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VARIATIONAL AND TOPOLOGICAL METHODS FOR PERIODIC ORBITS OF HAMILTONIAN SYSTEMS

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1.1 INTRODUCTION

The search for periodic solutions of hamiltonian systems, for which one of the main sources of inspiration was the field of celestial mechanics, has a long history.

During the past few years there has been a renewed interest in it and a considerable amount of progress: many new ideas and methods of solution developed.

We don't intend here to give a systematic survey of the results obtained in time. For this we refer to the paper |54| of Rabinowitz or to the lecture notes |71| of Zehnder.

Rather our goal is to consider certain particular problems in this ambit and to compare different techniques, which allow to approach and solve them; to show how new tools were introduced to get rid of the encountered difficulties.

We will devote our attention to problems and methods, which are of interest also in other sectors of mathematical physics. Precisely we'll be mainly concerned with the search of critical points of a functional (problem to which our original will be reconduced).

To such an investigation one is led starting from a variety of different questions. Everybody knows the role of variational principles in classical mechanics: let us just recall the principle of minimum potential energy, from which the equations of equilibrium of elastic bodies can be deduced, or the Hamilton's principle as a substitute to the equations governing dynamical systems. Also nonlinear Dirichlet problems admit a variational formulation.

Roughly: if $\Omega \subset R^n$ is a bounded domain with "smooth" boundary ,
suppose we want to find a solution of the problem

$$(1.1.1) \qquad \begin{cases} -\Delta u = g(u) & u \in \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian operator, $g:R \to R$ is a possibly nonlinear function.

Let us put $G(u) = \int_0^u g(v)dv$.

Then if $f(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(u)$, where u is a function in an opportune space (which we have to choose) and u vanishes on $\partial\Omega$, formally it is $f'(u) = -\Delta u - g(u)$. Therefore $f'(u) = 0$ if and only if u is solution of (1.1.1).

Clearly this has to be rigorously formalized. For example we should prove that f is differentiable on its domain of definition.

Finally, to mention some more "modern" example (to others, and they are really many, we will refer as we will explain the methods), we recall that the differential equations of a classical gauge theory are in many cases the formal variational equations of a functional on a topologically non trivial space.

In this connection to establish the existence of non trivial solutions for the Yang-Mills-Higgs equation a min-max procedure is successfully used in |68|, while in |7| the Yang-Mills functional over a compact Riemann surface is studied by means of the equivariant Morse theory.

We will present both min-max principle and equivariant Morse theory.

A nice application of the equivariant Morse theory is also in |51| and |52|, where it is used to give an estimate of the minimal number of central configurations in the n-body problem in R^3.

1.2 PRELIMINARS

Let $p,\ q \in R^n$ and $h = h(p,q) \in C^1(R^{2n},R)$.

An autonomous hamiltonian system has the form

$$(1.2.1) \qquad \begin{cases} \dot{p}_i = -\dfrac{\partial h(p,q)}{\partial q_i} \\ \dot{q}_i = \dfrac{\partial h(p,q)}{\partial p_i} \end{cases} \qquad i = 1,\ldots n$$

where $\dot{}$ denotes $\frac{d}{dt}$. This system can be represented more coincisely:

$$(1.2.2) \qquad \dot{z} = J\,\nabla h(z)$$

where $z = (p,q) \in R^{2n}$, $J = \begin{vmatrix} 0 & -I \\ I & 0 \end{vmatrix}$ (with I the identity in R^n) is the standard symplectic structure on R^{2n}, and ∇h is the gradient of h. If the hamiltonian h depends in an explicit and periodic fashion on time, we have the system

$$(1.2.3) \qquad \dot{z} = J\,\nabla h(t,z).$$

We approach the study of the existence of periodic orbits of (1.2.2) and (1.2.3) with the consciousness that it is a minor aspect in the understanding of the global phase portrait, but nevertheless a first step. After stationary points in fact periodic solutions are the simplest objects in the qualitative theory of dynamical systems. Despite of this even the solution of the problem of the existence of periodic trajectories is often highly non trivial and requires the use of topological methods. And we are indeed fascinated from the variety of problems and methods, to which this study gave rise.

There are different questions which can be considered, local or global.

One set of questions has been motivated by the fact that h is a constant of motion for (1.2.2). Thus one can look for solutions of (1.2.2) having a prescribed energy, and also ask what geometrical properties must an energy surface possess in order to ensure the existence of periodic orbits of (1.2.2) on it.

Other questions of interest are connected with the search of orbits of (1.2.2) having a prescribed period and orbits of (1.2.3) having the period of the hamiltonian. One motivation for such a problem is that if we simply know of existence of some periodic orbit, whose period is unknown, the doubt could remain that it be very large, and in this case our information would be lacking in interest.

We will concentrate on the second one of these two general problems, which is often considered also as a tool to solve the first one.

The underlying theme in the recent treatment of such problems has been the use of calculus of variations, in finding solutions as critical points of a functional.

The techniques used belong to global analysis and the results obtained are global.

Sometimes it is of interest to have local results: to find periodic solutions in a neighborhood, for example, of an equilibrium solution. In section 5.1 we will adopt this point of view; we will shortly mention some classical results and some very recent ones.

1.3 PRESCRIBED PERIOD CASE: THE VARIATIONAL PROBLEM. COMPARISON OF DIFFERENT TECHNIQUES AND PROGRAM

Let us consider the system

$$\dot{z} = J \nabla h(t,z)$$

(1.3.1) $$h(t,z) \in C^2(R \times R^{2n}, R)$$

$$h(t+T,z) = h(t,z)$$

and let us look for T-periodic solutions of (1.3.1). We reformulate the problem as an abstract variational problem for a functional in a loop space.

A natural function space is the real Hilbert space $H = L^2(0,T;R^{2n})$.

Let us define in H the linear operator

$A: dom(A) \subset H \dashrightarrow H$, by setting

$dom(A) = \{u \in H^1(0,T;R^{2n}) \mid u(T) = u(0)\}$,

as $Au \doteq -J\dot{u}$,

(H^1 is the Sobolev space of absolutely continuos functions, whose first derivative is in L^2).

Then define the continuos operator

$F: H \dashrightarrow H$, by

$F(u)(t) \doteq \nabla h(t,u(t))$.

Writing (1.3.1) in the form

$$- J\dot{z} = \nabla h(t,z)$$

we see that every solution $u \in dom(A)$ of the equation

(1.3.2) $$Au = F(u)$$

defines a classical T-periodic solution of (1.3.1). Conversely a T-periodic solution of (1.3.1) defines by restriction a solution u of (1.3.2).

Equation (1.3.2) is the Euler equation of the variational problem

$$\text{extr}\,\{f(u) \mid u \in \text{dom}(A)\},$$

where $f(u) = \frac{1}{2} \langle Au,u \rangle_H - \phi(u)$,

$$\phi(u) = \int_0^T h(t,u(t)dt.$$

In classical notations

$$f(u) = \int_0^T \left[\frac{1}{2} (\dot{u},Ju) - h(t,u(t)) \right] dt.$$

Therefore in order to find the required solutions of (1.3.2) we can just as well look for critical points of the action functional f.

The point now is that this functional is not bounded from above nor from below, even modulo compact perturbations. We express this by saying that it is indefinite.

This property, which is peculiar also to other variational problems of mathematical physics, increases the difficulties. Till few years ago no method was known to proceed in a direct way in this case in the setting of variational calculus. Direct methods like the min-max principle were in fact useful to garantue existence of critical points only for functionals bounded from below, or at least which became such after have been compactly perturbed.

Recently, as we will see, a direct way was found by Benci,starting from the pioneering work of Ambrosetti-Rabinowitz (|5|) and Benci-Rabinowitz (|15|), which allows to overcome the obstacle (see |12|, |13|, |16| and |10|).

Our program now is the following:

first of all we will recall the Lyusternik-Schnirelman criticalpoint theory for smooth functions on smooth finite dimensional manifolds. In this setting we will learn the basic ideas, from which variational methods draw inspiration. After that we will show how this theory has been generalized to manifolds modeled on Hilbert spaces. We will emphasize the meaning of the assumptions, which are needed for such a generalization (substantially the Palais-Smale condition for the functional). And we will also show how the violation of the Palais-Smale condition, which turns out to characterize some physical problem, can be in certain cases by-passed. We will then, remembering our starting point, expose the principal lines of the theory (due to Benci), by means of which min-max techniques can be applied also to indefinite functionals.

Actually when it was not known how to find critical points of indefinite functionals in any direct fashion, other methods were used.

We mention at this purpose the papers of Rabinowitz (|55|, |56|), who proved, in correspondence to every T, the existence of a T-periodic solution for hamiltonian systems, with hamiltonian superquadratic at infinity,by means of an approximation procedure: (a) restricting the functional to a finite dimensional subspace; (b) hence using a min-max argument for the finite dimensional problem, and then (c) passing to a limit, in order to find the solution of the original problem. The last part is very technical: in going to the limit one has to make sure that the stationary point obtained in step (b) does not run off to infinity nor to the origin; and this is achieved with the complicity of laborious estimates.

More recently Amann and Zehnder (|2|, |3|) treated the variational problem arising from hamiltonian equations [as well as the analogous one, arising from the search of periodic orbits of the wave equation $u_{tt} - u_{xx} = f(t,x,u)$] by means of a finite dimensional reduction, which is possible if the hessian of the hamiltonian is bounded [if a particular boundedness condition for the derivative f_u of the nonlinearity is satisfied], and then by means of a generalization of Morse theory, due to Conley. Their results have then be strenghtened by Chang (|26|), who also simplified the second part of their proof, substituting an approach via the classical Morse theory. With respect to the boundedness of the hessian of the hamiltonian, this is automatically satisfied when an hamiltonian vectorfield with $h(t,z) \in C^2(RxR^{2n},R)$, 1-periodic is given on the torus T^{2n}. In |31| this situation is studied from Conley and Zehnder. In the same paper the authors prove a conjecture of V. I. Arnold: every measure preserving C^1 diffeomorphism of T^2, which is of the form : $x \text{---}> x + f(x)$, $x \in R^2$, f periodic with meanvalue over the torus [f] = 0, has at least three fixed points.

Many results were also obtained by means of dual methods, developed by convex analysts (Clarke,Ekeland)(see |27| and |28|). Such methods apply to hamiltonian systems with the hamiltonian having certain convexity properties. Part of the idea of Clarke is to employ a Legendre transformation to convert the problem to a simpler one: in contrast to the Legendre transformation usually encountered in mechanics, here all the variables are transformed and a "dual" functional is obtained (whose critical points have to be looked for), which is bounded from below and hence much easier to study. The dual action principle has also been applied in existence problems of nonlinear partial differential equations (|14|, |23|).

We remark that if this appears as a nice trick from a mathematical point of view, it carries a relatively strong limitation. It is in fact applicable only in case the convexity assumptions are satisfied. Since we would like to give answer to problems as they naturally arise, we should try to avoid at most at possible restrictions on the cases of applicability.
Also: concerning the results of Rabinowitz contained in |55| and |56|, as we already told, they apply to superquadratic hamiltonians. They don't cover hence most of the classical mechanical problems. In fact for a mechanical system with constraints not depending on time, imbedded in a conservative field of forces, the hamiltonian has the form

$$h(p,q) = \sum_{i,j=1}^{n} a_{ij}(q)p_i p_j + V(q),$$

where $[a_{ij}(q)]$ is a positive definite matrix for every $q \in R^n$.
And such an hamiltonian is quadratic in p!
It is therefore important to be able to handle the problem in a direct way; it is in this spirit that we present the recent results of Benci.

Now: in a parallel way we will present (together with the Lyusternik-Schnirelman theory) the Morse theory for smooth functions on smooth compact finite dimensional manifolds.

This theory too can be extended to manifolds modeled on Hilbert spaces.

But the generalizations we will give here are of a different kind: we will illustrate the Conley's index theory and the equivariant Morse theory.

The first one can be viewed as a generalization of Morse theory for flows other than gradient flows on locally compact metric spaces. It is a useful tool in problems of nonlinear functional analysis. It was used to find special shocks in|66|, to prove existence and multiplicity results for systems of nonlinear elliptic boundary value problems in |2|, the existence of heteroclinic orbits for semilinear parabolic equations in |58|. In |30| and |31| the Conley's index theory allowed to find periodic solutions of time dependent hamiltonian systems.

The second one comprises the necessary modifications of the Morse theory when a symmetry group is present.

2.1 CRITICAL POINT THEORIES IN FINITE DIMENSIONS

Morse theory furnishes very detailed informations on the number of critical points of a C^2 function f on a compact manifold by means of homological methods. But it requires an hypothesis, which is in concrete problems rather difficult, if not impossible, to verify: it must in fact be a priori known that the eventual critical points are nondegenerate.

Lyusternik-Schnirelman theory does not present this difficulty. Moreover: the regularity of the function may be lower. It suffices in fact for f to be C^1 instead of C^2. But the results furnished are less specific than those of Morse Theory.

2.2 LYUSTERNIK-SCHNIRELMAN THEORY

The critical point theory of Lyusternik-Schnirelman is based on determining a topological analogue for the min-max principle, which characterizes the eigenvalues of a self-adjoint compact operator L. If the positive eigenvalues of L are denoted λ_1^+, λ_2^+, ..., arranged in decreasing order and counted according the multiplicity, then

$$\lambda_n^+ = \sup_{[S_{n-1}]} \min_{x \in S_{n-1}} (Lx,x) \tag{2.2.1}$$

where S_{n-1} denotes the unit sphere in an arbitrary n dimensional linear subspace Σ of H, and $[S_{n-1}]$ denotes the class of such spheres as Σ varies in H. Since the eigenvalues of L are precisely the critical values of the functional (Lx,x) on the unit sphere $\partial\Sigma_1 = \{x \mid |x| = 1\}$ of H, it is natural to try to extend (2.2.1) to general smooth functionals $F(x)$ by finding topological analogues for the sets S_{n-1} and $[S_{n-1}]$.

We start considering the case of a C^∞ real valued function f on a compact C^∞ Riemannian manifold M.

The basic technique will be the one of gradient flow, also called the method of "steepest descent".

Let ϕ_t be the flow generated by the C^∞ vectorfield $-\nabla f$. $\phi_t(p)$ is called the path of steepest descent of the point p. Note that

$$\frac{d}{dt} f(\phi_t(p)) = \langle \nabla f_{\phi_t(p)}, -\nabla f_{\phi_t(p)} \rangle = -|\nabla f_{\phi_t(p)}|^2. \tag{2.2.2}$$

Thus there are exactly two possibilities: either $|\nabla f_p|$ is 0, i.e. p is a critical point of f, (in which case $\phi_t(p) = p \ \forall t$), or else

$|\nabla f_p| \neq 0$, in which case $f(\phi_t(p))$ is less than (greater than) $f(p)$ for $t \geqslant 0$ (for $t \leqslant 0$), so that $\phi_t(p) \neq p$ for $t \neq 0$.

Let $K = K(f)$ denote the set of critical points of f, clearly a closed and hence compact subset of M. The compact set of real numbers $f(K)$ is called the set of critical values of f; and the complementary open set of real numbers is called the set of regular values of f. Thus $c \in R$ is a critical value of f if the level $f^{-1}(c)$ contains at least one critical point of f and it is a regular value if $f^{-1}(c) = \emptyset$ or contains only regular values of f.

The min-max principle is a very general method for locating critical values of f. It is a consequence of the deformation theorem, which we will give next.

We'll denote $f^c = f^{-1}((-\infty,c])$ and $K_c = K \cap f^{-1}(c)$.

Lemma 2.2.3: Let $x_o \in M$ be a regular point of f and let $f(x_o) = c$. Then there is an $\varepsilon > 0$ and a neighborhood V of x_o such that $\phi_1(V) \subset f^{c-\varepsilon}$.

