{2} \right)$$

are the Dirac operators projected on the subspaces of the spinor with definite chirality:

$$\gamma_5 \psi_{R,L} = \pm \psi_{R,L}$$

"γ_5" is the chirality operator in d dimensions (d is even).

Since

$$\not{D}_R = (-\not{D}_L)^{+}$$

the index of $\not{D}$ can be written as:

$$\begin{aligned} \text{index}\, \not{D} &= \dim \mathrm{Ker}(L) - \dim \mathrm{Ker}(L^{+}) = \\ &= \text{index}(L) \end{aligned}$$

with:

$$L \equiv i \not{D}_L$$

Following the prescription (a), we will build a supersymmetrical system with supersymmetry charge

$$Q = \begin{pmatrix} 0 & L^{+} \\ L & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\not{D}_R \\ i\not{D}_L & 0 \end{pmatrix}$$

The Hamiltonian will be:

$$2H = Q^2 = \begin{pmatrix} -\not{D}_L^{+}\not{D}_L & 0 \\ 0 & -\not{D}_R^{+}\not{D}_R \end{pmatrix} = (i\not{D})^2 \tag{79}$$

(79) can be thought of as a concrete representation of the Hamiltonian of an abstract quantum mechanical system. Such a system contains d bosonic variables X_μ whose conjugate momentum are represented in (79) by derivatives respect to X_μ :

$$i\partial_\mu \longleftrightarrow \Pi_\mu \tag{80}$$

The system contains also d fermionic real variables, satisfying the canonical anticommutation relations:

$$\{\psi^a, \psi^b\} = 2\delta^{ab}$$

ψ^a are represented in (79) by the gamma matrices:

$$\gamma^a \longleftrightarrow \psi^a \tag{81}$$

To take in account the gauge fields, further fermionic degrees of freedom have to be introduced. Let $\bar{\eta}^A$, η^A, A =1,...,m be fermionic creation and annihilation operators satisfying the anticommutation relations

$$\{\bar{\eta}^A, \eta^B\} = \delta^{AB}$$

Then the quantum mechanical operator

$$\bar{\eta}^A (T^\alpha)_{AB} \eta^B A^\alpha_\mu \tag{82}$$

is represented in the subspace of one-particle states $\bar{\eta}^A |0\rangle$, by the matrix

$$A^\alpha_\mu (T^\alpha)_{AB}$$

With the identifications (80),(81),(82), the Hamiltonian (79) can be rewritten as:

$$2H = [\psi^\mu(\Pi_\mu + \tfrac{i}{2}\omega^{ab}_\mu \sigma_{ab} + \bar{\eta} A_\mu \eta)]^2 = \tag{83}$$

$$= (\Pi_\mu + \tfrac{i}{2}\omega^{ab}_\mu \sigma_{ab} + \bar{\eta} A_\mu \eta)^2 + \tfrac{i}{2}\psi^\mu \psi^\nu \bar{\eta} F_{\mu\nu} \eta$$

The corresponding Lagrangian is:

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu} + \frac{i}{2} \psi^a (D_t^g \psi)^b + i \bar{\eta}^A (D_t^A)^{AB} \eta^B$$
$$- \frac{1}{2} \bar{\eta}^A F_{\mu\nu}^{AB} \eta^B \psi^{\mu} \psi^{\nu} \tag{84}$$

$$D_t^A \equiv \partial_t + \dot{x}^{\mu} A_{\mu} \quad ; \quad D_t^g \equiv \partial_t + \omega_{\mu}^{ab} \dot{x}^{\mu}$$

(84) describes a N=1/2 supersymmetric non linear σ-model defined on the riemannian compact manifold M and coupled to the external gauge fields A_{μ} .

Our aim is to compute the superpartition function of (84):

$$\text{index}\, \not{D} = \operatorname{tr}_1 e^{-\beta H} (-1)^F \tag{85}$$

where tr_1 means that the trace is taken over the subspace of the one-particles states

$$\bar{\eta}^A |0\rangle$$

To get (85), one can use the following trick. One computes:

$$\Delta(\lambda) \equiv \operatorname{tr} (-1)^F e^{-\beta H + i\lambda N} \tag{86}$$

where λ is a parameter and N is the "color" number operator:

$$N \equiv \bar{\eta}^A \eta^A$$

In (86) the trace is taken in the "big" Hilbert space which includes the multiparticles states

$$\bar{\eta}^{A_1}\bar{\eta}^{A_2}|0\rangle \; ; \; \dots \qquad ; \; \bar{\eta}^{A_1}\dots\bar{\eta}^{A_m}|0\rangle$$

This trace can be splitted in the subtraces on the different multi-particles subspaces labelled by the eigenvalue of N:

$$\Delta(\lambda) = \sum_{k=0}^{m} tr_k\left[(-1)^F e^{-\beta H}\right] e^{i\lambda k} \tag{87}$$

where tr_k indicates the trace on the subspaces of the k-particles states (k=0,...,m). Thus, to obtain the trace (85) in which we are interested, we have to extract from (87) the piece proportional to $e^{i\lambda}$.

