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FIBRE BUNDLE DESCRIPTION OF GAUGE AND SPINOR FIELDS AND OF THE KALUZA-KLEIN UNIFICATION

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Introduction

The idea of "gauge" symmetry is fundamental in physics. Electromagnetism is a gauge theory with U(1) as "gauge group" [1]. The idea was generalized by Yang and Mills [2] who introduced gauge fields corresponding to the spin isotopic group SU(2) and successively by Utiyama [3] who considered an arbitrary Lie group. The same idea, in conjunction with that of spontaneous symmetry breaking and in the context of quantum field theory, is at the basis of the Weinberg-Salam unification of electromagnetism with weak interactions and of quantum chromodynamics [4].

At the same time it has became more and more apparent that the appropriate language for describing classical gauge theories is that of principal fibre bundles with connections [5-12]. This framework is essential to the analysis of global properties.

The description of topologically non trivial gauge configurations such as the magnetic pole solution of Maxwell's equations [13] or the Belavin-Polyakov-Schwartz-Tyupkin solution of the sourceless Yang-Mills equations [14] is an example of the importance of fibre bundle techniques in physics.

Einstein's theory of gravitation may be considered as a gauge theory as well [3,15]. The principal bundle that matters here is that of linear frames over space-time M . However , there are some differences between gravitation and gauge theories [16,17] . First of all , the bundle of frames is "soldered" to its base M while for gauge theories the bundle is weakly connected to M . Moreover , the Einstein Lagrangian for gravitation is linear in the curvature while the Lagrangian for gauge theories is quadratic .

One may try to unify gravity with gauge fields by exploiting their similarities . An alternative approach consists in deriving the gauge fields from general relativity in more than four dimensions . This idea has its roots in the unified description of gravity with electromagnetism , in the framework of general relativity in five dimensions , proposed by Kaluza [18] and Klein [19] .

The Kaluza-Klein unification can be formulated in the contex of fibre bundles . Given a principal bundle P over the space-time M , with a connection ω describing gauge fields , and a Riemannian metric g on M which gives the gravitational field , a Riemannian metric can be naturally defined on P . Then , the Einstein Lagrangian for P decomposes in the Einstein Lagrangian for M plus the Lagrangian for the gauge fields and the curvature for the gauge group .

Fibre bundle techniques may be very useful for the quantization of gauge theories too and may provide interesting insights into the quantization procedures .

The most suitable approach to quantization of gauge fields is by means of Feynman path-integral [4] . At first sight one may think of integrating over the functional space $\mathcal{C}$ of all connections defined on a principal bundle P whose base is the space-time M . However , owing to the gauge freedom , such an integral is overdetermined . In fact , one should integrate over the set of all families of gauge related connections . If $\mathrm{Aut}_V P$ is the group of gauge transformations (see paragraph 2.5) , this set is the space $\mathcal{C} / \mathrm{Aut}_V P$ of the orbits of the action of $\mathrm{Aut}_V P$ on $\mathcal{C}$. Nevertheless , there are difficulties even in the definition of $\mathcal{C} / \mathrm{Aut}_V P$ [20,10] .

A possible way out could be to choose one connection on each orbit (gauge fixing) and to introduce a weight factor in the integral . Such a choice is equivalent to the construction of a mapping $s : \mathcal{C} / \mathrm{Aut}_V P \rightarrow \mathcal{C}$ so that $\pi \circ s = id$, where $\pi : \mathcal{C} \longrightarrow \mathcal{C} / \mathrm{Aut}_V P$ is the canonical projection . In general the map s is defined only locally and it is not possible to prolong it to a global one . The fact the no continuous choice of exactly one connection in each orbit can be made is known as Gribov ambiguity [20] .

This thesis is organized as follows . Section 1. containes the geometrical preliminaries . In particular the notions of principal bundle and of associated bundle are introduced . In section 2. the concepts of connection and covariant derivative are analysed and it is shown how gauge theories may be formulated in the framework of principal fibre bundles with connection . Section 3. deals with geometrical aspects of Kaluza-Klein type theories . Both the cases in which the internal space is a group manifold and , the more general one in which it is a homogeneous space are considered . In section 4. gauge theories over Stiefel bundles are studied and the importance of them as universal bundles is stressed . It is shown how Hopf bundles , which are particular cases of Stiefel bundles , describe topologically non trivial gauge configurations . Finally , in section 5. the notion of spinor structure is introduced . Particular cases of spinor structures over oriented spheres are shown to describe relevant physical systems . In particular , we prove that the solution of Yang-Mills equations in eight dimensions recently found by Grossman et al. [21] is a spinor connection over the eight dimensional sphere .

We shall not discuss the problem of quantization . This is a good problem for future investigations .

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1. Geometrical framework

1.1 Principal fibre bundles [23]

Let P, M be manifolds and G a Lie group. We say that $P = P(M, G, \pi)$ is a <u>principal fibre bundle</u> over the <u>base space</u> M with <u>structure group</u> G if:

1. there exists a smooth projection $\pi : P \longrightarrow M$,

2. G acts freely on P on the right

$$P \times G \ni (p, a) \longrightarrow R_a(p) = pa$$

with $R_{ab} = R_b \circ R_a$ for any $a, b \in G$; $R_e(p) = p$ for any $p \in P$ where e is the unit element of G; and $R_a(p) = p$ for some $p \in P$ only if $a = e$ (free action);

3. the equivalence relation induced by the action of G is the same as that induced by π, i.e.

$$\pi(pa) = \pi(p) \qquad \text{for any } p \in P \text{ and any } a \in G;$$

4. P is locally trivial, i.e. every $x \in M$ has a neighbourhood $U \subset M$ such that $\pi^{-1}(U)$ is isomorphic to $U \times G$ in the sense that there exists a diffeomorphism

$$\psi : \pi^{-1}(U) \longrightarrow U \times G, \quad p \longrightarrow \psi(p) = (\pi(p), \varphi(p))$$

with φ a mapping from $\pi^{-1}(U)$ into G such that

$$\varphi(pa) = (\varphi(p))a \quad \text{for any } p \in \pi^{-1}(U) \text{ and any } a \in G.$$

In words : G acts freely and transitively on the <u>fibre</u> $\pi^{-1}(x)$ attached to $x \in M$ and any fibre is isomorphic to G itself .

Transitive action of G on the fibres means that given any two points p' and p in the same fibre there exists an element $a \in G$ such that $p' = pa$.

The space P is the <u>total space</u> of the principal bundle .

The principal bundle is <u>trivial</u> if the diffeomorphism of the point 4. is globally defined i.e. $\psi : P \longrightarrow M \times G$.

1.2 Sections

A section of the principal bundle $P(M, G, \pi)$ is a smooth map $s : M \longrightarrow P$ such that $\pi \circ s = id_M$. The existence of local sections is equivalent to the local trivialization of the bundle . Given the local trivialization of P described in the point 3. , a set of local sections $s_u : U \rightarrow \pi^{-1}(U)$, $U \subset M$, is defined by

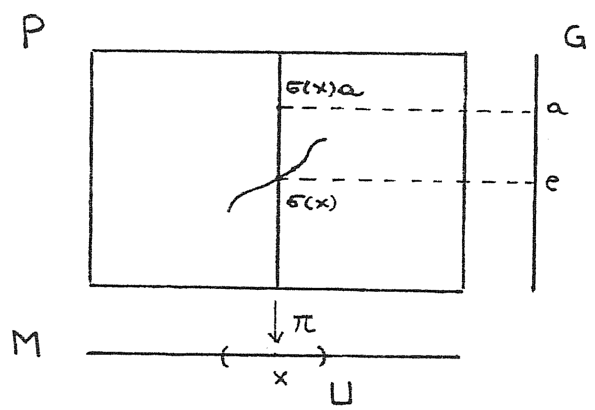
$$s_u(x) = p\,(\varphi(p))^{-1} \quad , \quad p \in \pi^{-1}(x)$$

and this expression is clearly independent of the point p in the fibre . On the other hand , given the local sections $s_u : U \rightarrow \pi^{-1}(U)$, $U \subset M$, for any $p \in \pi^{-1}(x)$, $x \in U$, there is a unique $a \in G$ such that $(s(x))a = p$; the maps

$$\pi^{-1}(U) \ni p \longrightarrow (\pi(p), a) \in U \times G$$

give a local trivialization of the principal bundle P .

A local section $s : U \longrightarrow P$ provides an identification of the identity of G with the submanifold of $\pi^{-1}(U)$ corresponding to U via s .



It is now obvious that P is trivial if and only if there exists a global section $s : M \longrightarrow P$.

1.3 Transition functions

Let $\{U_\alpha\}$ be an open covering of the base space M .

Consider the sections

$$s_\alpha : U_\alpha \rightarrow P \; ; \qquad s_\alpha(x) = p\,(\varphi_\alpha(p))^{-1}\,,\; p \in \pi^{-1}(x) \tag{1.1}$$

where φ_α is the mapping from $\pi^{-1}(U_\alpha)$ to G which describes the local trivialization of the principal bundle P . Since $\varphi_\alpha(pa) = (\varphi_\alpha(p))a$ for any $p \in \pi^{-1}(U_\alpha)$ and any $a \in G$, the sections (1.1) do not depend on the point p . If $x \in U_\alpha \cap U_\beta$ then

$$s_\beta(x) = p\,(\varphi_\beta(p))^{-1} = s_\alpha(x)\;\varphi_\alpha(p)\,(\varphi_\beta(p))^{-1} \tag{1.2}$$

and the quantity $\varphi_\alpha(p)\,(\varphi_\beta(p))^{-1}$ depends only on $\pi(p)$, not on p;

it defines a mapping

$$\psi_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow G \,, \quad \psi_{\alpha\beta}(\pi(p)) = \varphi_\alpha(p)\,(\varphi_\beta(p))^{-1} .$$

The mappings $\psi_{\alpha\beta}$ are the <u>transition functions</u> of the bundle P corresponding to the covering $\{U_\alpha\}$ of its base space M. They are such that [23,24]

$$\psi_{\alpha\gamma}(x) = \psi_{\alpha\beta}(x)\,\psi_{\beta\gamma}(x) \qquad \text{for any} \quad x \in U_\alpha \cap U_\beta \cap U_\gamma \quad . \tag{1.3}$$

An open covering $\{U_\alpha\}$ of M, toghether with a family of mappings $\psi_{\alpha\beta} : U_\alpha \cap U_\beta \quad G$ which have the property (1.3), uniquely characterize the principal bundle $P(M, G, \pi)$ [23,24].

1.4 Associated bundle

Let $P = P(M, G, \pi)$ be a principal fibre bundle. Suppose G acts on the left in a manifold V (in particular a vector space)

$$\rho : G \times V \longrightarrow V \quad , \quad (a,v) \longrightarrow \rho_a(v) \doteq \rho(a,v) \quad .$$

On the product $P \times V$ there is a right action of G defined by

$$(p,v)a = (pa, \rho_{a^{-1}}(v)) \quad .$$

If E is the quotient of $P \times V$ by this action, then

1. E may be given a differentiable structure;
2. there is a smooth projection

$$\pi_E : E \longrightarrow M \,, \quad \pi_E(\text{equivalence class of } (p,v)) = \pi(p) \quad ;$$

3. E is locally trivial, i.e. every $x \in M$ has a neighbourood $U \subset M$ such that $\pi_E^{-1}(U)$ is diffeomorphic to the product $U \times V$.

The bundle $E = E(M, V, G, P)$ is the bundle associated with P by the action ρ. Every fibre $\pi_E^{-1}(x)$, $x \in M$, is diffeomorphic to V.

The notion of associated bundles is very important in physics: they describe generalized matter fields. A matter field of type (ρ, V) is a section of a bundle E of typical fibre V associated to some principal bundle P by the action ρ.

