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THE PROBLEM OF FERMION MASSES IN GRAND UNIFIED THEORIES

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C O N T E N T S

INTRODUCTION

The following thesis is devoted to the analysis of the complex problem of the fermion masses and mixing angles. In particular, Chapter 0. is a general introduction to the subject. In Chapters 1. and 2. we will briefly recover the usual Renormalization Group analysis of the fermion masses, while in Chapters 3. and 4. we will analyze the effect of the presence of heavy right-handed Majorana neutrinos, due to the working of the Gell-Mann-Ramond-Slansky mechanism [20], beyond the tree-level [26], finding relevant modifications of the standard Renormalization Group equations. In the last chapter, we will discuss a tree-level natural model [27] which recovers the positive aspects both of the Georgi-Jarslkog model [15] [16], and the Fritzsch model [17]. The predictions for masses and mixing angles are in agreement with the phenomenology.

0. GENERAL ASPECTS OF THE FERMION MASS MATRIX

One of the fundamental problems of particle physics concerns the explanation of fermion masses and the mixing angles in terms of elementary principles.

The general framework of elementary particle physics is the framework of gauge theories[1] in which it is possible to construct unifying models of the elementary interactions which explain the phenomenological ratios of the different gauge coupling constants[2] (strong, weak and electromagnetic). However, the situation is much more difficult for the fermion masses and mixing angles which are fundamental parameters in defining weak and strong interactions of every elementary particle. Inside the framework of gauge theories, the purpose is to predict successful relations between masses and mixings whose validity is not destroyed by higher order effects [3] (Weinberg's naturalness). As we will see, this requisite is very stringent and it is difficult to find a natural model which can explain the entire spectrum of fermion masses.

In the Standard Model[4] (S.M.) there is no relation between fermion masses, because the Yukawa coupling constants for up, down and leptons are all independent. Moreover, the neutrino has zero mass[5], so the mixing angles in the lepton sector are zero and there is no possibility of neutrino oscillations. In the framework of Grand Unification Theories[2] (GUTs) we have a link between quarks and leptons so that there is the possibility to have some relations between fermion masses, at least inside every family, and non-zero neutrino masses, with mixings also in the leptonic sector.

First of all we will analize the phenomenological status of the problem and then we will present some phenomenological model or anszatz which should explain different aspects of the general problem. The final goal is to combine these models in a unitary theory having, in this way, the biggest possible number of predictions for fermion masses and mixing angles.

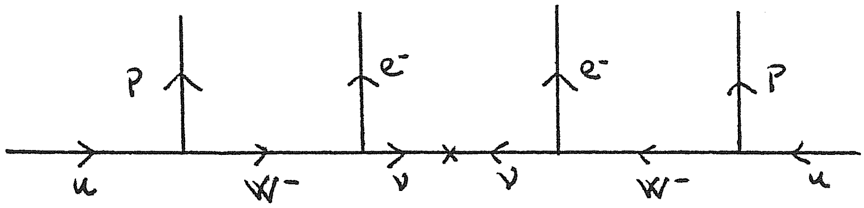
0.1 Phenomenology of masses

First of all we can observe that the phenomenological fermionic spectrum goes from 0.5 MeV (electron) to 40 GeV (top, if confirmed), with a very large spread of $10^3 \div 10^4$ orders of magnitude. However, it is possible to extract a general structure if we look at the fermions family, per family; then it is possible to see a jump of $10^1 \div 10^2$ orders of magnitude between one family and the following. This fact seems strongly to suggest a great relevance of radiative corrections which can be originated by the third family as a sort of radiative cascade(6). In this sense we could produce fermion masses in terms of the Fermi scale (being the top mass very close to the W mass).

However, it is very difficult to realize such a mechanism of mass generation by radiative corrections in a way compatible with renormalization (this is the condition of naturalness(3)). The spectrum of charged fermions is essentially known, except for the top quark. UA1 has a few events of the type $W \rightarrow t + \bar{b}$ which, if confirmed, would imply the existence of the top with a mass around 40 GeV.

The situation for neutrinos is much more different. Up to now it has been believed that their masses are exactly zero because people have strongly believed in the exact conservation of lepton number. However, from the theoretical point of view, there is no reason to consider lepton number, which is a global symmetry, as a fundamental symmetry. From this point of view, neutrinos can have very small Majorana masses at the breaking scale of lepton number (7). More generally, also the existence of right-handed neutrinos can be considered, re-establishing the parity symmetry of the theory, and so the possibility of a very light Dirac mass, but as we will see, the case of Majorana masses is more directly and naturally implemented in modern theories of elementary particles.

From the experimental point of view the existence of a double β-decay would imply the Majorana nature of neutrinos(8); in fact, it is described by the diagram:



which is non-zero only if $\nu_L \nu_L^T \neq 0$. The actual experiments are based on the possible reaction[9]:

$$^{76}Ge \rightarrow {}^{76}Se + 2\ e^-$$

for which at the moment we have only a bound $T > 10^{23}$ ys on the decay time of the reaction[9].

The single β-decay cannot distinguish the nature (Majorana or Dirac) of neutrino mass, in any case, at this moment there is an evidence for a non-zero mass of the electron neutrino. More precisely, Lyubimov and collaborators[10] studying the reaction

$$^{3}H \rightarrow {}^{3}He + e^- + \nu$$

give us the following limits:

$$14\ eV < m_{\nu_e} < 46\ eV$$

However, the estimation of the nuclear contributions to the end of the electron spectrum in the momentum is not yet completely clear, and so at the moment this data also is not very significative.

There are also many experiments on the oscillations of neutrinos[11]; the greater evidence for oscillations has been found for solar neutrinos[7], for which we have a reduction of the solar flux received on the Earth of a factor of 3. This is in agreement with oscillations with $\Delta m_\nu^2 > 10^{-10}$ eV^2. However, also this experiment cannot be accepted without criticism, because the extracted data are based on a theoretical model of solar emission which is not yet completely confirmed. There are also many reactor and atmosphere experiments which, however, do not have conclusive data except for a very recent experiment which must be confirmed, which has measured[9]

$$\left\{ \begin{array}{l} \Delta m_\nu^2 \sim 0.2\ eV^2 \\ \\ \sin^2 2\theta \sim 0.25 \end{array} \right.$$

The conclusion is that the only certain experimental data are:

$$m_{\nu_e} < 35\ eV \quad , \quad m_{\nu_\mu} < 510\ KeV \quad , \quad m_{\nu_\tau} < 100\ MeV$$

for the rest, we have many different experimental (Lyubimov, solar neutrinos) and cosmological[12] ($\sum_\nu m_\nu < 100$ eV) indications, for which light neutrinos could exist with a mass $m_\nu \lesssim 1 eV$.

At this point, the situation of fermion masses seems to be much more puzzling, with charged fermions in the range 1MeV $\div$ 10GeV and the neutrinos placed at the hierarchical suppressed scale of 1 eV or smaller.

0.2 Phenomenology of mixings

From a phenomenological point of view, after symmetry breaking via the Higgs mechanism, we will have a non-diagonal fermion mass matrix, which once diagonalised, will imply the existence of generation mixings represented by unitary matrices (GIM mechanism), in the charged currents of the theory. As we have said, there is no precise information at the moment for the leptonic mixings. They are compatible with zero, but the situation is clearer for quark mixings. They are represented by a unitary 3 x 3 matrix[12] U_c (Kobayashi-Maskawa matrix) which is represented by three angles θ_i (i = 1,2,3) and only one phase δ (using the phase freedom of quark fields). θ_1 corresponds essentially to the Cabibbo angle, while δ may be responsible for the breaking of CP symmetry[13].

Let us write U_c in the K.M. parametrization:

$$U_c = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ s_1 s_2 & -c_1 s_2 c_3 - c_2 s_3 e^{i\delta} & -c_1 s_2 s_3 + c_2 c_3 e^{i\delta} \end{pmatrix} \qquad (0.1)$$

where $c_i \equiv \cos\theta_i$, $s_i \equiv \sin\theta_i$.

From the measurement of $|U_{ud}|$ from β-decays, we extract the value : $s_1 = 0.227$; the determination of $|U_{us}|$ from strange particle decays connected with the request of unitarity gives us the bound: $s_3 < 0.5$; another important date is $|U_{cb}| > 0.032$ extracted from the b lifetime: $\tau_b < 1.4 \times 10^{-12}$ s. So we have a sharp measurement only for θ_1, however, some general phenomenological aspects may, in any case, be extracted. The principal fact is that all mixings are small, so U_c can be parametrized, as suggested by Wolfenstein[14], in the following way:

$$U_c = \begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & \lambda^3 A(\rho - i\eta) \\ -\lambda & 1-\frac{\lambda^2}{2} & A\lambda^2 \\ \lambda^3 A(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \qquad (0.2)$$

with $\lambda = 0.227$ as an expansion parameter and A, ρ, η are of order 1.

It is now evident that not only all mixings are small, but the mixing becomes weaker as the generation gap increases. Physically, this gives the cascade pattern of decay as the observed decay $b \to c \to s$, or the predicted one $t \to b \to c \to s$.

While we do have the approximate range of values of the quark mixing elements matrix, we do not yet have sharp information about the angles θ_2, θ_3 and especially about the phase δ. In any case, the general phenomenological structure of U_c is clear enough.

The principal difficulty in the estimation of δ is that the connection of this parameter with the measured ones ε, ε'/ε in the CP violation sector is affected by big errors in the estimation of other parameters (such as the B hadronic parameter). The experimental situation gives:

$$\mathrm{Re}\,\varepsilon = (1.536 \pm 0.062) \times 10^{-3}$$

$$\frac{\varepsilon'}{\varepsilon} = -0.0046 \pm 0.0053 \qquad \text{(Chicago-Saclay)}$$

$$\frac{\varepsilon'}{\varepsilon} = +0.0017 \pm 0.0084 \qquad \text{(Brookhaven-Yale)}$$

for the measurement of CP violation effects in the K-K system. However, this experimental information cannot say anything about δ [13] due to some theoretical uncertainties (B-parameter, penguin diagrams, long-range contributions) in the calculation of ε and ε'/ε. If new measurements of ε'/ε will decrease its upper bound, then the K.M. phase will not be sufficient to explain the CP violating effects and also some other mechanism must be responsible for CP violation such as Higgs scalars, or more generally, new particle interactions.

0.3 Models for charged fermion masses

One of the most exciting relation for fermion masses concerns the ratio of the b-quark mass to the τ -lepton mass[15]. If we rescale the values of these masses from low energy to the Grand Unification point, via the standard equations of the renormalization group, we find the very simple relation $b/\tau \simeq 1$.

These calculations can be extended to down-quarks and leptons of the other generations having larger errors than the previous case because of the infrared behaviour of QCD. However the indications are that[15]: $s/\mu \simeq 1/3$, $d/e \simeq 3$. We can add another interesting relation to these ones connecting the Cabibbo angle with the ratio of down and strange masses, the so-called Oakes relation: $\theta_c \simeq \sqrt{d/s}$.

Georgi and Jarlskög (GJ) have constructed a Grand Unification model which predicts all these phenomeological relations[15]. More presicely, the GJ model constructed in SU(5) gives the relation $\theta_c \simeq \eta\sqrt{d/s}$ where η is a free parameter of the theory, but if the GJ model is constructed in SO(10) (as done by Georgi and Nanopoulos[16]) we predict the Oakes relation because it must be $\eta = 1$ in SO(10), with a full realization of naturality. The Yukawa coupling of scalar Higgs in generation space is:

$$\begin{pmatrix} 0 & 10_1 & 0 \\ 10_1 & 126 & 0 \\ 0 & 0 & 10_2 \end{pmatrix} \qquad (0.3)$$

and the factor 3 in the GJ relations for the first and second generation has a geometrical origin, being produced by the presence of <u>126</u>. This model predicts the GJ down-lepton mass relations and the Oakes relation. However, it also predicts $\theta_2 = \theta_3 = \delta = 0$, so no mixing in the third generation sector (the b quark would be stable!), and no CP violation effect due to the K.M. mechanism. So the GJ model cannot be sufficient by itself and it must be extended to include CP violation effects in the K.M. matrix and mixings between the third generation and the others. An attempt to obtain a relation between mixing angles and masses in order to provide a satisfactory derivation and a generalization of the Oakes relation (leaving open the problem of mass relations) has been done by Fritzsch[17]. The model in its first form does not necessarily need the support of a Grand Unification Group such as SU(5) or SO(10), so it is at a more phenomenological level and it is based on the simple observation that the 2 x 2 matrix

$$\begin{pmatrix} 0 & a \\ a & b \end{pmatrix} \tag{0.4}$$

with a<<b has eigenvalues b, $-a^2/b$, and is diagonalized by an orthogonal matrix with angle $\theta \simeq a/b$. For the case of three generations Fritzsch has proposed to use a matrix of the form:

$$a \ll b \ll c \,, \qquad \begin{pmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & c \end{pmatrix} \tag{0.5}$$

both for down and up quarks. So he can predict some interesting relations for quark mixing parameters, in particular:

$$\theta_1 \simeq \left| \sqrt{\frac{d}{s}} + \sqrt{\frac{u}{c}} \cdot e^{i\delta} \right| \tag{0.6a}$$

$$\theta_3 \simeq \sqrt{\frac{us}{cd}} \cdot \theta_2 \tag{0.6b}$$

which are in good agreement with the present phenomenological status, if $\delta = \pm\pi/2$ is chosen. (Clearly this is an external input.)

The Fritzsch model is unable to say anything about fermion masses and if it is implemented in a GUT model (SU(5) or SO(10)) it can reproduce $\underline{b} = \underline{\tau}$, but it necessarily will give wrong predictions for the other down-lepton mass ratios.

In the last chapter we will see how to construct a natural SO(10) model which could combine the Fritzsch model with the GJ model to recover both the GJ mass relations, both a K.M. mixing matrix of the Fritzsch type. We wish to conclude this discussion by mentioning a very amusing observation done by Glashow[18]. He suggests that the following determinant must be zero:

$$\begin{vmatrix} u & c & t \\ d & s & b \\ e & \mu & \tau \end{vmatrix} = 0 \qquad (0.7)$$

first of all we note that (0.7) can be written in a renormalization group invariant way:

$$\frac{1}{cs\mu} \cdot \begin{vmatrix} u & t & c \\ d & b & s \\ e & \tau & \mu \end{vmatrix} = 0 \qquad (0.8)$$

and then that (0.8) is simply a phenomenological prediction for the t-quark. In any case we consider that (0.7) must be valid at the Grand Unification point. It is interesting to implement (0.7) through a more simple relation, such as a proportionality relation between up and down quarks:

$$\frac{t}{b} = \frac{c}{s} = \frac{u}{d} \qquad (0.9)$$

(0.9) is essentially renormalization group invariant and it predicts (using Leutwyler's running masses[19]) a top with a mass of 45 GeV at the scale of

1 GeV in good agreement with the present UA1 data. However, (0.9) fails on the up-quark because it predicts a mass of 69 MeV against the true value of 5 MeV attributed to the running mass of the u quark at the scale of 1 GeV. Another difficulty of (0.9) is that it is very hard to extract it from a satisfactory theoretical model. At present, no one has succeeded in this (the simpler implementation of (0.9) in GUT's models should imply $U_c = 1$). However, this relation seems to be a good prediction for top and the fact that up and down masses are proportional at the Grand Unification point is not so irrelevant. It is interesting to think that (0.9) is valid at the Grand Unification point, but that it is not valid at low energy scales, especially for the case of the first generation. This situation may be realised if (0.9) is not invariant under renormalization group rescaling, and that may happen if the standard renormalization group equations are modified by radiative corrections not yet considered. We will see that this is precisely the situation in SO(10), due to the presence of very heavy Majorana neutrinos. The possible effect of these particles on the standard renormalization group equations is one of the principal reasons to explore the role of right-handed Majorana neutrinos beyond the tree level.

0.4 Neutrino masses

We have seen how interesting it is to construct models for which neutrino masses are non-zero, or better, predictable. The more characterizing aspect of neutrino masses is the enormous gap between them and the charged fermion masses. This aspect has a natural explanation in left-right symmetric models due to the presence of the right-handed neutrinos and the working of the Gell-Mann-Ramond-Slansky (GRS) mechanism[20].