(86) can be expressed as the path integral:

$$\Delta(\lambda) = \int [dx_\mu d\psi_a d\bar{\eta} d\eta] \exp\Big(-\int_0^\beta dt\Big[\tfrac{1}{2}\dot{x}_\mu^2 + \tfrac{i}{2}\psi^a(D_t\psi)^a + i\bar{\eta}(D_t+\lambda)\eta - \tfrac{1}{2}\bar{\eta}F_{\mu\nu}\eta\psi^\mu\psi^\nu\Big]\Big) \tag{88}$$

For $\beta \to 0$ (88) is dominated by the constant configurations. Expanding around the constant modes

$$x_\mu(t) = x_\mu^0 + \tilde{x}_\mu(t) \quad ; \quad \psi^a(t) = \psi_0^a + \hat{\psi}^a(t) \quad ; \quad \eta^A = \eta_0^A + \hat{\eta}^A$$

and keeping only the terms quadratic in the fields, we obtain:

$$\Delta(\lambda) = \int \prod_{i=1}^{d} (d\psi_\mu^o\, dx_\mu^o) \int^1 [d\hat{x}_\mu]\, e^{-\int dt\, \frac{1}{2}[\dot{x}_\mu^2 + \dot{x}_\mu \mathcal{R}_\mu{}^\nu(0) x_\nu]} \times$$

$$\times \int^1 [d\hat{\psi}_\mu] \exp\left(-\frac{1}{2}\int \hat{\psi}_\mu \dot{\hat{\psi}}_\mu\, dt\right) \times$$

$$\times \int [d\bar{\eta}\, d\eta] \exp\left[-\int dt\left[\bar{\eta}\dot{\eta} + \bar{\eta}\left(-\frac{1}{2}\mathcal{F} + i\lambda\right)\eta\right]\right] \tag{89}$$

where:

$$\mathcal{R}_\mu{}^\nu(0) \equiv R_\mu{}^\nu{}_{ab}(x_o)\, \psi_o^a \psi_o^b \; ; \; \mathcal{F} \equiv F_{\mu\nu}(x_o)\, \psi_o^\mu \psi_o^\nu$$

and $R_\mu{}^\nu{}_{ab}$ is the Riemann tensor of the manifold M.

Performing the standard gaussian integrations in (81), we get:

$$\Delta(\lambda) = \int \prod_{i=1}^{d} (dx_\mu^o\, d\psi_\mu^o)\left(\frac{1}{2\pi}\right)^{d/2} (-1)^{d/2} \cdot$$

$$\cdot \det{}^{-1/2}\left[\frac{\sinh \mathcal{R}^\mu{}_\nu/2}{\mathcal{R}^\mu{}_\nu/2}\right] \times \det\left(1 - e^{\frac{1}{2}\mathcal{F} + i\lambda}\right) \tag{90}$$

Integrating out the fermions, (90) becomes:

$$\Delta(\lambda) = \left(\frac{i}{2\pi}\right)^{d/2} \int_M \det{}^{-1/2}\left[\frac{\sinh \tilde{\mathcal{R}}^\mu{}_\nu/2}{\tilde{\mathcal{R}}^\mu{}_\nu/2}\right] \det\left(1 - e^{\frac{1}{2}\tilde{\mathcal{F}} - i\lambda}\right) \tag{91}$$

where the two-forms on themanifolds M have been introduced:

$$\tilde{R}^\mu{}_\nu \equiv R^\mu{}_{\nu\lambda\sigma}\, dx^\lambda \wedge dx^\sigma \quad ; \quad \tilde{F} \equiv F_{\mu\nu}\, dx^\mu \wedge dx^\nu$$

(85) is obtained extracting from (91) the term proportional to $e^{i\lambda}$:

$$\operatorname{index}(\not{D}) = \left(\frac{i}{2\pi}\right)^{d/2} \int_M \det{}^{-1/2}\left[\frac{\sinh \tilde{R}^\mu{}_\nu/2}{\tilde{R}^\mu{}_\nu/2}\right] \cdot \operatorname{tr} e^{\tilde{F}/2} \tag{92}$$

which is the Atiyah-Singer formula for the index of the Dirac operator (78).[24].

From (92) one can derive the the expressions for the Euler characteristic and the Hirzebruch signature.