By definition a section of E is a mapping $s : M \to P$ such that $\pi_E \circ s = id_M$. There is a bijective correspondence between sections of E and equivariant functions φ from P to V, that is functions $\varphi : P \to V$ such that $\varphi(pa) = \rho_{a^{-1}} \circ \varphi(p)$ for any $p \in P$ and any $a \in G$ [25]. One says that the field φ at the point $\pi(p) \in M$, $p \in P$, has components $\varphi(p)$ with respect to the generalized frame $p \in P$. By changing frame $p \to pa$, the components trasform according to the rappresentation ρ.

1.5 Morphisms of principal bundles

Let $P_i = P_i(M_i, G_i, \pi_i)$ be two principal fibre bundles and let $R_i : P_i \times G_i \to P_i$ denotes the right action of the group G_i on P_i.

A <u>morphism</u> [25,26] of P_1 and P_2 is a triple (h, k, f)

of mappings $h : P_1 \to P_2$, $k : G_1 \to G_2$, $f : M_1 \to M_2$, with k a Lie groups homomorphism and such that the following diagram

$$\begin{array}{ccc} P_1 \times G_1 & \xrightarrow{h \times k} & P_2 \times G_2 \\ \downarrow R_1 & & \downarrow R_2 \\ P_1 & \xrightarrow{h} & P_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

commutes . If one denotes $R_i(p_i, a_i) = p_i a_i$, then h maps each fibre of P_1 into a fibre of P_2 with $h(pa) = h(p)k(a)$ for any $p \in P_1$ and any $a \in G_1$.

Two principal bundles P_1 and P_2 having the same base space M and the same structure group G are isomorphic if and only if there exists a mapping $h : P_1 \to P_2$ such that (h, id_G, id_M) is a morphism of P_1 and P_2 . The map h is then forced to be an isomorphism and the bundles P_1 and P_2 may be identified [26] . The set of equivalence classes of isomorphic principal G-bundles over M is denoted by $\mathcal{B}_G(M)$.

Let (h, k, id_M) be a morphism of the principal bundles $P_1 = P_1(M, G_1, \pi_1)$ and $P_2 = P_2(M, G_2, \pi_2)$.

If both h and k (and their derived mappings) are injective, then P_1 is a restriction of P_2 or P_2 is an extension of P_1 [25].

If both h and k (and their derived mappings) are surjective, then P_1 is a prolongation of P_2 or P_2 is a reduction of P_1 [25].

1.6 Induced principal bundle . Universal bundles

Let $P = P(M, G, \pi)$ be a principal fibre bundle .

For any mapping f from a manifold M' into M it is possible to construct a principal bundle $P' = f^* P$ over M' with structure group G . The bundle $f^* P$ is the bundle induced from P by f [23,26] .

The total space P' is defined as

$$P' = \{ (y,p) \in M' \times P \mid f(y) = \pi(p) \}$$

and G acts freely on P' by

$$(y,p) \longrightarrow (y,p)a = (y,pa) \quad \text{for} \quad (y,p) \in P' \text{ and } a \in G .$$

The projection $\pi' : P' \longrightarrow M'$ is given by $\pi'(y,p) = y$.

If $h : P' \longrightarrow P$ is defined by $h(y,p) = p$, then the triple (h, id_G, f) is a morphism of P' and P .

Suppose f and g are two mappings from M' into M which are homotopic . Then the principal G-bundles f^*P and g^*P are isomorphic bundles over M' [26] .

A principal bundle $P = P(M, G, \pi)$ is n-universal for the group G if and only if the following conditions are true [26] :

1. for each principal G-bundle $P' = P'(M', G, \pi')$ over M', with $\dim M' \leqslant n$, there exists a mapping $f : M' \to M$ such that P' and f^*P are isomorphic,

2. if $f, g : M' \to M$ are any two mappings such that f^*P and g^*P are isomorphic, then f and g are homotopic.

Thus, these conditions mean that there is a bijective correspondence between the set $\mathcal{B}_G(M')$ of equivalence classes of isomorphic G-bundles over M' and the set $[M', M]$ of homotopy classes of mappings $f : M' \to M$.

The following theorem is essential to recognise the universality of a principal bundle [24,26].

<u>Theorem 1.1</u> A principal bundle $P = P(M, G, \pi)$ is n-universal if and only if $\pi_i(P) = 0$ for $0 \leqslant i \leqslant n$. Here $\pi_i(P)$ is the i-th homotopy group of P.

2.5. Fundamental vector fields

Let $P = P(M, G, \pi)$ be a principal fibre bundle.

To each element ξ of the Lie algebra $\mathcal{G}$ of G we may linearly assign a vector field ξ^* on P called the <u>fundamental vector field</u> corresponding to ξ ; its value at the point $p \in P$ is defined by

$$\xi^*(p) = \frac{d}{dt} R_{\exp t\xi}(p)\Big|_{t=0}. \tag{2.1}$$

Since the action of G sends any fibre into itself, the vector $\xi^*(p)$ at each point p is tangent to the fibre through p. Moreover, as G acts freely on P, $\xi^*(p)$ never vanishes (unless $\xi = 0$). For any $p \in P$, the mapping $\xi \longrightarrow \xi^*(p)$ is an isomorphism of $\mathcal{G}$ with the subspace $V_p \subset T_p P$ of vectors which are tangent to the fibre through p.
The vectors of V_p are the vertical vectors at p. Any $v \in V_p$ is characterized by the condition $T_p\pi(v) = 0$ where $T_p\pi$ is the derived map of π at p.

2.2 Connection on a principal fibre bundle

Let $P = P(M, G, \pi)$ be a principal fibre bundle.

A <u>connection</u> on P is a smooth distribution of vector spaces

$$P \ni p \longrightarrow H_p \subset T_p P$$

with the properties

1. $T_p\pi : H_p \to T_{\pi(p)}M$ is an isomorphism for any $p \in P$,

2. $T_pR_a(H_p) = H_{pa}$.

The subspace H_p is the horizontal subspace at p .

Since $V_p \cap H_p = \{\underline{0}\}$, any vector Y of T_pP may be uniquely decomposed as a sum

$$Y = hY + vY \quad , \quad hY \in H_p \quad , \quad vY \in V_p \quad .$$

Given the vector $X \in T_xM$, its horizontal lift at $p \in \pi^{-1}(x)$ is the unique (by 1.) vector $\hat{X} \in H_p$ such that $T_p\pi\ (\hat{X}) = X$.

Given the curve $u : [0,1] \to M$, its lift to P , $\hat{u} : [0,1] \to P$, through the point $p \in \pi^{-1}(u(0))$, is defined by :

a. $\hat{u}(0) = p$

b. $\pi \circ \hat{u} = u$

c $\hat{u}$ is horizontal , i.e. , its tangent vector at $u(t)$ belongs to $H_{\hat{u}(t)}$ for any $t \in [0,1]$.

One says that $\hat{u}(1)$ is obtained from $p = \hat{u}(0)$ by parallel trasport along the curve u .

As we saw in the paragraph (2.1) , the Lie algebra $\mathcal{G}$ is isomorphic to the vertical subspace V_p at any $p \in P$. Using this isomorphism it is possible to characterize any connection on P by a $\mathcal{G}$ -value 1-form ω on P called the connection 1-form and defined by

$$T_pP \ni X \longrightarrow \omega_p(X) = \text{the unique element in } \mathcal{G} \text{ that corresponds to } vX .$$

The properties of ω are :

3. $X \in H_p \iff \omega(X) = 0$,

4. $\omega(\xi^*) = \xi$ for any $\xi \in \mathcal{G}$,

5. $(R_a)^* \omega \doteq \omega \circ T R_a = Ad_{a^{-1}} \circ \omega$ for any $a \in G$.

Here Ad is the adjoint representation of the group G in its Lie algebra $\mathcal{G}$. Conversely , any $\mathcal{G}$ -valued 1-form on P with the properties 3. , 4. and 5. determines a unique connection on P by defining

$$H_p = \ker \omega_p = \{ X \in T_pP \mid \omega_p(X) = 0 \} .$$

2.3 Exterior covariant derivative . Curvature form .

Let $P = P(M , G , \pi)$ be a principal fibre bundle and ω a connection 1-form on it . Let ρ be a representation of G in the vector space V .

A <u>pseudotensorial k-form of type ρ</u> is a V-valued k-form α on P such that $R_a^* \alpha = \rho_{a^{-1}} \circ \alpha$ for any $a \in G$. Such a form α is a <u>tensorial form</u> if it is horizontal , i. e. , if $\alpha(X_1, \ldots, X_k) = 0$ whenever at least one of the vectors X_i is vertical . The set of all V-valued tensorial k-forms of type ρ on P is denoted by $\Lambda^{(k)}(P, V, \rho)$

and it may be given the structure of a vector space .

Let α be a pseudotensorial k-form of type ρ . Its <u>exterior covariant derivative</u> $D\alpha$ is defined by

$$D\alpha\ (X_1, \ldots, X_{k+1}) \doteq \text{hor}\ d\alpha\ (X_1, \ldots, X_{k+1}) = d\alpha\ (hX_1, \ldots, hX_{k+1}) \quad . \qquad (2.2)$$

The form $D\alpha$ is a tensorial (k+1)-form of type ρ .

Let $\rho' : \mathcal{G} \longrightarrow L(V)$ be the derived mapping of Lie algebras, $L(V)$ being the Lie algebra of endomorphism of V . Then the exterior covariant derivative of a tensorial k-form of type ρ is [25]

$$D\alpha = d\alpha + (\rho' \circ \omega) \wedge \alpha \qquad (2.3)$$

where the symbol $(\rho' \circ \omega) \wedge \alpha$ means exterior product of forms and evaluation of $\rho' \circ \omega$, as an element of $L(V)$, on α as an element of V. In particular , if φ is a 0-form of type ρ , that is a matter field of type ρ , its exterior covariant derivative will be

$$D\varphi = d\varphi + (\rho' \circ \omega) \cdot \varphi \qquad (2.4)$$

The connection 1-form ω is a pseudotensorial form of type Ad . Then $\Omega = D\omega$ is a tensorial form of type Ad and it is called the <u>curvature form</u> of ω . It is explicitly given by the following <u>structure equation</u>

$$\Omega = d\omega + \omega \wedge \omega = d\omega + \frac{1}{2} [\omega, \omega] \qquad (2.5)$$

Note that this expression does not follow from (2.3) because ω is not tensorial .

The following theorem is due to Frobenius [26] .

<u>Theorem 2.1.</u> The curvature form Ω vanishes if and only if the connection is flat or completely integrable , i. e. , if and only if $[X, Y]$ is a horizontal vector field on P for any two horizontal vector fields X and Y on P .

The curvature form always satisfies the <u>Bianchi identity</u>

$$D\Omega = 0 \quad . \tag{2.6}$$

2.4 Pulling back by sections : gauge potentials and field strengths .

Let $P = P(M, G, \pi)$ be a principal fibre bundle and ω a connection 1-form on it . For any local section $s : U \longrightarrow P$, $U \subset M$, it is possible to define a $\mathcal{G}$ -valued 1-form on M

$$A^{(s)} = s^* \omega \tag{2.7}$$

and a $\mathcal{G}$ -valued 2-form on M

$$F^{(s)} = s^* \omega \quad . \tag{2.8}$$

They are related by the structure equation (2.5)

$$F^{(s)} = dA^{(s)} + \frac{1}{2}[A^{(s)}, A^{(s)}] \quad . \tag{2.9}$$

If one takes another section $s' : U \longrightarrow P$, there will exist a mapping $g : U \longrightarrow G$ such that

$$s'(x) = s(x)\, g(x) \qquad \text{for any} \quad x \in U \quad . \tag{2.10}$$

The new $A^{(s')}$ and the new $F^{(s')}$ are related to $A^{(s)}$ and $F^{(s)}$ by

$$A^{(s')} = Ad_{g^{-1}} \circ A^{(s)} + \tilde{\omega} \circ Tg \tag{2.11}$$

$$F^{(s')} = Ad_{g^{-1}} \, F^{(s)} \tag{2.12}$$

Here $\tilde{\omega}$ is the Maurer-Cartan form on G

$$\tilde{\omega} : TG \longrightarrow \mathcal{G}$$

$$T_g G \ni B \longrightarrow \tilde{\omega}(B) = T\,L_{g^{-1}}(B) \in \mathcal{G} \quad ,$$

L_g being the left action of G onto itself. If G is a group of matrices, the entries of the matrices may be taken as coordinates and $\tilde{\omega}_g = g^{-1} dg$.