First of all, let us examine the situation in SU(5). Every family is a $\underline{\bar{5}}$ + $\underline{10}$ and the right-handed neutrino can be introduced as a singlet, so increasing the number of Yukawa constants of the theory. But the big problem is that the neutrino Dirac mass must be at a much smaller scale than the W mass, increasing in this way the non-naturalness of the models. If we do not introduce the right-handed neutrino, we can produce a Majorana mass for the left-handed neutrino via the explicit breaking of the global B-L symmetry through the introduction of a scalar 15. But again, we will have a problem of naturalness and the neutrino mass would acquire a deeper meaning if B-L became a local symmetry of the theory. It is not possible to realize this idea in SU(5) since $Tr(B-L) \neq 0$; so we must enlarge the rank of the group. The simplest solution is given by SO(10), for which B-L is an element of the Cartan subalgebra[5] and every family is an irreducible 16, where the new state is identified as a right-handed Majorana neutrino[21].

The principal goal of SO(10) is that it is possible to predict a mass scale m_ν for the left-handed Majorana neutrinos of order $m_\nu \sim \frac{m}{M_R} \cdot m$, being M_R the mass scale for right-handed Majorana neutrinos and m the typical mass scale of charged fermions. In this way, looking at M_R as the breaking scale of B-L symmetry, more or less close to the Grand Unification scale, we have a natural and suggestive explanation of the hierarchical gap between the neutrino mass scale and the charged fermion mass scale.

This project is directly implemented in SO(10) throught the so called GRS mechanism[22]. It is based on the fact that a 126 of Higgses can give a heavy Majorana mass M_R to the right-handed neutrino. Every 10 or 126 of Higgses will give Dirac masses m to neutrinos of the same order of the corresponding up masses (due to the nature of the SO(10) symmetry). Thus we can write the following mass matrix:

$$\begin{pmatrix} M_R & m \\ m & 0 \end{pmatrix} \qquad (0.10)$$

whose diagonalization on the pure Majorana states will give:

$$\begin{pmatrix} M_R & 0 \\ 0 & -(m/M_R)m \end{pmatrix} \qquad (0.11)$$

The left-handed neutrino acquires a Majorana mass suppressed of m/M_R with respect to the typical mass m of charged fermions. This argument is not affected by the possible direct contribution of 126 to a Majorana mass for left-handed neutrinos as shown with a general argument by Maag and Wetterich[23].

Before concluding, we wish to note that the GRS mechanism may also be implemented if the scalar 126 is not present. In fact, Witten[24] has shown that it is possible to have a Majorana mass M_R for the right-handed neutrino as a radiative effect at the two-loop level if in the SO(10) model a scalar 16 is present which breaks the B-L symmetry.

0.5 Conclusions and purposes

To conclude, we wish to recall the essential points of the fermion mass spectrum:

i) $b = \tau$; strong support for SU(5)

ii) $\mu = 3s$; $e = 1/3\ d$

iii) relations between angles and mass ratios: $\theta_c \simeq \sqrt{\frac{m_d}{m_s}}$ and extentions of this one.

iv) Hierarchically suppressed neutrino masses:

$$m_\nu \lesssim 10^{-9} m_u$$

It is now interesting to note that the theoretical models (or mechanisms) previously presented solve only some of points i) to iv); in particular:

- Minimal SU(5): i) O.K.; ii), iii), iv) NO
- GJ model: i), ii) O.K.; iii) O.K. for θ_c, NO for θ_2, θ_3, δ; iv) difficult
- Fritzsch model: i), iii) O.K.; ii) NO; iv) difficult

- GRS mechanism: iv) O.K.; not yet incorporated in a natural and satisfactory way in models attempting to explain i), ii), iii).

Because of difficulties of finding a unitary model explaining i) to iv) perhaps the fundamental fermion mass spectrum at the Grand Unification point is different from the standard extrapolation of the renormalization group equations[(25)] or has significant corrections due to additional interactions (perhaps one has to use a different group rather than SU(5) or SO(10)) or because the symmetry breaking pattern implies additional significant corrections to the tree level. Due to the success of the GRS mechanism in explaining questions of principle (point iv)), we will examine it further, beyond the tree level.

In particular, we will explore the role of right-handed neutrinos beyond the tree level. Because of their heavy masses and of the presence of opportune gauge interactions with up-quarks in the framework of SO(10) (or else larger groups such as E_6), we will have a not negligible radiative contribution to the standard renormalization group equations for up-quarks masses due to the presence of right-handed Majorana neutrinos[(26)], implying the possibility to have simpler mass relations between up-quarks and the other charged fermions at the Grand Unification point.

Another aim is the construction of a tree order SO(10) model which incorporates in an unitary scheme, the positive aspects and predictions of the GJ and Fritzsch models, leaving open, however, the possibility that some free parameters of the model could be explained through radiative corrections in a more fundamental scheme[(27)].

Our point of view is to look, with the aim of obtaining a unified picture, at the various promising mechanisms presented (GJ, Fritzsch, GRS) and to look for possible relevant effects of radiative corrections to enlighten the complex problem of fermion masses and mixings.

1. GENERAL ASPECTS OF THE RENORMALIZATION GROUP

A general quantum field theory is essentially characterized by a coupling constant g and the physical amplitudes are calculated as power expansions in g. This in general, produces divergences which must be treated in such a way as to extract finite results.

This leads to different kinds[28] of quantum field theories. A field theory is called non-renormalizable if the number of divergent Green's functions is infinite and they cannot be made finite (order by order) only with a redefinition of the Lagrangian parameters. It is called renormalizable if only a finite number of Green's functions gives rise to overall divergences and the theory may be made finite (order by order) only with a redefinition of the Lagrangian parameters. If the number of divergent Green's functions is finite, the theory is called super-renormalizable.

However, the renormalization procedure introduce an arbitrary mass scale μ in the theory, or more generally, every renormalization scheme will produce different renormalized Lagrangians, but the physical observables must be independent of the particular renormalization scheme used. This kind of invariance is called Renormalization Group Invariance[29] (RGI).

1.1 The Callan-Symanzik equation

Let us now consider the implications of RGI for a general gauge theory. Given Γ (Γ_0) a renormalized (bare) 1PI Green's function, we will have:

$$\Gamma(p;\alpha,\xi,m,\mu) = Z(\mu,\varepsilon)\cdot\Gamma_0(p;\alpha_0,\xi_0,m_0,\varepsilon) \qquad (1.1)$$

with μ the renormalization scale; ε is a regulator for the bare theory; $\alpha = g^2/4\pi$ with g gauge gauge coupling; ξ is the gauge fixing parameter. RGI implies that Γ_0 must be independent of μ , i.e.

$$\mu\frac{d}{d\mu}\Gamma_0(p;\alpha_0,\xi_0,m_0,\varepsilon) = 0 \qquad (1.2)$$

So from (1.1) and (1.2):

$$\left(\mu\frac{\partial}{\partial\mu}+\mu\frac{d\alpha}{d\mu}\frac{\partial}{\partial\alpha}+\mu\frac{d\xi}{d\mu}\frac{\partial}{\partial\xi}+\frac{\mu}{m}\frac{dm}{d\mu}m\frac{\partial}{\partial m}\right)\Gamma = \tag{1.3}$$

$$=\frac{1}{Z}\mu\frac{dZ}{d\mu}\Gamma$$

with the following definitions:

$$\mu\frac{d\alpha}{d\mu}=\alpha\beta\left(\alpha,\frac{m}{\mu},\xi,\varepsilon\right) \tag{1.4a}$$

$$\frac{\mu}{m}\frac{dm}{d\mu}=-\gamma\left(\alpha,\frac{m}{\mu},\xi,\varepsilon\right) \tag{1.4b}$$

$$\mu\frac{d\xi}{d\mu}=\delta\left(\alpha,\frac{m}{\mu},\xi,\varepsilon\right) \tag{1.4c}$$

eq. (1.3) can be written:

$$\left(\mu\frac{\partial}{\partial\mu}+\alpha\beta\frac{\partial}{\partial\alpha}+\delta\frac{\partial}{\partial\xi}-m\gamma\frac{\partial}{\partial m}\right)\Gamma=\frac{1}{Z}\mu\frac{dZ}{d\mu}\Gamma \tag{1.5}$$

and it is called the Callan-Symanzik[29] eq. Then the RGI implies that <u>every</u> 1PI Green's functions must satisfy (1.5), so in this sense a gauge theory is characterized by the universal functions $\beta, \gamma, \delta, \frac{1}{Z}\mu\frac{dZ}{d\mu}$.

This set of functions is independent of ε since it is possible to write (1.5) for the set of renormalised Green's functions with primitive divergences and then solving the system of algebraic equations. Moreover, from the non-renormalizability of the gauge-fixing term in the Faddeev-Popov effective Lagrangian(30 it is easy to derive:

$$\delta = -\xi\left(\frac{1}{Z_3}\mu\frac{dZ_3}{d\mu}\right) \tag{1.6}$$

in which Z_3 is the wave function renormalization factor for gauge bosons, so $\delta = 0$ for $\xi = 0$ which is called the Landau gauge. In this gauge (1.5) will assume the simplest form

$$\left(\mu\frac{\partial}{\partial\mu} + \alpha\beta\frac{\partial}{\partial\alpha} - m\gamma\frac{\partial}{\partial m} - \gamma_\Gamma\right)\Gamma = 0 \tag{1.7}$$

with $\gamma_\Gamma = \frac{1}{Z}\mu\frac{dZ}{d\mu}$ depending on Γ.

1.2 Effective parameters

The form of the functions $\beta, \gamma, \delta, \gamma_\Gamma$ depends on the renormalization scheme used. In the μ-subtraction scheme, they will depend in a non-trivial way, on the ξ and m/μ variables, while in the t'Hooft minimal subtraction scheme, the universal functions do not depend on m/μ, and are also independent of ξ (29).

Given a $\Gamma(p;\alpha,\xi,m,\mu)$ function of dimension D, it must be true only for dimensional reasons, that

$$\left(\lambda\frac{\partial}{\partial\lambda} + m\frac{\partial}{\partial m} + \mu\frac{\partial}{\partial\mu} - D\right)\Gamma(\lambda p;\alpha,\xi,m,\mu) = 0 \tag{1.8}$$

defining $t = \ln\lambda$ and subtracting (1.8) from (1.5):

$$\left(-\frac{\partial}{\partial t} + \alpha\beta\frac{\partial}{\partial\alpha} + \delta\frac{\partial}{\partial\xi} - (1+\gamma)m\frac{\partial}{\partial m} + \right. \tag{1.9}$$

$$\left. + D - \gamma_\Gamma\right)\Gamma(e^t p;\alpha,\xi,m,\mu) = 0$$

this eq. governs the behaviour of Green's functions when all momenta are scaled up with a common factor, at fixed μ. Given now the solutions of the following equations (called effective parameters):

$$\frac{d\bar{\alpha}(t)}{dt} = \bar{\alpha}\beta \quad , \quad \bar{\alpha}(0) = \alpha \tag{1.10a}$$

$$\frac{d\bar{\xi}(t)}{dt} = \delta \quad , \quad \bar{\xi}(0) = \xi \tag{1.10b}$$

$$\frac{d\bar{m}(t)}{dt} = -(1+\gamma)\bar{m} \quad , \quad \bar{m}(0) = m \tag{1.10c}$$

we can obtain the solution for (1.9):

$$\Gamma(\lambda p; \alpha, \xi, m, \mu) = \lambda^D \, \Gamma(p; \bar{\alpha}, \bar{\xi}, \bar{m}, \mu) \cdot \\ \cdot \exp\left\{ - \int_0^t d\tau \, \gamma_\Gamma(\bar{\alpha}(\tau)) \right\} \tag{1.11}$$

so the behaviour of Γ is governed by the flow of the effective parameters and by a change in the overall scaling factor:

$$\lambda^D \to \exp\left\{ tD - \int_0^t d\tau \, \gamma_\Gamma(\bar{\alpha}(\tau)) \right\}$$

for this reason γ_Γ is called the anomalous dimension of Γ .

From another point of view, if the renormalization point μ_1 is changed into $\mu_2 = \mu_1 e^{t_2}$ defining $\mu = \mu_1 e^{t}$ and $\alpha(\mu) \equiv \bar{\alpha}(t)$, we will have:

$$\alpha\beta = \mu \frac{d\alpha(\mu)}{d\mu} = \mu_0 e^t \frac{d\bar{\alpha}(t)}{\mu_0 e^t dt} = \frac{d\bar{\alpha}(t)}{dt} = \bar{\alpha}\beta$$

So $\bar{\alpha}(t)$ is exactly the effective coupling constant of (1.10a) and $\bar{\alpha}(t_2) = \alpha(\mu_2)$, so given the value α at a certain renormalization point μ, the value α' at another point $\mu' = \mu\, e^t$ is simply given by $\bar{\alpha}(t)$. In this sense we introduce the <u>renormalization group transformation:</u>

$$T : \mu \rightarrow \mu\, e^t$$

and we can consider $\alpha \rightarrow \bar{\alpha}(t)$ as a certain representation of the renormalization group elements. For the mass it will be: $\bar{m}(t_2) = m(\mu_2)\, e^{-t_2}$. if we are interested in the study of $m(\mu)$ we can consider the (1.4b) eq.:

$$\frac{\mu}{m} \cdot \frac{dm(\mu)}{d\mu} = -\gamma\left(\alpha, \frac{m}{\mu}, \xi\right) \qquad (1.4b)$$

as we will see, in the Standard Model γ and β do not depend on ξ and m, even in the μ-renormalization scheme if $m/\mu \ll 1$. In this case:

$$\int_{m(M)}^{m(\mu)} \frac{dm}{m} = -\int_{M}^{\mu} \gamma(\alpha(\mu'))\, \frac{d\mu'}{\mu'} \qquad (1.12)$$

and for (1.4a):

$$m(\mu) = m(M) \exp\left\{ -\int_{\alpha(M)}^{\alpha(\mu)} \frac{\gamma(\alpha)}{\beta(\alpha)} \cdot \frac{d\alpha}{\alpha} \right\} \qquad (1.13)$$

The value of $\beta(\alpha)$ for a SU(N) gauge theory is[30]:

$$\beta(\alpha) = -\frac{1}{2\pi}\left(\frac{11}{3} N - \frac{2}{3} f\right)\alpha \qquad (1.14)$$

with f being the number of fermion fundamental representations.

2. RENORMALIZATION GROUP FOR THE STANDARD MODEL

The general expression for the ε-regularized fermion propagator is

$$\frac{i}{\not{p} - m - \Sigma(\not{p}, \varepsilon)}$$

with $\Sigma(\not{p}, \varepsilon)$ the 1PI fermion self-energy which has the gereral structure

$$\Sigma(\not{p}, \varepsilon) = \Sigma_1(p^2, \varepsilon) + (\not{p} - m)\Sigma_2(p^2, \varepsilon) \tag{2.1}$$

In the μ-renormalization scheme $m(\mu)$ is defined in such a way that the fermion propagator at $p^2 = -\mu^2$ is of the form(31)

$$\left.\frac{i}{\not{p} - m(\mu)}\right|_{p^2 = -\mu^2}$$

so:

$$m(\mu) = \lim_{\varepsilon \to 0} \left(m(\varepsilon) + \Sigma_1(p^2 = -\mu^2, \varepsilon) \right) \tag{2.2}$$

with Σ_1 containing $m(\mu)$ instead of m.