3. The Callias-Bott-Seeley theorem

The Callias-Bott-Seeley theorem deals with linear differential operators of the type:

$$L \equiv i\gamma^\mu \partial_\mu \otimes 1\!\!1_m + i\, 1\!\!1_p \otimes \Phi(x) \tag{93}$$

where $\Phi(X)$ is a mxm matrix unitary and hermitian. γ^μ are the gamma matrices in d dimensions (d odd). L is defined on the non-compact space R^d. The theorem states that[4]:

$$\operatorname{tr}\left(\frac{z}{z + L^\dagger L} - \frac{z}{z + L L^\dagger}\right) = \frac{1}{(1+z)^{d/2}} \operatorname{index} L \tag{94}$$

and:

$$\text{index } L = \frac{1}{2\left(\frac{d-1}{2}\right)!}\left(\frac{i}{8\pi}\right)^{\frac{d-2}{2}} \int_{\mathbb{R}^d} \mathrm{tr}\,[d\,\phi(x)]^n \tag{94'}$$

Introducing the definitions:

$$Q \equiv \begin{pmatrix} 0 & L^+ \\ L & 0 \end{pmatrix} \quad ; \quad 2H \equiv Q^2$$

$$(-1)^F = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{95}$$

(94) can be rewritten as:

$$\Delta(z) \equiv \mathrm{tr}\, \frac{z}{z+H}(-1)^F = \frac{1}{(1+z)^{d/2}}\,\text{index}(L) \tag{96}$$

The left-hand-side of the equation (96) is proportional to the Laplace trasform of the superpartition function of the system defined in (95):

$$\Delta(z) = z\int_0^\infty e^{-\beta z} Z_s(\beta)\, d\beta$$

$$Z_s(\beta) \equiv \mathrm{tr}\,(-1)^F e^{-\beta H} \tag{97}$$

Δ (z) depends non-trivially on z, and so $Z_s(\beta)$ is β-dependent. This implies that the fermionic and bosonic densities of energy eigenstates of the Hamiltonian in (95) do not match. We have studied an Hamiltonian of this type in II.3.(iii), the Wess-Zumino supersymmetric quantum mechanics with an arbitrary number d of degrees of fredom:

$$2H = p_\mu^2 + W_{,\mu}^{2} - i\,\epsilon_{AB}\,\chi_\mu^A\,\chi_\nu^B\,W_{,\mu\nu} \tag{98}$$

We found for the superpartition function of (98):

$$Z_{IS}(\beta) = N\,\Omega^{(d)} \int_0^{\sqrt{\frac{\beta}{2}}\,v} \frac{dt}{(\pi)^{d/2}}\, t^{d-1} e^{-t^2} \tag{99}$$

The superpotential $W(X_\mu)$ is chose to satisfy the asymptotic conditions:

$$|W_{,\mu}(x)|^2 \xrightarrow[|x|\to\infty]{} v \tag{100}$$

We would like to show that, for d odd, (99) is a derivation of the Callias-Bott-Seeley theorem with a specific choice of the potential $\Phi(X)$. For d even (99) provides a new trace formula, not of the type (94).

The supersymmetry generators of the system (98) are:

$$\left.\begin{aligned} Q_1 &= \frac{1}{\sqrt{2}}\left(p^\mu \chi_1^\mu + W_{,\mu}\chi_2^\mu\right) \\ Q_2 &= \frac{1}{\sqrt{2}}\left(p^\mu \chi_2^\mu - W_{,\mu}\chi_1^\mu\right) \end{aligned}\right\} \tag{101}$$

In a basis in which the fermionic number is diagonal, the supersymmetry generators take the form:

$$Q_1 = \begin{pmatrix} 0 & L \\ L^\dagger & 0 \end{pmatrix}$$

Let us construct L in the two dindinct case d odd and d even.

d odd

It can be checked that an explicit representation for the fermionic operators χ_A^μ ,obeying the 2d-dimensional Clifford algebra, is:

$$\begin{aligned} \chi_1^\mu &= \gamma^\mu \otimes 1\!\!1_p \otimes \sigma_1 \\ \chi_2^\mu &= 1\!\!1_p \otimes \gamma^\mu \otimes \sigma_2 \end{aligned} \tag{102}$$

where γ_1^μ μ =1,...,d are the p= $2^{\frac{d-1}{2}}$ dimensional matrices representing the d-dimensional Clifford algebra. σ_1 and σ_2 are the 2x2 Pauli matrices. $(-1)^F$, anticommuting with the fermionic variables χ_A^μ , is represented by:

$$(-1)^F = -i\,\chi_1^1 \cdots \chi_1^d\ \chi_2^1 \cdots \chi_2^d = 1\!\!1_p \otimes 1\!\!1_p \otimes \sigma_3$$

In this basis:

$$Q_1 = \frac{1}{\sqrt{2}}\left(\gamma^\mu p_\mu \otimes 1\!\!1_p \otimes \sigma_1 + W_{,\mu}\, 1\!\!1_p \otimes \gamma^\mu \otimes \sigma_2\right)$$

from which it follows:

$$L = \gamma^\mu p_\mu \otimes 1\!\!1_p + i\, 1\!\!1_p \otimes \gamma^\mu W_{,\mu} \tag{103}$$

