SYMMETRY BREAKING AND RESTORATION IN CURVED SPACETIME

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I. INTRODUCTION

The effective potential has received a great deal of attention because it allows one to study spontaneous symmetry breaking beyond the tree - graph level, meanwhile the spontaneous symmetry breaking has proved to be extremely fruitful in mamy areas of physics. Since one cannot evaluate the effective potential exactly when the interaction exists, the usual approach is to resort to a loop expansion, i.e. , a perturbative method in which the functional method is widely used. In virtue of effective potential, the effects of non - Minkowskian spacetime topologic structure and spacetime curvature upon symmetry breaking have been treated in many articles recently. Both in finite temperature and compact spatial axis cases, the spontaneous symmetry breaking may be restored above a critical temperature or below a critical length, the mass pole being shifted when one loop radiative correction is taken into consideration. The restoration may occur even at the classical level when a field coupled with a background gravitational field. The vacuum instability or in slightly different terms, the phase transition from the symmetric phase to the broken symmetry phase and opposite process are dealt with in various situations.

The outline of this paper is as follows. In Ch.II the usual path integral formulation has been summarized by which one can know how to calculate effective potential up to one loop correction that is useful to discuss spontaneous symmetry breaking above classical level. In Ch.III the calculation of effective potential in non - Minkowskian topology is given where both finite temperature and compact spatial axis cases are considered and Zeta function, dimentional regularization as well as other mathematical methods are used. For the sake of convenience, only the example of self interacting scalar field is provided. The restoration of the spontaneous symmetry breaking beyond a critical temperature and critical length is discussed in terms of the effective potential approach. In Ch.IV the effects of curvature and temperature in the closed and open Einstein universes are considered. The critical radius of the static Einstein closed universe is computed. Both zero and finite temperature theory in static open Einstein universe are provided where Zeta function prescription and heat kernal mothod are used. In appendix technical tools are reviewd

II. EFFECTIVE POTENTIAL IN FLAT SPACETIME.

1. Path Integral Expression of Quantum Field Theory [1]–[5]

For simplicity, we start with an examination of the familiar case of a single scalar field, ϕ , whose dynamics are described by a Lagrangian density, $\mathcal{L}(\phi, \partial_\mu\phi)$, The generalization to more complicated cases is trivial. Consider the effect of adding to the Lagtangian density a linear coupling of ϕ with an external real c -number source J(x), the action becomes

$$S[\phi, J] = \int d^4x\, (\mathcal{L} + J(x)\phi(x)) \tag{2.1}$$

Define the generating functional Z[J], which is the vacuum transition amplitude in the presense of external source

$$Z[J] = \langle O_{out} | O_{in} \rangle_J = e^{\frac{i}{\hbar}W[J]} = N\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}S[\phi, J]} \tag{2.2}$$

The derivative of Z[J] give rise to the Green functions with n external legs

$$G^n(x_1, \cdots, x_n) := \langle 0 | T\phi(x_1)\cdots\phi(x_n) | 0 \rangle = \left(\frac{\hbar}{i}\right)^n \frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)} \tag{2.3}$$

Meanwhile, the generating functional of connected Green function W[J] can be expanded into a functional Talor series

$$W[J] = \sum_n \frac{1}{n!}\int d^4x_1 \cdots d^4x_n\, G_c^n(x_1 \cdots x_n)\, J(x_1)\cdots J(x_n) \tag{2.4}$$

The successive coefficients in this series are the connected Green functions, G_c^n , which is the sum of all connected Feynman diagrams with n external legs and can be derived inversely

$$G_c^n(x_1 \cdots x_n) = \left(\frac{\hbar}{i}\right)^{n-1} \frac{\delta^n W[J]}{\delta J(x_1)\cdots\delta J(x_n)} \tag{2.5}$$

The field induced by the presence of a given source J(x) is the classical background field in the sense that

$$\bar{\phi}(x) := \frac{\delta W[J]}{\delta J(x)} = \frac{\langle O_{out} | \phi(x) | O_{in} \rangle}{\langle O_{out} | O_{in} \rangle} \tag{2.6}$$

The effective action $\Gamma[\bar{\phi}]$ is defined through the Legendre transformation

$$\Gamma[\bar{\phi}] := W[J] - \int d^4x\, J(x)\bar{\phi}(x) \tag{2.7}$$

From this definition, it follows directly

$$\frac{\delta\Gamma[\bar{\phi}]}{\delta\bar{\phi}(x)} = -J(x) \tag{2.8}$$

The effective action can be expanded in a manner similar to that of eq.(2.4)

$$\Gamma[\bar{\phi}] = \sum_n \frac{1}{n!}\int d^4x_1 \cdots d^4x_n\, \Gamma^{(n)}(x_1, \cdots x_n)\bar{\phi}(x_1)\cdots\bar{\phi}(x_n) \tag{2.9}$$

The successive coefficients in this series are the 1PI Green functions (or proper vertices); $\Gamma^{(n)}$ is the sum of all one particle irreducible Feynman graphs with n external legs amputated. 1PI means such a kind connected graph that cannot be disconnected by cutting a single internal line. There is an alternative way to expand the effective action. Instead of expanding in powers of $\bar{\phi}$, one can expand it in powers of momentum (about the point where all external momenta vanish) . In configuration space, such an expansion looks like

$$\Gamma[\bar{\phi}] = \int d^4x \left[-V(\bar{\phi}) + \frac{1}{2}(\partial_\mu\bar{\phi})^2 Z(\bar{\phi}) + \cdots \right] \tag{2.10}$$

$V(\bar{\phi})$ is called the effective potential . Comparing the expansions (2.9) with (2.10), one is easy to know that the nth derivative of $V(\bar{\phi})$ is the sum of all 1PI Feynman graphs with n external legs of zero momentum. $\bar{\phi}(x)$ is arbitrary here, however in order to compute effective potential only, one can choose $\bar{\phi}(x)$ as a constant field $\hat{\phi}$

$$\Gamma[\hat{\phi}] = -V_{eff}(\hat{\phi})\int d^4x \tag{2.11}$$

or

$$V_{eff}(\hat{\phi}) = -\frac{1}{\Omega}\Gamma[\hat{\phi}] \tag{2.12}$$

where $\Omega = (2\pi)^4\delta^4(0)$ is the volume of spacetime.

Furthermore, the Fourier transformation of classical background field is

$$\bar{\Phi}(x) = \int \frac{d^4k}{(2\pi)^4} e^{-ikx} \tilde{\Phi}(k)$$

where $\tilde{\Phi}(k)$ is the Fourier components of $\bar{\Phi}(x)$. For constant field, one gets

$$\tilde{\Phi} = (2\pi)^4 \delta(0) \hat{\Phi} \tag{2.13}$$

The effective action is

$$\Gamma[\bar{\Phi}] = \sum_n \frac{1}{n!} \int d^4x_1 \cdots dx_n^4 \, \Gamma^{n}(x_1 \cdots x_n) \bar{\Phi}(x_1) \cdots \bar{\Phi}(x_n)$$

$$= \sum_n \frac{1}{n!} \int \frac{d^4k_1}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4} (2\pi)^4 \delta(\Sigma k_i) \tilde{\Gamma}^{(n)}(k_1 \cdots k_n) \tilde{\Phi}(k_1) \cdots \tilde{\Phi}(k_n) \tag{2.14}$$

For constant field

$$\Gamma[\hat{\Phi}] = \sum_n \frac{\hat{\Phi}^n}{n!} \tilde{\Gamma}^{n}(0) (2\pi)^4 \delta(0) \tag{2.15}$$

in which, $\tilde{\Gamma}^{(n)}(0)$ are the 1PI Green functions with all the vanishing external momentums , therefore they correspond to vacuum graphs. The effective potential can be expressed in the form

$$V_{eff}(\hat{\Phi}) = -\sum_{n=0}^{\infty} \frac{\hat{\Phi}^n}{n!} \Gamma^{(n)}(0) \tag{2.16}$$

In another words, effective potential is the generating functions of one particle irreducible Green functions at zero external momentum in the sense that

$$\tilde{\Gamma}^{(n)}(0) = - \frac{d^n V_{eff}(\hat{\Phi})}{d\hat{\Phi}^n} \Big|_{\hat{\Phi}=0} \tag{2.17}$$

As an example, let us consider a self interacting ϕ^4 scalar theory, the corresponding Feynman graphs are

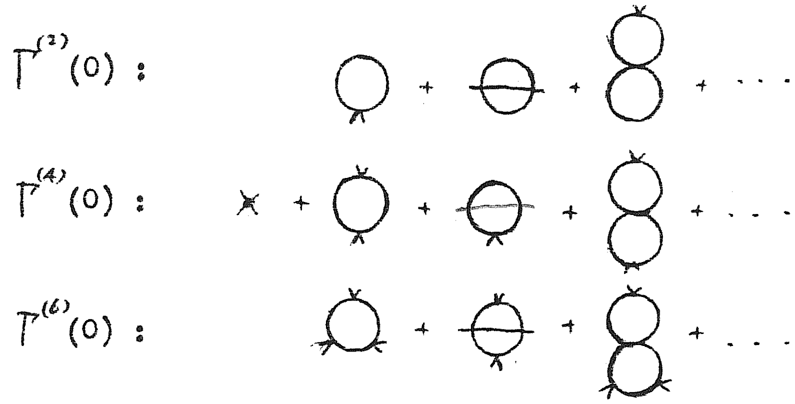


Fig. 1

If one expands V_{eff} in the powers of $\hbar$, i.e. loop expanding, one may classify all the vacuum graphs according to the loop number as shown in Fig.2

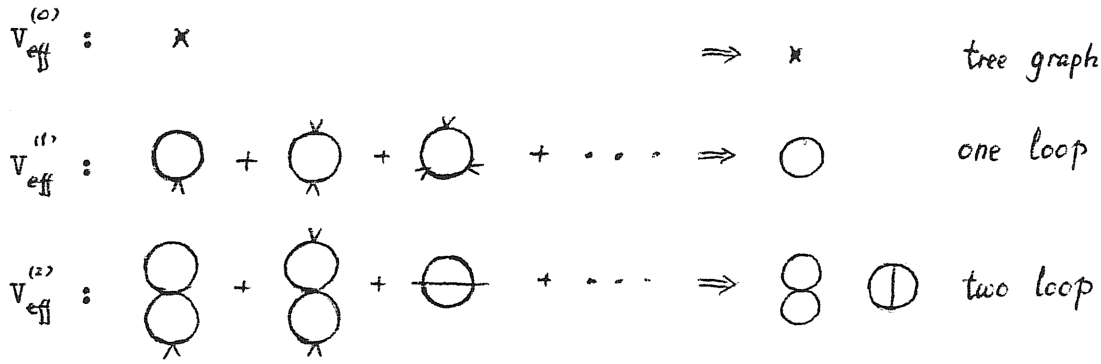


Fig.2

2. Effective Potential with One Loop Radiative Corrections

Using steepest - descent method, the saddle point ϕ_c is defined as

$$\left.\frac{\delta S[\phi]}{\delta\phi(x)}\right|_{\phi=\phi_c} = -J(x) \tag{2.18}$$

$\phi_c(x)$ is a functional of $J(x)$. It is the classical solution when the source term J vanishes. In the case of scalar field, one has

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - V(\phi)$$

$$(\Box + m^2)\phi_c + V'(\phi_c) = J(x) \tag{2.19}$$

then, one can shift the field $\phi(x) \to \phi(x) + \phi_c(x)$ and expand $Z[J]$ at the saddle point ϕ_c

$$Z[J] = N\int \mathcal{D}\varphi\, e^{\frac{i}{\hbar}S[\phi_c, J]}\, e^{\frac{i}{\hbar}\{\frac{1}{2}\int \frac{\delta^2 S}{\delta\phi\delta\phi}|_{\phi_c}\phi^2 + \sum_{P\geq 3}\int \frac{\phi^P}{P!}V^{(P)}(\phi)|_{\phi_c}\}} \tag{2.20}$$

where $V^{(P)}$ is the Pth derivative of potential. Using $i\Delta$ denote the prepergator and simplified notation in following way

$$\frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)} := \Delta^{-1}_{\phi_c}(x, y)$$

$$\int d^4x\, d^4y\, \frac{\delta^2 S}{\delta\phi(x)\delta\phi(y)}\phi(x)\phi(y) = \phi * \Delta^{-1}_{\phi_c} * \phi$$

one can rewrite (2.20) as follows

$$e^{\frac{i}{\hbar}W[J]} = e^{\frac{i}{\hbar}S[\phi_c J]} \cdot N\int \mathcal{D}\varphi\, e^{\frac{i}{\hbar}\{\frac{1}{2}\phi * \Delta^{-1}_{\phi_c} * \phi + \sum_{P\geq 3}\frac{\phi^P}{P!} * V^{(P)}(\phi)|_{\phi_c}\}} \tag{2.21}$$

in which only even p terms have contribution due to Wick integral,

$$\int \mathcal{D}\varphi\, e^{\frac{i}{\hbar}\frac{1}{2}\phi * \Delta^{-1}_{\phi_c} * \phi} = (\det \Delta^{-1})^{\frac{1}{2}} \tag{2.22}$$

By suitably defining N to absorb all constants, one gets

$$W[J] = S[\phi_c J] + \frac{i}{2}\hbar \ln \det \Delta^{-1}_{\phi_c} + W_2[J] \tag{2.23}$$

where

$$W_2[J] = -i\hbar \ln \frac{\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}\{\frac{1}{2}\phi * \Delta^{-1}_{\phi_c} * \phi + \sum_{P\geq 3}\frac{\phi^P}{P!}V^{(P)}|_{\phi_c}\}}}{\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}\frac{1}{2}\phi * \Delta^{-1}_{\phi_c} * \phi}}$$

If one rescales ϕ by putting $\phi \to \hbar^{\frac{1}{2}}\phi$, it is easy[1] to see $W_2[J]$ is of the order $\hbar^2$

$$W[J] = S[\phi_c J] + \frac{i\hbar}{2}\ln \mathrm{Det}\, \Delta^{-1}_{\phi_c} + O(\hbar^2) \tag{2.24}$$

and it can be proved $S[\bar{\phi}, J] - S[\phi_c J] = O(\hbar^2)$, after a cumbersome calculation which is omitted here, one can rewrite $W[J]$ as follows

$$W[J] = S[\bar{\phi}, J] + \frac{i\hbar}{2}\ln \mathrm{Det}\, \Delta^{-1}_{\bar{\phi}} + O(\hbar^2) \tag{2.25}$$

where $\bar{\Phi}$ is substituded for ϕ_c. By substraction $J(x)\bar{\Phi}(x)$ from $W[J]$, one can obtain effective action

$$\Gamma[\bar{\Phi}(x)] = S[\bar{\Phi}(x)] + \frac{i\hbar}{2} \ln \mathrm{Det}\, \Delta_{\bar{\Phi}}^{-1} + O(\hbar^2) \tag{2.26}$$

As mentioned above, the effective potential can be derived from $\Gamma[\bar{\Phi}]$ by setting $\bar{\Phi}(x)$ to be a constant field $\hat{\Phi}$

$$V_{eff}(\hat{\Phi}) = -\frac{1}{\Omega}\Gamma[\hat{\Phi}]$$

On the other hand, one can expand V_{eff} according to the powers of $\hbar$

$$V_{eff}(\hat{\Phi}) = V_{eff}^{(0)} + V_{eff}^{(1)} + V_{eff}^{(2)} + \cdots \tag{2.27}$$

Comparing (2.27) with (2.26), it is easy to get the effective potential at tree graph level as follows

$$V_{eff}^{(0)}(\hat{\Phi}) = -\mathcal{L}(\hat{\Phi}) = \frac{1}{2} m^2 \hat{\Phi}^2 + V(\hat{\Phi}) \tag{2.28}$$

Notice $\ln \mathrm{Det}\, \Delta_{\hat{\Phi}}^{-1} = \mathrm{Tr}\, \ln \Delta_{\hat{\Phi}}^{-1}$, where the capital letters are used to remind us both determinant and trace are taken with respect to momentum index as well as internal index. Due to translation invariance, the propergator $i\Delta_{\hat{\Phi}}$ is diagonalized in momentum space $\langle k' | \Delta_{\hat{\Phi}}^{-1} | k \rangle = \langle k | \Delta_{\hat{\Phi}}^{-1} | k \rangle \delta(k - k')$, and we know that so far as diagonalized matrix is concerned, the relation below is valid $\langle k | \ln \Delta_{\hat{\Phi}}^{-1} | k \rangle = \ln \langle k | \Delta_{\hat{\Phi}}^{-1} | k \rangle = \ln \Delta_{\hat{\Phi}}^{-1}(k)$. Therefore the functional determinant may easily be evaluated

$$\begin{aligned} \ln \mathrm{Det}\, \Delta_{\hat{\Phi}}^{-1} &= \mathrm{Tr}\ \delta(k-k') \ln \Delta_{\hat{\Phi}}^{-1}(k) \\ &= \int d^4k\, \delta(k-k')\Big|_{k'=k} \mathrm{tr} \ln \Delta_{\hat{\Phi}}^{-1} \end{aligned}$$

where the trace sums up the internal index only. Isolating the spacetime volume $(2\pi)^4 \delta^4(0)$ in $\Gamma[\hat{\Phi}]$, the second term of $\Gamma[\hat{\Phi}]$ turns out to be

$$\frac{i\hbar}{2} \int \frac{d^4k}{(2\pi)^4} \ln \det \Delta_{\hat{\Phi}}^{-1}(k) \cdot (2\pi)^4 \delta(0) \tag{2.29}$$

Finally, the one loop contribution to the effective potential may be derived in the form

$$V^{(1)}_{eff}(\hat{\phi}) = -\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ \ln\det\Delta^{-1}_{\hat{\phi}}(k) \tag{2.30}$$