Proof:

it is evident from (2.2.2) that $f(\phi_t(x_o))$ is monotone nonincreasing. Moreover at $t = 0$ the derivative is $-|\nabla f_{x_0}|^2$, strictly negative and hence $f(\phi_1(x_o))$ is less than $f(\phi_o(x_o)) = f(x_o) = c$. So for some $\varepsilon > 0$ $f(\phi_1(x_o)) < c - \varepsilon$ and also for x in a neighborhood V of x_o $f(\phi_1(x)) < c - \varepsilon$.

Deformation theorem 2.2.4: Given $c \in R$, let U be a neighborhood of K_c in M. Then there is an $\varepsilon > 0$ s.t. $\phi_1(f^{c+\varepsilon} - U) \subseteq f^{c-\varepsilon}$. In particular if c is a regular value of f, then there is an $\varepsilon > 0$ s.t. $\phi_1(f^{c+\varepsilon}) \subseteq f^{c-\varepsilon}$.

Proof:

for each x in the compact set $X = f^{-1}(c) - U$ choose a neighborhood V_x of x in M and a $\delta_x > 0$ s.t. $\phi_1(V_x) \subseteq f^{c-\delta_x}$.
Let $V_{x_1} \cup \ldots \cup V_{x_m}$ cover X and let $\delta = \min(\delta_{x_1}, \ldots \delta_{x_m})$, so that if $\varepsilon < \delta$, then $\phi_1(V_{x_1} \cup \ldots \cup V_{x_m}) \subseteq f^{c-\varepsilon}$. Since M is compact and $\vartheta = U \cup V_{x_1} \cup \ldots \cup V_{x_m}$ is a neighborhood of $f^{-1}(c)$, there is an $\varepsilon > 0$, which we can assume less than δ, s.t. $f^{-1}([c-\varepsilon, c+\varepsilon]) \subseteq \vartheta$.
Then since $f^{c+\varepsilon} \subseteq f^{c-\varepsilon} \cup f^{-1}([c-\varepsilon, c+\varepsilon])$
and $\vartheta - U \subseteq V_{x_1} \cup \ldots \cup V_{x_m}$,

$$f^{c+\varepsilon} - U \subseteq f^{c-\varepsilon} \cup (V_{x_1} \cup \ldots \cup V_{x_m}).$$

But both $\phi_1(f^{c-\varepsilon})$ and $\phi_1(V_{x_1} \cup \ldots \cup V_{x_m})$ are included in $f^{c-\varepsilon}$.

We are now in the position to formulate and prove the min-max principle.

Let $\mathfrak{F}$ be a family of subsets of M. We define the min-max of f over $\mathfrak{F}$ by

$$\text{min-max}\,(f,\mathfrak{F}) = \inf_{F\in\mathfrak{F}} \sup_{x\in F} f(x).$$

It is easily seen that an equivalent definition is

$$\text{min-max}\,(f,\mathfrak{F}) = \inf \{c \in R \mid \exists F \in \mathfrak{F} \text{ with } F \subseteq f^c\}.$$

The family $\mathfrak{F}$ is called an ambient isotopy invariant if given an isotopy g_t of M (i.e. a C^∞ map : $(t,x) \dashrightarrow g_t(x)$ of $[0,1] \times M$ into M with each g_t a diffeomorphism of M onto M and g_o the identity) and $F \in \mathfrak{F}$, it follows that $g_1(F) \in \mathfrak{F}$.

<u>Min-max principle 2.2.5</u>: If $\mathfrak{F}$ is an ambient isotopy invariant family of subsets of M, then min-max $(f,\mathfrak{F})$ is a critical value of f.

Proof:

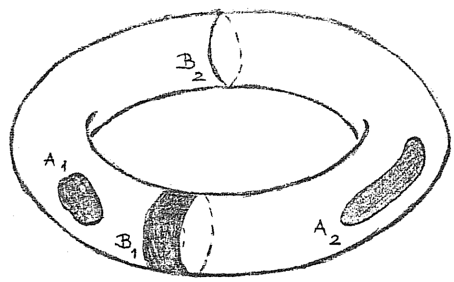
suppose c = min-max $(f,\mathfrak{F})$ is a regular value of f and let $F \in \mathfrak{F}$ with $F \subseteq f^{c+\varepsilon}$, where $\varepsilon > 0$ is chosen as in the deformation theorem. Since $(\phi_t)_{0 \le t \le 1}$ is an isotopy of M, it follows that $\phi_1(F) \in \mathfrak{F}$.

But $\phi_1(F) \subseteq \phi_1(f^{c+\varepsilon}) \subseteq f^{c-\varepsilon}$. So min-max $(f,\mathfrak{F}) \leqslant c - \varepsilon$, a contradiction.

One of the most important applications of the min-max principle is the derivation of the Lyusternik-Schnirelman critical point theory, which we will now recall.

Definition 2.2.6: A subset A of M is said to have Lyusternik-Schnirelman category m in M (and we write cat (A;M) = m) if A can be covered by m, but not fewer, closed subsets of M, each of which is "contractible" to a point in M. This means that the inclusion map into M is homotopic, as a map into M, to a constant map.

E.g. on the torus T^2 there are sets (like A_1, A_2) with cat $(A_i;T^2) = 1$ and sets (like B_1, B_2) with cat $(B_i;T^2) = 2$. The torus



itself, it can be showed, has category cat $(T^2;T^2) \doteq \text{cat}(T^2) = 3$.

Next we give some properties of the function cat(.,M):

i) cat $(A;M) = 0 \iff A = \emptyset$;

ii) cat $(A;M) = 1 \iff \bar{A}$ is contractible in M;

iii) cat $(A;M) = \text{cat}(\bar{A};M)$;

iv) if A is closed in M, then cat $(A;M) \leqslant m \iff$ A is the union of m closed sets, each contractible in M;

v) $A \subseteq B \subseteq M \implies \text{cat}(A;M) \leqslant \text{cat}(B;M)$;

vi) cat $(A \cup B;M) \leqslant \text{cat}(A;M) + \text{cat}(B;M)$;

vii) if A is closed and deformable through M into B (i.e. if the inclusion map of A into M is homotopic as a map into M with a map of A into B), then cat $(A;M) \leqslant \text{cat}(B;M)$;

viii) if h is an homeomorphism of M onto itself, then

$$\text{cat}\ (h(A);M) = \text{cat}\ (A;M).$$

Now: given a positive integer $m \leqslant \text{cat}(M)$, let us define $\mathfrak{F}_m$ to be the family of subsets F of M s.t. cat $(F;M) \geqslant m$. It is immediate from property viii) that $\mathfrak{F}_m$ is ambient isotopy invariant.

It follows that if we define $c_m = c_m(f)$ by

$$c_m = \text{min-max}\ (f, \mathfrak{F}_m) = \\ = \inf_{\text{cat}(A;M) \geq m}\ \sup_{x \in A} f(x) = \\ = \inf\ \{c \in R \mid \exists\, A \subseteq f^c \text{ with } \text{cat}\ (A;M) \geqslant m\},$$

then by the min-max principle for each positive integer $m \leqslant \text{cat}(M)$, $c_m(f)$ is a critical value of f. We note that by the monotonicity of cat (.;M) an equivalent definition of $c_m(f)$ is

$$c_m(f) = \inf\ \{c \in R \mid \text{cat}\ (f^c;M) \geqslant m\}.$$

Since cat $(f^c;M) \geqslant m + 1$ implies: cat $(f^c;M) \geqslant m$, it is clear that $c_m(f) \leqslant c_{m+1}(f)$. It can of course happen that equality occurs. For example if f is constant, all the $c_m(f)$ are equal. However if equality should occur, then it would be made up for by there being more than one critical point on the critical level. In fact we have the following remarkable:

<u>Lyusternik-Schnirelman multiplicity theorem 2.2.7</u>: If $c = c_{n+1}(f) = = c_{n+2}(f) = \ldots = c_{n+k}(f)$, thenf has at least k critical points on the critical level c. Hence if $1 \leqslant m \leqslant \text{cat}\ (M)$, then f has at leat m critical points at or below the level $c_m(f)$, and in particular f has at least cat (M) critical points alltogether.

Proof:

we can assume that there are only a finite number of critical points on the level c, say $x_1, \ldots x_r$ and we must prove that $r \geq k$. Choose open neighborhoods V_i of x_i, whose closures are disjoint closed balls, so that if $\vartheta = V_1 \cup \ldots \cup V_r$, then clearly cat $(\vartheta;M) \leq r$. By the deformation theorem, for some $\varepsilon > 0$, $f^{c+\varepsilon} - \vartheta$ can be deformed into $f^{c-\varepsilon}$ and, since

$$c - \varepsilon < c_{n+1}(f) = \inf \{a \in R \mid \text{cat } (f^a;M) \geq n+1\},$$

it follows from property vii) of cat (.;M) that

$$\text{cat } (f^{c+\varepsilon} - \vartheta\,;M) \leq n.$$

By the monotonicity and subadditivity of cat (.;M):

$$\text{cat } (f^{c+\varepsilon};M) \leq \text{cat } ((f^{c+\varepsilon} - \vartheta) \cup \vartheta\,;M) \leq n + r,$$

and hence

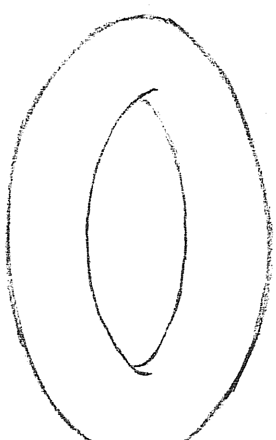
$$c < c + \varepsilon < \inf \{a \in R \mid \text{cat } (f^a;M) \geq n + r + 1\} =$$

$$= c_{n+r+1},$$

so that $n + r + 1 \geq n + k$ and therefore $r \geq k$.

Let us consider finally the following

Example 2.2.8: Let M be the bidimensional torus, f the function height of the point p on the plane, where M is posed. And let us consider situations 1. and 2. In both cases the Lyusternik-Schnirelman theory predicts, according to theorem 2.2.7 at least 3 critical points of f.



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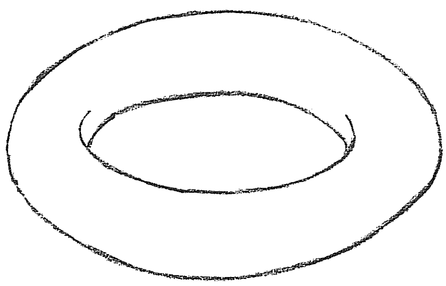


2

2.3 MORSE THEORY

Let f be a smooth function on a smooth compact manifold (smooth will mean differentiable of class C^{∞}, but actually it suffices for f to be C^2). A critical point for f is a point p at which the differential df vanishes. At such a point the hessian $H_p f$ is a symmetric, bilinear form on $T_p M$, the tangent space to M at p. In local coordinates $\{x_i\}$, centered at p, the matrix $H_p f$, relative to the basis $\frac{\partial}{\partial x_i}$ at p is given by

$$[H_p f]_{ij} = \left[\frac{\partial^2 f}{\partial x_i \partial x_j} \right].$$

p is called a nondegenerate critical point of f if det $H_p f \neq 0$. The dimension of the maximal subspace of $T_p M$, on which $H_p f$ is negative definite (that is the number of negative eigenvalues of $H_p f$) is independent of the local coordinates used. It is denoted by $\lambda_p(f)$ and it is called the index of f at p. Independency of the coordinates is also a property of the nullity of the hessian, which is defined as the dimension of the subspace of all $v \in T_p M$, s.t. $\forall w \in T_p M$, $H_p f(v,w) = 0$.

If p is a nondegenerate critical point of f, the Morse Lemma asserts that there exists a suitable coordinate system $x_1, \ldots x_n$ about p, s.t. near p

$$f = f(p) - x_1^2 - \ldots - x_\lambda^2 + x_{\lambda+1}^2 + \ldots + x_n^2,$$

with $\lambda = \lambda_p$, the index of f at p.
Hence the behaviour of f at p can be completely described by this index. As an immediate consequence of the Morse Lemma it follows that a nondegenerate critical point of f is isolated in the set K of critical points. Functions, whose critical points are all non-

degenerate are called Morse functions. They are generic, in the sense that in the vicinity of every function one may find a nondegenerate one. For them K is discrete and since K is closed in M, it is also compact. Therefore a Morse function has only a finite number of critical points.

We can count the critical points of a fixed index k to obtain an integer, m_k, called the kth type number of f. And we can introduce the "Morse polynomial" of f

$$\mathcal{M}_t(f) \doteq \sum_k m_k t^k \doteq \sum_p t^{\lambda} p,$$

where the last sum is extended over the critical points of f.

This polynomial gives a quantitative measure of the critical behaviour of f. A lower bound for such a behaviour is furnished by the homology of M. In fact if we indicate

$$P_t(M) \doteq \sum_i \dim H_i(M;K)\ t^i,$$

(where $H_i(M;K)$ denotes the ith homology group of M with coefficients in the field K), then the following holds true:

<u>Theorem 2.3.1</u>: If f is a nondegenerate function on the compact n-manifold M, then there exists a polynomial $Q_t(f) = q_0 + q_1 t + \ldots$, with nonnegative coefficients s.t.

(2.3.2) $$\mathcal{M}_t(f) = P_t(M) + (1 + t)Q_t(f).$$

It follows then:

$$\mathcal{M}_t(f) \geqq P_t(M).$$

This inequality implies that $\mathcal{M}_t(f)$ majorizes $P_t(M)$ coefficient by coefficient.

Indicated by $b_k = \dim H_k(M;K)$ the kth Betti number of M, this means:

(2.3.3) $$m_k \geqq b_k.$$

(2.3.3) can also be seen as a consequence of the Morse inequalities, which are sometimes written as:

$$m_0 \geqq b_0$$

$$m_0 - m_1 \leqq b_0 - b_1$$

(2.3.4) $$m_0 - m_1 + m_2 \geqq b_0 - b_1 + b_2$$

$$\text{-----------------}$$

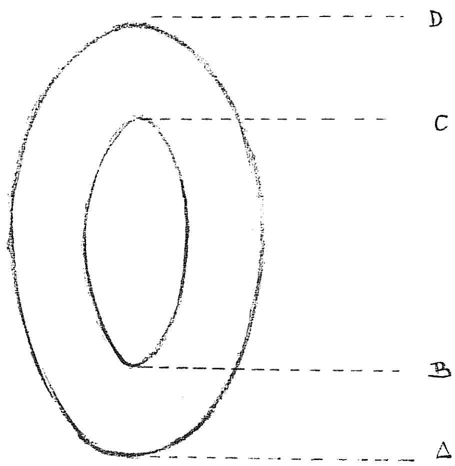
$$m_0 - m_1 + \ldots + (-1)^n m_n = b_0 - b_1 + \ldots + (-1)^n b_n.$$

We observe that one could think of these inequalities as an extension of the minimum principle: in the present context it asserts that $m_0 \geqq b_0$.

The inequalities (2.3.4) can be deduced as a corollary of the next theorems 2.3.8 and 2.3.9, the same from which, as we'll see, theorem 2.3.1 follows.

Before presenting the announced theorems, let us give a classical example:

Example 2.3.5: We consider the bidimensional torus $T^2 = M$, as we did in the precedent paragraph. But now we must put it in ready-to-roll position, since the position corresponding to situation 2 (in 2.2) would comprise degenerate critical points for the function which gives the height of the point p above the plane.



The Betti numbers for the torus are:

$b_0 = 1$ $(H_0(M;R) = R)$

$b_1 = 2$ $(H_1(M;R) = R \oplus R)$

$b_2 = 1$ $(H_2(M;R) = R)$

Therefore the Morse theory tells us that there are at least 4 critical points for a Morse function on M: one of index 0, two of index 1 and one of index 2. In this case we already know that the height has the four critical points A, B, C, D of idex respectively 0, 1, 1, 2. Hence $Q_t(f) = 0$ for this function: $\mathfrak{M}_t(f)$ is exactly equal to $P_t(M)$

Acriteron which suggests cases when this happens is the following:

Lacunary principle 2.3.6: If in $\mathfrak{M}_t(f)$ all products of two consecutive coefficients vanish (i.e. $m_i m_{i+1} = 0 \; \forall i$), then

(2.3.7) $$\mathfrak{M}_t(f) = P_t(M)$$

for any coefficient field.