In physics [5,11] the base space M of the bundle is the space-time and the quantities $A^{(s)}$ and $F^{(s)}$ are respectively the gauge potential and the field strength in the local gauge s. By changing the local gauge as in (2.10), they trasform according to the gauge trasformation of the second kind (2.11) and (2.12).

If $\varphi : P \longrightarrow V$ is a matter field of type ρ its expression in the gauge s will be

$$\Phi^{(s)} = \varphi \circ s : U \longrightarrow V \quad . \tag{2.13}$$

By changing the section as in (2.10) the local expressions $\Phi^{(s)}$ and $\Phi^{(s')}$ will be related by

$$\Phi^{(s')} = \rho_{g^{-1}} \circ \Phi^{(s)} \tag{2.14}$$

By defining $D\Phi^{(s')} = s^* D\varphi$, the local expression of the covariant derivative (2.6) of the matter field Φ is, dropping the label $^{(s)}$,

$$D\Phi = d\Phi + (\rho' \circ A)\,\Phi \quad . \tag{2.15}$$

Given a basis $\{ \xi_i \}$, $(i = 1, \ldots, n ,\ n = \dim \mathcal{G})$, for the Lie algebra $\mathcal{G}$, the fundamental vector fields $\{ \xi_i^* \}$, $(i = 1, \ldots, n)$, give a basis for the vertical spaces V_p, for any $p \in P$.

If $\{ \xi_\mu \doteq \partial_\mu \}$, $(\mu = 1, \ldots, 4)$ is a local basis for the tangent bundle of $U \subset M$, the lifted vectors $\{ \hat{\xi}_\mu \}$, $(\mu = 1, \ldots, 4)$ give a basis for the horizontal space H_p, for any $p \in \pi^{-1}(U)$.

The $4+n$ linearly independent vector fields $\{ \xi_i^* , \hat{\xi}_\mu \}$ are a basis for the tangent bundle of $\pi^{-1}(U)$.

One may decompose the connection form ω in the basis $\{ \xi_i \}$ of $\mathcal{G}$

$$\omega = \omega^i \xi_i \tag{2.16}$$

where ω^i, $(i = 1, \ldots, n)$, are now real-valued 1-forms on P.

By the properties of ω it follows that

$$\begin{aligned} \omega^i (\xi_k^*) &= \delta^i_k & & i , k = 1 , \ldots , n \\ \omega^i (\hat{\xi}_\mu) &= 0 & & i = 1 , \ldots , n \ ; \ \mu = 1 , \ldots , 4 \end{aligned} \tag{2.17}$$

so that the forms ω^i are dual of the vector fields $\hat{\xi}^*_i$. Note that ω^i and ξ_i^* are globally defined.

It is also possible to decompose the forms A and F

$$A = A^i \xi_i \tag{2.18}$$

$$F = F^i \xi_i \tag{2.19}$$

Since A^i and F^i are real-valued forms on M, using the dual basis $\{ dx^\mu \}$ of $\{ \partial_\mu \}$ one writes

$$A = A^i_\mu \, dx^\mu \, \xi_i \tag{2.20}$$

$$F = \frac{1}{2} F^i_{\mu\nu} \, dx^\mu \wedge dx^\nu \, \xi_i \tag{2.21}$$

and the structure equation (2.9) implies

$$F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + f^i_{jk} A^j_\mu A^k_\nu \tag{2.22}$$

with f^i_{jk} the structure constants of the group G

$$[\xi_i , \xi_j] = f^k_{ij} \, \xi_k \tag{2.23}$$

Suppose $\{ e_A \}$, (A = 1 , ... , dimV), is a basis of the vector space V . Then it is possible to express in this basis the matter field Φ

$$\Phi = \Phi^A e_A \tag{2.24}$$

and its covariant derivative

$$D \Phi = (D_\mu \Phi^A) dx^\mu e_A \tag{2.25}$$

The explicit transcription of the covariant derivative (2.15) is given by

$$D_\mu \Phi^A = \partial_\mu \Phi^A + \rho^A_{Bi} A^i_\mu \Phi^B \tag{2.26}$$

with the quantities ρ^A_{Bi} defined by

$$\rho'(\xi_i) \cdot \xi_A \doteq \frac{d}{dt} \rho_{\exp t \xi_i} (\xi_A) \Big|_{t=0} \doteq \rho^B_{Ai} \, \xi_B \tag{2.27}$$

2.5 The group of gauge transformations [22,12]

In the previous paragraph we have seen how a change of local sections corresponds to a gauge transformation . Equivalently , such transformations may be viewed as global automorphisms of the principal bundle .

Let $P = P(M, G, \pi)$ be a principal fibre bundle .

An automorphism of P is a diffeomorphism $f : P \longrightarrow P$ such that $f(pa) = f(p)a$ for any $p \in P$ and any $a \in G$. Any such f induces a diffeomorphism $\tilde{f} : M \longrightarrow M$ given by $\tilde{f}(\pi(p)) = \pi(f(p))$. The set $\mathrm{Aut}\, P$ of all automorphisms of P form a group under composition . A vertical automorphism is an automorphism for which $\tilde{f} = id_M$. The set $\mathrm{Aut}_V P$ of all vertical automorphisms of P is a normal subgroup of P . Any $f \in \mathrm{Aut}_V P$ maps each fibre into itself . There exists , therefore , a map $\tau : P \longrightarrow G$ such that

$$f(p) = p\,\tau(p) \quad , \quad \tau(pa) = a^{-1}\tau(p)a \tag{2.28}$$

for any $p \in P$ and any $a \in G$.

Let $\mathcal{C}$ be the set of all connection 1-forms on P . If ω and ω' are any two elements of $\mathcal{C}$, their difference $\omega - \omega'$ is a $\mathcal{G}$-valued tensorial 1-form of type Ad on P , i. e. , an element of $\Lambda^{(1)}(P, \mathcal{G}, \mathrm{Ad})$ (see paragraph 2.3) . Since $\Lambda^{(1)}(P, \mathcal{G}, \mathrm{Ad})$ is a vector space , the set $\mathcal{C}$ may be given the structure of an affine space .

There is an action of the group $\mathrm{Aut}_V P$ on $\mathcal{C}$

$$\mathrm{Aut}_V P \times \mathcal{C} \ni (f, \omega) \longrightarrow \omega' = f^*\omega \in \mathcal{C} \quad . \tag{2.29}$$

The connection ω' is given explicitly by

$$\omega' = \mathrm{Ad}_{\tau^{-1}} \circ \omega + \tilde{\omega} \circ T\tau \tag{2.30}$$

while the curvature 2-form corresponding to ω' is

$$\Omega' = Ad_{\tau^{-1}} \circ \Omega \quad . \tag{2.31}$$

Suppose $s : U \longrightarrow P$, $U \subset M$, is a section of P . Then $s' = f \circ s : U \longrightarrow P$ is a section over U as well . By defining the gauge potential $A^{(s)}$ and the field strenght $F^{(s)}$ as in (2.7) and (2.8) , the gauge transformed quantities

$$A^{(s')} = s'^{*} \omega \quad = s^{*} \omega' \tag{2.32}$$

$$F^{(s')} = s'^{*} \omega \quad = s^{*} \omega' \tag{2.33}$$

may be interpreted as being either the pullbacks by the transformed section s' (passive viewpoint) or the pullbacks by the original section s (active viewpoint) . Putting $g = \tau \circ s$, one obtaines the expressions (2.11) and (2.12) .

Owing to the action (2.29) , the group $Aut_{V} P$ is called the group of gauge transformations .

2.5 Lagrangian and field equations

In the previous paragraph we have shown how the quantities which characterize a gauge configuration on the space-time M may be described in terms of geometrical quantities . This has been done by constructing a principal bundle $P \longrightarrow M$ whose structure group is the gauge group G and by giving a connection on P .

One needs next the equations which are satisfied by the gauge fields and by the matter fields that eventually interact with them . These equations restrict the class of possible connections which can be introduced on P . In general they are derived from a lagrangian .

The general form of the gauge invariant Lagrangian for a system of gauge fields interacting with matter fields has been described by Utiyama [3] . In The framework of fibre bundles the Lagrangian is taken to be the following 4-form on M

$$\mathcal{L} = h({}^*F \wedge F) + k({}^*D\Phi \wedge D\Phi) + U(k(\Phi , \Phi)) \eta \qquad (2.34)$$

where h is a biinvariant , i. e. left and right invariant , metric on G and k a G-invariant metric on V . The $*$ operator and the volume element η are defined by means of a Riemannian metric on M .

In the case of Maxwell and Yang-Mills theories , usually $V = \mathcal{G}$, $\rho = Ad$ and k and h are both the Killing metric .

The field equations for the gauge fields are of the form

$$D\,{}^*F = - {}^*J \qquad (2.35)$$

where J is a $\mathcal{G}$-valued current 1-form .

The Bianchi identity (2.8) provided the homogeneous part of field equations

$$DF = 0 \qquad (2.36)$$

A part from the equations for the gauge fields there are , of course , also the equations for the matter field Φ .

3. Geometry of Kaluza-Klein type theories

3.1 Kaluza-Klein unification

In 1921 Kaluza [18] proposed a model to describe in a unified fashion electromagnetism and gravity. He considered a space-time M of five dimensions endowed with a particular Lorentz metric

$$\gamma = \gamma_{MN}\, dx^M dx^N \qquad M, N = 1, \ldots, 5 \quad . \tag{3.1}$$

The metric was assumed to be "cylindrical", i. e., such that there exists a particular frame in which the components γ_{MN} do not depend on x^5

$$\partial_5\, \gamma_{MN} = 0 \quad . \tag{3.2}$$

Moreover Kaluza assumed the fifth dimension to be space-like and in particular he took

$$\gamma_{55} = 1 \quad . \tag{3.3}$$

The metric (1.1) may then be written as

$$\gamma = g_{\mu\nu}\, dx^\mu dx^\nu + (dx^5 + A_\mu dx^\mu)(dx^5 + A_\mu dx^\mu) \quad . \tag{3.4}$$

Under the particular transformation of coordinates

$$\begin{aligned} x^\mu &\to x'^\mu = x^\mu \qquad \mu = 1, \ldots, 4 \\ x^\mu &\to x'^5 = x^5 + \Lambda(x^1, \ldots, x^4) \end{aligned} \tag{3.5}$$

$g_{\mu\nu}$ is invariant while A_μ transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \Lambda \qquad \mu = 1, \ldots, 4 \quad . \tag{3.6}$$

This is just a gauge transformation if one identifies A_μ with the electromagnetic potential . As a consequence , the gauge transformations in four dimensions are part of the transformations of coordinates in the five dimensional world .

One assumes next that , without sources , the Lagrangian density is

$$\mathcal{L} = \sqrt{|\gamma|} \quad R \quad . \tag{3.7}$$

Here R is the scalar curvature of the Levi-Civita connection of the metric γ while $|\gamma|$ is the absolute value of its determinant .

By explicit calculations

$$R = R^{(4)} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{3.8}$$

where $R^{(4)}$ is the scalar curvature of the Levi-Civita connection of the metric $g = g_{\mu\nu} dx^\mu dx^\nu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength and $F^{\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F_{\rho\sigma}$.