By definition:

$$m(\mu) = m(\mu_0) + \Sigma_1(\mu^2, \mu_0^2) \tag{2.3}$$

where $\Sigma_1(\mu^2, \mu_0^2) = \Sigma_1(p^2 = -\mu^2, \varepsilon) - \Sigma_1(p^2 = -\mu_0^2, \varepsilon)$ is a finite quantity. For γ we will have

$$\gamma = -\frac{\mu}{m(\mu)}\frac{d}{d\mu}m(\mu) = -\frac{\mu}{m(\mu)}\frac{d}{d\mu}\Sigma_1(p^2=-\mu^2,\varepsilon) =$$

$$= -\mu\frac{d}{d\mu}\left(\frac{1}{m(\mu)}\Sigma_1(p^2=-\mu^2,\varepsilon)\right) + \mu\Sigma_1\frac{d}{d\mu}\frac{1}{m(\mu)} =$$

$$= -\mu\frac{d}{d\mu}\left(\frac{1}{m(\mu)}\Sigma_1(p^2=-\mu^2,\varepsilon)\right) + \frac{\Sigma_1}{m(\mu)}\cdot\gamma \tag{2.4}$$

so if Σ_1 is evaluated at the 1-loop order we can write:

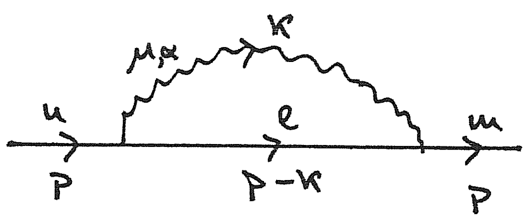
$$\gamma = -\mu\frac{d}{d\mu}\left(\frac{1}{m(\mu)}\Sigma_1(p^2=-\mu^2,\varepsilon)\right) \tag{2.5}$$

2.1 RG for fermion masses in SU(N)

Let us consider a SU(N) theory with the fermions in the fundamental representation. The interaction term is:

$$\mathcal{L}_I = -g\bar{\psi}\,t^\alpha\gamma^\mu\psi A_{\mu\alpha} \tag{2.6}$$

with t^α representing the group generators in the fermion representation. We wish to evaluate the following loop:

$-i\Sigma \quad =$ 

It is:

$$-i\Sigma(\not{p},\varepsilon) = M^{\varepsilon}\int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}}(-ig)^2\, t^{\alpha}_{m\ell}\,\gamma^{\mu}\frac{i}{\not{p}-\not{k}-m}\, t^{\alpha}_{\ell n}\,\gamma^{\nu}\cdot$$

$$\cdot\,\frac{-i}{k^2}\left(g_{\mu\nu}-(1-\xi)\frac{k_{\mu}k_{\nu}}{k^2}\right) \qquad (2.7)$$

where we perform the integral in $4-\varepsilon$ dimensions to regularize the divergences, and M is an arbitrary mass scale which appears when g is taken dimensionless.

So,

$$-i\Sigma = -g^2\,\frac{N^2-1}{2N}\,\delta_{mn}\,M^{\varepsilon}\int\frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}}\,\gamma^{\mu}(\not{p}-\not{k}+m)\cdot \qquad (2.8)$$

$$\cdot\,\gamma_{\mu}\frac{1}{[(p-k)^2-m^2]}\cdot\frac{1}{k^2} + (1-\xi)g^2\,\frac{N^2-1}{2N}\,\delta_{mn}\,M^{\varepsilon}\int\frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}}\,\not{k}(\not{p}-\not{k}+m)\not{k}\cdot$$

$$\cdot\,\frac{1}{[(p-k)^2-m^2](k^2)^2}$$

since

$$\not{k}(\not{p}-\not{k}+m)\not{k} = k^2(m-\not{p}-\not{k}) + 2(k\cdot p)\not{k} \qquad (2.9)$$

and introducing the Feynman parametrization, we can write:

$$-i\Sigma = -g^2\,\frac{N^2-1}{2N}\,M^{\varepsilon}\int\frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}}\,\gamma^{\mu}(\not{p}-\not{k}+m)\gamma_{\mu}\int_0^1 dx\,\frac{1}{[(k-\ell)^2-R^2]^2} +$$

$$+(1-\xi)g^2\,\frac{N^2-1}{2N}\,M^{\varepsilon}\int\frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}}(m-\not{p}-\not{k})\int_0^1 dx\,\frac{1}{[(k-\ell)^2-R^2]^2} +$$

$$+(1-\xi)g^2\,\frac{N^2-1}{2N}\,M^{\varepsilon}\int\frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}}\,2(k\cdot p)\not{k}\,2\int_0^1 dx\,\frac{x}{[(k-\ell)^2-R^2]^3}$$

(2.10)

where

$$\ell = p(1-x) \quad , \quad R^2 = (1-x)(m^2 - x p^2) \tag{2.11}$$

The translation $k \to k + \ell$ in $4 - \varepsilon$ dimensions does not give rise to surface terms, so:

$$-i\Sigma = -g^2 \frac{N^2-1}{2N} \int_0^1 dx\, \mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{1}{(k^2-R^2)^2} \cdot \Big\{ \gamma^\mu (x\not{p} - \not{k} + m)\gamma_\mu +$$
$$+ (1-\xi)\Big[(\not{k} + \not{p}(2-x) - m) - \frac{2x}{k^2-R^2}\, 2\big((p\cdot k)\not{k} + (p\cdot k)\not{p}(1-x) +$$
$$+ p^2 \not{k}(1-x) + p^2 \not{p}(1-x)^2\big)\Big]\Big\} \tag{2.12}$$

and after Wick rotation and symmetric integration:

$$-i\Sigma = -i g^2 \frac{N^2-1}{2N} \int_0^1 dx\, \mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{1}{(k_E^2+R^2)^2} \cdot \Big\{ (-2+\varepsilon) x \not{p} +$$
$$+ (4-\varepsilon) m + (1-\xi)\Big[(\not{p}(2-x) - m) + \frac{x}{k_E^2+R^2} \Big(4p^2(1-x)^2 \not{p} -$$
$$- \not{p}\, k_E^2 \frac{4}{4-\varepsilon} \Big)\Big]\Big\} \tag{2.13}$$

Using now the general formula:

$$\int \frac{d^d k_E}{(k_E^2+R^2)^\alpha} = \pi^{d/2} (R^2)^{\frac{d}{2}-\alpha} \cdot \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \tag{2.14}$$

we obtain:

$$\int \frac{d^{4-\varepsilon} k_E}{(k_E^2+R^2)^2} = \pi^{2-\varepsilon/2} (R^2)^{-\varepsilon/2} \cdot \Gamma\left(\frac{\varepsilon}{2}\right) \tag{2.15a}$$

$$\int \frac{d^{4-\varepsilon} k_E}{(k_E^2+R^2)^3} = \frac{1}{2} \pi^{2-\varepsilon/2} (R^2)^{-1-\varepsilon/2} \cdot \Gamma\left(1+\frac{\varepsilon}{2}\right) \tag{2.15b}$$

$$\int \frac{d^{4-\varepsilon} k_E}{(k_E^2+R^2)^3} k_E^2 = \pi^{2-\varepsilon/2} (R^2)^{-\varepsilon/2} \cdot \frac{4-\varepsilon}{4} \Gamma\left(\frac{\varepsilon}{2}\right) \tag{2.15c}$$

inserting (2.15)'s in(2.13) after some algebraic manipulation

$$\Sigma(\not p, \varepsilon) = g^2 \frac{N^2-1}{2N} \frac{1}{(16\pi^2)^{1-\varepsilon/4}} \cdot \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx \left(\frac{M^2}{R^2}\right)^{\varepsilon/2} \cdot$$

$$\cdot \Big\{ \left[(-2+\varepsilon) x \not p + (4-\varepsilon) m + (1-\xi)\big((2-x)\not p - m\big)\right] +$$

$$+ (1-\xi)\, \varepsilon \frac{p^2}{R^2} x(1-x)^2 \not p - (1-\xi) x \not p \Big\} \tag{2.16}$$

Writing (2.16) in the (2.1) form we can extract Σ_1 :

$$\Sigma_1(p^2, \varepsilon) = g^2 \frac{N^2-1}{2N} \frac{1}{(16\pi^2)^{1-\varepsilon/4}} \cdot \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx \left(\frac{M^2}{R^2}\right)^{\varepsilon/2} \cdot$$

$$\cdot \Big\{ 2m(2-x) - m\varepsilon(1-x) + (1-\xi) m(1-2x) +$$

$$+ (1-\xi) x \varepsilon (1-x) \frac{p^2}{m^2 - x p^2} m \Big\} \tag{2.17}$$

expanding Σ_1 in powers of ε :

$$\Sigma_1(p^2,\varepsilon) = g^2 \, \frac{N^2-1}{2N} \, \frac{1}{16\pi^2} \int_0^1 dx \left(\frac{2}{\varepsilon} + \ln \frac{M^2}{R^2} + \ln 4\pi - \gamma_E + O(\varepsilon) \right) \cdot$$

$$\cdot \Big\{ 2m(2-x) - m\varepsilon(1-x) + (1-\zeta)\, m(1-2x) +$$

$$+ (1-\zeta)\, \varepsilon\, x(1-x) \frac{p^2}{m^2 - x p^2}\, m \Big\} \tag{2.18}$$

where γ_E is the Euler's constant. We must now remember that to extract γ we must replace $m \rightarrow m(\mu)$ in (2.18), so:

$$\frac{1}{m(\mu)} \cdot \Sigma_1(p^2 = -\mu^2, \varepsilon) = \frac{\alpha(\mu)}{4\pi} \, \frac{N^2-1}{2N} \int_0^1 dx \left(\frac{2}{\varepsilon} + \ln \frac{M^2}{(1-x)(m(\mu)^2 + x\mu^2)} + \right.$$

$$\left. + \ln 4\pi - \gamma_E \right) \cdot \Big\{ 2(2-x) - \varepsilon(1-x) + (1-\zeta)(1-2x) +$$

$$+ (1-\zeta)\, \varepsilon\, x(1-x) \, \frac{-\mu^2}{m(\mu)^2 + x\mu^2} \Big\} \tag{2.19}$$

where $\alpha = g^2/4\pi$.

Now using (2.5), ignoring higher orders in $\alpha(\mu)$ and performing the $\varepsilon \rightarrow 0$ limit, we obtain:

$$\gamma = \frac{\alpha(\mu)}{\pi} \, \frac{N^2-1}{2N} \int_0^1 dx \, \Big\{ \frac{\mu^2 x}{m^2 + x\mu^2}(2-x) + \frac{1}{2}(1-\zeta)\, \frac{\mu^2 x}{m^2 + x\mu^2}(1-2x) +$$

$$+ (1-\zeta)\, x(1-x) \, \frac{\mu^2 m^2}{(m^2 + x\mu^2)^2} \Big\} \tag{2.20}$$

and the final expression is:

$$\gamma = \frac{\alpha(\mu)}{\pi} \frac{N^2-1}{2N} \Big\{ \frac{3}{2} + \frac{m^2}{\mu^2} - \frac{m^2}{\mu^2}\left(2+\frac{m^2}{\mu^2}\right) \ln\left(1+\frac{\mu^2}{m^2}\right) +$$

$$+ (1-\xi)\left[-\frac{m^2}{\mu^2} + \frac{1}{2}\frac{m^2}{\mu^2}\left(1+2\frac{m^2}{\mu^2}\right)\ln\left(1+\frac{\mu^2}{m^2}\right)\right]\Big\} \qquad (2.21)$$

In particular in the Landau gauge $\xi = 0$:

$$\gamma = \frac{\alpha}{\pi} \frac{N^2-1}{2N} \cdot \frac{3}{2}\left(1 - \frac{m^2}{\mu^2} \ln\left(1+\frac{\mu^2}{m^2}\right)\right) \qquad (2.22)$$

but in any case, if $m(\mu)/\mu \ll 1$:

$$\gamma = \frac{\alpha}{\pi} \frac{N^2-1}{2N}\left(\frac{3}{2} + \frac{3+\xi}{2}\frac{m^2}{\mu^2} \ln \frac{m^2}{\mu^2}\right) + O\left(\frac{m^2}{\mu^2}\right) \qquad (2.23)$$

so the leading term

$$\gamma = \frac{\alpha}{\pi} \frac{N^2-1}{2N} \cdot \frac{3}{2} \qquad (2.24)$$

does not depend on m and ξ in such approximation.

In particular, for the $SU(3)_c$ theory we find

$$\gamma = \frac{2\alpha_s(\mu)}{\pi} \qquad (2.25)$$

and we can now extract the RG eqs. for masses in QCD. Using (1.13):

$$m(\mu) = m(M) \exp\left\{ - \int_{\alpha_s(M)}^{\alpha_s(\mu)} \frac{-4}{11 - \frac{2}{3} f} \cdot \frac{d\alpha_s}{\alpha_s} \right\} \tag{2.26}$$

so:

$$m(\mu) = m(M) \left(\frac{\alpha_s(\mu)}{\alpha_G} \right)^{\frac{4}{11 - \frac{2}{3} f}} \tag{2.27}$$

where choosing for M the Grand Unification point , $\alpha_s(M)$ must be exactly $\alpha_G = g_G^2/4\pi$ where g_G is the Grand Unification coupling constant[25]. The (2.27) represents a sort of resummation of a series in $\alpha_s(\mu)$ whose leading term is

$$m(\mu) = m(M)\left(1 + \frac{\alpha_s(\mu)}{\pi} \ln \frac{M^2}{\mu^2}\right) \tag{2.28}$$

2.2 RG for fermion masses in $SU(N)_L$

Let us now consider the case of a gauge group SU(N) for which the left-handed fermions are in the fundamental representation and the right-handed fermions are singlets. The interaction term is:

$$\mathcal{L}_I = -g \bar{\psi}_L t^\alpha \gamma^\mu \psi_L A_{\mu\alpha} = \tag{2.29}$$

$$= -g \bar{\psi} t^\alpha \gamma^\mu \frac{1+\gamma_5}{2} \psi A_{\mu\alpha}$$

where the same notation of the previous case has been used. Since it is

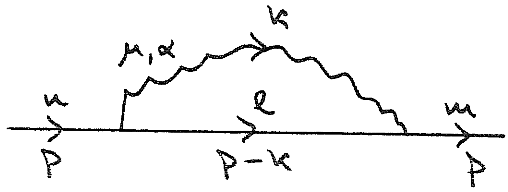
$$-i\Sigma = \frac{1}{2}\Big(g^2\, \bar{\psi}_L t^\alpha \gamma^\mu \psi_L \bar{\psi}_L t^\beta \gamma^\nu \psi_L A_{\mu\alpha} A_{\nu\beta} +$$

$$+ g^2\, \bar{\psi}_L t^\alpha \gamma^\mu \psi_L \bar{\psi}_L t^\beta \gamma^\nu \psi_L A_{\mu\alpha} A_{\nu\beta} \Big) \equiv$$

$$\equiv \frac{1}{2}\left(-i\Delta_L - i\Delta_R\right) \qquad (2.30)$$

Δ_R is obtained from Δ_L simply replacing $\psi_L \to \psi_R^c$ and $t^\alpha \to t^{\alpha *}$. Now:

$$-i\Delta_L =$$



$$-i\Delta_L = \mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} (-ig)^2\, t^\alpha_{ml}\, \gamma^\mu \frac{1+\gamma_5}{2} \frac{i}{\not p - \not k - m} t^\alpha_{ln}\, \gamma^\nu \frac{1+\gamma_5}{2} \cdot$$

$$\cdot \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2} \right) \qquad (2.31)$$

$$-i\Delta_L = -g^2 \frac{N^2-1}{2N} \delta_{mn} \mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \gamma^\mu \frac{1+\gamma_5}{2} \frac{\not p - \not k + m}{(p-k)^2 - m^2} \gamma^\nu \frac{1+\gamma_5}{2} \cdot$$

$$\cdot \frac{1}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2} \right) \qquad (2.32)$$

The calculation is very similar to the previous one except for the fact that we will not have m in the numerator; so from (2.16):

$$\Delta_L = g^2 \frac{N^2-1}{2N} \frac{1-\gamma_5}{2} \frac{1}{(16\pi^2)^{1-\varepsilon/4}} \cdot \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx \left(\frac{M^2}{R^2}\right)^{\varepsilon/2} \cdot$$

$$\cdot \left\{ \left[(-2+\varepsilon) x \not{p} + (1-\xi)(2-x)\not{p}\right] + (1-\xi)\varepsilon \frac{p^2}{R^2} x(1-x)^2 \not{p} - \right.$$

$$\left. - (1-\xi) x \not{p} \right\}$$

(2.33)

Δ_R is now obtained simply replacing $\gamma_5 \to -\gamma_5$ in Δ_L so:

$$\Sigma(\not{p}, \varepsilon) = \frac{g^2}{2} \frac{N^2-1}{2N} \frac{1}{(16\pi^2)^{1-\varepsilon/4}} \cdot \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx \left(\frac{M^2}{R^2}\right)^{\varepsilon/2} \cdot$$

$$\cdot \left\{ \left[(-2+\varepsilon) x \not{p} + (1-\xi)(2-x)\not{p}\right] + (1-\xi)\varepsilon \frac{p^2}{R^2} x(1-x)^2 \not{p} - \right.$$

$$\left. - (1-\xi) x \not{p} \right\}$$

(2.34)