This belongs to the class of operators considered by Callias, with:

$$\Phi(x) = \gamma^\mu W_{,\mu} \tag{104}$$

Taking the Laplace trasform of $Z_\zeta(\beta)$ in (99), one obtains:

$$\Delta(z) = z \int_0^{\infty} d\beta\, e^{-\beta z}\, Z_s(\beta) =$$

$$= z \int_0^{\infty} d\beta\, e^{-\beta z}\, N \Omega^{(d)} \int_0^{\sqrt{\frac{\beta}{2}}\nu} \frac{dt}{(\pi)^{d/2}}\, t^{d-1} e^{-t^2}$$

$$= \frac{\nu}{(\nu^2 + z)^{d/2}}\, N \tag{105}$$

we recover the Callias-Bott-Seeley trace formula (94) (in (94) is set equal to 1), since the winding number is equal to the index of L:

$$\mathrm{index}(L) = \frac{1}{2\left(\frac{d-1}{2}\right)!}\left(\frac{i}{8\pi}\right)^{\frac{d-2}{2}} \int tr\left(\gamma^{\mu}\, dW_{1\mu}\right)^{n}$$

$$= \frac{d}{\Omega^{(d)}} \int d\hat{W}_1 \wedge \cdots \wedge d\hat{W}_d = N$$

d even

A representation of the 2d-dimensional Clifford algebra,for even d, in which $(-1)^F$ is diagonal, is:

$$\chi_1^{\mu} = \Gamma^{\mu} \otimes \sigma_1 \qquad \mu = 1,\ldots,d$$

$$\chi_2^{\mu} = \begin{cases} \Gamma^{\mu+d} \otimes \sigma_1 & \mu = 1,\ldots,d-1 \\ 1\!\!1_q \otimes \sigma_2 & \mu = d \end{cases}$$

where Γ'^{a},a=1,...,2d-1 are the $q = 2^{d-1}$ dimensional matrices satisfying the 2d-1 dimensional Clifford algebra.In this representation

$$(-1)^F = -i\chi_1^1 \cdots \chi_2^d = 1\!\!1_q \otimes \sigma_3$$

and:

$$L = \sum_{\mu=1}^{d} \Gamma^\mu p_\mu + \sum_{\mu=1}^{d-1} \Gamma^{\mu+d} W_{,\mu} + i\, W_{,d}\, 1\!\!1_q \qquad (106)$$

This operator does not belong to the class considered by Callias: in any case our result (99) correctly gives its trace and the related trace.

One may think to derive the formula (94) in the general case when $\phi(x)$ is not of the form (104), but it is a generic hermitian (unitary) mxm matrix. Building the corresponding supersymmetric Lagrangian is quite strightforward. It is not of the Wess-Zumino type (98). Thus, the ultra-local prescription to compute $Z_S(\beta)$ from the path integral does not apply. Nevertheless the path integral can be reduced to gaussian type and, essentially; the expression (94) is recovered. However to determine the correct normalization factor appears to be tricky[33].

CHAPTER IV

1. Index theorems and supersymmetry in the soliton sector

The relevance of the index theorems discussed in the previous chapter is not limited to quantum mechanics.

The role played by the Atiyah-Singer index theorem in instantons calculation or in the axial anomaly problem has been, by now, quite well understood.

When operators defined on non-compact spaces appear, the Atiyah-Singer theorem is not of great usefulness: one needs alternative theorems like the Callias trace formula or others[(7),(34)].

Recently these theorems have found several applications to study field theoretical phenomena like fermion fractionization, chiral symmetry breaking, etc.[(35),(36)]. Because of their topological character, one expects that trace formulae like (94) become useful in the soliton or monopole sectors of a field theory. In fact, we will illustrate in the following how to use trace formulae for operators on open spaces to compute quantum corrections to the mass of supersymmetrical solitons. For sake of simplicity we will consider the 1+1 dimensional case[(11)]. An analogous technique is valid for four dimensional supersymmetric field theories admitting monopoles (as N=2 or N=4 super-Yang-Mills)[(12),(16)]; in this case one makes use of a trace formula derived in ref.(7).

Consider the supersymmetric theory in two dimensions

$$S=\int d^2x\left[\frac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi+\frac{i}{2}\bar{\Psi}\not{\partial}\Psi-\frac{1}{2}W'(\varphi)^2-\frac{1}{2}W''(\varphi)\bar{\Psi}\Psi\right] \tag{107}$$

where φ is a real scalar field and ψ a Majorana spinor. $W(\varphi)$ is a superpotential chosen such that the above theory admits topological solitons, and the prime denotes a derivatives respect to the argument.