3. An Example : Calculation of Effective Potential in ϕ^4 Theory

Let us summarize what have been said above: in order to get one loop correction of effective potential, first, one should shift field $\phi(x)\to\phi(x)+\hat{\phi}$ where $\hat{\phi}$ is a saddle point and chosen to be a constant field, so the Lagrangian $\mathcal{L}(\phi,\partial_\mu\phi)$ is shifted to $\mathcal{L}(\phi+\hat{\phi},\partial_\mu\phi)$; second, one can expand action around $\hat{\phi}$

$$S(\phi+\hat{\phi}) = S(\hat{\phi}) + \frac{\delta S}{\delta\phi}\Big|_{\hat{\phi}}\phi + S_0\{\hat{\phi},\phi\} + S_{int}\{\hat{\phi},\phi\} \tag{2.31}$$

to obtain the propergator $i\Delta(\hat{\phi}, x\ y)$ of the shifted field

$$\Delta^{-1}\{\hat{\phi}, x\ y\} = \frac{\delta^2 S_0\{\hat{\phi},\phi\}}{\delta\phi(x)\,\delta\phi(y)} \xrightarrow[\text{in momentum space}]{} \Delta^{-1}_{\hat{\phi}}(k) \tag{2.32}$$

where $S_0\{\hat{\phi}.\phi\}$ is the quadratic part of it; finally, the tree graph and one loop contribution of the effective potential are

$$V^{(0)}_{eff}(\hat{\phi}) = -\mathcal{L}(\hat{\phi})$$
$$V^{(1)}_{eff}(\hat{\phi}) = -\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ \ln\det\Delta^{-1}_{\hat{\phi}}(k) \tag{2.33}$$

Imitating these procedue, in the case of self- interacting scalar field with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

after shifting field $\phi\to\phi+\hat{\phi}$, one gets shifted Lagrangian

$$\mathcal{L}\{\hat{\phi},\phi\} = \mathcal{L}(\phi+\hat{\phi}) - \mathcal{L}(\hat{\phi}) - \frac{\partial\mathcal{L}}{\partial\phi}\Big|_{\hat{\phi}}\phi$$

$$= \frac{1}{2}\partial^{\mu}\phi\,\partial_{\mu}\phi - \frac{1}{2}(m^2+\frac{\lambda}{2}\hat{\phi}^2)\phi^2 - \frac{\lambda}{6}\hat{\phi}\phi^3 - \frac{\lambda}{4!}\phi^4 \qquad (2.34)$$

which means the constant terms and ϕ term are dropped from it. The inverse of the propergator is

$$\Delta^{-1}_{\hat{\phi}}(x,y) = \frac{\delta^2 S_c\{\hat{\phi}\,\phi\}}{\delta\phi(x)\delta\phi(y)} = (-\Box - M^2)\delta(x-y) \qquad (2.35)$$

$$M^2 := m^2 + \frac{\lambda}{2}\hat{\phi}^2$$

in momentum space, which yields

$$\Delta^{-1}_{\hat{\phi}}(k) = k^2 - M^2 \qquad (2.36)$$

The effective potential at classical level and one loop quantum correction are respectively

$$V^{(0)}_{eff}(\hat{\phi}) = \frac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4$$

$$V^{(1)}_{eff}(\hat{\phi}) = -\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ \ln\det\,(k^2 - M^2) \qquad (2.37)$$

For single neutraly scalar field, $\ln\det A = \mathrm{Tr}\ln A = \ln A$. By rotating the integral contour, eg.(2.37) may be integrated in Euclidean space and regularized by introducing a cutoff $k^2 = \Lambda^2$,

$$V^{(1)} = \frac{\hbar}{2}\,\frac{2\pi^2}{2}\int_0^{\Lambda^2}\frac{k_E^2\,dk_E^2}{(2\pi)^4}\ \ln\,(k_E^2 + M^2)$$

where unimportant constant is dropped. It is much easier to compute the integral

$$\frac{\hbar}{32\pi^2}\int_{M^2}^{\Lambda^2+M^2}(x\ln x - M^2\ln x)\,dx$$

The final result is

$$V^{(1)}_{eff} = \frac{\hbar}{64\pi^2}\left[\Lambda^4\ln(1+\frac{M^2}{\Lambda^2}) - M^4\ln(\frac{\Lambda^2}{M^2}+1) + M^2\Lambda^2\right] + O(\Lambda^4) \qquad (2.38)$$

4. Renormalization of the Effective Potential

Renormalizability means [6] that the divergences occured in perturbation theory can be cancelled by adding counterterms to $\mathcal{L}(x)$, of same forms of terms already in $\mathcal{L}(x)$. The counterterms can therefore be absorbed by a redefinition of the scale and of the parameters. In $\lambda\phi^4$ theory, the overall degree of divergence of $\Gamma^{(n)}$ is positive only for $\Gamma^{(2)}$ and $\Gamma^{(4)}$ and is furthermore a function only of the number of external legs and not of the order λ, so that it is renormalizable. In such a case, one can renormalize the theory by adding to $\mathcal{L}$ counterterms

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \mathcal{L}_{counterterms}$$

$$\mathcal{L}_{counter} = \frac{1}{2}(Z-1)\partial_\mu\phi\,\partial^\mu\phi - \frac{\delta m^2}{2}\phi^2 - \frac{\delta\lambda}{4!}\phi^4 \qquad (2.39)$$

where Z is $\hbar^2$ order because the lowest order graph in $\lambda\phi^4$ theory is two loops (fig.3),

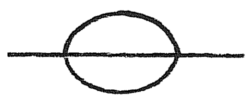


Fig. 3

hence the wave function renormalization counterterm can be ignored up to $O(\hbar)$ order.

Consequently, the renormalized one loop effective potential is

$$V_{eff}(\hat{\phi}) = \frac{1}{2}m^2\hat{\phi}^2 + \frac{1}{4!}\lambda\hat{\phi}^4 + \frac{\hbar}{64\pi^2}\left[\Lambda^4\ln\left(1+\frac{M^2}{\Lambda^2}\right) - M^2\ln\left(\frac{\Lambda^2}{M^2}+1\right) + \Lambda^2 M^2\right]$$

$$+ \frac{1}{2}\delta m^2\hat{\phi}^2 + \frac{1}{4!}\delta\lambda\,\hat{\phi}^4 + \delta C \qquad (2.40)$$

where δC is required for keeping V_{eff} finite when $\Lambda \to \infty$. The other counterterms are chosen so as to secure the normalization conditions

$$\left.\frac{dV_{eff}}{d\hat{\phi}}\right|_{\langle\phi\rangle} = 0$$
$$\left.\frac{d^2V_{eff}}{d\hat{\phi}^2}\right|_{\langle\phi\rangle} = m^2$$
$$\left.\frac{d^4V_{eff}}{d\hat{\phi}^4}\right|_{\langle\phi\rangle} = \lambda \tag{2.41}$$

where $\langle\phi\rangle$ is the normalization point and is conventionly taken to be the extremum of the effective potential. These conditions result from the constraints on $\Gamma^{(2)}$ and $\Gamma^{(4)}$ which are the only divergent loop diagrams in $\lambda\phi^4$ theory.

To introduce counterterms is equivalent to subtract divergent terms from V_{eff} which exist as a result from $\Gamma^{(2)}(0)$ and $\Gamma^{(4)}(0)$ (fig. 4).

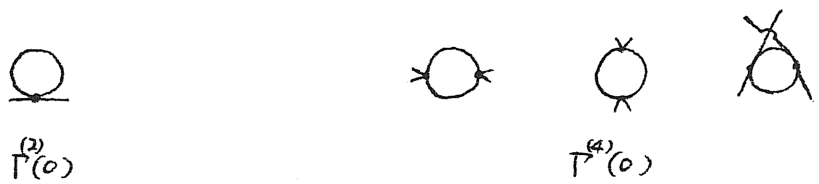


Fig. 4

Basing on eq.(2.16) and rewriting (2.37) as follows

$$V_{eff}^{(1)} = -\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\ln\left(1-\frac{\lambda\phi^2/2}{k^2-m^2}\right) \tag{2.42}$$

one can get the renormalized one loop correction of effective potential

$$V_{eff}^{(1)} = -\frac{i\hbar}{2}\int\frac{d^4k}{(2\pi)^4}\left[\ln\left(1-\frac{\lambda\hat{\phi}^2/2}{k^2-m^2}\right)+\frac{\hat{\phi}^2}{2}\frac{\lambda}{k^2-m^2}+\frac{\hat{\phi}^4}{4!}\frac{\lambda^2}{(k^2-m^2)^2}\times 3\right] \tag{2.43}$$

By performing Wick rotation $k_0 \to ik_4$, the integral may be evaluated in Euclidean space with the cutoff $k_E^2 = \Lambda^2$,

$$V_{eff}^{(1)} = \frac{\hbar}{2}\int\frac{d^4k_E}{(2\pi)^4}\left[\ln\left(1+\frac{\lambda\hat{\phi}^2/2}{k_E^2+m^2}\right)-\frac{\lambda\hat{\phi}^2/2}{k_E^2+m^2}+\frac{1}{2}\frac{(\lambda\hat{\phi}^2/2)^2}{(k_E^2+m^2)^2}\right] \tag{2.44}$$

One will find that the cutoff dependent terms being exactly cancelled each other . Dropping some unimportant constants, namely, cutoff and $\hat{\phi}^2$ independent terms, the final renormalized one loop contribution to the effective potential is[1][7]

$$V_{eff}^{(1)} = -\frac{\hbar}{64\pi^2}\left[M^4 \ln \frac{M^2}{m^2} - \frac{3}{2}(M^2 - \frac{2}{3}m^2)^2\right] \qquad M^2 = m^2 + \frac{\lambda\hat{\phi}^2}{2} \qquad (2.45)$$

If one adapt a little different convention of normalization conditions

$$\left.\frac{d^4V(\hat{\phi})}{d\hat{\phi}^4}\right|_{\hat{\phi}=M} = \lambda \qquad M \neq 0 \qquad (2.46)$$

where M being a nonvanishing normalization point, for a massless scalar field, the resulting $V_{eff}(\hat{\phi})$ is

$$V_{eff}(\hat{\phi}) = \frac{\lambda\hat{\phi}^4}{4!} + \frac{1}{64\pi^2}\left(\frac{\lambda\hat{\phi}^2}{2}\right)^2\left[\ln\frac{\hat{\phi}^2}{M^2} - \frac{25}{6}\right] + \cdots \qquad (2.47)$$

in agreement with ref. 8.

III. SYMMETRY RESTORATION AND MASS GENERATION DUE TO SPACETIME TOPOLOGY

1. Spontaneus Symmetry Breaking [9][10]

Generally speaking, for a system, the symmetry of the Lagrangian may not be just the same as that of the vacuum state. If the symmetry of the vacuum is smaller than that of the Lagrangian, the spontaneous symmetry breaking will occur.

Let us investigate spontaneous symmetry breaking in the case of scalar field $\phi(x)$, with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$$

and Hamiltonian

$$\mathcal{H} = \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + V(\phi) \tag{3.1}$$

in which the state of the lowest energy is a constant state denoted by $\langle\phi\rangle$. The value of $\langle\phi\rangle$ is determined by the location of the minima of the potential, V, so called the vacuum expectation value. If the potential is

$$V = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4 \tag{3.2}$$

where the coupling constant $\lambda > 0$, but the mass square of the field can be positive or negative. This Lagrangian admits the symmetry

$$\phi \longrightarrow -\phi$$

If $\mu^2 > 0$, the potential is as shown in fig.5a, the vacuum state locates at $\langle\phi\rangle = 0$. Since the vacuum possesses the same symmetry as the Lagrangian does, the theory have no spontaneous symmetry breaking.

If $\mu^2 < 0$, the potential is as shown in fig.5b, the derivation of it is

$$V' = \frac{\lambda}{6}\phi^3 + \mu^2\phi$$

which gives the minima as

$$\langle\phi\rangle = \pm\sqrt{\frac{-6\mu^2}{\lambda}} \tag{3.3}$$

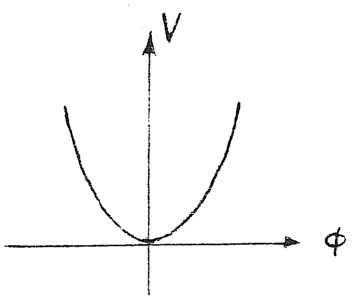
Let one define a new field by a translation

$$\phi' = \phi - \langle\phi\rangle$$

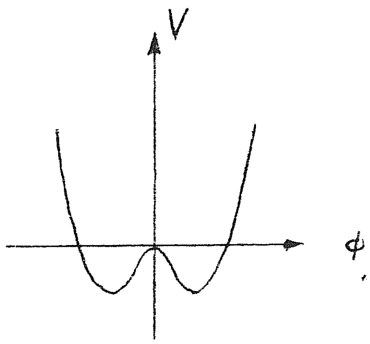
In terms of the new field, the potential is

$$U = \frac{\lambda}{6}\langle\phi\rangle^2\phi'^2 - \frac{\lambda}{6}\langle\phi\rangle\phi'^3 + \frac{\lambda}{4!}\phi'^4 \tag{3.4}$$

therefore, the true mass is $\frac{\lambda}{3}\langle\phi\rangle^2$ and the U has no the symmetry of $\phi \longrightarrow -\phi$ any longer, Since the vacuum state have no summetry $\phi \longrightarrow -\phi$, it leads to symmetry breaking spontaneously.



(a)



(b)

Fig. 5

2. Finite Temperature Field Theory [11]–[13]

The path integral which has been presented in Ch. II, provides an indefinite integral representation of the defferential equations of field theory. But the path integral itself does not contain a specification of the boundary conditions. The path integral prescription in which the corresponding Green functions of the differential equations satisfy the causal boundary conditions at $t = \pm\infty$ is usually called zero-temperature field theory. For example, the causal Green function in free scalar field theory is

$$\Delta(x-y) = \langle O_{out} | T\phi(x)\phi(y) | O_{in} \rangle = \frac{\int \mathcal{D}\phi\, \phi(x)\phi(y)\, e^{\frac{i}{\hbar}\int d^4x\, \mathcal{L}}}{\int \mathcal{D}\phi\, e^{\frac{i}{\hbar}\int d^4x\, \mathcal{L}}}$$

where $|0_{in}\rangle$ and $|0_{out}\rangle$ are the vacuum states corresponding to $t \to -\infty$ and $t \to +\infty$ respectively.

In order to carry out the calculation and improve the convergence property of the generating function (2.2) , it is custQmary to rotate the time integration contour of the generating function on imaginary time axis $x_0 \to -ix_4$ (or. $t \to -i\tau$) so that

$$\mathcal{L} \xrightarrow[t \to -i\tau]{} -\mathcal{H}_E$$

$$\frac{i}{\hbar} S[\phi] \xrightarrow[t \to -i\tau]{} \frac{1}{\hbar} S_E[\phi] \tag{3.5}$$

where

$$S_E[\phi] \equiv \int d^4x_E\, \mathcal{L}_E[\phi] = -\int d\tau\, dx^3\, \mathcal{H}_E \tag{3.6}$$

In Euclidean section, the generating functional of Eq.(2.2) can be read off

$$Z[J] = \int_{b.c} \mathcal{D}\phi\, e^{\frac{1}{\hbar}\left(S_E[\phi] + \int d\tau\, dx^3\, J\cdot\phi \right)} \tag{3.7}$$

where b.c. denotes a certain boundary condition. If we restrict the time integration in (3.6), (3.7) to finite interval $0 \leq \tau \leq \beta$, we shall get the transition amplitude between the vacuum state at $\tau = 0$ and the vacuum state at $\tau = \beta$. Imposing on the field periodic or antiperiodic boundary conditions: $\phi(\tau,\vec{x}) = \pm\, \phi(\tau \pm \beta\, ,\vec{x})$ according to boson or fermion field respectively, one can interpret (3.7) as the partition function for an equilibrium system of particles at the temperature β^{-1},

$$Z^{\beta}[J] = \int_{p.b.c} \mathcal{D}\phi \; e^{\int_0^{\beta} d\tau \int d^3x \, [-\mathcal{H}_E + J\cdot\phi]}$$

$$= T_r \left[e^{-\beta H} e^{\int_0^{\beta} d\tau \int dx^3 \, J\cdot\phi} \right] \tag{3.8}$$

where the natural unit $\hbar = c = 1$ is used and trace comes from the periodic boundary condition (p.b.c.) The corresponding Green function

$$G_{\beta}(x_1 \cdots x_n) = \frac{T_r [e^{-\beta H} \; T\phi(x_1) \cdots \phi(x_n)]}{T_r \, e^{-\beta H}} \tag{3.9}$$

and hence

$$W_E^{\beta}[J] = -\ln Z^{\beta}[J] \tag{3.10a}$$

$$\bar{\Phi}(x) = \frac{\delta W_E^{\beta}[J]}{\delta J(x)} \tag{3.10b}$$

$$\Gamma_E^{\beta}[\bar{\Phi}] = W_E^{\beta}[J] - \int d^4x_E \, \bar{\Phi}(x) J(x) \tag{3.10c}$$

where E denotes Euclidean spacetime and $W_E[J] = i\,W[J]$, $\Gamma_E[\phi] = i\,\Gamma[\phi]$ being understood. $\bar{\phi}(x)$ is the thermodynamic average of the field $\phi(x)$ when the external source J=0

$$\bar{\Phi}(x)\Big|_{J=0} = \frac{T_r \, e^{-\beta H} \, \phi(x)}{T_r \, e^{-\beta H}} \tag{3.11}$$

The effective potential is defined at a constant field $\hat{\phi}$

$$V_{\beta}(\hat{\Phi}) = -\frac{1}{\Omega_E} \Gamma_E^{\beta}(\hat{\Phi}) \tag{3.12}$$

where Ω_E is the valume of Eclidean spacetime manifold. Therefore the field theory with the compactified imaginary time axis, in which the wave functions satisfy the periodic boundary condition, equivalent to finite temperature field theory in the sence that the inverse of imaginary time period interval β^{-1} corresponds to the temperature.

It is now clear that for both theoris at zero and finite- temperatures we can use the same integral formulation but different boundary conditions, giving rise to different propergators. The periodic condition of the propergator in free scalar field theory $i\Delta_\beta(x) = \langle T\phi(x)\phi(0)\rangle$ is

$$i\Delta_\beta(x)\Big|_{\tau=0} = i\Delta_\beta(x)\Big|_{\tau=\beta} \tag{3.13}$$

Its Fourier components can be derived throuth Fourier series and integrals

$$i\Delta_\beta(x) = -\frac{1}{i\beta}\sum_n e^{-i\omega_n x_0}\int\frac{d^3k}{(2\pi)^3}e^{-i\vec{k}\cdot\vec{x}}\, i\Delta_\beta(\omega_n,\vec{k}) \tag{3.14a}$$

where the frequences

$$\omega_n = \frac{2\pi n}{-i\beta}, \qquad n = 0, \pm 1, \pm 2, \cdots \tag{3.14b}$$

are restricted by the boundary condition (3.13) , which implies that, in the field theory at finite temperature , not all solutions of the motion equation are permitted, only those wave functions which satisfy the periodic boundary condition are allowed. In other words, the periodic boundary condition selects some particularly wave functions from the whole set of the solutions of the motion equation, which reveals the influence of global topological structure of manifold to quantum effects.

