SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
STUDI AVANZATI
TRIESTE
Strada Costiera 11

ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

TESI

DIPLOMA DI PERFEZIONAMENTO
"MAGISTER PHILOSOPHIAE"

Generalized Hopf bifurcation

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Anno Accademico 1982-83

TRIESTE

GENERALIZED HOPF BIFURCATION

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INTRODUCTION.

Let

(1) $\dot{x} = f(x)$ $f(0) = 0$

be an autonomous ordinary differential equation, where $x\ R^n$, $f \in C^{\infty}(B_r(0), R^n)$, and $B_r(0)$ is an open ball of radius r centered at 0. We assume that the Jacobian matrix $f'(0)$ has a complex conjugate pair of eigenvalues $\pm\lambda i$, and that no other eigenvalue is an integer multiple of λi (non-resonance condition). From now on (1) will be called the unperturbed equation, and the corrisponding dynamical system, the unperturbed system.

Let us consider a smooth curve

$$\mu \mapsto f_{\mu}$$

in the space of vector fields defined in $\overline{B_r(0)}$, such that $f_o = f$ and $f_{\mu}(0)=0$ $\forall \mu \in]-\bar{\mu}, \bar{\mu}[$. Moreover let $\alpha(\mu) \pm i\beta(\mu)$ be curves of complex conjugate eigenvalues of $f'(0)$ passing through $\pm\lambda i$. The classical Hopf Bifurcation Theorem gives a solution to the problem of existence of periodic solutions of:

$$\dot{x} = f(x)$$

contained in a neighborhood of 0, with period near to 2π, and such that for $\mu \to 0$, the closed orbits tend to the equilibrium point. Since $\mu \in R$, it can be called one-parameter Hopf Bifurcation. By substituting a k-dimensional smooth surface to the curve just considered, we obtain the definition of multiparameter Hopf Bifurcation.

It is possible to carry on further the generalization: let us introduce a topology in the space of C^{∞} vector fields on $\overline{B^r}(0)$ in the following way: Define the real function $|||\cdot|||$ in the of C^{∞} vector fields

$$|||f||| = \sum_{0}^{\infty}{}_{\ell} \frac{\|f\|^{(\ell)}}{2^{\ell}(1+\|f\|^{(\ell)})}$$

where $||f||^{(1)}$ is the usual C^1 norm of f. $|||\cdot|||$ defines a distance by the relation

$$d(f,g) \doteq |||f - g|||$$

Generalized Hopf Bifurcation is concerned with the existence of periodic solutions with period near to 2π, contained in neighborhood of 0, for an equation:

Generalized Hopf Bifurcation is concerned with the existence of periodic solutions with period near to 2π, contained in a neighborhood of 0, for the equation

$$\dot{x} = g(x)$$

with g close to f in the sense of $|||\cdot|||$.

The goal of this thesis is to expose some techniques, essentially based on the stability properties of the involved dynamical systems, which recently proved to be useful tools to treat Generalized Hopf Bifurcation (from now on, only GHB). Moreover , some theorems can be generalized to obtain information about bifurcation of families of compact invariant sets for abstract dynamical systems; an interesting consequence is the possibility to give a purely topological proof of the classical Hopf' theorem in R^2 or when a bidimensional invariant manifold exists, in addition to a more accurate study of the relationship between the stability properties of the unperturbed system and the perturbed ones.

In § 1 the classical theorem by Poincarè-Andronov-Hopf is reported, in a recent version due to Ruelle and Takens (30) , together with Chafee's results about bifurcation without transversality, which can be considered as intermediate steps in the passage from classical to GHB.

In § 2 the study of bifurcation of families of compact invariant sets is motivated, and the second method of Liapunov used to prove that a switch in asymptotic stability of a dynamical system leads to bifurcation phenomena. A similar result in which total stability partially substitutes asymptotic one is reported, with an analysis of the effect of this substitution on the attractivity of bifurcated orbits.

The third section is devoted to an exposition of the h-asymptotic stability approach to GHB in R^n.

An appendix has been attached, in which the so called Poincarè method for the analisys of stability in presence of purely imaginary eigenvalues is summarized, in view of the fundamental role it plays in the theorems of the previous section.

Finally, I wish to thank proff. Cellina and Moauro for their courtesy.

§ 1. CLASSICAL HOPF BIFURCATION

The classical theory of Hopf bifurcation starts with Poincarè's work whose ideas were developed by Andronov (2) in the thirties for bidimensional systems and by Hopf(21), who extended previous results to R^n. Recent proofs with further information about the stability and attractivity of bifurcating orbits have been obtained by Sacker , Ruelle and Takens(30), and others. The following statement is due to Ruelle and Takens.

Theorem 1:

Let f_μ be a one-parameter family of C^k vector fields on R^n ($k \geq 4$), such that $f_\mu(0)=0$ for each μ and $F=(f_\mu,0)$ is also C^k. Suppose that $df_\mu(0)$ has two distinct, comlex conjugate eigenvalues $\lambda(\mu),\overline{\lambda(\mu)}$ and that the rest of the spectrum is distinct from $\lambda(\mu),\overline{\lambda(\mu)}$ .Moreover, let $\left.\frac{d(Re\ \lambda(\mu))}{d\mu}\right|_{\mu=0} > 0$

Then:

i)there is a C^{k-2} function $\mu:(-\varepsilon,\varepsilon)\to R$ such that $(x_1,0,\mu(x_1))$ is on a closed orbit of period close to $2\pi/|\lambda(0)|$ and radius growing like $\sqrt{\mu}$ for $x_1\neq 0$ and such that $\mu(0) = 0$.

ii) if the other eigenvalues do not cross the imaginary axis as μ crosses zero, then there exists a neighborhood U of the origin R^3 such that any closed orbit in U is one of those above.

iii) if the origin is a "vague attractor", then $\mu(x_1) > 0$ for each $x_1\neq 0$ and the orbits are attracting.

To explain what "vague attractor" means, it is necessary to define the so-called displacement function. In our hypothesis, a trajectory starting at a point on the x^1-axis will meet in a finite time the same line in a second point, say y_1, distinct from the origin, which is supposed to be an equilibrium point. The map P which associates y_1 to x_1 is called Poincarè map or first return map. The displacement function is defined as the difference between $P(x_1)$ and x_1:

$$V(x_1,\mu) = P(x_1,\mu) - x_1$$

In the hypothesis of theorem 1 it is possible to prove that the third derivative of v with respect to x_1 exists; the origin is a vague attractor if:

$$\frac{\partial^3 V(0)}{\partial x_1^3} < 0$$

and coordinates are chosen so that

$$dX_0(0) = \begin{pmatrix} 0 & |\lambda(0)| & d_3 X^1(0) \\ -|\lambda(0)| & 0 & d_3 X^2(0) \\ 0 & 0 & d_3 X^3(0) \end{pmatrix} \qquad \lambda(0) \notin \sigma\left(d_3 X^3(0)\right)$$

Marsden and McCracken gave an algorithm to compute the derivatives of V by means of those ones of the unperturbed vector field: by point iii) of theorem 1 , it can be considered as a computational method to determine the asymptotic stability of bifurcated orbits.

Hopf theorem allows us todeduce the existence of periodic orbits by a transversal crossing of the imaginary axis by a pair of complex conjugate eigenvalues. Chafee (11) studied the case of non-transversal crossing, obtaining a weaker statement: in fact, he does not succeed in finding bifurcating orbits, but only invariant sets.

<u>Theorem 2</u>

If

$$\dot{x} = f_\mu(x) = P(\mu)x + X(x,\mu) \qquad f_\mu(0) = 0$$

where P is a real n×n matrix, and the following hypothesis are verified:

i) ε_0, $r_0 > 0$ exist such that P is continuous on $[0,\varepsilon_0]$ nad X is continuous on $B_r(0) \times [0,\varepsilon_0]$.

ii) for each r in $[0,\varepsilon_0]$ there exists $k(r)>0$ such that X is ~~k(r)~~-uniformly lipschitzian with respect to x on $B_r(0)\times[0,\varepsilon_0]$ and k(r) is infinitesimal as r goes to 0.

iii) 0 is an asymptotically stable equilibrium point for

$$\dot{x} = f_0(x)$$

iv) there exists a pair of simple conjugate eigenvalues $a(\mu) \pm ib(\mu)$ such that:

$$a(0) = 0 \quad , \quad a(\mu) > 0 \quad \text{for} \quad \mu \in]0, \varepsilon_0]$$
$$b(\mu) > 0 \quad \text{for} \quad \mu \in [0; \varepsilon_0] \quad ;$$

all the other eigenvalues lie in the left-hand complex half space.

Then there exist r_1, r_2 ($0 < r_2 \leq r_1$) and $\varepsilon_1 \in]0, \varepsilon_0]$ such that:

C_1) for every $\mu \in]0, \varepsilon_1]$ there are two closed orbits, not necessarily distinct, $\gamma_1(\mu)$ and $\gamma_2(\mu)$, contained in a neighborhood $B_{r(\mu)}(0)$ where $0 < r(\mu) \leq r_2$; $\gamma_1(\mu)$ and $\gamma_2(\mu)$ lie on a local integral manifold $M^2(\mu)$ homeomorphic to a bidimensional open disk and passing through the origin. When $\gamma_1(\mu) \neq \gamma_2(\mu)$, they are concentric about the origin.