It is now possible to extract Σ_1 :

$$\Sigma_1(p^2, \varepsilon) = \frac{g^2}{2} \frac{N^2-1}{2N} \frac{1}{16\pi^2} \int_0^1 dx \left(\frac{2}{\varepsilon} + \ln \frac{M^2}{R^2} + \ln 4\pi - \gamma_E\right) \cdot$$

$$\cdot \left\{ -2mx + \varepsilon m x + (1-\xi) 2m(1-x) + (1-\xi)\varepsilon m x(1-x) \frac{p^2}{m^2 - x p^2} \right\}$$

(2.35)

After some calculation, we obtain:

$$\gamma = \frac{\alpha(\mu)}{2\pi}\,\frac{N^2-1}{2N}\int_0^1 dx \left\{ -\frac{\mu^2 x^2}{m^2+x\mu^2} + (1-\xi)\,\frac{\mu^2 x(1-x)}{m^2+x\mu^2} + \right.$$

$$\left. + (1-\xi)\,x(1-x)\,\frac{\mu^2 m^2}{(m^2+x\mu^2)^2} \right\} \tag{2.36}$$

So:

$$\gamma = \frac{\alpha(\mu)}{2\pi}\,\frac{N^2-1}{2N}\left\{ -\frac{1}{2} + \frac{m^2}{\mu^2} - \left(\frac{m^2}{\mu^2}\right)^2 \ln\left(1+\frac{\mu^2}{m^2}\right) + \right.$$

$$\left. + (1-\xi)\left[\frac{1}{2} - \frac{m^2}{\mu^2} + \left(\frac{m^2}{\mu^2}\right)^2 \ln\left(1+\frac{\mu^2}{m^2}\right)\right]\right\} =$$

$$= \xi\,\frac{\alpha(\mu)}{2\pi}\,\frac{N^2-1}{2N}\left[-\frac{1}{2} + \frac{m^2}{\mu^2} - \left(\frac{m^2}{\mu^2}\right)^2 \ln\left(1+\frac{\mu^2}{m^2}\right)\right] \tag{2.37}$$

and in the Landau gauge we obtain $\gamma = 0$. This fact is easily understood if we take the divergent part of Σ_1:

$$\Sigma_1^{div.} \propto \frac{1}{\varepsilon}\cdot\xi$$

so Σ_1 is finite at the 1-loop order in the Landau gauge and we do not have mass renormalization. This fact is not absurd because the previous calculation has been performed in the presence of a mass term which breaks explicitly the $SU(2)_L$ symmetry. One must also consider the scalar Higgses which spontaneously break the $SU(2)_L$. However, as we will see, their effect is smaller than $O(\alpha)$ effects in the Landau gauge. So we can conclude that $\gamma = 0$ at the order $O(\alpha)$ in the Landau gauge(16). In particular the calculation has been performed for $m_u = m_d$, but it is easy to see that $\gamma = 0$ in the Landau gauge at order $O(\alpha)$ also if more generally $m_u \neq m_d$.

2.3 RG for fermion masses in U(1) theories

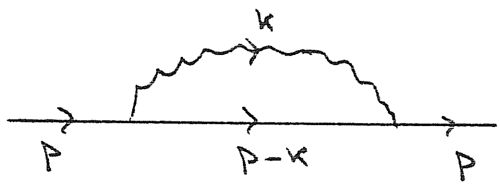
Let us take a U(1) gauge group with, in general, different quantum numbers for left-handed and right-handed fermions. The interaction is of the form:

$$\mathcal{L}_I = -g\bar{\psi}\gamma^\mu\left(\frac{1+\gamma_5}{2}Y_L + \frac{1-\gamma_5}{2}Y_R\right)\psi A_\mu \qquad (2.38)$$

where Y_L (Y_R) is the left-handed (right-handed) quantum number.

Again:

$$-i\Sigma = \frac{1}{2}(-i\Delta_L - i\Delta_R) \qquad (2.39)$$

$-i\Delta_L =$ 

$$-i\Delta_L = \mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}}(-ig)^2\gamma^\mu\left(\frac{1+\gamma_5}{2}Y_L + \frac{1-\gamma_5}{2}Y_R\right)\frac{i}{\not{p}-\not{k}-m}\cdot$$

$$\cdot \gamma^{\nu}\left(\frac{1+\gamma_5}{2}Y_L + \frac{1-\gamma_5}{2}Y_R\right) \cdot \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \tag{2.40}$$

$$-i\Delta_L = -g^2 Y_L Y_R \mu^{\varepsilon} \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \left[(4-\varepsilon)m - (1-\xi)m\right] \frac{1}{\left[(p-k)^2 - m^2\right]k^2} +$$

$$+ \mu^{\varepsilon} \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} (-ig)^2 \gamma^{\mu} \frac{1+\gamma_5}{2} Y_L \frac{i}{\not p - \not k - m} \gamma^{\nu} \frac{1+\gamma_5}{2} Y_L \cdot$$

$$\cdot \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) + \mu^{\varepsilon} \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} (-ig)^2 \gamma^{\mu} \frac{1-\gamma_5}{2} Y_R \cdot$$

$$\cdot \frac{i}{\not p - \not k - m} \gamma^{\nu} \frac{1-\gamma_5}{2} Y_R \cdot \frac{-i}{k^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right) \tag{2.41}$$

So:

$$\Delta_L = g^2 Y_L Y_R \frac{1}{(16\pi^2)^{1-\varepsilon/4}} \int_0^1 dx \left(\frac{\mu^2}{R^2}\right)^{\varepsilon/2} \cdot \Gamma\left(\frac{\varepsilon}{2}\right)\left[(4-\varepsilon)m - (1-\xi)m\right] +$$

$$+ g^2\left(Y_L^2 \frac{1-\gamma_5}{2} + Y_R^2 \frac{1+\gamma_5}{2}\right) \frac{1}{(16\pi^2)^{1-\varepsilon/4}} \cdot \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx \left(\frac{\mu^2}{R^2}\right)^{\varepsilon/2} \cdot \Big\{\left[(-2+\varepsilon)x\not p +\right.$$

$$\left. + (1-\xi)(2-x)\not p\right] + (1-\xi)\,\varepsilon \frac{p^2}{R^2} x(1-x)^2 \not p - (1-\xi)x\not p \Big\} \tag{2.42}$$

Δ_R is obtained replacing $\gamma_5 \to -\gamma_5$ so:

$$\Sigma(\not p, \varepsilon) = g^2 Y_L Y_R \frac{1}{(16\pi^2)^{1-\varepsilon/4}} \cdot \int_0^1 dx \left(\frac{\mu^2}{R^2}\right)^{\varepsilon/2} \cdot \Gamma\left(\frac{\varepsilon}{2}\right)\left[(4-\varepsilon)m - (1-\xi)m\right] +$$

$$+ \frac{g^2}{2}\left(Y_L^2 + Y_R^2\right) \frac{1}{(16\pi^2)^{1-\varepsilon/4}} \cdot \Gamma\left(\frac{\varepsilon}{2}\right) \int_0^1 dx \left(\frac{\mu^2}{R^2}\right)^{\varepsilon/2} \cdot \Big\{ \left[(-2+\varepsilon)x\not p +\right.$$

$$+(1-\xi)(2-x)\not{p}\Big]+(1-\xi)\varepsilon\frac{p^2}{R^2}x(1-x)^2\not{p}-(1-\xi)x\not{p}\Big\} \tag{2.43}$$

So γ is given by:

$$\gamma=-\mu\frac{d}{d\mu}\Big\{g^2 Y_L Y_R\frac{1}{16\pi^2}\int_0^1 dx\left(\frac{2}{\varepsilon}+\ln\frac{\mu^2}{R^2}+\ln 4\pi-\gamma_E\right)\cdot$$

$$\cdot(3-\varepsilon+\xi)\Big\}+\xi\frac{\alpha}{2\pi}(Y_L^2+Y_R^2)\left[-\frac{1}{2}+\frac{m^2}{\mu^2}-\left(\frac{m^2}{\mu^2}\right)^2\ln\left(1+\frac{\mu^2}{m^2}\right)\right] \tag{2.44}$$

So finally:

$$\gamma=\frac{\alpha(\mu)}{2\pi}Y_L Y_R(3+\xi)\left[1-\frac{m^2}{\mu^2}\ln\left(1+\frac{\mu^2}{m^2}\right)\right]+$$

$$+\frac{\alpha(\mu)}{2\pi}(Y_L^2+Y_R^2)\,\xi\left[-\frac{1}{2}+\frac{m^2}{\mu^2}-\left(\frac{m^2}{\mu^2}\right)^2\ln\left(1+\frac{\mu^2}{m^2}\right)\right] \tag{2.45}$$

In the Landau gauge

$$\gamma=\frac{3\alpha(\mu)}{2\pi}Y_L Y_R\left[1-\frac{m^2}{\mu^2}\ln\left(1+\frac{\mu^2}{m^2}\right)\right] \tag{2.46}$$

the dominant term in the limit $m(\mu)/\mu \ll 1$ being:

$$\gamma=\frac{3\alpha(\mu)}{2\pi}Y_L Y_R \tag{2.47}$$

The β-function for a U(1) theory is

$$\beta(\alpha) = -\frac{1}{2\pi}\left(-\frac{2}{3}f\right)\alpha \tag{2.48}$$

if the U(1) generator is normalized to $\frac{f}{2}$ such as $SU(3)_c$. In this case

$$m(\mu) = m(M)\exp\left\{-\int_{\alpha_G}^{\alpha_1(\mu)} \frac{9Y_LY_R}{2f}\cdot\frac{d\alpha_1}{\alpha_1}\right. \tag{2.49}$$

so:

$$m(\mu) = m(M)\cdot\left(\frac{\alpha_1(\mu)}{\alpha_G}\right)^{-\frac{9}{2f}Y_LY_R} \tag{2.50}$$

in fact since M is chosen as the Grand Unification point we will have $\alpha_1(M) = \alpha_s(M) = \alpha_G$ because of the good normalization chosen for the U(1) generator.

2.4 RG for fermion masses in the Standard Model

The Standard Model (SM) is a gauge theory of fundamental interaction based on the group $SU(3)_c \times SU(2)_L \times U(1)$. The breaking to $SU(3)_c \times U(1)_{em}$ is realized via Higgs mechanism so at least one physical scalar is needed. If moreover, the SM is used as an effective theory between low energy scales and the Grand Unification scale we will have a lot of scalars responsible for the breaking of the Grand Unification group. We can, however, ignore these contributions in a first approximation since in the Landau gauge they are generally of order h^2 (with h being the Yukawa coupling) and so negligible respect to the gauge contributions[3] if we adopt the idea that $h \ll g$, for example $h \sim g^2$.

This is the strongest argument to choose the Landau gauge in our calculation. To this one, we can add the fact that in this gauge, we do not have renormalization for the ξ parameter ($\delta = 0$), at last we have showed that $\gamma = 0$ for $SU(2)_L$ and the other γ's are depending only on the coupling constants in the limit $m(\mu)/\mu \ll 1$. For all these reasons, we will perform all calculations in the Landau gauge.

We wish to remember that we have used a space-like renormalization point. However, the possible time-like correction for the b-quark is $\Delta m_b/m_b \sim 4\%$ (see Buras 1977) so we do not worry about this problem[25].

The previous calculations have been performed for gauge theories without symmetry breaking, therefore with zero mass for vector bosons. This is not true for the $SU(2)_L \times U(1)$ sector of the SM. However since we are interested in extrapolating the RG to the Grand Unification point $\mu = M$, we may in general consider $M_{W,Z}/\mu \ll 1$, and so we put $M_{W,Z} \simeq 0$ in our calculation. Moreover, since the Cabibbo mixing angle can be present only in the $SU(2)_L$ part of γ, and since $\gamma = 0$ for $SU(2)_L$ in the Landau gauge, we will not have the presence of Cabibbo mixing in RG eqs. in the Landau gauge.

Finally, we may write the RG eqs. for the fermion masses in the SM:

$$\frac{m_u(\mu)}{m_d(\mu)} = \frac{m_u(M)}{m_d(M)} \cdot \left(\frac{\alpha_1(\mu)}{\alpha_G}\right)^{-9/20f} \qquad (2.51a)$$

$$\frac{m_d(\mu)}{m_e(\mu)} = \frac{m_d(M)}{m_e(M)} \cdot \left(\frac{\alpha_s(\mu)}{\alpha_G}\right)^{\frac{4}{11-\frac{2}{3}f}} \cdot \left(\frac{\alpha_1(\mu)}{\alpha_G}\right)^{3/2f} \qquad (2.51b)$$

where we have used $Y_{L,R} = \sqrt{3/5}\,.(T_{3\,L,R} - Q)$ and the U(1) normalization factor $\sqrt{3/5}$ is such that $\alpha_s(M) = \alpha_1(M) = \alpha_G$.

We will see that the (2.51a) will be modified by the presence of heavy Majorana neutrinos, admitted in SO(10) or E_6[26]. Since the (2.51b) will remain essentially unchanged, we will again have the Georgi-Jarlskog mass relations for the down-lepton sector:

$$\frac{m_b(M)}{m_\tau(M)} = 1 \qquad \frac{m_s(M)}{m_\mu(M)} = \frac{1}{3} \qquad \frac{m_d(M)}{m_e(M)} = 3$$

which if rescaled to low energy will give down-lepton mass ratios very close to the phenomenological ones(15).

3. MAJORANA NEUTRINOS RADIATIVE CORRECTIONS TO THE STANDARD RENORMALIZATION GROUP EQUATIONS

The very suggestive idea that the fundamental interactions have a common origin at a scale $M \sim 10^{15}$ GeV is successfully implemented in the Grand Unification Theories (GUT's) which establish a connection between quarks and leptons because they are in the same irreducible representation of the simple Grand Unified Group which contains $SU(3)_c \times SU(2)_L \times U(1)$ as a subgroup(2)(21).

The unpleasant aspect of this kind of theories is the big number of scalars needed to break the symmetries and to give mass to the fermions. But we must remember that the Higgs mechanism is only a particularly simple mechanism to break symmetries and it is possible that it is only a mimic of a more fundamental non-perturbative mechanism because of which we must look at the scalars as classical fields representing the phenomena of fermion condensation (32) (something similar to what happens in a superconductor in the BCS theory).

In the minimal SU(5) model parity is explicitly broken and the fermions are assigned in a reducible representation leaving unsolved the problem of cancellation of anomalies. Moreover there is no satisfactory explanation for non-vanishing neutrino masses and B-L breaking.

On the contrary, in a SO(10) theory parity is spontaneously broken and every family is in an irreducible representation and the anomaly cancellation is understood simply in terms of group reasons(33). The B-L breaking is connected to the presence of Majorana masses for right-handed neutrinos. In particular SO(10) breaking patterns like to link together the up-quark masses m_u and the neutrino masses m_ν and it becomes non-trivial to explain the hierarchical ratios $m_\nu / m_u \sim 10^{-5}$. This strongly points in the direction of the Gell-Mann-Ramond-Slansky (GRS) mechanism which naturally predicts a neutrino mass $m_\nu \sim m_u^2/M_R$, where M_R is the Majorana mass scale for the right-handed neutrinos, not very far from the grand unification scale.

We will exploit the GRS mechanism beyond the tree approximation. In particular, we will be interested in the radiative corrections to the renormalization group equations for fermion masses induced by the (right-handed) Majorana neutrinos of mass M_R [26].

3.1 The currents

The adjoint representation of SO(10) is a 45 which under SU(5) has the following branching rule [34]:

$$\underline{45} = \underline{24} + \underline{10} + \overline{\underline{10}} + \underline{1}$$

so we have a SU(5) 10 of leptoquarks whose corresponding currents are [26] (classified according to $SU(3)_c$ and $SU(2)_L$):

$(\bar{3},1)$, B–L = –4/3, Q = –2/3

$$J_\mu^i = \frac{1}{\sqrt{2}}\left[\bar{\nu}_L\gamma_\mu u_L^i - \bar{u}_L^{ci}\gamma_\mu \nu_L^c + \bar{e}_L\gamma_\mu d_L^i - \bar{d}_L^{ci}\gamma_\mu e_L^c\right] \qquad (3.1a)$$

(3,2), B–L = –2/3, Q = 2/3

$$J_\mu^i = \frac{1}{\sqrt{2}}\left[\bar{\nu}_L\gamma_\mu u_L^{ci} - \bar{u}_L^i\gamma_\mu \nu_L^c - \varepsilon_{ijk}\bar{d}_L^{cj}\gamma_\mu d_L^k\right] \qquad (3.1b)$$

(3,2), B–L = –2/3, Q = –1/3

$$J_\mu^i = \frac{1}{\sqrt{2}}\left[-\bar{d}_L^i\gamma_\mu \nu_L^c + \bar{e}_L\gamma_\mu u_L^{ci} + \varepsilon_{ijk}\bar{d}_L^{cj}\gamma_\mu u_L^k\right] \qquad (3.1c)$$

(1,1), B–L = 0, Q = 1

$$J_\mu = \frac{1}{\sqrt{2}}\left[\bar{d}_L^{ci}\gamma_\mu u_L^{ci} + \bar{e}_L^c\gamma_\mu \nu_L^c\right] \qquad (3.1d)$$

and a $\overline{10}$ associated with the hermitian conjugates of these currents ("i" is a colour index).