The classical soliton (antisoliton) $\varphi_s(x)$ satisfies the Bogomolny equation

$$\frac{d\varphi_s(x)}{dx} = \underset{(+)}{-}\; W'(\varphi_s(x)) \tag{108}$$

The classical soliton mass is:

$$\begin{aligned} M_0 &= \int_{-\infty}^{+\infty} dx \left[\frac{1}{2}\left(\frac{\partial\varphi_s}{\partial x}\right)^2 + \frac{1}{2} W'(\varphi_s(x))^2 \right] = -\int_{-\infty}^{+\infty} dx\, \frac{\partial\varphi_s}{\partial x} W'(\varphi_s(x)) = \\ &= -\int_{-\infty}^{+\infty} \frac{\partial W(\varphi_s(x))}{\partial x} dx = -\Big[W(\varphi_s(x)) \Big]_{-\infty}^{+\infty} \end{aligned} \tag{109}$$

The $O(\hbar)$ correction to the soliton mass is given by:

$$\Delta M_1 = \frac{\hbar}{2}\left(\sum \omega_B - \sum{}' \omega_F\right) \tag{110}$$

apart from renormalization counterterms. ω_B and ω_F are the eigenvalues of the differential operators coming from the quadratic expansion of the action(107) around the classical soliton.

The bosonic fluctuations $\xi(x)$ satisfy:

$$L^{+} L\, \xi(x) = \omega_B^2\, \xi(x) \tag{111}$$

while the fermionic eigenfunctions $\psi(x) = \begin{pmatrix} u_+(x) \\ u_-(x) \end{pmatrix}$ satisfy:

$$L\, u_+(x) = \omega_F\, u_-(x)$$
$$L^+ u_-(x) = \omega_F\, u_+(x) \tag{112}$$

where we defined the linear differential operator on R :

$$L = i \frac{d}{dx} + i\, W''(\varphi_s(x)) \tag{113}$$

(L^+ is the adjoint of L). From (112) one obtains the decoupled equations:

$$L^+ L\, u_+(x) = \omega_F^2\, u_+(x)$$
$$L L^+ u_-(x) = \omega_F^2\, u_-(x) \tag{114}$$

The quantities in which we are interested are the densities n_B and n_F of the eigenfunctions of the bosonic and fermionic fluctuations equations,(111) and (112).Let us define n_+ (n_-) as the densities of eigenfunctions of the operators L^+L (LL^+). Then, from (111), it is clear that:

$$n_B = n_+ \tag{115}$$

It is not difficult to convince oneself that:

$$n_F = \frac{1}{2} (n_+ + n_-) \tag{116}$$

Therefore:

$$n_B - n_F = \frac{1}{2}(n_+ - n_-) \tag{117}$$

One can evaluate the right-hand-side of this equation using the Callias-Bott-Seeley trace formula (94) which, in this case, gives:

$$\mathrm{tr}\left(\frac{z}{z+L^+L} - \frac{z}{z+LL^+}\right) = \frac{1}{2}\left[\frac{a_+}{\sqrt{z+a_+^2}} - \frac{a_-}{\sqrt{z+a_-^2}}\right] \tag{118}$$

where $a_\pm = W''(\varphi_s(\pm\infty))$ .Trasforming the trace into an integral, (118) can be rewritten as:

$$\int_0^\infty \left[\frac{dn_+}{dE}(E) - \frac{dn_-}{dE}(E)\right]\frac{dE}{z+E} = f(z)$$

where
$$f(z) \equiv \frac{1}{2z}\left[\frac{a_+}{\sqrt{a_+^2+z}} - \frac{a_-}{\sqrt{a_-^2+z}}\right]$$

This has the form of a dispersion relation, from which it follows that

$$-2\pi i\left(\frac{dn_+}{dE} - \frac{dn_-}{dE}\right) = \lim_{\epsilon\to 0^+}\Big(f(-E+i\epsilon) - f(-E-i\epsilon)\Big) - 2\pi i\,\delta(E$$

that is:

$$\frac{dn_+}{dE} - \frac{dn_-}{dE} = -\frac{1}{2\pi E}\left[\frac{a_+\theta(E-a_+^2)}{\sqrt{E-a_+^2}} - \frac{a_-\theta(E-a_-^2)}{\sqrt{E-a_-^2}}\right] + \delta(E)$$

Thus, the soliton mass correction is

$$\Delta M_1 = \frac{\hbar}{2} \sum (\omega_B - \omega_F) =$$
$$= \frac{\hbar}{4} \int_0 dE \left(\frac{dn_+}{dE} - \frac{dn_-}{dE} \right) \sqrt{E}$$

which,after a change of variables, gives the result:

$$\Delta M_1 = - \frac{\hbar}{4\pi} \int dk \left[\frac{a_+}{\sqrt{k^2 + a_+^2}} - \frac{a_-}{\sqrt{k^2 + a_-^2}} \right] \tag{119}$$

(apart renormalization counterterms).

One sees from (119) that the non-vanishing of the mass correction is intimately linked to the topologically non-triviality of the classical soliton solution - the same formula evidently gives zero for the energy correction to the vacuum in O($\hbar$).

