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"RINORMALIZZAZIONE IN SPAZI-TEMPO CURVI:

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# « Renormalization in Curved Space-Time : a Brief Revue and Applications »

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§1. Introduction

The aim of this thesis is to revue some ideas that have been developped in the last twenty years by researchers working on quantum field theory in a curved geometry. A particular attention will be payed to topics which have some cosmological relevance.

The first and most relevant problem one is faced with, when field quantization is performed in curved space time, is the appearance of infinities in any computed physical quantities.
The presence of such infinities, far from being an annoying feature of the theory is an unavoidable consequence of quantization when an external field is present. A similar problem arises in quantum electrodynamics and techniques have been developped many years ago (see e.g. Schwinger, 1951) which turn out to be very useful in the present context.

In quantum electrodynamics it is well known that the field strength, in the vacuum state, « undergoes random fluctuations completely analogous to the zero-point oscillations of harmonic oscillators, and that when couplings to the electron field are taken into account these fluctuations are accompanied by pair creation and annihilation events » (De Witt, 1979).

The fact that the vacuum state has such interesting properties seems rather puzzling. One would expect naively the vacuum to be a state simply with no properties at all! In quantum field theory, however, one must define the vacuum state to be the state without any particles, but the actual properties of this state are dictated by the laws of physics « and we can not insist ... that the simplest possible situation be as simple as we wish » (Zel'dovich, 1981).

In our case we deal with quantum fields which live in a curved geometry, that is, in an external gravitational field. In such a world the notion of vacuum, or no-particle state, is inherently ambiguous: it depends on which point of the space-time one defines it (in this sense is a local concept); it depends on the observer who makes measurements (in this sense is an observer or frame-dependent concept); it depends on the topology of the space-time on the whole (in this sense is a global concept).

The situation is so much paradoxical that all informations about the physical system can be derived from the knowledge of how the probability for the field to remain in the vacuum state changes as the external field (the "source") is changed.

These statements are formulated more precisely in § 2 where we define the effective action which is the natural logarithm of the vacuum-to-vacuum transition amplitude. By functionally differentiating the effective action one obtains (a suitable expectation value of) the stress tensor, that is, the quantity which appears at the r.h.s. of the Einstein's equations telling the space-time (as a back-reaction effect) how to curve.

As in strong, or rapidly varying, electric fields, particles are created, in pairs, when strong curvatures, or rapidly varying curvatures, are involved; still the pair creation probability is obtained from the vacuum-to-vacuum transition amplitude.

As we said at the beginning, there are infinities in the theory, e.g. in the stress tensor, just coming from these vacuum fluctuations; the problem then is how to remove diverging quantities without removing or altering such physical properties of the vacuum: this is "renormalization" and is described in §§ 3 and 4. It turns out that infinities can be expressed by means of invariants of the metric (that is of the "source") with infinite constants of proportionality. Then they have all the qualities for going in the r.h.s. of Einstein's equation, where they change the initial ("bare") values of the cosmological constant, the gravitational constant and other, non-classical, coupling constants.

This fact is so impressive that many physicists (see the review by Adler, 1982) were led to speculate that gravity itself (namely the gravitational constant) is a consequence of vacuum fluctuations. This idea was first formulated by Sakharov who said « The presence of the [gravitational] action leads to a metrical elasticity of space, i.e., to generalized forces which oppose the curving of space.... Here we consider the hypothesis which identifies the [gravitational] action with the change in the action of quantum fluctuations of the vacuum if space is curved » ( Sakharov, 1967).

We shall not deal in this thesis with such ideas which now bring the name of "induced gravity", rather we show the reader how to calculate the effects of vacuum fluctuations, at least in the most simple cases. Amongst such effects there are the so-called "trace anomalies", that is a non-vanishing trace of the renormalized stress tensor in cases when a particular invariance (the "conformal invariance") of the classical theory predicts a zero value. These are discussed in §§ 5 and 6. Trace anomalies which are expressed by curvature squared terms can be very relevant in the early Universe at scales much longer than the Planck scale, $\lambda_{Pl} \sim 10^{-33}$ cm ( see e.g. Ginzburg et al., 1971).

In § 7 the possibility of treating a quantum field in a curved geometry as a thermodynamical equilibrium system is briefly discussed. It seems that these concepts are rather difficult to be used in cosmology where the Universe expansion may sensibly alter initially thermal distribution functions (in this respect see Hu, 1982, and Bonometto and Matarrese, 1982).

Finally in §§ 8 and 9 a renormalization program is applied to the rather simple case of a massive vector (Proca) field in a Friedmannian universe. Finite expressions are derived for the stress tensor and, in § 9, the "one-loop" corrections to the bare cosmological and gravitational constants (and to a third constant 'b') coming from this kind of field are calculated. These last two sections mainly follow the line of papers (Matarrese and Bonometto, 1981, 1982).

Units $c = \hbar = G = k_B = 1$ are used wherever; sign conventions of De Witt, 1964, are used, apart §§ 8 and 9. The symbol $\nabla_\mu$ indicates Einstein covariant derivative; $\Box \equiv \nabla_\mu \nabla^\mu$.

## § 2 The effective action

In order to understand the general theory of renormalization in curved space-time we shall restrict ourselves to the simplest case of a neutral scalar field $\phi$ described by the action (we use the signature $-+++$; $g \equiv -\det g_{\mu\nu}$)

$$S[\phi] = -\frac{1}{2}\int g^{1/2} d^4x \left[ \nabla_\mu \phi \nabla^\mu \phi - (m^2 + \xi R)\phi^2 \right] \quad (2.1)$$

($\nabla_\mu$ indicates the covariant derivative) which, by the Gauss theorem, can be also written in the form

$$S[\phi] = \frac{1}{2}\int d^4x \, \phi F \phi \quad (2.2)$$

being $F$ the differential operator

$$F = g^{1/2}\left( \Box - m^2 - \xi R \right) \quad (2.3)$$

We require the system to have some initial "in-region" and final "out-region" on which we define vacuum states $|in, vac\rangle$ and $|out, vac\rangle$ respectively. All dynamics occurs in the region separating these in and out regions. It was argued by De Witt

(1975) that the actual existence of such two regions is inessential to the final results.

The most important quantity to be defined is the so-called "vacuum-to-vacuum" or "vacuum persistence amplitude" defined by

$$\langle out, vac \mid in, vac \rangle \equiv e^{i\mathcal{W}} \tag{2,4}$$

We shall show that all informations about the system can be gained from the quantity $\mathcal{W}$ which is commonly called the "effective action".

If we make an infinitesimal change $\delta g_{\mu\nu}$ in the background geometry, in the region of dynamical interest, the quantity $\mathcal{W}$, thought as a functional of the metric, suffers a change

$$\delta\mathcal{W} = -i e^{-i\mathcal{W}} \delta e^{i\mathcal{W}} = -i e^{-i\mathcal{W}} \delta \langle out, vac \mid in, vac \rangle \tag{2,5}$$

The Schwinger variational principle (Schwinger, 1951) states that the change in the background field induces a change in the classical action such that

$$\delta \langle out, vac \mid in, vac \rangle = i \langle out, vac \mid \delta S \mid in, vac \rangle \tag{2,6}$$

thus, from (2.5), (2.6)

$$\delta W = e^{-iW} \langle out, vac | \delta S | in, vac \rangle \tag{2.7}$$

The expression (2.6) is actually an infinitesimal substitute to the functional integral (this can be found in many textbooks, see e.g. Itzykson and Zuber, 1980, chapter IX)

$$\langle out, vac | in, vac \rangle = e^{iW} = \int d[\phi]\, e^{iS[\phi]} \tag{2.8}$$

$d[\phi]$ being a measure over the space of all field configurations $\phi$ satisfying the given boundary conditions ($d[\phi]$ is intended to include also normalizing factors). We do not need here to specify better the meaning of the "path integral" (2.8).

Let us return now to eq. (2.7); remembering the definition of the classical stress tensor

$$T_{\mu\nu} = 2 g^{-1/2} \frac{\delta S}{\delta g^{\mu\nu}} \tag{2.9}$$

one immediately realizes that

$$\langle T_{\mu\nu} \rangle_{matrix} = 2 g^{-1/2} \frac{\delta W}{\delta g^{\mu\nu}} \tag{2.10}$$

where we used the definition

$$\langle \mathcal{O} \rangle_{matrix} \equiv \frac{\langle out, vac | \mathcal{O} | in, vac \rangle}{\langle out, vac | in, vac \rangle} \qquad (2.11)$$

for the operator $\mathcal{O}$.

The quantity $\langle T_{\mu\nu} \rangle_{matrix}$ is actually a matrix element; it can be related to a true expectation value like

$$\langle T_{\mu\nu} \rangle_{vac} \equiv \frac{\langle in, vac | T_{\mu\nu} | in, vac \rangle}{\langle in, vac | in, vac \rangle} \qquad (2.12)$$

if one knows how to express $|out, vac\rangle$ in terms of $|in, vac\rangle$ which requires a complete knowledge of the "S-matrix" with allowance of the particle production rate.