Finally , since $|\gamma| = |g|$, one gets

$$\mathcal{L} = \sqrt{|g|} \, (R^{(4)} - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}) \tag{3.9}$$

This is just the Lagrangian density for the gravitational field $g_{\mu\nu}$ in presence of an electromagnetic field A_μ .

By considereing independent variations of $g_{\mu\nu}$ and A_μ , the Euler-Lagrange equations for $\mathcal{L}$ are

$$G^{(4)}_{\mu\nu} = \frac{1}{2} \left[F_{\mu\rho} F_{\nu\sigma} g^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right] \tag{3.10}$$

$$D_\mu F^{\nu\mu} = 0 \tag{3.11}$$

Here $G^{(4)}_{\mu\nu}$ is the Einstein tensor of the metric $g_{\mu\nu}$ while D_μ

denotes covariant derivatives with respect to the Levi-Civita connection of $g_{\mu\nu}$. If one took $\gamma_{55} = -1$, then the second term in the right-hand side of (3.10) would have the opposite sign and this would produce an unphysical negative electromagnetic energy .

Klein [19] proposed a generalization of the previous ideas . He gave up the condition (3.2) assuming a periodic dependence of γ_{MN} on x^5.

Later , Einstein and Bergmann [27] suggested that the fifth dimension could be physically real . In order to explain the four dimentional character of the macroscopic world , they assumed that the fifth dimension closes into itself to form a small circle . The circle is sufficiently small so that the variation of a physical quantity around it is small when compared with its variations along the other four dimensions .

Jordan and Thiry [28] considered another possible generalization . They maintained the condition (3.2) and assumed that γ_{55} is a scalar field $\gamma_{55} = \varphi(x^1, \ldots, x^4)$.

One of the consequences of this assumption is that the gravitational constant becomes space-time dependent .

Fifteen years later De Witt [29] proposed to extend Kaluza-Klein ideas to non abelian gauge fields by considering a space-time of 4+n dimensions endowed with a suitable Lorentz metric . This has been subsequently developed in the framework of fibre bundle theories

[5,30,31] resulting in a very elegant geometric setting .

By considering the Dirac equation in the context of Kaluza-Klein theory in five dimensions , Thirring [32] obtained CP-violation .

In the last five years there has been an increasing interest in theories of kaluza-Klein type [33,34] . The starting point is general relativity in 4+n dimensions with the scalar curvature in 4+n dimensions as Lagrangian . It is then assumed that the "ground state" is a product $M^{(4)} \times B$ with $M^{(4)}$ the Minkowski space of 4 dimensions and B a compact space of n dimensions . The group of gauge symmetries is just the group of isometries of B . Then , all relevant quantities are expanded in eigenfunctions of differential operators on B (harmonic expansion) to get the full spectrum of an effective theory in four dimensions .

3.2 Special case : the internal space is a group manifold

The possibility of describing Kaluza-Klein unification of gravity with electromagnetism and its generalization to non abelian gauge fields by means of fibre bundles has been first analysed by Trautman [5] , Kerner [30] and Cho [31] .

However , the idea of costructing a Riemannian metric on a principal bundle with connection , once given a Riemannian metric on the base of the bundle , was already present in the mathematical literature [35] .

Let $P = P(M, G, \pi)$ be a principal fibre bundle over the space-time M with structure group G. Assume there is a connection on P described by a $\mathcal{G}$-valued 1-form ω on it. Moreover, let g be a Riemannian metric on M and h an Ad-invariant scalar product in $\mathcal{G}$ (this is equivalent to taking a bi-invariant metric on G [23]). A Riemannian metric γ on P is defined by

$$\gamma_p(X, Y) = g_{\pi(p)}(T\pi(X), T\pi(Y)) + h(\omega(X), \omega(Y)) \qquad (3.12)$$

for any $X, Y \in T_p P$, $p \in P$.

The metric γ is invariant under the action of G, $R_a^* \gamma = \gamma$ for any $a \in G$, so that any fundamental vector field ξ^* is a Killing vector field for γ.

With the metric (3.12), at any point p the vertical vector space is orthogonal to the horizontal space.

Conversely, if γ is a G-invariant metric on P it determines

1. a connection on P : $H_p \in T_p P$ is the vector space of all vectors at $p \in P$ which are orthogonal to V_p ;

2. a metric g on M : $g_x(X, Y) = \gamma_p(\hat{X}, \hat{Y})$ for any $X, Y \in T_x M$, with $\pi(p) = x$ and $\hat{X}$ and $\hat{Y}$ the horizontal lifts of X and Y to p. Due to the invariance of γ the value of $g_x(X, Y)$ does not depend onthe point $p \in \pi^{-1}(x)$;

3. an Ad-invariant scalar product on $\mathcal{G}$: $h(\xi, \eta) = \gamma(\xi^*, \eta^*)$

for any $\xi , \eta \in \mathcal{G}$, where ξ^* and η^* are the fundamental vector fields associated to ξ and η .

The form h is Ad-invariant because γ is G-invariant and $(Ad_a \xi)^* = T R_a(\xi^*)$ for any $\xi \in \mathcal{G}$ and any $a \in G$.

In the case in which G is semisimple one may take for h "minus" the Killing form of G

$$h(\xi , \eta) = - Tr (Ad'_\xi \circ Ad'_\eta) \qquad \text{for any} \quad \xi , \eta \in \mathcal{G} \quad (2.13)$$

where $Ad'_\xi (\eta) \doteq \frac{d}{dt} Ad_{\exp t\xi} (\eta)|_{t=0} = [\xi , \eta]$. In terms of structure constants of G , $h_{ij} \doteq h(\xi_i , \xi_j) = f^m_{ik} f^k_{mj}$.

For compact groups the Killing form is negatively defined and h is positively defined .

For a discussion on the possible gauge groups and their metrics see Kopczyński [36] .

As we saw in the paragraph 2.4 , the vectors $\{ \xi^*_i , \hat{\xi}_\mu \}$, $(i = 1 , ... , n ; \mu = 1 , ... , 4)$, form a basis for the tangent bundle of P .

Here ξ^*_i are the fundamental vector fields associated to the elements ξ_i of a basis of $\mathcal{G}$ while $\hat{\xi}_\mu$ are the horizontal lifts of the local basis $\xi_\mu = \partial_\mu$ of the tangent bundle of M .

In this basis the components of γ are given by

$$\gamma (\hat{\xi}_\mu , \hat{\xi}_\nu) = g(\partial_\mu , \partial_\nu) = g_{\mu\nu}$$

$$\gamma (\hat{\xi}_\mu , \xi^*_i) = 0$$

$$\gamma\,(\xi^*_i, \xi^*_j) = h\,(\xi_i, \xi_j) = h_{ij} \tag{3.14}$$

We introduce now a new basis in order to show how the gauge potentials enter the metric γ [31] . Once given an arbitrary section $s : U \longrightarrow P$, $U \subset M$, it is possible to cover the whole of $\pi^{-1}(U)$ with the family of sections $s_a = R_a \circ s$, $a \in G$.

A basis $\{\tilde{\xi}_A\}$, (A = 1 , ... , n+4) , for the tangent bundle of $\pi^{-1}(U)$ is given as follows

1. choose $\tilde{\xi}_i = \xi^*_i \qquad i = 5, \ldots, n+4$

2. at the point $p \in \pi^{-1}(U)$ such that $p = s_a(x)$, $x \in U$, define $\tilde{\xi}_\mu$, (μ = 1 , ... 4) , as $\tilde{\xi}_\mu = T\, s_a(\partial_\mu)$.

The vectors $\tilde{\xi}_A$ obey the following commutation relations [31]

$$[\tilde{\xi}_i, \tilde{\xi}_j] = f^{\,k}_{ij}\,\tilde{\xi}_k$$
$$[\tilde{\xi}_i, \tilde{\xi}_\mu] = 0$$
$$[\tilde{\xi}_\mu, \tilde{\xi}_\nu] = 0 \tag{3.15}$$

Observe that $\tilde{\xi}_\mu \neq \hat{\xi}_\mu$ because in general $[\hat{\xi}_\mu, \hat{\xi}_\nu] \neq 0$. The commutator $[\hat{\xi}_\mu, \hat{\xi}_\nu]$ is not even horizontal and it may be shown that

$$[\hat{\xi}_\mu, \hat{\xi}_\nu] = -F^{\,k}_{\mu\nu}\,\xi^*_k \tag{3.16}$$

where $F^{\,k}_{\mu\nu}$ are the components of the gauge field strength .

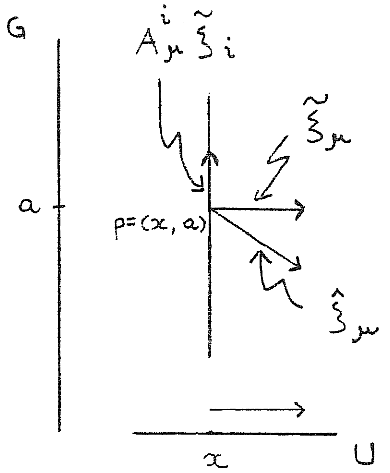
In the basis $\{\tilde{\xi}_A\}$ the vectors $\hat{\xi}_\mu$ are given by

$$\hat{\xi}_\mu = \tilde{\xi}_\mu - A^i_\mu(a, x)\,\tilde{\xi}_i \quad , \quad \mu = 1, \ldots, 4$$

and $A^i_\mu(a, x)$ are the components of the gauge potential

$$A^i_\mu(a, x) = (s_a(x))^* \omega^i \tag{3.17}$$

Pictorially the situation is as follows



In the basis $\{\tilde{\xi}_A\}$ the metric γ is given by

$$\gamma(\tilde{\xi}_\mu, \tilde{\xi}_\nu) = g_{\mu\nu} + h_{ij} A^i_\mu A^j_\nu$$

$$\gamma(\tilde{\xi}_\mu, \tilde{\xi}_i) = A^j_\mu h_{ij}$$

$$\gamma(\tilde{\xi}_i, \tilde{\xi}_j) = h_{ij}$$

$$\gamma = \left(\begin{array}{c|c} g_{\mu\nu} + h_{ij} A^i_\mu A^j_\nu & A^i_\mu h_{ij} \\ \hline h_{ij} A^j_\mu & h_{ij} \end{array}\right) \tag{3.18}$$

and its inverse is

$$\gamma^{-1} = \left(\begin{array}{c|c} g^{\mu\nu} & -g^{\mu\nu} A^i_\nu \\ \hline -A^i_\mu g^{\mu\nu} & h^{ij} + g^{\mu\nu} A^i_\mu A^j_\nu \end{array}\right) \tag{3.19}$$

with $g^{\mu\nu} g_{\nu\rho} = \delta^\mu_\rho$ and $h^{ij} h_{jk} = \delta^i_k$.

For electromagnetism $G = U(1)$ and taking $h_{55} = 1$ one gets the Kaluza metric [18]

$$\gamma = \left(\begin{array}{c|c} g_{\mu\nu} + A_\mu A_\nu & A_\mu \\ \hline A_\nu & 1 \end{array}\right) \tag{3.20}$$

In the spirit of Kaluza-Klein unification one assumes as Lagrangian

$$\mathcal{L} = R(\gamma)\eta \tag{3.21}$$

where $R(\gamma)$ is the scalar curvature of the Levi-Civita connection compatible with the metric γ while η is a horizontal volume element on P. The Euler-Lagrange for $\mathcal{L}$ are

$$(R_{AB} - \frac{1}{2}\gamma_{AB} R)\,\delta\gamma^{AB} = 0 \ . \tag{3.22}$$

Now the variations $\delta\gamma^{AB}$ are not arbitrary. One varies $g_{\mu\nu}$ and A^i_μ arbitrarly preserving the particular structure (3.12) of the metric γ.