The same situation happens in Casimir effect , i.e, the electromagnetic effects between two parallel neutral conducting plates at a distance L in vacuum. In this case, there exist many modes of vacuum state due to the quantum fluctuctions of the electromagnectic field, but only some of them which satisfy the vanishing boundary conditions are allowed. Such vacuum fluctuations produce an attractive force between the wall of the conderser in the slab shaped case and an repulsive force for a spherical shaped condenser. This phenomenon was studied in 1948 by H.G.Casimir and verified experimentally by Sparnay in 1958 first, later by the others. Therefore, the Casimir effects provide us a powerful and observational evidence for the influence from global spacetime structure on quantum effects.

Return now to scalar field case, we rewrite Eq.(3.14) in the form of compact notation

$$\Delta_\beta(x) = \int_k e^{-ikx} \Delta_\beta(k) \tag{3.15a}$$

$$k = \left(\frac{2\pi n}{-i\beta} , \vec{k} \right) \tag{3.15b}$$

$$\int_k = -\frac{1}{i\beta} \sum_n \int \frac{d^3k}{(2\pi)^3} \tag{3.15c}$$

in momentum space

$$i\Delta_\beta(k) = \frac{i}{k^2 - m^2} = \frac{-i}{\frac{4\pi^2 n^2}{\beta^2} + \vec{k}^2 + m^2} . \tag{3.16}$$

Furthermore, consider the self-interacting scalar field

$$\mathcal{L}(\phi) = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{\lambda}{4!} \phi^4 .$$

To compute effective potential $V_\beta(\hat{\phi})$, one first shifts the field $\phi(x) \rightarrow \phi(x)+\hat{\phi}$ and then gets the shifted Lagrangian

$$\begin{aligned} \mathcal{L}\{\hat{\phi},\phi\} &= \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{1}{2}(m^2 + \frac{\lambda}{2}\hat{\phi}^2)\phi^2 - \frac{1}{6}\lambda\hat{\phi}\phi^3 \\ &= \mathcal{L}_0\{\hat{\phi},\phi\} + \mathcal{L}_{int}\{\hat{\phi},\phi\} \end{aligned} \tag{3.17}$$

where $\mathcal{L}_0\{\hat{\phi},\phi\}$ is the quadratic part of the shifted Lagrangian and $\mathcal{L}_{int}\{\hat{\phi},\phi\}$ is the remainder part. The inverse of the propergator is

$$\Delta_\beta^{-1}(\hat{\phi}\ x\ y) = \frac{\delta^2 S_0\{\hat{\phi}\ \phi\}}{\delta\phi(x)\,\delta\phi(y)} = (-\Box - M^2)\,\delta(x-y)$$

$$\Delta_\beta^{-1}(\hat{\phi},k) = k^2 - M^2 \qquad M^2 = m^2 + \frac{\lambda}{2}\hat{\phi}^2 \tag{3.18}$$

The effective potential is given by

$$V_\beta(\hat{\phi}) = V_\beta^{(0)}(\hat{\phi}) + V_\beta^{(1)}(\hat{\phi}) - i\langle \exp\frac{i}{\hbar}\int d^4x\,\mathcal{L}_{int}\{\hat{\phi},\phi\}\rangle$$

$$V_\beta^{(0)}(\hat{\phi}) = \frac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 \tag{3.19}$$

$$V_\beta^{(1)}(\hat{\phi}) = -\frac{i\hbar}{2}\int_k \ln\det\Delta_\beta^{-1}(\hat{\phi},k) = \frac{\hbar}{2\beta}\sum_n\int\frac{d^3k}{(2\pi)^3}\ln\left(-\frac{4\pi^2n^2}{\beta^2} - E_M^2\right) \tag{3.20}$$

where
$$E_M^2 = \vec{k}^2 + M^2$$

The summation over n is divergent, which can be carried out in the following way. Defining

$$v(E) = \sum_n \ln\left(\frac{4\pi^2n^2}{\beta^2} + E_M^2\right) \tag{3.21}$$

$$\frac{\partial v(E)}{\partial E} = \sum_n \frac{2E}{4\pi^2n^2/\beta^2 + E^2} \qquad n = 0, \pm 1, \cdots \tag{3.22}$$

and using the following formular which can be proved by an infinite product of $\operatorname{sh}\pi y$

$$\sum_{n=1}^{\infty}\frac{y}{y^2+n^2} = -\frac{1}{2y} + \frac{\pi}{2}\coth\pi y \tag{3.23}$$

one can deduce that

$$\frac{\partial \upsilon(E)}{\partial E} = \frac{\beta}{\pi}\left(\pi \coth \pi \cdot \frac{E\beta}{2\pi}\right) = 2\beta\left(\frac{1}{2} + \frac{1}{e^{E\beta}-1}\right)$$

$$\upsilon(E) = 2\beta\left(\frac{E}{2} + \frac{1}{\beta}\ln(1-e^{-E\beta})\right) + \text{term indep. of } E \tag{3.24}$$

Correspondingly, one has

$$\begin{aligned} V_\beta^{(1)}(\hat{\phi}^2) &= \frac{\hbar}{2\beta}\int\frac{d^3k}{(2\pi)^3}\,\upsilon(E) + \text{infinite costants} \\ &= \hbar\int\frac{d^3k}{(2\pi)^3}\left(\frac{E_M}{2} + \frac{1}{\beta}\ln(1-e^{-E\beta})\right) \\ &= V_0^{(1)}(\hat{\phi}) + \overline{V}_\beta^{(1)}(\hat{\phi}) \end{aligned} \tag{3.25}$$

$$V_0^{(1)}(\hat{\phi}) = \hbar\int\frac{d^3k}{(2\pi)^3}\,\frac{E_M}{2}$$

$$\begin{aligned} \overline{V}_\beta^{(1)}(\hat{\phi}) &= \frac{\hbar}{\beta}\int\frac{d^3k}{(2\pi)^3}\ln(1-e^{-E\beta}) \\ &= \frac{\hbar}{2\pi^2\beta^4}\int_0^\infty dx\, x^2 \ln\left(1-e^{-(x^2+\beta^2M^2)^{1/2}}\right) \end{aligned}$$

where $V_0^{(1)}(\hat{\phi})$ is recognized as the zero-temperature one loop contribution basing on (2.37), because

$$-\frac{i}{2}\int_{-\infty}^{\infty}\frac{dk_0}{2\pi}\ln(-k_0^2+\vec{k}^2+M^2) = \frac{E}{2}$$

$V_0^{(1)}(\hat{\phi})$ in (3.25) is the one-loop potential at zero temperature, exactly the same as Eq.(2.45). The one loop potential $\overline{V}_\beta^{(1)}$ at a finite temperature can be evaluated in many ways. In order to compute β_c , we expand Eq.(3.26) with respect to small β ,

$$\overline{V}_\beta^{(1)}(\hat{\phi}^2) = -\frac{\pi^2}{90\beta^4} + \frac{M^2}{24\beta^2} - \frac{1}{12\pi}\frac{M^3}{\beta} - \frac{M^4}{64\pi^2}\ln M^2\beta^2 + \frac{M^4}{64\pi^2}\left(\frac{3}{2} - 2\gamma + 2\ln 4\pi\right) + O(M^6\beta^2) \quad (3.26)$$

[14]

where γ is Euler constant , defined by

$$\gamma = -\psi(1) = \lim_{n\to\infty}\left[\sum_{k=1}^{n-1}\frac{1}{k} - \ln n\right] \quad (3.27)$$

If a system possesses a spontaneous symmetry breaking at zero temprature, the zero temperature effective potential has a minimum at $\phi \neq 0$

$$\left.\frac{\partial V_c(\hat{\phi})}{\partial\hat{\phi}}\right|_{\hat{\phi}=\langle\phi\rangle} = 0$$

which is called a symmetry breaking phase. If the effective potential at a finite temperature has a minimum only at $\hat{\phi} = 0$

$$\frac{\partial V_\beta(\hat{\phi}^2)}{\partial\hat{\phi}} = 2\hat{\phi}\,\frac{\partial V^\beta(\hat{\phi}^2)}{\partial\hat{\phi}^2} = 0 \qquad \text{if } \hat{\phi} = 0$$

the broken symmetry will be restored when temperature becomes higher, at which the phase transition from symmetry breaking phase to symmetry one will ocuur. Symmetry breaking is absent when $\left.\frac{\partial V_\beta(\hat{\phi}^2)}{\partial\hat{\phi}^2}\right|_{\langle\phi\rangle} \neq 0$,and $\langle\phi\rangle \neq 0$. The restoration of symmetry *is assumed to require*

$$\left.\frac{\partial V_\beta(\hat{\phi}^2)}{\partial\hat{\phi}^2}\right|_{\hat{\phi}\neq 0} > 0 \quad (3.28)$$

Decomposing $V_\beta(\hat{\phi}^2)$ into $V_c(\hat{\phi}^2) + \overline{V}_\beta(\hat{\phi}^2)$, a zero temperature comtribution and finite temperature contribution, the necessary condition for symmetric restored becomes

$$\frac{\partial V_0(\hat{\phi}^2)}{\partial \hat{\phi}^2}\bigg|_{\hat{\phi}=0} + \frac{\partial \overline{V}_\beta(\hat{\phi}^2)}{\partial \hat{\phi}^2}\bigg|_{\hat{\phi}=0} \geqslant 0 \tag{3.29}$$

Noticing the mass term

$$m^2 = \frac{\partial^2 V_0(\hat{\phi}^2)}{\partial \hat{\phi}^2}\bigg|_{\hat{\phi}=0} = 2\frac{\partial V_0(\hat{\phi})}{\partial \hat{\phi}^2}\bigg|_{\hat{\phi}=0}$$

we get the following inequality from (3.29)

$$\frac{\partial \overline{V}_\beta(\hat{\phi}^2)}{\partial \hat{\phi}^2}\bigg|_{\hat{\phi}=0} \geqslant -\frac{m^2}{2} \tag{3.30}$$

and the critical temperature β_c^{-1}

$$\frac{\partial \overline{V}_\beta(\hat{\phi}^2)}{\partial \hat{\phi}^2}\bigg|_{\substack{\hat{\phi}=0\\ \beta=\beta_c}} = -\frac{m^2}{2} \quad , \tag{3.31}$$

which means that the symmetry is restored when the temperature β^{-1} exceedes β_c^{-1}. It follows that the critical temperature in the self-interacting scalar field is given by

$$\frac{\partial \overline{V}_\beta^{(1)}(\hat{\phi}^2)}{\partial \hat{\phi}^2}\bigg|_{\substack{\hat{\phi}=0\\ \beta=\beta_c}} = -\frac{m^2}{2} = \frac{\frac{\lambda}{2}}{24\,\beta_c^{\,2}} \tag{3.32}$$

so that

$$\frac{1}{\beta_c^{\,2}} = -\frac{24\,m^2}{\lambda} \tag{3.33}$$

We now turn to discuss the temperature dependence of the quartic self-coupling constant in ϕ^4 theory. The temperature correction to λ can be evaluate from the expresions (3.25) and (3.26). In fact, if $m^2 > 0$,

$$\lambda(\beta) = \left. \frac{d^4 V(\hat{\varphi}, \beta)}{d\hat{\varphi}^4} \right|_{\hat{\varphi}=c}$$

$$= \lambda - \frac{3\lambda^2}{16\pi\beta m} - \frac{3\lambda^2}{32\pi^2} \ln \frac{\beta^2 m_o^2}{16\pi^2} \tag{3.34}$$

where $m_o^2 = m^2 e^{2\gamma}$. If temperature is very high, we can neglect the logarithmic term, thereby $\lambda(\beta)$ goes to vanish and the theory is asymptotically free when the temperature near

$$T_c \equiv \frac{1}{\beta_c} \cong 16\pi m / 3\lambda \tag{3.35}$$

If $m^2 < 0$, spontaneous broken may ocuur at the tree level. In order to avoid the trouble connecting the imaginary terms in (3.34) , it would be better to adopt alternatively an off-shell renormalization point ϕ_o .

$$\lambda(\beta) = \left. \frac{d^4 V(\hat{\varphi}, \beta)}{d\hat{\varphi}^4} \right|_{\hat{\varphi}=\varphi_c} \tag{3.36}$$

where ϕ_c is chosen such that $\lambda\phi_o^2 >> |m^2|$. The one loop temperature dependence is finally found to be

$$\lambda(\beta) = \lambda + \frac{3\lambda^2}{32\pi^2} \ln \frac{16\pi^2}{\beta^2\sigma^2} \tag{3.37}$$

where $\varphi_o^2 = \sigma^2 \exp(-2\gamma - \frac{8}{3})$. It is therefore apprent that λ increases with temperature β^{-1} .

3. Field theory on $S^1 \times R^3$ manifold, compact spatial axis [15]-[25]

Following the line of the finite temperature case, we can also compactify one or more dimensions , namely, the spatical dimensions extend for a finite length and the extremes are considered coincident. If only one spatial axis is compactified, the manifold has a $S^1 \times R^3$ topology, which likes Minkowskian space locally,but different globally.

The quantum field theory with the compactifying time coordinate is usually called the finite temperature quantum field theory in the sence of its mathematic prescriptions being exactly the same as those in a finite temperature quantum field theory. We also find a close analogy between quantum field theory with a compactified time axis and one with a compactified space axis. This can be easily understood by means of Wick rotation $t \to -i\tau$. Hence the Euclidean Green function in $S^1 \times R^3$ manifold used here has the same properties as the thermal Green functions given above.

For a single self-interacting scalar field theory the Lagrangian density is as follows

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4$$

The flat spacetime with $S^1 \times R^3$ topology has the metric [26]

$$ds^2 = dt^2 - dx_1^2 - dx_2^2 - dx_3^2$$

where $-\infty < t,\ x_2,\ x_3 < +\infty$, whereas $0 \le x_1 \le L$ and the two points $x_1 = 0$, $x_2 = L$ is asummed to be identified. The $\phi(x)$ is a periodic function of S^1 so that the four-momentum k_μ has a discrete component $k_1 = \frac{2\pi n}{L}$, with $n = 0, \pm 1, \pm 2, \ldots$. The effective potential can be computed by the usual way given in Ch.II since the local dynamics of the field is unchanged by the closure of the x_1 axis. The effective potential up to one loop approximation is

$$V(\hat{\phi}) = V_0(\hat{\phi}) + V_1(\hat{\phi})$$

$$V_0(\hat{\phi}) = \frac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4$$

$$V_1(\hat{\phi}) = -\frac{i\hbar}{2}\int_k \ln(k^2 - m^2 - \frac{\lambda}{2}\hat{\phi}^2) , \quad \int_k = \frac{1}{(2\pi)^3 L}\sum_n \int dk_0 dk_1 dk_2 \tag{3.38}$$

performing Wick rotation $t \to -i\tau$, correspondingly $k_0 \to ik_4$, the one loop correction to the potential is

$$V_1(\hat{\phi}) = \frac{\hbar}{(2\pi)^3 2L}\sum_n \int d^3\tilde{k} \ln(k^2 + M^2) \tag{3.39}$$

where $d^3\tilde{k} = dk_2 dk_3 dk_4$, $k^2 = (\frac{2\pi n}{L})^2 + k_2^2 + k_3^2 + k_4^2$, $M^2 \equiv m^2 + \frac{\lambda}{2}\hat{\phi}^2$.
What we interest is the small limit, $L^2M^2 << 1$, which is analogue of the high temperature limit in finite temperature field theory. We can expand $V_1(\hat{\phi})$ into Taylor series around $L^2M^2 = 0$ up to first order [21]

$$V_1^L(\hat{\phi}) = \mathcal{V}_1^L(\hat{\phi}) + M^2L^2\,\mathcal{V}_2^L(\hat{\phi}) + O(L^4M^4) \tag{3.40}$$

where $\mathcal{V}_1^L$ and $\mathcal{V}_2^L$ are the expanding coeficents of following integrals

$$\mathcal{V}_1^L(\hat{\phi}) = V_1^L(\hat{\phi})\Big|_{L^2M^2=0} = \frac{1}{(2\pi)^3 2L}\sum_n \int d^3\tilde{k} \ln k^2$$

$$\mathcal{V}_2^L(\hat{\phi}) = \frac{\partial V_1^L(\hat{\phi})}{\partial(L^2M^2)}\Big|_{L^2M^2=0} = \frac{1}{(2\pi)^3 2L}\sum_n \int d^3\tilde{k}\, \frac{1}{k^2+M^2}\Big|_{L^2M^2=0} \tag{3.41}$$

Using dimensional regularization [27] [28] , these divergent integrals can be calculated in a D + 1 dimensional space by a substitution

$$\sum_n \int d^3\tilde{k} \longrightarrow \sum_n \int d^D k_E$$

Using following notation [18] [25]

$$I(D, \alpha, q, H) = \sum_n \int d^D k_E \frac{1}{(k_E^2 + 2q_E \cdot k_E + H + i\varepsilon)^\alpha} \tag{3.42}$$

where both k_μ and q_μ are D + 1 dimention vectors, $k^2 = k_{D+1}^2 + (\frac{2\pi n}{L})^2 + k_2^2 + \cdots + k_D^2$

and the dimension regularezation formular

$$I(\alpha,\beta) = \int d^D k_E \frac{(k_E^2)^\beta}{(k_E^2+H)^\alpha} = \frac{\pi^{D/2}\Gamma(\frac{D}{2}+\beta)\Gamma(\alpha-\beta-\frac{D}{2})}{\Gamma(\frac{D}{2})\Gamma(\alpha)} \frac{1}{H^{\alpha-\beta-\frac{D}{2}}} \tag{3.43}$$

then shifting coordinate origion in momentum space $k_\mu + q_\mu = k'_\mu$ and saparating discrete components $k^1 = \frac{2\pi n}{L}$ and q^1 from others, the integral can be evaluated as follows

$$I(D,\alpha,q\,H) = \frac{\pi^{D/2}\Gamma(\alpha-\frac{D}{2})}{\Gamma(\alpha)} \sum_n \frac{1}{[(\frac{2\pi n}{L}+q^1)^2+H-q^2]^{\alpha-D/2}}$$

$$= \pi^{D/2}\frac{\Gamma(\alpha-D/2)}{\Gamma(\alpha)}\left(\frac{2\pi}{L}\right)^{2(\frac{D}{2}-\alpha)} F(\alpha-\tfrac{D}{2};a,b) \tag{3.44}$$

(for the definition of $F(\lambda;a,b)$ and other details, see appendix A)

In our case, $a = \frac{LM}{2\pi}$, $b = 0$. By virtue of above notations $V_2^L(\hat{\Phi})$ can be expressed as

$$V_2^L(\hat{\Phi}) = \frac{1}{(2\pi)^3 2L} I(D,1,0,M^2) \tag{3.45}$$

The final result is (Appendix A)

$$L^2 M^2 V_2^L(\hat{\Phi}) = \frac{M^2}{24L^2} \tag{3.46}$$

In order to compute $V_1^L(\hat{\Phi})$, one may consider its derivative with respect to L

$$\frac{\partial}{\partial L}\sum_n \int d\tilde{k}\, \ln(\tilde{k}+M^2) = -\frac{2}{L}\sum_n \int d^3\tilde{k}\left(\frac{2\pi n}{L}\right)^2 \frac{1}{\tilde{k}^2+(\frac{2\pi n}{L})^2}$$

$$= -\frac{2}{L} I_{11}(D,1,0\,0) = -\frac{6(2\pi)^3}{L^4}\cdot\frac{\pi^2}{90} \tag{3.47}$$

Integration of (3.47) gives rise to

$$V_1^L(\hat{\Phi}) = \frac{\pi^2}{90L^4} + \frac{C}{L} \tag{3.48}$$

where C is a L independent integration constant and can be negligible based on the dimensional consideration. So the one loop effective potential in small L limit turns out to be

$$V_1^L(\hat{\phi}) = -\frac{\pi^2}{90L^4} + \frac{1}{24L^2}(m^2 + \frac{\lambda}{2}\hat{\phi}^2) + O(L^4M^4) \tag{3.49}$$

The first term depends on the global spacetime geometry, which corresponds to zero point energy density of free energy. The second term tells us that if the field is massless at tree graph level the one loop radiative correction will generate a mass owing to the nontrivial spacetime topology[15][16][22].

$$m^2 = \frac{\lambda}{24L^2} \tag{3.50}$$

In such a case, there exists no spontaneous breaking at tree level because it is massless. The one loop radiative correction may generate a positive mass for $\lambda > 0$, so that the symmetry is reserved even at higher loop and the vacuum $\hat{\phi} = 0$ is stable.