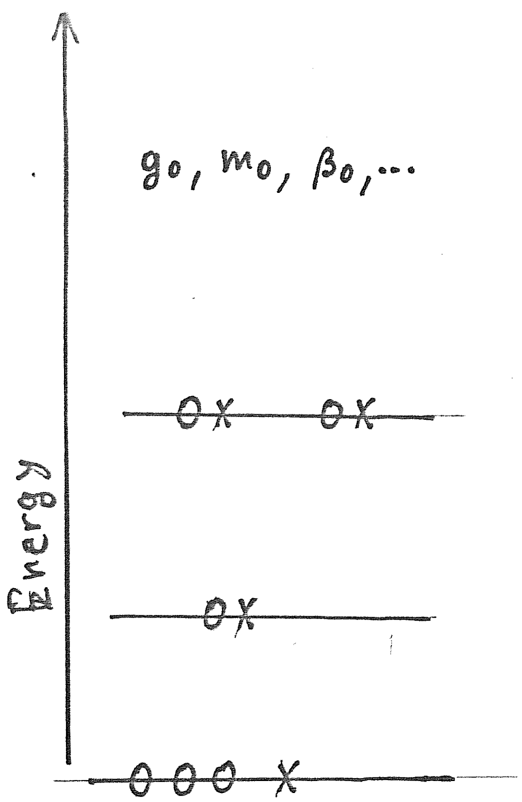


fig. 2a

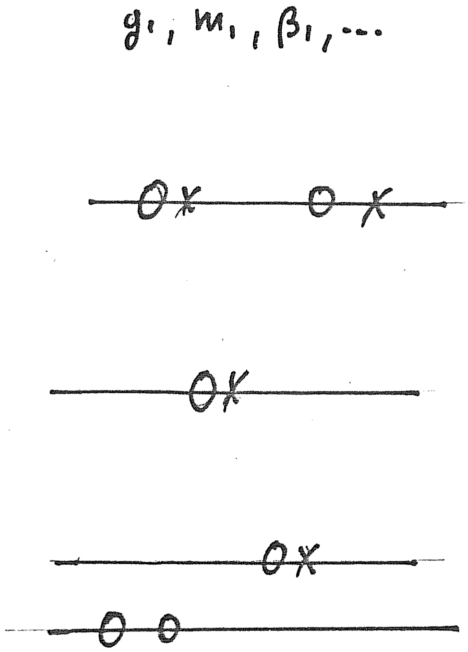


fig. 2b

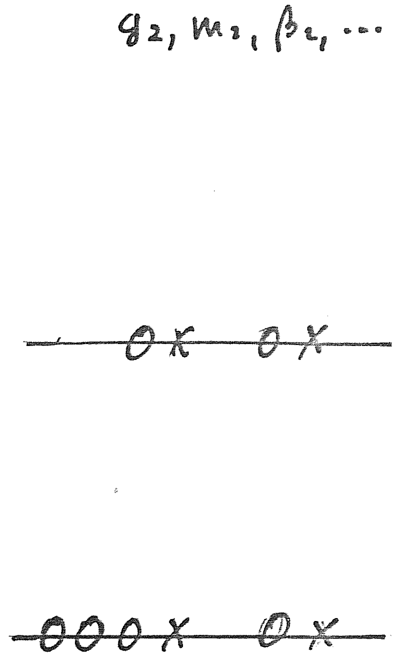
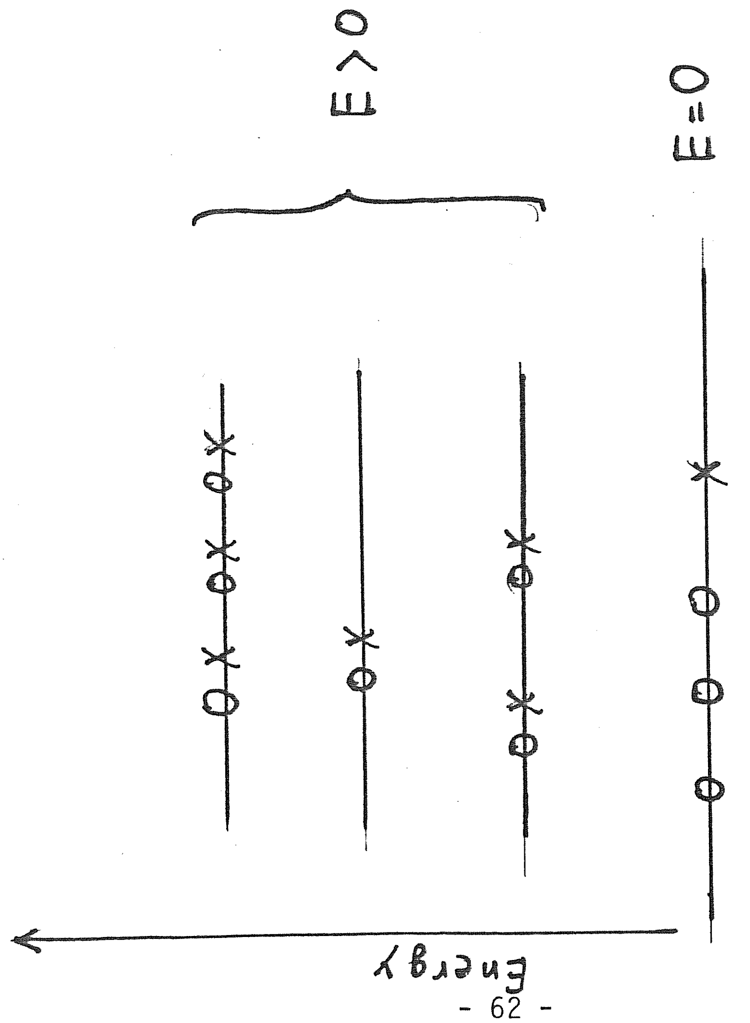