C_2) the part of $M^2(\mu)$ which lies inside the inner one of the two curves, say $\gamma_1(\mu)$, is filled with solutions whose positive and negative limit sets are, respectively, $\gamma_1(\mu)$ and the origin (exept for the origin, that is an equilibrium point). NO other solutions remain in $B_{r_1}(0)$ for all $t < 0$.

C_3) for each $\mu \in]0, \varepsilon_1]$ that part of $M^2(\mu)$ exterior to $\gamma_2(\mu)$ and contained in $B_{r_2}(0)$ is filled by solutions which remain in $M^2(\mu) \cap B_{r_1}(0)$ for all $t > 0$, whose positive limit set is $\gamma_2(\mu)$.

C_4) for each $\mu \in]0, \varepsilon_1]$ there exist solutions approaching the origin as t diverges; they fill an invariant manifold $M^{n-2}(\mu)$ homeomorphic to an open ball in R^{n-2} and containing the origin.

C_5) if $x_0 \in B_{r_2}(0)$, then the solution passing through it, $x(t, x_0, \mu)$ remains in $B_{r_1}(0)$ for all $t > 0$; if it does not tend to the origin as t diverges, then it approaches the closed invariant set $Q(\mu)$ consisting of those points in $M^2(\mu)$ which lie in the annulus defined by $\gamma_1(\mu)$ and $\gamma_2(\mu)$; it contains in its positive limit set one or more closed orbits

Chafee showed also that without a transversality condition it is not possible to predict how many distinct famil~~i~~s of closed orbits bifurcate from the origin. On the other side, to renounce transversality allows us to obtain information about all systems close to a given one in a suitable topology; the following statement (Chafee (14)) is a partial answer to the problem of GHB:

Let us consider the space U consisting of all the functions of $C^{\infty}(B_r(0), R^n)$ whose derivatives are bounded on $\bar{B}_r(0)$, endowed with the topology of uniform convergence over all the derivatives, and its subspace U_1, , characterized in the following way:

$$f \in U_1 \iff \begin{cases} f(x) = Ax + X(x) \quad ; \quad f(0) = 0 \\ A \text{ is a real nxn matrix with two complex} \\ \text{conjugate eigenvalues; } X'(0) = 0 \end{cases}$$

In what follows $BU_1(f,r)$ will denote the ball of radius r centered at f. Chafee applies the alternative method (15) to obtain a bifurcation function $\psi(\xi, f)$, where ξ measures the amplitude of the orbit involved, and examines the multiplicity of the null solution of ψ .

Theorem 3

Let $f_o \in U$, such that $f_o(0) = 0$, $f_o'(0)$ has a complex conjugate pair of simple eigenvalues $\pm i$ and no other eigenvalue is a multiple of $\pm i$.

Then there exists a bifurcation function ψ such that if zero is a root of multiplicity k of $\psi(\xi,f)=0$, then there exist three positive numbers d_1, r_1, μ_1 verifying:

i) for any f in $BU_1(f_o,d_1)$, the equation

$$\dot{x} = f(x) \qquad (*)$$

has no more than k nontrivial closed orbits in $B_{r_1}(0)$ with period contained in $]2\pi - \mu_1, 2\pi + \mu_1[$

ii)for each integer $j \in [0,k]$ and for any $r_2 \in]0,r_1]$, $d_2 \in]0,d_1]$, $\mu_2 \in]0,\mu_1]$ there exists a function $f \in BU_1(f_o,d_2)$ suchthat (*) has exactly j nontrivial or-exists a function $f \in BU_1(f_0,d_2)$ such that (*) has exactly j nontrivial closed orbits with periods in $]\ell\pi-\mu_\ell, \ell\pi+\mu_\ell[$, lying in $B_{\mu_\ell}(0)$

iii) for any $r \in]0,r_1]$, $\mu \in]0,\mu_1]$, there exists a number $d \in]0,d_1]$ such that, if $f \in BU_1(f_0,d)$, if Γ is a periodic orbit of (*) contained in $B_{r_1}(0)$ with period T in $]\ell\pi-\mu_1, \ell\pi+\mu_1[$ then Γ lies in $B_r(0)$ and T belongs to $]\ell\pi-\mu, \ell\pi+\mu[$

iv) let S_j the set of all functions in $BU_1(f_0,d_1)$ such that (*) has exactly j nontrivial closed orbits whose periods satisfy: $\ell\pi-\mu_1 < T < \ell\pi+\mu_1$; then for each integer $j \in [0,k]$, S_j has a nonempty interior and f_0 lies in the boundary of that interior; moreover:

$$BU_1(f_0,d_1) = S_0 \cup S_1 \cup .. \cup S_k$$

Theorem 3 is not concerned either with stability or attractivity properties of bifurcating orbits, or with computational methods to determine the majorization involved. IN section 3 and the Appendix it will be shown how it is possible to provide a successful algorithm even for GHB, by choosing a different approach to the question.

§ 2. BIFURCATION FOR FAMILIES OF COMPACT INVARIANT SETS.

Most of the theorems recorded in this thesis are strictly connected with applications to physical problems (it is sufficient to consider, for example, the importance that oscillation theory has for physics).
It is well known the importance of stability in the construction of a mathematical model to describe biological, economical or physical systems: a non-stable model risks not to be a model at all. In fact, to represent in an abstract way a real phenomenon, it is necessary to neglect a lot of factors considered "irrelevant"; this features may be regarded as perturbations of our equations (if the model is supposed to be deterministic and differentiable) so the only traits of our model we may consider as "realistic" are those which are preserved by small perturbations of the vector field: in other mords, we request it to be structurally stable. In some situations we are not interested in the fine topological structure of the system: concrete models proposed to study turbulence (30) are characterized by the arising of hhgher and higher dimensional invariant tori in the phase space, as the complexity of the motion of the particles in the fluid increases. This phenomenon is associated to switches in stability of invariant sets: when a subset passes from asymptotic stability to complete instability, a new invariant subset appears. In this case we are only interested in the preservation of the stability of invariant sets under small perturbations: that is, in total stability. A relevant fact is that asymptotically stable sets are authomatically totally stable, as proved in (37) and, for abstract dynamical systems, in (25). This makes them far easier to use, because it is not necessary to examine the behaviour of all the systems close to the given one to obtain information of structural type: asymptotically stable sets are always phisically relevant.

This section is mainly concerned with abstract dynamical systems (both discrete and continuous ones; I shall write I to mean Z or R); for standard definitions, see (6). I recall the definition of total stability for abstract dynamical systems given in (25), extending that one given by Dubosin (16), as proved in (6). In what follows (E, ϱ) is a locally compact metric space with distance ϱ, $\mathcal{C}$ the family of nonempty, compact, proper subsets of E, $\mathcal{S}$ the set of dynamical systems on E, p, π elements of $\mathcal{S}$.

Definition 1.

Let $M \in \mathcal{C}$; for $\pi, p \in \mathcal{S}$ and $\varepsilon > 0$, $t \in I^+ \setminus \{0\}$ we set

$$D(p, \pi; \varepsilon, F) = \sup \{ \varrho(p(t,x), \pi(t,x)) : 0 \le t \le F,\ x \in S(M, \varepsilon) \}$$

we say that M is -totally stable if:

$$(\forall \varepsilon > 0)(\forall F \in I^+ \setminus \{0\})(\exists \delta_1, \delta_2 > 0)(\forall p \in \mathcal{S} : D(\pi, p; \varepsilon, F) < \delta_2) :$$

$$\gamma_p^+ (S(M, \delta_1)) \subset S(M, \varepsilon)$$

A first step to the main bifurcation result is the following theorem, in which we use the fact that asymptotic stability implies total stability (see (6)). First I premise the

Definition 2.

Let $\mu \mapsto p_\mu$ be a map from $[0, \bar{\mu}[$ into $\mathcal{S}$, such that $p : [0, \bar{\mu}[\times I \times E \to E$ is continuous; $\{p_\mu\}_\mu$ will be said a one-parameter family of flows. Let us associate to it a new map $\mu \mapsto M_\mu$ from $[0, \bar{\mu}[\to \mathcal{C}$, for which

(i) for each $\mu \in [0, \bar{\mu}[$, the set M_μ is p_μ-invariant;

(ii) $\max \{ \varrho(x, M_0) : x \in M_\mu \} \to 0$ as $\mu \to 0$;

For such families, holds the following:

Theorem 4.

Let M be p_o-asymptotically stable. Then there exist a $\mu^* \in]0,\bar{\mu}[$ and a function $\nu : R^+ \to R^+$ of class K such that for each $\mu \in]0,\mu^*[$ the set $P_\mu = \gamma^+_{p_\mu}(S(M_o,\nu(\mu)))$ satisfies the following conditions:

(a) P_μ is a p_μ-asymptotically ~~stable~~ compact ~~set~~;

(b) there exists a neighborhood $\mathcal{N}$ of M for which

$$(\exists T \in I^+)(\forall x \in \mathcal{N})(\forall t \geq T) : p_\mu(t,x) \in P_\mu$$

Moreover:

(c) $P_\mu \to M_o$ as $\mu \to 0$, in the Hausdorff metric.