For the massive field case,[22] (3.49) shows that the one loop correction generates a shift mass owing to the self interaction and spacetime topology

$$\Delta m^2 = \frac{\lambda}{24L^2}$$

If $m^2 < 0$, there exists spontaneous symmetry breaking at tree level. As it is well known, the spontaneous symmetry breaking can be restored as soon as one loop correction is taken into account when the compact length is smaller than the critical length L_c, which is derived by

$$\left.\frac{\partial V^L(\hat{\phi}^2)}{\partial \hat{\phi}^2}\right|_{\substack{\hat{\phi}=0 \\ L=L_c}} = -\frac{m^2}{2}, \qquad L_c^2 = -\frac{\lambda}{24m^2} \tag{3.51}$$

(3.52)

This implies when the peoriodic interval of compact space axis is below L_c, the positive correction $\Delta m^2 = \frac{\lambda}{24L^2}$ becomes larger so that it may cancell the negative mass sguare term $m^2 < 0$, resulting a positive effective mass $m_R^2 = m^2 + \frac{\lambda}{24L^2} > 0$, consequantly spontaneous symmetry breaking disappears.

As a result, the vacuum state at tree level $\hat{\phi} = \pm\sqrt{\frac{-6m^2}{\lambda}}$ is no longer stable. The vacuum state will locate at $\hat{\phi} = 0$ instead of $\hat{\phi} \neq 0$ if the one loop correction is taken into consideration.

By compareing with the finite temperature field theory, one finds the extremely analogy between the inverse temperature β and L. Replacing L_c by β_c in (3.52), the (3.33) is again recovered.

4. Zeta Function Regulization

In order to calculate one or higher loop contributions to the effective potential, one must adopt a regularization precedure to remove the divergence. Hawking's ζ function regularized method is powerful as far as compact manifold is concerned. For the sake of convenience, it would be better to work on the Euclidean manifold at beginning. The action of a massless self interacting scalar field in euclidean manifold is

$$
\begin{aligned}
&\frac{i}{\hbar} S[\phi] \xrightarrow[t \to -i\tau]{} \frac{1}{\hbar} S_E[\phi] \\
&S_E[\phi] = \int_M \mathcal{L}_E(x)\, d^4x_E \\
&\mathcal{L}_E = -\frac{1}{2}\partial_a\phi\partial^a\phi - \frac{\lambda}{4!}\phi^4
\end{aligned}
\tag{3.53}
$$

The generating function and effective action is

$$
\begin{aligned}
&Z_E[J] = \int \mathcal{D}\phi\, e^{\frac{1}{\hbar}[S_E(\phi) + \int d^4x_E J(x)\phi(x)]} \\
&W_E[J] = iW[J] = -\hbar \ln Z_E[J] \\
&\Gamma_E[\bar{\phi}] = W_E[J] - \int_M d^4x_E J(x)\bar{\phi}(x) \\
&\frac{\delta W_E[J]}{\delta J} = \bar{\phi}(x)
\end{aligned}
\tag{3.54}
$$

The effective potential is

$$V(\hat{\phi}) = -\frac{1}{\mathrm{Vol}\,M}\Gamma_E[\hat{\phi}] = V^{(0)}(\hat{\phi}) + V^{(1)}(\hat{\phi}) + O(\hbar^2)$$

$$V^{(0)}(\hat{\phi}) = \frac{\lambda}{4!}\hat{\phi}^4$$

$$V^{(1)}(\hat{\phi}) = -\frac{\hbar}{\mathrm{Vol}\,M}\ln\int \mathcal{D}\phi\, e^{-\frac{1}{2}\int_M d^4x_E \int d^4y_E\, \phi(x)A(xy)\phi(y)}$$

(3.55)

$$A(xy) = \left(-\Box_x + \frac{\lambda}{2}\hat{\phi}^2\right)\delta(x-y)$$

(3.55)

When the background metric is real and positive , i.e. Euclidean, the operator A will be real elliptic and self - adjoint, therefore it has a complete spectrum of eigenvectors ψ_n with real eigenvalues λ_n

$$A\,\psi_n = a_n\psi_n \quad , \quad \int d^4x_E\, \psi_n\psi_m = \delta_{nm}$$

(3.57)

so that the field ϕ(x) can be expanded in terms of the ψ(x)

$$\phi(x) = \sum_n C_n\psi_n(x)$$

(3.58)

The measure $\mathcal{D}\phi$ on the space of all field ϕ can be expressed in terms of the coefficients C_n

$$\mathcal{D}\phi = \prod_n \frac{\mu}{\sqrt{2\pi}}\,dC_n \tag{3.59}$$

where μ is some normalization constant with mass dimention. From (3.37) and (3.59), it follows that

$$\begin{aligned} Z[\phi] &= \int \mathcal{D}\phi\, e^{-\frac{1}{2}\int d^4x_E\, \phi A\phi} \\ &= \prod_N \frac{\mu}{\sqrt{2\pi}}\,dC_n\, e^{-\frac{1}{2}a_n C_n^2} \\ &= \prod_n \mu\, a_n^{-\frac{1}{2}} \end{aligned}$$

(3.60)

and

$$V^{(1)}(\hat{\phi}) = \frac{1}{2\,\mathrm{Vol}\,M} \sum_N \ln \frac{a_N}{\mu} , \tag{3.61}$$

which is divergent, because the eigenvalues a_N increase without bound. In order to regularized it, one defines a generalized ζ function from the eigenvalues of the operator A

$$\zeta(s) = \sum_N a_N^{-s} \tag{3.62}$$

In four dimensions this will converge for Re s > 2, but it can be extended to whole s plane by analytical continuation, with poles only at s = 2 and s = 1 , in particular, it is analytic at s = 0. The gradient of Zeta at s = 0 is formally equal to $-\sum_N \log a_N$, thus the regularized expression of (3.61) turns out to be

$$V^{(1)}(\hat{\phi}) = -\frac{1}{2\,\mathrm{Vol}\,M}\left[\zeta'(0) + \zeta(0)\ln\mu^2\right] \tag{3.63}$$

Consider[17] flat spacetime with a given topology of $S^1 \times S^1 \times S^1 \times S^1$ by making periodic identification in each of the coordinates. The periodecities in the time and space coordinates are taken to be β, L_1 , L_2 , L_3 respectively; then the volume of the manifold is Vol M = $\beta L_1 L_2 L_3$. The eigenvalues a_N of operator A appeared in (3.62) are

$$a_N = \frac{\lambda}{2}\hat{\phi}^2 + \left(\frac{2\pi n_1}{L_1}\right)^2 + \left(\frac{2\pi n_2}{L_2}\right)^2 + \left(\frac{2\pi n_3}{L_3}\right)^2 + \left(\frac{2\pi n_4}{L_4}\right)^2 \tag{3.64}$$

where N stands for the set (n_1, n_2, n_3, n_4) which takes on all integer values respectively.

The effective potential in the case of $S^1 \times R^3$ manifold, may be easily obtained from $S^1 \times S^1 \times S^1 \times S^1$ case by assuming β, L_2 , L_3 >> L_1 and taking the limit β , L_2 , $L_3 \longrightarrow \infty$. Therefore, the ζ function in $S^1 \times R^3$ manifold turns out to be

$$\zeta(s) = \left(\frac{L_2}{2\pi}\right)\left(\frac{L_3}{2\pi}\right)\left(\frac{\beta}{2\pi}\right)\int d^3\tilde{k}\left[\frac{\lambda}{2}\hat{\phi}^2 + \left(\frac{2\pi n}{L_1}\right)^2 + \tilde{k}^2\right]^{-s} . \tag{3.65}$$

where $\tilde{k}^2 = k_2^2 + k_3^2 + k_4^2$. Introduing D + 1 dimension momentum space and using (3.43), one obtains

$$\int d^D k \frac{1}{(k^2+a^2)^S} = \frac{\pi^{D/2}}{2^D} \frac{\Gamma(S-D/2)}{\Gamma(S)} \frac{1}{a^{2S-D}}$$

$$a^2 = \frac{\lambda}{2}\hat{\phi}^2 + \left(\frac{2\pi n}{L_1}\right)^2 .$$

Finally, we get

$$\begin{aligned} \zeta(s) &= \frac{L_2 L_3 \beta}{4\pi^2} \frac{\Gamma(\frac{3}{2})\Gamma(s-\frac{3}{2})}{\Gamma(s)} \left(\frac{L_1}{2\pi}\right)^{2s-3} \sum_{n=-\infty}^{\infty} \frac{1}{(n^2 + \lambda L_1^2 \hat{\phi}^2/8\pi^2)^{s-3/2}} \\ &= \frac{L_2 L_3 \beta}{4\pi^2} \left(\frac{2\pi}{L_1}\right)^{3-2s} \frac{\Gamma(\frac{3}{2})\Gamma(s-\frac{3}{2})}{\Gamma(s)} D(s-\tfrac{3}{2}, \nu) \end{aligned} \tag{3.66}$$

where $\nu^2 = \frac{\lambda L_1^2 \hat{\phi}^2}{8\pi^2}$ (for $D(s,\nu)$, see appendix B).

To calculate $\zeta(0)$ and $\zeta'(0)$, we first calculate $\zeta(\varepsilon)$ in the limit $\varepsilon \to 0$ (see (3.68)) and then compare it with the following expansion,

$$\zeta(\varepsilon) = \zeta(0) + \zeta'(0)\cdot\varepsilon \tag{3.67}$$

The $\zeta(\varepsilon)$ is given by

$$\zeta(\varepsilon) = \frac{L_2 L_3 \beta}{4\pi^2} \Gamma(\tfrac{3}{2}) \frac{(\frac{2\pi}{L_1})^3 (1 - 2\varepsilon \ln\frac{2\pi}{L_1}) \Gamma(-\frac{3}{2})(1+\psi(-\frac{3}{2})\varepsilon)}{\Gamma(\varepsilon)} \left(\frac{D_1(2,\nu)}{\varepsilon} + D_0(2,\nu) + O(\varepsilon)\right) \tag{3.68}$$

It results in

$$\zeta(0) = \frac{L_2 L_3 \beta}{4\pi^2} \left(\frac{2\pi}{L_1}\right)^3 \Gamma(\tfrac{3}{2})\Gamma(-\tfrac{3}{2}) D_1(2,\nu) = \frac{L_1 L_2 L_3 \beta}{128\pi^2} \lambda^2 \hat{\phi}^4 \tag{3.69}$$

$$\zeta'(0) = \frac{4\pi^2 L_2 L_3 \beta}{3 L_1^3} \left[D_0(2,\nu) + \left(\frac{8}{3} - \ln\frac{16\pi^2}{L_1^2}\right) \frac{3\lambda^2 L_1^4}{512\pi^4} \hat{\phi}^4 \right] \tag{3.70}$$

Substituting (3.69), (3.70) into (3.63), the renormalized effective potential up to one loop correction is

$$V(\hat{\phi}) = \frac{\lambda+\delta\lambda}{4!}\hat{\phi}^4 - \frac{1}{64\pi^2}\left(\frac{\lambda\hat{\phi}^2}{2}\right)^2 \ln\mu^2 - \frac{1}{64\pi^2}\left(\frac{\lambda\hat{\phi}^2}{2}\right)^2\left[\frac{8}{3} - \ln\frac{16\pi^2}{L^2}\right] - \frac{2\pi^2}{3L_1^4} D_0(2;\nu) \tag{3.71}$$

The divergent term in (3.71) is L_1 independent, so that the renormalization procedue can be carried out as in Minkowskian spacetime in the limit $L_1 \rightarrow \infty$, by virtue of (B.17)

$$V(\hat{\phi}) \xrightarrow[L_1\rightarrow\infty]{} \frac{\lambda+\delta\lambda}{4!}\hat{\phi}^4 - \frac{1}{64\pi^2}\left(\frac{\lambda\hat{\phi}^2}{2}\right)^2 \ln\mu^2 + \frac{1}{64\pi^2}\left(\frac{\lambda\hat{\phi}^2}{2}\right)\left(\ln\frac{\lambda}{2}\hat{\phi}^2 - \frac{1}{2}\right) \tag{3.72}$$

Then normalization condition (2.46) can be imposed to (3.72) to fix counter term

$$\frac{\delta\lambda}{4!} = \frac{3\lambda^2}{512\pi^2} - \frac{\lambda^2}{256\pi^2}\left[\ln\frac{\frac{\lambda}{2}M^2}{\mu^2} + \frac{25}{6}\right] \tag{3.73}$$

Inserting (3.73) back into (3.71), the renormalized effective potential turns out to be

$$V(\hat{\phi}) = \frac{\lambda}{4!}\hat{\phi}^4 - \frac{\lambda^2}{48\pi^2}\phi^4 - \frac{2\pi^2}{3L_1^4}D_0(2;\omega) - \frac{\lambda^2}{256\pi^2}\phi^4\ln\frac{\lambda L_1^2 M^2}{32\pi^2} \tag{3.74}$$

The energy density of the vacuum state $\hat{\phi}=0$ can be derived imediately from (3.74)

$$V(\hat{\phi}=0) = -\frac{\pi^2}{90L_1^4} \tag{3.75}$$

and the topological mass is

$$m^2 = \left.\frac{d^2V(\hat{\phi})}{d\hat{\phi}^2}\right|_{\hat{\phi}=0} = \frac{\lambda}{24L_1^2} \tag{3.76}$$

So everything is in accordance with what we have obtained in precious section.

For a massive self interacting scalar field [22], by using the same procedue as in massless case a similar result can be obtained. The renormalized one loop contribution to the effective potential , which bears a resemblance to (3.71),is

$$V(\hat{\phi}) = \frac{1}{2}(m^2+\delta m^2)\hat{\phi}^2 + \frac{1}{4!}(\lambda+\delta\lambda)\hat{\phi}^4 - \frac{1}{64\pi^2}\left(m^2+\frac{\lambda\hat{\phi}^2}{2}\right)^2 \ln\mu^2$$

$$-\frac{1}{64\pi^2}\left(m^2+\frac{\lambda}{2}\hat{\phi}^2\right)^2\left(\frac{8}{3}-\ln\frac{16\pi^2}{L^2}\right) - \frac{2\pi^2}{3L^4}D_0\left(2;\frac{L}{2\pi}\sqrt{m^2+\frac{\lambda\hat{\phi}^2}{2}}\right) + \delta C$$

(3.77)

where δC is a global counterterm to regularize energy density of $\hat{\phi} = 0$ state finite as before. Imposing normalization condition

$$\left.\frac{dV}{d\hat{\phi}}\right|_{\hat{\phi}=v} = 0$$

$$\left.\frac{d^4V}{d\hat{\phi}^4}\right|_{\hat{\phi}=v} = \lambda$$

$$\left.\frac{d^2V}{d\hat{\phi}^2}\right|_{\hat{\phi}=v} = -2m^2$$

(3.78)

where the normalization point is chosen to be one of the minima of potential $V(\hat{\phi})$

$$v = \sqrt{\frac{-6m^2}{\lambda}}$$

(3.79)

the counterterms can be fixed and the renormalized effective potential turns out to be

$$V(\hat{\phi}, L) = V(\hat{\phi}) - \frac{M^4}{64\pi^2}\left(\ln L^2M^2 + \frac{7}{6} - 2\ln 4\pi\right) - \frac{2\pi^2}{3L^4}D_0\left(2;\frac{LM}{2\pi}\right) \qquad (3.80)$$

$$V(\hat{\phi}) = \frac{1}{2}m^2\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 + \frac{M^4}{64\pi^2}\ln\frac{M^2}{-2m^2} - \frac{\lambda m^2}{64\pi^2}\hat{\phi}^2 - \frac{1}{64\pi^2}\frac{3}{2}M^2$$

(3.81)

where $M^2 = m^2 + \frac{\lambda}{2}\hat{\phi}^2$, $V(\hat{\phi})$ is the usual one loop effective potential which is independent of the non - trivial spacetime topology.