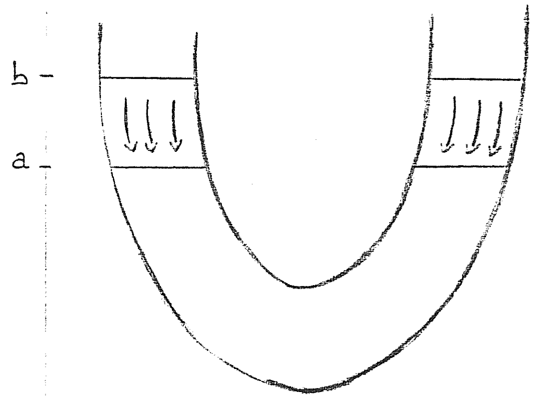
Thus for such a function f, $\mathfrak{M}_t(f)$ computes the Poincarè polynomial of M.

We call functions satisfying (2.3.7) perfect Morse functions. They allow to turn the Morse theory around and use it as a computational tool.

In order to see how (2.3.2) is obtained, we give now two fundamental results:

Theorem 2.3.8: Suppose that the nondegenerate function f has no critical points in the region $a \leqq f \leqq b$. Then if M_t denotes the "halfspace" where $f \leqq t$ on M, there is a diffeomorphism of M_a with M_b: $M_a \cong M_b$. Furthermore M_a is a deformation retract of M_b, so that the inclusion map $M_a \hookrightarrow M_b$ is an homotopy equivalence.

The idea of the proof is to push M_b down to M_a along the orthogonal trajectories of the hypersurfaces f = const. (that is along the gradient of f) and of course uses the com-



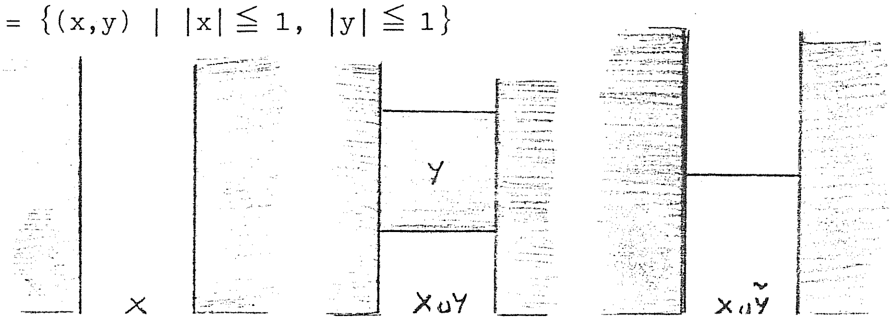
pactness of the M_i's.

The next result goes to the heart of the matter of what happens to M_t as t passes a critical value.

Let us recall the concept of attaching a thickened cell to a manifold. The underlying geometrical idea is best gleaned from the following diagram, where we've attached the thickened 1-cell Y to X.

Here $X = \{(x,y) \mid |x| \geqq 1\}$

$Y = \{(x,y) \mid |x| \leqq 1, \ |y| \leqq 1\}$



and the terminology arises from the fact that homotopically X U Y is quite equivalent to the space $X \cup \hat{Y}$ (where $\hat{Y}$ is the interval $|x| \leqq 1$ on the x-axis).

Thus as far as the glueing of part of the boundary of Y into the boundary of X, the two factors of Y play quite distinct roles; i.e. the second one just play the role of a "thickening".

Quite generally now one says that X' is obtained from X by attaching a thickened k-cell if X' is obtained from the disjont union

$$X' = X \cup e^k \times e^{n-k}$$

by glueing "half" the boundary $\partial e^k \times e^{n-k}$ of the cell $e^k \times e^{n-k}$ into the boundary ∂X by a diffeomorphism

$$\alpha : \partial e^k \times e^{n-k} \dashrightarrow \partial X.$$

The resulting manifold is then also often denoted by $X \cup_\alpha e^k \times e^{n-k}$.

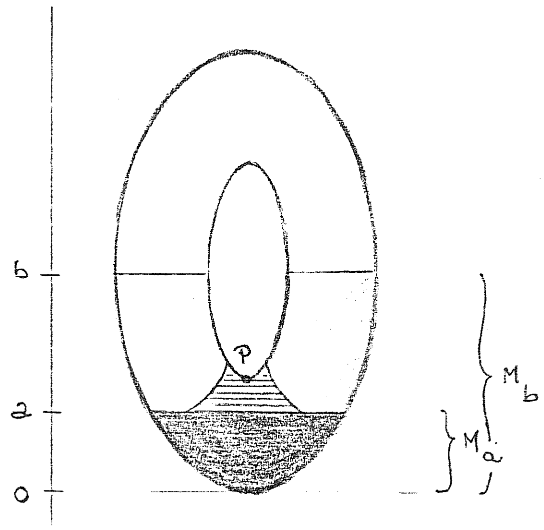
With this terminology we are in the position to state:

<u>Theorem 2.3.9</u>: Suppose that f has only one nondegenerate critical

point p of index λ in the range $a \leqq f \leqq b$ and that $a < f(p) < b$. Then $M_b \cong M_a \cup_\alpha e^\lambda \times e^{n-\lambda}$, that is: M_b is diffeomorphic to M_a with a thickened λ-cell attached.

We indicate the idea of the proof of this theorem for the special case of the height function on a torus.

The region $M_a = f^{-1}((-\infty,a])$ is heavily shaped. We introduce a new function $F : M \dashrightarrow R$, which coincides with the height function f except that $F < f$ in a small neighborhood of p. Then the region $F^{-1}((-\infty,a])$ will consist of M_a together with a region H near p. In the diagram H is the horizontal shaped region. Choosing a suitable cell $e^\lambda \subset H$, a direct argument (pushing in along the horizontal lines) will show that $M_\lambda \cup e^\lambda$ is a deformation retract of $M_a \cup H$.



Finally, by applying theorem 2.3.8 to the function F and the region $F^{-1}([a,b])$, we will see that $M_a \cup H$ is a deformation retract of M_b.

The relation (2.3.2) and the Morse inequalities (2.3.4) follow from theorem 2.3.8 and 2.3.9 by procedures of algebraic topology. In passing from M_a to M_b the change in $\mathfrak{M}_t(f)$ is given by t^λ. On the other hand the change $\Delta P_t = P_t(M_b) - P_t(M_a)$ can be twofold:

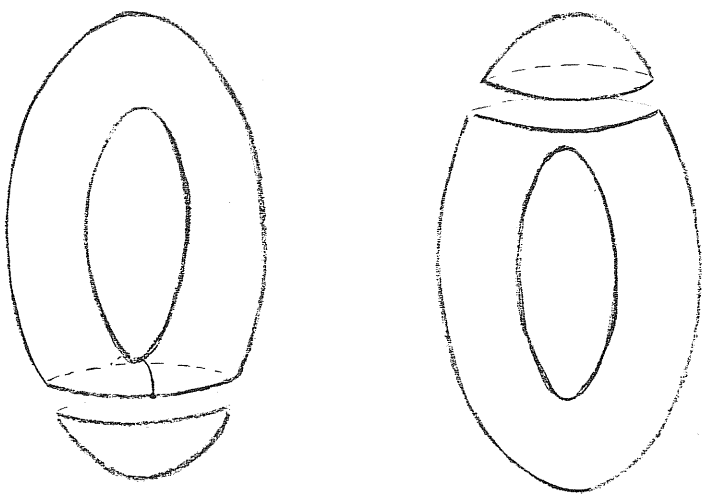
$$\Delta P_t = \begin{cases} t^\lambda \text{ or} \\ -t^{\lambda-1} \end{cases}$$

Once this is granted, the inequality: $\Delta \mathfrak{M}_t - \Delta P_t$ can be:

$$\Delta \mathfrak{M}_t - \Delta P_t = \begin{cases} 0 \text{ or} \\ t^{\lambda-1}(1+t) \end{cases}$$

Proceeding inductively, we see that there exists a polynomial Q(t) with nonnegative coefficients : $Q(t) = q_0 + q_1 t + \ldots$, $q_i \geqq 0$, s.t. $\mathfrak{M}_t(f) - P_t(M) = (1+t)Q(t)$.

The crucial step is therefore the alternative for ΔP_t above, and this is a standard result in homology theory. It is also a very intuitive one. Consider the boundary ∂e_λ of the attaching cell. It is a λ-1-sphere $S_p^{\lambda-1}$ in M_a. The cycle carried by this sphere either bounds a chain in M_a or not. In the first case we cap the chain bounded by $S_p^{\lambda-1}$ with e_λ to create a new nontrivial homology class in M_b. This corresponds to the alternative $\Delta P_t = t^\lambda$. In the second case e^λ manifestly has as boundary the nontrivial cycle $S^{\lambda-1}$ in M_a. Hence in this situation P_t decreases by a $t^{\lambda-1}$. An example of the twofold possibility is illustrated in the figure:



Note that to ascertain which alternative is valid involves a global analysis of M_a.

We remark finally that in the original papers of M. Morse the homological part was couched in the language of the analysis situs of the time.

3.1 PASSAGE TO THE INFINITE DIMENSIONAL CASE: VARIOUS REMARKS

The Morse and the Lyusternik-Schnirelman theory admit both a generalization to the case of C^r functions $(r = 1,2)$ on infinite dimensional Hilbert (and also Banach) manifolds. The key step in proving it is to generalize the technique of "steepest descent", which gave the deformations that are at the hearth of the proofs of the two basic theorems: the "deformation theorem" (2.2.4) and the "theorem on passing a nondegenerate critical level" (2.3.9). For Hilbert manifolds one can define the gradient vectorfield of a function (relative to a Riemannian metric) just as in finite dimensions. For arbitrary Banach manifolds there is no real gradient field; however one can define "pseudo-gradient" fields, which have all the essential properties of a gradient field. (v is called a "pseudo-gradient" vector for ϕ at $u \in U$ if i) $|v| \leqq 2|\phi'(u)|$, and ii) $\langle \phi'(u), v \rangle \geqq |\phi'(u)|^2$). The real problem in both cases lies in the fact that, since the manifold is no longer compact, the flow generated by a vectorfield is no longer a compact flow (i.e. a one-parameter group) and it is only by making certain compactness assumptions on the manifold, or by replacing them by some kind of compactness assumption relative to the function (like the Palais-Smale condition) that one can show that the flow generated by the pseudo-gradient field really "descends to the critical point set" in a strongh enough sense to carry out the proofs of the above theorems. Hence a natural idea is to restrict the attention to pairs (M,f) satisfying the following Palais-Smale condition:

given a sequence $\{x_n\} \subset M$ s.t. $f(x_n)$ is bounded and $|df_{x_n}| \longrightarrow 0$, there is a subsequence of $\{x_n\}$ converging (automatically, by the continuity of $|df|$, to a critical point of f).
From now on we will simply write (PS) to designate this condition.

The (PS) is met in many important calculus of variation problems, and this justifies its assumption in proving abstract theorems. But it has to be noticed that it is not the last word!

When it is not satisfied maybe other way can be found, to solve the problem. In some cases this was successfully done.

For example this problem camed out in |24|. Here periodic solutions of a Lagrangian system with bounded potential V are sought as critical points of a C^1 action functional on $H^1(S^1, R^n)$, for which (due to the boundedness of V) the (PS) condition is not satisfied in general. (This allows the set of critical points to be unbounded).

As another example let us consider the problem of finding nontrivial 2π-periodic solutions of the nonlinear wave equation

$$(3.1.1) \qquad \begin{cases} u_{tt} - u_{xx} = g(u) & \text{in }]0, \pi[\times R \\ u(0,t) = u(\pi, t) & t \in R \end{cases}$$

with $g : R \longrightarrow R$ a continuos function.
This problem too can be read as a variational one. Precisely the 2π-periodic solutions of (3.1.1) can be found as critical points of the functional

$$f(u) = \int_0^{2\pi}\int_0^{\pi}\left\{\frac{1}{2}(u_x^{\,2} - u_t^{\,2}) - G(u)\right\}dxdt$$

where $G(u) = {}_0\int^u g(s)ds$,

defined on an opportune domain A in an opportune function space H. With a right choiche of A and H, f can be written in the form

$$f(u) = \frac{1}{2}\langle Lu,u\rangle_h + \Phi(u)$$

with L linear operator, selfadjoint on A,...

Well: the Kernel of L is an infinite dimensional subspace of H and, because of this, the problem lacks the compactness properties required in proving the (PS) condition.

To overcome this difficulty without doing specific requirements on g (many authors suppose monotonicity for g and solve the problem with dual methods) there is a trick, recently used by Coron (see |32|). The idea is to consider the functional restricted to a suitable subspace of functions $\hat{H}$ of H, satisfying some symmetry property, invariant under L and s.t. the intersection of $\hat{H}$ with the Kernel of L is reduced to 0.

Thanks to this idea Coron proves that

if $g \in C^2(R,R)$, $g'(0) \neq 0$, $\lim \frac{g(u)}{u} = 0$ when $|u| \to \infty$, and there exist real numbers α and β s.t. $-1 < \alpha \leq g'(s) \leq \beta \;\; \forall s \in R$, then

for every integer l there is some T_o s.t. if $T > T_o$ and T has the form $T = 2\pi\frac{b}{a}$, a integer, b an odd integer, there exist at least l C^3 solutions of (3.1.1), which are nonconstant in time and geometrically distinct.

Moreover, combining the Coron's idea with the use of the Benci's geometrical index for the S^1-group (|13|), in |11| it is shown that the boundedness assumption on g' can be avoided.

This idea, which is also used in |24|, to consider the functional resticted to a subspace (of functions) having all the symmetries of the problem, in order to obtain compactness, is explained in |40| (see also |17| and |18|). In this paper the importance of a symmetry in the solution is explicitely seen in the study of the Choquard equation, for which optimal existence and multiplicity results are obtained, despite the fact that the functional at issue does not satisfy the (PS) condition in the space H.

Other problems where the symmetries are essential to get compactness arise: in astrophysics (in the determination of equilibrium configurations of axisymmetric rotating fluids, for instance stars), in quantum mechanics (in Thomas-Fermi theory), in the search for certain kinds of solitary waves (stationary states) in nonlinear equations of the Klein-Gordon or Schrödinger type and, generally, in the solution of nonlinear scalar field equations.

For further examples and details we refer to |40|, |17| and |18|, and the relative bibliographies.

3.2 GENERALIZATION OF THE MIN-MAX PRINCIPLE

But now let us assume the compactness condition (PS) to be satisfied.

In the present and following section we will see why indefinite functionals are difficult to study and also how this difficulty can be bypassed.

How the Lyusternik-Schnirelman theory is generalized to any pair (M,f), satisfying the (PS) condition, is clearly showed, for example, in |64|. It is done, following the steps in the proof for the finite dimensional case, and proving an analogue of the Lemmas and of the deformation theorem, given in such case.
In particular in the present setting the (PS) condition assures that $K_c = \{x \in M \mid f(x) = c, \ (\nabla f)(x) = 0\}$ is compact. This property (which we could suppose certainly satisfied in the finite dimensional case) is crucial in the proof of the following theorem.

Theorem 3.2.1: Let (M,f) satisfy (PS) and let $c_m(f)$ be a sequence with

$$c_m(f) = \inf_{cat(A) \leq m} \ \sup_{x \in A} f(x).$$

Suppose that $m \leq n$ and $-\infty < c = c_m(f) = \ldots = c_n(f) < +\infty$. Then the set K_c of critical points is of category $n - m + 1$ at least; moreover, even if $m = n$, the set K_c is not empty.

The application of the theorem is facilitated by the

Lemma 3.2.2: Let (M,f) satisfy (PS) and let someone of the constants $c_m(f)$ be equal to $+\infty$. Then f assumes arbitrary large values on the set of its critical points (which is consequently infinite).