Proof:

The asymptotic stability of M_o implies its total stability. There exist positive λ and F such that $S[M_o,\lambda]$ is a compact subset of $A_{p_o}(M_o)$. By definition of total stability and continuity of $p_\mu(t,x)$ there exist two maps h,k of class K such that:

$$\gamma^+_{p_\mu}(S(M_o,h(\varepsilon))) \subset S(M_o,\varepsilon) \qquad \forall \varepsilon \in [0,\lambda] \quad \forall \mu \in [0,k(\varepsilon)]$$

provided $K(\lambda) \leq \bar{\mu}$. The asymptotical stability of M is equivalent to the existence of a scalar function V defined in its region of attraction (see (6)), positive definite on $A_{p_o}(M_o) \setminus M_o$ and strictly decreasing on trajectories. For such a function and for each $\varepsilon \in]0,\lambda]$ there is $L(\varepsilon) \in]1,+\infty[$ such that (see (25), proof of Thm. 3.1):

$$|V(y) - V(z)| \leq L\varrho(y,z) + \frac{c(h(\varepsilon))}{4} \qquad \forall y,z \in S[M_o,\lambda]$$

where c is of class K for which:

$$V(p_o(\bar{F},x)) - V(x) \leq -c(\rho(x,M_o))$$

Moreover, the continuity of p allows to prove the existence of a ψ of class K such that:

$$\rho(p(\mu,t,x),p(0,t,x)) \leq \frac{c(h(\varepsilon))}{4L} \qquad \text{for all } \mu \in [0,\psi(\varepsilon)],$$
$$(t,x) \in [0,\bar{t}] \times S[M_o,\lambda]$$

Consider a function χ of class K such that $\chi(\varepsilon) \leq \min\{k(\varepsilon),\psi(\varepsilon)\}$ for every $\varepsilon \in [0,\lambda]$. Let $\nu = h \circ \chi^{-1}$ and $\mu^* = \chi(h^2(\lambda))$. Then:

$$\mu^* = \chi(h^2(\lambda)) \leq \chi(\lambda) \leq k(\lambda) \leq \bar{\mu}$$

$\forall \mu \in]0,\mu^*]$: $S(M_o,h^2(\lambda))$ is an open neighborhood of $P_\mu = \overline{\gamma_p^+(S(M_o,\nu(\mu)))}$ because it contains the region of attraction of P_μ : for each $x \in S(M_o,h^2(\lambda))$ there exists $\tau_x \in I^+$ such that $p_\mu(\tau_x,x) \in S(M_o,\nu(\mu))$. If not, let N be a positive integer and $x_n = p_\mu(n\bar{t},x)$ for $n \in \{0,\dots,N\}$; we have x_n, $p_o(\bar{t},x_n) \in S(M_o,\lambda)$. By using elementary properties of stability theory one proves that $V(x_{n+1}) - V(x_n) \leq -c(\nu(\mu))/2$ for $0 \leq n \leq N$ which implies $V(p_\mu(N\bar{t},x)) - V(x) \leq -N(c(\nu(\mu)))/2$, which contradicts the positiveness of V. Now, to prove (b) it is sufficient to notice that $S(M_o,\nu(\mu)) \subseteq P_\mu$ which is positively invariant; if we choose as τ_x any number greater than

$$\max\left\{\frac{2\nu(x)}{c(\nu(\mu))} : x \in S[M_o,h^2(\lambda)]\right\}$$

the result is achieved. An immediate consequence is that P_μ is a compact p_μ-uniform attractor, hence asymptotically stable (point (a)). Finally, we have:

$$M_o \subset P_\mu \subset \overline{S(M_o,\chi^{-1}(\mu))}$$

which implies point (c).

For one-parameter families of elements of $\mathcal{C}$ it is possible to give a generalized definition of bifurcation in the following way:

Definition 3.

Let M_μ be a map satisfying the conditions of Definition 1.
0 is said to be a bifurcation point for M_μ if there exists a $\mu^* \in]0,\bar{\mu}[$ and a second map $M'_\mu :]0,\mu^*[\to \mathcal{C}$ such that:

(α) $\forall \mu \in]0,\mu^*[$ M'_μ is p_μ-invariant and $M_\mu \cap M'_\mu = \emptyset$

(β) $\max \{ p(x,M_0) : x \in M'_\mu \} \to 0$ as $\mu \to 0$

Next theorem shows that for compact invariant subsets, an inversion in asymptotical stability actually ensures bifurcation.

Theorem 5.

Let E be connected, $\bar{\mu} > 0$, $M: [0,\bar{\mu}[\to \mathcal{C}$ a map like in Def.1.
If M_0 is asymptotically stable and M_μ is completely unstable for $\mu \in]0,\bar{\mu}[$, then 0 is a bifurcation point for M_μ. Moreover, μ^* and M' can be determined in such a way that $\forall \mu \in]0,\mu^*[$:

(a) M'_μ is p_μ-asymptotically stable

(b) $M'_\mu = \tilde{P}_\mu \setminus A^-_{p_\mu}(M_\mu)$, where $\tilde{P}_\mu$ is a suitable compact set containing $A^-_{p_\mu}(M_\mu)$.

Proof:

Functions h, χ and ν are the same as in Thm.4, χ being subject to an additional condition to be introduced later in the proof.

The compactedness of M_μ and condition (ii) (see Def.2) ensures the existence of a function σ of class K such that:

$$M_\mu \leq S(M_0,h(\varepsilon)) \qquad \forall \mu \in [0,\sigma(\varepsilon)] \quad , \quad \forall \varepsilon \in [0,\lambda]$$

λ satisfying $S[M_0,\lambda] \subsetneqq A_{p_0}(M_0)$. We request χ to be not greater than σ in the whole interval $[0,\lambda]$. Let us set $\mu^* = \chi(h^2(\lambda))$:
for each $\mu \in]0,\mu^*]$, $P_\mu = \overline{\gamma^+_{p_\mu}(S(M_0,\nu(\mu))}$ is compact and satisfies statements (a),(b),(c) of Thm.4. Let $\tilde{P}_\mu$ be the union of all invariant subsets of P_μ : it is a compact p_μ-uniform attractor with respect to P_μ ,

hence, by statement (b) of Thm.4, $\widetilde{P}_\mu$ is p_μ-asymptotically stable.

To prove that it contains properly the region of p_μ-negative attraction of M_μ, let us take $y \in A^-_{P_\mu}(M_\mu)$. There exists $r_\mu > 0$ such that $S(M_o, r_\mu) \subsetneqq S(M_o, \nu(\mu))$; M_μ is p_μ-negatively asymptotically stable, so there exists $t' \in I^-$ for which $p_\mu(t', y) \in S(M_\mu, r_\mu) \subset S(M_o, \nu(\mu)) \subset P_\mu$.

P_μ is positively invariant, $-t' \in I^+$ and we may write:

$$y = p(-t', p_\mu(t', y)) \in P_\mu$$

that is, $A^-_{P_\mu}(M_\mu) \subset P_\mu$; being $A^-_{P_\mu}(M_\mu)$ p_μ-invariant, and $\widetilde{P}_\mu$ is the union of invariant subsets of P_μ, we have

$$A^-_{P_\mu}(M_\mu) \subset \widetilde{P}_\mu$$

The inclusion is proper, because $A^-_{P_\mu}(M_\mu)$ is open, $\widetilde{P}_\mu$ compact, F connected and $\widetilde{P}_\mu$ contained in $S[M_o, \lambda]$ which is a proper subset of E.

Now let us consider the compact set $M'_\mu = \widetilde{P}_\mu \setminus A^-_{P_\mu}(M_\mu)$. By the elementary theory of dynamical systems, it is p_μ-invariant; moreover, it is p_μ-asymptotically stable. It is sufficient to show (see (6) or (25),I.1.2.3) that

$$\{x \in E: \emptyset \neq J^*_{P_\mu}(x) \subset M\}$$

is a neighborhood of M. Let x be in $A^-_{P_\mu}(M_\mu) \setminus M_\mu$; since $A^-_{P_\mu}(M_\mu)$ is compact and p_μ-invariant, $J^*_{P_\mu}(x)$ is a non empty set contained in $\overline{A}^-_{P_\mu}(M_\mu)$. If $y \in A^-_{P_\mu}(M_\mu) \cap J^*_{P_\mu}(x)$ we have $\emptyset \neq J^-_{P_\mu}(y) \subset M_\mu$ which implies $x \in J^+_{P_\mu}(y)$ and $x \in M_\mu$, that is in contradiction whith our choice of x. So

$$\left.\begin{array}{l} x \in A^-_{P_\mu} \setminus M_\mu \\ J^+_{P_\mu}(x) \neq \emptyset \end{array}\right. \quad \Longrightarrow \quad J^+_{P_\mu}(x) \subset \partial A^-_\mu$$

and our partial result is achieved, because invariant p_μ-uniform attractors are p_μ-asymptotically stable.

It remains only to prove condition (β) in Def.3: it easily comes from statement (c) in Thm.4 and from

$$M'_\mu \subset \widetilde{P}_\mu \subset P_\mu$$

Now it is interesting to notice how it is possible to apply Thm.5. to dynamical systems defined by differential equations in R^2 to obtain result similar to Chafee's one (see (11) or §1, for the statement).

Let $\bar{\mu} > 0$ and $f: [0,\bar{\mu}[\times R^2 \to R^2$ be a continuous map such that $\forall \mu \in [0,\bar{\mu}[$

(i) $f(\mu,0) = 0$

(ii) $\dot{x} = f(\mu,x)$ (*)

defines a flow p_μ on R^2

(iii) Equation (*) has no equilibrium points distinct from the origin in $B_r(0)$, $r > 0$.

In the next theorem, which I report without proof (see (25)), the set $\{0\}$ will play the role of M_μ:

Theorem 6.