In order to know the behaviour of $V(\hat{\phi}\,; L)$ for small L, we expand $D_0(2, \nu)$ as well as $V(\hat{\phi}\,; L)$ in small L limit ($\nu \ll 1$)

$$V(\hat{\phi}; L) \approx V(\hat{\phi}) - \frac{\pi^2}{90L^4} + \frac{1}{24\lambda^2}M^2 - \frac{M^3}{12\pi L} + \frac{M^4}{64\pi^2}\left(\ln\left(\frac{4\pi}{LM}\right)^2 + \frac{3}{2} - .2\gamma\right)$$

(3.82)

where γ is the Euler constant. (3.82)shows that the minimum of the effective potential occurs at $\hat{\phi} = 0$ again in small L limit even if it locates at $\hat{\phi} \neq 0$ at tree level when $m^2 < 0$, that is to say, the spontaneous symmetry breaking is restored by the radiative corrections, i.e., a phase transition from spontaneous symmetry breaking phase to symmetry phase occurs. The renormalized mass term can be defined as

$$\left.\frac{d^2 V(\hat{\phi}; L)}{d\hat{\phi}^2}\right|_{\hat{\phi}=0} = m_R^2 = m^2 + \frac{\lambda}{24L^2} \tag{3.83}$$

where the last term corresponds to the topological mass as mentioned before. The critical value L_c which characterizes the transition between breaking symmetry phasé and symmetry preserving phase can be decided in such a way that the effective mass term vanishes

$$m_R^2 = 0$$

which gives rise to

$$L_c^2 = -\frac{\lambda}{24m^2} \tag{3.84}$$

in accordance with what we have got before once again.

Repeating the same precedue and calculations in previous section, we can also investigate the spatial periodic interval L dependence of the coupling constant $\lambda(L)$: if $m^2 < 0$, we will find $\lambda(L)$increasing while the L decreases.

IV. QUANTUM FIELD THEORY IN CURVED SPACETIME

1.Static Closed Einstein Universe [16] [31]-[34]

The static Einstein universe is an example in which both the curvature and non-Minkowskian topology are present. Let us consider a more general case of a conformally coupled scalar field theory with the following Lagrangian

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{12} R\phi^2 - V(\phi)$$

$$V(\phi) = \frac{\lambda}{4!}\phi^4 + \frac{\delta\lambda}{4!}\phi^4 \qquad (4.1)$$

where R is the Ricci scalar of the classical background geometry. Such a massless theory is conformally invariant when the potential $V(\phi) = 0$. In this case, we may need to add a $\frac{R}{12}\hat{\phi}^2$ term into the effective potential given in the previous chapter.

$$Z[J] = \int \mathcal{D}\phi \, e^{\frac{1}{\hbar}\int d^4x \sqrt{g}\,(\mathcal{L} + J\phi)}$$

$$V^{(1)}(\hat{\phi}) = -\frac{1}{V_0 \rho M} \ln \int \mathcal{D}\phi \, e^{-\frac{1}{2}\int d^4x\sqrt{g}\int d^4y\sqrt{g}\,\phi(x) A(x,y)\phi(y)} \qquad (4.2)$$

$$A(x,y) = \left(-\Box + \frac{\lambda}{2}\hat{\phi}^2 + \frac{1}{6}R\right)\delta(x-y)$$

The generalized ζ function can be defined by $\zeta(s) = \sum_N a_N^{-s}$,where a_N are the eigenvalues of operator A(x,y). The effective potential then turns out to be

$$V(\hat{\phi}) = \frac{R}{12}\hat{\phi}^2 + \frac{\lambda+\delta\lambda}{4!}\hat{\phi}^4 - \frac{1}{2V_0\rho M}\left[\zeta'(0) + \zeta(0)\ln\mu^2\right] \qquad (4.3)$$

The metric of the static closed Einstein universe in Friedman coordinates is

$$ds^2 = -dt^2 + a^2\left[d\chi^2 + \sin^2\chi\,(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \tag{4.4}$$

where $0 \leq \chi \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$; a being a constant scalar factor. The manifold of a constant positive curvature possesses a $R^1 x S^3$ topology, i.e, the t = 0 hypersurface is a 3-sphere of radius a. If we work on finite temperature, after Wick rotation $t \to -i\tau$, the topology becomes $S^1 x S^3$ with the volume of manifold

$$Vol\, M = 2\pi^2 \beta a^3 \tag{4.5}$$

and measure

$$d^4x\sqrt{g} = d\tau\, d\chi\, d\theta\, d\varphi\; a^3 \sin^2\chi\, \sin\theta \tag{4.6}$$

The Ricci scalar in static closed Einstein universe is

$$R = \frac{6}{a^2} \tag{4.7}$$

The eigenfunctions of operator A can be chosen as

$$\psi_N(x) \propto e^{-\frac{2\pi i n}{\beta} t}\, Y_{\ell m}(\theta, \varphi) \sin^\ell\chi \cdot C^{\ell+1}_{k-\ell}(\cos\chi) \tag{4.8}$$

where $C^{\alpha}_{\beta}(x)$ is a Gegenbauer polynomial [35] [36], and N stands for the quantum numbers (n,k,l,m) , $n=0,\pm1,\ldots$, $k=0,1,2,\ldots$, $l=0,1,\ldots,k$, $m=-\ell,-\ell+1,\cdots$ $\ldots,\ell$. The eigenvalues a_N are

$$a_N = \frac{\lambda}{2}\hat{\phi}^2 + \left(\frac{2\pi n}{\beta}\right)^2 + \frac{(k+1)^2}{a^2} \tag{4.9}$$

which are $(k+1)^2$ degenerate as $\sum_{\ell=0}^{k} \sum_{m=-\ell}^{\ell} = (k+1)^2$.

The one loop effective potential is

$$V^{(1)}(\hat{\Phi}) = -\frac{1}{4\pi^2 a^3 \beta}\left[\zeta'(0) + \zeta(0)\ln\mu^2\right] \tag{4.10}$$

In large β limit(i.e, low temperature), the generalized ζ function is

$$\zeta(s) = \frac{\beta}{2\pi}\int_{-\infty}^{\infty} dk_4 \sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=-l}^{l}\left[\frac{\lambda}{2}\hat{\Phi}^2 + \frac{(k+1)^2}{a^2} + k_4^2\right]^{-s} \tag{4.11}$$

Following the standared dimensional regularization procedue, (4.11) becomes

$$\zeta(s) = \frac{\beta}{2\pi}\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)}\sum_{k=0}\sum_{l}\sum_{m}\left(\frac{\lambda\hat{\Phi}^2}{2} + \frac{(k+1)^2}{a^2}\right)^{-s} \tag{4.12}$$

Introducing k'= k+1, the summation can be symplified as

$$\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=-l}^{l}\left(\frac{(k+1)^2}{a^2} + \frac{\lambda\hat{\Phi}^2}{2}\right)^{-s+\frac{1}{2}} = a^{2s-1}\sum_{k'=1}^{\infty}\frac{k'^2}{(k'^2+\nu^2)^{s-1/2}}$$

$$= a^{2s-1}\sum_{k'}\left[\frac{1}{(k'^2+\nu^2)^{s-3/2}} - \frac{\nu^2}{(k'^2-\nu^2)^{s-1/2}}\right]$$

where $\nu^2 = \frac{\lambda\hat{\Phi}^2 a^2}{2}$. Finally the ζ function is found as

$$\zeta(s) = \frac{\beta}{2\pi}\frac{\Gamma(\frac{1}{2})\Gamma(s-\frac{1}{2})}{\Gamma(s)} a^{2s-1}\left[F(s-\tfrac{3}{2},\nu) - \nu^2 F(s-\tfrac{1}{2},\nu)\right] \tag{4.13}$$

where F function is defined in appendix B . $\zeta(0)$ and $\zeta'(0)$ can be obtained by computing $\zeta(\varepsilon)$ and comparing with the expansion $\zeta(\varepsilon) = \zeta(0) + \zeta'(0)\,\varepsilon$.

The renormalized effective potential is

$$V(\hat{\phi}) = \frac{\lambda+\delta\lambda}{4!}\hat{\phi}^4 + \frac{1}{2a^2}\hat{\phi}^2 - \frac{\lambda}{256\pi^2}\hat{\phi}^4 \ln\frac{\mu^2 a^4}{4} - \frac{\lambda^2}{128\pi^2}\hat{\phi}^4 + \frac{1}{4\pi^2 a^4}\left[F_c(2,\nu) - \nu^2 F_c(1,\nu)\right] \tag{4.14}$$

Since the divergent term in Eq.(4.14) does not dependent on the curvature, the renormaliration can be carried out as usual way in Minkowskian spacetime. Taking L $\rightarrow\infty$ and imposing the normalizetion conditions , we get the counterterm

$$\frac{\delta\lambda}{4!} = \frac{3\lambda^2}{512\pi^2} + \frac{\lambda^2}{256\pi^2}\ln\mu^2 - \frac{\lambda^2}{256\pi^2}\left[\ln\frac{\lambda\phi_0^2}{2} + \frac{25}{6}\right] \tag{4.15}$$

Where ϕ_0 is renormalization point given by $\frac{d^4V}{d\hat{\phi}^4}\Big|_{\hat{\phi}=\phi_c} = \lambda$.

The renormalized effective potential up to one loop correction is estimated as

$$V(\hat{\phi}) = \frac{\lambda}{4!}\hat{\phi}^4 + \frac{1}{2a^2}\hat{\phi}^2 - \frac{7\lambda^2}{384\pi^2}\hat{\phi}^4 - \frac{\lambda^2}{256\pi^2}\hat{\phi}^4\ln\left(\frac{\lambda}{8}a^2\phi_c^2\right) + \frac{1}{4\pi^2a^4}\left[F_c(2,\nu) - \nu^2F_c(1,\nu)\right] \tag{4.16}$$

It would be easy to generalize above result to a massive field case by the substitution of $\frac{\lambda\hat{\phi}^2}{2} \rightarrow M^2 = m^2 + \frac{\lambda\hat{\phi}^2}{2}$ in one loop correction (4.14) . The renormalized effective potential is given by

$$V(\hat{\phi}) = \frac{\lambda+\delta\lambda}{4!}\hat{\phi}^4 + \frac{1}{2}\left(m^2+\delta m^2+\frac{1}{a^2}\right)\hat{\phi}^2 - \frac{M^4}{64\pi^2}\ln\frac{\mu^2 a^2}{4} - \frac{M^2}{32\pi^2} + \frac{1}{4\pi^2a^4}\left[F_c(2,\nu) - \nu^2F_c(1,\nu)\right] + \delta C \tag{4.17}$$

where $\nu^2 = a^2M^2$, δC is a global counterterm independed on $\hat{\phi}$ and a, which is necessery for normalizing the vacuum energy of the Einstein universe in the flat spacetime limit. This means that δC would be fixed by $V(\hat{\phi}=c)\underset{a\rightarrow\infty}{=} C$

$$\delta C = -\frac{m^4}{64\pi^2}\ln\frac{m^2}{\mu^2} + \frac{3}{2}\frac{m^4}{64\pi^2} \qquad (4.18)$$

Meanwhile, normalizetion condition (2.41) may fix $\delta\lambda$ and δm^2 in $V(\hat{\phi})\Big|_{a\to\infty}$,

$$\delta\lambda = -\frac{16\lambda^2}{64\pi^2} - \frac{6\lambda^2}{64\pi^2}\ln\frac{m^2+\frac{\lambda}{2}\phi_0^2}{\mu^2}$$

$$\delta m^2 = \frac{\lambda m^2}{64\pi^2} - \frac{\lambda}{64\pi^2}\ln\frac{m^2}{\mu^2} \qquad (4.19)$$

where ϕ_0 is the normalization point. Finally, the renormalized effective potential in static closed Einstein universe is

$$V(\hat{\phi},a) = V(\hat{\phi}) + \frac{1}{2a^2}\hat{\phi}^2 - \frac{M^4}{64\pi^2}\left(\ln\frac{M^2a^2}{4}+\frac{1}{2}\right) + \frac{1}{4\pi^2a^4}\left[F_c(3,\nu) - \nu^2F_c(1,\nu)\right]$$

$$V(\hat{\phi}) = \frac{m^2}{2}\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 - \frac{M^4}{64\pi^2}\left(\ln\frac{m^2}{M^2}-\frac{3}{2}\right) + \frac{\lambda^2\hat{\phi}^4}{4}\frac{1}{64\pi^2}\ln\frac{\lambda\phi_0^2}{2m^2} \qquad (4.20)$$

$$-\frac{1}{64\pi^2}\left(\frac{3}{2}m^4 + \frac{2}{3}\lambda^2\hat{\phi}^4 - m^2\lambda\hat{\phi}^2\right)$$

where $V(\phi)$ is the flat space part, *which is separated from the terms induced by the curvature.*

In large curvature case, namely $a^2M^2 \ll 1$, $V(\phi,a)$ can be expanded as follows

$$V(\hat{\phi},a)\Big|_{a^2M^2\ll 1} = V(\hat{\phi}) + \frac{1}{2a^2}\hat{\phi}^2 - \frac{M^4}{64\pi^2}\left(\ln\frac{M^2a^2}{4}+\frac{1}{2}\right) + \frac{1}{480\pi^2a^2} - \frac{M^2}{96\pi^2a^2} - \frac{M^4}{32\pi^2}\gamma \qquad (4.21)$$

It is easily found from (4.21) that $\hat{\phi} = 0$ is a minimum of $V(\hat{\phi},a)$, when the curvature becomes large enough . In such a case , the vacuum energy is given by

$$V(\hat{\phi}=0,a) = \frac{1}{480\pi^2a^4} - \frac{m^2}{96\pi^2a^2} - \frac{7m^4}{32\pi^2} - \frac{m^4}{64\pi^2}\left(\ln\frac{m^2a^2}{4}+\frac{1}{2}\right) \qquad (4.22)$$

and the renormalized mass is

$$m_R^2 = \frac{d^2V}{d\hat{\phi}^2}\bigg|_{\hat{\phi}=c} \approx m^2 + \frac{1}{a^2}\left(1 - \frac{\lambda}{96\pi^2}\right) \tag{4.23}$$

where $O(m^4)$ and $O(m^4 \ln m^2 a^2)$ are neglected. If $m^2 + \frac{1}{a^2} < 0$, there exists spontaneous symmetry breaking at tree graph level. Taking one loop quantum correction into account, when the curvature is larger enough and exceeds the criticle value, the $\langle\phi\rangle \neq 0$ vacuum state will be replaced by $\langle\phi\rangle = 0$, and the spontaneous symmetry breaking is restored. As for the criticle radius of the closed Einstein universe or the criticle curvature, it can be decided from the vanishing renormalized mass term $m_R^2 = 0$

$$a_c^2 = -\frac{1}{m^2}\left(1 - \frac{\lambda}{96\pi^2}\right) \tag{4.24}$$

$$R_c \approx -6m^2\left(1 + \frac{\lambda}{96\pi^2}\right) \tag{4.25}$$

By aid of preceding analysis, it shows that only if the Einstein universe has a small spatial extension comparable with the size of elemental particles just as the early stage of the universe, can the phase transition occur, meanwhile it is found that only the symmetric phase can exist in a closed universe during the early stage of the cosmological expansion.

We now concentrate our attention to the curvature dependence of the coupling constant. In virtue of (4.21), if $m^2 > 0$, the $\lambda(R)$ can be obtained by

$$\lambda(R) := \frac{d^4 V(\hat{\phi}; a)}{d\hat{\phi}^4}\bigg|_{\hat{\phi}=0} = \lambda + \frac{3\lambda^2}{32\pi^2} \ln \frac{2R}{3m_c^2} \tag{4.26}$$

where $m^2 \equiv m_c^2 \exp\left[-2\gamma - 2\right]$, which clearly shows that λ increases with R. If $m^2 < 0$, there exists spontaneously symmetry breaking at tree level. According to the same reason mentioned in Ch.III, λ must be defined off shell like in (3.36)

$$\lambda(R) := \left. \frac{d^4 V(\hat{\phi}; a)}{d\hat{\phi}^4} \right|_{\hat{\phi}=\phi_0} = \lambda + \frac{3\lambda^2}{32\pi^2} \ln \frac{4R}{3\lambda\sigma^2} \tag{4.27}$$

where $\phi_0^2 = \sigma^2 \exp[-2(\frac{7}{3}+\gamma)]$. The result (4.27) is quite similar to (4.26) Therfore the λ always increases with R both in $m^2 > 0$ and $m^2 < 0$, which is different from the finite temperature case in flat spacetime.

2. Zero Temperature Field Theory in Static Open Einstein Universe [37] [38]

The manifold of open Einstein universe has $R^1 \times H^3$ topology whose t = 0 hypersurface is a 3 - hyperboloid. Consider a case of self interacting scalar field conformally coupled with the background gravitational field, through $\frac{R}{12}\phi^2$ term. The Lagrangian is conformal invariant when the mass term missing

$$\mathcal{L} = -\frac{1}{2} g_{\mu\nu} \partial^\mu \phi \, \partial^\nu \phi - \frac{m^2}{2}\phi^2 - \frac{R}{12}\phi^2 - \frac{1}{4!}\phi^4 \tag{4.28}$$

Its metric has a negative curvature in Friedman coordinates

$$ds^2 = g_{\mu\nu}(x) dx^\mu dx^\nu = -dt^2 + a^2[d\chi^2 + \sinh^2\chi(d\theta^2 + \sin^2\theta \, d\phi^2)] \tag{4.29}$$

where a is a scale factor , while the Ricci scalar is $R = -\frac{6}{a^2}$. Because the classical potential is $\frac{1}{2}(m^2 - \frac{1}{a^2})\phi^2 + \frac{\lambda}{4!}\phi^4$, the conformal coupling provides the field with an effective mass $\mu^2 = m^2 - \frac{1}{a^2}$. If $m^2 \geq \frac{1}{a^2}$, the minimum of potential lies at $\langle\phi\rangle = 0$ and the theory is symmetric under reflection transformation. If $0 < m < \frac{1}{a}$, the minima locate at $\langle\phi\rangle = \pm\sqrt{\frac{6}{\lambda}(\frac{1}{a^2} - m^2)}$, so that the field has spontaneous symmetry breaking even if $m^2 > 0$, due to the negative curvature of the background. Increasing curvature may lead to a transition from symmetry phase to symmetry breaking phase as soon as the critical curcature a_c is reached ($a_c = \frac{1}{m}$).

As for massless field , spontaneous symmetry breaking exists already at tree graph level since the minima of potential locate at $\langle\phi\rangle = \pm\sqrt{\frac{6}{\lambda a}}$, different from zero.