It follows then:

<u>Corollary 3.2.3</u>: If f is bounded from below, then f has at least cat (M) critical points.

Proof:

if f is bounded from below, then: $-\infty < c_1(f)$. Now: let k be a positive integer, $k \leq$ cat (M).

If $c_k < +\infty$, then f has at least k critical points in $\{x \mid f(x) \leqq c_k + \varepsilon\}$ as a consequence of theorem 3.2.1.

Therefore:

if $c_k < +\infty$ for all $k \leq$ cat (M), there are at least cat (M) critical points of f; if for some k, $c_k = +\infty$, f has by Lemma 3.2.2 infinitely critical points. In any case f has at least cat (M) critical points alltogether.

<u>Remark 3.2.4</u>: In certain cases this corollary is too general to be fruithfully applied for the existence of solutions. This happens when the topology of the manifold M is trivial. In such a case the corollary does not give great informations, as is shown by the following example.

Suppose A(x) and B(x) are C^2 functionals on a reflexive Banach space X, satisfying (PS) and B(x) bounded from below. Suppose we are seeking solutions (x,λ) X x R of

$$(3.2.5) \qquad \begin{cases} A'(x) = \lambda B'(x) \\ A(x) = c = \text{const.} \end{cases}$$

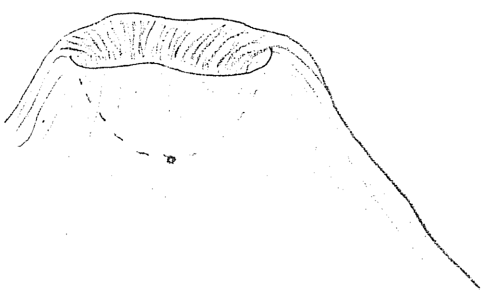
The solutions of (3.2.4) are contained in the set of critical points of the functional B(x) on the level set $\{x \in X \mid A(x) = c\}$.

If this level set is for example equal to the unit sphere in X $S = \{ x \in X \mid |x| = 1 \}$, which is known to be contractible (since X is infinite dimensional), it is cat (S) = 1. And then the theorem gives no additional critical points besides the minimum, which we already know to exist.

Bur the principal limitation of the corollary is given by the fact that it applies only to functionals bounded from below.

We will introduce now the announced method to treat also unbounded functionals.

As a last remark before do it, we want to point out that the first result concerning indefinite functionals, and the one from which what follows draw inspiration, is in the paper |5| of Ambrosetti-Rabinowitz. We refer to the famous "mountain pass theorem". Let us describe it for a real function f defined on the plane, representing the height of the land above sea level over the point $x \in R^2$. Suppose the origin lies in a valley surrounded by a ring of mountains; i.e. there is an open neighborhood Ω of the origin s.t. for $x \in \partial\Omega$: $f(x) \geq c_o > f(0)$. Suppose that there is some point x_o outside, i.e. $x_o \notin \overline{\Omega}$, s.t. $f(x_o) < c_o$. We wish to walk from x_o to 0, climbing as little as possible. Naturally the way to do this is to take a path crossing the mountain over the lowest mountain pass. The top of the mountain correspond to a stationary point of f and there the value of f, i.e. the stationary value, equals



$$c = \inf_{P} \max_{x \in P} f(x) \geq c_o.$$

Here P represents any continuos path from x_o to 0 and the inf is taken with respect to all these paths. Clearly every such path has to intersect $\partial\Omega$ and so

$$\max_{x \in P} f(x) \geq c_o.$$

The assertion of the mountain pass theorem is that the number c so defined is a stationary value of f. Note that c will be in general less than sup f.

This formulation is not quite correct, since the plane is not compact and one should add a compactness condition, for example (PS). The theorem is then true and holds even in Banach spaces.

The exact formulation is:

Mountain pass theorem 3.2.6: Let f be a C^1 real function defined on a Banach space X and satisfying (PS). Assume there is an open neighborhood Ω of 0 and a point $x_o \notin \overline{\Omega}$, s.t.

$$f(0),\ f(x_o) < c_o \leq \inf f.$$

Then the following number is a critical value of f:

$$c = \inf_{P} \max_{x \in P} f(x) \geq c_o,$$

where P represents any continuos path from x_o to 0.

3.3 INDEX AND PSEUDOINDEX THEORIES

We present first the abstract framework in which the theory will be constructed.

Let M be a Riemannian manifold modeled on a Hilbert space H. For $A \subset M$, $C^k(A)$ denotes the space of k times Frèchet differentiable maps from A to R. For A, $B \subset M$, $C^k(A,B)$ denotes the set of k times Frèchet differentiable maps from A to B. Id denotes the identity map. Moreover we set $N_\delta(A) = \{ x \in M \mid \mathrm{dist}(x,A) \leq \delta \}$.

Definition 3.3.1: An index theory I on M is a triplet $\{\Sigma, \mathcal{M}, i\}$, which fullfills the following properties:

(1-1) Σ is a family of closed subsets of M, s.t. $A \cup B$, $A \cap B$, $\overline{A \setminus B} \in \Sigma$ whenever $A, B \in \Sigma$;

(1-2) $\mathcal{M}$ is a set of continuos mappings : $\Sigma \to \Sigma$, containing the identity and closed under composition;

(1-3) $\forall A \in \Sigma$ and $\forall h \in \mathcal{M}$, $\overline{h(A)} \in \Sigma$;

(1-4) $i : \Sigma \to \mathbb{N} \cup \{+\infty\}$ is a mapping satisfying:

(i-1) $i(A) = 0 \iff A = \emptyset$,

(i-2) monotonicity: if $A \subset B$, then $i(A) \leq i(B)$ $\forall A, B \in \Sigma$,

(i-3) subadditivity: $i(A \cup B) \leq i(A) + i(B)$ $\forall A, B \in \Sigma$,

(i-4) continuity: if $A \in \Sigma$ is a compact set, then there exists a $\delta > 0$ s.t. $i(N_\delta(A)) = i(A)$,

(i-5) superinvariancy: $i(A) \leq i(\overline{h(A)})$ $\forall A$ and $\forall h \in \mathcal{M}$.

As an example: if $\tilde{\Sigma}$ is the family of all closed subsets of M, $\tilde{\mathcal{M}}$ is the family of the continuos mappings of M homotopic to a constant map, and cat_M is the Lyusternik-Schnirelman category, then it is immediate to check that $\{\tilde{\Sigma}, \tilde{\mathcal{M}}, \mathrm{cat}_M\}$ defines an index theory on M.

When we deal with indefinite functionals the existence of an index theory may be not sufficient to garantue the existence of critical points. In fact if we simply use the Lyusternik-Schnirelman theory (or some other index theory) in a direct way, when we put

$$c_k = \inf_{i(A)\geq k} \sup_{x\in A} f(x),$$

it may happen that $c_k = -\infty$ (or $+\infty$) $\forall\, k \in N$.

Then the concept of "pseudoindex" turns out to be useful.

Definition 3.3.2: Let M and $I = \{\Sigma, \mathfrak{M}, i\}$ be as in Def. 3.3.1.

A pseudoindex theory I^* is a couple $\{\mathfrak{M}^*, i^*\}$, which satisfies the following assumptions:

(1-5) $\mathfrak{M}^* \subset \mathfrak{M}$ is a group of homeomorphisms of M onto M;

(1-6) $i^* : \Sigma \to N \cup \{+\infty\}$ is a map with the following properties:

(i^*-1) $i^*(A) \leq i(A)$ $\forall A \in \Sigma$,

(i^*-2) if $A \subset B$, then $i^*(A) \leq i^*(B)$ $\forall A, B \in \Sigma$,

(i^*-3) $i^*(\overline{A\setminus B}) \leq i^*(A) - i(B)$ $\forall A, B \in \Sigma$,

(i^*-4) $i^*(h(A)) = i^*(A)$ $\forall h \in \mathfrak{M}^*$, $A \in \Sigma$.

A method to construct pseudoindex theories is described by the following

Proposition 3.3.3: Let $I = \{\Sigma, \mathfrak{M}, i\}$ be an index theory on the Riemannian manifold M. Let $\mathfrak{M}^* \subset \mathfrak{M}$ be a group of homeomorphisms on M. Given $Q \in \Sigma$, we set $i^*(A) = \min_{h \in \mathfrak{M}^*} i(h(A) \cap Q)$ for each $A \in \Sigma$.

Then $I^* = \{\mathfrak{M}^*, i^*\}$ is a pseudoindex theory.

We shall now show how the concept of pseudoindex can be applied to the search of critical points of a functional $f \in C^1(M)$.

For each $c \in R$ we set

(3.3.4)
$$\mathcal{Q}_c = \{ x \in M \mid f(x) \leq c \}$$
$$K_c = \{ x \in M \mid f(x) = c,\ f'(x) = 0 \}$$

Definition 3.3.5: If $f \in C^1(M)$ and c_0, $c_\infty \in R$ (with $c_0 < c_\infty$), we say that the triplet $\{ f, c_0, c_\infty \}$ satisfies the property (P) with respect to the couple $\{ \Sigma , \mathfrak{M}^* \}$ if

(3.3.5)
(a) $\mathcal{Q}_c$, $K_c \in \Sigma$ and K_c is compact $\forall c \in [c_0, c_\infty]$

(b) $\forall c \in [c_0, c_\infty]$, $\forall N = N_\delta(K_c)$

there exist $\varepsilon > 0$ and $\eta \in \mathfrak{M}^*$ s.t.

$\eta(\mathcal{Q}_{c+\varepsilon} \setminus N) \subset \eta(\mathcal{Q}_{c-\varepsilon})$

In concrete cases, as we will see, the property (P) is streactly related to the (PS) condition.

The following theorem holds:

Theorem 3.3.7: Let M be a Riemannian manifold with an index theory $I = \{ \Sigma, \mathfrak{M}, i \}$ and a pseudoindex theory $I^* = \{ \mathfrak{M}^*, i^* \}$. Suppose that $f \in C^1(M)$ is a functional s.t.

(3.3.8)
(a) there exist constants c_0, $c_\infty \in R$ s.t. $\{ f, c_0, c_\infty \}$ satisfies property (P) with respect to $\{ \Sigma , \mathfrak{M}^* \}$

(b) $i^*(A) = 0$ $\forall A \in \Sigma$ s.t. $A \subset \mathcal{Q}_{c_0}$

(c) there exists $\hat{A} \in \Sigma$ s.t. $\hat{A} \subset \mathcal{Q}_{c_\infty}$ and $i^*(\hat{A}) \geq \hat{k} \geq 1$

Then the real numbers

(3.3.9)
$$c_k = \inf_{i^*(A) \geq k} \sup_{x \in A} f(x) \qquad k = 1, \ldots \hat{k}$$

are critical values of f and

(3.3.10)
$$c_0 \leqq c_1 \leqq \ldots \leqq c_{\hat{k}} \leqq c_\infty .$$

Moreover if $c = c_k = \ldots = c_{k+r}$ with $k \geq 1$ and

(3.3.11) $k + r \leqq \hat{k}$,

then

(3.3.12) $i(K_c) \geqq r + 1$.

Proof:

we shall prove initially that

$$c_0 \leqq c_k \leqq c_\infty \qquad k = 1,\dots\hat{k}.$$

In particular we deduce that the numbers c_k's are well defined.

We argue by contraddiction. In fact if $c_k < c_0$, then there exists $\hat{A} \subset f^{-1}(]-\infty,c_0[)$ s.t. $i^*(\hat{A}) \geqq k$. On the other hand (recall Proposition 3.3.3)

$i^*(\hat{A}) = \min_{h\in\mathfrak{M}^*} i(h(\hat{A}) \cap Q) \leqq i(\hat{A} \cap Q) = 0$ by assumption (b).

So we get a contraddiction.

If $c_k > c_\infty$, $\sup f(u) > c_\infty$ for any $A \in \Sigma_k$ ($\Sigma_k = \{A \mid i^*(A) \geq k\}$) and this contraddicts (c).

Now: obviously we have $c_k \leqq c_{k+1}$ ($k = 1,\dots\hat{k}-1$).

Therefore (3.3.10) is proved.

To prove that the c_k's are critical values, it will be sufficient to show the sharper multiplicity statement (3.3.12). We argue indirectly .

Suppose $i(K_c) \leqq r$ and observe that (a) implies that K_c is compact. By continuity property (i-4) there is a neighborhood N of K_c, $N \in \Sigma$ with

(3.2.13) $i(N) \leqq r$.

Moreover it can be seen that there is a $\delta > 0$ with $c - \delta > a$, $c + \delta < b$ and an equivariant homeomorphism η of M onto M s.t.

$$\eta(\mathcal{Q}_{c+\delta}\setminus N) \subset \mathcal{Q}_{c-\delta}$$

and $\eta(u) = u \quad \forall u \notin f^{-1}(]a,b[)$.

Also if $c = c_{k+r}$, there exists $A \in \Sigma_{k+r}$ s.t. $A \subset \mathcal{Q}_{c+\delta}$. Then

$$\eta(A\backslash N) \subset \mathcal{Q}_{c-\delta}$$

and

(3.3.14) $$i^*(A) \geqq k + r.$$

Now, since

(3.3.15) $$i^*(\overline{A\backslash N}) \geqq i^*(A) - i(N)$$

using (3.3.13) and (3.3.14) we get

(3.3.16) $$i^*(\overline{A\backslash N}) \geqq k + r - r = k.$$

Observe that $\eta \in \mathfrak{M}^*$; thus

(3.3.17) $$i^*(\eta(\overline{A\backslash N})) = i^*(\overline{A\backslash N}).$$

By (3.3.16) and (3.3.17) we get: $\eta(\overline{A\backslash N}) \in \Sigma_k$.

Therefore, since $c = c_k$, $\sup f(\eta(\overline{A\backslash N})) \geqq c$, which contraddicts (3.3.14).

Remark 3.3.18: If in the assumption (c) we know that $i^*(\hat{A}) = +\infty$, then clearly (3.3.10) defines critical values for each $k \in N^+$.

In order to apply theorem (3.3.7) in concrete situations it is necessary to construct an appropriate pseudoindex theory. This depends of course on the functional f, whose critical points we seek. Essentially the problem is to determine the right class of homeomorphisms $\mathfrak{M}^*$. This class should be big enough in order to contain a function η s.t. (3.3.6) (b) is satisfied. But if $\mathfrak{M}^*$ is too big, it may happen that $i^*(A) = 0$ or 1 for each $A \in \Sigma$. For example if every $h \in \mathfrak{M}^*$ is of the form $h = Id + \text{compact}$, $\mathfrak{M}^*$ is too small and the property (P) is not satisfied. On the other hand if $\mathfrak{M}^* = \{$group of all equivariant homeomorphisms on $\Sigma\}$, then the group is too large and $i^*(A)$ vanishes $\forall A \in \Sigma$. Another "delicate" point is the choiche of Q (when the pseudoindex theory is constructed according to Proposition 3.3.3).

To have an intuitive and roughly understanding, the pseudoindex is a tool which allows to "feel" only those subsets, where the functional assumes values greater than some real number c_o, in such a way that the risk to have $-\infty$ in evaluating $c_m(f)$ vanishes.

We shall consider functionals symmetric with respect to the action of some compact Lie group G; because of this we'll study index and pseudoindex theories on a real Hilbert space H, on which a unitary representation T_g of the group G acts.

A functional $f \in C^1(H)$ is said to be T_g-invariant if

$$f(T_g u) = T_g(fu) \qquad \forall u \in A, \ \forall g \in G.$$

Amap $h \in C(H,H)$ is said to be T_g-equivariant if

$$h(T_g u) = T_g(hu) \qquad \forall u \in H, \ \forall g \in G.$$

If f is in $C^1(H)$, then $f' \in C^o(H,H)$, since we identify H with its dual; and if f is T_g-invariant, f' is T_g-equivariant.