3.2 Mixing of leptoquarks

First of all we want to examine the mixing of the previous leptoquarks. The (3.1a) leptoquark and (3.1b) may mix if B–L and $SU(2)_L$

are broken; if, in particular, ϕ is a 16 or 126, we will have a mixing μ^2_{12} between the vector bosons W_1 associated to the (3.1a) and the vector bosons W_2 associated to the hermitian conjugate of (3.1b). In particular, we will have

$$\mu^2_{12} = c\, g^2 \langle\phi_1\rangle^* \langle\phi_w\rangle = c\, g^2 |\langle\phi_1\rangle| \cdot |\langle\phi_w\rangle|\, e^{-i\gamma} \qquad (3.2)$$

where ϕ_1, ϕ_w denote the SU(5) singlet component and the SU(2) breaking component of ϕ, respectively. For the 16 representation c=1/2, for the 126, c = 4. Denoting by μ_1, μ_2 the eigen-values of the W_1-W_2 mass matrix and by ζ the mixing angle, we will have

$$\sin\zeta = \frac{|\mu^2_{12}|}{\mu^2_1 - \mu^2_2} \qquad (3.3)$$

If we write

$$\mathcal{L}_m \sim (W_1^+ \;\; W_2^+) \begin{pmatrix} \mu^2_1 & \mu^2_{12} \\ \mu^{2*}_{12} & \mu^2_2 \end{pmatrix} \begin{pmatrix} W_1 \\ W_2 \end{pmatrix}$$

where it has been considered that $|\mu^2_{12}| \ll \mu^2_{12}$, and we can diagonalize the mass matrix with the transformation:

$$\begin{pmatrix} W_1 \\ W_2 \end{pmatrix} = \begin{pmatrix} \cos\zeta & -\sin\zeta \\ \sin\zeta \cdot e^{i\gamma} & \cos\zeta \cdot e^{i\gamma} \end{pmatrix} \begin{pmatrix} \overline{W}_1 \\ \overline{W}_2 \end{pmatrix} \qquad (3.4)$$

where $\sin\zeta$ is given by (3.3) and $\overline{W}_{1,2}$ are vector bosons mass eigen-states. So it is very easy to see:

$$\underbrace{W_1 W_2^+} = \sin\zeta\cos\zeta \cdot e^{-i\gamma} \left(\underbrace{\overline{W}_1 \overline{W}_1^+} - \underbrace{\overline{W}_2 \overline{W}_2^+} \right) \qquad (3.5)$$

Before concluding this part, we write down the masses μ^2_1, μ^2_2 in terms of ϕ_{45} and ϕ_{126} :

$$\mu^2_1 = 8 g^2 \left(-\frac{2}{3}\phi_{45}(1) + \frac{1}{6}\phi_{45}(24) \right)^2 + g^2 |\phi_{126}(1)|^2 \qquad (3.6a)$$

and:

$$\mu_2^2 = 8g^2\left(-\frac{2}{3}\phi_{45}(1) + \phi_{45}(24)\right)^2 + g^2\left|\phi_{126}(1)\right|^2 \qquad (3.6b)$$

and clearly, there is a degeneracy in the limit in which $\phi(24) \ll \phi(1)$, because in this limit we have dominant breaking SO(10)$\rightarrow$SU(5) x U(1) and W_1 and W_2 will be degenerate in mass being both in a <u>10</u> of SU(5).

3.3 <u>Neutrino dynamics</u>

For the reasons previously discussed, we adopt the GRS scheme at the tree level, yielding a neutrino mass matrix of the form

$$\mathcal{M}^{(\nu)} = \begin{pmatrix} M_R & m \\ m^t & m_L \end{pmatrix} \qquad (3.7)$$

in the Weyl basis ($\nu_L^c = \sigma_2 \nu_R^*$, ν_L), where t stands for the transpose, M_R, m, m_L are matrices in generation space, M_R (m_L) is the Majorana mass for right-(left) handed neutrinos and m is the Dirac neutrino mass, in general related (by SO(10) breaking) to the up-quark mass.

If det $M_R \neq 0$ by a unitary congruence $\mathcal{M}^{(\nu)}$ can be brought to the form

$$\begin{pmatrix} M_R & 0 \\ 0 & m_L - m^t M_R^{-1} m \end{pmatrix}$$

This kind of diagonalization will produce a mixing of order m/M_R, this could give, from (3.1a) contracting with itself, a 1-loop contribution to up masses of order $\frac{m}{M_R}$ L, while the contraction of (3.1a) with (3.1b) will give

$$\sin\zeta \cdot L \sim \frac{g^2 \phi_1 \phi_w}{\mu_1^2 - \mu_2^2} \cdot L \sim \frac{\phi_w}{\phi_{45}(24)} \cdot L \gg \qquad (3.8)$$

$$\gg \frac{m}{M_R} \cdot L$$

in the limit $\phi_{45}(24) \ll \phi_{126}(1)$ in which the dominant breaking is $SO(10) \to SU(5) \times U(1)$. So, in general, we will ignore this kind of mixing.

To this end, we must consider that, in general, M_R and m^u are non-diagonal matrices in generation space. Their diagonalization will produce some unitary matrices in the (3.1) currents. For simplicity we will not consider these matrices as non-diagonal, since first of all, we are interested in evaluating the essence of the Majorana neutrino contribution to the RG for light fermions. In general, the presence of these unitary matrices will give a Σ_1 of the form:

$$\Sigma_1 = \frac{1+\gamma_5}{2}\Sigma_{1RL} + \frac{1-\gamma_5}{2}\Sigma_{1LR} \qquad (3.9)$$

in which Σ_{1RL} refers to the term $\bar{\Psi}_R \Sigma_1 \Psi_L$ and $\Sigma_{1LR} = \Sigma^{+}_{1RL}$. So the most general Σ_1 is a matrix obeying the rule $\Sigma_1^{+} = \gamma_0 \Sigma_1 \gamma_0$, then, in general, we will have not diagonal counter-terms (3) and so, defining not diagonal anomalous dimensions γ, not diagonal RG equations in the generation space. Only at the end can we extract from the non diagonal $m(\mu)$ matrix the real eigen-values which are the physical masses.

For the moment, we want to avoid this kind of technical complication and we will consider M_R and m^u real and diagonal, and at the same time, $\gamma = 0$ in the (3.2) and (3.5). So we are making the unnatural assumption that the 126 has only real vacuum expectation values with the same sign. This is only a technical simplification, not a fundamental hypothesis; in the end it will be easy to see what happens in the general case.

3.4 Neutrino exchange

The only possible interactions of light fermions with heavy Majorana neutrinos (ν_L^c in Weyl basis) are in the 10 and $\overline{10}$ since the 24 gives the standard SU(5) interactions. For the up-quarks there are three types of 1-loop diagrams, with a virtual Majorana neutrino, involving the propagators $\underbrace{W_1 W_2^+}$, $\underbrace{W_1 W_1^+}$ and $\underbrace{W_2 W_2^+}$. The last two contributions involve vertices with the same chirality, right-right and left-left respectively, while the first one involves different chiralities.

For the down-quarks and leptons we will have only contributions with the same chirality as we can see from (3.1c) and (3.1d). We will see that the same chirality contributions is negligible with respect to the standard contributions, and so the only interesting case is given by the different chirality contribution present only for the up-quarks.

3.5 Majorana neutrino radiative corrections to standard R.G. for down-quarks

As we have said, the only interesting contribution is given by the interaction term:

$$\begin{aligned} \mathcal{L}_I &= -\frac{1}{\sqrt{2}} g \bar{d}_L \gamma_\mu \nu_L^c X^\mu + h.c. = \\ &= -\frac{1}{\sqrt{2}} g \bar{d} \gamma_\mu \frac{1+\gamma_5}{2} \eta X^\mu + h.c. \end{aligned} \tag{3.11}$$

where we have defined a "right-handed" Majorana neutrino η:

$$\eta \equiv \begin{pmatrix} \nu_L^c \\ -\sigma_2 \nu_L^{c*} \end{pmatrix} \tag{3.12}$$

It is convenient to use the Majorana (four components) fields rather than Weyl (two components) fields, because then the propagators have the same form as the Dirac propagators.

It is:

$$-i\Sigma = \frac{1}{2}\left(\frac{1}{2} f^2 \bar d \gamma_\mu \frac{1+\gamma_5}{2} \eta \bar\eta \gamma_\nu \frac{1+\gamma_5}{2} d\, X^\mu X^{\nu\dagger} + \right.$$

$$\left. + \frac{1}{2} f^2 \bar\eta \gamma_\mu \frac{1+\gamma_5}{2} d \bar d \gamma_\nu \frac{1+\gamma_5}{2} \eta\, X^\mu X^{\nu\dagger}\right) =$$

$$= \frac{1}{2}\left(-i\Delta_L - i\Delta_R\right) \tag{3.13}$$

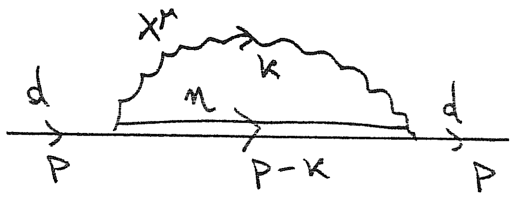
where Δ_R is obtained from Δ_L replacing $\gamma_5 \to -\gamma_5$ and $d \to d^c$.

So

$$-i\Delta_L = \mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \left(\frac{-if}{\sqrt{2}}\right)^2 \gamma_\mu \frac{1+\gamma_5}{2} \frac{i}{\not p - \not k - M_R} \gamma_\nu \frac{1+\gamma_5}{2} \cdot$$

$$\cdot \frac{-i}{k^2-\mu_3^2}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2-\xi\mu_3^2}\right) \tag{3.14}$$

where

$-i\Delta_L =$ 

Choosing the Landau gauge for the same reasons of Section 2, we have:

$$-i\Delta_L = -\frac{f^2}{2}\mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \gamma_\mu \frac{1+\gamma_5}{2} \frac{\not p - \not k + M_R}{(p-k)^2 - M_R^2} \gamma_\nu \frac{1+\gamma_5}{2} \cdot$$

$$\cdot \frac{1}{k^2-\mu_3^2}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \tag{3.15}$$

and so:

$$-i\Delta_L = -\frac{f^2}{2}\frac{1-\gamma_5}{2}\mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \left[\gamma_\mu(\not p - \not k)\gamma^\mu - \right.$$

$$\left. - \frac{\not k(\not p - \not k)\not k}{k^2}\right] \cdot \frac{1}{[(p-k)^2 - M_R^2](k^2-\mu_3^2)} \tag{3.16}$$

defining:

$$\ell = p(1-x) \quad , \quad R^2 = (1-x)(M_R^2 - x p^2) + \mu_3^2 x \qquad (3.17)$$

we can write:

$$-i\Delta_L = -\frac{g^2}{2}\frac{1-\gamma_5}{2}\mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \left[3\not k - \not p + \varepsilon(\not p - \not k)\right] \int_0^1 dx \frac{1}{[(k-\ell)^2 - R^2]^2} +$$

$$+ \frac{g^2}{2}\frac{1-\gamma_5}{2}\mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} 2(p\cdot k)\not k \frac{1}{[(p-k)^2 - M_R^2](k^2-\mu_3^2)k^2} \qquad (3.18)$$

Using again Feynman parametrization:

$$-i\Delta_L = \frac{g^2}{2}\frac{1-\gamma_5}{2}\mu^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \left\{ \left[\not p - 3\not k + \varepsilon(\not k - \not p)\right] \int_0^1 dx \frac{1}{[(k-\ell)^2 - R^2]^2} + \right.$$

$$+ 2(p\cdot k)\not k\, 2 \int_0^1 dx \int_0^x dy \frac{1}{[(k-\ell)^2 - R^2 + \mu_3^2 y]^3} \qquad (3.19)$$

shifting $k \to k+\ell$ and remembering (3.13) :

$$-i\Sigma(\not p, \varepsilon) = i\frac{g^2}{4}\mu^\varepsilon \int \frac{d^{4-\varepsilon}k_E}{(2\pi)^{4-\varepsilon}} \left\{ \int_0^1 dx \left[\not p(3x-2) - \varepsilon x \not p\right] \frac{1}{(k_E^2 + R^2)^2} - \right.$$

$$\left. - \int_0^1 dx \int_0^x dy\, 4\left(\not p \frac{-1}{4-\varepsilon} k_E^2 + \not p p^2 (1-x)^2\right) \frac{1}{(k_E^2 + R^2 - \mu_3^2 y)^3} \right\} \qquad (3.20)$$

having done the symmetric integration and the Wick rotation. Integrating in k_E with the usual (2.15) formulas:

$$\Sigma(\not p, \varepsilon) = -\frac{g^2}{4}\frac{1}{(16\pi^2)^{1-\varepsilon/4}} \cdot \Gamma\left(\frac{\varepsilon}{2}\right) \left\{ \int_0^1 dx (3x - 2 - \varepsilon x)\not p \left(\frac{\mu^2}{R^2}\right)^{\varepsilon/2} + \right.$$

$$+ \int_0^1 dx \int_0^x dy\, \not p \left(\frac{\mu^2}{R^2 - \mu_3^2 y}\right)^{\varepsilon/2} - \int_0^1 dx \int_0^x dy\, \not p \frac{p^2}{R^2 - \mu_3^2 y}(1-x)^2 \varepsilon \cdot \qquad (3.21)$$

$$\left. \cdot \left(\frac{\mu^2}{R^2 - \mu_3^2 y}\right)^{\varepsilon/2} \right\}$$

using (2.1):

$$\Sigma_1(p^2,\varepsilon) = -\frac{f^2}{4}\frac{m}{16\pi^2}\Big\{\int_0^1 dx\,(3x-2-\varepsilon x)\Big(\frac{2}{\varepsilon}+\ln\frac{M^2}{R^2}+\ln 4\pi-\gamma_E\Big)-$$

$$-\int_0^1 dx\int_0^x dy\,(1-x)^2\varepsilon\frac{p^2}{R^2-\mu_3^2 y}\Big(\frac{2}{\varepsilon}+\ln\frac{M^2}{R^2-\mu_3^2 y}+\ln 4\pi-\gamma_E\Big)+$$

$$+\int_0^1 dx\int_0^x dy\Big(\frac{2}{\varepsilon}+\ln\frac{M^2}{R^2-\mu_3^2 y}+\ln 4\pi-\gamma_E\Big)\Big\} \qquad (3.22)$$

using (2.5) (ignoring higher orders in f):

$$\gamma = -\frac{f^2}{4}\frac{1}{16\pi^2}\Big\{\int_0^1 dx\,(3x-2)\,\frac{2\mu^2 x(1-x)}{R^2}+\int_0^1 dx\int_0^x dy\,\frac{2\mu^2 x(1-x)}{R^2-\mu_3^2 y}-$$

$$-\mu\frac{d}{d\mu}\int_0^1 dx\int_0^x dy\,2(1-x)^2\frac{\mu^2}{R^2-\mu_3^2 y}\Big\} \qquad (3.23)$$

and so the contribution of (3.23) to the RG for down-quarks is:

$$\ln\frac{m(\mu)}{m(M)} = -\int_M^\mu \gamma\,\frac{d\mu'}{\mu'} = \frac{1}{m(\mu)}\Big(\Sigma_1(p^2=-\mu^2,\varepsilon)-\Sigma_1(p^2=-M^2,\varepsilon)\Big) \qquad (3.24)$$

with M being the Grand Unification point:

$$\ln\frac{m(\mu)}{m(M)} = -\frac{f^2}{4}\frac{1}{16\pi^2}\Big\{\int_0^1 dx\,(3x-2)\ln\frac{(1-x)(M_R^2+xM^2)+\mu_3^2 x}{(1-x)(M_R^2+x\mu^2)+\mu_3^2 x}+$$

$$+\int_0^1 dx\int_0^x dy\,\ln\frac{(1-x)(M_R^2+xM^2)+\mu_3^2(x-y)}{(1-x)(M_R^2+x\mu^2)+\mu_3^2(x-y)}-$$

$$-\int_0^1 dx\int_0^x dy\,2(1-x)^2\Big[\frac{\mu^2}{(1-x)(M_R^2+x\mu^2)+\mu_3^2(x-y)}-\frac{M^2}{(1-x)(M_R^2+xM^2)+\mu_3^2(x-y)}\Big]\Big\} \qquad (3.25)$$

ignoring higher orders in g.
Choosing μ at low energy, so $\mu << \mu_3, M_R$, and writing $\frac{g^2}{4\pi} = \alpha_q = \frac{8}{3}\alpha$:

$$\ln \frac{m(\mu)}{m(M)} \simeq -\frac{\alpha}{6\pi}\left\{ \int_0^1 dx\,(3x-2)\ln\frac{M^2}{(1-x)M_R^2+\mu_3^2 x} + \right.$$
$$\left. + \int_0^1 dx \int_0^x dy \ln \frac{M^2}{(1-x)M_R^2+\mu_3^2(x-y)} + \int_0^1 dx \int_0^x dy\; 2(1-x)^2 \frac{\mu^2}{(1-x)M_R^2+\mu_3^2(x-y)} + \cdots \right\}$$

(3.26)

where we have considered the limit $M >> M_R, \mu_3$ as a sort of step approximation. This kind of contribution is of order $\alpha/6\pi$ and so negligible for down and up for which the S.M. gives a contribution of order one. For the leptons the S.M. contribution is

$$-\frac{27}{20f}\ln\frac{\alpha_1(\mu)}{\alpha_q} = -\frac{27}{20f}\ln\frac{5\alpha}{8\alpha_q} \simeq \frac{5}{12}$$

again dominant with respect $\alpha/6\pi$.
So the radiative contributions of the type with <u>the same chirality</u> are not relevant. However, the only contribution of the type with <u>different chirality</u> is present only for up-quarks.[26] We will now calculate this kind of contribution.