When the quantum corrections enter in consideration, the effective potential of one loop correction is

$$\int d^4x\sqrt{g}\, V^{(1)}(\hat{\phi}) = \hbar \ln \int \mathcal{D}\phi\, e^{-\frac{1}{2}\int d^4x\sqrt{g}\,\phi(x) A(x\,x)\phi(x)} \tag{4.30a}$$

$$A(x\,y) = A(x\,x)\,\delta(x\,y) = (-\Box_c - \frac{1}{a^2} + M^2)\,\delta(x,y) \tag{4.30b}$$

where the measure $d^4x\sqrt{g} = a^3 \sinh^2\chi \sin\theta\, d\tau\, d\chi\, d\theta\, d\phi$, and $\tau = i t$, the covariant d'Alembertian [39] $\Box_c = g^{-1/2}\partial_\mu g^{1/2} g^{\mu\nu}\partial_\nu$, $M^2 = m^2 + \frac{\lambda}{2}\hat{\phi}^2$.

Performing integral of (4.30), one gets

$$\int d^4x\sqrt{g}\, V^{(1)}(\hat{\phi}) = -\frac{\hbar}{2} T_r \ln A(x,y) \tag{4.31}$$

The Green's function of differential operator A is defined by following relation

$$A\, G(x\,y) = \delta^4(x\,y) \tag{4.32}$$

which has a formal solution [40] [41] [42]

$$G(x\,y) = A^{-1}\delta(x\,y) = \int_0^\infty ds\, e^{-As} = \int_0^\infty ds\, K(x\,y\,s) \tag{4.33}$$

where $K(x,y,s)$ is called heat kernal which satisfies the heat equation [43]-[44]

$$A\, K(x\,y\,s) = -\frac{\partial K(x\,y\,s)}{\partial s} \tag{4.34a}$$

and boundary condition

$$K(x\,y\,0) = \delta(x\,y) \tag{4.34b}$$

(in detail , see appendix C)

The heat kernal expression of one loop contribution to effective potential is given by (see appendix C)

$$V^{(1)} = -\frac{\hbar}{2}\int_0^\infty \frac{ds}{s}\, K(x\; x\; s\;) \tag{4.35}$$

where $K(x\; x;\; s) = \lim_{y\to x} K(x,y,s)$, and the integral is divergent . In order to regularized it , one can resort to dimensional regularization method.

In 2ω dimension momentum space, the Green function[42] of flat spacetime can be derived as

$$G(x) = \int \frac{ds}{(4\pi s)^\omega}\, e^{-M^2 s - \frac{x^2}{4s}}$$

so that the heat kernal in flat spacetime is

$$K_\infty(x, y, s, 2\omega) = \frac{e^{-M^2 s}}{(4\pi s)^\omega}\, e^{-\sigma^2(x,y)/4s} \tag{4.36}$$

where the $\sigma(x\; y)$ is the geodesic interval between x and y. While in curved spacetime the heat kernal can be written as follows[37][47]

$$K(x\; y\; s, 2\omega\;) = \frac{e^{-M^2 s}}{(4\pi s)^\omega}\, \frac{\sigma(x\; y)}{a}\, \mathrm{cosech}\left(\frac{\sigma(x\; y)}{a}\right) e^{-\sigma^2(x\; y)/4s} \;. \tag{4.37}$$

As a result , the dimensionally regularized one loop effective potential is

$$V^{(1)}(\hat\phi) = -\frac{\hbar}{2(4\pi)^\omega}\int_0^\infty \frac{ds}{s^{1+\omega}}\, e^{-M^2 s} \tag{4.38}$$

The renormalized form of effective potential is

$$V(\hat\phi) = \frac{m^2+\delta m^2}{2}\hat\phi^2 - \frac{1}{2a^2}\hat\phi^2 + \frac{\lambda+\delta\lambda}{4!}\hat\phi^4 - \frac{\hbar}{2(4\pi)^\omega}\int_0^\infty \frac{ds}{s^{1+\omega}}\, e^{-M^2 s} + \delta C \tag{4.39}$$

where δC is the global counterterm which normalizes the energy density of the vacuum state being zero in the flat spacetime limit, namely, $V(\hat\phi=0) \underset{a\to\infty}{=} 0$.

$$\delta C = \frac{\hbar}{2(4\pi)^{\omega}} \int_0^{\infty} \frac{ds}{s^{1+\omega}}\, e^{-m^2 s} \tag{4.40}$$

The mass and coupling constant counterterms δm^2, $\delta\lambda$ can be fixed by normalization condition

$$\begin{aligned}
\frac{\partial V(\hat{\phi})}{\partial \hat{\phi}}\Big|_{\phi_0} &= 0 \\
\frac{\partial^4 V(\hat{\phi})}{\partial \hat{\phi}^4}\Big|_{\phi_0} &= \lambda \\
\frac{\partial^2 V(\hat{\phi})}{\partial \hat{\phi}^2}\Big|_{\phi_0} &= m^2 - \frac{1}{a^2} + \frac{\lambda}{2}\phi_0^2
\end{aligned} \tag{4.41}$$

As a consequence, one gets

$$\begin{aligned}
\delta\lambda &= \frac{\hbar}{2(4\pi)^{\omega}} \int_0^{\infty} \frac{ds}{s^{1+\omega}} \left(3\lambda^2 s - 6\lambda^3 s^3 \phi_0^2 + \lambda^4 s^4 \phi_0^3\right) e^{-M_0^2 s} \\
\delta m^2 &= -\frac{\hbar}{2(4\pi)^{\omega}} \int_0^{\infty} \frac{ds}{s^{1+\omega}} \left(\lambda s + \frac{1}{2}\lambda^2 s^2 \phi_0^2\right) e^{-M_0^2 s} + O(\lambda^3)
\end{aligned} \tag{4.42}$$

Inserting δC, δm^2, $\delta\lambda$ back into (4.39) and carrying out the integral, then letting $\omega \rightarrow 2$, we will find that the divergent terms may cancell each other. The final result is

$$\begin{aligned}
V(\hat{\phi}) = \frac{1}{2}\left(m^2 - \frac{1}{a^2}\right)\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 + \frac{\hbar}{4(4\pi)^2}\Big[& M^4 \ln\frac{M^2}{M_0^2} - \frac{3\lambda m^2}{2}\hat{\phi}^2 + \\
& + \lambda M_0^2 \hat{\phi}^2 - \frac{3}{8}\lambda\hat{\phi}^4 + m^4 \ln\frac{M_0^2}{m^2} \Big]
\end{aligned} \tag{4.43}$$

Although the quantum correction of $V(\hat{\phi})$ in curved spacetime has the same form as it has in flat spacetime, it is curvature dependent because the normalization point ϕ_0 which is the minimum of $V(\hat{\phi})$ is a dependent.

3. Finite Temperature Field Theory in Static Open Einstein Universe

The heat kernal formalism of the one loop effective potential has many advantages, one of which is that the effect of temperature on the propagator is just to add a θ-function correction. In consequense, the one loop effective potential at finite temperature can be easily written as [37] [39] [24] [42]

$$V^{(1)}(\hat{\phi};\beta) = -\frac{\hbar}{2(4\pi)^{\omega}}\int_0^{\infty}\frac{ds}{s^{1+\omega}}\, e^{-M^2 s}\,\theta_3\left(0, -\frac{\beta^2}{4\pi i s}\right) \tag{4.44}$$

where $\theta_3(0, t)$ is the third Jacobi theta function [48] [49] (in detail, see appendix D) which has the Jacobi transformation relation

$$\theta_3(0, t) = \sqrt{\frac{i}{t}}\;\theta_3\left(0, -\frac{1}{t}\right) \tag{4.45}$$

(4.45) would be useful later as it allows a duality transformation between the high and low temperature regimes.

$$V^{(1)}(\hat{\phi}, \beta) = -\frac{\hbar}{2\beta(4\pi)^{\omega-\frac{1}{2}}}\int_0^{\infty}\frac{ds}{s^{\omega+\frac{1}{2}}}\, e^{-M^2 s}\,\theta_3\left(0, \frac{4\pi i s}{\beta^2}\right) \tag{4.46}$$

which is suitable for high temperature expansion. The renormalized one loop contribution to the effective potential is

$$V(\hat{\phi};\beta) = \frac{1}{2}(m^2+\delta m^2)\hat{\phi}^2 + \frac{\lambda+\delta\lambda}{4!}\hat{\phi}^4 - \frac{\hbar}{2\beta(4\pi)^{\omega-\frac{1}{2}}}\int_0^{\infty}\frac{ds}{s^{\omega+\frac{1}{2}}}\, e^{-M^2 s}\,\theta_3\left(0, \frac{4\pi i s}{\beta^2}\right) + \delta C \tag{4.47}$$

In high temperature limit, $M^2 << \frac{1}{\beta}$, $e^{-M^2 s}$ can be expanded into Talor series, consequently, the one loop correction is give by

$$V^{(1)}(\hat{\phi};\beta) = -\frac{\hbar}{2\beta(4\pi)^{\omega-\frac{1}{2}}}\sum_{0}^{\infty}{}_{n}\frac{(-1)^n}{n!}M^{2n}\int_0^{\infty}\frac{ds}{s^{\omega+\frac{1}{2}-n}}\,\theta_3'\left(0, \frac{4\pi i s}{\beta^2}\right) - \frac{\hbar}{2\beta(4\pi)^{\omega-\frac{1}{2}}}\int_0^{\infty}\frac{ds}{s^{\omega+\frac{1}{2}}}\, e^{-M^2 s} \tag{4.48}$$

where $\theta'(0,t)$ denotes the summation $\sum_{n=-\infty}^{\infty}{}' e^{n^2\pi i t}$ without $n = 0$ term, while the last one is the $n = o$ piece.

By means of following integration relations

$$\int_0^\infty \frac{ds}{s^{\omega+1/2}} e^{-M^2 s} = M^{2\omega-1}\Gamma(\tfrac{1}{2}-\omega) \tag{4.49a}$$

$$\int_0^\infty \frac{ds}{s^{\omega+1/2-n}} \theta_3'(0,\frac{4\pi i s}{\beta^2}) = \sum_{-\infty}^{\infty}{}' \int_0^\infty \frac{ds}{s^{\omega+1/2-n}} e^{-\frac{4 l^2\pi^2}{\beta^2}s}$$

$$= (\tfrac{2\pi}{\beta})^{2\omega-2n-1} \sum_{-\infty}^{\infty}{}' \frac{1}{l^{2n-2\omega+1}} \Gamma(n-\omega+\tfrac{1}{2})$$

$$= (4\pi^2)^{\omega-n-\frac{1}{2}} \beta^{2n+1-2\omega}\, 2\cdot \zeta_R(2n-2\omega+1)\Gamma(n-\omega+\tfrac{1}{2}) \tag{4.49b}$$

the one loop effective potential can be evaluated as

$$V^{(1)} = -\frac{\hbar(4\pi)^{1/2-\omega}}{2\beta}\Big[\Gamma(\tfrac{1}{2}-\omega)M^{2(\omega-1/2)} + \frac{\Gamma(\frac{1}{2}-\omega)}{(4\pi^2)^{\frac{1}{2}-\omega}} 2\beta^{1-2\omega}\zeta_R(1-2\omega)$$

$$- \frac{M^2\Gamma(\frac{3}{2}-\omega)}{(4\pi^2)^{3/2-\omega}} 2\beta^{3-2\omega} \zeta_R(3-2\omega) + 2\sum_{n=2}^{\infty}\frac{(-1)^n}{n!}\frac{M^{2n}\Gamma(n-\omega+\frac{1}{2})}{(4\pi^2)^{n-\omega+\frac{1}{2}}}\beta^{2n-2\omega+1}\zeta_R(2n-2\omega+1)\Big] \tag{4.50}$$

where ζ_R is the Riemann zeta function[49], defined as $\zeta_R(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$, which is analytic in whole s plane except for a simple pole at $s = 1$.

When $\omega = 2$, the first three terms in (4.~~48~~ 50) can be computed imediately, the $n = 2$ piece in the last term is divergent because it contains $\zeta_R(1)$. Letting $n = 2 + \varepsilon$ and expanding it around $n = 2$, we can separately get the finite part and a pole term. Finally the potential is

$$V^{(1)}(\hat{\phi},\beta) = -\frac{\hbar\pi^2}{90\beta^4} + \frac{\hbar M^2}{24\beta^2} - \frac{\hbar M^3}{12\pi\beta} - \frac{\hbar M^4}{4(4\pi)^2}\frac{1}{\varepsilon} - \frac{\hbar M^4}{4(4\pi)^2}\left(\ln(\tfrac{\beta M}{4\pi})^2 - \tfrac{3}{2} + 2\gamma\right)$$

$$- \frac{\hbar(4\pi)^{3/2}}{\beta}\sum_{n=3}^{\infty}\frac{(-1)^n M^{2n}\Gamma(n-\frac{3}{2})\beta^{2n-3}}{\Gamma(n+1)(4\pi^2)^{n-3/2}} \tag{4.51}$$

The divergent part $\frac{-\hbar M^4}{4(4\pi)^2}\frac{1}{\varepsilon}$ is β independent, just the same as that in zero temperature case , hence the renormalization takes place through the same counterterms as the zero temperature case which can be fixed by normalization conditions. Finally the renormalized effective potential at high temperature limit is

$$V_R(\hat{\Phi},\beta) = \frac{1}{2}(m^2-\frac{1}{a^2})\hat{\Phi}^2+\frac{\lambda}{4!}\hat{\Phi}^4+V_R^{(1)}(\hat{\Phi})-\frac{\hbar\pi^2}{90\beta^4}-\frac{\hbar M^2}{24\beta^2}-\frac{\hbar M^3}{12\pi\beta}$$
$$-\frac{\hbar M^4}{4(4\pi)^2}\left[\ln\left(\frac{\beta M}{4\pi}\right)^2-\frac{3}{2}+2\gamma\right]-\frac{\hbar}{(4\pi)^{3/2}}\sum_{3}^{\infty}\frac{(-1)^{\ell}}{\ell!}\frac{M^{2\ell}\Gamma(\ell-\frac{3}{2})}{(4\pi^2)^{\ell-3/2}}\beta^{2\ell-4}\zeta_R(2\ell-3) \tag{4.52}$$

where $V^{(1)}(\hat{\Phi})$ is the β independent part and is the same as the last term given in (4.43)

The critical temperature at which the symmetry is restored can be obtained by making the effective mass vanishing,

$$m_R^2 := \left.\frac{\partial^2 V_R(\hat{\Phi},\beta)}{\partial\hat{\Phi}^2}\right|_{\hat{\Phi}=0} = m^2-\frac{1}{a^2}+\frac{\lambda\hbar}{24\beta_c^2} \tag{4.53}$$

thereby,

$$\beta_c^2 = -\frac{\hbar\lambda}{24}\frac{1}{m^2-\frac{1}{a^2}} \tag{4.54}$$

which gives rise to the relation between the critical temperature and the curvature. It shows that the effects of the curvature in static Einstein universe is equvalent to shifting the mass by the conformal coupling.

In high temperature case , the last term in (4.52) is neglegible, then the coupling constant may be obtained in the form

$$\lambda(\beta, R) := \left.\frac{d^4 V_R(\hat{\phi}\,\beta)}{d\hat{\phi}^4}\right|_{\hat{\phi}=0}$$

$$= \lambda - \frac{3\lambda^2\hbar}{16\pi\beta m^2} + \frac{3\lambda^2\hbar}{32\pi^2} \ln \frac{16\pi^2}{\tilde{M}^2\beta^2} \tag{4.55}$$

with

$$\tilde{M}^2 = (m^2 + \tfrac{\lambda}{2}\phi_c^2)\, e^{2\gamma}$$

where γ is Euler constant, ϕ_c is normalization point, which we have chosen in (4.41) to be the minimum of the effective potential at zero temperature.

With the aid of (4.43) and (4.41), we may evaluate the derivative of $V_R(\hat{\phi})$ with respect to $\hat{\phi}$ and realize that the quantum corrections will not affect the extremes of the classical potential, so the ϕ_c is still located at

$$\phi_c = \sqrt{-\frac{6}{\lambda}\left(m^2 - \frac{1}{a^2}\right)} \tag{4.56}$$

In the case of $0 < m^2 < \frac{1}{a^2}$, (4.55) shows that the effective coupling constant $\lambda(\beta, R)$ will decrease so long as the temperature increases while the curvature remains unchangable and for temperature near $\beta^{-1} \approx 16\pi m/3\lambda$ λ vanishes and theory becomes asymptotically free.

However λ will always decrease logrithmusly if the curvature increases while the temperature remains unchangable.

4. Low Curvature Limit in Non - minimal Coupling Case

The general expressions of a renormalizable scalar field theory in a curved spacetime are

$$\mathcal{L} = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2}(m^2 + \xi R)\phi^2 - \frac{\lambda}{4!}\phi^4$$

$$\Gamma[\bar{\Phi}(x)] = S[\bar{\Phi}] - \frac{1}{2} \ln \mathrm{Det} A \equiv \Gamma^{0}[\bar{\Phi}] + \Gamma^{(1)}[\bar{\Phi}]$$

$$A = -\Box_c + \xi R + M^2 \tag{4.57}$$

where ξ is coupling constant between matter field and gravitational background.