A subset $A \subset H$ is said to be T_g-invariant if

$$T_g A = A \qquad \forall g \in G.$$

We set

$$\Sigma(T_g) = \{A \subset H \mid A \text{ is closed and } T_g\text{-invariant}\},$$

$$\mathfrak{M}(T_g) = \{h \subset C(H,H) \mid h \text{ is } T_g\text{-equivariant}\}.$$

We shall say that an index theory $\{\Sigma, \mathfrak{M}, i\}$ is related to a representation T if $\Sigma = \Sigma(T_g)$, $\mathfrak{M} = \mathfrak{M}(T_g)$.

In |16| the following result has been obtained by means of pseudoindex theory:

Theorem 3.3.19: Let H be a real Hilbert space, on which a unitary representation T_g of the group S^1 acts. Let $f \in C^1(H,R)$ be a fun - ctional on H, satisfying the following assumptions:

f_1) $f(u) = \frac{1}{2}<Lu,u> - \Phi(u)$, where $< , >$ is the inner product in H, L is a bounded, selfadjoint operator and $\Phi \in C^1(H,R)$, $\Phi(0) = 0$, is a functional, whose Frèchet derivative is compact. We suppose that both L and Φ' are S^1-equivariant;

f_2) 0 is a regular value for L or it is an isolated eigenvalue of L of finite multiplicity;

f_3) every sequence $\{u_n\} \in H$, for which $f(u_n) \dashrightarrow c \in (0,+\infty)$ and $|f'(u_n)||u_n| \dashrightarrow 0$, posseses a bounded subsequence;

f_4) there are two closed, T_g-invariant subspaces V, $W \subset H$ and R, $c_o > 0$, s.t.:

a) W is L-invariant (LW = W),

b) $Fix(T_g) \subset V$ or $Fix\ T_g \subset W$,

c) $f(u) < c_o$ for $u \in Fix(T_g)$ s.t. $f'(u) = 0$,

d) f is bounded from above on W,

e) $f(u) \geq c_o$ for $u \in V$ s.t. $|u| = R$,

f) $codim(V + W) < +\infty$, $dim(V \cap W) < +\infty$;

Under the above assumptions there exist at least $\frac{1}{2}(dim(V \cap W) - codim(V + W))$ orbits of critical points, with critical values grater or equal to c_o.

Outline of the proof:

let $\mathcal{U}$ be a class of equivariant homeomorphisms $U : H \dashrightarrow H$ of the form $U = e^{\alpha(.)L}$, where $\alpha : H \dashrightarrow R$ is an S^1-invariant bounded functional.

We denote by B the class of continuos equivariant maps $b : H \dashrightarrow H$, s.t. for every bounded set $\Omega \subset H$, there exists a finite dimensional space $E_n \subset H$, spanned by a finite number of eigenvectors of L, s.t. $b(\Omega) \subset E_n$.

Finally we set

$$\mathfrak{M}^* = \left\{ \begin{array}{l} h : H \dashrightarrow H \mid h \text{ is an equivariant homeo-} \\ \text{morphism s.t. } h(0) = 0,\ h = U_1 + b_1 \text{ and} \\ h^{-1} = U_2 + b_2, \text{ with } U_1,\ U_2 \in \mathcal{U},\ b_1,\ b_2 \in B. \end{array} \right.$$

It is not difficult to prove that $\mathfrak{M}^*$ is a group. By virtue of (f_1), (f_2), (f_3) it is possible to prove that $\{f,(0,+\infty)\}$ satisfies the property (P) with respect to $\{\Sigma, \mathfrak{M}^*\}$ by constructing an homeomorphism η as in definition 3.3.5.

Moreover by using the assumption (f_4) we can estimate the pseudoindex $i^*(.,.,\ \ *)$ by means of $(\dim(V + W) + \operatorname{codim}(V \cap W))$.

Both the construction of η and the proof of this estimate are very involved and we will not give them here.

Using theorem 3.3.7 it follows that: the c_k's defined as in (3.3.9) are critical values and they are bigger than c_o. If they are all different from each other, then the conclusion follows. If not, for some k we have $i(K_{c_k}) \geqq 2$. By (f_4) (c) $K_{c_k} \cap \operatorname{Fix}(T_g) = \emptyset$. Then K_{c_k} consists of infinitely many independent critical points.

3.4 APPLICATIONS TO HAMILTONIAN SYSTEMS

This abstract critcal point theorem, proved together with some slightly modified versions in |16| (see also |12|), is in the same paper applied to the search of periodic orbits for hamiltonian systems.

We mention here two of the results obtained.

The first one concernes an autonomous hamiltonian system with hamiltonian h of the form

$$h(p,q) = \sum_{i,j} a_{i,j}(q)p_i p_j + \sum_i b_i(q)p_i + V(q)$$

(as is the case for mechanical problems).
If some technical assumptions are satisfied for V(q) (in particular it has to be superquadratic at infinity; for the other assumptions we refer to the original paper), then the existence of infinitely many, nonconstant T-periodic solutions for every prescribed period T is proved.

The second result concerns the case, in which h is asymptotically quadratic:

Theorem 3.4.1: Suppose $h \in C^1(R^{2n},R)$, but h(z) twice differentiable for z = 0. Suppose there exists a linear operator $h_{zz}(\infty): R^{2n} \longrightarrow R^{2n}$, s.t.

$$h_z(z) = h_{zz}(\infty)z + o(z)$$
$$\text{where } \frac{o(z)}{z} \longrightarrow 0 \text{ as } |z| \longrightarrow \infty ,$$

and the following nonresonance condition holds true:

$$\sigma(i\omega J h_{zz}(\infty)) \cap Z = \emptyset,$$

where $\sigma(A)$ denotes the spectrum of the matrix A.

$h_{zz}(\infty)$ is supposed to be positive definite and

$$h(z) > 0 \quad \forall z \in R^{2n}, \text{ s.t. } h_z(0) = 0.$$

Given two hermitian operators $A, B : R^{2n} \longrightarrow R^{2n}$,

let the integer number $\Theta(A,B)$ be defined as

$$\Theta(A,B) \doteq \sum_{k \in Z} \{N(ikJ + A) - \overline{N}(ikJ + B)\},$$

where

$$N(A) = \text{number of negative eigenvalues of } A,$$

$$\overline{N}(A) = \text{number of nonpositive eigenvalues of } A.$$

With the above hypotheses there are for the system

$$\dot{z} = J \nabla h(z)$$

at least $\frac{1}{2}\Theta(\omega h_{zz}(\infty), \omega h_{zz}(0))$ nonconstant $2\pi\omega$ -periodic solutions, whenever $\Theta(\omega h_{zz}(\infty), \omega h_{zz}(0)) > 0$.

The proofs of both results are done by translating our equations into the abstract setting of theorem 3.3.19.
That is to say: a natural Hilbert space is found, on which the functional relative to our problem verifies the required hypotheses.
Indeed, in practice, when a functional f is given, for which we seek solutions of $f'(u) = 0$, the topological space, on which one should work, is not given. It has to been choosen in such a way that on it f is of class C^1 and satisfying possibly the (PS) condition.

In |16| the choiche of a suitable space is done in the following manner, in order to prove the first mentioned result (and in order to obtain the second one there are little differences).

If $t \geq 1$, we set

$$L^t = L^t(S^1, R^{2n}).$$

If $s \in R$, we set

$$W^s = \left\{ u \in L^2 \mid \sum_{\substack{j \in Z \\ k=1,\dots 2n}} (1 + |j|^2)^s |u_{jk}|^2 < \infty \right\},$$

where u_{jk} are the Fourier components of u with respect to the basis (in L^2)

$$\Phi_{jk} = e^{itJ}\phi_k = \cos(jt)\phi_k + J\mathrm{sen}(jt)\phi_k$$

where

$\{\phi_k\}$ (k = 1,...2n) is the standard basis in R^{2n}.

W^s, equipped with the inner product

$$(u,v)_{W^s} = \sum_{j,k} (1 + |j|^2)^s u_{jk} v_{jk},$$

is a Hilbert space.

We recall that the embedding $W^s \hookrightarrow L^t$ is compact if $\frac{1}{t} > \frac{1}{2} - s$.

So in particular $W^{1/2}$ is compactly embedded in L^t for any $t \geq 1$.

Now, making the change of variable $t \longrightarrow \frac{2\pi t}{T}$,

(3.4.2) $\qquad - J\dot{z} = \nabla h(z)$

becomes

(3.4.3) $\qquad - J\dot{z} = \omega \nabla h(z), \qquad \omega = \frac{T}{2\pi}$.

The 2π-periodic solutions of (3.4.3) correspond to the T-periodic solutions of (3.4.2).

In order to construct the action functional, whose critical points are the 2π-periodic solutions of (3.4.3), we introduce the following bilinear form:

$$a(u,v) = \sum_{j \in Z} \sum_{k=1}^{2n} j u_{jk} v_{jk} \qquad u,v \in W^{1/2},$$

where u_{jk}, v_{jk} are the Fourier components of u, v with respect to the basis $\{\Phi_{jk}\}$. The bilinear form a(.,.) is symmetric and continuos in $W^{1/2}$. Let L: $W^{1/2} \longrightarrow W^{1/2}$ be the selfadjoint, continuos operator defined by

$$(Lu|v)_{W^{1/2}} = a(u,v) \qquad u,\ v \in W^{1/2}.$$

Observe that if $u,v \in C^1(S^1,R^{2n})$

$$(Lu|v)_{W^{1/2}} = \int_0^{2\pi}(-J\dot{u},v)\,dt.$$

If there are positive constants c_1, c_2, s s.t.

$$(3.4.4) \qquad |h_z(z)| \leq c_1 + c_2|z|^s \qquad \forall\ z,$$

standard arguments show that the functional

$$f(z) = \frac{1}{2}(Lz|z)_{W^{1/2}} - \omega\int_0^{2\pi} h(z)\,dt \qquad z \in W^{1/2}$$

is Frèchet differentiable and its critical points correspond to the 2π-periodic solutions of (3.4.3).

Let us call $\Phi(z) = \int_0^{2\pi}\omega\, h(z)dt$. Since $W^{1/2}$ is compactly embedded in L^t for any $t \geq 1$, by (3.4.4) we have that the map $z \dashrightarrow \omega h_z(z)$ is compact from $W^{1/2}$ to $W^{-1/2}$; then Φ' is compact.

It is easy to verify that the spectrum of L consists of the limit points -1 and 1, and of the eigenvalues

$$\lambda_j = \frac{j}{(1+j^2)^{1/2}} \qquad j \in Z$$

and each eigenvalue λ_j has multiplicity 2n.

Thenthe functional f(z) is strongly indefinite, and it can be seen that it satisfies assumptions (f_1), (f_2), (f_3) of theorem 3.3.19.

We are not interested to give the details of the proof here. Simply we observe that probably one of the major difficulties in applications like this is to individuate the spaces V and W verifying the geometrical assumption (f_4).

4.1 CONLEY'S INDEX THEORY

The homotopy index or Conley's index theory, introduced in 1978 in a pioonering work (|29|) by C.C. Conley, generalizes the Morse theory for flows, which are not necessarily gradient flows.

In order to briefly outline this theory, let us consider a continuos two-sided flow on a locally compact and metric space X. We recall that a flow is a map

$$\pi : X \times R \longrightarrow X$$

$$\pi : (x,t) \longrightarrow x \cdot t,$$

which satisfies:

$$x \cdot 0 = x \quad \forall\, x \in X$$

$$x \cdot (t + s) = (x \cdot t) \cdot s \quad x \in X;\ t,\ s \in R.$$

Also: a subset $A \subset X$ is said to be invariant if $A \cdot R = A$, where $A \cdot R = \{x \cdot t \mid x \in A,\ t \in R\}$.

A compact and invariant subset $S \subset X$ is called isolated if it admits a compact neighborhood N, s.t. S is the maximal invariant subset, which is contained in N.

With such an isolated invariant set S, a pair (N_1, N_0) can be associated, where:

i) $N_0 \subset N_1$ and $cl(N_1 \setminus N_0)$ is an isolating neighborhood for S;

ii) N_0 is positively invariant relative to N_1 (which means: $x \in N_0$ and $x \cdot [0,t] \subset N_0$ imply $x \cdot [0,t] \subset N_1$);

iii) if $x \in N_1$ and $x \cdot R^+ \not\subset N_1$, then there is a $t \geq 0$ s.t. $x \cdot [0,t] \subset N_1$ and $x \cdot t \in N_0$.

Roughly speaking: N_0 is the "exit set" of N_1.

Such a pair will be called an index pair.

Now: to an isolated invariant set S we can assigne an algebraic invariant, which is actually an invariant for an index pair for S. Let us in fact denote by h(S) the homotopy type of the pointed space $(N_1/N_0,[N_0])$; that is let us put $h(S) = [(N_1/N_0,[N_0])]$. It turns out that the homotopy type h(S) does not depend on the particular choiche of the index pair (N_1,N_0) (what is crucial for the definition to be a good definition); h(S) only depends on the way S sits in the local flow on X.

Now define the algebraic invariant

$$p(t,h(S)) \doteq \sum_{j\geq 0} \gamma_j t^j,$$

where $\gamma_j = \text{rank } H^j(N_1,N_0)$.

The setting for the generalized Morse theory is described by the following definition.

<u>Definition 4.1.1</u>: A Morse decomposition of S is an ordered family $\{M_1,\dots M_m\}$ of disjoint, compact and invariant subsets of S, s.t. for every $x\in S\setminus \bigcup_{j=1}^{m} M_j$ there is a pair of indices $i < j$ for which

$$\omega(x) \subset M_i \text{ and } \omega^*(x) \subset M_j,$$

where the positive limit set $\omega(x)$ and the negative limit set $\omega^*(x)$ are defined as the maximal invariant sets in the closure of $x\cdot[0,+\infty)$ and $x\cdot(-\infty,0]$ respectively.

The relation between the invariants of S and the local invariants of a Morse decomposition of S is given by the following identity

$$\sum_{j=1}^{n} p(t,h(M_j)) = p(t,h(S)) + (1+t)Q(t), \tag{4.1.2}$$

where Q(t) is a formal power series,having nonnegative coefficients. We refer to |30| for a rigorous proof of (4.1.2).

Rather here we want to emphasize why the above identity can be viewed as a generalization of the classical identity in Morse theory.

Let then X = S = M be a d-dimensional compact manifold and f a C^2 function on M. Let us consider the gradient flow $\dot{x} = - \nabla f(x)$ on M. Assume the critical points are isolated; then the family $(x_1, \dots x_n)$ of all critical points is a Morse decomposition of the manifold M if we order them in such a way that $f(x_i) \leqq f(x_j)$ for $i \leqq j$. This is an immediate consequence of the gradient structure of the flow. Since the critical points $\{x_j\}$ are compact and isolated invariant sets, we conclude from (4.1.2) that the following equation holds true:

$$\sum_{j=1}^{n} p(t,h(\{x_j\})) = p(t,h(M)) + (1+t)Q(t). \tag{4.1.3}$$

As $(M,\emptyset)$ is an index pair for the invariant set M, the first term on the right hand side is the Poincarè polynomial

$$p(t,h(M)) = \sum_{k=0}^{d} \beta_k t^k,$$

the β_k being the Betti numbers of the manifold M. If we assume now, in addition, the critical points to be nondegenerate, then the manifold M is the union of the stable and unstable invariant manifolds of the critical points.

Observe that in this case the only local topological invariant of a critical point x_j, which is an hyperbolic equilibrium point of the flow, is the dimension of the unstable invariant manifold; and this is equal to the Morse index d_j of the critical point x_j. It is easy to show (see |30| for instance) that the Conley's index of the set $\{x_j\}$ is given by $h(\{x_j\}) = [(S^{d_j},p)]$, where p is a distinguished point of the d^j-dimensional sphere S^{d_j}. Therefore $p(t,h(\{x_j\})) = t^{d_j}$.

Summarizing, we find for the Morse decomposition of the manifold M indeed the classical equation of Morse theory

$$\sum_{j=1}^{n} t^{d_j} = \sum_{k=0}^{d} \beta_k t^k + (1 + t)Q(t),$$

Q(t) being a polynomial having nonnegative integer coefficients only.