fig. 2c

fig. 2



bosonic state

fermionic state

fig. 1

fig. 3

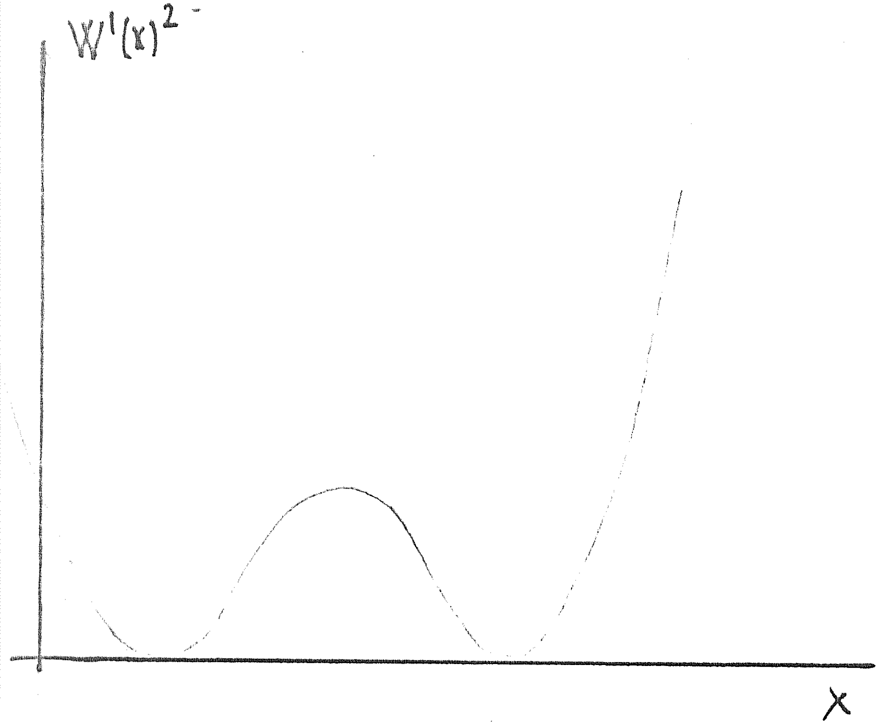


fig. 3a

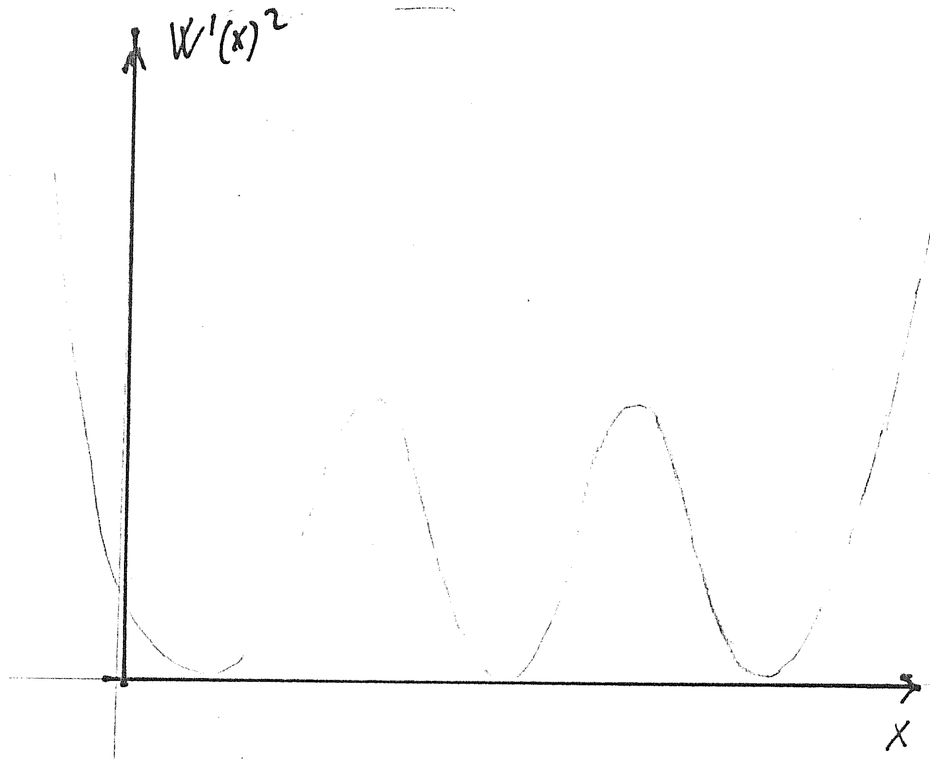


fig.3b

REFERENCES

1. B. Zumino, Nucl.Phys. B89 (1975),535

   P. West, Nucl.Phys. B106 (1976),219

   D.M. Capper and M. Ramon-Medrano, J.Phys. 62 (1976),269

   W. Lang, Nucl.Phys. B114 (1976), 123

   M.T. Grisaru, W. Siegel and M. Rocek, Nucl.Phys. B159 (1979),429

2. E. Witten, Nucl. Phys. B202 (1982),253
3. S. Cecotti and L. Girardello, Phys. Lett. 110B (1982),39
4. C. Callias, Commun.Math.Phys. 62 (1978),213

   R. Bott and R. Seeley, Commun.Math.Phys. 62 (1978), 235

5. M. Hirayama, Prog.Theor.Phys. 70 (1983),1444
6. S. Goldeberg, Unbounded linear operators (McGraw-Hill, New York, 1966)
7. E. Weinberg, Phys. Rev. D20 (1979),936
8. L. Alvarez-Gaumê, Commun.Math.Phys. 90 (1983),161
9. L. Girardello, C. Imbimbo and S. Mukhi, Phys. Lett. 132B (1983),69
10. C. Imbimbo and S. Mukhi, Nucl. Phys. B242 (1984),81
11. C. Imbimbo and S. Mukhi, Nucl.Phys. B247 (1984),471
12. C. Imbimbo and S. Mukhi, Bombay preprint, TIFR/TH-84-23,to appear in Nucl.Phys. B
13. E. Witten, Nucl.Phys. B185 (1981), 513
14. F. Cooper and B. Freedman, Ann. Phys. 146 (1983), 262

    P. Salomonson and J.van Holten, Nucl.Phys. B190 (1982),509

    R.D. Carlitz, Pittsburgh preprint, PITT-19-84, (1984)

15. C. Imbimbo and S. Mukhi, unpublished
16. R.K. Kaul, Phys.Lett. 143B (1984),427

17. R. Feynman, Statistical mechanics (Benjamin, Reading, MA)
R. Feynman and A. Hibbs, Quantum mechanics and path integrals (McGraw-Hill, New York, 1965)