If the origin of R^2 is p_o-asymptotically stable and p_μ-completely unstable for $\mu > 0$, then $\mu = 0$ is a bifurcation point. The number μ^* and the map M'_μ can be determined in such a way that for each μ in $]0,\mu^*[$:

(a) M'_μ is p_μ-asymptotically stable;

(b) M'_μ is the compact annulus having as boundary two cycles C_μ, C'_μ of the dynamical system concentric with respect to 0, the inner one being equal to $\partial A^-_{p_\mu}(0)$.

In Theorem 5 it is possible to weaken hypothesis for dynamical systems defined in R^n by taking M_o only totally stable, using a characterization of total stability for compact sets obtained by Seibert (34):

Theorem 7.

A compact subset M of R^n is totally stable with respect to the flow defined by a differential eqaution if and only if M has a fundamental family of asymptotically stable compact neighborhoods.

In the following statement we have once again $M_\mu = \{0\}$; p_μ denotes a flow defined by a differential equation in R^n, such that a is an equilibrium point of p_μ for each μ.

Theorem 8.

Let the origin be totally stable with respect to ($*$) for $\mu = 0$ and completely unstable for ($*$) for each $\mu > 0$. Then there exists $\mu^* > 0$ such that for every μ in $]0, \mu^*[$ there exists a compact subset M'_μ satisfying:

(a) M'_μ is p_μ-invariant and asymptotically stable;

(b) $M'_\mu \cap \{0\} = \emptyset$ and $\max \{ \|x\| : x \in M'_\mu \} \to 0$ as $\mu \to 0$

Proof:

By Seibert's theorem, 0 has a fundamental system of p_0-asymptotically stable compact neighborhoods. For each $\varepsilon \in]0, \lambda]$, λ positive, let A_ε be one of them, contained in $S(0,\varepsilon)$. Asymptotic stability of A_ε is equivalent to the existence of compact neighborhoods N_ε of A_ε and a real $\mathcal{C}^1$ function V_ε , positive definite in N_ε with respect to A_ε (see (6)) for which:

$$\dot{V}_\varepsilon(x) \leq c(\rho(x, A_\varepsilon)) \qquad \forall\, x \in N_\varepsilon$$

where $\dot{V}_\varepsilon$ is the time derivative of V_ε along the solutions of ($*$) for $\mu = 0$ and c is of class K. Moreover, A_ε is p_0-totally stable, so, like in the proof of Thm.5, there exist three functions h, h_1, k of class K such that:

(α) $\overline{\gamma^+_\mu(S(A_\varepsilon,h_1(\varepsilon)))} \subset N_\varepsilon \qquad \forall \mu \in [0,k(\varepsilon)]$

(β) $\gamma^+_\mu(S(A_\varepsilon,h\ (\varepsilon))) \subset S(A_\varepsilon,h_1(\varepsilon)) \quad \forall \mu \in [0,k(\varepsilon)]$

where k has been chosen in such a way that $k(A) < \bar{\mu}$. The compactedness of N_ε implies the existence of a positive constant L_ε for which $\|\operatorname{grad} V_\varepsilon\| \leq L_\varepsilon$ in N_ε ; for the continuity of $f(\mu,x)$ there exists ψ of class K such that:

(γ) $\|f(\mu,x) - f(0,x)\| \leq \dfrac{c(h(\varepsilon))}{2L_\varepsilon} \qquad \forall (\mu,x) \in [0,\psi(\varepsilon)] \times N_\varepsilon$

If we define $\chi(\varepsilon)$ as $\min\{k(\varepsilon),\psi(\varepsilon)\}$, it is of class K and ($\alpha$), ($\beta$), ($\gamma$) are true for μ in $[0,\chi(\varepsilon)]$. Now set $\mu^* = \chi(A)$ and for μ in $]0,\mu^*[$, let $\varepsilon = \chi^{-1}(\mu)$. We have:

$$P_\mu := \overline{\gamma^+(S(A_\varepsilon,h(\varepsilon)))} \subset S(A_\varepsilon,h_1(\varepsilon)) \subset S(0,h_1(\varepsilon)+\varepsilon)$$

hence, for $\mu \to 0$: $\max\{\|x\|: x \in P_\mu\} \to 0$. Let $\tilde{P}_\mu$ be the union of all invariant subsets of P_μ : from now on on the proof goes on just as in that one of Thm.5, and the thesis is completely achieved.

Further improvements of the stability analysis of classical bidimensional Hopf bifurcation can be deduced by the previous statemnt. The next theorems will be concerned with equations (*) where f is supposed to be $\mathcal{C}^4$ and a pair of complex eigenvalues to cross transversally the imaginary axis.

Theorem 9.

Let the origin of R^2 be p_0-stable but not p_0-asymptotically stable. Then there exists a sequence $\{\gamma_n\}_{n\in N}$ of closed orbits around the origin such that $\max\{\|x\| : x \in \gamma_n\} \to 0$ as $n \to \infty$.

Proof.

Since the origin is p_o-stable, it has a fundamental family of positively invariant compact neighborhoods. Choose one of them, say $W_\varepsilon \subset S(0, \varepsilon)$; for $x \in W_\varepsilon$, $\Lambda^+(x) \subset W_\varepsilon$. Moreover, the origin is supposed to be asymptotically stable, so there exists $\bar{x} \in W_\varepsilon$ such that $0 \notin \Lambda^+(x)$. By the transversality condition 0 is an isolated fixed point of p_o, and we may choose ε so that in $S(0, \varepsilon)$ there are no other fixed points. Bendixson's theorem implies that $\Lambda^+(\bar{x})$ is a closed orbit around the origin.

Theorem 9 is used to show that total stability is not sufficient to guarantee attractivity of bifurcated orbits:

Theorem 10.

Let the origin of R^2 be p_o-totally stable but not p_o-asymptotically stable. Then, there exists $\mu' \in]0,\bar{\mu}[$ such that for $\mu \in]0,\mu'[$ there are bifurcating orbits which are not attractive.

Proof.

The hypotheses of Thm.8 are satisfied, so we may state the existence of a family of bifurcating sets M'_μ verifying conditions (a) and(b) of that theorem.. Moreover, by theorem 6, M'_μ is an annulus bounded by two closed orbits of p_μ, which are bifurcating orbits for μ small enough, in virtue of (b) of Thm.8. Hence, the bifurcation function $\mu(c)$ (see Thm.1) has positive values for points c arbitrarily close to zero. For this reason, and for Thm.9, there exists $c' > 0$ such that $\mu(c') = 0$. Let us set $\mu'' = \sup\{\mu(c): c \in [0,c']\}$, μ'' is strictly positive and for each μ in $]0,\bar{\mu}[$ there are at least two bifurcating orbits which cannot be both attractive.

In next section it will be shown that the asymptotic stability of the origin is not even sufficient to ensure attractivity of bifurcating orbits.

§ 3. h-STABILITY AND GENERALIZED HOPF BIFURCATION.

The first part of this section is concerned with classical Hopf bifurcation in R^2. It is showed how the use of suitable stability techniques allows us to study the attractivity properties of bifurcated periodic orbits. The method exposed leads also to an algorithm for the determination of the bifurcated orbits by an inspection of the unperturbed system. In the second part the more difficult problem of GHB in R^n is treated.

In what follows, I refer to an equation

$$\begin{aligned} \dot{z} &= f\ (z) \\ f\ (0) &= 0 \end{aligned} \qquad (*)$$

where $f \in \mathscr{C}^{k+1}(]-\bar{\mu},\bar{\mu}[\times D_a, R^2)$, $k \geq 3$, $\bar{\mu} > 0$, $a > 0$, D_a open disk in R^2 centered at the origin. The jacobian matrix $f'(0)$ is supposed to have two eigenvalues $\alpha(\mu) \pm i\beta(\mu)$, with $\alpha\ (0) = 0$, $\alpha'(0) > 0$, $\beta(\mu) \neq 0$, $\forall \mu$. Equation $(*)$ will also be written in the form:

$$\begin{aligned} \dot{x} &= \alpha(\mu)x - \beta(\mu)y + X(\mu,x,y) \\ \dot{y} &= \alpha(\mu)y + \beta(\mu)x + Y(\mu,x,y) \end{aligned} \qquad (*)'$$

with X,Y real functions of order ≥ 2, that is, whose Taylor's formula starts with terms of order $\geq$ 2. The set of $\mathscr{C}^k$ functions of order h ($h \leq k$) will be denoted by $\mathscr{C}^k_h$. For $\mu = 0$, setting $\lambda = \beta\ (0)$, the unperturbed system appears in this form:

$$\begin{aligned} \dot{x} &= -\lambda y + X(x,y) \\ \dot{y} &= \lambda x + Y(x,y) \end{aligned} \qquad (*_o)$$

with obvious meanings for X and Y.

By X_i and Y_i will be denoted homogeneous polynomials of degree i in Taylor's formula for X and Y.

Definition 4.

Let h be an integer such that $2 \leq h \leq k$. The null solution of $(*_o)$ is h-asymptotically stable (h-completely unstable) if:

(i) $\forall \xi, \tau \in \mathcal{C}(D_a, R)$ of order $> h$, such that $\xi(0,0) = \tau(0,0) = 0$ the zero solution of

$$\begin{aligned} \dot{x} &= -\lambda y + X_2(x,y) + \ldots + X_h(x,y) + \xi(x,y) \\ \dot{y} &= \lambda x + Y_2(x,y) + \ldots + Y_h(x,y) + \tau(x,y) \end{aligned} \qquad (**)$$

is asymptotically stable (completely unstable)

(ii) h is minimal with respect to (i).

The previous definition has an obvious generalization to n-dimensional systems.