We remark at this point that since the two-sidedness of the flow and the compactness of the underlying space are an essential assumption in Conley's index theory, this can be applied essentially only to ordinary differential equations in finite dimensions. Applications to other kinds of equations require some reduction procedure. So, for example, a reduction procedure is carried on in |30| to deduce existence of periodic orbits for a time-dependent hamiltonian system.

Before to illustrate how this is done, we just want to recall that the Conley's theory already was extended by Rybakowsky to infinite-dimensionals semiflows. Using this "extended" homotopy index, variational equations of the form Lx = N(x) (L a linear operator, N nonlinear) can be treated. In particular in |59| periodic solutions of second order gradient systems are obtained, without passing through finite-dimensional reduction.

4.2 APPLICATION TO HAMILTONIAN SYSTEMS OF CONLEY'S INDEX THEORY

But the result, on which we want to dedicate our attention now, is the following theorem 4.2.3, obtained in |30|. We need, before stating it, a definition.

Definition 4.2.1: A T-periodic solution $x_o(t)$ of $\dot{x} = J\nabla h(t,x)$ is called nondegenerate if, when considered the linear equation

(4.2.2) $$\dot{y} = Jh''(t,x_o(t))y \doteq JA(t)y,$$

the fundamental solution $X(t)$ of (4.2.2), for which $X(0) = 1$, is such that no eigenvalue of $X(T)$ is equal to 1.

To every nondegenerate solution an index ($\in Z$) can be associated, which is roughly the signature of the hessian of f at the corresponding critical point. For the precise definition we refer to |30|.

Theorem 4.2.3: Let $h = h(t,x) \in C^2(R \times R^{2n}, R)$, $n \geq 2$ be periodic in time of period T. Assume:

i) the hessian of h is bounded: $-\beta \leq h''(t,x) \leq \beta$ $\forall (t,x) \in R \times R^{2n}$ and for some constant β;

ii) the hamiltonian vectorfield is asymptotically linear:

$J\nabla h(t,x) = JA_\infty(t)x + o(|x|)$ as $|x| \to +\infty$

uniformly in t, where $A_\infty(t) = A_\infty(t + T)$ is a continuos loop of symmetric matrices;

iii) the trivial solution of the equation $\dot{x} = JA_\infty(t)x$ is nondegenerate; denote its index by j_∞.

Then the following statement holds:

1) there exists a periodic solution of period T for the system

(4.2.4) $$\dot{x} = J\nabla h(t,x).$$

If this periodic solution is nondegenerate with index j_o, then there is a second T-periodic solution, provided $j_o \neq j_\infty$. Moreover: if there are two nondegenerate periodic solutions, there is also a third periodic solution;

2) assume all the periodic solutions are nondegenerate; then there are only finitely many of them, and their number is odd.

If j_k, $1 \leqq k \leqq m$ denote their indices, we have the following identity:

$$\sum_{k=1}^{m} t^{-j_k} = t^{-j_\infty} + t^{-d}(1 + t)Q_d(t),$$

where $d > 0$ is an integer and where $Q_d(t)$ is a polynomial having nonnegative integer coefficients.

We also give an interesting special case of the above statement, which can be viewed as a generalization to higher dimensions of the Poincarè-Birkhoff fixed point theorem for mappings in the plane. This wellknown theorem states that a measure-preserving homeomorphism of an anulus, which twists the two boundaries in opposite directions has at least two fixed points.

Corollary 4.2.5: Let $h = h(t,x) \in C^2(R \times R^{2n}, R)$, $n \geqq 2$ be periodic, $h(t + T,x) = h(t,x)$ and let the hessian of h be bounded. Assume:

$$J\nabla h(t,x) = JA_\infty(t)x + o(|x|) \quad \text{as } |x| \to \infty,$$

$$J\nabla h(t,x) = JA_o(t)x + o(|x|) \quad \text{as } |x| \to 0$$

uniformly in t for two continuos loops $A_o(t + T) = A_o(t)$ and $A_\infty(t + T) = A_\infty(t)$. Assume that the two linear systems

$$\dot{x} = JA_\infty(t)x$$

and $$\dot{x} = JA_o(t)x$$

do not admit any nontrivial periodic solution, and denote by j_∞ and j_o the indices of these two linear systems (i.e. of the respective trivial solutions). If $j_\infty \neq j_o$, then there exists a nontrivial periodic solution of (4.2.4). Moreover, if this periodic solution is also nondegenerate, then there is a second T-periodic solution.

In other words if the two linear systems with $A_o(t)$ and $A_\infty(t)$ cannot be continuosly deformed into eachother within the set of the matrices, whose corresponding solutions are nondegenerate, then we conclude the existence of a T-periodic orbit.

The proof of the theorem proceeds as follows: first the problem is formulated as an abstract variational problem for a functional in the loop space. And this is done exactly as in section 1.3 of this thesis. At this point the assumption (i) allows the application of an analytical device, due to Amann (see |1|) and used also in |2| and |3|, which reduces the study of critical points of f to the study of critical points of a related functional, defined on a finite dimensional space Z.
The idea is simply the following one (we will refer now for the symbols to section 1.3): observe that the operator A is selfadjoint, $A^* = A$. It has a closed range and a compact resolvent. The spectrum of A, $\sigma(A)$, is a pure point spectrum: $\sigma(A) = 2\pi Z$. Every eigenvalue $\lambda \in \sigma(A)$ has multiplicity 2n and the eigenspace $E(\lambda) \doteq Ker(\lambda - A)$ is spanned by the orthogonal basis given by the loops:

$$t \longrightarrow e^{t\lambda J} e_k = (\cos\lambda t)e_k + (\sin\lambda t)Je_k, \quad k = 1,\dots 2n,$$

where $\{e_k\}$ is the standard basis in R^{2n}. In particular $Ker(A) = R^{2n}$.

Denoting by $\{E_\lambda \mid \lambda \in R\}$ the spectral resolution of A, we define the orthogonal projection $P \in \mathcal{L}(H)$ by

$$P = \int_{-2\beta}^{2\beta} dE_\lambda$$

where $\beta \notin 2\pi Z$. Let $P^\perp = 1 - P$, and set $Z = P(H)$, $Y = P^\perp(H)$. Then $H = Z \oplus Y$ and $\dim(Z) < \infty$. With these notations the equation $Au = F(u)$ for $u \in \mathrm{dom}(A)$ is equivalent to the pair of equations

$$\begin{aligned} &APu - PF(u) = 0 \\ (4.2.6)\qquad & \\ &AP^\perp(u) - P^\perp F(u) = 0. \end{aligned}$$

Now writing $u = Pu + P^\perp u = z + y \in Z \oplus Y$, we shall solve for fixed $z \in Z$ the second equation of (4.2.6), which becomes

$$Ay - P^\perp F(z + y) = 0.$$

With $A_o \doteq A|_Y$ this equation is equivalent to

(4.2.7) $$y = A_o^{-1} P^\perp F(z + y).$$

Observe that $|A_o^{-1}| \leqq \frac{1}{2\beta}$ and $|P^\perp| = 1$. Also, from assumption (i) we conclude that

$$|F(u) - F(v)| \leqq \beta |u - v| \qquad \forall\, u, v \in H.$$

Conseguently the right hand side of (4.2.7) is a contraction operator in H, having contraction constant $\frac{1}{2}$. We conclude for fixed $z \in Z$ that the equation (4.2.7) has a unique solution $y = v(z) \in Y$. Since $(A_o^{-1} y)(t) = \int_0^t Jy(s)ds$, we have $A_o^{-1}(Y) \subset H^1$ and therefore $v(z) \in \mathrm{dom}(A)$. Moreover the map $z \dashrightarrow v(z)$ from Z into Y is Lipschitz continuos. In fact we have

$$|v(z_1) - v(z_2)| \leqq \frac{1}{2} \left\{ |z_1 - z_2| + |v(z_1) - v(z_2)| \right\}.$$

Setting $u(z) = z + v(z)$

we now have to solve the first equation of (4.2.6), namely $Az - PF(u(z)) = 0$, which in view of (4.2.7) is equivalent to the equation

$$Au(z) = F(u(z)).$$

One verifies readily that

(4.2.8) $$\nabla g(z) = Az - PF(u(z)) \quad \text{with } g(z) \doteq f(u(z)).$$

It remains to find critical points of the function g, which is defined on the finite dimensional space Z.

Now to the gradient flow

$$\dot{z} = \nabla g(z)$$

the Morse type index theory for flows is applied. And the application is as follows: it can be seen that, due to the assumptions (ii) and (iii), the set S of bounded solutions of this gradient flow is compact. Hence for it (as for any isolated invariant set of a local flow) an index is defined. Using the invariance of the index under deformations crucially, this index is computed to be the homotopy type of a pointed sphere:

$$h(S) = [\dot{S}^{m_\infty}], \quad m_\infty = \frac{1}{2}\dim Z - j_\infty .$$

Here $\dot{S}^{m_\infty}$ denotes a sphere of dimension m_∞ with a distinguished point. Hence $p(t,h(S)) = t^{m_\infty}$. This is not the index of the empty set, which is a pointed one-point space and has hence the homotopy type $[\{p\},p]$ for an arbitrary point p. Therefore $S \neq \emptyset$ and, since the limit set of a bounded orbit of a gradient system consists of critical points, the function g possesses at least one critical point, and consequently the hamiltonian equation admits at least one T-periodic solution.

If the periodic orbit found above is nondegenerate, it has an index, denoted by $j \in Z$. The corresponding critical point z of g is then (it can be shown) an isolated invariant set with index

$h(\{z\}) = [\dot{S}^m]$ where $m = d - j$. Assume z is the only critical point of g; then $S = \{z\}$, since we are dealing with a gradient system, and therefore $h(S) = [\dot{S}^m]$, which on the other hand is equal to $[\dot{S}^{m_\infty}]$. Consequently $m = m_\infty$. Therefore if $j \neq j_\infty$ and hence $m \neq m_\infty$ there must be more than one critical point of g.

Assume now that the hamiltonian system possesses two nondegenerate periodic orbits, having indices j_1 and j_2. We claim that there is at least a third periodic orbit. In fact if this is not the case, then the isolated invariant set S contains precisely two isolated critical point z_1 and z_2, with indices $h(\{z_1\}) = [\dot{S}^{m_1}]$, $m_1 = d - j_1$ and $h(\{z_2\}) = [\dot{S}^{m_2}]$, $m_2 = d - j_2$ ($d = \frac{1}{2}\dim Z$). If we label them such that $a(z_1) \leqq a(z_2)$, then $\{z_1, z_2\}$ is an admissible Morse decomposition of S. From (4.1.2) we deduce the identity

$$p(t,h(\{z_1\})) + p(t,h(\{z_2\})) = p(t,h(S)) + (1 + t)Q(t),$$

which leads to the identity

$$t^{m_1} + t^{m_2} = t^{m_\infty} + (1 + t)Q(t).$$

Setting $t = 1$, we find $2 = 1 + 2Q(1)$, with $Q(1)$ nonnegative integer. This is a nonsense; hence our assumption was wrong and we must have at least three critical points of g.

Assume finally all the periodic solutions to be nondegenerate and denote their indices by j_k, $k = 1, 2, \ldots$. They correspond to the critical points of g, which are isolated. Since S is compact, there are only finitely many of them, say $z_1, \ldots z_n$. We order them s.t. $a(z_i) \leqq a(z_j)$ if $i \leqq j$. Then $(z_1, \ldots z_n)$ is an admissible ordering of a Morse decomposition of S, and we have

$$\sum_{k=1}^{n} p(t,h(z_k)) = t^{m_\infty} + (1 + t)Q(t)$$

with $m_\infty = d - j_\infty$.

By assumption the periodic solutions are nondegenerate, hence we know that

$$p(t,h(z_k)) = t^{m_k}, \quad m_k = d - j_k,$$

so that

$$\sum_{k=1}^{n} t^{m_k} = t^{m_\infty} + (1 + t)Q(t),$$

which, after multiplication by t^{-d}, becomes the advertized identity in theorem 4.2.3. We conclude that there is at least one periodic solution of index j_∞. Also, setting $t = 1$, we find $n = 1 + 2Q(1)$. Hence the number of periodic solutions is odd. This finidhes the proof of theorem 4.2.3.

We want to emphasize that the advantage of the generalized Morse theory is that it does not require the critical points of the function g to be nondegenerate; and it allows to define an index not only for nondegenerate critical points, but also for compact invariant sets.

Finally, as an illustration of the theorem 4.2.3, let us assume there are precisely three nondegenerate periodic solutions with indices j_k, $1 \leqslant k \leqslant 3$. Then:

$$t^{m_1} + t^{m_2} + t^{m_3} = t^{m_\infty} + (1 + t)Q(t).$$

Hence $Q(1) = 1$, and therefore $Q(t) = t^r$ for some integer r. We conclude that one of the j_k's agrees with j_∞, say $j_3 = j_\infty$. The remaining indices are therefore bounded to satisfy $|j_2 - j_1| = 1$.

It would be interesting to have an example of an hamiltonian system realizing this rather special situation.

4.3 EQUIVARIANT MORSE THEORY

When we consider an autonomou hamiltonian system: $\dot{z} = J\nabla h(z)$, ... the functional $f(z) = {}_0\int^T[\frac{1}{2}(\dot{z},Jz) - h(z)]dt$, defined as in section 1.3 on $D(f) = \left\{ z \in H^1((0,T);R^{2n}) \mid z(T) = z(0) \right\}$ is invariant under the action of the group S^1. This has as a consequence the fact that the critical points appear in manifolds and hence they cannot be nondegenerate.

Actually a symmetry group is present in many other physical problems. We would like to "adapt" the methods of Morse theory also to these situations. In other words we are led to the question: how is the Morse theory to be altered, to take into account a priori symmetries of a function f under the action of a compact Lie group on a manifold?

This will be the topic of the present and the following section.

We start recalling first of all the notion of group action.

Let X be a topological space and G a group, with the multiplicative notation. We'll denote by Aut(X) the group under composition of homeomorphisms from X to itself.

Definition 4.3.1: An action of G on X is an homomorphism $\phi : G \longrightarrow Aut(X)$; the homeomorphism corresponding to an element $g \in G$ is usually denoted by $\phi(g)(x) = g(x) \quad x \in X$.

When G is a topological group there is another way of defining an action on X, which also considers the topology on G. It distinguishes between left and right actions:

Definition 4.3.2: A left action of G on X is a map

$$\mu : G \times X \dashrightarrow X$$

$$\mu(g,x) = gx,$$

satisfying the following properties:

1) $1x = x \qquad 1 \in G,\ x \in X$

2) $g_1(g_2x) = (g_1g_2)x \qquad g_1,\ g_2 \in G,\ x \in X.$

The difference between left and right actions is not just a matter of notation, since properties 1) and 2) give a different order in applying g_1 and g_2. Hence, if the group is not commutative, a left action is not generally a right action.

Given $x \in X$, we denote by $O(x)$ the orbit of x, that is the set of those points in X, which can be obtained from x, using the action of the group:

$$O(x) = \left\{ gx \mid g \in G \right\}.$$

Then the quotient space X/G represents the set of all orbits.
The set

$$G_x = \left\{ g \in G \mid gx = x \right\}$$

will be called the isotropy group of x; it is the set of elements in G, which leave x fixed.
If G is a compact topological group, then G_x is a closed subgroup of G.

Definition 4.3.3: The action of G on X is said to be free if $g \in G$, $g \neq 1 \implies gx \neq x$ for evry $x \in X$; that is $G_x = 1$ for all x. If $x \in X$ and $\phi : G \dashrightarrow O(x)$ is the map given by $\phi(g) = gx$, then ϕ is surjective. If the action is free, ϕ is also injective. This implies that, when the action is free, every orbit looks like G.

Definition 4.3.4: The action of G on X is said to be effective if

$$\bigcap_{x \in X} G_x = 1.$$

We also define the trivial action of G as the one, which leaves everything fixed, that is $\forall x$: $G_x = G$.