The components of the Ricci tensor R_{AB} in the basis $\{\tilde{\xi}_A\}$ are [31]

$$\tilde{R}_{ij} = R_{ij} + \frac{1}{4} h_{ik} h_{j\ell} g^{\mu\alpha} g^{\nu\beta} F^k_{\mu\nu} F^\ell_{\alpha\beta}$$

$$\tilde{R}_{\mu i} = \frac{1}{4} h_{ij} A^j_\mu + \frac{1}{4} h_{ij} h_{k\ell} g^{\alpha\gamma} g^{\beta\delta} A^k_\mu F^\ell_{\alpha\beta} F^j_{\gamma\delta} + \frac{1}{2} h_{ij} g^{\alpha\beta} D_\alpha F^j_{\mu\beta}$$

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} + \frac{1}{4} h_{ij} A^i_\mu A^j_\nu + \frac{1}{4} h_{ij} h_{k\ell} g^{\alpha\gamma} g^{\beta\delta} A^i_\mu A^k_\nu F^j_{\alpha\beta} F^\ell_{\gamma\delta} +$$

$$- \frac{1}{2} g^{\alpha\beta} h_{ij} F^i_{\mu\alpha} F^j_{\nu\beta} + \frac{1}{2} h_{ij} g^{\alpha\beta} (A^i_\mu D_\alpha F^j_{\nu\beta} + A^i_\nu D_\alpha F^j_{\mu\beta}) \tag{3.23}$$

Here R_{ij} and $R_{\mu\nu}$ are respectively the Ricci tensors of G and M

with the metric h_{ij} and $g_{\mu\nu}$, and D_α is the covariant derivative

$$D_\alpha F^i_{\mu\nu} = \partial_\alpha F^i_{\mu\nu} - \Gamma^\beta_{\alpha\mu} F^i_{\beta\nu} - \Gamma^\beta_{\alpha\nu} F^i_{\mu\beta} + f^i_{kj} A^k_\alpha F^j_{\mu\nu}$$

with $\Gamma^\lambda_{\mu\nu}$ the Christoffel symbols of M .

The scalar curvature is

$$\tilde{R} = g^{\mu\nu} R_{\mu\nu} + h^{ij} R_{ij} - \frac{1}{4} h_{ij} g^{\mu\alpha} g^{\nu\beta} F^i_{\mu\nu} F^j_{\alpha\beta}$$

$$= R_M + R_G - \frac{1}{4} F^2 \qquad (3.24)$$

From (3.22) one gets two sets of equations

1. the Einstein field equations for the metric $g_{\mu\nu}$ in presence of a gauge field

$$R_{\mu\nu} - \frac{1}{2} R_M g_{\mu\nu} - \frac{1}{2} R_G g_{\mu\nu} = T_{\mu\nu} \qquad (3.25)$$

with

$$T_{\mu\nu} = \frac{1}{2} h_{ij} (F^i_{\mu\rho} F^j_{\nu\sigma} g^{\rho\sigma} - \frac{1}{4} g_{\mu\nu} F^i_{\alpha\rho} F^j_{\beta\sigma} g^{\alpha\beta} g^{\rho\sigma}) \qquad (3.26)$$

2. the vacuum Yang-Mills equations in a gravitational field

$$D_\mu F^{i\mu\nu} = 0 \qquad (3.27)$$

with $F^{i\mu\nu} = g^{\mu\rho} g^{\nu\sigma} F^i_{\rho\sigma}$.

We can see that the scalar curvature R_G plays the role of a cosmological constant ; a calculation for it gives something like 10^{120} times the actual extimations [31,36] .

Kopczyński [36] describes a way to get rid of the cosmological term by constructing a particular linear connection , i. e., a connection on the bundle of linear frame of P , which is fully and uniquely determined by the metric γ although different from its Levi-Civita connection . This is possible because a principal bundle has a richer structure than an ordinary manifold .

It is possible to generalize the previous scheme by considering a space-time dependence of the quantities h_{ij} which then give a positive defined metric on each fibre $\pi(x)$, $x \in M$. This generalization provides a Jordan-Thiry version of the theory where the $h_{ij}(x)$ became scalar fields . It has been considered by Cho and Freund [37] .

3.3 The general case : the internal space is a homogeneous space

In the previous paragraph we have reviewed the geometric description of higher dimensional Kaluza-Klein unification in the case in which the internal space is the group manifold of the gauge group G .

In order to lower the number of extra dimensions necessary to implement a gauge symmetry G , one consider an homogeneous space G/H as internal space . Here H is a subgroup of G and there is a natural transitive action of G on G/H [33, 34] .

Descriptions of models with homogeneous space as internal spaces , in the framework of fibre bundle , have been provided by several authors [38-43] .

In particular Coquereaux and Jadczyk [40-42] describe a very elegant way to obtain a fibration of a manifold E, taken as the multidimensional universe, on which a global symmetry compact Lie group G acts from the right. The action of G is taken to be effective, i. e., the only transformation which leaves the all of E unchanged is the one corresponding to the identity of G, and simple, i. e., all the isotropy groups of elements of E are conjugated to a standard one H.

The space of orbits $M = E/G$ is the space-time and π is the canonical projection

$$\pi : E \longrightarrow M \quad , \quad u \longrightarrow G(u) = \text{orbit of } G \text{ through } u \quad .$$

Let P be the submanifold of E consisting of all points whose isotropy group is just H. Then π restricts to a projection $\pi_P : P \longrightarrow M$.

Let $N(H)$ be the normalizer of H in G, i. e., the biggest subgroup of G in which H is normal $N(H) = \{ g \in G \mid g H = H g \}$. It is possible to show that

1. the manifold P is a principal fibre bundle over M with $N(H)|H = K$ as structure group.

2. E is a bundle associated to P whose typical fibre is G/H. Here $G/H = \{ H g \mid g \in G \}$ is the set of right cosects.

The action of K on G/H which provides this assosiation is characterized by the following theorem.

<u>Theorem 3.1</u> The group K is the group of all the invertible mappings of G/H into itself which commute with the righ action of G in G/H .

Explicitly , K is the group of all invertible mappings $\alpha : G/H \longrightarrow G/H$ such that $\alpha([a]g) = (\alpha([a]))g$ for any $[a] \in G/H$ and any $g \in G$. Any mapping α is of the form $[a] \longrightarrow \alpha_{[n]}([a]) \doteq [na]$ for some $[n] \in K$.

It is possible to provide G/H with a Riemannian geometry . Assume G and H are compact connected Lie groups . The compactness of G implies that it admits a bi-invariant Riemannian metric , and this is equivalent to the existence of an Ad_G -invariant positive defined scalar product (,) in the Lie algebra $\mathcal{G}$ of G [23] .

Let $\mathcal{H}$ be the Lie algebra of H and $\mathcal{S}$ its orthogonal complement in $\mathcal{G}$

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{S} \qquad (3.28)$$

Since (,) is in particular Ad_H -invariant , the homogeneous space G/H is reductive [44] , i. e.,

$$Ad_H(\mathcal{S}) \subset \mathcal{S} \qquad (3.29)$$

and , H being connected , this is equivalent to

$$[\mathcal{H} , \mathcal{S}] \subset \mathcal{S} \qquad (3.30)$$

It is possible to identify $\mathcal{S}$ with the vector space tangent to $S \doteq G/H$

at the origin [44] .

If $\mathcal{N}$ is the Lie algebra of N and $\mathcal{K}$ is the orthogonal complement of $\mathcal{H}$ in $\mathcal{N}$, then

$$\mathcal{N} = \mathcal{H} \oplus \mathcal{K} \tag{3.31}$$

and N|H is reductive as well

$$Ad_H(\mathcal{K}) \subset \mathcal{K} \tag{3.32}$$

This is equivalent to $[\mathcal{H}, \mathcal{K}] \subset \mathcal{K}$ but, H being normal in N , $\mathcal{H}$ is an ideal of $\mathcal{N}$ so that $[\mathcal{H}, \mathcal{K}] \subset \mathcal{H}$, and

$$[\mathcal{H}, \mathcal{K}] = 0 \tag{3.33}$$

Since N|H is a group it is possible to identify $\mathcal{H}$ with its Lie algebra . Finally , let $\mathcal{L}$ be the orthogonal complement of $\mathcal{N}$ in $\mathcal{G}$

$$\mathcal{G} = \mathcal{N} \oplus \mathcal{L} \quad , \quad Ad_N(\mathcal{L}) = \mathcal{L} \quad . \tag{3.34}$$

Using (3.28) and (3.31) we have also

$$\mathcal{S} = \mathcal{K} \oplus \mathcal{L} \quad , \quad Ad_H(\mathcal{L}) = \mathcal{L} \quad . \tag{3.35}$$

It may be shown that $\mathcal{L}$ and $\mathcal{K}$ are orthogonal with respect to any Ad_H-invariant scalar producr in $\mathcal{S}$ [40] .

Let $\{\xi_i\}$, (i = i , ... , n ; n = dim G), be a basis of $\mathcal{G}$: $[\xi_i, \xi_j] = f^k_{ij}\, \xi_k$. This basis is adapted to the decomposition

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{S} = \mathcal{H} \oplus \mathcal{K} \oplus \mathcal{L}$$

$$\{\xi_i\} = \{\xi_{\hat{a}} \in \mathcal{H} \, , \, \xi_\alpha \in \mathcal{S}\}$$

$$\{\xi_\alpha\} = \{\xi_{\hat{a}} \in \mathcal{K} \, , \, \xi_a \in \mathcal{L}\} \tag{3.36}$$

We know that the fundamental vector fields corresponding to the action of G on E are tangent to the fibres at any point of E : they are vertical . In particular , the vectors $\{\xi_i^*\}$ corresponding to the basis $\{\xi_i\}$ of $\mathcal{G}$ are vertical .

Since the dimension of $\mathcal{G}$ is greater than the dimension of the vertical space , which is in turn isomorphic to $T_o(G/H)$, there are too many ξ_i^* , that is they are linearly dependent .

On the other hand , for any $p \in P$, $Hp = p$, then $\xi_{\hat{\alpha}}^*(p) = 0$ on P . The vectors $\xi_\alpha^*(p)$, $p \in P$ are linearly independent as vectors of $T_p E$; then they are linearly independent in some neighbourhood U of P in E where they constitute a frame for the tangent space of the fibres .

Once given a connection $p \longrightarrow H_p$ on P , a Riemannian metric g on M and a family of G-invariant metrics h_x , $x \in M$, on the fibres E_x of E , it is possible to define a G-invariant Riemannian metric γ on E .

Given $X , Y \in T_u E$, let $p \in P$ be such that $u = pa$ for some $a \in G$; let $T\pi(X)$ and $T\pi(Y)$ be the projections of X and Y to M , $\hat{X}$ and $\hat{Y}$ the horizontal lifts of $T\pi(X)$ and $T\pi(Y)$ to H_p . Since $TR_a(\hat{X})$ and $TR_a(\hat{Y})$ belong to $H_{pa} = H_u$, the vectors $X - TR_a(\hat{X})$ and $Y - TR_a(\hat{Y})$ are vertical .

The scalar product is defined as

$$\gamma_u(X,Y) = g_x(T\pi(X), T\pi(Y)) + h_x(X - TR_a(\hat{X}), Y - TR_a(\hat{Y})) \quad . \quad (3.37)$$

Conversely, any G-invariant metric γ on E determines

1. a G-invariant metric h_x on any fibre E_x of E : h_x is the restriction of γ to the fibre E_x . The metric h_x defines an Ad_H-invariant scalar product on $T_o(G/H) \simeq \mathcal{S}$ [44] ;

2. a G-invariant horizontal distribution $u \longrightarrow H_u$ on E or equivalenly, a principal connection on P : the horizontal space $H_u \subset TE$ consists of all vectors at $u \in E$ which are orthogonal to the space V_u of vertical vectors at u ; G-invariance of γ implies that $TR_a(H_u) = H_{ua}$ for any $a \in G$.