In effect when particle production occurs the in and out vacua will differ by more than a phase factor (see (2.4)); the total particle production probability $P$ is

$$1 - |\langle out, vac | in, vac \rangle|^2 . \qquad (2.13)$$

In order to understand this point one must realize that $|e^{i\mathcal{W}}|^2$ is just the probability that no real particles are produced during the dynamical history of the field. Then for small $P$ one simply has

$$P \approx 2\,\mathrm{Im}\,W \,. \tag{2.14}$$

It is then customary to express such a quantity in terms of the so-called Bogoliubov coefficients (see e.g. De Witt, 1975) but we shall not be concerned here with this problem.

Let us return to the stress-energy tensor: in a semiclassical approach to the Einstein's equation it is a quantity like $\langle T_{\mu\nu}\rangle_{vac}$ (2.12) which enters self-consistently as a source in its r.h.s.; however $\langle T_{\mu\nu}\rangle_{vac}$ is a diverging quantity and a convenient renormalization prescription should be used in order that it acquires its correct physical meaning.

We shall have in general

$$\langle T_{\mu\nu}\rangle_{vac} = 2g^{-1/2}\frac{\delta W}{\delta g^{\mu\nu}} + \vartheta_{\mu\nu} \;; \tag{2.15}$$

the tensor $\vartheta_{\mu\nu}$ is certainly zero when particle production is absent, it is also finite as De Witt

(1975), has shown, and obeys the conservation equation

$$\nabla^{\mu} \vartheta_{\mu\nu} = 0 \quad . \tag{2.16}$$

Thus all divergences appearing in $\langle T_{\mu\nu} \rangle_{vac}$ come from the diverging part of $W$. A gauge-invariant (which in this case means generally covariant) prescription for isolating such infinities will be the subject of §§ 3 ÷ 5.

## § 3. The heat kernel expansion

In equation (2,11) we defined $\langle \Theta \rangle_{matrix}$ : this will be called the "Schwinger average" of the operator $\Theta$. With a Schwinger average of the chronological product $[T(\ldots)]$ of $\phi(x)$ and $\phi(y)$ one defines the Feynman propagator $G(x,y)$ relative to the in and out regions :

$$e^{-iW} \langle out, vac | T(\phi(x)\phi(y)) | in, vac \rangle = -iG(x,y) \quad ; \quad (3,1)$$

it obeys the equation

$$F G(x,y) = -\delta(x,y) \tag{3.2}$$

with $F$ the differential operator (2,3) acting on the first argument of $G$ and such that $F\phi = 0$. If we adopt a symbolic notation with 1 standing for $\delta(x,y)$ equation (3.2) has the formal solution

$$G = -F^{-1} \tag{3.3}$$

which can be also written as

$$g^{1/4} G g^{1/4} = i \int_0^\infty \exp(i g^{-1/4} F g^{-1/4} s)\, ds \tag{3.4}$$

having used the identity

$$\int_0^\infty e^{-i(z-i\epsilon)s}\,ds = \frac{1}{i(z-i\epsilon)}$$

where $\epsilon$ is a small positive quantity which must be inserted in order to avoid the singularities of $F^{-1}$ and removed by a limit procedure at the end of calculations.

Equation (3.4) is known as the Schwinger - De Witt representation of the Feynman propagator; the parameter $s$ is called the "Schwinger proper time". Eq. (3.4) is also equivalent to

$$G(x,x') = i\int_0^\infty g^{-1/4}(x)\, K(x,x';s)\, g^{-1/4}(x')\,ds \qquad (3.5)$$

with the kernel K satisfying the first order differential equation

$$\frac{\partial}{i\partial s} K(x,x';s) = g^{-1/4}(x)\, F\, g^{-1/4}\, K(x,x';s) \qquad (3.6)$$

with the initial condition

$$K(x,x';0) = \delta(x,x'). \qquad (3.7)$$

Equation (3.6) is known to have a nice interpretation which turns out to be very useful; if one puts $is = \varpi$ and assumes an Euclidean signature $(++++)$ — this can be achieved by Wick rotating the time coordinate — then eq. (3.6) becomes the "heat" or "diffusion equation" and the "heat kernel" $K$ describes the diffusion over the spacetime in a "time" $\varpi$ of a unit quantity of heat placed at the point $x'$ when $\varpi = 0$ (see Hawking, 1977). This equation has been extensively studied by mathematicians and many important properties have been derived (see e.g. Gilkey, 1974).

A suitable expression of $K$ can be written in analogy with the Minkowskian expression for $G_1$ (see De Witt, 1964) which reads

$$G_1(x,x') = \frac{1}{(4\pi)^2} \int_0^\infty \frac{1}{s^2} \exp\left[ i\frac{(x-x')^2}{4s} - im^2 s \right] .$$

One introduces the biscalar (*) $\sigma(x,x')$ equal to

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(*) We recall that an $n$-tensor $T_{\mu_1 \dots \mu_e, \nu_1' \dots \nu_m', \sigma_1'' \dots \sigma_n''}$ is a quantity which transforms like the product of $n$ tensors one at each space-time point:

$$A_{\mu_1 \dots \mu_e}(x)\, B_{\nu_1 \dots \nu_m}(x') \cdots F_{\sigma_1 \dots \sigma_n}(x''') .$$

one half of the square of the geodetic distance between $x$ and $x'$ ($\sigma$ is called "geodetic interval" or "world function"), and the Van Vleck - Morette determinant

$$D(x,x') = -\det(-\nabla_{\nu'}\nabla_{\mu}\sigma(x,x')) .$$

The following properties of $D$ and $\sigma$ have been proved (see e.g. Christensen, 1976)

$$\nabla_{\mu}\sigma\nabla^{\mu}\sigma = \nabla_{\mu'}\sigma\nabla^{\mu'}\sigma = 2\sigma \tag{3.8}$$

$$2\nabla_{\mu}D^{1/2}\nabla^{\mu}\sigma + D^{1/2}\Box\sigma = 4D^{1/2} \tag{3.9}$$

then one can write

$$K(x,x',s) = \frac{i\,D^{1/2}(x,x')}{(4\pi i s)^2}\,\Omega(x,x',s)\exp\left[\frac{i}{2s}\sigma(x,x') - i m^2 s\right] \tag{3.10}$$

with the initial condition

$$\Omega(x,x',0) = 1 \quad , \tag{3.11}$$

coming from (3.7).

Inserting this expression into eq. (3.6) and using (3.8), (3.9) we get

$$\frac{\partial}{i\partial s}\Omega + \frac{1}{is}\nabla_\rho \sigma \nabla^\rho \sigma = \Delta^{-1/2}(\Box - \xi R)(\Delta^{1/2}\Omega) \qquad (3.12)$$

$\Delta$ being the biscalar

$$\Delta(x,x') = g^{-1/2}(x)\, D(x,x')\, g^{-1/2}(x') \qquad (3.13)$$

When $x'$ lies very close to $x$ and $s$ is small one can write $\Omega$ as a power expansion

$$\Omega(x,x';s) \sim \sum_{r=0}^{\infty} a_r(x,x')(is)^r \qquad (3.14)$$

The expression (3.10) with $\Omega$ written in the form (3.14) is known as the "heat kernel expansion". When substitud into (3.12) the expansion (3.14) yields the recursive relation

$$g^{\mu\nu}\partial_\mu \sigma \partial_\nu a_{r+1} + (r+1)a_{r+1} = \Delta^{-1/2}(\Box - \xi R)(\Delta^{1/2}a_r)$$
$$r = 0,1,2,.. \qquad (3.15)$$

with

$$a_0(x,x') = \Omega(x,x',0) = 1 \qquad (3.16)$$

In many renormielization methods one is really interested only in the coincidence limits ($\lim_{x\to x'}$)

$$a_r(x,x) = \lim_{x \to x'} a_r(x,x')$$

that can be obtained from eq. (3.15) and its first derivatives using the coincidence limits of (3.8), (3.9) and their derivatives (these expressions can be found e.g. in Christensen, 1976).

It is clear that the $a_r(x,x)$ are built only with the metric tensor and its derivatives at the point $x$ in a coordinate-independent way.

We can formally integrate $K(x,x,s)$ obtaining

$$K(s) \equiv \int K(x,x,s)\, d^4x \sim \frac{i e^{-im^2 s}}{(4\pi i s)^2} \sum_{r=0}^{\infty} A_r (is)^r \tag{3.17}$$

with

$$A_r[g] = \int a_r(x,x)\, g^{1/2} d^4x \tag{3.18}$$

When the space-time manifold has a non-null boundary the integrated heat kernel $K(s)$ also contains boundary terms which are responsible for many interesting effects like thermal properties of black-holes etc.. (see e.g. Kennedy, 1981, and Denardo and Spallucci, 1980)

We shall not at all be concerned in the fol lowing with such terms that have been widely stu died in the literature (see e.g. Mc Kean and Singer, 1967); thus for simplicity we shall assume that no boundary exists.