Next theorem (see (28)) shows how to recognize the h-stability of the involved system by suitable truncations of the Taylor's formula associated to $(*_o)$ or by the derivatives of the displacement function V, as defined in § 1.

Theorem 11.

The following propositions are equivalent:

(a) The null solution of $(*_o)$ is h-asymptotically stable (h-completely unstable)

(b) the Poincaré index of the system

$$\begin{aligned} \dot{x} &= -\lambda y + X_2(x,y) + \ldots + X_h(x,y) \\ \dot{y} &= \lambda x + Y_2(x,y) + \ldots + X_h(x,y) \end{aligned} \qquad (**)$$

is h+1 and the relative constant negative (positive).

(c) $$\frac{\partial^i V}{\partial c^i}(0,0) = 0 \qquad i = 1,\ldots,h-1$$

$$\frac{\partial^h V}{\partial c^h}(0,0) < 0 \qquad (>0)$$

Moreover, if (a) holds, then h is odd.

Proof.

I report only the equivalence (a)$\Leftrightarrow$(b) in the case of h-asymptotic stability.

(a) $\Rightarrow$(b). When we set $\xi \equiv \tau \equiv 0$, the origin is asymptotically stable, so there exists a polynomial F (see Appendix) whose time derivative along the solutions of ($**$) satisfies:

$$\dot{F}(x,y) = G(x^2 + y^2)^{\frac{1}{2}M} + o(x^2 + y^2)^{\frac{1}{2}M}$$

where G is negative and M is an even integer. By (ii) in Def.4 $M \geqslant h+1$. It is not possible that $M \neq h+1$. In fact, let us set

$$\xi = ax(x^2 + y^2)^{\frac{1}{2}M}$$

$$\tau = ay(x^2 + y^2)^{\frac{1}{2}M}$$

with $a > -\frac{1}{2}G$. Now the time derivative along the solutions of ($*$)' is:

$$\dot{F}(x,y) = (2a + G)(x^2 + y^2)^{\frac{1}{2}M} + o(x^2 + y^2)^{\frac{1}{2}M}$$

hence the origin should be completely unstable: contradiction.

(b) $\Rightarrow$(a). By the hypothesis, a polynomial F exists such that:

$$\dot{F}(x,y) = G(x^2 + y^2)^{\frac{1}{2}(h+1)} + o(x^2 + y^2)^{\frac{1}{2}(h+1)} \qquad G < 0$$

Condition (i) of Def.2 is satisfied, and h is minimal: if not, the previous steep of this proof would generate a contradiction.

In what follows, I shall frequently refer to bifurcating periodic orbits by means of the bifurcation function μ as defined in § 1 in the statement of Thm.1. An orbit will be identified by a couple $(c, \mu(c))$, where c is allowed to vary on $]0, \varepsilon[$, $\varepsilon > 0$.

Next lemma introduces the theorem announced in § 2:

Lemma 1.

If 0 is asymptotically stable (completely unstable), then the bifurcating periodic orbits are attracting (repulsing) if and only if there exists $\varepsilon^* \in]0,\varepsilon[$ such that:

(i) the restriction of the bifurcation function is injective;

(ii) $\mu(c)\alpha'(0) > 0$ $\quad (\mu(c)\alpha'(0) < 0)$ on $]0,\varepsilon^*[$

Proof.

Let us consider the case of 0 asyptocally stable and $\alpha'(0) > 0$. $\mu(c)\alpha'(0) > 0$ implies $\mu(c) > 0$ in $]0,\varepsilon^*[$. Moreover 0 is completely unstable for $(*)$, because $\alpha'(0) > 0$ implies $\alpha(\mu) > 0$ for $\mu > 0$. Hence (see (11), (25),or §1 and §2) there exist two closed orbits of $(*)$, ~~for $0 <$~~ $\mu < \mu_1$, boundaries of two compact sets K_1 and K_2, for which holds:

$$0 \in \mathring{K}_1 \subset K_2 \subset B_{\varepsilon^*}(0)$$

and such that $K_1 \setminus \mathring{K}_2$ is compact, invariant, asymptotically stable. We may choose $\varepsilon_1 \in]0,\varepsilon^*[$ in such a way that

$$\sup\{\mu(c),\ c \in]0,\varepsilon_1[\} < \mu_1$$

If γ is the closed orbit passing through $(c_1,\mu(c))$, then $\gamma = \gamma_1 = \gamma_2$ because they are contained in $B_{\varepsilon^*}(0)$ and μ is one-one. By applying the results of previous sections we obtain the asymptotic stability of γ.

Now let us suppose the bifurcating orbits for $\mu \in]0,\varepsilon^*[$ asymptotically stable, and at the same time there exist $c_1 < c_2$ in $]0,\varepsilon^*[$ such that $\mu(c_1) = \mu(c_2) =: \mu$ and no other points in $]c_1,c_2[$ verifies $\mu(\bar{c}) = \mu$

Let us denote by γ_1 and γ_2 the orbits corresponding to $(c_1,\mu(c_1))$, $(c_2,\mu(c_2))$. For an arbitrary x in the annulus defined by γ_1 and γ_2,

$\Lambda^-(x)$ is a periodic orbit for $(*)$, distinct by γ_1 and γ_2, because $\Lambda^-(x)$ is repulsive: a contradiction, because γ_1 and γ_2 are consecutive.

Finally, let us suppose that μ has a zero in $]0, \varepsilon^*[$. Since $\alpha(\mu)$ is increasing in a neighborhood of 0, for $\alpha'(0) > 0$, the origin is asymptotically stable even for μ negative and small. Repeating the work performed above, we obtain the existence of a bifurcated orbit non asymptotically stable, which is against our hypothesis.

It is possible to give a definition of h-attractivity for sets, that is, an attractivity property which is not destryed by perturbations of order $>$ h.

Definition 5.

Let us denote by $S_h = S(X_2,\ldots,X_h,Y_2,\ldots,Y_h)$ the set of couples (P,Q) of funtions in $\mathcal{C}^{k+1}[(-\bar{\mu},\bar{\mu}) \times D_a, R]$ of order ≥ 2, such that:

$$[P(0,x,y)]_i = X_i(x,y) \qquad\qquad i = 2,\ldots,h$$
$$[Q(0,x,y)]_i = Y_i(x,y)$$

for $(P,Q) \in S_h$, $V_{P,Q}$, $\mu_{P,Q}$ will be the displacement and bifurcation functions for:

$$\begin{aligned} \dot{x} &= \alpha(\mu)x - \beta(\mu)y + P(\mu,x,y) \\ \dot{y} &= \alpha(\mu)y + \beta(\mu)x + Q(\mu,x,y) \end{aligned} \qquad (*)_{P,Q}$$

Let h be an integer, $3 \leq h \leq k$. The bifurcating periodic orbits are said to be h-attracting (h-repulsing) if:

(a) for any $(P,Q) \in S_h$ the periodic orbits of $(*)_{P,Q}$ are attracting;

(b) h is minimal with respect to (a).

Theorem 12.

The bifurcating periodic orbits of (*) are h-attracting (h-repulsing) if and only if 0 is asymptotically stable (h-completely unstable) for $\mu = 0$.

Proof. (sketch).

In this proof $\alpha'(0)$ is supposed to be positive. I report only the case of h-attractivity.

By Thm.   the zero solutions of

$$\dot{x} = -\lambda y + P(0,x,y)$$
$$\dot{y} = \lambda x + Q(0,x,y) \qquad (*_{o})_{P,Q}$$

where $(P,Q) \in S_h$ is h-asymptotically stable. Let us write μ for $\mu_{P,Q}$.

We have:

$$\mu(0) = \frac{\partial \mu}{\partial c}(0) = 0$$

Form the identity

$$V_{P,Q}(c, \mu(c)) = 0$$

we obtain

$$\frac{\partial^{s+1} V_{P,Q}}{\partial c^{s+1}}(0,0) = -(s+1)\frac{\partial^2 V_{P,Q}}{\partial c}(0,0)\,\mu^{(s)}(0)$$

It is possible to prove, applying Thm.   , that

$$\frac{\partial^i \mu}{\partial c^{s+1}}(0) = 0 \qquad i = 1,\dots,h-2$$

$$\frac{\partial^{h-1}}{\partial c^{h-1}}(0) > 0$$

Then μ is strictly increasing and such that $\mu(c)\alpha'(0) > 0$ on $]0,\varepsilon[$. By Lemma 1 the bifurcating orbita are attracting. Finally, by working as in Thm.11(part (a) $\Rightarrow$ (b)), we prove that for each odd integer j less than h-1 it is possible to find a couple $(P,Q) \in S_j$ such that 0 is completely unstable for $(*_{o})_{P,Q}$.

To prove the opposite implication we use the argument of the sufficiency: if Poincaré index of $(*_o)_{P,Q}$ is , then there exists $(P,Q) \in S_j$ such that the origin is j-completely unstable for $(*)_{P,Q}$. By the first part of this proof, the bifurcating closed orbits could be h-completely unstable, which is not possible. If the Poincaré index is $M < \infty$, then the origin is either asymptotically stable or completely unstable; by arguments similar to those ones used in the sufficiency we obtain $M = h+1$ and determine the sign of Poincaré's constant.

If the unperturbed system is defined by an analitic vector field, it is possible to improve the statement of the previous thm.:

Corollary 1.