In Ch.III and Ch. IV, we have provided generalized ζ function prescription and the heat kernal methods which are powerful for computing effective action

$$\Gamma^{(1)}[\bar{\Phi}] = -\frac{1}{2}\left[\zeta'(0) + \zeta(0) \ln \mu^2\right]$$

$$\Gamma^{(1)}[\bar{\Phi}] = \frac{1}{2} Tr \int_0^\infty \frac{d\tau}{\tau} e^{-M^2\tau} K(x, x, \tau) \tag{4.58}$$

The former is suitable whenever the manifold $\mathcal{M}$ is compact so that operator A has a discrete spectrum; the later is quite general but requires the explicit form of $K(x\ y\ \tau)$ which may be difficult to write up in many cases. However , one may resort to the relation between $K(x, y, \tau)$ and $\zeta(s)$ which is connected by a Mellin transform[50]

$$\zeta(s) = \frac{1}{\Gamma(s)} Tr \int_0^\infty d\tau\ \tau^{s-1} K(x\ x\ \tau)\, e^{-M^2\tau} \tag{4.59}$$

where $\Gamma(s)$ is the Γ-function. For small τ , $K(x, x, \tau)$ can be expanded as follows

$$Tr\, K(x\, x\, \tau) = \frac{1}{(4\pi)^2} \sum_{n=0}^{\infty} B_n \tau^{n/2-2}$$

(4.60)

$$B_n = \int_{\mathcal{M}} d^4x \sqrt{g}\; b_n$$

(4.61)

where the b_n's are the Hamidew * coefficients and are n- degree polynomials in the curvature tensors of manifold. The first three non vanishing coefficients are given in appendix C

In (4.59) and (4.60), the non vanishing contributions to $\zeta(0)$ come only from n = 0, 2, 4 because only these divergent integrals yield $\Gamma(0)$, $\Gamma(-1)$, $\Gamma(-2)$ poles which compensate with the infinite denominator $\Gamma(0)$ to give finite result

$$\zeta(0) = \frac{1}{(4\pi)^2 \Gamma(0)} \sum_{n=0}^{\infty} B_n (M^2)^{2-n/2} \Gamma(\tfrac{n}{2}-2)$$

$$= \frac{1}{(4\pi)^2} \int_{\mathcal{M}} d^4x \sqrt{g} \left[\frac{1}{2} b_0 M^4 - b_2 M^2 + b_4 \right] \tag{4.62}$$

The derivative of $\zeta(s)$ at s = 0 is computed as

$$\zeta'(0) = \frac{1}{(4\pi)^2 \Gamma(0)} \sum_n B_n (M^2)^{2-n/2} \left[-\psi(0) + \psi(\tfrac{n}{2}-2) - \ln M^2 \right] \Gamma(\tfrac{n}{2}-2).$$

where ψ function is the log derivatives of Γ function. Discarding terms which van sh at s = 0 or finite but at least cubic an the curvature invariants, i.e. n $\geqslant$ 6 terms, one obtains

$$\zeta'(0) = \frac{1}{(4\pi)^2} \int_{\mathcal{M}} d^4x \sqrt{g} \left[\frac{M^4}{2} b_0 (\tfrac{3}{2} - \ln M^2) + b_2 M^2 (\ln M^2 - 1) - b_4 \ln M^2 \right] \tag{4.63}$$

Combining (4.62) and (4.63), the one loop effective action can be derived

$$\Gamma[\Phi] = S[\Phi] + \frac{1}{2(4\pi)^2} \int_{\mathcal{M}} d^4x \sqrt{g} \left[\frac{b_0}{2} M^4 (\tfrac{3}{2} - \ln \tfrac{M^2}{\mu^2}) - b_2 M^2 (1 - \ln \tfrac{M^2}{\mu^2}) - b_4 \ln \tfrac{M^2}{\mu^2} \right]$$

(4.64)

which is ultraviolet regularized . Above result is reliable as far as the curvature is small compared with M^2. The first term of the integral is the usual one loop contribution to effective action in flat spacetime , while the second and third terms contain linear and quadratic corredtions in the curvature.

For a static constant curvature, homogeneous isotropic manifold the classical field $\bar{\phi}(x)$ can be chosen to constant $\hat{\phi}$ so that the effective potential can be defined

$$V(\hat{\phi}) = V^{(0)}(\hat{\phi}) - \frac{1}{2(4\pi)^2}\left[\frac{1}{2}M^4\left(\frac{3}{2} - \ln\frac{M^2}{\Lambda^2}\right) - b_2 M^2\left(1 - \ln\frac{M^2}{\Lambda^2}\right) - b_4 \ln\frac{M^2}{\Lambda^2}\right] \tag{4.65}$$

We now are in position to descuss the renormalization of the effective potential . For the sake of simplicity , suppose the minimum of classical potential locates at $\hat{\phi} = 0$. The renormalized effective potential is

$$\begin{aligned} V_R(\hat{\phi}) = {} & \frac{1}{2}(m^2 + \xi R + \delta m^2)\hat{\phi}^2 + \frac{\lambda + \delta\lambda}{4!}\hat{\phi}^4 + \delta C \\ & - \frac{1}{2(4\pi)^2}\left[\frac{1}{2}M^4\left(\frac{3}{2} - \ln\frac{M^2}{\Lambda^2}\right) + b_2 M^2\left(1 - \ln\frac{M^2}{\Lambda^2}\right) - b_4 \ln\frac{M^2}{\Lambda^2}\right] \end{aligned} \tag{4.66}$$

The counterterms δm^2, $\delta\lambda$, δC can be fixed by normalization condition

$$\begin{aligned} \left.\frac{dV^{(c)}(\hat{\phi})}{d\hat{\phi}}\right|_{\hat{\phi}=0} &= 0 \\ \left.\frac{d^2 V_R(\hat{\phi})}{d\hat{\phi}^2}\right|_{\hat{\phi}=0} &= m^2 + \xi R \\ \left.\frac{d^4 V_R(\hat{\phi})}{d\hat{\phi}^4}\right|_{\hat{\phi}=\phi_0} &= \lambda \end{aligned} \tag{4.67}$$

where ϕ_0 is a nonzero normalization point . Letting $M^2 = m^2 + \frac{\lambda}{2}\hat{\phi}^2$, $M_0^2 = m^2 + \frac{\lambda}{2}\phi_0^2$, finally the one loop renormalized effective potential is

$$\begin{aligned} V_R(\hat{\phi}) = {} & \frac{1}{2}(m^2 + \xi R)\hat{\phi}^2 + \frac{\lambda}{4!}\hat{\phi}^4 - \frac{1}{4(4\pi)^2}\left[\frac{25}{24}\lambda^2\hat{\phi}^4 + \frac{\lambda m^2}{2}\hat{\phi}^2 - M^4 \ln\frac{M^2}{m^2} + \frac{\lambda^2\hat{\phi}^4}{4}\ln\frac{M_0^2}{m^2}\right] \\ & - \frac{1}{2(4\pi)^2}\left[-\frac{\lambda\hat{\phi}^2}{2}b_2 + b_2 M^2 \ln\frac{M^2}{m^2} + \frac{\lambda b_4}{2m^2}\hat{\phi}^2 - b_4 \ln\frac{M^2}{m^2}\right] \end{aligned} \tag{4.68}$$

which give the general form of renormalized effective potential of a massive ϕ^4 theory non minimally coupled to a static constant courvature background of which the last term is the gravitational corrections.

Basing on (4.68) , some particular case can be easily discussed. For instance, a conformally coupling theory where $\xi = \frac{1}{6}$, $b_2 = 0$, $b_4 = -\frac{1}{12}\frac{R^2}{180}$ the renormalized effective potential is reduced to be

$$V_R(\hat{\phi}) = \frac{1}{2}(m^2+\frac{R}{6})\hat{\phi}^2+\frac{\lambda}{4!}\hat{\phi}^4 - \frac{1}{4(4\pi)^2}\left[\frac{25}{24}\lambda^2\hat{\phi}^4+\frac{\lambda m^2}{2}\hat{\phi}^2 - M^4 \ln\frac{M^2}{m^2} - \frac{\lambda^2\hat{\phi}^4}{4}\ln\frac{M_c^2}{m^2}\right]$$
$$+\frac{1}{2(4\pi)^2}\left[b_4 \ln\frac{M^2}{m^2} - \frac{\lambda b_4}{2m^2}\hat{\phi}^2\right] \qquad (4.69)$$

where the curvature enters linearly at classical level , and quadratically at the quantum level. In positive constant curvature case $R>0$, $\hat{\phi}=0$ state is the classical minimum which is stable against the quantum corrections.

However, if Lagrangian is not the conformal invariant, b_2 would be non- zero as $b_2 = (\frac{1}{6} - \xi)R$ (see apendix C). Hence, if $b_2 \neq 0$, the local scale invariance would be spontaneously broken even at tree level.

V. CONCLUSION

Since path integral prescription of quantum field theory and effective potential is very useful to study spontaneous symmetry breaking beyond the tree level, first it is worthwhile to derive detailed expressions in flat spacetime, next ,the generalization of such a theory to finite temperature and compact spatial axis are discussed. It has been found that the broken symmetry will be restored above a critical temperature or below a critical length so far as the one loop radiative correction is taken into account. Both the global feature of the spacetime manifold of the non - Minkowskian topology and the local feature of the spacetime curvature will give new contents to physics.

Spontaneous symmetry breaking is very important in the unified theory and it is wellknown, our universe just was in extremely high temperature configuration and possessed very large curvature in the early evolution stage, shortly after big bang, so that the studies of symmetry breaking and its restoration at high temperature and large curvature is of great significance to the physics on the point that only after we know much more about the early stage of universe, can we understand the nowadays universe much better.

In this paper, the self - interacting neutral scalar field is dealt with as examples in different problems. As a matter of fact, O(N)symmetry self interacting scalar field,[5] spinor field[51] gauge field[51] and scalar QED[11],[52]-[54] have been discussed in many papers.

Appendix A

The definition of $F(\lambda;a,b)$ is following summation

$$F(\lambda;a,b) = \sum_{n=-\infty}^{\infty} \left[(n+b)^2 + a^2 \right]^{-\lambda} \tag{A.1}$$

where a, b, is real. This series is convergent provided that $\mathrm{Re}\,\lambda > \frac{1}{2}$, but it can be defined in whole λ plane by analytic continuation, aside from poles at $\lambda = \frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}, \cdots$, through following integral representation definition.

$$F(\lambda;a,b) = \pi^{1/2} a^{1-2\lambda} \frac{\Gamma(\lambda-\frac{1}{2})}{\Gamma(\lambda)} + 4\sin\pi\lambda\, f_\lambda(a,b)$$

$$f_\lambda(a,b) = \int_a^\infty (u^2-a^2)^{-\lambda}\, \mathrm{Re}\left(e^{2\pi(u+ib)} - 1\right)^{-1} du \tag{A.2}$$

As $a,b \to 0$, one has

$$f_\lambda(0,0) = (2\pi)^{2\lambda-1}\,\Gamma(1-2\lambda)\,\zeta(1-2\lambda) \qquad \mathrm{Re}\,\lambda < 0 \tag{A.3}$$

$$f_{-\frac{1}{2}}(0,0) = (2\pi)^{-2}\,\Gamma(2)\,\zeta(2) = \frac{1}{4\pi^2}\cdot\frac{\pi^2}{6} = \frac{1}{24} \tag{A.4}$$

$$f_{-\frac{3}{2}}(0,0) = (2\pi)^{-4}\,\Gamma(4)\,\zeta(4) = \frac{6}{(2\pi)^4}\frac{\pi^4}{96} = \frac{1}{240} \tag{A.5}$$

Derivatives of f give rise to

$$\frac{\partial}{\partial a} f_\lambda(a,b) = 2\lambda a\, f_{\lambda+1}(a,b) \tag{A.6}$$

$$\frac{\partial^2}{\partial b^2} f_\lambda(a,b) = -2\lambda(2\lambda+1) f_{\lambda+1}(a,b) - 4\lambda(\lambda+1)a^2 f_{\lambda+2}(a,b) \tag{A.7}$$

Define integral

$$I(D,\alpha;\varrho,H) = \sum_n \int d^D k_E \frac{1}{(k_E^2+2\varrho\cdot k+H+i\varepsilon)^\alpha} \tag{A.8}$$

$$I_\mu(D,\alpha;\varrho,H) = \sum_n \int d^D k_E \frac{k_\mu}{(k_E^2+2\varrho\cdot k+H+i\varepsilon)^\alpha} \tag{A.9}$$

$$I_{\mu\nu}(D,\alpha;\varrho,H) = \sum_n \int d^D k_E \frac{k_\mu k_\nu}{(k_E^2+2\varrho\cdot k+H+i\varepsilon)^\alpha} \tag{A.10}$$

The dimensional regulazation formular gives rise to

$$I(D,\alpha,\varrho,H) = \pi^{D/2} \frac{\Gamma(\alpha-D/2)}{\Gamma(\alpha)} \left(\frac{2\pi}{L}\right)^{2(\frac{D}{2}-\alpha)} F(\alpha-\tfrac{D}{2};a,b)\,.$$

where

$$a = \frac{L}{2\pi}(H - q^2)^{1/2}$$

$$b = \frac{L}{2\pi} q^1$$

q^1 is the first component of q_μ which is a D + 1 dimension Euclidean vector. If $q^2=0$, $H = M^2$, then $a = \frac{LM}{2\pi}$, $b = 0$, one has

$$\begin{aligned} I(D,1,0,M^2)\Big|_{\substack{L^2M^2\to 0\\ a\to 3}} &= \pi^{D/2}\Gamma(1-D/2)\left(\frac{2\pi}{L}\right)^{D-2} F(1-D/2, a, 0)\Big|_{\substack{a=0\\ D\to 3}} \\ &= -2\pi^2 \frac{2\pi}{L} 4\cdot\sin\left(-\frac{\pi}{2}\right)\cdot f_{-\frac{1}{2}}(0,0) \\ &= \frac{2\pi^3}{3L} \end{aligned} \tag{A.11}$$

so that eq.(3.4) may be derived , yielding,

$$L^2 M^2 U_2^L(\hat{\phi}) = \frac{L^2M^2}{(2\pi)^3 2L} I(D,1,0,M^2) = \frac{M^2}{24L^2}$$

The I_μ and $I_{\mu\nu}$ can be derived by differentiation of I with respect to k_μ

$$I_\mu(D,\alpha,k,H) = \pi^{D/2}\frac{\Gamma(\alpha-1-\frac{D}{2})}{2\Gamma(\alpha)}\left(\frac{2\pi}{L}\right)^{D-2\alpha+1}\left[(k^2-H)^{-\frac{1}{2}} k_\mu \frac{\partial}{\partial a}F(\alpha-1-\tfrac{D}{2}, a, b) + n_\mu\frac{\partial}{\partial b}F(\alpha-1-\tfrac{D}{2}, a, b)\right] \tag{A.12}$$

$$\begin{aligned} I_{\mu\nu}(D,\alpha,k,H) = \pi^{D/2}\frac{\Gamma(\alpha-2-\frac{D}{2})}{4\Gamma(\alpha)}\left(\frac{2\pi}{L}\right)^{D-2\alpha+2}\Big[g_{\mu\nu}\frac{1}{a}\frac{\partial}{\partial a} + \left(\frac{L}{2\pi}\right)^2\frac{1}{a^2}k_\mu k_\nu\left(\frac{\partial^2}{\partial a^2} - \frac{1}{a}\frac{\partial}{\partial a}\right) \\ + \frac{L}{2\pi}\frac{1}{a}(k_\mu n_\nu + k_\nu n_\mu)\frac{\partial^2}{\partial a\partial b} + n_\mu n_\nu\frac{\partial^2}{\partial b^2}\Big] F(\alpha-2+\tfrac{D}{2}, a, b) \end{aligned} \tag{A.13}$$

where sign $g_{\mu\nu} = (-1,-1, \cdots)$, $n_\mu = (0,1,0,0 \cdots)$. In particular $\mu = \nu = 1$, and $\alpha = 1$, $k^1 = 0$, one has

$$I_{11}(D\ 1\ 0\ 0\ ) = \pi^{D/2}\Gamma(-1-\tfrac{D}{2})\left(\frac{2\pi}{L}\right)^D\left[-\frac{1}{a}\frac{\partial}{\partial a} + \frac{\partial^2}{\partial b^2}\right]F(-1-\tfrac{D}{2}, a, b)\Big|_{\substack{a\to 0\\ b\to 0}} \tag{A.14}$$

In virtue of (A.6) and (A.7), (A.14) may be evaluated in the form

$$\begin{aligned} I_{11}(D\,1\,0\,0)\Big|_{D=3} &= \frac{\pi}{4}^{D/2}\Gamma(-1-\tfrac{D}{2})\left(\frac{2\pi}{L}\right)^D\left[-2\lambda f_{\lambda+1}(a,b) - 2\lambda(2\lambda+1)f_{\lambda+1}(a,b)\right]_{\substack{a,b\to 0\\ \lambda=-5/2}}\cdot 4\sin\pi\lambda \\ &= \left(\frac{\pi}{4}\right)^{3/2}\Gamma(-\tfrac{5}{2})\left(\frac{2\pi}{L}\right)^3\cdot -15\cdot f_{-3/2}\cdot 4 = \frac{\pi^5}{L^3}\frac{8}{30} \end{aligned}$$

hence, the first term in the expansion of (3.40) can be derived

$$U_1^L(\hat{\phi}) = \frac{1}{(2\pi)^3 2L}\int dL \cdot \frac{-2}{L} I_{11}(3,1,0,0) = \frac{\pi^2}{90L^4} + \frac{C}{L}$$

Appendix B

Define for Re $s>\frac{1}{2}$ a series $F(s;\nu)$ by

$$F(s;\nu) = \sum_{n=1}^{\infty} (n^2+\nu^2)^{-s} \tag{B.1}$$

which may be analytically continued throughout the complex s plane to give a function which is analytic everywhere except at $s = \frac{1}{2} - n$, for $n = 0,1,2,\cdots$ where simple poles occur;

$$F(s;\nu) = \frac{1}{2}\nu^{1-2s}\Gamma(\tfrac{1}{2})\frac{\Gamma(s-\frac{1}{2})}{\Gamma(s)} + \frac{1}{2}(1+\nu^2)^{-s} - \int_0^1 (\nu^2+x^2)^{-s}dx$$
$$+ i\int_0^{\infty} \frac{[(1+ix)^2+\nu^2]^{-s} - [(1-ix)^2+\nu^2]^{-s}}{e^{2\pi x}-1} \tag{B.2}$$

The only singularities of $F(s;\nu)$ are simple poles coming from the first term in (B.2) given by $s = \frac{1}{2} - n$, for $n = 0,1,2,\cdots$ while the last integral is analytic in s.