## § 4 The Schwinger prescription

Let us return to equation (2,7) that in the new notation (2,11) reads

$$\delta \mathcal{W} = \langle \delta S \rangle_{matrix} \quad ; \qquad (4.1)$$

the change $\delta g_{\mu\nu}$ produces a change $\delta F$ in $S$ (see (2,2)) such that

$$\delta \mathcal{W} = \frac{1}{2} \int d^4x \langle \phi \, \delta F \phi \rangle_{matrix} \quad ; \qquad (4.2)$$

De Witt (1980) has shown that the following relation holds

$$\delta \mathcal{W} = -\frac{i}{2} \mathrm{Tr}\left[ g^{1/4} G \, g^{1/4} \delta \left( g^{-1/4} F g^{-1/4} \right) \right] \quad . \qquad (4.3)$$

where the trace symbol involves the integration over the space-time as well as a coincidence limit (from a space-like direction) of the arguments $x$ and $x'$. Recalling the Schwinger - De Witt representation (3.4) one gets the variational equation

$$\delta \mathcal{W} = \delta \left[ \frac{1}{2} \mathrm{Tr} \int_0^\infty \frac{1}{is} \exp\left( i g^{-1/4} F g^{-1/4} s \right) ds \right] \qquad (4.4)$$

sion is thus incapable of yielding the imaginary part of $\mathcal{W}$ but it is enough for the evaluation of divergences.

The Schwinger prescription now is to remove from the effective action the first three terms in (4.6)

$$\mathcal{W}_{div} \equiv \frac{1}{32\pi^2}\left(\frac{1}{2} m^4 A_0 - m^2 A_1 + A_2\right)\Gamma(0) \qquad (4.7)$$

(having used the property $\Gamma(-n) = (-1)^n \Gamma(0)/n!$).
The behaviour of $\Gamma(x)$ in a neighbourhood of the origin is given by the formula

$$\Gamma(\epsilon) \simeq \frac{1}{\epsilon} - \gamma + O(\epsilon) \qquad (4.8)$$

($\gamma \simeq 0.5772$ is the Euler constant) telling us that $\mathcal{W}$ contains an infinite contribution but also a finite remainder.

The renormalized effective action $\mathcal{W}_{ren}$ then simply is

$$\mathcal{W}_{ren}[g] \equiv \mathcal{W}[g] - \mathcal{W}_{div}[g] \qquad (4.9)$$

and the renormalized v.e.v. (vacuum expectation value) of the stress-tensor is obtained from (2.15) and (4.9)

$$\langle T_{\mu\nu} \rangle_{vac,ren} = 2 g^{-1/2} \frac{\delta W}{\delta g^{\mu\nu}} + \vartheta_{\mu\nu} \tag{4.10}$$

The subtracted diverging part of $\langle T_{\mu\nu} \rangle_{vac}$ must find suitable counterterms in the l.h.s. of the Einstein's equation, or, if you like, in the gravitational action.

As we discussed at the end of § 3 the three coefficients $A_0, A_1, A_2$ are built with local invariants (scalars) of the metric. It is an easy task to deduce that the only invariants of suitable canonical dimensions are

$$\int g^{1/2} d^4x \,, \int g^{1/2} R \, d^4x \,, \int g^{1/2} R^2 d^4x \,,$$

$$\int g^{1/2} R_{\mu\nu} R^{\mu\nu} d^4x \,, \int g^{1/2} R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} d^4x \,. \tag{4.11}$$

However the number of independent invariants which are quadratic in the curvature is reduced by one because of the Gauss-Bonnet theorem in 4-dimensions that can be put in the form

$$\frac{\delta}{\delta g^{\alpha\beta}} \int g^{1/2} G \, d^4x \,,$$

$$G = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2 . \tag{4.12}$$

This holds for changes in the metric which do not alter the global topology of the manifold; actually the quantity $G$ in (4.12) is the Euler topological invariant in four dimensions, apart from a multiplicative constant.

Then one is led to renormalize the bare cosmological constant, the bare gravitational constant as well as two other constants multiplying the curvature squared invariants. The final, finite value, of these constants is not fixed by the theory, it must be gained from experiments.

## § 5 The stress tensor trace anomaly

Let us consider the following infinitesimal conformal (Weyl) transformations

$$\delta g_{\mu\nu} = g_{\mu\nu}\,\delta\lambda \tag{5.1}$$

$$\delta\phi = -\frac{1}{2}\phi\,\delta\lambda$$

with $\delta\lambda(x)$ an arbitrary infinitesimal scalar function; the change induced on the effective action $W$, thought as a functional of the metric, is, by definition (see also (2.10))

$$\delta W[g] \equiv \int d^4x \frac{\delta W}{\delta g_{\mu\nu}}\,\delta g_{\mu\nu}$$

$$= \frac{1}{2}\int d^4x\, g^{1/2} \langle T_\mu{}^\mu \rangle_{matrix}\,\delta\lambda \tag{5.2}$$

The invariance of the classical action under transformations (5.1), (5.2) is achieved in the scalar case by putting $\xi = 1/6$ and $m = 0$ in (2.1).

Equation (2.7) shows us that $W$ mantains such an invariance of $S$. Then (5.3) proves that if the classical action is invariant under (5.1), (5.2) then the

Schwinger averaged trace of the stress tensor vanishes:

$$\langle T_\mu{}^\mu \rangle_{matrix} = 0 \qquad (5.4)$$

Looking at (2.10), (4.7) we are led to define in an obvious manner

$$\langle T_{\mu\nu} \rangle_{matrix, ren} = \langle T_{\mu\nu} \rangle_{matrix} - \langle T_{\mu\nu} \rangle_{matrix, div} \qquad (5.5)$$

As is clear from the expansion (4.8) the regularization scheme involves a limit procedure on some parameter $\epsilon$ for $\epsilon \to 0$. As $\epsilon \to 0$ the diverging part of $\langle T_{\mu\nu} \rangle$ goes to infinity; however if the theory is conformally invariant equation (5.4) holds, and from (5.5) one gets (Bunch, 1979)

$$g^{\mu\nu} \langle T_{\mu\nu} \rangle_{matrix, ren} = - \lim_{\epsilon \to 0} g^{\mu\nu} \langle T_{\mu\nu}(\epsilon) \rangle_{matrix, div} \qquad (5.6)$$

It is clear that the r.h.s. of (5.6), which must be finite – in spite of its name –, cannot be simply evaluated by putting $m = 0$ into (4.7). In fact when $m = 0$ the factor $e^{-im^2 s}$ of $K(s)$ (see (3.17)) desappears: as $s \to \infty$ nothing con trols the convergence of (4.5), troubles come out from the $r=2$ term of (3.17) leading to infrared divergences.

We shall show in §6 that the r.h.s. of (5.6) when correctly evaluated is finite and different from zero: a "conformal anomaly" (violation of conformal invariance) is present in the renormalized theory, giving rise to a non-vanishing trace of $\langle T_{\mu\nu} \rangle_{matrix, ren}$, the so-called "trace anomaly".

Let us quote here the final form of the anomalous trace (see e.g. Gibbons, 1979)

$$\langle T_\mu{}^\mu \rangle_{anomalous} \equiv \langle T_\mu{}^\mu \rangle_{matrix, ren} = (4\pi)^{-2} a_2(x,x) \qquad (5.7)$$

$a_2(x,x)$ being the coincidence limit of the $r=2$ term in (3.14). These coefficients have been evaluated for different fields (see Christensen, 1978); the general expression for $a_2$ is (Duff, 1977)

$$a_2(x,x) = (1/180)\left[\alpha R^2 + \beta R_{\alpha\beta} R^{\alpha\beta} + \gamma R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + \right.$$
$$\left. + \eta \Box R \right] \qquad (5.8)$$

where

$$\alpha = 0 \quad , \quad \beta = -1 \quad , \quad \gamma = 1 \quad , \quad \eta = 1 \tag{5.9}$$

for conformally invariant scalars,

$$\alpha = -\frac{5}{4} \quad , \quad \beta = 2 \quad , \quad \gamma = \frac{7}{4} \quad , \quad \eta = 3 \tag{5.10}$$

for massless two-component spinors

$$\alpha = -25 \quad , \quad \beta = 88 \quad , \quad \gamma = -13 \quad , \quad \eta = -18 \tag{5.11}$$

for photons (Adler and Lieberman, 1978). The photon trace anomaly also includes contributions from the Faddeev-Popov ghosts (Dowker and Critchley, 1977). The coefficient $\eta$ in (5.11) has a different determination from $\eta' = 12$ given by dimensional regularization (Capper and Duff, 1974), the difference probably arising from the absence of a conformally invariant Maxwell theory in $N \neq 4$ dimensions.

The formula (5.8) can be simplified making use of the topological invariant $\mathcal{G}$ of (4.12) and of the conformal invariant

$$\mathcal{I} = C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 2 R_{\mu\nu} R^{\mu\nu} + \frac{1}{3} R^2 \tag{5.12}$$

$C_{\mu\nu\rho\sigma}$ being the conformal invariant Weyl tensor.