18. D.Z. Freedman and P.K. Townsend, Nucl.Phys. B177 (1981),282

19. B. Zumino, Phys.Lett. 87B (1979), 203

20. S.S. Chern, Ann. Math. 45 (1944),747

21. A.D. Davis, A.J. Macfarl ane and J.W.van Holten, Nucl.Phys. B216 (1983),493

22. L. Alvarez-Gaumê, Harvard University prerpint, HUTP-83-A035

23. P. Windey to appear in Acta Physica Polonica (1984)
D. Friedan and P. Windey, to appear in Nucl. Phys. B (1984)

24. M. Atiyah and I. Singer, Ann. Math 87 (1968),485
M. Atiyah and G. Segal, Ann.Math 87 (1968), 531
M. Atiyah and I. Singer, Ann. Math. 87 (1968), 546 and 93 (1971), 119 and 93 (1971),139

25. H. Nicolai, Phys.Laett 89B (1980), 341

26. G. Parisi and N. Sourlas, Nucl.Phys. B206 (1982),321

27. S. Cecotti and L; Girardello, Ann.of Phys. 145 (1983),81

28. S. Cecotti, Nuovo Cimento, (1983)

29. S. Cecotti and L. Girardello, Nucl. Phys. B239 (1984),573

30. E. Cohen and C. Gomez,Phys.Lett. 135B (1984),99

31. F. Palumbo, CERN preprint, Ref.TH.3874, (1984)

32. R. Akhoury and A. Comtet, UCLA preprint 84-TEP-1, (1984)

33. C. Imbimbo and S. Mukhi, unpublished

34. M. Atiyah, V. Patodi and I. Singer, Bull.London.Math.Soc. 5 (1973),229 and Proc.Cambridge Philos.Soc. 77 (1975),42 and 78 (1975),405 and

79 (1976),71

35. M. Hirayama and T. Torii, Prog. Theor.Phys. 68 (1982),1354

36. B. Paranjape and W. Semenoff, Phys.Lett. 132B (1983), 369

A.J. Niemi and W. Semenoff, Phys. Rev. D (1984),809

A.J. Niemi, I.A.S.,Princeton prepprint, (1984), May

A.J. Niemi,B. Paranjape and W. Semenoff, Phys. Rev. 53 (1984),515

A.J. Niemi, I.A.S., Princeton preprint, (1984),April