In order to have a principal connection on P, one must show that H_p is tangent to P for any $p \in P$. This follows by observing that H_p is orthogonal to the vectors $\xi^*_\alpha(p)$, which correspond to $\xi_\alpha \in \mathcal{L}$, and which span the orthogonal complement of T_pP , and by remembering that $\mathcal{H}$ and $\mathcal{L}$ are orthogonal one to the other with respect to any Ad_H-invariant metric in $\mathcal{S}$. Moreover, $TR_{[n]}(H_p) = H_{p[n]}$ for any $[n] \in N|H$. Therefore, $p \longrightarrow H_p$ is a principal connection on P .

3. a metric g on M : $g_x(V,W) = \gamma_u(\hat{V},\hat{W})$ for any $V, W \in T_xM$, with $\pi(u) = x$ and $\hat{V}$ and $\hat{W}$ the horizontal lifts of V and W to u . Owing to G-invariance of γ , the value of $g_x(V,W)$ does not depend on the point $u \in \pi^{-1}(x)$.

As usual a connection on P may be described by a $\mathcal{H}$-valued 1-form ω on P or by a family $\{ A^{(s)} = s^* \omega$, $s : U \subset M \to P$ a local section of P $\}$ of $\mathcal{H}$-valued 1-forms on M (remember that $\mathcal{H}$ is the Lie algebra of N|H).

Any 1-form $A^{(s)}$ may, in turn, be written as (by dropping the label $^{(s)}$)

$$A = A^{\hat{a}} \xi_{\hat{a}} = A^{\hat{a}}_{\mu} dx^{\mu} \xi_{\hat{a}} = A_{\mu} dx^{\mu} \tag{3.38}$$

where $\{x^{\mu}\}$ are local coordinates for M and $\{\xi_{\hat{a}}\}$ is a basis for $\mathcal{H}$.

In the spirit of Kaluza-Klein unification the Lagrangian is taken to be

$$\mathcal{L} = R(\gamma)\,\eta \tag{3.39}$$

where $R(\gamma)$ is the scalar curvature of the Levi-Civita connection of the metric γ while η is a horizontal volume element on E .

The field equations are obtained by varying $\mathcal{L}$ with respect to γ while keeping the bundle structure of γ fixed .

The scalar curvature is given by [41]

$$R(\gamma) = R_M + R_{G/H} - \frac{1}{4} h_{\hat{a}\hat{b}} g^{\mu\rho} g^{\nu\sigma} F^{\hat{a}}_{\mu\nu} F^{\hat{b}}_{\rho\sigma} + \\ - \frac{1}{4} h^{\alpha\beta} h^{\gamma\delta} g^{\mu\nu} (D_{\mu} h_{\alpha\gamma} D_{\nu} h_{\beta\delta} + D_{\mu} h_{\alpha\beta} D_{\nu} h_{\gamma\delta}) - g^{\mu\nu} \nabla_{\mu} (h^{\alpha\beta} D_{\nu} h_{\alpha\beta}) \tag{3.40}$$

Here R_M is the scalar curvature of the Levi-Civita connection ∇_{μ} of $g_{\mu\nu}$; D_{μ} is the local expression of the covariant derivative of the connection $A_\mu^{\hat{\alpha}}$

$$D_\mu h_{\alpha\beta} = \partial_\mu h_{\alpha\beta} + f_{\alpha\hat{\gamma}}^{\delta} A_\mu^{\hat{\gamma}} h_{\delta\beta} + f_{\beta\hat{\gamma}}^{\delta} A_\mu^{\hat{\gamma}} h_{\alpha\delta}, \qquad (3.41)$$

while $F_{\mu\nu}^{\hat{\alpha}}$ is the local expression of its curvature ; $R_{G/H}$ is the scalar curvature of the metric $h_{\alpha\beta}$ on the copy of G/H over any point of M

$$R_{G/H} = -h^{\alpha\alpha'}\left(\frac{1}{2} f_{\alpha\beta}^{\gamma} f_{\alpha'\gamma}^{\beta} + \frac{1}{4} h^{\beta\beta'} h_{\gamma\delta'} f_{\alpha\beta}^{\gamma} f_{\alpha'\beta'}^{\delta'} + f_{\alpha\beta}^{\hat{\alpha}} f_{\alpha'\hat{\alpha}}^{\beta}\right) \qquad (3.42)$$

The quantities $A_\mu^{\hat{\alpha}}$ and $F_{\mu\nu}^{\hat{\alpha}}$ are gauge potentials and gauge field strengths corresponding to the group $N|H$ while $h_{\alpha\beta}$ play the role of scalar fields .

The curvature $R(\gamma)$ is the sum of

1. R_M : the curvature of space-time metric $g_{\mu\nu}$,

2. $R_{G/H}$: the curvature of the metric $h_{\alpha\beta}$ which gives the potential term for the scalar $h_{\alpha\beta}$,

3. Yang-Mills Lagrangian of $A_\mu^{\hat{\alpha}}$,

4. kinetic term for the scalars $h_{\alpha\beta}$.

4. Gauge theory over Stiefel bundles

4.1. Stiefel bundles as universal bundles [26]

Let F denote the field R of real numbers or the field C of complex numbers or the division algebra H of quaternions.

Let F^n be the right vector space of n-ples of n elements of F . any element of F^n is of the form

$$\underline{z} = (z_1, \dots, z_n) = z_\alpha e_\alpha \quad , \quad z_\alpha \in F \tag{4.1}$$

with $\{e_\alpha\}$ a frame of unit vectors in F^n .

An inner product is defined in F^n as follows

$$(\underline{v}, \underline{z}) = \bar{v}_\alpha z_\alpha \tag{4.2}$$

where $\bar{z}_\alpha$ is the conjugate of z_α ; if $F = R$ then $\bar{z}_\alpha = z_\alpha$.

If $U_n(F)$ is the Lie group of linear transformations of F^n which preserve the inner product, then $a \in U_n(F) \Longleftrightarrow a^+ a = I$, with a^+ the transpose conjugate of a , and

$$U_n(F) = \begin{cases} O(n) & \text{for } F = R \\ U(n) & \text{for } F = C \\ Sp(n) & \text{for } F = H \end{cases}$$

For $k = 1, \dots, n$, a k-frame $\mathcal{Z}$ in F is an ordered set $(\underline{z}_1, \dots, \underline{z}_k) = \mathcal{Z}$ of k orthonormal vectors

$$(\underline{z}_i, \underline{z}_j) = \delta_{ij} \qquad i, j = 1, \dots, k .$$

The set of all k-frame in F^n is a manifold. The connected component of $(\underline{e}_1, \dots, \underline{e}_k)$ is the Stiefel space $V_{n,k}(F)$.

There is a transitive left action of $U_n(F)$ on $V_{n,k}(F)$

$$U_n \times V_{n,k} \ni (a, \mathcal{Z}) \longrightarrow \mathcal{Z}' = a\,\mathcal{Z} = (a\underline{z}_1, \dots, a\underline{z}_k) \quad .$$

The stability group of $(e_1, \dots, e_k)$ is the subgroup $I_k \times U_{(n-k)}(F) \simeq U_{(n-k)}(F)$ of $U_n(F)$ consisting of all matrices of the form

$$\begin{pmatrix} \delta_{ij} & 0 \\ 0 & a_{AB} \end{pmatrix} \qquad (a_{AB}) \in U_{(n-k)}(F) \quad , \quad A,B = k+1, \dots, n \; .$$

The space $V_{n,k}(F)$ may be identified with a left coset space

$$V_{n,k}(F) = U_n(F) / U_{(n-k)}(F) \quad . \tag{4.3}$$

To the matrix of $U_n(F)$

$$a = \begin{pmatrix} a_{ij} & a_{iB} \\ a_{Aj} & a_{AB} \end{pmatrix} \qquad \text{mod.} \quad I_k \times U_{(n-k)}(F) \; ,$$

there coresponds the k-frame $\mathcal{Z} = (\underline{z}_1, \dots, \underline{z}_k)$ given by

$$\underline{z}_i = \sum_\alpha a_{\alpha i} \underline{e}_\alpha \qquad i = 1, \dots, k \quad . \tag{4.4}$$

Consider the $(n \times k)$ matrix

$$S = (a_{\alpha i})_{\substack{\alpha = 1, \dots, n \\ i = 1, \dots, k}} \quad , \tag{4.5}$$

the identification (4.3) associates the matrix S to the unique k-frame $\mathcal{Z}$ determined by (4.4). Since the elements of $\mathcal{Z}$ are orthonormal vectors it follows that $S^+ S = I_k$.

There is a right action of the group $U_k(F)$ on the space $V_{n,k}(F)$

$$(Z, a) \longrightarrow R_a(Z) = Z a = Z' \quad , \quad z'_i = \sum_j z_j a_{ji} \quad .$$

The quotient of $V_{n,k}(F)$ by this action may be given the structure of a differentiable manifold . This manifold is the Grassmann manifold $G_{n,k}(F)$: it is the set of all k-dimensional subspace of F^n . Two k-frames Z , $Z' \in V_{n,k}(F)$ span the same k-subspace if and only if there is an $a \in U_k(F)$ such that $Z' = Z a$.

The Stiefel bundle

$$\pi : V_{n,k}(F) \longrightarrow G_{n,k}(F) \tag{4.6}$$

is a principal $U_k(F)$-bundle .

The manifold $G_{n,k}(F)$ may be identified with a coset space

$$G_{n,k}(F) = U_n(F) / U_k(F) \times U_{(n-k)}(F) \tag{4.7}$$

On the Stiefel bundles there is a natural connection . Narasimhan and Ramanan [45] showed that if S is the matrix given by (4.5) and characterizing the identification (4.3) , the differential form $\omega_o = S^+ dS$ takes values in the Lie algebra of $U_n(F)$ and defines a connection form on the bundle $V_{n,k}(F)$. Moreover , the connection is invariant under the left action of $U_n(F)$.

The connection ω_o gives sourceless gauge fields over the Grasssman manifolds $G_{n,k}(F)$ [46] .

The Stiefel bundles are important because they are universal .

In fact, it is well known that [26]

$$\pi_i(V_{n,k}(F)) = 0 \qquad \text{for} \quad 0 \leq i \leq (n-k+1)c - 2$$

with $c = \dim F$; by theorem 1.1 this implies that the bundle (4.6) is $[(n-k+1)c-2]$ -universal .

Given any manifold M of real dimensions $\leq (n-k+1)c-2$, any $U_k(F)$-bundle P over M may be obtained as the induced bundle $f^* V_{n,k}(F)$ where $f : M \longrightarrow G_{n,k}(F)$.

Moreover , the connection form ω_0 is itself universal [45] : any connection form on P may be induced from it .

However , in general the induced connection will not give a sourceless gauge field [46] .

4.2 Hopf bundles and topologically non trivial gauge configurations

We specialize the Stiefel bundle (4.6) to the case $k = 1$. Since

$$V_{n+1,1}(F) = S^n(F) = \left\{ \underline{u} \in F^{n+1} \;\middle|\; (\underline{u}, \underline{u}) = 1 \right\}$$

and

$$G_{n+1,1}(F) = FP^n = \text{the set of all lines in } F^{n+1} \text{ through the origin},$$

we obtain the Hopf bundles over the projective spaces with structure groups $U_1(F)$

$$S^n(F) \longrightarrow FP^n \qquad (4.8)$$

Explicitly ,

$$S^n(F) = \begin{cases} S^n & \text{if} \quad F = R \\ S^{2n+1} & \text{if} \quad F = C \\ S^{4n+3} & \text{if} \quad F = H \end{cases}$$

while

$$U_1(F) = \begin{cases} O(1) \simeq Z_2 & \text{if} \quad F = R \\ U(1) & \text{if} \quad F = C \\ Sp(1) \simeq SU(2) & \text{if} \quad F = H \end{cases} \quad .$$

Physically the interesting cases are those of complex and quaternionic numbers for which the structure groups $U(1)$ and $SU(2)$ are respectively the electromagnetic and Yang-Mills gauge groups .