One gets

$$\langle T_\mu{}^\mu \rangle_{anomalous} = (4\pi)^{-2}(1/180)\{[\alpha + \tfrac{1}{3}(\beta+\gamma)]R^2 +$$

$$+ (\tfrac{1}{2}\beta + 2\gamma)\mathcal{F} - (\tfrac{1}{2}\beta + \gamma)\mathcal{G} + \eta\Box R\} \qquad (5.13)$$

One might ask whether the trace anomaly can be altered by the addition of finite counterterms to the effective action. It is easy to verify using (4.12) and the two relations

$$g^{-1/2} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int d^4x\, g^{1/2}\mathcal{F} = 0 \quad , \qquad (5.14)$$

$$g^{-1/2} g^{\mu\nu} \frac{\delta}{\delta g^{\mu\nu}} \int d^4x\, g^{1/2} R^2 = -6\Box R \quad , \qquad (5.15)$$

that the coefficient $\eta$ is indeed the only coefficient being sensitive to finite counterterms in the effective action (Parker, 1978)

One may easily verify that the coefficients in (5.9), (5.10) and (5.11) (using the $\eta'$ determination) are not at all independent but satisfy the two relations (Duff, 1977)

$$4\alpha + \beta = \alpha - \gamma = -\eta \qquad . \qquad (5.16)$$

then (5,13) becomes

$$\langle T_\mu{}^\mu \rangle_{\text{anomalous}} = (4\pi)^{-2}(1/120)\{\eta(\mathcal{F} + \tfrac{2}{3}\Box R) +$$

$$+ (\eta + \tfrac{2}{3}\gamma)\mathcal{G}\} \qquad (5.17)$$

Relations like (5,16) have of course a deep theoretical meaning relying on the so-called "gravitational index theorems" (see e.g. Christensen and Duff, 1978). A trace anomaly also occurs for non-conformally invariant theories, but there is a question there on which part of the resulting renormalized trace of the stress tensor must be regarded as anomalous; examples of this kind are the graviton trace anomaly (Critchley, 1978) and a trace anomaly in the massive vector case (Matarrese and Bonometto, 1982; see also § 8).

A relevant result showing the importance of trace anomalies has been proved by Brown and Cassidy (1977). Let us change infinitesimally the metric as in (5.1): the response of the renormalized Schwinger averaged stress tensor to such a variation is completely determined by its trace, as is shown by the equation (see e.g. De Witt, 1980)

$$\delta\left(g^{-1/2}\langle T_\mu{}^\nu(x)\rangle_{matrix,\,ren}\right) =$$

$$= g_{\mu\sigma}(x)\frac{\delta}{\delta g_{\nu\sigma}(x)}\int\langle T_\rho{}^\rho(x')\rangle_{matrix,\,ren}\,\delta\lambda(x')\,g^{1/2}(x')\,d^4x'\,; \qquad (5.18)$$

as a consequence for a conformally invariant field, the change in $\langle T_{\mu\nu}\rangle_{matrix,\,ren}$ for a conformal transformation of the metric is completely governed by its trace anomaly.

## § 6 The generalized Riemann zeta function

Let us consider the differential operator (see (2.3))

$$\hat{\Delta} \equiv -g^{-1/2} F - m^2 . = -\Box + \xi R \qquad (6.1)$$

$\hat{\Delta}$ is called the "Laplacian" for a neutral scalar field (this name comes by analogy with the ordinary Laplacian in three dimensional Euclidean space). Laplacians for higher spin fields can be found in the literature (see Christensen and Duff, 1979). If the space-time has an Euclidean signature $(++++)$ then $\hat{\Delta}$ is self-adjoint; in non Euclidean metrics $\hat{\Delta}$ will not be so, in general; however one can assume (Hawking, 1977) that it admits a complete set of eigenfunctions $\{\phi_n\}$ with eigenvalues $\lambda_n$; without loss of generality we can choose the $\phi_n$ to be real. Then we have

$$\hat{\Delta} \phi_n(x) = \lambda_n \phi_n(x) \qquad (6.2)$$

and the orthonormality relations

$$\int g^{1/2}(x) \phi_e(x) \phi_n(x) d^4x = \delta_{en} \qquad (6.3)$$

$$\sum_n g^{1/4}(x) \phi_n(x) \phi_n(x') g^{1/4}(x') = \delta(x,x') . \qquad (6.4)$$

The set $\{\phi_n\}$ gives us a possible representation of the heat kernel $K(x,x';s)$. In fact it is easily seen that the ansatz (see De Witt, 1979)

$$K(x,x';s) = \sum_n g^{1/4}(x)\,\phi_n(x)\,\phi_n(x')\,g^{1/4}(x')\cdot$$

$$\cdot \exp\left[-i(\lambda_n + m^2)s\right] \tag{6.5}$$

gives a (formal) solution of the equation (3.6) consistent with the initial condition (3.7).

From (6.5) one obtains an expression for the integrated heat kernel $K(s)$ defined in (3.17)

$$K(s) = \sum_n \exp\left[-i(\lambda_n + m^2)s\right] . \tag{6.6}$$

One can express the field itself by means of the eigenfunctions of $\hat{\Delta}$

$$\phi = \sum_n a_n \phi_n \tag{6.7}$$

(where the $a_n$ are real coefficients). This gives us a possible realization of the path integral (2.8), the measure $d[\phi]$ being expressible in terms of the coefficients $a_n$:

$$d[\phi] = \prod_n \mu \, da_n \qquad ; \tag{6.8}$$

where $\mu$ is a normalization constant having the dimensions of a mass. From (6.1) – (6.3) and (6.7) it follows

$$\int d^4x \, \phi F \phi = - \int d^4x \, g^{1/2} \sum_{n,l} (\lambda_n + m^2) a_l a_n \phi_l \phi_n$$

$$= - \sum_n (\lambda_n + m^2) a_n^2 \; . \tag{6.9}$$

Then from (2.8), (6.8) and (6.9) one gets

$$W[g] = - i \ln \prod_n \int \mu \, da_n \exp\left[ - \frac{i}{2} (\lambda_n + m^2) a_n^2 \right] \tag{6.10}$$

which, after performing a gaussian integral becomes

$$W[g] = \frac{i}{2} \sum_n \ln (\lambda_n + m^2) + \text{const} \tag{6.11}$$

One can define a generalization of the Riemann zeta function ($\zeta_R(z;q) \equiv \sum_{n=0}^{\infty} (n+q)^{-z}$) by means of the eigenvalues of $\hat{\Delta}$ (Dowker and Critchley, 1976; Hawking, 1977):

$$\zeta(z; m^2) \equiv \sum_n (\lambda_n + m^2)^{-z} \tag{6.12}$$

so that eq. (6.11) reads

$$\mathcal{W}[g] = -\frac{i}{2}\frac{d}{dz}\zeta(0;m^2) + \text{const} \tag{6.13}$$

The zeta function is related to the integrated heat kernel $K(s)$ (see (6.6)) via an inverse Mellin transform in the variable $w = is$

$$\zeta(z;m^2) = \frac{1}{\Gamma(z)}\int_0^\infty K(w)\, w^{z-1}\, dw \tag{6.14}$$

Using the heat kernel expansion (3.17) we see that $\zeta$ has simple poles at $z=2$, $z=1$ with residues proportional to $A_0$ and $A_1$ respectively. The logarithmic divergence of the integral at $z=0$ is cancelled by the pole in $\Gamma(0)$ leaving a finite residue proportional to $A_2$.

Let us consider now a scale transformation of the metric, that is a transformation of the kind (5.1) with the restriction $\delta\lambda = k-1 = \text{const}$. The eigenvalues $\lambda_n$ of $\hat{\Delta}$ will become $\tilde{\lambda}_n = k^{-1}\lambda_n$ (this results from (6.3), (6.2) and the scaling properties of $\Delta$). Thus, if we consider the massless case, $\zeta$ will transform into $\tilde{\zeta}$ according to the rule

$$\tilde{\zeta}(z;0) = k^z\,\zeta(z;0) \tag{6.15}$$

which implies

$$\frac{d}{dz}\tilde{\zeta}(0;0) = \ln k\,\zeta(0;0) + \frac{d}{dz}\zeta(0;0) \qquad (6.16)$$

Thus for a scale transformation the effective action (6.13) transforms as follows (for $m = 0$):

$$\tilde{W}[kg] = W[g] - \frac{i}{2}\ln k \cdot \zeta(0;0) \qquad (6.17)$$

having assumed that $\mu$ and then the constant in (6.13) is uneffected by transformations of the metric (Hawking, 1977).