3.6 Majorana neutrino radiative corrections to standard RG for up-quarks

We have to calculate only the <u>different chirality</u> contributions, so we consider:

$$\mathcal{L}_I = -\frac{1}{\sqrt{2}} g_1 \bar{u}^c_L \gamma_\mu \nu^c_L W_1^\mu - \frac{1}{\sqrt{2}} g_2 \bar{u}_L \gamma_\mu \nu^c_L W_2^{\mu +} + h.c. =$$

$$= \frac{1}{\sqrt{2}} g_1 \bar{\eta}\gamma_\mu \frac{1-\gamma_5}{2} u W_1^\mu - \frac{1}{\sqrt{2}} g_2 \bar{u}\gamma_\mu \frac{1+\gamma_5}{2}\eta W_2^{\mu +} + h.c. \qquad (3.27)$$

where we have used the relation:

$$\bar{u}^c_L \gamma_\mu \nu^c_L = -\bar{\nu}_R \gamma_\mu u_R \qquad (3.28)$$

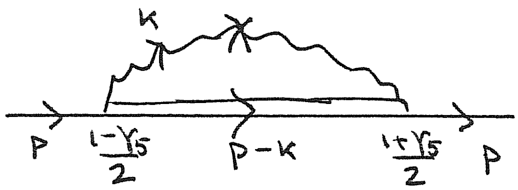
Since we are interested only in the contractions of the type $W_1 W_2^+$ we can write:

$$-i\Sigma = \frac{1}{2}\Big(-\frac{f_1 f_2}{2}\bar{u}\gamma_\mu \frac{1+\gamma_5}{2}\eta\bar{\eta}\gamma_\nu \frac{1-\gamma_5}{2} u\, W_1^\mu W_2^{\nu +} + \frac{-f_1 f_2}{2}\bar{\eta}\gamma_\mu \frac{1-\gamma_5}{2} u\bar{u}\gamma_\nu \frac{1+\gamma_5}{2}\eta\, W_1^\mu W_2^{\nu +} + \frac{-f_1 f_2}{2}\bar{u}\gamma_\mu \frac{1-\gamma_5}{2}\cdot \eta\bar{\eta}\gamma_\nu \frac{1+\gamma_5}{2} u\, W_1^{\mu +} W_2^{\nu} + \frac{-f_1 f_2}{2}\bar{\eta}\gamma_\mu \frac{1+\gamma_5}{2} u\bar{u}\gamma_\nu \frac{1-\gamma_5}{2}\eta\, W_1^{\mu +} W_2^{\nu}\Big) =$$

$$= \frac{1}{2}\left(-i\Delta_a - i\Delta_b - i\Delta_c - i\Delta_d\right) \qquad (3.29)$$

It is:

So:

$$-i\Delta_a =$$ 

$$-i\Delta_a = M^\varepsilon \int \frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}} \frac{f_1 f_2}{2}\gamma_\mu \frac{1+\gamma_5}{2}\frac{i}{\not{p}-\not{k}-M_R}\gamma_\nu \frac{1-\gamma_5}{2}\cdot$$

$$\cdot \sin\xi\cos\xi\left(\frac{-i}{k^2-\mu_1^2} - \frac{-i}{k^2-\mu_2^2}\right)\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right) \qquad (3.30)$$

where we have chosen the Landau gauge and, in the notation, we have redefined the ζ angle as ξ .

We will have:

$$-i\Delta_a = \frac{1-\gamma_5}{2}\sin\xi\cos\xi\,\frac{f_1 f_2}{2}(3-\varepsilon)M_R\int\frac{d^{4-\varepsilon}k}{(2\pi)^{4-\varepsilon}}\int_0^1 dx\,\cdot$$

$$\cdot\left\{\frac{1}{[(k-\ell)^2-R_1^2]^2} - \frac{1}{[(k-\ell)^2-R_2^2]^2}\right\} \qquad (3.31)$$

where:

$$\ell = p(1-x)\;,\quad R_{1,2}^2 = (1-x)(M_R^2 - x p^2) + \mu_{1,2}^2\, x \qquad (3.32)$$

using a shifting $k \to k+\ell$ and a Wick rotation:

$$-i\Delta_e = \frac{1-\gamma_5}{2}\, i \sin\xi\cos\xi\, \frac{3-\varepsilon}{2} f_1 f_2 M_R \int_0^1 dx \int \frac{d^{4-\varepsilon}k_E}{(2\pi)^{4-\varepsilon}} \cdot \left\{ \frac{1}{(k_E^2+R_1^2)^2} - \frac{1}{(k_E^2+R_2^2)^2} \right\} \qquad (3.33)$$

Remembering (3.29), it is easy to understand that:

$$\Sigma_1(p^2) = -\frac{3-\varepsilon}{2} f_1 f_2 \sin\xi\cos\xi\, M_R \int_0^1 dx \int \frac{d^{4-\varepsilon}k_E}{(2\pi)^{4-\varepsilon}} \cdot \left\{ \frac{1}{(k_E^2+R_1^2)^2} - \frac{1}{(k_E^2+R_2^2)^2} \right\} \qquad (3.34)$$

Taking the limit, we obtain($\varepsilon \to 0$):

$$\Sigma_1(p^2) = -\frac{3}{32\pi^2} f_1 f_2 \sin\xi\cos\xi\, M_R \int_0^1 dx\, \ln \frac{(1-x)(M_R^2 - x p^2) + M_2^2 x}{(1-x)(M_R^2 - x p^2) + M_1^2 x} \qquad (3.35)$$

and so the Majorana neutrino radiative contribution to the up-quarks mass is finite [26], and, at the point $p^2 = -\mu^2$ given by:

$$\Sigma_1(p^2=-\mu^2) = -\frac{3}{32\pi^2} f_1 f_2 \sin\xi\cos\xi\, M_R \int_0^1 dx\, \ln \frac{(1-x)(M_R^2 + x\mu^2) + M_2^2 x}{(1-x)(M_R^2 + x\mu^2) + M_1^2 x} \qquad (3.36)$$

Taking the derivative of $\Sigma_1(p^2=-\mu^2)$ with respect to μ we observe that $df_{1,2}/d\mu \propto f_{1,2}^2$,so we will ignore these terms (as previously done for α_s) and we will assume a very slow dependence on μ for M_R and ξ . We can write:

$$\Sigma_1(p^2=-\mu^2) = \frac{3}{32\pi^2} f_1(\mu) f_2(\mu) \sin\xi\cos\xi\, M_R \cdot H(\mu) \qquad (3.37)$$

where we have used the definition:

$$H(\mu) \equiv \int_0^1 dx\, \ln \frac{(1-x)(M_R^2 + x\mu^2) + M_1^2 x}{(1-x)(M_R^2 + x\mu^2) + M_2^2 x} \qquad (3.38)$$

Remembering that:

$$\gamma(\mu) = -\frac{\mu}{m(\mu)} \frac{d}{d\mu} \Sigma_1(p^2=-\mu^2) \tag{3.39}$$

we will have

$$\gamma = \frac{3}{32\pi^2} f_1(\mu) f_2(\mu) \sin\xi \cos\xi \frac{M_R}{m(\mu)} \cdot I(\mu) \tag{3.40}$$

where we have defined

$$I(\mu) = -\mu \frac{d}{d\mu} H(\mu) \tag{3.41}$$

and so:

$$I(\mu) = \int_0^1 dx \frac{2\mu^2 x^2 (1-x)(\mu_1^2 - \mu_2^2)}{[(1-x)(M_R^2 + x\mu^2) + \mu_1^2 x] \cdot [(1-x)(M_R^2 + x\mu^2) + \mu_2^2 x]} \tag{3.42}$$

and we have assumed a slow variation of M_R and ξ in the μ variable. To evaluate $I(\mu)$ we can introduce the functions $J_i(\mu)$ ($i = 1,2$) defined by:

$$J_i(\mu) = a_i \ln \frac{\Delta_i + a_i + 1}{\Delta_i + a_i - 1} + \frac{1}{2\Delta_i}(a_i \Delta_i - a_i^2 - \tau_i) \cdot \ln \frac{1 + \tau_i - \Delta_i}{1 + \tau_i + \Delta_i} \tag{3.43}$$

where:

$$\Delta_i = [a_i^2 + 2\tau_i + 1]^{1/2}, \quad a_i = \frac{M_R^2 - \mu_i^2}{\mu^2}, \quad \tau_i = \frac{M_R^2 + \mu_i^2}{\mu^2} \tag{3.44}$$

and $I(\mu)$ is expressed in terms of the $J_i(\mu)$ through the relation:

$$I(\mu) = 2\left(J_1(\mu) - J_2(\mu)\right) \tag{3.45}$$

Let us now consider the case in which $M_R^2 \ll \mu_i^2$. In fact, looking at the (3.6) and assuming that the Yukawa coupling h is such that $h \leq g$, it is obvious that $M_R \leq \mu_i$, and so it is reasonable that $M_R \sim (10^{-2} \div 10^{-3})\ \mu_i$ otherwise we should have too small neutrino masses via the GRS mechanism. So the previous assumption is very reasonable.

In this case

$$\begin{cases} a_i \simeq \tau_i = -\dfrac{\mu_i^2}{\mu^2} \\ \Delta_i \simeq a_i - 1 \end{cases} \tag{3.46}$$

and:

$$J_i(\mu) \simeq \tau_i \ln\left(1 + \frac{1}{\tau_i}\right) \tag{3.47}$$

In this case we will write:

$$\gamma(\mu) = \frac{3}{32\pi^2} f_1(\mu) f_2(\mu) \sin\zeta\cos\zeta \frac{M_R}{m(\mu)} \cdot I(\mu) \tag{3.48}$$

where:

$$I(\mu) \simeq F_1(\mu) - F_2(\mu). \tag{3.49}$$

and:

$$F_i(\mu) \equiv 2\frac{\mu_i^2}{\mu^2} \ln\left(1 + \frac{\mu^2}{\mu_i^2}\right) \tag{3.50}$$

4. CORRECTIONS TO STANDARD RG EQUATIONS FOR UP-QUARKS

We want to consider the effect of the Majorana neutrinos, previously calculated in terms of γ , in the RG equations for the up-quarks. We have to consider that for (3.42)

$$\int_{\mu}^{M} I(\mu') \frac{d\mu'}{\mu'} = - H(M) + H(\mu) \simeq H(\mu) \tag{4.1}$$

since M is the Grand Unification point for which (i.e. in the step approximation) H(M) = 0. Remembering that:

$$\frac{\mu}{m(\mu)} \cdot \frac{dm(\mu)}{d\mu} = - \gamma(\mu) \tag{4.2}$$

where in γ we have the sum of all the possible 1-loop contribution for the up-quarks, we have:

$$\frac{\mu}{m(\mu)} \frac{dm(\mu)}{d\mu} = - \frac{2}{\pi} \alpha_S(\mu) - \frac{1}{10\pi} \alpha_1(\mu) - \\ - \frac{3}{32\pi^2} f_1(\mu) f_2(\mu) \sin\xi \cos\xi \frac{M_R}{m(\mu)} \cdot I(\mu) \tag{4.3}$$

where obviously again we will consider very slow the dependence of ξ and M_R in μ. The (4.3) can be written in the form:

$$\frac{dm(\mu)}{d\mu} + \left(\frac{2}{\pi} \alpha_S(\mu) + \frac{1}{10\pi} \alpha_1(\mu)\right) \frac{m(\mu)}{\mu} = \\ = - \frac{3}{32\pi^2} f_1(\mu) f_2(\mu) \sin\xi \cos\xi \frac{M_R}{\mu} \cdot I(\mu) \tag{4.4}$$

The (4.4) is a differential equation of the type:

$$y' + e(x)\, y = f(x) \tag{4.5}$$

whose solution is:

$$y(x) = e^{-A(x)} \left(c + \int_0^x f(t)\, e^{A(t)}\, dt \right) \tag{4.6}$$

where

$$A(x) \equiv \int_0^x e(t)\, dt \tag{4.7}$$

and C is an arbitrary constant. In our case we will have:

$$m(\mu) = e^{-A(\mu)} \left(m(M) - \int_M^\mu \frac{3}{32\pi^2} f_1(\mu')\, f_2(\mu')\, \sin\xi \cos\xi \cdot \right.$$
$$\left. \cdot \frac{M_R}{\mu'} \cdot I(\mu')\, e^{A(\mu')}\, d\mu' \right) \tag{4.8a}$$

where:

$$A(\mu) \equiv \int_M^\mu \left(\frac{2}{\pi} \alpha_s(\mu') + \frac{1}{10\pi} \alpha_1(\mu') \right) \frac{d\mu'}{\mu'} \tag{4.9}$$

is the standard term, so:

$$A(\mu) = -\frac{4}{11 - \frac{2}{3} f} \ln \frac{\alpha_s(\mu)}{\alpha_G} + \frac{3}{10 f} \ln \frac{\alpha_1(\mu)}{\alpha_G} \tag{4.10}$$

Considering that:

$$\int_{\Pi}^{\mu} g_1(\mu')g_2(\mu')\,I(\mu')\,e^{A(\mu')}\,\frac{d\mu'}{\mu'} = -\,g_1(\mu')g_2(\mu')\,H(\mu')\,e^{A(\mu')}\Big|_{\Pi}^{\mu} +$$

$$+\int_{\Pi}^{\mu} H(\mu')\,e^{A(\mu')}\left[g_1(\mu')g_2(\mu')\,\frac{dA(\mu')}{d\mu'} + \frac{d}{d\mu'}\left(g_1(\mu')g_2(\mu')\right)\right]d\mu'$$

(4.11a)We can write:

$$\int_{\Pi}^{\mu} g_1(\mu')g_2(\mu')\,I(\mu')\,e^{A(\mu')}\,\frac{d\mu'}{\mu'} = -\,g_1(\mu)g_2(\mu)\,H(\mu)\,e^{A(\mu)} +$$

$$+\,g_1(\Pi)g_2(\Pi)\,H(\Pi) + \int_{\Pi}^{\mu} H(\mu')\,e^{A(\mu')}\cdot O(g^4)\,d\mu'$$

(4.11b)

and since $H(\mu')$ is a regular function (typically $0 < H(\mu') < \ln\frac{\mu_1^2}{\mu_2^2}$), we can consider only the dominant terms and we will write:

$$m(\mu) = e^{-A(\mu)}\Big(m(\Pi) + \frac{3}{32\pi^2}\,g_1(\mu)g_2(\mu)\,H(\mu)\,e^{A(\mu)}\sin\xi\cos\xi\,\Pi_R -$$

$$-\,\frac{3}{32\pi^2}\,g_1(\Pi)g_2(\Pi)\sin\xi\cos\xi\,\Pi_R\cdot H(\Pi)\Big)$$

(4.8b)

or better:

$$m(\mu) - \frac{3}{32\pi^2}\,g_1(\mu)\,g_2(\mu)\sin\xi\cos\xi\,\Pi_R\cdot H(\mu) =$$

$$= e^{-A(\mu)}\cdot\Big(m(\Pi) - \frac{3}{32\pi^2}\,g_1(\Pi)\,g_2(\Pi)\sin\xi\cos\xi\,\Pi_R\cdot H(\Pi)\Big)$$

(4.8c)

To obtain a more simple formula for numerical estimations, we remember that H(M) $\simeq$ 0, and only at this point, we will use the following approximation:

$$\frac{g_1(\mu)\, g_2(\mu)}{4\pi} \simeq \frac{g_1(M)\, g_2(M)}{4\pi} = \alpha_G = \frac{8}{3}\alpha$$

At the end we obtain

$$m(\mu) = m(M)\cdot e^{-A(\mu)} + \frac{\alpha}{\pi} \sin\xi \cos\xi\, M_R \cdot H(\mu) \qquad (4.12)$$

Finally, we can write the complete RG equations for up-quarks in which the standard contribution in the Landau gauge (the first term[35]) is modified by the Majorana neutrino contribution (the second term):

$$m(\mu) = m(M)\cdot\left(\frac{\alpha_s(\mu)}{\alpha_G}\right)^{\frac{4}{11-\frac{2}{3}f}} \cdot \left(\frac{\alpha(\mu)}{\alpha_G}\right)^{-3/10f} +$$

$$+ \frac{\alpha}{\pi} \sin\xi \cos\xi\, M_R \cdot H(\mu) \qquad (4.13)$$

In the next section, we will see that the Majorana neutrino contribution is not negligible, at least for the first generation, and can be a strong support for very simple mass relations between up and down at the Grand Unification point.