Following Trautman [47] we will show how the Hopf bundles (4.8) for the cases $F = C$ or H describe topologically non trivial solutions of the sourceless Maxwell and Yang-Mills equations .

The crucial point is the introduction of a Kaluza-Klein type metric (see eq . 3.1) on the bundles (4.8) .

Any point $\underline{z} = (z_0, z_1, \ldots, z_n) \in S^n(F)$ is such that $(\underline{z}, \underline{z}) = 1$.

If $z_0 \neq 0$, local coordinates for the sphere are the following

$$z_0 = \rho\, u \quad , \quad u \in U_1(F) \quad , \text{ i. e.} \quad |u|^2 = 1 \quad ,$$

$$z_a = \xi_a z_0 \qquad a = 1, \ldots, n \quad ;$$

then $\rho = |z_0| > 0$ and $\rho^{-2} = 1 + \sum_{a=1}^{n} \bar{\xi}_a \xi_a$.

The set $\{\xi_a\}$ gives a local coordinate system for the projective space

FP^n , while the coordinates $\{u, \xi_a\}$ give a local trivialization for the bundle (4.8) .

The natural riemannian metric on the sphere is

$$d\ell^2 = \sum_{\alpha=0}^{n} d\bar{z}_\alpha \, dz_\alpha \tag{4.9}$$

This metric may be written as

$$d\ell^2 = ds^2 - \omega^2 \tag{4.10}$$

where

$$\omega = u^{-1}du + \frac{1}{2}\rho^2 u^{-1} \sum_a [\bar{\xi}_a d\xi_a - (d\bar{\xi}_a)\xi_a] u \tag{4.11}$$

$$ds^2 = \rho^2 \sum_a d\bar{\xi}_a d\xi_a + $$

$$- \frac{1}{2}\rho^4 \sum_{ab} [\bar{\xi}_a (d\xi_a)(d\bar{\xi}_b \xi_b + (d\bar{\xi}_a)\xi_a \bar{\xi}_b d\xi_b] . \tag{4.12}$$

The line element ds^2 defines a Riemannian metric on FP^n . When $F = C$, ds^2 is just the Fubini-Study metric of CP^n [48]

$$ds^2 = \rho^2 \sum_a d\bar{\xi}_a d\xi_a - \rho^4 \sum_{ab} \bar{\xi}_a \xi_b d\xi_a d\bar{\xi}_b \tag{4.13}$$

The 1-form ω is a connection 1-form on the Hopf bundle (4.8) . It takes values in the Lie algebra of $U_1(F)$ hence is pure imaginary . Its curvature is

$$\Omega = d\omega + \omega\wedge\omega = u^{-1}\left[\sum_{ab} d\bar{\xi}_a \wedge h_{ab} d\xi_b\right] u \tag{4.14}$$

where

$$h_{ab} = \rho^2 \delta_{ab} - \rho^4 \xi_a \bar{\xi}_b \quad , \quad \bar{h}_{ab} = h_{ba} \tag{4.15}$$

Since Ω is a horizontal $U_1(F)$-invariant 2-form , it projects to a 2-form on FP^n denoted by the same symbol . Then Ω is a sourceless gauge field on FP^n [47]

$$D^*\Omega = 0 \quad . \tag{4.16}$$

The dual is evaluated using the metric ds^2 and a suitable volume element on FP^n .

The magnetic pole solution of lower strength [13] corresponds to $F = C$ and $n = 1$. The Hopf bundle is

$$S^3 \xrightarrow{U(1)} CP^1 = S^2$$

with connection

$$\omega = u^{-1}du + \frac{1}{2}u^{-1}\frac{\bar{\xi}d\xi - \xi d\bar{\xi}}{1+\bar{\xi}\xi}u \tag{4.17}$$

while the metric on S^2 is given by

$$ds^2 = \frac{d\bar{\xi}d\xi}{1+\bar{\xi}\xi} \tag{4.18}$$

The previous is the standard metric on a sphere of radius 1/2 when expressed in stereographic coordinates .

If S^3 is parametrized by means of Euler angles

$$z_0 = e^{\frac{i}{2}(\chi+\varphi)}\cos\frac{\theta}{2} \quad , \quad z_1 = e^{\frac{i}{2}(\chi-\varphi)}\sin\frac{\theta}{2}$$

the stereographic coordinates for S^2 are

$$\xi = e^{-i\varphi}\,tg\,\frac{\theta}{2}$$

which gives for u and ρ

$$u = e^{\frac{i}{2}(\chi+\varphi)} \quad , \quad \rho = \cos\frac{\theta}{2}$$

The local section $u = 1$ gives

$$z_0 = \cos\frac{\theta}{2} \quad , \quad z_1 = e^{-i\varphi}\sin\frac{\theta}{2}$$

and it is defined everywhere but the south pole $\theta = \pi$. It lead to the potential

$$A = -i\omega = \frac{1}{2}(1 - \cos\theta)\, d\varphi$$

which describes a magnetic pole of strength $g = \frac{1}{2}$ (in units such that $\frac{e}{\hbar c} = 1$) and it is defined everywhere on S^2 but the south pole .

The BPST solution of the Yang-Mills equations [14] corresponds to $F = H$ and $n = 1$. The Hopf bundle is

$$S^7 \xrightarrow{SU(2)} HP^1 = S^4$$

with connection ω on S^7 and metric ds^2 on S^4 the same as in (4.17) and (4.18) but now $u \in U_1(H)$ and $\xi \in H$.

The local gauge $u = 1$ gives the following gauge potential on S^4

$$A = \frac{1}{2}\,\frac{\bar{\xi}\, d\xi - (d\bar{\xi})\,\xi}{1 + \bar{\xi}\xi} \qquad (4.19)$$

Using Cartesian coordinates (t, x, y, z) of R^4 ,one may writes

$$\xi = t + ix + jy + kz$$

where i, j and k are the quaternionic units . If the latters are expressed in terms of Pauli matrices

$$i = \sqrt{-1}\,\sigma_1 , \quad j = \sqrt{-1}\,\sigma_2 , \quad k = \sqrt{-1}\,\sigma_3 ,$$

the metric on S^4 is given by

$$ds = \frac{dt^2 + dx^2 + dy^2 + dz^2}{(1 + r^2)^2} \qquad r = t^2 + x^2 + y^2 + z^2$$

while the gauge potential is $(i = \sqrt{-1})$

$$A = \frac{i}{(1+r^2)^2}\Big[(t\,dx - x\,dt + z\,dy - y\,dz)\,\sigma_1 + (t\,dy - y\,dt + x\,dz - z\,dx)\,\sigma_2 + (t\,dz - z\,dt + y\,dx - x\,dy)\,\sigma_3 \Big]$$

which is just the local expression of the BPST solution defined everywhere but the south pole [48,49] .

5. Spinor structures

5.1 Definition of spinor structure

In order to define spinor fields on a manifold M one needs to introduce a spinor structure on it.

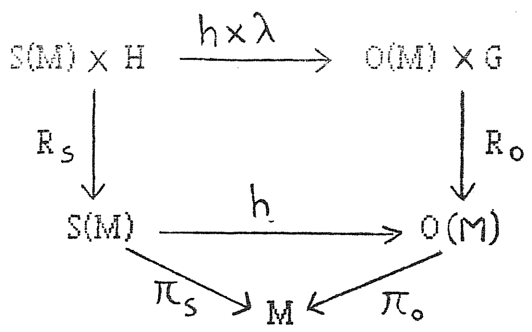
We shall consider only the case when M is space and time orientable.

Let G denote the orthogonal group $SO(n)$ or the proper Lorentz group $SO(1, n-1)$ with $n = \dim M$.

The first step consists in considering a restriction $O(M)$ (see paragraph 1.5), if it exists, of the frame bundle $F(M)$ to the group G. The principal bundle $O(M)$ is as $SO(n)$-bunde [$SO(1, n-1)$-bundle] of oriented orthonormal frames. Giving such a restriction is equivalent to giving a globally defined Riemannian [Lorentzian] metric on M together with a choice of space [space and time] orientation [49].

Let $\lambda : H \longrightarrow G$ be the standard covering homomorphism of the group G. Of course $H = Spin(n)$ for $G = SO(n)$ while $H = Spin(1, n-1)$ for $G = SO(1, n-1)$.

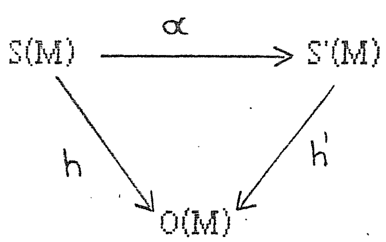
A (proper) spinor structure [50,51] on M, if it exists, is a prolongation (see paragraph 1.5) of $O(M))$ to H, i. e., a principal bundle $S(M)$ over M with structure group H together with a surjective submersion $h : S(M) \rightarrow O(M)$ such that the following diagram



commutes . A necessary and sufficient condition for the existence of a spinor structure on a manifold M is the vanishing of the second Stifel-Whitney class $w_2 \in H^2(M, Z_2)$ of M [50,51] .

Geroch [52] has proved that for a SO(1,3)-bundle of oriented orthonormal frames over a noncompact four-dimensional manifold M a spinor structure exists if and only if the bundle is trivial , i. e. , admits a global section ; moreover , the spinor bundle itself is then trivial .

On a given manifold M there may be more than one spinor structure . If (S , h) and (S' , h') are two such structures , they are said to be equivalent if there exists an isomorphism $\alpha : S \longrightarrow S'$ (see paragraph 1.5) so that the following diagram



commutes . The number of inequivalent spinor structures on M is equal to the number of elements in the cohomology group $H^1(M, Z_2)$ [51] .

5.2 Spinor fields

Let $h : S(M) \longrightarrow O(M)$ be a spinor structure over M and ρ a representation of H in a vector space V . A spinor field ψ of type (ρ, V) on M is a section of a vector bundle of typical fibre V , associated with $S(M)$ via the representation ρ . Equivalently , one may define a spinor field ψ of type (ρ, V) as an equivariant mapping $\psi : S(M) \longrightarrow V$ such that $\psi(u\Lambda) = \rho(\Lambda^{-1}) \circ \psi(u)$ for any $u \in S(M)$ and any $\Lambda \in H$.

5.3 Spinor connection

Let $h : S(M) \longrightarrow O(M)$ be a spinor structure over M . A spinor connection is a connection on the principal bundle [53,54] . Such a connection is determined by a connection on $O(M)$. Although h is surjective , its derivative h' is an isomorphism on each fibre and it may be used to lift the horizontal space H_p , $p \in O(M)$ to a horizontal space at $u \in S(M)$ such that $p = h(u)$.

If $\omega_o : T\,O(M) \longrightarrow \mathcal{G}$ is the connection 1-form on $O(M)$, the corresponding connection 1-form ω_s on $S(M)$

$$\omega_s : T\,S(M) \longrightarrow \mathcal{H}$$

is defined by

$$\omega_s \doteq \lambda'^{-1} \circ h^* \omega_o \qquad (5.1)$$

where $\lambda' : \mathcal{H} \longrightarrow \mathcal{G}$ is the derived mapping of Lie algebras ;

λ' is an isomorphism since $\ker \lambda = Z_2$ is discrete .

In general the connection ω_o is any metric connection on M .

Let $s : U \longrightarrow S(M)$, $U \subset M$, be a local section of $S(M)$; then $h \circ s : U \longrightarrow O(M)$ is a local section of $O(M)$.

If $A = s^* \omega_s$ and $\Gamma = (h \circ s)^* \omega_o$ are the corresponding pullbacks of ω_s and ω_o to M , they are related by λ'

$$A = \lambda'^{-1} \circ \Gamma \qquad (5.2)$$

Having the connection 1-form it is possible to define the covariant derivative .