For $\delta\lambda = k - 1$ one has from (5.3)

$$\tilde{W}[g + \delta g] = W[g] + \frac{1}{2}(k-1)\int d^4x\, g^{1/2} \langle T_\mu{}^\mu \rangle_{matrix} \qquad (6.18)$$

but it is also (Parker, 1979)

$$\tilde{W}[g + \delta g] = \tilde{W}[kg] = W[g] + (k-1)\frac{\partial}{\partial k}\tilde{W}[kg]\Big|_{k=1} \qquad (6.19)$$

then

$$\int d^4x\, g^{1/2} \langle T_\mu{}^\mu \rangle_{matrix} = 2\frac{\partial}{\partial k}\tilde{W}[kg]\Big|_{k=1} . \qquad (6.20)$$

According to the previous discussion the expression (6.13) for $W$ does not contain logarithmic divergences. Then in the massless conformal case one can write $\langle T_\mu{}^\mu \rangle_{anomalous}$ instead of $\langle T_\mu{}^\mu \rangle_{matrix}$ in eq. (6.20) and from (6.20) and (6.17) one gets

$$\int d^4x\, g^{1/2} \langle T_\mu{}^\mu \rangle_{anomalous} = -i\,\zeta(0;0) \quad , \qquad (6.21)$$

Now from (6.14), (3.17) and (3.18) one has

$$-i\,\zeta(0;0) = (4\pi)^{-2} \int d^4x\, g^{1/2} a_2(x,x) \qquad (6.22)$$

which implies the relation

$$\langle T_\mu{}^\mu \rangle_{anomalous} = (4\pi)^{-2} a_2(x,x) + 4\text{-divergence} \qquad (6.23)$$

The only covariant 4-divergence with the right canonical dimensions will be of the form

$$c_1 \Box R + c_2 \nabla_\mu \nabla_\rho R^{\mu\rho} = (c_1 + \tfrac{1}{2} c_2) \Box R \qquad (6.23)$$

(having used the contracted Bianchi identities). Thus we have proved (5.7) up to a $\Box R$ term which can be eliminated by a finite counterterm (see §5).

The complete demonstration of (5,7) uses further properties of $\zeta(z;0)$ and can be found in the literature (see e.g. Hawking, 1977; Parker, 1978; De Witt, 1979); we shall not deal here with such a problem.

## § 7. The partition function

We already knew in § 2 that the transition amplitude $\langle out, vac \mid in, vac \rangle$ can be represented as a path integral. Now let us consider the amplitude to go from a state with metric $g_1$, generical matter fields $\phi_1$ on a surface $\Sigma_1$, to another metric $g_2$, matter fields $\phi_2$ on a surface $\Sigma_2$ ($\Sigma_1$ and $\Sigma_2$ can be e.g. the in and out regions of § 2); this can be written in analogous manner (Hawking, 1979)

$$\langle g_2, \phi_2, \Sigma_2 \mid g_1, \phi_1, \Sigma_1 \rangle = Z = \int d[g]\, d[\phi] \exp\left(i S[g,\phi]\right) \tag{7.1}$$

$d[g]$ and $d[\phi]$ being suitable measures over the spaces of metrics and matter fields respectively and the integral being taken over all fields which have given values on $\Sigma_1$ and $\Sigma_2$. The quantity $Z$ is commonly called the "generating functional". The action functional $S[g,\phi]$ is normally taken to be

$$S[g,\phi] = (16\pi G_0)^{-1} \int (R - 2\Lambda_0)\, g^{1/2} d^4x + a_0 \int \left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right) g^{1/2} d^4x + b_0 \int R^2 g^{1/2} d^4x + S[\phi] \tag{7.2}$$

Much more terms are needed in (7.2) when dealing with quantum gravity (see Fradkin and Tseytlin, 1982) but this is not our aim. Since a boundary is present boundary terms are needed in (7.2), these can be found in the literature (see Hawking, 1978) and we shall omit them here.

The dominant contribution to the functional integral (7.1) comes from field configurations $g_0$, $\phi_0$ which satisfy the given boundary conditions and extremise the action, that is from solutions of the classical field equations

$$\left.\frac{\delta S[g,\phi]}{\delta g^{\mu\nu}}\right|_{g_0,\phi_0} = 0 \qquad (7.3)$$

and

$$\left.\frac{\delta S[g,\phi]}{\delta \phi}\right|_{g_0,\phi_0} = 0 \qquad (7.4)$$

Let us notice that (7.3) are just the Einstein's equations (with a cosmological and curvature squared terms), while (7.4) are the matter field equations.

We can define the "quantum fluctuations" $\bar{g}$, $\bar{\phi}$

$$\bar{g} = g - g_0 \qquad , \quad \bar{\phi} = \phi - \phi_0 \tag{7.5}$$

about the "background fields" $g_0$ and $\phi_0$. We are in a position now to expand $S[g,\phi]$ as a functional Taylor series (see e.g. Abers and Lee, 1973) about the background fields:

$$S[g,\phi] = S[g_0,\phi_0] + S_2[\bar{g}] + S_2[\bar{\phi}] + \text{higher order terms} \tag{7.6}$$

The terms $S_2$ are quadratic in the fluctuations (7.5); terms linear in $\bar{g}$ and $\bar{\phi}$ are absent because of (7.3), (7.4). Replacing (7.6) into (7.1) and neglecting higher order terms one has (since $d[g] = d[\bar{g}]$ and $d[\phi] = d[\bar{\phi}]$)

$$\ln Z \simeq i S[g_0,\phi_0] + \ln \int d[\bar{g}] \exp i S_2[\bar{g}] + \cdots$$

$$+ \ln \int d[\bar{\phi}] \exp i S_2[\bar{\phi}] \tag{7.7}$$

This is known as the "one-loop approximation (also called "W.K.B." or "stationary phase" approximation); the first term is the contribution of background fields to $Z$, the second term comes from quantum gravitational fluctuations ("gravitons"), met_ ter fluctuations yield the third term.

Neglecting quantum gravitational effects in (7.7) one has the external field approach which we are adopting in this thesis. Such an approximation is justified as long as we are dealing with scales much larger than the Planck length $\lambda_{Pl} \sim 10^{-33}$ cm.

Let us return to the term $S_2[\bar{\phi}]$ which is defined by

$$S_2[\bar{\phi}] = \frac{1}{2}\int d^4x\, d^4y\, \bar{\phi}(x) \left.\frac{\delta^2 S[g,\phi]}{\delta\phi(x)\delta\phi(y)}\right|_{g_0,\phi_0} \bar{\phi}(y) \qquad (7.8)$$

If the matter field action $S[\phi]$ is of the form (2.2) one simply has

$$\frac{\delta^2 S[g,\phi]}{\delta\phi(x)\delta\phi(y)} = F\,\delta(x,y)$$

so that (6.8) takes the form

$$S_2[\bar{\phi}] = \frac{1}{2}\int d^4x\, \bar{\phi}(x)\, F\, \bar{\phi}(y) \qquad (7.8)$$

Let us notice that the metric tensor appearing in F (see e.g. (2.3)) must be $g_0$. From (7.8) the one-loop generating functional for matter fields is obtained

$$Z_{\bar\phi} = \int d[\bar\phi]\, \exp\left(\frac{i}{2}\int d^4x\, \bar\phi F \bar\phi\right) \qquad (7.10)$$

Up to now he have dealt with a Lorentzian metric (i.e. with signature $-+++$); the matter action $S_2[\phi]$ will be real so making the path integral (7.10) to oscillate and not to converge. In flat space-time one solves such a problem by Wick rotating the time axis, namely one replaces $t$ by $-i\tau$. This rotation introduces a factor $-i$ into the integral of the action; then one defines an "Euclidean action" $\hat S = -iS$ which is greater than or equal to zero for all fields $\phi$ that are real on the Euclidean section ($\tau$ real). One has

$$Z_{\bar\phi} = \int d[\bar\phi]\, \exp\left(-\hat S_2[\bar\phi]\right) ; \qquad (7.11)$$

the path integral is performed over all field configurations $\bar\phi = \phi - \phi_0$ which die away on the given boundary; the factor $\exp(-\hat S_2)$ produces an exponential damping of the integral and makes it to converge.

Moreover there are situations when periodic boundary conditions with some period $\beta$ in the Euclidean time coordinate must be imposed on the boundary: the time integration in $\hat{S}$ is restricted to the finite interval $0 \le \tau \le \beta$. In such a case $Z$ is known from statistical mechanics to be the "partition function" for a canonical ensemble at the temperature $T \equiv 1/\beta$ (assuming $k_B = 1$).

What one should like to do is to extend the previous analysis to a curved space-time; much work has been done in this direction (see e.g. Hawking, 1979; Gibbons, 1979), however there are difficulties like the presence of singularities, the possibility of coordinate transformations which alter the boundary conditions, which must be dealt case by case. However at present it is believed that "thermal properties" of black holes (the "Hawking effect") and of accelerating frames (see e.g. Sciama, Candelas and Deutsch, 1981) can be understood in this context (see e.g. Denardo and Spallucci, 1980).

## § 8 Regularization methods in a cosmological background

In 1971 Zel'dovich and Starobinsky introduced a method of regularization, the "n-wave method", which is suitable for space-times with special symmetries like the Friedmann models and the Bianchi type I space-time. They used their method to eliminate divergences in the case of a conformally coupled, $\xi = 1/6$, scalar field in a Bianchi type I metric. Later the same method was applied by Mamaev and Mostepanenko to scalar and spinor fields in the homogeneous and isotropic case (see Grib, Mamaev and Mostepanenko, 1980, for a revue). Matarrese and Bonometto (1982) applied this method to a massive vector (Proca) field in a Friedmannian universe; the same problem was studied by Grib and Nesteruk (1982) in a radiation dominated universe.