4.1 Approximations concerning the Majorana contribution

Let us now consider an explicit form of the function $H(\mu)$ in the case $\mu^2/\mu_i^2,\ \mu^2/M_R^2 \ll 1$ which is the case of interest for our discussion; it is:

$$H(\mu) = \int_0^1 dx \ln \frac{(1-x)\frac{M_R^2}{\mu_1^2} + x(1-x)\frac{\mu^2}{\mu_1^2} + x}{(1-x)\frac{M_R^2}{\mu_2^2} + x(1-x)\frac{\mu^2}{\mu_2^2} + x} + \ln \frac{\mu_1^2}{\mu_2^2}$$

(4.14)

We can write:

$$H(\mu) = P_1(\mu) - P_2(\mu) + \ln \frac{\mu_1^2}{\mu_2^2} \tag{4.15}$$

where:

$$P_i(\mu) = \int_0^1 dx \ln\left((1-x)\frac{M_R^2}{\mu_i^2} + x + x(1-x)\frac{\mu^2}{\mu_i^2}\right) \tag{4.16}$$

and so:

$$P_i(\mu) \simeq \int_0^1 dx \ln\left((1-x)\frac{M_R^2}{\mu_i^2} + x\right) + \frac{\mu^2}{\mu_i^2}\int_0^1 dx \frac{x(1-x)}{(1-x)\frac{M_R^2}{\mu_i^2} + x} \tag{4.17}$$

it is now easy to extract

$$P_i(\mu) = \frac{1}{1 - \frac{M_R^2}{\mu_i^2}}\left(-1 - \frac{M_R^2}{\mu_i^2}\ln\frac{M_R^2}{\mu_i^2} + \frac{M_R^2}{\mu_i^2}\right) + O\left(\frac{\mu^2}{\mu_i^2}\right)$$

(4.18)

and since $M_R^2 \ll \mu_i^2$ (as seen previously) and $\mu^2 \ll M_R^2$, the leading term is:

$$P_i(\mu) \simeq -1 - \frac{M_R^2}{\mu_i^2} \ln \frac{M_R^2}{\mu_i^2} \tag{4.19}$$

and finally:

$$H(\mu) \simeq \frac{M_R^2}{\mu_2^2} \ln \frac{M_R^2}{\mu_2^2} - \frac{M_R^2}{\mu_1^2} \ln \frac{M_R^2}{\mu_1^2} + \ln \frac{\mu_1^2}{\mu_2^2} \tag{4.20}$$

Remembering equations (3.6) we can write:

$$\frac{\mu_1^2}{\mu_2^2} \simeq 1 + \varepsilon \tag{4.21}$$

where

$$\varepsilon = \frac{80}{32 + 9 \frac{|\phi_{126}(1)|^2}{\phi_{45}(1)^2}} \cdot \frac{\phi_{45}(24)}{\phi_{45}(1)} \ll 1 \tag{4.22}$$

in the case $\phi_{45}(24) \ll \phi_{45}(1)$, in which the dominant breaking is SO(10)$\rightarrow$ SU(5) x U(1)[36]. In this case, the leading term in H(μ) is:

$$H(\mu) \simeq \ln \frac{\mu_1^2}{\mu_2^2} + \varepsilon \frac{M_R^2}{\mu_2^2} \ln \frac{M_R^2}{\mu_2^2} \tag{4.23}$$

or more simple:

$$H(\mu) \simeq \ln \frac{\mu_1^2}{\mu_2^2} \tag{4.24}$$

because $M_R^2 \ll \mu_2^2$.

The (4.24) is general, in the sense that it is valid also if $\frac{\mu_1^2}{\mu_2^2} - 1 \sim 1$ as it is easily seen from (4.20). So we can use (4.24) without any particular restriction on the breaking pattern of SO(10).

However, in the case in which the following approximations are valid:

$$\frac{\mu^2}{\mu_1^2} \ll 1, \quad \frac{M_R^2}{\mu_1^2} \ll 1, \quad \frac{\mu^2}{M_R^2} \ll 1, \quad \frac{\mu_1^2 - \mu_2^2}{\mu_2^2} \ll 1 \tag{4.25}$$

we can write:

$$H(\mu) = \ln \frac{\mu_1^2}{\mu_2^2} \simeq \frac{\mu_1^2 - \mu_2^2}{\mu_2^2} \tag{4.26}$$

and the (4.13) becomes:

$$m(\mu) = m(M) \cdot \left(\frac{\alpha_s(\mu)}{\alpha_G}\right)^{\frac{4}{11 - \frac{2}{3}f}} \cdot \left(\frac{\alpha_1(\mu)}{\alpha_G}\right)^{-3/10f} + \\ + \frac{\alpha}{\pi} \sin\xi \cos\xi \, M_R \cdot \frac{\mu_1^2 - \mu_2^2}{\mu_2^2} \tag{4.27}$$

4.2 Numerical estimation

Let us call for simplicity δm the radiative contribution of Majorana neutrinos to the standard RG equations for up-quarks. So, by definition:

$$\delta m = \frac{\alpha}{\pi} M_R \frac{|\mu_{12}^2|}{\mu_2^2} \tag{4.28}$$

where we have used (3.3) and the fact that $\sin \xi \ll 1$. Remembering (3.2) and assuming that $g\phi_1 \sim \mu_2$ and $g\phi_w \sim M_w$ we can write:

$$\delta m \simeq c \frac{\alpha}{\pi} M_w . \frac{M_R}{\mu_2} \tag{4.29}$$

where M_w represents the typical scale of the mass of the W. If M_R is close enough to μ_2 and so to the Grand Unification point ($M_R \sim 10^{-1}\mu_2$), we will have that the typical scale for δm is $\delta m \sim 100$ MeV. So we can conclude that the effect of Majorana neutrinos in the standard RG equations is, in general, negligible, except for the up-quark of the first generation, in the case in which M_R is sufficiently close to the Grand Unification point. In this case the (2.51a) for the first generation must be modified in the following way:

$$\frac{m_u(\mu)}{m_d(\mu)} \simeq \frac{m_u(M)}{m_d(M)} \cdot \left(\frac{\alpha_1(\mu)}{\alpha_G}\right)^{-9/20f} \pm c \frac{\alpha}{\pi} \cdot \frac{M_w}{m_d(\mu)} \cdot \frac{M_R}{\mu_2} \tag{4.30}$$

with the sign plus in the case in which $\gamma = 0$ in the (3.2) (ϕ_1 and ϕ_w with the same sign), and the sign minus in the case in which $\gamma = \pi$ in the (3.2) (ϕ_1 and ϕ_w with opposite sign). A particular example which shows in practice the relevance of this radiative effect is given by the previously discussed relation:

$$\frac{u}{d} = \frac{c}{s} = \frac{t}{b} \tag{4.31}$$

in fact we can now assume that (4.31) is valid at the Grand Unification point without implying its validity also at low energy. In particular, at low energy, we will still have $c/s \simeq t/b$, which is in good agreement with the phenomenological situation, but it will not be true that $u/d \simeq c/s$ due to the presence of the term $\delta m \sim 100$ MeV. In particular, it is not excluded that the new RG equations could reproduce the correct mass of the up-quark at low energies.

Since the typical scale of δm is 100 MeV, it is possible that the effect of Majorana neutrino radiative corrections is to lower the big value of the up-quark mass at low energies derived from (4.31) and the standard RG equations, to the phenomenologically correct one.

4.3 Conclusions

In conclusion, we want to recall the essential points concerning radiative effects mediated by right-handed Majorana neutrinos:

(i) We have corrections of the same order of the typical masses of the first generation, and the radiative contribution for the second one is not completely negligible.

(ii) The radiative effect is essentially relevant only for up-quarks while it is negligible for the down-quarks and leptons, and so the GJ mass relations can be considered valid also in the presence of radiative effects due to the presence of heavy Majorana neutrinos. In particular, the $b = \tau$ remains still valid.

(iii) Another important aspect is that the 1-loop correction term for up-quarks is a finite term [26] and so it could be interesting to prospect the possibility of having a radiative generation of up-quark masses.

(iv) The radiative correction term for down-quark masses is completely different. The diagonalization of up and down mass matrices is thus modified by the radiative corrections and one expects in general the generation of non-trivial, but small, Cabibbo angles, depending on the Majorana mass matrix of right-handed neutrinos.[26]

However, the realization of point (iv) in a way compatible with the constraint of naturalness is not so easy It is then interesting to explore tree level models with the maximum number of phenomenological predictions. Once a satisfacotry model of this kind has been found, one can prospect the possibility that radiative effects could reduce the number of arbitrary parameters, and so the possibility to derive a more fundamental model explaining the structure of the fermion mass matrix in a deeper way.

5. A NATURAL SO(10) MODEL OF THE FERMION MASS MATRIX

We now want to explore the possibility of constructing a model which, at the tree level, shares the positive features of the GJ model and of the Fritzsch model.

As we have discussed, the GJ model [15] [16] gives very appealing mass relations in the down-lepton sector together with the Oakes relation for the Cabibbo angle, but it yields $\theta_2 = \theta_3 = \delta = 0$ predicting a stable b-quark and no CP violation phase in the KM matrix. A phenomenologically attractive ansatz about up and down mass matrices has been made by Fritzsch [17], yielding the mixing angles as functions of the masses. However, the construction of a natural SO(10) model of the Fritzsch type seems to meet non-trivial difficulties (in particular for the down-lepton mass relations). For a recent revival of the Fritzsch model, see the papers by the Harvard group [37].

In this chapter, we will suggest a natural SO(10) model [27] which reproduces the GJ mass relations, it predicts relations between the KM mixing angles and the masses, similarly to the Fritzsch mass matrix , without the difficulties of the Fritzsch model, and it involves a rather economical Higgs structure (only two complex 10's and one 126 representation).

5.1 The model and mass relations

The idea is to extend the GJ model in a natural way to produce non-zero mixings between the third family and the first two. More precisely, we consider the following Higgs fermion coupling matrix:

$$\begin{pmatrix} 0 & 10_1 & 0 \\ 10_1 & \overline{126} & \overline{10}_1 \\ 0 & \overline{10}_1 & 10_2 \end{pmatrix} \tag{5.1}$$

which can be naturally obtained by imposing, for example, the following discrete symmetry:

$$16_1 \to 16_1\, e^{i(4p-u)\frac{\pi}{4}} \qquad 16_2 \to 16_2\, e^{iu\frac{\pi}{4}} \qquad 16_3 \to 16_3\, e^{-i(4p+u)\frac{\pi}{4}}$$

$$10_1 \to 10_1\, e^{-i(4p)\frac{\pi}{4}} \qquad 10_2 \to 10_2\, e^{i2(4p+u)\frac{\pi}{4}}$$

$$\overline{126} \to \overline{126}\, e^{-i(2u)\frac{\pi}{4}} \qquad 16_H \to 16_H\, e^{i(2p)\frac{\pi}{4}} \tag{5.2}$$

where $16_i\ (i=1,2,3)$ are the fermion families, and 16_H is a scalar representation.

This allows the presence of the following terms: $(\overline{126})^4$, $(16_H)^4$, $16_H 16_H 10_1$ in the scalar potential. The 16 representation may allow to exploit the Witten mechanism (24) to produce hierarchically suppressed neutrino masses.

The success of the model for the charged fermion sector will be an encouragement to explore in a more detailed way the presence of possible predictions for the neutrino sector. For the moment, we will dedicate our attention to the charged fermion sector.

The corresponding mass matrices for down-quarks and leptons are:

$$M_d = m \cdot \begin{pmatrix} 0 & \lambda a e^{i\alpha} & 0 \\ \lambda a e^{i\alpha} & b e^{i\beta} & a e^{-i\alpha} \\ 0 & a e^{-i\alpha} & e^{i\gamma} \end{pmatrix} \tag{5.3a}$$

$$M_e = m \cdot \begin{pmatrix} 0 & \lambda a e^{i\alpha} & 0 \\ \lambda a e^{i\alpha} & -3b e^{i\beta} & a e^{-i\alpha} \\ 0 & a e^{-i\alpha} & e^{i\gamma} \end{pmatrix} \tag{5.3b}$$

with a,b, m > 0; we have chosen real Yukawa coupling constants to have a spontaneous CP breaking. M_u has the same form as that of M_d with different elements $(m \to m', a \to a', b \to b', \alpha \to \alpha', \beta \to \beta', \gamma \to \gamma')$:

$$M_u = m' \cdot \begin{pmatrix} 0 & \lambda a' e^{i\alpha'} & 0 \\ \lambda a' e^{i\alpha'} & b' e^{i\beta'} & a' e^{-i\alpha'} \\ 0 & a' e^{-i\alpha'} & e^{i\gamma'} \end{pmatrix} \qquad (5.3c)$$

We can write the mass term in the form:

$$\mathcal{L}_m = \cdots + \bar{d}_R M_d d_L + \cdots = \cdots + \bar{d}_R^{ph.} U_R^{d\dagger} M_d U_L^d d_L^{ph.} + \cdots \qquad (5.4)$$

and to search for a biunitary transformation such that:

$$\begin{cases} d_L = U_L^d d_L^{ph.} \\ d_R = U_R^d d_R^{ph.} \end{cases}, \quad U_R^{d\dagger} M_d U_L^d = \mathcal{M}_D^d \qquad (5.5)$$

where $\mathcal{M}_D^d$ is the diagonal and positive down mass matrix. To simplify the diagonalization we will consider that:

$$m A \equiv K_R^{d*} M_d K_L^d = m \cdot \begin{pmatrix} 0 & \lambda a & 0 \\ \lambda a & b & a \\ 0 & a & e^{i\varphi} \end{pmatrix} \qquad (5.6),$$

where $\varphi = \beta + 2\alpha + \gamma$ and:

$$K_R^d = \begin{pmatrix} 1 & & \\ & e^{i(\beta-\alpha)} & \\ & & e^{-2i\alpha} \end{pmatrix}, \quad K_L^d = \begin{pmatrix} e^{i(\beta-2\alpha)} & & \\ & e^{-i\alpha} & \\ & & e^{i\beta} \end{pmatrix} \tag{5.7}$$