If $f(\mu,x,y) \in \mathcal{C}^\infty(]-\bar{\mu},\bar{\mu}[\times D_a(0),R^2)$, $f(0,x,y)$ is (x,y)-analytic and $\alpha'(0) > 0$, (< 0), then:

(a) If 0 is asymptotically stable for $\mu = 0$, then the bifurcating closed orbits are attracting and exist only for μ positive (negative)

(b) If 0 is completely unstable for $\mu = 0$ then the bifurcating closed orbits are repulsing and exist only for μ negative (positive)

(c) 0 is stable but not attracting for $\mu=0$: the bifurcating closed orbits are stable but not attracting and exist only for $\mu = 0$.

If the unperturbed system is not analytic, the asymptotic stability of the origin for the perturbed system is not sufficient for the attractivity of the bifurcating orbits. A counterexample is given by:

$$\begin{aligned} \dot{x} &= \mu x - y - x(x^2 + y^2)f(x,y) \\ \dot{y} &= x + \mu y - y(x^2 + y^2)f(x,y) \end{aligned} \qquad (C_\mu)$$

where

$$f(x,y) = \begin{cases} e^{-\frac{1}{x^2+y^2}}(\sin(x^2 + y^2)^{-2} + 1) & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$$

The eigenvalue of (C_μ) satisfies the hypothesis of classical Hopf theorem with transversality, so we are sure of the existence of bifurcated periodic orbits.

A suitable Liapunov function for (C_μ) is

$$V(x,y) = \tfrac{1}{2}(x^2 + y^2)$$

whose time derivative along the solutions of (C_μ) is

$$\dot{V}(x,y) = (x^2 + y^2)(\mu - (x^2 + y^2)f(x,y))$$

which is negative definite in a neighborhood of 0. Hence the origin is asymptotically stable for $\mu = 0$. The bifurcated closed orbits are circumferences of radius c : $x^2 + y^2 = c^2$.

THe bifurcation equation associated to (C_μ) is

$$\mu(c) = c^2 e^{-\frac{1}{c^2}}(\sin^2 c^{-4} + 1)$$

whose derivative is

$$\mu'(c) = 2c^{-\frac{1}{c^2}-3}\,(c^4\sin^2 c^{-4} + c^4 + c^2\sin^2 c^{-4} + c^2 - 2\sin 2c^{-4})$$

In the points $c_n = \sqrt{2}\,(\pi(4n+1))^{-\frac{1}{6}}$ the derivative is negative.
Since 0 is a cluster point for the set $\{c_n\}_n$ and μ is positive on each right neighborhood of the origin, μ cannot be injective in $[0,\varepsilon[$ for any positive ε. Hence, bifurcating orbits cannot be asymptotically stable.

Let us pass to the problem of GHB in R^n. The objects of next theorems will be the "unperturbed" equation

$$\dot{z} = f_o(z) \qquad (+)$$

where $f_o \in \mathcal{C}^\infty(\overline{B_{a_o}(0)},R^n)$, $f_o(0) = 0$,

and the "perturbed" one

$$\dot{z} = f(z) \qquad (++)$$

in which f is closed to f_o in the topology of uniform convergence for all $\mathcal{C}^\ell$ norms. We suppose that $f_o'(0)$ has two purely imaginary eigenvalues $\pm i$ and no other eigenvalue is an integer multiple of $\pm i$. The problem of determining the number of bifurcated closed orbits contained in a neighborhood of 0 with period near to 2π was solved by Andronov, Gordon, Leontovich and Mayer (2) using stability techniques, helped in this by the theory of Poincaré-Bendixson. Their theory may be applied to n-dimensional problems when there exists a single pair of imaginary eigenvalues, because in this case there exists a bidimensional invariant manifold. In the general case, the problem can be faced by using suitable "quasi-invariant" manifolds as exposed in (7).

Before stating the main theorem I introduce precisely the definition of quasi-invariant manifold. After a linear change of coordinates, the systems (+) and (++) may be written in this way:

$$\dot{x} = -y + X_o(x,y,p)$$
$$\dot{y} = x + Y_o(x,y,p) \qquad (+)'$$
$$\dot{p} = A_o p + P_o(x,y,p)$$

$$\dot{x} = \alpha x - \beta y + X(x,y,p,f)$$
$$\dot{y} = \alpha y + \beta x + Y(x,y,p,f) \qquad (++)'$$
$$\dot{p} = Ap + P(x,y,f,p)$$

where X,Y,P are of order $\geqslant 2$ in x,y,p, and with obvious meanings for all the other terms involved.

Let

$$\Phi^{(h)}(x,y) = \Phi_1(x,y) + \dots + \Phi_h(x,y)$$

be a (n-2)-dimensional polynomial, whose homogeneous term of degree j is $\Phi_j(x,y)$. Let us set

$$\Psi(x,y,p) = p - \Phi^{(h)}(x,y)$$

and let us evaluate the time derivative of Ψ along the solutions of (+)' in the points of the manifold

$$p = \Phi^{(h)}(x,y)$$

The problem consists of determining $\Phi^{(h)}(x,y)$ in such a way that

$$\frac{d}{dt}\Psi(x,y,p)\Big|_{p=\Phi^{(h)}(x,y)} = o(x^2 + y^2)^{h/2}$$

It is possible to prove that such a polynomial exists and is unique, by using arguments very similar to those ones applied in the proof of Poincaré's procedure (see Appendix). I emphasize that there exists a recursive relationship between $\Phi^{(h)}$ and $\Phi^{(h+1)}$:

$$\Phi^{(h+1)} = \Phi^{(h)} + \Phi_{h+1}$$

Definition 6.

The bidimensional manifold

$$p = \Phi^{(h)}(x,y)$$

will be called quasi-invariant manifold of order h.

Consider now the reduced system

$$\begin{aligned} \dot{x} &= -y + X_o(x,y,\Phi^{(h)}(x,y)) \\ \dot{y} &= x + Y_o(x,y,\Phi^{(h)}(x,y)) \end{aligned} \qquad (S_h)$$

which may lead to two possible results:

I) an odd integer $h > 1$ exists, such that (S_h) is h-asymptotically stable or h-completely unstable;

II) case (I) does not happen.

This two occurences determine two opposite qualitative behaviours of the system in a neighborhood of the origin:

Theorem 13.

If (I) is true, then, setting $k = \frac{h-1}{2}$

(a_i) there exist $a_1 > 0$, $\delta_1 > 0$, N_1 neighborhood of f_o such that for each $f \in N_1$ there exist not more than K closed non trivial orbits contained in $B_{a_1}(0)$, with period $[2\pi - \delta_1, 2\pi + \delta_1]$.

(a_{ii}) for each integer $j \in [0,K]$, for each $a_2 \in]0,a_1[$, $\forall \delta_2 \in]0,\delta_1[$, $\forall N_2$ neighborhood of f_o, $N_2 \subset N_1$ there exists $f \in N_2$ with exactly j nontrivial orbits contained in $B_{a_2}(0)$, having period in $[2\pi - \delta_2, 2\pi + \delta_2]$.

(a_{iii}) $\forall \bar{a} \in]0, a_1[$, $\forall \bar{\delta} \in]0, \delta_1[$ there exists a neighborhood $\bar{N}$ of f_o, $\bar{N} \subset N_1$, such that if $f \in \bar{N}$ and γ is a closed orbit of (++) contained in $B_{a_1}(0)$ with period in $[2\pi - \delta_1, 2\pi + \delta_1]$, then it lies in $B_{\bar{a}}(0)$ and its period belongs to $[2\pi - \bar{\delta}, 2\pi + \bar{\delta}]$

If (II) is true, then

(A) for any integer $j > 0$, $\forall\, a > 0$, $\forall\, \delta > 0$ and for any neighborhood N_o of f_o, there exists $f \in N$ such that (++) has exactly j orbits contained in $B_a(0)$ with period $[2\pi - \delta, 2\pi + \delta]$.

The proof of this statement is rather involved, so it is divided in a sequence of proposition. Let us pave the way for the first of them.

By performing the change of coordinates

$$s = p - \phi^{(h)}(x,y)$$

(+)' and (++)' become:

$$\begin{aligned} \dot{x} &= -y + X_o^{(h)}(x,y,s) \\ \dot{y} &= x + Y_o^{(h)}(x,y,s) \qquad\qquad (+)'' \\ \dot{s} &= A_o s + W_o^{(h)}(x,y,s) \end{aligned}$$

$$\begin{aligned} \dot{x} &= \alpha x - \beta y + X^{(h)}(x,y,s,f) \\ \dot{y} &= \beta x + \alpha y + Y^{(h)}(x,y,s,f) \qquad\qquad (++)'' \\ \dot{s} &= As + W^{(h)}(x,y,s,f) \end{aligned}$$

where A and A_o are matrices and $X_o^{(h)}$, $X^{(h)}$, $Y_o^{(h)}$, $Y^{(h)}$, $W_o^{(h)}$, $W^{(h)}$ are terms of order $\geqslant 2$. In these new coordinates (S_h) appears as:

$$\dot{x} = -y + X_o^{(h)}(x,y,0)$$
$$\dot{y} = x + Y_o^{(h)}(x,y,0) \qquad (S_h)'$$

To control the period of bifurcating orbits we may change the scale of time for perturbed systems: we can replace t with t, to obtain:

$$\dot{x} = \varepsilon\left[\alpha x - \beta y + X^{(h)}(x,y,s,f)\right]$$
$$\dot{y} = \varepsilon\left[\beta x + \alpha y + Y^{(h)}(x,y,s,f)\right] \qquad (++)'''$$
$$\dot{s} = \varepsilon\left[As + W^{(h)}(x,y,s,f)\right]$$

There exist $a^* \in]0,a_o[$, $\delta^* > 0$, N^* neighborhood of f_o, for which:

(i) $\det(I - e^{2\pi\varepsilon A}) \neq 0 \qquad \forall\, \varepsilon \in]1-\delta^*,1+\delta^*[\,,\ \forall\, f \in N^*$

(ii) the solutions of $(++)'''$ $(x(t,x_o,y_o,s_o,\varepsilon,f),y(\dots),s(\dots))$ starting in $B_{a^*}(0)$ remain in $B_{a_o}(0)$ for any $\varepsilon \in]1-\delta^*,1+\delta^*[$, $f \in N^*$, $t \in [0,2\pi]$.