Parker and Fulling (1974) have shown this method to be completely equivalent to their "adiabatic regularization" approach. Finally De Witt (1975) proved the n-wave method to be essentially equivalent to the Schwinger-De Witt proper-time approach (this is in essence the method explained in §§ 2–5).

In this section we shall display such a

method by applying it to the case of a Proca field: this case presents some interesting peculiarities which deserve some attention.

In this section and in § 8 we use the signature $(+---)$ which is the most used one in cosmological cases.

It has been shown (see Matarrese and Bonometto, 1981, and Marchetti and Matarrese, 1981) that a lagrangian formulation for a massive vector field $a_\mu$ can start from the lagrangian density ($g \equiv -\det g_{\mu\nu}$)

$$\mathcal{L} = -g^{1/2}\{\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}/4 - m^2 a_\mu a^\mu/2\}\ , \qquad (8,1)$$

with

$$\mathcal{F}_{\mu\nu} \equiv \nabla_\mu a_\nu - \nabla_\nu a_\mu = \partial_\mu a_\nu - \partial_\nu a_\mu\ . \qquad (8,2)$$

In what follows we consider only a Friedmann metric with $k = 0$ spatial curvature:

$$ds^2 = a^2(\eta)(d\eta^2 - dl^2) \qquad (8,3)$$

with $\eta$ the "conformal time" related to the proper time $t$ of comoving observers by

$$a(\eta)\,d\eta = dt \quad , \quad a(\eta) = a(t) \tag{8.4}$$

The space-time (8.3) is conformally flat ($C_{\mu\nu\rho\sigma} = 0$), then the previous formulation is equivalent to taking

$$d\tilde{s}^2 = d\eta^2 - dl^2 \tag{8.5}$$

for $\eta \geq \eta_0$, $\eta_0$ being the initial singularity, and new field variables

$$A_\mu = \mathcal{A}_\mu \ , \ A^\mu = a^2(\eta)\,\mathcal{A}^\mu \ , \ F_{\mu\nu} = \mathcal{F}_{\mu\nu} \ , \ F^{\mu\nu} = a^2(\eta)\,\mathcal{F}^{\mu\nu} \tag{8.6}$$

fulfilling the equation of motion

$$\partial_\mu F^{\mu\nu} + m^2 a^2(\eta) A^\nu = 0 \ , \tag{8.7}$$

$$\partial_\nu A^\nu = -2 A^0 \,\dot{a}(\eta)/a(\eta) \ , \tag{8.8}$$

(dots indicate $\eta$-derivatives). Because of the mass term in (8.1) the theory lacks of conformal invariance : from this the $\eta$-dependence of the mass term in (8.7) and the peculiar form of (8.8), replacing the ordinary Lorentz condition, which connects the longitudinal and time components of the field.

The passage between covariant and contravariant indices will be performed by means of the Minkowski tensor $\eta_{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$.

The stress tensor is

$$T_{\mu\nu} = a^{-2}(\eta)\{F_{\mu\rho}F^{\mu\rho} + \eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}/4 +$$

$$+ m^2 a^2(\eta)(A_\mu A_\nu - \eta_{\mu\nu}A_\rho A^\rho/2)\} \tag{8.9}$$

In a cosmological metric like (8.3) one is able to select a momentum variable k labelling the basis functions; a three-dimensional Fourier transform of the field variable can be performed

$$A_\mu(x) = (2\pi)^{-3/2}\, 2^{-1/2}\int d^3k\,[\exp(-i\vec{k}\cdot\vec{x})\,q_\mu^*(\vec{k},\eta) + \mathrm{h.c.}] \tag{8.10}$$

(h.c. means hermitean conjugate) where the momentum-space field variables $q_\mu(\vec{k},\eta)$ can be written as

$$q_\mu(\vec{k},\eta) = \sum_{\sigma=0}^{3} e_{\mu\sigma}\, v_\sigma(k,\eta)\, a_\sigma(\vec{k}). \tag{8.11}$$

Here $e_{00}=1$, $e_{0i}=e_{i0}=0$, $e_{i3}=k_i/k$ while $e_{i1}$ and

$e_{i\lambda}$ are the components of unit vectors mutually orthogonal and normal to $\vec{k}$ (notice that latin indices label space components only thus extending from 1 to 3); the $a_i(\vec{k})$ are operators obeying the commutation relations

$$[a_i(\vec{k}), a_j^*(\vec{k}')] = \delta_{ij}\, \delta^{(3)}(\vec{k}, \vec{k}') \tag{8.12}$$

From (8.8) it follows that

$$a_0(\vec{k}) = (k/m)\, a_3(\vec{k}) \tag{8.13}$$

The $\eta$ dependence of $A_\mu$ is all contained in the c-number functions $v_\sigma(k,\eta)$ which, owing to (8.7), (8.8) obey the equations

$$\ddot{v}(k,\eta) + \omega^2(k,\eta)\, v(k,\eta) = 0 \tag{8.14}$$

where $v$ can be either $v_1$ or $v_2$,

$$\ddot{h}(k,\eta) + [\omega^2(k,\eta) + Q(\eta)]\, h(k,\eta) = 0 \tag{8.15}$$

$h$ being related to $v_3$ and $v_0$ through

$$v_3 = -(i/ma^2)\frac{\partial}{\partial\eta}(a h)$$

$$v_0 = h/a \qquad (8.16)$$

In (8.14), (8.15)

$$\omega(k,\eta) = [k^2 + m^2 a^2(\eta)]^{1/2} ,$$

$$Q(\eta) = \ddot{a}(\eta)/a(\eta) - 2[\dot{a}(\eta)/a(\eta)]^2 \qquad (8.17)$$

The Wronskian conditions

$$\mathrm{Im}(h\dot{h}^*) = \mathrm{Im}(v\dot{v}^*) = -1 \qquad (8.18)$$

can be imposed thanks to the form of the equations (8.14), (8.15); the r.h.s. constant value is dictated by the normalization implicit in (8.12).

In order to define the states of the system and the constants appearing in $v$ and $h$, boundary conditions are to be imposed; these can be conveniently set at $\eta \to +\infty$ (in order to avoid the passage through the singularity). We shall require

$$\lim_{\eta\to+\infty} d^n a(\eta)/d\eta^n = 0 \quad , \quad \lim_{\eta\to+\infty} a(\eta) = a_0 \qquad (8.19)$$

with any necessary value of $n$; in the limit $\eta \to +\infty$

then the space-time is flat. It is then convenient to assume

$$\lim_{\eta \to +\infty} v(k,\eta),\, h(k,\eta) = \left[ k^2 + m^2 a_0^2 \right]^{-1/4} \qquad (8.20)$$

$$\lim_{\eta \to +\infty} \dot{v}(k,\eta),\, \dot{h}(k,\eta) = i \left[ k^2 + m^2 a_0^2 \right]^{1/4} \qquad (8.21)$$

The operators $a_i^{(*)}(\vec{k})$ become annihilation (creation) operators <u>at this time</u>, and a vacuum state

$$a_i(\vec{k}) | 0 \rangle = 0 \qquad (8.22)$$

can be defined. At any other time $|0\rangle$ does not represent the vacuum state (in this connection see Marchetti and Matarrese, 1981).

We are now interested in the v.e.v of $T_{\mu\nu}$; the only non-vanishing components of $\langle 0 | T_{\mu\nu} | 0 \rangle$ are

$$\rho(\eta) = a^{-2}(\eta) \langle 0 | T_{00} | 0 \rangle \qquad (8.23)$$

$$P(\eta) = a^{-2}(\eta) \langle 0 | T_{i(i)} | 0 \rangle \qquad (8.24)$$

(no summation over $i$ in (8.24)), where $\rho$ is the energy density and $P$ is the (isotropic) pressure density.

It will be convenient to distinguish between the part of $T_{\mu\nu}$ related to the operators $a_r(\vec{k})$ $(r=1,2)$, the "transversal" part, and the one related to $a_3(\vec{k})$ or $a_0(\vec{k})$, the "longitudinal" part. We shall have

$$T_{\mu\nu} = 2\,T^{tr}_{\mu\nu} + T^{\ell}_{\mu\nu} \tag{8,25}$$

("tr" = transversal, "$\ell$" = longitudinal).