If G is a compact Lie group, acting freely on a manifold X, then X/G is a manifold. However if the action is not free or the group is not compact this need not be the case.

Once clarified these concepts, let us suppose that there is a left action of a group G on a manifold M. We say that a flow on M is equivariant if

$$(g \cdot x) \cdot t = g \cdot (x \cdot t) \qquad x \in M,\ g \in G,\ t \in R.$$

If we have a gradient flow on a G-invariant compact manifold, then it is equivariant if the function f is G-invariant, that is $f(gx) = f(x)$, for $x \in M$, $g \in G$.

To study an equivariant flow, the most natural thing would be to look at the quotient space M/G. It is in fact obvious that a flow can be defined on M/G in the following way:

$[x] \cdot t = [x \cdot t], \quad [x] \quad M/G,\ t \in R,$

where $[x]$ is the orbit (equivalence class) of x under the action of G. The flow above is well defined: if x and x' belong to the same equivalence class, then $x' = gx$ for some g and consequently

$[x'] \cdot t = [x' \cdot t] = [(g \cdot x) \cdot t] = [g \cdot (x \cdot t)] = [x \cdot t] = [x] \cdot t.$

But, as we already told, if the action of G on M is not free, M/G fails in general to be a manifold and therefore, to apply the Morse theory, we should first of all extend it intelligently to nonmanifolds.

A different approach is given by the equivariant Morse theory, which is a natural extension of the free case.

It consists of extending the flow to the space M x E, where E is a contractible space, on which G acts freely, and then obtaining the Morse inequalities in the quotient space (M x E)/G, replacing the cohomology of the spaces involved in the equality

$$\sum_{j=1}^{n} p_t(h(M_j)) = p_t(h(M)) + (1+t)Q(t) \tag{4.3.5}$$

with their "equivariant cohomology", which we'll present next.

We recall we are supposing there is an equivariant flow, defined on a Haussdorff topological space Γ, and S is G-invariant, G being a topological group acting on Γ.

If G is compact (see |50| and the references of algebraic topology there) there is a universal G-bundle characterized by having its total space E contractible:

$$\begin{array}{c} G \\ \downarrow \\ E \\ \downarrow \\ E/G = BG \end{array} \tag{4.3.6}$$

The space BG is called the classifying space of G. The action of G on E is free and E is unique up to homotopy. Since the action of G on E is free, the diagonal action of G on the product S x E is free too. Here diagonal action means:

$$g(\gamma, e) = (g\gamma, ge) \qquad g \in G,\ \gamma \in S,\ e \in E.$$

We can extend the flow to S x E in the trivial way:

$$(\gamma \cdot e) \cdot t = (\gamma \cdot t, e) \qquad t \in R.$$

We can project this flow on the quotient space $(S \times E)/G = S_G$.

It is obvious that if I is a G-invariant, invariant set for the flow on S, then $(I \times E)/G = I_G$ is an invariant set for the quotient flow in S_G.

To obtain an analogue of the Morse relation (4.3.5) for this quotient flow, using equivariant cohomology, we need some compactness condition. In fact in obtaining (4.3.5) compact pairs have been used. Also the definition of isolated invariant set requires the presence of a compact isolating neighborhood. But in the bundle (4.3.6), usually, E and BG are realized as infinite dimensional manifold; so all compactness is lost in S x E and S_G. This difficulty can be overcomed, by means of a limit procedure as follows.

When G is a compact topological group, E and BG can be obtained as limit of finite dimensional compact spaces:

$$E = \lim_{k \to \infty} E_k \qquad BG = \lim_{k \to \infty} B_k G$$

related to the bundles

$$\begin{array}{c} G \\ \downarrow \\ E_k \\ \downarrow \\ E_k/G \;=\; B_k G. \end{array}$$

The action of G on E_k is free.

So the Morse relation is obtained for each k, and we pass to the limit using the stabilizing properties of cohomology:

if $\{M_1, \ldots M_n\}$ is an admissible ordering of a Morse decomposition of S and each M_j is G-invariant, then

$$\left\{(M_1 \times E_k)/G, \ldots\ldots (M_n \times E_k)/G\right\}$$

is a Morse decomposition for the isolated invariant set $(S \times E_k)/G$.

Observe that the flow in $S \times E_k$ is defined in the trivial way, as in S x E. Also: if (N,N^-) is an index pair with N and N^- G-invariant for the G-invariant isolated invariant set I, then

$$((N \times E_k)/G, (N^- \times E_k)/G) = (N_k, N^-_k)$$

is an index pair for $(I \times E_k)/G$.

So, if we denote by $h_k(I)$ the (homotopy) index associated to any pair (N_k, N^-_k) of $(I \times E_k)/G$, we obtain:

(4.3.7) $$\sum_{j=1}^{n} p_t(h_k(M_j)) = p_t(h_k(S)) + (1 + t)Q_t^{k}, \quad k = 1, 2, \ldots$$

We pass now to the limit in (4.3.7) for $k \to \infty$, using the stabilization of the cohomology for the classifying space (we still refer to |50| and relative references); that is:

for $E = \lim E_k$ and $BG = \lim E_k/G$, for each $i \in N$ there exists $m(i) \in N$ s.t.:

$$k \geqq m(i) \Longrightarrow H^i(E) \cong H^i(E_k) \text{ and }$$
$$H^i(BG) \cong H^i(E_k/G).$$

Hence we obtain

(4.3.8) $$\sum_{j=1}^{n} p_t^{G}(h(M_j)) = p_t^{G}(h(S)) + (1 + t)\, Q_t^{G},$$

where the Poincarè series $p_t^G(h(S))$ (resp. $p_t^G(h(M_j))$) represents the cohomology of the pair $((N \times E)/G, (N^- \times E)/G)$ if (N,N^-) is a G-invariant index pair for S (resp. for M_j), that is the equivariant cohomology of (N,N^-).

If G acts on a space X and E is defined by (4.3.6), hten the equivariant cohomology of X, $H^*_G(X)$ is: $H^*_G(X) = H^*(X_G)$, where $X_G = (X \times E)/G$.

If $x = \{x_o\}$, then $H^*_G(x_o) = H^*(BG)$, that is $H^*(BG)$ is the equivariant cohomology of a point.

If G acts freely on X, then the map

$$p : X_G \longrightarrow X/G$$
$$p([(x,e)]) = [x]$$

is an homotopy equivalence. Hence

$$H^*_G(X) = H^*(X/G),$$

that is the equivariant cohomology of X is the cohomology of the quotient space X/G.

Now: the homotopy type of the pair $((N \times E)/G, (N^- \times E)/G)$ will be denoted by $h_G(I)$ and called the equivariant (homotopy) index of I. With this understood, (4.3.8) becomes

(4.3.9) $$\sum_{j=1}^{n} p_t(h_G(M_j)) = p_t(h_G(S)) + (1 + t)Q_t^G.$$

4.4 FORCED OSCILLATIONS OF PERTURBED SYSTEMS VIA MORSE THEORY

We want to spend now some more words on the Morse theory for critical submanifolds, which allows to treat the case of functions having nondegenerate critical orbits. (Recall that alternatively the study of equivariant gradient flows in presence of particular symmetries is exposed in |13| and |12|, using a geometrical index as a replacement for the Lyusternik-Schnirelman category).

Definition 4.4.1: We say that a connected submanifold $T \subset M$ is an isolated critical manifold if

i) each point $p \in T$ is a critical point of f,

ii) T is isolated as a critical point set.

From i) and ii) it follows that T is an isolated invariant set for the gradient flow $\dot{x} = -\nabla f(x)$.

Then T has an homotopy index h(T) as always. We will next see how this can be computed in the case when T is "nondegenerate". Non-degenerate means that i) is satisfied and also

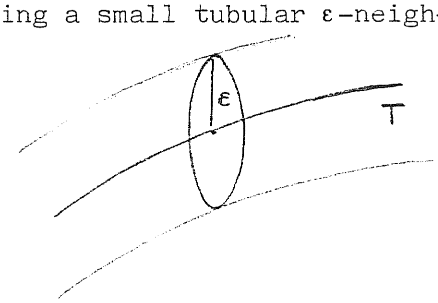
ii') the hessian of f is nondegenerate in the normal direction to T.

ii') means that if $(x_1,\ldots x_k, x_{k+1},\ldots x_n)$ is a system of local co-ordinate in M, centered at p, s.t. near $p \in T$ is given by the $n-k$ equations: $x_{k+1} = 0,\ldots x_n = 0$, then:

$$\det \left[\frac{\partial^2 f}{\partial x_i \partial x_j}\right]_p \neq 0$$

$$i, j = k+1,\ldots n.$$

Another way of saying this is considering a small tubular ε-neighborhood $M_\varepsilon(T)$, fibered over T by the normal discs to T, relative to some Riemannian



structure on M. Thus ii') means that f, restricted to each normal disc, is nondegenerate.

Moreover ii') implies ii); that is: each nondegenerate critical manifold is also isolated.

We denote by $\nu(T)$ the normal bundle of T endowed with a Riemann metric and by $H_T f$ the hessian of f on $\nu(T)$.

If we set $(A_T x,y) = H_T f(x,y)$, with x, $y \in \nu(T)$, then we define a selfadjoint endomorphism A_T from $\nu(T)$ to $\nu(T)$. Hypothesis ii') implies that A_T does not have zero as an eigenvalue and hence $\nu(T)$ can be decomposed into the direct sum

$$\nu(T) = \nu^+(T) \oplus \nu^-(T)$$

where $\nu^+(T)$ and $\nu^-(T)$ are spanned respectively by the positive and negative eigenvectors of A_T.

Definition 4.4.2: The fiber dimension λ_T of $\nu^-(T)$ will be called the index of T as a critical submanifold of f.

We want to write the Morse identity (4.3.5) for a smooth function, whose critical setes are only nondegenerate critical manifolds.

The contribution in the Morse polynomial of the critical manifold T is

$$\mathcal{M}_t(T) = \Sigma_i t^i \text{rank } H_c^i\left\{\nu^-(T)\right\}, \tag{4.4.3}$$

where H_c^i denotes the compactly supported cohomology. (If X is a locally compact topological space, $H_c^i(X) = H^i(\hat{X})$, i = 1, 2..., where $\hat{X}$ is the one-point compactification of X).

By the Thom isomorphism

$$H_c^i\left\{\nu^-(T)\right\} = H^{i-\lambda_T}(T, \vartheta^- \otimes K),$$

where K is a ring, ϑ^- is the orientation bundle of $\nu^-(T)$ and

$H^*(T, \vartheta^- \otimes K)$ is the cohomology with local coefficients. Hence (4.4.3) becomes

$$\mathcal{M}_t(T) = t^{\lambda_T} P_t(T, \vartheta^- \otimes K).$$

In particular when the bundle $\nu^-(T)$ is orientable, $P_t(T, \vartheta^- \otimes K) = P_t(T,K)$. Then, if we consider a Morse decomposition of M, given by the nondegenerate critical manifolds of f, (4.3.5) becomes

$$\Sigma\, t^{\lambda_T} P_t(T, \vartheta^- \otimes K) = P_t(M) + (1 + t)Q_t,$$

$$(\Longrightarrow \mathcal{M}_t(f) \geqq P_t(M)),$$

where the sum is taken over all the critical manifolds of f.

This theory (developed by Bott (see for instance |22|) and then adapted to Hilbert manifolds by Wassermann (|69|)) together with the perturbation methods in critical point theory, developed by Marino-Prodi (|43|) are at the basis of the papers |6| and |33|.

In |6| the existence of critical points is studied for perturbations f_ε of functionals f, whose critical points appear in manifolds. This is the case when f is invariant under the action of a continuos group and the perturbation f_ε breaks such a symmetry.

The main motivation for such an investigation is in |6| the study of hamiltonian systems such as

$$(H_\varepsilon) \qquad \begin{cases} \dot{p} = -\dfrac{\partial h}{\partial q} + \varepsilon h_1(t) \\[2ex] \dot{q} = \dfrac{\partial h}{\partial p} + \varepsilon h_2(t) \end{cases}$$

where $(p,q) \in R^{2n}$, h_1 and h_2 are T-periodic and we look for T-periodic solutions of (H_ε) for $\varepsilon > 0$ small.

Here the umperturbed system (H_0) is autonomous and hence S^1-invariant, while this is no more true for (H_ε), $\varepsilon > 0$, in view of the forcing terms h_1 and h_2.

In the paper two theorems are proved in the abstract setting of critical point theory.

In the first theorem it is supposed that the umperturbed functional f has a minimum consisting of a manifold of critical points Z. Assuming that Z has two nontrivial homology groups, it is shown that f_ε has at least two critical points at levels near f(Z). The proof does not require any nondegeneracy assumption on Z.

In the second theorem the case when Z is a compact, connected manifold of critical points of f, possibly not at the minimum level, is considered. Assuming a suitable nondegeneracy (which turns out to be the most general one in the framework of applications to (H_ε)) it is proved that f_ε has, for ε small enough, at least cat (Z) critical points near Z.

These results are for hamiltonian systems of the form (H_ε), with the hamiltonian satisfying some technichal assumptions, among which convexity.

It is seen that near a nondegenerate orbit of (H_0) there are at least two forced oscillations of (H_ε). Moreover if (H_0) is completely integrable then there are at least n + 1 forced oscillations.

In |33|, using the same techniques and tools, the existence of 2π-periodic solutions of second order, this time not convex, hamiltonian systems

$$- \ddot{x} = \nabla_x V_\varepsilon(t,x)$$

is studied, where $V_\varepsilon : S^1 \times R^n \longrightarrow R$ is of class C^2, asymptotically quadratic in x, and $V_0(t,x) = V(x)$ is time independent.

On min-max techniques and Morse theory methods rely also the detailed papers of Bahri and Berestycki |8| and |9|, concerning forced vibrations.

In |8| the existence of T-periodic solutions for hamiltonian systems of type

(4.4.4) $\dot{z} = J\nabla h(z) + f(t)$

is studied, with $h = h(z) \in C^2(R^{2n},R)$, $f : R \longrightarrow R^{2n}$ of class C^1, T-periodic; h is supposed superquadratic and verifying an additional growth condition.

The main result asserts that with these hypotheses the system (4.4.4) has infinitely many T-periodic solutions $\{z_k\}_{k\in N}$, and $|z_k|_{L^\infty} \longrightarrow +\infty$ as $k \longrightarrow +\infty$.

The first step in proving it is to construct critical values for the Lagrangian functional associated with the autonomous system

(4.4.5) $\dot{z} = J\nabla h(z)$.

This construction is based on a min-max principle, which relies on the S^1-invariance and an approximation of the space. Then the critical values are shown to be "stable" in a topological sense: the homotopy groups of the level sets associated with those values are seen to be not trivial and to remain so under "small" perturbations. Sharp estimates on the growth of the critical values are required. Finally, combining the preceding results and using Morse theory, the existence of infinitely many critical values for some perturbation of the autonomous system is obtained.

In |9| the second order system of nonlinear ordinary differential equations of the form

$$(4.4.6) \qquad \ddot{x} + \nabla V(x) = f(t)$$

$x : R \longrightarrow R^n$, $V \in C^2(R^n,R)$, $f : R \longrightarrow R^n$ a T-periodic forcing term, are studied, and forced vibrations (of period T) are seeked.

The main result is that, if V is superquadratic at ∞ , for any given $f \in L^2_{loc}(R,R^n)$ which is T-periodic the system (4.4.6) admits infinitely many T-periodic solutions.

This is achieved by a procedure similar to the one in |8|: first a sequence of critical values $\{c_k\}_{k\in N}$ for the autonomous system

$$(4.4.7) \qquad \ddot{x} + \nabla V(x) = 0$$

is constructed by means of min-max principles. Hence the level sets of the functional associated with (4.4.6), corresponding to the numbers c_k, are shown to have some topological property, which is in some sense stable under perturbations. It is also required a sharp estimate from below on the growth of the c_k's as $k \longrightarrow +\infty$.