It is possible to find (x_o,y_o,s_o) in $B_{a^*}(0)$ such that

$$s(2\pi,x_o,y_o,s_o,\varepsilon,f) = s_o \qquad (s_1)$$

for each $\varepsilon \in]1-\delta^*,1+\delta^*[$, $f \in N^*$. In fact (s_1) is equivalent, by $(++)'''$, to:

$$F(x_o,y_o,s_o,\varepsilon,f) = 0 \qquad (F_1)$$

where

$$F(x_o,y_o,s_o) = (e^{2\pi\varepsilon A} - I)s_o + \varepsilon\int_0^{2\pi} e^{\varepsilon A(2\pi-r)}W^{(h)}(x(r),y(r),s(r),f)\,dr$$

and

$$F(0,0,0,1,f_o) = 0$$

because $W^{(h)}$ has order ≥ 2 with respect to (x,y,s).

Now, since

$$D_s F(0,0,0,1,f_o) = \det (e^{2\pi A_o} - I) \neq 0$$

we may apply the theorem of implicit function to obtain $a' \in]0,a^*[$, $\delta' \in]0,\delta^*[$, $N' \subset N^*$ and $\sigma \in \mathcal{C}(D_{a'}(0) \times]1-\delta',1+\delta'[\times N', R^{n-2})$, where $D_{a'}(0)$ is a disk in R^2, such that

$$\sigma(0,0,1,f_o) = 0$$

and in $B_{a'}(0) \times]1-\delta',1+\delta'[\times N'$ (s_1) is verified if and only if:

$$s_o = \sigma(x_o,y_o,\varepsilon,f)$$

Since we are looking fir periodic solutions of (++), we may restrict our research of solutions verifying relation (s_1), which will be called $(2\pi,s)$ solutions. Next step of the proof is the following:

Lemma 2.

In a neighborhood of 0, the origin is inner to the projection of a periodic orbit of (++)''' on the (x,y)-plane and touches each of the two axes in only two points.

Proof.

In the (x,y)-plane we may pass to polar coordinates (ρ,ϑ): the solutions being periodic, we must assume all values in $]0,2\pi[$ and the thesis will be proved. We have:

$$\dot{\vartheta} = \frac{x\dot{y} - y\dot{x}}{r^2} = \varepsilon\beta + \frac{\varepsilon x X^h(x,y,s,f) - \varepsilon y Y^{(h)}(x,y,s,f)}{r^2}$$

$\varepsilon\beta$ is strictly positive and the quotient at its right is infinitesimal, as r tends to 0, because the numerator is of order $\geq$ 3: this is sufficient to say that $\dot{\vartheta}$ is strictly positive in a neighborhood of 0.

Now, by a suitable application of the variation of constants to the first two equations in the system (++)"' and of the theorem of implicit function (I jump over this parte of the proof) we may prove the existence of $a'' \in \,]0,a'[$, $N'' \subset N'$ and a $\mathcal{C}^{\infty}$ real function ε defined on $B_{a''}(0) \times N''$ such that the periodic orbits of (++)"' are those solutions satisfying

$$x(2\pi, c, \varepsilon(c,f), f) = c$$

$c \in B_{a''}(0)$. So we may define the displacement function

$V: B_{a''}(0) \times N'' \longrightarrow R$

$$V(c,f) = x(2\pi, c, \varepsilon(c,f), f) = c$$

and look for its zeros.

Since the solutions of (++)"' are $\mathcal{C}^{\infty}$, we may write their Taylor's formula up to terms of arbitrary order:

$$\begin{aligned} x(t,c,f) &= u_1(t,f)c + u_2(c,f)c^2 + \dots + u_h(t,f)c^h + o(c^h) \\ y(t,c,f) &= v_1(t,f)c + v_2(c,f)c^2 + \dots + v_h(t,f)c^h + o(c^h) \\ s(t,c,f) &= w_1(t,f)c + w_2(t,f)c^2 + \dots + w_h(t,f)c^h + o(c^h) \end{aligned} \tag{0}$$

where $u_1(0,f) = 1$, $u_2(0,f) = \dots = u_h(0,f) = v_1(0,f) = \dots = v_h(0,f) = 0$ and $w_j(0,f) = w_j(2\pi,f)$ for $j \geq 1$. Since ε has continuous derivatives with respect to c, we have also:

$$\varepsilon(c,f) = \varepsilon_o(f) + \varepsilon_1(f)c + \dots + \varepsilon_h(f)c^h + o(c^h)$$

with $\varepsilon_o(f_o) = 1$.

It is possible to determine the first degree coefficients in c. In fact we have:

$$\frac{\partial w_1}{\partial t} = A_o w_1$$

hence:

$$w_1(t,f_o) = w_1(0,f_o)e^{A_o t}$$

and

$$w_1(2\pi,0) = w_1(0,f_o)e^{2\pi A_o} = w_1(0,f_o) = 0$$

that implies

$$w_1(t,f_o) = 0$$

By this, and recalling that $W^{(h)}(x,y,0,f_o)$ is of order greater than h, we have:

$$\frac{\partial w_2}{\partial t} = A_o w_2$$

by which, analogously to what done before:

$$w_2(t,f) = 0$$

Proceeding in the same way one arrives at the following relation:

$$w_1(t,f) = \dots = w_h(t,f) = 0$$

Now, to determine u_i and v_j we put s = 0 for $f_o = 0$ and obtain the system:

$$\begin{aligned} \dot{x} &= \varepsilon\,[-y + X_o^{(h)}(x,y,0)] \\ \dot{y} &= \varepsilon\,[\,x + Y_o^{(h)}(x,y,0)] \end{aligned} \qquad (S_h)''$$

which is $(S_h)'$ after the rescaling of time.

If the zero solution of (S_h) is h-asymptotically stable, it is possible to determine a polynomial (see Appendix):

$$F(x,y) = x^2 + y^2 + F_3(x,y) + \dots + F_{h+1}(x,y)$$

whose time-derivative along the solutions of (S_h)" is:

$$\dot{F} = \varepsilon\, G_{h+1}(x^2 + y^2)^{(h+1)/2} + o(x^2 + y^2)^{(h+1)/2}$$

with $G_{h+1} < 0$. Integrating over the interval $[0,2\pi]$ this last formula along the solution passing through the point $(x_o,y_o) = (c,0)$ we obtain:

$$F(x(2\pi,c,f_o),0) - F(c,0) = \int_0^{2\pi} \left[\varepsilon\, G_{h+1}(x^2(t)+y^2(t))^{\frac{1}{2}(h+1)} + o(x^2+y^2)^{\frac{1}{2}(h+1)}\right] dt$$

x and y can be substituted by the corresponding expressions in (0); by comparing terms of the same degree we have:

$$v_1(2\pi,f_o) = 1$$

$$v_j(2\pi,f_o) = 0 \qquad\qquad j = 2,\dots,h-1$$

$$v_h(2\pi,f_o) = \pi\, G_{h+1}$$

by which we may deduce the derivatives of the displacement function with respect to c:

$$\frac{\partial^i V}{\partial c^j}(0,f_o) = 0 \qquad\qquad j = 1,\dots,h-1$$

$$\frac{\partial^h V}{\partial c^h}(0,f_o) = h!\,\pi\, G_h < 0$$

By the continuity of $\partial^h V/\partial c^h$, there exist $a_1 \in\,]0,a''[$, $N_1 \subset N''$ such that $(\partial^h V/\partial c^h)(c,f) < 0$ on $B_{a_1}(0) \times N_1$. Rolle's theorem implies that $V(c,f)$ cannot have more than h-1 zeros in $B_{a_1}(0)$ distinct from the null one, since the origin is an equilibrium point of (++) for each f. To each periodic orbit there correspond two zeros of V, so (++)"' can have at most $\frac{1}{2}(h-1)$ closed orbits of period 2 , corresponde to close orbits of (++) with period near to 2π: (a_i) is proved.