If $\psi : S(M) \longrightarrow V$ is a spinor field of type (ρ , V) , its covariant derivative is given by (2.3)

$$D\psi = d\psi + (\rho' \circ \omega_s)\psi \qquad (5.3)$$

The local expression of ψ in the gauge s is

$$\Psi = \psi \circ s : U \longrightarrow V \quad .$$

By defining $D\Psi = s^* D\psi$, the local expression of the covariant derivative (5.3) is given by

$$D\Psi = d\Psi + (\rho' \circ A)\Psi \qquad (5.4)$$

5.4 Spinor structures over oriented spheres give gauge fields

Let S^n be the oriented n-dimensional sphere with its standard metric. Remembering that its bundle of oriented orthonormal frames is diffeomorphic to $SO(n+1)$, one can easily show that the spinor bundle may be identified with $Spin(n+1)$ [54],

$$Spin(n+1) \longrightarrow SO(n+1) \longrightarrow S^n \tag{5.5}$$

Here the bundle map $h : S(M) \longrightarrow O(M)$ coincides with λ.

Particular cases of the sequence (5.5) describe relevant physical systems [54].

For $n = 2$, $Spin(3) \simeq SU(2) \simeq S^3$ and $Spin(2) \simeq U(1)$; the resulting spinor bundle is the Hopf bundle $S^3 \xrightarrow{U(1)} S^2$ and describes the magnetic pole of strength $g = \frac{\hbar c}{2e}$ [13].

For $n = 3$, $Spin(4) \simeq SU(2) \times SU(2) \simeq S^3 \times SU(2)$ and $Spin(3) \simeq SU(2)$ [55]; the spinor bundle $S^3 \times SU(2) \xrightarrow{SU(2)} S^3$ with the corresponding connection ω_S gives the meron solution of Yang-Mills equations [56].

For $n = 4$, $Spin(5) \simeq Sp(2)$ and $Spin(4) \simeq Sp(1) \times Sp(1) \simeq SU(2) \times SU(2)$ [55]; since $spin(4) \simeq su(2) \oplus su(2)$, ω_S may be split in two components both of which project to $Sp(2)/Sp(1) \simeq S^7$ which is the total space of the Hopf bundle $S^7 \xrightarrow{SU(2)} S^4$. These projections give the instanton and anti-instanton solutions of Yang-Mills equations [57,58] .

Another interesting case is $n = 8$. The resulting spinor structure gives the solution of the Yang-Mills equations in eight dimensions found by Grossman , Kephart and Stasheff [59] .

The GKS solution consists of a gauge field on S^8 with gauge group Spin(8) . Moreover , it is invariant under the action of Spin(9) on S^8 [21] . We shall show that this gauge field is of the form (5.2) with the connection Γ chosen to be the Levi-Civita connection of S^8 .

In stereographic coordinates , obtained by projecting from the north or south pole onto the equatorial plane identified with R^8 , the restriction of the flat metric of R^9 is given by

$$g = \frac{4}{(1+x^2)^2} \sum_{\mu=1}^{8} (dx^\mu)^2 \ , \quad x^2 = \sum_{\mu=1}^{8} (x^\mu)^2 \tag{5.6}$$

By using the orthonormal basis

$$e^\mu = \frac{2}{1+x^2} dx^\mu \qquad \mu = 1, \dots, 8 \tag{5.7}$$

one gets the Levi-Civita connection 1-forms [60]

$$\Gamma^{\mu\nu} = x^\mu e^\nu - x^\nu e^\mu = \frac{2}{1+x^2}(x^\mu dx^\nu - x^\nu dx^\mu) \tag{5.8}$$

and the corresponding curvature 2-forms

$$R^{\mu\nu} = e^\mu \wedge e^\nu = \frac{4}{(1+x^2)^2} dx^\mu \wedge dx^\nu \ . \tag{5.9}$$

Let $\{ \hat{S}_{\mu\nu} \}$, $(\mu, \nu = 1, \dots, 8 \ ; \ \hat{S}^T_{\mu\nu} = - \hat{S}_{\mu\nu} = \hat{S}_{\nu\mu})$, be a basis of so(8) . The connection and the curvature may be put in a matrix form

$$\hat{\Gamma} = \frac{1}{2} \Gamma^{\mu\nu} \hat{S}_{\mu\nu} = \hat{\Gamma}_{\mu} \, dx^{\mu} \tag{5.10}$$

$$\hat{R} = \frac{1}{2} R^{\mu\nu} \hat{S}_{\mu\nu} = \frac{1}{2} \hat{R}_{\mu\nu} \, dx^{\mu} \wedge dx^{\nu} \tag{5.11}$$

where the quantities with hat are matrices .

From the expressions (5.8) and (5.9) one obtains

$$\hat{\Gamma}_{\mu} = - \frac{2}{1+x^2} \hat{S}_{\mu\nu} \, dx^{\nu} \tag{5.12}$$

$$\hat{R}_{\mu\nu} = \frac{4}{(1+x^2)^2} \hat{S}_{\mu\nu} \tag{5.13}$$

If $\lambda'^{-1} : so(8) \longrightarrow spin(8)$ is given by $\lambda'^{-1}(\hat{S}_{\mu\nu}) = i \, \hat{\Sigma}_{\mu\nu}$,

then the matrices $\hat{\Sigma}_{\mu\nu}$, $(\mu , \nu = 1 , \ldots , 8 \; ; \; \hat{\Sigma}_{\mu\nu} + \hat{\Sigma}_{\nu\mu} = 0)$,

are hermitian and the set $\{ i \, \Sigma_{\mu\nu} \}$ gives a basis of spin(8) .

The spinor connection obtained by lifting (5.12) is

$$\hat{A}_{\mu} = - \frac{2}{1+x^2} \, i \, \hat{\Sigma}_{\mu\nu} \, x^{\nu} \tag{5.14}$$

and coincides with the gauge potential derived in [21] . The corresponding field strenght is

$$\hat{F}_{\mu\nu} = \frac{4}{(1+x^2)^2} \, i \, \hat{\Sigma}_{\mu\nu} \tag{5.15}$$

Grossman , Kephart and Stasheff characterize their solution with a topological charge which they identify with the Euler number χ_S of the Spin(8)-bundle over S^8 .

The Euler number χ_S is defined by [61]

$$\chi_S = \int_{S^8} Pf\left(\frac{\hat{F}}{2\pi} \right) \tag{5.16}$$

where Pf is the Pfaffian and $\hat{F} = \frac{1}{2} F_{\mu\nu} \, dx^{\mu} \wedge dx^{\nu}$.

Since the Pfaffian is defined only for antisymmetric matrices , formula

(5.16) makes sense only because it is possible to embed spin(8) in so(8) . This possibility is related to the Cartan's triality principle [62] and we shall show how to realize it explicitly .

Let $\hat{\Gamma}_\mu$, $\mu = 1, \dots, 8$, be eight anticommuting hermitian matrices

$$\hat{\Gamma}_\mu \hat{\Gamma}_\nu + \hat{\Gamma}_\nu \hat{\Gamma}_\mu = 2\,\delta_{\mu\nu} \qquad \mu,\nu = 1, \dots, 8 \quad . \tag{5.17}$$

Then the matrices $i\,\hat{\Sigma}_{\mu\nu}$ are given by

$$i\,\hat{\Sigma}_{\mu\nu} = \frac{1}{4}\left[\hat{\Gamma}_\mu , \hat{\Gamma}_\nu\right] = \frac{1}{2}\hat{\Gamma}_\mu \hat{\Gamma}_\nu \quad . \tag{5.18}$$

If the matrices $\hat{\Gamma}_\mu$ are real , the matrices $i\,\hat{\Sigma}_{\mu\nu}$ will be antisymmetric and will provide a representstion of spin(8) in terms of antisymmetric matrices . A possible choice for the $\hat{\Gamma}_\mu$'s is the following [63]

$$\begin{aligned} &\hat{\Gamma}_1 = \lambda_{1000} , \quad \hat{\Gamma}_2 = \lambda_{2200} , \quad \hat{\Gamma}_3 = \lambda_{2120} , \quad \hat{\Gamma}_4 = \lambda_{2112} \\ &\hat{\Gamma}_5 = \lambda_{2132} , \quad \hat{\Gamma}_6 = \lambda_{2302} , \quad \hat{\Gamma}_7 = \lambda_{2321} , \quad \hat{\Gamma}_8 = \lambda_{2323} \end{aligned} \tag{5.19}$$

Here

$$\lambda_{ijk\ell} = \lambda_i \otimes \lambda_j \otimes \lambda_k \otimes \lambda_\ell \tag{5.20}$$

with $\lambda_1 = \sigma_1$, $\lambda_2 = -i\,\sigma_2$, $\lambda_3 = \sigma_3$ and σ_1 , σ_2 , σ_3 the three Pauli matrices ; λ_0 is the 2×2 unit matrix .

The matrix (5.19) are of the form

$$\hat{\Gamma}_1 = \begin{pmatrix} 0 & I_8 \\ I_8 & 0 \end{pmatrix}$$

$$\hat{\Gamma}_\mu = \begin{pmatrix} 0 & -\gamma_\mu \\ \gamma_\mu & 0 \end{pmatrix} \qquad , \quad \mu = 2, \ldots, 8 \tag{5.21}$$

and the γ_μ's are seven 8×8 anticommuting matrices .

The matrices $i\,\hat{\Sigma}_{\mu\nu}$ are

$$i\,\hat{\Sigma}_{\mu\nu} = \begin{pmatrix} i\,\hat{\Sigma}^{(+)}_{\mu\nu} & 0 \\ 0 & i\,\hat{\Sigma}^{(-)}_{\mu\nu} \end{pmatrix} \tag{5.22}$$

with

$$i\,\hat{\Sigma}^{(+)}_{\mu\nu} = \begin{cases} \frac{1}{2}\gamma_\nu & \mu = 1 \ , \quad \nu = 2, \ldots, 8 \\ -\frac{1}{2}\gamma_\mu\gamma_\nu & \mu, \nu = 2, \ldots, 8 \end{cases} \tag{5.23}$$

and

$$i\,\hat{\Sigma}^{(-)}_{\mu\nu} = \begin{cases} -\frac{1}{2}\gamma_\nu & \mu = 1 \ , \quad \nu = 2, \ldots, 8 \\ -\frac{1}{2}\gamma_\mu\gamma_\nu & \mu, \nu = 2, \ldots, 8 \end{cases} \tag{5.24}$$

The two sets $\{ i\,\hat{\Sigma}^{(+)}_{\mu\nu} \}$ and $\{ i\,\hat{\Sigma}^{(-)}_{\mu\nu} \}$ of 8×8 antisymmetric metrices give the two semispinor representations of spin(8) .

Owing to (5.15) , it is possible to break the solution (5.15) in two pieces

$$\hat{F}^{(\pm)}_{\mu\nu} = \frac{4}{(1+x^2)^2}\, i\,\hat{\Sigma}^{(\pm)}_{\mu\nu} \tag{5.25}$$

and we shall see that the two pieces are characterized by opposite values of the Euler number .

The situation here is analogous to the one in four dimensions where it is possible to put together the instanton and anti-instanton solution of the

Yang-Mills equations by extending the gauge group to

$Spin(4) \simeq SU(2) \times SU(2)$ [57] .

By using the formulae for the Pfaffian developped by Caianiello [64] ,

after some algebra one gets

$$Pf\left(\frac{\hat{F}^{(\pm)}}{2\pi}\right) = \frac{840}{(4\pi)^4} e^1 \wedge \dots \wedge e^8 \tag{5.27}$$

and this gives

$$\chi_S^{(\pm)} = \int_{S^8} Pf\left(\frac{\hat{F}^{(\pm)}}{2\pi}\right) = \frac{840}{(4\pi)^4} \, vol(S^8) = \pm 1 \quad . \tag{5.28}$$

These values for the Euler numbers have to be compared with the Euler number of the tangent bundle of S^8 which is $\chi = 2$.

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