The conclusion uses a perturbation argument on the autonomous functional.

5.1 LOCAL QUESTIONS

In conclusion we will concern ourselves with some local results: we will discuss the problem of finding periodic solutions of an hamiltonian system

(5.1.1) $\dot{z} = J\nabla h(z)$,

$z = (q_1,\dots q_n,p_1,\dots p_n) \in R^{2n}$, $h \in C^2(R^{2n},R)$, $h(0) = 0$,

in a neighborhood of the origin of R^{2n}, which we assume to be an equilibrium point.

If the linearized equation is

(5.1.2) $\dot{z} = Az$, $A \in L(R^{2n})$,

the presence of purely imaginary eigenvalues of A is clearly necessary for our goal. In fact, if A would be nonsingular with none of its eigenvalues purely imaginary, then the rest point $z = 0$ would be hyperbolic for (5.1.2) as well as for (5.1.1). (The flow of (5.1.1) would be, close to 0, topologically conjugate to those of the linearized system). No periodic solution, except the trivial one, could then exist in a neighborhood of 0.

Well: we will assume A diagonalizable with all its eigenvalues $(\lambda_1,\dots\lambda_n,-\lambda_1,\dots-\lambda_n)$ imaginary.

In such a case there exist for (5.1.2) at least n families of periodic solutions (called normal modes), and the problem becomes to see if they survive after the perturbation including the nonlinear terms.

If the λ_i, $i = 1,\dots n$, do not satisfy resonance condition, in the sense that $\lambda_i/\lambda_j \neq k$, $k \in Z$, $\forall\, i \neq j$, then a well known theorem due to Liapunov assures that (5.1.1) has at least n periodic solutions with periods close to those of the normal modes on every energy surface $h^{-1}(c)$ with c sufficiently small and positive. Hence (5.1.2) still has at least n one-parameter famulies of periodic solutions in a neighborhood of the origin (the energy being the parameter).

What the Liapunov theorem tells us is more precisely that, if $\lambda_i/\lambda_j \notin Z \; \forall \; i \neq j$, then the existence of a one-parameter family of closed orbits is garantued, with period $\sim \frac{2\pi i}{\lambda_j}$.

The nonresonance assumption is in general necessary for the validity of the theorem, as it is shown by the following example, taken from |65|:

Example 5.1.3: With the cubic polynomial

$$h = \frac{1}{2}(x_1^2 + y_1^2) - (x_2^2 + y_2^2) + x_1y_1x_2 + \frac{1}{2}(x_1^2 - y_1^2)y_2$$

as hamiltonian function, the corresponding system becomes

$$(5.1.4) \qquad \begin{cases} \dot{x}_1 = y_1 + x_1x_2 - y_1y_2 \\ \dot{x}_2 = -2y_2 + \frac{1}{2}(x_1^2 - y_1^2) \\ \dot{y}_1 = -x_1 - y_1x_2 - x_1y_2 \\ \dot{y}_2 = 2x_2 - x_1y_1 \end{cases}$$

with the obvious equilibrium solution $x_1 = x_2 = y_1 = y_2 = 0$ and eigenvalues of the linearized system: i, 2i, -i, -2i.

$\lambda_1/\lambda_2 = 1/2$, which is not an integer, and this implies the existence of a one-parameter family of periodic solutions with approximate period $\frac{2\pi i}{2i} = \pi$. These solutions can be readily exhibited: indeed the uniqueness theorem for O.D.E. implies that for initial values $x_1 = y_1 = 0$, the general solution is: $x_1 = 0$, $y_1 = 0$, $x_2 = \alpha\cos 2t - \beta\sin 2t$, $y_2 = \alpha\sin 2t + \beta\cos 2t$, with α and β constants, i.e. a circle in the x_2-y_2 plane, traversed in time π , with no dependence on the radius. In this case the period is equal to π exactly, rather than only in first approximation.

On the other hand $\lambda_2/\lambda_1 = 2$; hence the condition for the existence theorem is violated. As we have just seen all solutions with initial values $x_1 = y_1 = 0$ have period and we show now that there are no other periodic solutions. By the uniqueness theorem for O.D.E. the quantity $p = x_1^2 + y_1^2$ remains strictly positive throughout for the other solutions. Calling $q = x_2^2 + y_2^2$, a simple calculation aided by a suitable combination of the terms in complex form leads from (5.1.4) to the differential equation: $\ddot{p} = 4pq + p^2$.
Since $p^2 > 0$, $4pq \geqq 0$, it follows that p is a stictly convex function of t and therefore certainly not periodic. Thus apart from the given circular orbits there are no additional periodic orbits for this system.

In contrast to Liapunov theorem, A. Weinstein showed in |70| that, if the hamiltonian function is definite, e.g. positive definite, at the equilibrium point, then on every energy surface $h^{-1}(c)$ (c sufficiently small and positive) there are n orbits with period closed to the one of the normal modes. No nonresonance conditions are assumed. In stead the quadratic form h_2 in the Taylor expansion $h = h_0 + h_1 + h_2 + \dots$ (where $h_0 = h_1 = 0$) is asked to be positive definite.

Remark 5.1.5: these two results are interesting because they carry informations on the localization as well as on the periods, garantueing these to be not very large. On the contrary the hamiltonian closing lemma of Pugh garantues for almost all h the existence of infinitely many periodic orbits on each energy level, but it makes no estimate of the periods.

Remark 5.1.6: If we merely assume that 0 is a nondegenerate critical point of h, then we can get results like the mentioned one of Weinstein, with n replaced by the dimension of certain subspaces, on which the hessian of h is positive or negative definite.

The original proof of Weinstein uses algebraic topology (in particular the Lyusternik-Schnirelman category) and tools from the theory of Lagrangian submanifolds.

In |46| Moser gave a new proof, extending at the same time the applicability to systems not necessarily hamiltonian, possessing an integral G(z) with $G_z(0) = 0$ and positive definite hessian $G_{zz}(0)$. His proof relies on a variant of the method of Liapunov-Schmidt to reduce the problem to that of finding critical points of a C^1 function on a finite dimensional manifold. Once more topological properties of the manifold, Morse and Lyusternik-Schnirelman theory are the most natural ingredients for the proof. In another remarkable paper of Moser (|47|) the "averaging method" on a manifold is exposed and used to study vectrofields on a compact manifold, which are closed to one, having only periodic orbits. As an application a perturbation of the Kepler problem is considered. The restricted three-body problem can be seen as an example of such a perturbation.

Other results are disponible if one imposes conditions on the nonlinearity. But we don't intend to describe them there. Rather we want to mention some recent observations on this topic, due to Dell'Antonio-D'Onofrio and contained in |35|.

There an hamiltonian system in R^4 is considered, having the origin as isolated equilibrium point. The hamiltonian h, which has to be of class C^m $(m \geqslant 3)$, is assumed to have a diagonalizable quadratic part h_2 with frequencies ν_1 and ν_2, s.t. $\nu_2 = k\nu_1$, $k \in Z$.

(i.e, $h_2 = \nu_1(x_1^2 + y_1^2) + \nu_2(x_2^2 + y_2^2)$). A resonance condition is hence explicitely assumed, without requirements on the definitness of h_2. In the paper a lower bound is provided for the families of periodic solutions near the origin. Precisely the following is shown:

a) when h is of definite sign at the origin, there are at least two one-parameter familes of periodic solutions, with frequencies close to those of the linearized system. Their energy can be used as a parameter. And here there is nothing new with respect to the papers of Weinstein and Moser;

b) when h is not of definite sign at the origin, i.e. $\nu_1\nu_2 < 0$ (assume $|\nu_2| \geq |\nu_1|$), then

if $\nu_2 \neq -2\nu_1$, the same situation as in (a) holdes "generically",

if $\nu_2 = -2\nu_1$, "generically" there is only one family of periodic solutions with frequencies close to ν_2.

"Generic" refers to a set in the space of parameters, which can be specified by means of the normal form of h. Let us see it a little bit more carefully: a polyinomial p of order s is in normal form with respect to h_2 if it is invariant with respect to the group generated by h_2. Every polynomial p of order s in normal form is a sum of monomials in normal form. The coefficients of p with respect to a fixed basis of monomials can then be regarded as elements of a vector space Ξ. We denote c an element of Ξ.

We say that a property P of the hamiltonian system holds generically to order s if it is possible to find a set Σ in Ξ, of codimension one, s.t. P holds whenever one of the normal forms of $h_0 + \ldots + h_s$ correspond to an element $c \in \Sigma$. If we consider normal forms of least possible order $\geqslant 3$, we simply write "generically".

(We remark that to $h_0 + \dots + h_s$ there correspond several normal forms, obtained one from the other by a transformation in a particular group. For more details see |35|.)

In conclusion "generically" has to be understood roughly as something like "for almost all the cases", but it also means something more: the region where the property P holds (if it holds generically) is the complement of regular submanifolds and it contain large pieces.

The result above explains why the case $\nu_2 = -2\nu_1$ is the easiest to use to provide counterexamples to the existence of n families of closed orbits in resonance conditions.

It relies on a prevalently analytical approach: large use is done of the implicit function theorem; a central role is played by the properties of normal forms and by bifurcation arguments. We remark that more regularity is required for h, than is done in |70| and |46|.

In |36| the analysis, which in |35| was done in R^{2n}, n = 2, is extended to the case $n > 2$ (see also |34|). A geometric interpretation is furnished and a major use of topological methods is done. Also the theories described in the preceding sections are used: Morse theory for example, especially in its equivariant form. Often in fact the solution of the problem is reduced to the study of stationary points of a function on a manifold.

5.2 SOME REMARKS ABOUT STABILITY

Finally we recall that nonlinear oscillations near an equilibrium solution ca be seen as a bifurcation problem.

We want to mention the paper |25|, where the hamiltonian Hopf bifurcation in a resonance case is considered in a general treatment of families of periodic solutions of hamiltonian systems. An example of one system verifying the assumptions of this paper is given by the planar restricted three-body problem in Celestial Mechanics.

An alternative approach to bifurcation of periodic orbits is possible by use of stability theory. As showed for example in |42|, |20|, |61| in fact, stability arguments can be used not only for analyzing the qualitative behaviour of a flow near the bifurcation sets, which are in certain cases periodic orbits, but also to prove the existence itself of these sets.

Estimations on the number of bifurcating periodic orbits are given in |20|, |61|, |62| in terms of stability properties of the zero solution of a differential equation, appropriately associated to the umperturbed one in consideration.

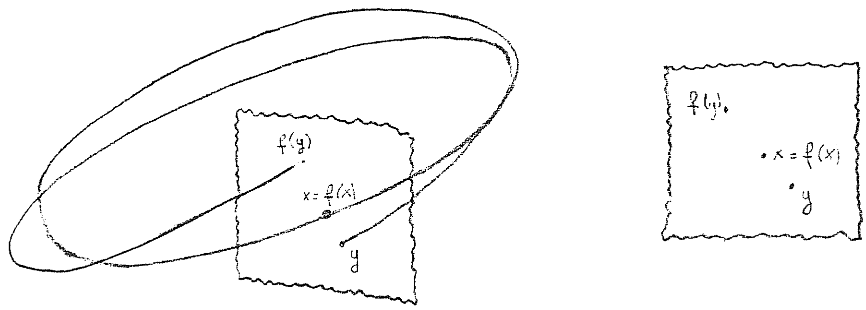
In the line of this approach also informations on the stability of the bifurcating orbits (a question of relevant physical interest) can be obtained. This is done in the paper |45|.
But these stability results hold true for systems, for which the trivial equilibrium solution is h-asymptotically stable or h-completely unstable, properties which cannot hold for autonomous hamiltonian system. (For the definitions of h-asymptotic stability

and h-complete instability we refer to the mentioned papers).

The desire arises naturally to be able to say something also about the stability of the periodic orbits, of which we deduced existence.

But we observe that for hamiltonian equations the orbit structure is indeed very complicated. Stable behaviour cannot be separated in general from unstable behaviour. And it seems absolutely hopeless to solve the initial value problem to get an insight into the longtime behaviour.

Just to clarify these statements, we recall that, for example, when studing the qualitative behaviour of a 4-dimensional hamiltonian system near a periodic orbit for all times, an idea is to study the iterates of the Poincarè mapping of a transversal section in the energy surface. This mapping is a local diffeomorphism f on the

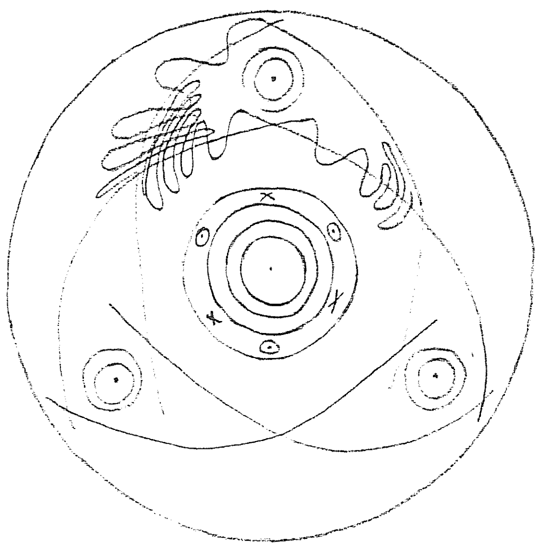


plane, defined in an open neighborhood of a fixed point (corresponding to the closed orbit) and it is area preserving.
The eigenvalues of the linear part $(df)_0$ at the point 0 (the fixed point) lie in the set $\{z \in C \mid |z| = 1\} \cup \{z \in C \mid \text{im } z = 0\}$.
If there is a real eigenvalue λ, the fixed point is called hyperbolic. And an hyperbolic fixed point is always unstable for an area preserving map: as a necessary condition for stability the fixed point has to be elliptic; the eigenvalues must all lie on the unit

circle. Let us suppose that this is the case; and let us exclude the nongeneric case: $\lambda = \pm 1$.

In these hypotheses it is known, after the work of Kolmogorov, Arnold, Moser, that, under smoothness assumptions on f, there exist, close to 0, many invariant curves on the transversal section (which are sections of invariant tori in the energy surface). They don't form a one-parameter family, but a rather complicated set, which is closed, nowhere dense and of positive measure.

If we "take a microscope" and focus on the region between two of these curves (the so called zone of instability), we see that the orbit structure is very irregular and unpredictable: there are elliptic an hyperbolic periodic points with homoclinic points, giving



rise to hyperbolic sets. It is proved in |72| that an elliptic fixed point of an even analytic diffeomorphism is "in general" a cluster point of homoclinic points.

Actually in the case of two degree of freedom a sort of stability for the closed orbit (corresponding to the elliptic fixed point) is obtained. In fact the mentioned invariant tori delimit invariant

regions in the three-dimensional energy surface; and the orbits starting in these regions cannot excape from them.
But the picture above is sufficient to give a "feeling" of how complicated the situation is in general.

Once understood that the kind of stability, which we can expect to find for closed orbits of hamiltonian systems, is different from the one which can hold for nonconservative systems, we recall a "stability" result contained in |6|.
There (for existence results we refer to section 4.4, where the paper was mentioned) forced oscillations are obtained substantially as bifurcating orbits of a system (H_ε), perturbed of an autonomous hamiltonian system (H_o). We add now that also some information is given on the stability of these forced odcillations. It is shown, for instance, the following: if the periodic orbit z_o of (H_o), from which the forced oscillations bifurcate, is nondegenerate and its Floquet multipliers $(1,1,\lambda_3,\dots\lambda_{2n})$ are s.t. $|\lambda_i| = 1 \;\forall i$, and all the λ_i for $i \geqslant 3$ are simple, then one of the bifurcating orbits (let us say z_ε^1) is "stable" (in the sense that $|\lambda_i^1(\varepsilon)| = 1 \;\forall i$), while anotherone (z_ε^2) is "unstable" (meaning by that that the condition $|\lambda_i^2(\varepsilon)| = 1 \;\forall i$ is not verified).

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