The problem has been reduced to the diagonalization of A (it will be the same technique for up-quarks and leptons). The diagonalization becomes rather simple in the range of parameters in which λa^2 is very small ($\lambda a^2 \ll 1$). This will actually turn out to be the region of parameters corresponding to the experimental mass ratios: $m_b \gg m_s \gg m_d$. So, as previously done by Fritzsch [17], in particular we will have:

$$S^T A S = \begin{pmatrix} 0 & (1-\frac{1}{2}|\varepsilon|^2)\lambda a & \lambda a \varepsilon \\ (1-\frac{1}{2}|\varepsilon|^2)\lambda a & b - a\varepsilon^* & 0 \\ \lambda a \varepsilon & 0 & e^{i\varphi} + a\varepsilon \end{pmatrix} \tag{5.8}$$

where $\varepsilon = a \dfrac{b + e^{i\varphi}}{1-b^2}$, and:

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\frac{1}{2}|\varepsilon|^2 & \varepsilon \\ 0 & -\varepsilon^* & 1-\frac{1}{2}|\varepsilon|^2 \end{pmatrix}, \quad S^+ S = S S^+ = 1 \tag{5.9}$$

Ignoring terms of order λa^2 we can write:

$$S^T A S = \begin{pmatrix} 0 & \lambda a & 0 \\ \lambda a & b e^{i\varphi} & 0 \\ 0 & 0 & e^{i\varphi} \end{pmatrix} \tag{5.10}$$

which can be easily reduced to the diagonal form:

$$T^T \begin{pmatrix} 1 & & \\ & e^{-i\varphi} & \\ & & 1 \end{pmatrix} S^T A S \begin{pmatrix} e^{i\varphi} & & \\ & 1 & \\ & & 1 \end{pmatrix} T = \begin{pmatrix} -\lambda^2 e^2/b & & \\ & b + \frac{\lambda^2 e^2}{b} & \\ & & e^{i\varphi} \end{pmatrix} \quad (5.11),$$

where:

$$T = \begin{pmatrix} \cos w_1 & \sin w_1 & 0 \\ -\sin w_1 & \cos w_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \sin w_1 \simeq \frac{\lambda e}{b} \quad (5.12)$$

Finally, with the biunitary transformation:

$$\begin{cases} U_L^d = K_L^d S \begin{pmatrix} e^{i\varphi} & & \\ & 1 & \\ & & 1 \end{pmatrix} T \\ U_R^d = K_R^d S^* \begin{pmatrix} 1 & & \\ & e^{i\varphi} & \\ & & 1 \end{pmatrix} T^* \begin{pmatrix} -1 & & \\ & 1 & \\ & & e^{i\varphi} \end{pmatrix} \end{cases} \quad (5.13)$$

we obtain the down-quark masses:

$$U_R^{d\dagger} \mathcal{M}_d U_L^d = m_D^d = m \cdot \begin{pmatrix} \lambda^2 e^2/b & & \\ & b + \frac{\lambda^2 e^2}{b} & \\ & & 1 \end{pmatrix} \quad (5.14)$$

This yields the GJ mass relations:

$$3m_e = m_d, \quad m_\mu - m_e = 3(m_s - m_d), \quad m_b = m_\tau \quad (5.15)$$

at the Grand Unification point.

5.2 The model and mixing angles

Since ε is a small number, we can write:

$$\varepsilon = e \, \frac{b + e^{i\varphi}}{1-b^2} \equiv |\varepsilon| e^{i\nu} \equiv \sin w_2 \cdot e^{i\nu} \tag{5.16}$$

then:

$$U_L^d = K_L^d \cdot \begin{pmatrix} c_1 e^{i\varphi} & s_1 e^{i\varphi} & 0 \\ -c_2 s_1 & c_2 c_1 & s_2 e^{i\nu} \\ s_1 s_2 e^{-i\nu} & -c_1 s_2 e^{-i\nu} & c_2 \end{pmatrix} \tag{5.17}$$

where $s_i \equiv \sin w_i$, $i = 1,2$.

The KM mixing matrix U_c is obtained from:

$$U_c = U_L^{u\dagger} U_L^d \tag{5.18}$$

To simplify the calculation we observe that it is possible to write:

$$U_c = J^{u*} V_L^{u\dagger} K \, V_L^d J^d \tag{5.19}$$

where:

$$J^d = \begin{pmatrix} 1 & & \\ & 1 & \\ & & e^{i\nu} \end{pmatrix}, \quad I^d = \begin{pmatrix} e^{-i\varphi} & & \\ & 1 & \\ & & e^{i\nu} \end{pmatrix}, \quad K = I^{u} \cdot K_L^{u*} \cdot K_L^d \cdot I^{d*}$$

$$V_L^d = \begin{pmatrix} c_1 & s_1 & 0 \\ -c_2 s_1 & c_2 c_1 & s_2 \\ s_1 s_2 & -c_1 s_2 & c_2 \end{pmatrix} \tag{5.20}$$

and similarly for the up-quarks. Ignoring the redefinition of quark fields we have:

$$U_c = V_L^{u\dagger} K \cdot V_L^{d} \tag{5.21}$$

where:

$$K = \begin{pmatrix} e^{i(\eta-2\theta+\rho)} & & \\ & e^{-i\theta} & \\ & & e^{i(\eta-\chi)} \end{pmatrix} \tag{5.22}$$

with: $\eta=\beta-\beta'$, $\chi=\nu-\nu'$, $\theta=\alpha-\alpha'$, $\rho=\gamma-\gamma'$.

If we use the new definitions:

$$\theta-\eta-\rho\equiv a\ , \quad 2\theta-\rho-\chi\equiv\tau, \quad \omega_i\equiv\nu_i\ , \quad \omega_i'\equiv\mu_i\ ,$$

we obtain U_c in the following form:

$$U_c = \begin{pmatrix} 1-\frac{1}{2}\mu_1^2-\frac{1}{2}\nu_1^2+\mu_1\nu_1 e^{ia} & -\nu_1+\mu_1 e^{ia} & \mu_1(\nu_2 e^{ia}-\mu_2 e^{i\tau}) \\ -\mu_1+\nu_1 e^{ia} & \begin{matrix}\mu_1\nu_1+e^{ia}(1-\frac{1}{2}\mu_1^2-\frac{1}{2}\nu_1^2- \\ -\frac{1}{2}\mu_2^2-\frac{1}{2}\nu_2^2)+e^{i\tau}\mu_2\nu_2\end{matrix} & \nu_2 e^{ia}-\mu_2 e^{i\tau} \\ \nu_1(\mu_2 e^{ia}-\nu_2 e^{i\tau}) & \mu_2 e^{ia}-\nu_2 e^{i\tau} & \mu_2\nu_2 e^{ia}+e^{i\tau}(1-\frac{1}{2}\mu_2^2-\frac{1}{2}\nu_2^2) \end{pmatrix} \tag{5.23}$$

which is of the same form of the Fritzsch's mixing matrix, but now μ_2, ν_2 are different functions of the masses.[17]

Redefining the quark fields in the following way:

$$U_c \to \begin{pmatrix} 1 & & \\ & e^{i(\zeta_1 - a + \xi_2 + \xi_1)} & \\ & & e^{i(\zeta_2 + \xi_2 - \gamma)} \end{pmatrix} \cdot U_c \cdot \begin{pmatrix} e^{-i\xi_2} & & \\ & e^{-i\xi_1} & \\ & & e^{-ix} \end{pmatrix} \tag{5.24}$$

with:

$$e^{i\zeta_1} = \frac{-\nu_1 + \mu_1 e^{ia}}{|-\nu_1 + \mu_1 e^{ia}|}\,, \qquad s_1 = |-\nu_1 + \mu_1 e^{ia}|$$

$$e^{ix} = \frac{\nu_2 e^{ia} - \mu_2 e^{i\tau}}{|\nu_2 e^{ia} - \mu_2 e^{i\tau}|}$$

$$e^{i\gamma} = \frac{\mu_2 e^{ia} - \nu_2 e^{i\tau}}{|\mu_2 e^{ia} - \nu_2 e^{i\tau}|}\,, \qquad r = |\mu_2 e^{ia} - \nu_2 e^{i\tau}|$$

$$e^{i\zeta_2} \simeq 1 + i\mu_1\nu_1 \sin a$$

$$e^{i\xi_1} \simeq 1 + i\left(\mu_2\nu_2 \sin(a-\tau) - \mu_1\nu_1 \frac{r^2}{s_1^2} \sin a\right)$$

$$e^{i\xi_2} \simeq 1 - i\left(\mu_1\nu_1 \sin a + \mu_2\nu_2 \sin(a-\tau) - \mu_1\nu_1 \frac{r^2}{s_1^2} \sin a\right) \tag{5.25}$$

we obtain U_c in the KM form:

$$U_c = \begin{pmatrix} 1 - \frac{\theta_1^2}{2} & \theta_1 & \theta_1\theta_3 \\ -\theta_1 & 1 - \frac{\theta_1^2}{2} - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2} - \theta_2\theta_3 e^{i\delta} & \theta_3 + \theta_2 e^{i\delta} \\ \theta_1\theta_2 & -\theta_2 - \theta_3 e^{i\delta} & -\theta_2\theta_3 + \left(1 - \frac{\theta_2^2}{2} - \frac{\theta_3^2}{2}\right) e^{i\delta} \end{pmatrix} \tag{5.26}$$

with:

$$\delta = \pi - \alpha \quad , \quad \theta_1 = |-\nu_1 + \mu_1 e^{i\alpha}|$$

$$\theta_1\theta_3 = \mu_1 |\nu_2 e^{i\alpha} - \mu_2 e^{i\tau}| \quad , \quad \theta_1\theta_2 = \nu_1 |\mu_2 e^{i\alpha} - \nu_2 e^{i\tau}|$$

(5.27)

where we have kept only terms up to second order, and:

$$\mu_1 = \left[\frac{m_u}{m_c - m_u}\right]^{1/2} \quad , \quad \nu_1 = \left[\frac{m_d}{m_s - m_d}\right]^{1/2}$$

$$\mu_2 = \lambda^{-1} \frac{[m_u(m_c - m_u)]^{1/2}}{m_t} \cdot \left|\frac{m_c - m_u}{m_t} + e^{i\varphi'}\right|$$

$$\nu_2 = \lambda^{-1} \frac{[m_d(m_s - m_d)]^{1/2}}{m_b} \cdot \left|\frac{m_s - m_d}{m_b} + e^{i\varphi}\right|$$

(5.28)

5.3 Predictions about the fermion mass matrix

We have just seen that our model has non-trivial predictions for the masses of down-quarks and leptons (GJ relations):

$$3m_e = m_d \; , \quad m_\mu - m_e = 3(m_s - m_d) \; , \quad m_b = m_\tau$$

(5.29)

at the Grand Unification point.

We also have a non-trivial mixing matrix with a non-zero CP violating phase

$$\delta = \pi - \alpha = \pi - \alpha + \alpha' + \beta - \beta' + \gamma - \gamma' \tag{5.30}$$

and non-zero mixing angles $\theta_1, \theta_2, \theta_3$. In particular, we predict the following relations:

$$\theta_1 = \left| \left[\frac{m_d}{m_s - m_d} \right]^{1/2} + \left[\frac{m_u}{m_c - m_u} \right]^{1/2} \cdot e^{i\delta} \right| \tag{5.31a}$$

$$\theta_3 = \left[\frac{m_u (m_s - m_d)}{m_d (m_c - m_u)} \right]^{1/2} \cdot \theta_2 \tag{5.31b}$$

which are the same relations predicted by Fritzsch.

From a phenomenological point of view, the above relation implies:

$$\frac{|V_{ub}|^2}{|V_{cb}|^2} = \frac{m_u}{m_c - m_u} = (4 \pm 1) \times 10^{-3} \tag{5.32}$$

and so:

$$R \equiv \frac{\Gamma(b \to u e \bar{\nu})}{\Gamma(b \to c e \bar{\nu})} \simeq (1.9 \pm 0.6) \times 10^{-3} \tag{5.33}$$

compatible with the experimental bound $R < 4 \times 10^{-2}$.

Depending on the magnitude of δ, the Cabibbo angle varies between 9.5° and 17°; the experimental value is well reproduced by $\delta = \pm\pi/2$, in this case:

$$\theta_1 = 13.8^\circ \pm 4.1^\circ \tag{5.34}$$

while $\theta_{1\,exp} = 13.2^\circ \pm 0.4^\circ$. The choice $\delta = \pm\pi/2$ was not free of problems in the Fritzsch model, in contrast with the present case.(17)

From the analysis for B lifetime we can derive:(38)

$$\frac{1}{\tau_B} = \frac{1}{28.2}\, m_b^5\, \gamma_1 (s_2^2 + s_3^2 + 2 s_2 s_3 c_\delta) \tag{5.35}$$

where m_b and τ_B are in GeV and psec respectively, $\gamma_1 = 3.2$ is the phase space factor with QCD corrections. For $\tau_B = 0.7$ psec, we have:

$$s_2 + s_3 c_\delta \leq (s_2^2 + s_3^2 + 2 s_2 s_3 c_\delta)^{1/2} = 0.078 \tag{5.36}$$

and thus $s_2 \lesssim 0.078$. Using B = 1 we obtain:

$$\frac{\varepsilon'}{\varepsilon} \gtrsim 10^{-3} \tag{5.37}$$

In the case $\delta = \pm\pi/2$, we will have a saturation of the previous bound, in particular:

$$\theta_2 \simeq 0.078 \tag{5.38}$$

Since in our model (27)

$$\theta_2 \simeq \lambda^{-1} \left| \frac{\sqrt{m_u m_c}}{m_t} e^{i\alpha} - \frac{\sqrt{m_d m_s}}{m_b} e^{i\tau} \right| \simeq$$

$$\simeq \lambda^{-1} \frac{\sqrt{m_d m_s}}{m_b} \simeq 0.007 \cdot \lambda^{-1} \tag{5.39}$$

the good value of (5.38) is reproduced if $\lambda \simeq$ 1/10, in this case we will have $\varepsilon'/\varepsilon \simeq 10^{-3}$ in agreement with the present experimental situation.

Finally, the above determination of the mixing angles yields a determination of the ε parameter in agreement with the experimental data. In fact, for $m_t \simeq 30$ GeV, $s_2 \simeq 0.07$, $m_c \simeq 1.4$ GeV, we get:[39]

$$|\varepsilon| \simeq 1.95 \times 10^{-3} \cdot B \tag{5.40}$$

and so for B = 1, we will have a good agreement with the experimental value $|\varepsilon_{exp}| \simeq 2.3 \times 10^{-3}$.

5.4 Conclusions

In conclusion, we have constructed a natural SO(10) model whose fundamental aspects are the following:

(i) Fermion mass predictions of the GJ type:

$$3m_e = m_d \qquad m_\mu - m_e = 3(m_s - m_d) \qquad m_b = m_\tau$$

(ii) Non-trivial CP violating phase δ and non-trivial mixing angles θ_1, θ_2, θ_3.

(iii) θ_3 is predicted to be very small throughout the relation:

$$\theta_3 = \left[\frac{m_u(m_s - m_d)}{m_d(m_c - m_u)}\right]^{1/2} \cdot \theta_2$$

(iv) If $\delta = \pm\pi/2$ we predict

$$\theta_1 = 13.8 \pm 4.1^\circ$$

in good agreement with the phenomenology.

(v) If we choose the free paramenter λ to be: $\lambda \simeq 1/10$, we have $\theta_2 \simeq 0.07$, and so the calculation of ε'/ε and ε in the model is in good agreement with the phenomenology; in fact, we will have:

$$\frac{\varepsilon'}{\varepsilon} \simeq 10^{-3} \quad , \quad |\varepsilon| \simeq 2 \times 10^{-3}$$

To conclude, it is interesting to speculate about the possibility that the value $\lambda \simeq 1/10$ could be explained by a two-loop radiative mechanism in a more fundamental model. This can be an important possibility to speculate about and, in fact, we have previously verified the relevance of other radiative mechanisms to understand the structure of the fermion mass matrix, such as the Witten mechanism or the radiative corrections to fermion masses mediated by the exchange of heavy Majorana neutrinos. However, this possibility cannot be supported at present by explicit arguments and it needs a more detailed analysis of the Yukawa couplings and of the fermion mass matrix.

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