$F(s;\nu)$ can be expanded in a Laurent series about the pole at $s = \frac{1}{2} - n$ as follows

$$F(s;\nu) = \frac{F_{-1}(n;\nu)}{s+n-\frac{1}{2}} + F_0(n;\nu) + O(s+n-\tfrac{1}{2}) \tag{B.3}$$

$$F_{-1}(n;\nu) = \frac{1}{2}\left(\frac{\nu^2}{4}\right)^n \frac{(2n)!}{(n!)^2} \tag{B.4}$$

$$F_0(n;\nu) = \frac{1}{2}\left(\frac{\nu^2}{4}\right)^n \frac{(2n)!}{(n!)^2}\left[\sum_{k=1}^{n}\frac{1}{k} - 2\sum_{k=1}^{n}\frac{1}{2k-1} - \ln\frac{\nu^2}{4}\right] + \frac{1}{2}(1+\nu^2)^{n-\frac{1}{2}}$$
$$- \int_0^1 (\nu^2+x^2)^{n-\frac{1}{2}}dx + i\int_0^{\infty}\frac{[(1+ix)^2+\nu^2]^{n-\frac{1}{2}} - [(1-ix)^2+\nu^2]^{n-\frac{1}{2}}}{e^{2\pi x}-1}dx \tag{B.5}$$

Of particular interest are the cases $n = 1, 2$. In the case of $n = 1$, one has

$$F_{-1}(1;\nu) = \frac{1}{4}\nu^2 \tag{B.6}$$

$$F_0(1;\nu) = -\frac{1}{4}\nu^2 + \frac{1}{4}\nu^2\ln 4 - \frac{1}{2}\nu^2\ln[1+(1+\nu^2)^{\frac{1}{2}}] + i\int_0^{\infty}\frac{[(1+ix)^2+\nu^2]^{\frac{1}{2}} - [(1-ix)^2+\nu^2]^{\frac{1}{2}}}{e^{2\pi x}-1}dx \tag{B.7}$$

for small ν

$$F_0(1;\nu) = -\frac{1}{12} + \frac{1}{2}(\gamma-1)\nu^2 - \frac{1}{6}\zeta_R(3)\nu^4 + \frac{1}{16}\zeta_R(5)\nu^6 + \cdots \tag{B.8}$$

for large ν

$$F_0(1;\nu) = -\frac{1}{4}\nu^2\ln\frac{\nu^2}{4} - \frac{1}{4}\nu^2 - \frac{1}{2}\nu \tag{B.9}$$

In the case n = 2, one has

$$F_{-1}(2;\nu) = \frac{3}{16}\nu^4 \tag{B.10}$$

$$F_0(2;\nu) = -\frac{7}{32}\nu^4 - \frac{1}{8}(1+\nu^2)^{3/2} + \frac{3}{8}(1+\nu^2)^{1/2} + \frac{3}{16}\nu^4 \ln 4 - \frac{3}{8}\nu^4 \ln\left[1+(1+\nu^2)^{1/2}\right]$$
$$+ i\int_0^\infty \frac{\left[(1+ix)^2+\nu^2\right]^{3/2} - \left[(1-ix)^2+\nu^2\right]^{3/2}}{e^{2\pi x}-1}\,dx \tag{B.11}$$

for small ν

$$F_0(2;\nu) = \frac{1}{120} - \frac{1}{8}\nu^2 + \left(\frac{3}{8}\gamma - \frac{1}{2}\right)\nu^4 - \frac{1}{16}\zeta_R(3)\nu^6 + \cdots \tag{B.12}$$

for large ν

$$F_0(2;\nu) = -\frac{3}{16}\nu^4 \ln\frac{\nu^2}{4} - \frac{7}{32}\nu^4 - \frac{1}{2}\nu^3 \tag{B.13}$$

A related series is

$$D(s;\nu) = \sum_{n=-\infty}^{\infty}(n^2+\nu^2)^{-s} = 2F(s;\nu) + \nu^{-2s} \tag{B.14}$$

which converges for Re s $>\frac{1}{2}$ and may be analytically continued throughout the complex s plane, where it will have the same analytic structure as $F(s;\nu)$.

Expanding

$$D(s;\nu) = \frac{D_{-1}(n;\nu)}{s+n-\frac{1}{2}} + D_0(n;\nu) + O(s+n-\tfrac{1}{2}) \tag{B.15}$$

for s in a neighberhood of $s = \frac{1}{2} - n$

$$D_{-1}(n;\nu) = 2F_{-1}(n;\nu) = \frac{3}{8}\nu^4 \tag{B.16}$$

$$D_0(n;\nu) = 2F_0(n;\nu) + \nu^{2n-1} \tag{B.17}$$

Appendix D Heat kernal method

For simplicity, let us start from scalar field

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \xi R \phi^2$$

whose equation of motion is

$$(-\Box_c + \xi R + m^2) \phi(x) = 0$$

The corresponding Green function obeys

$$(-\Box_c + \xi R + m^2) G(x\, y) = \frac{1}{\sqrt{g}} \delta(x-y) \tag{C.1}$$

Most of our calculations and analyses are based upon how to construct Green functions, but a general knowledge of Green functions is diffcut to obtain. Under some certain situation, the DeWitt- Schwinger proper time formalism is particular useful (i.e.heat kernal method). In order to explain it more clearly, let us introduce a Hilbert space of state vectors $|x\rangle$ which are eigenvectors of the position operator $\hat{x}$, $\hat{x}|x\rangle = x|x\rangle$, normalized according to

$$\langle x | y \rangle = \frac{1}{\sqrt{g}} \delta(x, y)$$

$$\int d^4x \sqrt{g}\, |x\rangle\langle x| = 1$$

If we adopt a symbolic notation where $\mathbb{1}$ stands for the delta - function, A a differential operation, then the Green function can be expressed as an average value of some operator $\hat{G}$ in this Hilbert space

and $A(x,y) = -\Box_c + \xi R + m^2$ can be expressed as follows

$$A(x\, y) = \langle x | A | y \rangle = A(x) g^{-1/2} \delta(x, y)$$

so (C.1) can be rewritten as

$$A\, G = \mathbb{1} \tag{C.2}$$

which has a formal solution in terms of a parameter s

$$G = A^{-1} = \int_0^\infty e^{-sA} ds \tag{C.3}$$

and hence

$$G(x\, y) = \int_0^\infty ds \langle x | e^{-sA} | y \rangle \tag{c.4}$$

Actually, s is only a parameter here, although it possesses double time dimension $[s] = [t^2]$ (in natural unit). If we regard s as a proper time, e^{-As} may be considered as an evolution operation and (C.4) may be explained as the propagation amplitude from x to y in a time s summed over all possible values of s.

Defining

$$|y,s\rangle = e^{sH}|y\rangle \quad , \quad \langle x,s| = \langle x|e^{-sH}$$

(C.4) may be rewritten in the form

$$G(x\,y) = \int_0^\infty ds \langle x,s|y,0\rangle := \int_0^\infty ds\, K(x,y;s) \tag{C.5}$$

where $K(x\,y;s) := \langle x,s|y,0\rangle$ is called heat kernal basing on the following reason: with the aid of (C.2), one may easily find $K(x,y,s)$ satisfies heat diffusion equation

$$A\,K(x\,y;s) = \frac{-\partial K(x\,y,s)}{\partial s} \tag{C.6a}$$

with the boundary conditions

$$K(x\,y,0) = g^{-\frac{1}{2}}\delta(x\,y) \tag{C.6B}$$

In flat spacetime, the solution of (C.6) is

$$K(x\,y,s) = \frac{e^{-m^2 s}}{(4\pi s)^2}\, e^{-\sigma^2(x\,y)/4s} \tag{C.7}$$

where $\sigma^2(x,y) = (X-y)^2$, σ is the geodesic distance between x and y. In curved space, the possible solution of (C.6) may be written in such a form

$$K(x,y,s) = \frac{e^{-m^2 s}}{(4\pi s)^2}\, \Delta^{\frac{1}{2}}(x,y)\, e^{-\frac{\sigma(x\,y)}{4s}}\, F(x\,y,s) \tag{C.8}$$

where $F(x,y,s)$ is a function presently to be determined and $\Delta^{1/2} = g^{-1/2}[-\det(-\frac{1}{2}\,\partial^2\sigma^2/\partial x\partial y)]\,g^{-1/2}$. Inserting (C.8) into (C.6) we obtain an equation for F which are not able to be solved in general. But we can resort to asymptotic expansion to represent F for $s \longrightarrow 0$

$$F(x\,y\,s) = \sum_{n=0}^{\infty} E_n(x\,y)\, s^{n/2} \tag{C.9}$$

and hence

$$K(x\,y\,s) \approx \frac{e^{-m^2 s}}{(4\pi s)^2} \Delta^{\frac{1}{2}}(x\,y)\, e^{-\frac{\sigma(x\,y)}{4s}} \sum_{n=0}^{\infty} E_n(x\,y)\, s^{n/2} \tag{C.10}$$

In the limit of $y \to x$, which is implied in the operation of trace for calculation of the effective action, $\sigma(xy) \to 0$, $\Delta^{\frac{1}{2}}(x\ y) \to 1$, we have

$$K(x\,x\,s) \approx \frac{e^{-m^2 s}}{(4\pi s)^2} \sum_{n=0}^{\infty} E_n(x)\, s^{n/2} \tag{C.11}$$

Return now to the expression of the effective action (4.3)

$$\Gamma^{(1)} = \int d^4x \sqrt{g}\, V^{(1)}(\hat{\phi}) = -\frac{1}{2} Tr\, \ln A$$

Considering the functional variation of $\Gamma^{(1)}$ with respect to mitric

$$\delta\Gamma^{(1)} = -\frac{1}{2} Tr\, \delta \ln A = -\frac{1}{2} Tr \frac{\delta A}{A} = -\frac{1}{2} Tr \int_0^{\infty} \delta A\, e^{-As} = -\frac{1}{2} Tr\left(-\delta \int_0^{\infty} \frac{ds}{s} e^{-As}\right)$$

the effective action is

$$\Gamma^{(1)} = \frac{1}{2} Tr \int_0^{\infty} \frac{ds}{s} e^{-As} \tag{C.12}$$

And it's average values between states in above Hilbert space give rise to heat kernal expression of the effective action

$$\Gamma^{(1)} = \int d^4x \sqrt{g}\, V^{(1)}(\phi) = \frac{1}{2} Tr \int_0^{\infty} \frac{ds}{s} K(x\,y\,s) \tag{C.13}$$

where Tr denotes functional trace over x and y

$$\Gamma^{(1)} = \int d^4x \sqrt{g}\, V^{(1)}(\phi) = \frac{1}{2} \int d^4x \sqrt{g} \int_0^{\infty} \frac{ds}{s} K(x\,x\,s) \tag{C.14}$$

whose asymptotic expantion is

$$\Gamma^{(1)} = \sum_n \frac{1}{2(4\pi)^2} \underbrace{\int_0^{\infty} ds\, s^{n/2-3} e^{-m^2 s}}_{\Gamma(\frac{n}{2}-2)(m^2)^{2-n/2}} \left\{ \int_M b_n(x) \sqrt{g}\, d^4x + \int_{\partial M} C_n(x)\, h^{\frac{1}{2}} d^3x \right\}$$

where $b_m(x)$ are local invarints of the curvature and $c_n(x)$ invariants of the intrinsic and extrinsic curvature of the boundary and $h^{\frac{1}{2}} d^3x$ the induced measure on the boudary ∂M ($h = \det h_{\mu\nu}$, $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, n_μ is the spacelike normal on the boundary). If the manifold M has no boundary $\partial M = \phi$, only the volume part b_n contribute and b_n vanishes for n = odd and the divergent terms only come from n = 0, 2, 4.

In the massless case m = 0, the first three b_n are

$$b_0 = 1$$
$$b_2 = \left(\frac{1}{6} - \xi\right) R$$
$$b_4 = \frac{1}{180}\left[R^{\lambda\mu\nu\rho} R_{\lambda\mu\nu\rho} - R^{\mu\nu} R_{\mu\nu}\right] + \frac{1}{2}\left(\frac{1}{6} - \xi\right)^2 R^2 + \frac{1}{6}\left(\frac{1}{5} - \xi\right) \Box R$$

Appendix D Theta function

Define a function

$$f(z) = \sum_{n=-\infty}^{\infty} e^{-n^2 t + 2niz}$$

which has a property

$$f(z+\pi) = f(z)$$

If $q = e^{\pi i t}$, we have the third type Θ function

$$\Theta_3(z, t) = 1 + 2q \cos 2z + 2q^4 \cos 4z + 2q^9 \cos 6z + \cdots$$

in particular,

$$\Theta_3(0, t) = \phi(t) = \sum_{-\infty}^{\infty}{}_n \, e^{n^2 \pi i t}$$

so that

$$\Theta_3\left(0, -\frac{\beta^2}{4\pi i s}\right) = \sum_{-\infty}^{\infty}{}_n \, e^{-\frac{n^2\beta^2}{4s}}$$

Appendix E

$$V^{(1)} = -\frac{\hbar(4\pi)^{\frac{1}{2}-\omega}}{2\beta}\Big[\Gamma(\tfrac{1}{2}-\omega)M^{2\omega-1} + \frac{\Gamma(\frac{1}{2}-\omega)2\beta^{1-2\omega}}{(4\pi^2)^{\frac{1}{2}-\omega}}\zeta_R(1-2\omega) - \frac{M^2\Gamma(\frac{3}{2}-\omega)}{(4\pi^2)^{\frac{3}{2}-\omega}}2\beta^{3-2\omega}\zeta_R(3-2\omega)$$

$$+2\sum_{\ell=2}^{\infty}\frac{(-1)^{\ell}}{\ell!}M^{2\ell}\frac{\Gamma(\ell+\frac{1}{2}-\omega)}{(4\pi^2)^{\ell+\frac{1}{2}-\omega}}\beta^{2\ell+1-2\omega}\zeta_R(2\ell-\omega-1)$$

$$\equiv V_A + V_B + V_C + V_D$$

If $\omega = 2$, the first two terms can be evaluated imediately

$$V_A = -\frac{\hbar(4\pi)^{-3/2}}{2\beta}\Gamma(-\tfrac{3}{2})M^3 = -\frac{\hbar M^3}{12\pi\beta} \tag{E.1}$$

$$V_B = -\frac{\hbar\Gamma(2)\zeta_R(4)}{\pi^2} = -\frac{\hbar\pi^2}{90\beta^4} \tag{E.2}$$

where following relations are used to derive (E.1) and (E.2)

$$\Gamma(-\tfrac{3}{2}) = \tfrac{4}{3}\sqrt{\pi}\ , \quad \zeta_R(4) = \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90}$$

In order to compute V_C, we may make use of the following duality relation

$$\frac{\Gamma(\frac{s}{2})\zeta_R(s)}{\pi^{s/2}} = \frac{\Gamma(\frac{1-s}{2})\zeta(1-s)}{\pi^{1-s/2}} \tag{E.3}$$

hence

$$V_C = \frac{\hbar(4\pi)^{\frac{1}{2}-\omega}}{\beta}\frac{M^2}{(4\pi)^{\frac{3}{2}-\omega}}\frac{\Gamma(\frac{3}{2}-\omega)\zeta_R(3-2\omega)}{\pi^{\frac{3}{2}-\omega}}\beta^{3-2\omega}$$

$$= \frac{\hbar M^2}{4\pi\beta^{2\omega-2}}\frac{\Gamma(\omega-1)\zeta_R(2\omega-2)}{\pi^{\omega-1}} \underset{\uparrow\,\omega=2}{=} \frac{\hbar M^2}{4\pi\beta^2}\frac{\zeta_R(2)}{\pi} = \frac{\hbar M^2}{24\beta^2} \tag{E.4}$$

The $\ell = 2$ term in the series V_D is divergent because it contains $\zeta_R(1)$, so we must put $\ell = 2+\varepsilon$ and expand it around $\ell = 2$,

$$V_D(\ell=2+\varepsilon) = -\frac{(-1)^{\ell}\hbar\pi^{3/2}}{\beta^4}(M^2\beta^2)^{\ell}\frac{\Gamma(\ell-\frac{3}{2})\zeta_R(2\ell-3)}{\Gamma(\ell+1)(4\pi^2)^{\ell}}\Bigg|_{\ell=2+\varepsilon} \tag{E.5}$$

In virtue of the expansion relationsbelow, when $\varepsilon \to 0$

$$(M^2\beta^2)^{2+\varepsilon} = (M^2\beta^2)^2 + \varepsilon(\beta^2M^2)^2\ln\beta^2M^2$$

$$(4\pi^2)^{2+\varepsilon} = (4\pi^2)^2 + \varepsilon(4\pi^2)^2\ln(4\pi^2)$$

$$\zeta_R(2\ell-3) = \zeta_R(1+2\varepsilon) = \frac{1}{2\varepsilon} + \gamma + O(2\varepsilon)$$

$$\Gamma(\ell-\tfrac{3}{2}) = \Gamma(\tfrac{1}{2}+\varepsilon) = \Gamma(\tfrac{1}{2})+\varepsilon\,\Gamma'(\tfrac{1}{2}) = \sqrt{\pi} - \varepsilon\sqrt{\pi}\,(\gamma+2\ln 2)$$

$$\Gamma(\ell+1) = \Gamma(3+\varepsilon) = \Gamma(3)+\varepsilon\,\Gamma'(3) = 2+\varepsilon(3-2\gamma)$$

$$\psi(n) = \frac{\Gamma'(z)}{\Gamma(z)}\Big|_{z=n} = -\gamma+\sum_{m=1}^{n-1}\frac{1}{m}$$

$$\Gamma'(1) = \gamma\,, \quad \Gamma'(3) = -2\gamma+3\,, \quad \Gamma'(\tfrac{1}{2}) = -\sqrt{\pi}\,(\gamma+2\ln 2) \tag{E.6}$$

the divergent part in V_D can be isolated, yielding

$$V_D(\ell=2+\varepsilon) = -\frac{\hbar\pi^{3/2}}{\beta^4}\Big\{\Big[\Big(\frac{M^2\beta^2}{4\pi^2}\Big)^2+\varepsilon\Big(\frac{M^2\beta^2}{4\pi^2}\Big)^2\ln\frac{\beta^2M^2}{4\pi^2}\Big]\Big[\sqrt{\pi}-\varepsilon\sqrt{\pi}(\gamma+2\ln 2)\Big]\frac{1}{\Gamma^2(3)}\Big[\Gamma(3)+\varepsilon(3-2\gamma)\Big]$$
$$\times\Big(\frac{1}{2\varepsilon}+\gamma+O(2\varepsilon)\Big)\Big\}$$
$$= -\frac{\hbar\pi^{3/2}}{\beta^4}\,\frac{M^4\beta^4\sqrt{\pi}}{2\cdot(4\pi^2)^2}\Big(\frac{1}{2\varepsilon}+\gamma\Big) - \frac{\hbar\pi^{3/2}}{\beta^4\cdot 2}\Big(\frac{M^2\beta^2}{4\pi^2}\Big)^2\Big[\ln\frac{\beta^2M^2}{4\pi^2}-\sqrt{\pi}(\gamma+2\ln 2)-\frac{3-2\gamma}{2}\Big]\cdot\frac{1}{2^2}$$
$$= -\frac{\hbar M^4}{4(4\pi)^2}\,\frac{1}{\varepsilon} - \frac{\hbar M^4}{4(4\pi)^2}\Big[\ln\frac{M^2\beta^2}{4\pi^2}+2\gamma-2\ln 2-\gamma-\frac{3}{2}+\gamma\Big]$$

$$= -\frac{\hbar M^4}{4(4\pi)^2}\,\frac{1}{\varepsilon} - \frac{\hbar M^4}{4(4\pi)^2}\ln M^2\beta^2 + \frac{\hbar M^4}{4(4\pi)^2}\Big[2\ln 2+\frac{3}{2}-2\gamma\Big] \tag{E.7}$$

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