Replacing the expansions (8,10), (8,11) into (8,9) one gets the final relations

$$\rho^{tr}(\eta) = (2\pi^2 a^4)^{-1}\int_0^\infty dk\, k^2 \left\{\omega(k,\eta)\left[\,|\beta^{tr}(k,\eta)|^2 + \frac{1}{2}\right]\right\} \tag{8,26}$$

$$\rho^{\ell}(\eta) = (2\pi^2 a^4)^{-1}\int_0^\infty dk\, k^2 \left\{\omega(k,\eta)\left[\,|\beta^{\ell}(k,\eta)|^2 + \frac{1}{2}\right]\right\} \tag{8,27}$$

$$P^{tr}(\eta) = (6\pi^2 a^4)^{-1}\int_0^\infty dk\, k^2 \Big\{\omega(k,\eta)\left[\,|\beta^{tr}(k,\eta)|^2 + \frac{1}{2}\right] -$$

$$- m^2 a^2\, |v(k,\eta)|^2/2\Big\} \tag{8,28}$$

$$P^{\ell}(\eta) = (6\pi^2 a^4)^{-1}\int_0^\infty dk\, k^2 \Big\{\omega(k,\eta)\left[\,|\beta^{\ell}(k,\eta)|^2 + \frac{1}{2}\right] +$$

$$+\left[k^2\, |h(k,\eta)|^2 - |\dot{h}(k,\eta) + (\dot{a}/a)\, h(k,\eta)|^2\right]/2\Big\} \tag{8,29}$$

where

$$|\beta^{tr}|^2 = (4\omega)^{-1}\{|\dot{v}|^2 + \omega^2|v|^2\} - 1/2 \qquad (8.30)$$

$$|\beta^{\ell}|^2 = (4\omega)^{-1}\{|\dot{h} + (\dot{a}/a)h|^2 + \omega^2|h|^2\} - 1/2 \qquad (8.31)$$

It is clear from (8.26) — (8.28) that, unless $|\beta|^2$ has a very peculiar dependence upon $k$, the large $k$ contributions will lead to ultraviolet divergences both for $\rho$ and P. These ultraviolet divergences can be removed from the theory in a covariant way by defining regularized v.e.v. of $T_{\mu\nu}$ as follows.

As can be seen from (8.26), (8.29) one can write in general

$$\langle 0|T_{\mu\nu}|0\rangle = \int_0^\infty d^3k\, \mathcal{T}_{\mu\nu}(k,m) \quad ; \qquad (8.32)$$

Zel'dovich and Starobinsky define the regularized quantity

$$\mathcal{T}^{reg}_{\mu\nu}(k,m) \equiv \lim_{\lambda\to 0}\left\{\mathcal{T}_{\mu\nu}(k,m) - \hat{D}(\lambda^2)\overset{(\lambda)}{\mathcal{T}}_{\mu\nu}(k,m)\right\} \qquad (8.33)$$

where

$$\overset{(\lambda)}{\mathcal{T}}_{\mu\nu}(k,m) \equiv \lambda\,\mathcal{T}_{\mu\nu}(k/\lambda, m/\lambda) \qquad (8.34)$$

$$\hat{D}(\lambda^2) \equiv 1 + \frac{\partial}{\partial(\lambda^2)} + \frac{1}{2}\frac{\partial^2}{\partial(\lambda^2)^2} \quad . \qquad (8.35)$$

The quantity $\tau_{\mu\nu}^{reg}(k,m)$ is such that the integral

$$\langle 0|T_{\mu\nu}|0\rangle_{reg} \equiv \int_0^\infty d^3k\, \tau_{\mu\nu}^{reg}(k,m) \tag{8,36}$$

is finite ; furthermore for its definition $\langle 0|T_{\mu\nu}|0\rangle_{reg}$ is a tensor and has vanishing 4-divergence

$$\nabla^\mu \langle 0|T_{\mu\nu}|0\rangle_{reg} = 0 \quad . \tag{8,37}$$

To operate the subtractions involved in (8,33) the large $k$ behaviour of $\tau_{\mu\nu}(k,m)$ must be detected. This can be done by using asymptotic expansions of $v$ and $h$ in powers of $(\lambda/\omega)$ (Zel'dovich and Starobinsky use a different parameter $n = 1/\lambda$ from which the name of "$n$-wave method").

It is useful to introduce the new functions $A_v(k,\eta)$, $A_h(k,\eta)$, $B_v(k,\eta)$, $B_h(k,\eta)$ instead of $v$ and $h$ through the relations

$$v, h = \omega^{-1/2}\left[A_{v,h}\, e_+ + B^*_{v,h}\, e_-\right] , \tag{8,38}$$

$$\dot{v}, \dot{h} = i\omega^{1/2}\left[A_{v,h}\, e_+ - B^*_{v,h}\, e_-\right] , \tag{8,38}$$

where

$$e_{\pm} = \exp\left[\pm i \int^{\eta} \omega(k,\tilde{\eta})\, d\tilde{\eta}\right] ; \qquad (8.40)$$

the Wronskian condition (8.18) implies

$$|A_{v,h}(k,\eta)|^2 - |B_{v,h}(k,\eta)|^2 = 1 \qquad (8.41)$$

Replacing (8.39) — (8.41) into (8.14) or (8.15) we have two systems of equations for $A_{v,h}$ and $B_{v,h}$; because of the functional relation (8.41) they can be conveniently written in terms of the three real unknown functions $s$ $u$ and $\tau$ ($s^0$, $u^0$ and $\tau^0$) defined by

$$s = |B_h|^2 \qquad s^0 = |B_v|^2$$

$$u = 2\,\mathrm{Re}\left[A_h B_h e_+^2\right] \qquad u^0 = 2\,\mathrm{Re}\left[A_v B_v e_+^2\right]$$

$$\tau = 2\,\mathrm{Im}\left[A_h B_h e_+^2\right] \qquad \tau^0 = 2\,\mathrm{Im}\left[A_v B_v e_+^2\right]. \qquad (8.42)$$

The equations for $s$, $u$ and $\tau$ read (Zel'dovich and Starobinsky, 1971)

$$\dot{s} = (u\,\varpi + \tau\,Q/\omega)/2$$

$$\dot{u} = (1+2s)\,\varpi - \tau\,(Q/\omega + 2\omega)$$

$$\dot{\tau} = (1+2s)\,Q/\omega + u\,(Q/\omega + 2\omega) \qquad (8.43)$$

where

$$w \equiv \dot{\omega}/\omega = m^2 \dot{a} a/\omega^2 .$$

The equations for $s^\circ, u^\circ, v^\circ$ are the same as (8.43) just with $Q = 0$. The boundary conditions (8.20), (8.21) on $v$ and $h$ imply that

$$\lim_{\eta \to +\infty} s, u, v = 0 \quad ; \quad \lim_{\eta \to +\infty} s^\circ, u^\circ, v^\circ = 0 \qquad (8.44)$$

The system of first-order differential equations (8.43) and initial conditions (8.44) is equivalent to the system of integral equations

$$u(\eta) = \int_{\eta_{in}}^{\eta} d\eta' (1 + 2s(\eta'))\Big[w(\eta') \cos\vartheta_\eta(\eta') -$$

$$- (Q(\eta')/\omega(\eta')) \sin\vartheta_\eta(\eta')\Big]$$

$$v(\eta) = \int_{\eta_{in}}^{\eta} d\eta' (1 + 2s(\eta'))\Big[w(\eta') \sin\vartheta_\eta(\eta') +$$

$$+ (Q(\eta')/\omega(\eta')) \cos\vartheta_\eta(\eta')\Big]$$

$$s(\eta) = \frac{1}{2} \int_{\eta_{in}}^{\eta} d\eta' \int_{\eta_{in}}^{\eta} d\eta'' (1 + 2s(\eta''))\Big\{w(\eta')\Big[w(\eta'') \cos\vartheta_{\eta'}(\eta'') -$$

$$- (Q(\eta'')/\omega(\eta'')) \sin\vartheta_{\eta'}(\eta'')\Big] + (Q(\eta')/\omega(\eta')) \cdot$$

$$\cdot \left[ w(\eta'') \sin \vartheta_{\eta'}(\eta'') + (\mathcal{Q}(\eta'')/\omega(\eta'')) \cos \vartheta_{\eta'}(\eta'') \right] \Big\} \qquad (8.45)$$

where

$$\vartheta_{\eta}(\eta') = \int_{\eta_{in}}^{\eta} d\tilde{\eta} \left[ 2\omega(\tilde{\eta}) + (\mathcal{Q}(\tilde{\eta})/\omega(\tilde{\eta})) \right] \qquad (8.46)$$

($\eta_{in}$ in our case is $+\infty$).

The integral eq.s for $u^0$, $\tau^0$ and $s^0$ are obtai_ned from (8.45), (8.46) by putting $\mathcal{Q} = 0$.

Equations (8.45) tell us that the form of $u$, $\tau$ and $s$ at $\eta$ actually depends on the whole histo_ry of $a(\eta)$ during the (future) interval $\eta \leq \eta' \leq +\infty$ (this strange feature depends on where we imposed ini_tial conditions) then $u$, $\tau$ and $s$ are essentially non-local functions of the metric. The eq.s (8.45) can be trivially integrated when $m = 0$ and $\mathcal{Q} = 0$, yielding $s = u = \tau = 0$ identically. In other cases one has no ge_neral solutions of them and must make use of so_me suitable approximations.

The large-$\omega$ behaviour of $s, u, \tau$ can be detected with the aid of asymptotic expansions

$$s(k/\lambda, \eta) \sim \sum_{r=0}^{\infty} (\lambda/\omega)^r \tilde{s}_r(\eta) = \sum_{r=0}^{\infty} \lambda^r s_r(\omega, \eta) \qquad (8.47)$$

and analogous expressions for $u$ and $\tau$.

From the initial conditions (8.44) one gets

$$s_0(\eta) = u_0(\eta) = \tau_0(\eta) = 0 \tag{8.47}$$

($s_0$, $u_0$ and $\tau_0$ must not be confused with $s^0$, $u^0$ and $\tau^0$).