As for (a_{ii}), let j be an integer in $[1,K]$ (case j=0 is trivial). Consider the following perturbation of (++)'''

$$\dot{x} = \varepsilon\left[-y + X_o^{(h)}(x,y,s) + \sum_1^j a_j x(X^2 + y^2)^{k-i}\right]$$
$$\dot{y} = \varepsilon\left[x + Y_o^{(h)}(x,y,s) + \sum_1^j a_j y(x^2 + y^2)^{k-i}\right] \qquad (++)^{iv}$$
$$\dot{s} = \varepsilon\left[A_o s + W_o^{(h)}(x,y,s)\right]$$

where the a_j's will be determined later. $V(c,a_1,\dots,a_j)$ denotes the displacement function associated to $(++)^{iv}$. By construction, we have:

$$V(c,0,\dots,0) = g_o c^h + o(c^h) \qquad g_o < 0$$

and by the continuity of V, there exists $\eta_1 > 0$ such that $|a_i| < \eta_1$, i=1,...,j, implies:

$$V(c_o,a_1,\dots,a_j) < 0$$

For j = 1, we may apply poincaré's procedure to the bidimensional system associated to $(++)^{iv}$ to determine a polynomial F_1, whose time derivative along the solutions of $(++)^{iv}$ starts with the term $(x^2 + y^2)^k$ of order h -1, with positive Poincaré constant g_1. So we may write, for $0 < a_1 < \eta_1$

$$V(c,a_1,0,\dots,0) = g_1 c^{h-2} + o(c^{h-2})$$

and, for a positive $c_1 < c_o$:

$$V(c_1,a_1,0,\dots,0) > 0$$

Iterating this procedure over all the indices, we prove the existence of a positive $\eta_2 < \eta_1$ such that $|a_i| < \eta_2$, i=2,...,j; we have:

$$V(c_1,a_1,\dots,a_j) > 0$$

So we may start again with the relation:

$$V(c,a_1,a_2,0,\dots,0) = g_2c^{h-4} + o(c^{h-4})$$

and go on in an analogous way, until we determine the existence of j positive numbers $\bar{c}_i$ such that

$$\begin{aligned} & c_i < \bar{c}_i < c_{i-1} \\ & V(\bar{c}_i,a_1,\dots,a_j) = 0 \end{aligned} \qquad i = 1,\dots,j$$

Each of them determine a couple of solutions of $V(c,a_1,\dots,a_j) = 0$

and, being 0 a root of order h-2j, we may state that they are the only positive solutions of $V(c,a_1,\dots,a_j) = 0$. Provided we choose c_o close enough to the origin, we have arbitrarily small $\bar{c}_i$'s. They correspond to the j periodic solutions of statement (a_{ii}), which is now proved.

Finally we have that, provided $\bar{c}$ is chosen sufficiently small, for any $c \quad B_{\bar{c}}(0)$:

$$|V(c,f_o)| \geq \mu c^h \qquad \mu > 0$$

By the continuity of V, $\forall\ c_1 \in B_c(0)$ there is a neighborhood $N_{c_1} \subset N_1$ such that all the zeros of V(c,f) contained in $B_{\bar{c}}(0)$ lie in $B_{c_1}(0)$, that is equivalent to (a_{iii}).

This part of the proof can be developed analogously in the case of h-complete instability .

It remains to prove the second part of the theorem. For a fixed positive integer j let us consider such a perturbation of (+)'

$$\dot{x} = -y + X_o^{(h)}(x,y,s) + bx(x^2+y^2)^{\frac{1}{2}(h-1)}$$
$$\dot{y} = x + Y_o^{(h)}(x,y,s) + by(x^2+y^2)^{\frac{1}{2}(h-1)} \qquad (++)^{v}$$
$$\dot{s} = A_o s + W_o^{(h)}(x,y,s)$$

where b is a constant, and h=2j+1. After a rescaling of time $(++)^{v}$ assumes the form:

$$\dot{x} = \varepsilon\,[-y + X_o^{(h)}(x,y,s) + bx(x^2+y^2)^{\frac{1}{2}(h-1)}]$$
$$\dot{y} = \varepsilon\,[x + Y_o^{(h)}(x,y,s) + by(x^2+y^2)^{\frac{1}{2}(h-1)}] \qquad (++)^{vi}$$
$$\dot{s} = \varepsilon\,[A_o s + W_o^{(h)}(x,y,s)]$$

For b=0 Poincaré's procedure applied to the bidimensional system associated to $(++)^{v}$ gains only null constants, because the system is neither asymptotically stable nor completely unstable; on the opposite side, for $b\neq 0$, there exists a $G_{h+1}\neq 0$ which determines the behaviour of the systems close to the given one. Hence we know that there exists a third system, close to $(++)^{vi}$ having exactly $j = \frac{1}{2}(h-1)$ periodic orbits in a neighborhood of its equilibrium point. This last system is the perturbation requested.

APPENDIX. POINCARÉ'S METHOD.

In his celebrated paper appeared in 1892 (23) Liapunov proved that the stability of an equilibrium point 0 can be recognized by an inspection of the spectrum of the linearized equation. In particular, if all the eigenvalues lie inthe left-hand half space of the complex plane, the equilibriumpoint is asymptotically stable; on the other hand, if one eigenvalue with positive real part exists, 0 is not even stable. The theorem of stability in the first approximation does not give us any information about the behaviour of the system in a neighborhood of 0 if one pair of imaginary eigenvalues exists. Poincarè's procedure allows usto carry on the analysis of stability even in this case, as it is shown by the following theorem.

Let us consider the equation

$$\dot{x} = -y + X(x,y)$$
$$\dot{y} = x + Y(x,y)$$

where X,Y are analytic functions of order ≥ 2 (that is, whose Taylor's formula starts with terms of degree ≥ 2; X_j, Y_j will denote their homogeneous terms of degree j). We define the function $V_m : U \to R$, U open set containing 0, by recurrence:

$$V = V_o = x^2 + y^2$$

$$V_{m+1} = V_m + f_{m+1}$$

where f_{m+1} is a homogeneous polynomial of degree m+1, such that $\dot{V}_{m+1}$, the time derivative of V_{m+1} along the solutions of (1) does not contain (m+1)-order terms; the following theorem is concerned with the existence and uniqueness of f_{m+1} and V_{m+1}. In what follows the symbol $[V]_j$ will denote the term of degree j in the Taylor's formula associated to V.

Theorem.

If V_m is such that:

$$\left[V_m\right]_2 = \dots = \left[V_m\right]_{m-1} = 0$$

then:

a) if m is odd, then there exists only one homogeneous form f_m such that:

$$\left[\dot{V}_m\right]_m = 0$$

b) if m is even, then there exist infinite forms f_m such that:

$$\left[\dot{V}_m\right]_m = G_m(x + y)^{m/2} :$$

Moreover, G_m is uniquely determined.

Sketch of the proof:

By induction. For m = 2 the statement is trivially true, so let us consider the case m 2. As a first step we look for an f_m (m-homogeneous) for which:

$$\left[\dot{V}_m\right]_m = 0$$

By the hypothesis of induction, we have already determined f_j, j=3,..,m-1 all verifying

$$\left[\dot{V}_j\right]_j = 0 \qquad j=1, \dots, m-1$$

So, when we derive V along the solutions of (1) and we impose V_m to be equal to the corrispondingterm of the right-hand side of (1), we obtain a partial differential equation of the following type:

$$\left(\frac{\partial f}{\partial y} x - \frac{\partial f}{\partial x} y \right) = W_m \qquad (2)$$

where W_m is an m-degree form. By substitutingin (2) a generic homogeneous polynomial of degree m we obtain a set of linear relations involving the unknown coefficients of f_m and those ones of W_m. Using linear algebra techniques it is possible to provethat the problem of existence of a solution for (2) is reduced to the analysis of the set of solutions os the algebraic equation:

$$D_m(\chi) = 0$$

where D_m is the detecminant of a system of linear algebraic equations. By combining algebraic and analytic considerations, it is easy to argue that if m is odd, then there exists only one f_m verifyng (2), while if m is even existence is not even assured. In the second case the next step is the research of a f_m which satisfies:

$$\left[\dot{V}_m\right]_m = G_m\ (x + y\ )^{m/2}$$

which leads to

$$\left(\frac{\partial f}{\partial y}\, x - \frac{\partial f}{\partial x}\, y\right) = W_m + G_m\ (x + y\ )m/2 \qquad (3)$$

Still passing through linear algebra one obtains the existence of a solution and a constant, G_m, uniquely defined by the algebraic algorithm, which allows also to deduce the existence of infinite polynomials verifying (3)..

If $G_m = 0$ the procedure goes on with higher order polynomials; if $G_m \neq 0$, our data are sufficient for an analysis of stability. It remains to see how to prove that G_m is indipendent of the choice of the f_j's, j=3,..,m-1, which are not uniqiely determined for even j's.

Let us suppose $G_m > G'_m > 0$ to be two constantsobtained by Poincare's method, associated to the polynomials V and V'. For V''= V, we may write:

$$V' = (G_m 1)(x + y\ ) + o(x + y\ ) \qquad (4)$$

$$V'' = G_m\ (x + y\ ) + o(x + y\ ) \qquad (5)$$

Since $G'_m > 0$, V satisfies Liapunov theorem about instability, so 0 is an unstable equilibrium point. On the other hand, if $(1, G'_m/G_m)$, the function V = V" - V' is positive definite, with time derivative negative definite, that implies the asymptotic stability of 0: a contradiction. By a similar argument it is possible to treat the case $G_m < G'_m < 0$.

By what precedes, two possibilities may happen:

A) The procedure is not finite, and we may write, formally:

$$V = x + y + f_3 + \ldots + f_m + \ldots .$$

which is a first integral of (1) (see (33)) . V satisfies the hypothesis of Liapunov' theorem on uniform stability in a neighborhood of 0.

B) The procedure is finite: there exists $m \in N$ and $G_m \neq 0$ such that for

$$V = V_m = x + y + f_3 + \dots + f_m$$

the derivatives along the solutions of (1) satisfies:

$$\dot{V} = G_m (x + y)^{m/2} + o(x + y)^{m/2}$$

Hence, if $G_m < 0$, (respectively $G_m > 0$) V and V verify the hypothesis of Liapunov theorem about asymptotic stability (complete instability).

In both cases we reach a conclusion about the behaviour of the system in a neighborhood of the equilibrium point.

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