By replacing (8.47) and (8.48) into (8.43) and by separating different powers of $\lambda$, equations giving $s_r$, $u_r$ and $\tau_r$ up to any suitable value of $r$ can be worked out. We only need to solve them up to $r = 4$. Recalling that $s^0$, $u^0$ and $\tau^0$ are the solutions of (8.43) with $Q = 0$ one can write

$$s = s^0 + s^Q \quad , \quad s_r = s_r^0 + s_r^Q \tag{8.48}$$

and analogous expressions for $u, \tau$ and $u_r, \tau_r$.

Up to $r = 4$ the non-vanishing coefficients of the asymptotic series are (Grib et al. 1980)

$$\tau_1^0 = \dot{\omega}/2\omega \quad , \quad s_2^0 = (\dot{\omega}/4\omega)^2 \quad , u_2^0 = (\dot{\omega}/\omega)^{\dot{}2}/4\omega, \tag{8.49}$$

$$\tau_3^0 = -\left[(\dot{\omega}/\omega)^{\cdot}/\omega\right]^{\cdot}/8\omega + (\dot{\omega}/\omega)^3/16 \ ,$$

$$s_4^0 = -\left[(\dot{\omega}/\omega)^{\cdot}/\omega\right]^{\cdot}(\dot{\omega}/\omega)^2/32\omega + \left[(\dot{\omega}/\omega)^{\cdot}/8\omega\right]^2 +$$

$$+ (\dot{\omega}/\omega)^4/128 \ ,$$

$$u_4^0 = -\{[(\dot{w}/\omega)^{\cdot}/\omega]^{\cdot}/\omega\}^{\cdot}/16\omega + 3(\dot{w}/\omega)^2(\dot{w}/\omega)^{\cdot}/32\omega \qquad (8.50)$$

and ( Matarrese and Bonometto, 1982)

$$u_2^Q = -\, Q/2\omega^2 \quad , \tau_3^Q = (Q/4\omega^3)^{\cdot} \quad ,$$

$$s_4^Q = [(Q/\omega^3)^{\cdot}(\dot{w}/\omega) - (Q/\omega^3)(\dot{w}/\omega)^{\cdot} + (Q/\omega^2)^2]/16$$

$$u_4^Q = (Q/\omega^3)^{\cdot\cdot}/8\omega - (Q/\omega^2)[(\dot{w}/4\omega)^2 + (\dot{w}/\omega)^{\cdot}/8\omega - \\ - Q/4\omega^2] \qquad (8.51)$$

Let us notice that the coefficients (8.50), (8.51) are <u>local</u> functions of the metric, then an asymptotic series like (8.47) is clearly unable to reproduce the non-local character of s, u and $\tau$ and then of $\langle 0|T_{\mu\nu}|0\rangle$; they cannot be used e.g. to evaluate, even approximately, the particle creation rate which depends on the global features of the space-time ( such as the existence of singularities or event horizons). The coefficients (8.50), (8.51) can be only used to remove the infinities from $\langle 0|T_{\mu\nu}|0\rangle$.

Now we are able to perform the regularization in (8.32) - (8.36). Recalling that the energy and pressure densities (8.26) - (8.29) are functions of $\nu$

and $h$ and then of $s, u, \tau$, one can see that the regularization amounts to write $\langle 0 | T_{\mu\nu} | 0 \rangle$ in terms of $s$, $u$ and $\tau$ and then to subtract the first terms of the asymptotic expansions (8.47) according to (8.33).

The final expressions will be

$$\rho^{tr, ref}(\eta) = (2\pi^2 a^4)^{-1} \int_0^\infty dk\, k^2 \omega (s^0 - s^0_2 - s^0_4) \tag{8.52}$$

$$\rho^{l, ref}(\eta) = \rho^{tr, ref} + (2\pi^2 a^4)^{-1} \int_0^\infty dk\, k^2 \omega \left[ s^Q - s^Q_4 + (\dot a / 2 a \omega)^2 (u + 2s - u_2 - 2 s_2) - (\dot a / 2 a \omega)(\tau - \tau_1 - \tau_3) \right] \tag{8.53}$$

$$P^{tr, ref}(\eta) = (6\pi^2 a^4)^{-1} \int_0^\infty dk\, k^2 \left[ (k^2/\omega)(s^0 - s^0_2 - s^0_4) - (m^2 a^2 / 2\omega)(u^0 - u^0_2 - u^0_4) \right] \tag{8.54}$$

$$P^{l, ref}(\eta) = P^{tr, ref} + (6\pi^2 a^4)^{-1} \int_0^\infty dk\, k^2 \left[ \omega (s^Q - s^Q_4 + u - u_2 - u_4) - (m^2 a^2 / 2\omega)(2 s^Q - 2 s^Q_4 + u^Q - u^Q_2 - u^Q_4) + (\dot a / 2 a)(\tau - \tau_1 - \tau_3) - (\dot a^2 / 4 \omega a^2)(u - u_2 + 2 s^0_2) \right] \tag{8.55}$$

The regularization program of Zel'dovich and

Starobinsky stops here, it does not involve a renormalization recipe, that is a rule for the cancellation of the infinities subtracted from $\langle 0|T_{\mu\nu}|0\rangle$ with counterterms in the l.h.s. of the Einstein's equation.

In the next section we shall describe a method to handle diverging integrals so to write them in terms of local invariants of the metric.

## §9 The interpretation of infinities

In §7 we introduced a classical action which is suitable for renormalization of matter fields in a curved space-time. Here we prefer to write down the Einstein's equation, with higher derivative terms, instead of the action from which it is obtained.

In the semiclassical approximation we have

$$g_{\mu\nu}\Lambda_0 + \left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right) + {}'b_0 {}'H_{\mu\nu} = -8\pi G_0 \langle T_{\mu\nu} \rangle \tag{9.1}$$

where (see Bunch, 1979) the tensor ${}'H_{\mu\nu}$ reads

$$\begin{aligned} {}'H_{\mu\nu} \equiv g^{-1/2} \frac{\delta}{\delta g^{\mu\nu}} \int g^{1/2} R^2 d^4x &= 2 \nabla_\mu \nabla_\nu R - 2 g_{\mu\nu} \Box R + \\ &+ \frac{1}{2} g_{\mu\nu} R^2 - 2 R R_{\mu\nu} \end{aligned} \tag{9.2}$$

(notice that from (9.2) equation (5.15) follows), and in our metric (8.3) it has non-vanishing components

$$\begin{aligned} {}'H_{00} &= 18(-2\ddot{c}c + \dot{c}^2 + 3c^4)a^{-2} \\ {}'H_{ij} &= \delta_{ij} 6(2\dddot{c} - 2\ddot{c}c + \dot{c}^2 - 12\dot{c}c^2 + 3c^4)a^{-2} \end{aligned} \tag{9.3}$$

where

$$c(\eta) = \dot{a}(\eta)/a(\eta) \tag{9.4}$$

In (9.1) $\Lambda_0$, ${}'b_0$ and $1/G_0$ are bare constants to be renormalized, they are the only infinite constants that we need in our case.

It is useful to introduce the further tensorial quantity

$$^{3}H_{\mu\nu} = -2C_{\alpha\mu\beta\nu}R^{\alpha\beta} + \frac{1}{2}g_{\mu\nu}R_{\alpha\beta}R^{\alpha\beta} - R_{\mu\alpha}R^{\alpha}{}_{\nu} + \frac{2}{3}RR_{\mu\nu} - \frac{1}{4}g_{\mu\nu}R^{2}, \tag{9.5}$$

which in our metric has non-vanishing components

$$\begin{aligned} {}^{3}H_{00} &= 3c^{4}a^{-2} \\ {}^{3}H_{ij} &= \delta_{ij}(-4\dot{c}c^{2}+c^{4})a^{-2}. \end{aligned} \tag{9.6}$$

As Ginzburg et al. (1971) first noticed, the tensor $^{3}H_{\mu\nu}$ has vanishing covariant divergence in conformally flat space times, then in our metric

$$\nabla^{\mu}\,{}^{3}H_{\mu\nu} = 0, \tag{9.7}$$

The parts of $\langle 0|T_{\mu\nu}|0\rangle$ which were subtracted in the regularization procedure and must find suitable counterterms read

$$\langle 0|T_{00}^{sub,tr}|0\rangle = (2\pi^{2}a^{2})^{-1}\int_{0}^{\infty}dk\,k^{2}\omega\left(\tfrac{1}{2}+s_{2}^{0}+s_{4}^{0}\right)$$

$$\langle 0|T_{00}^{sub,\ell}|0\rangle = \langle 0|T_{00}^{sub,tr}|0\rangle + (2\pi^{2}a^{2})^{-1}\int_{0}^{\infty}dk\,k^{2}\omega\left[s_{4}^{Q} + (c/2\omega)^{2}(1+u_{2}+2s_{2}) - (c/2\omega)(\tau_{1}+\tau_{3})\right]$$

$$\langle 0|T_{ij}^{sub,tr}|0\rangle = (6\pi^{2}a^{2})^{-1}\delta_{ij}\int_{0}^{\infty}dk\,k^{2}\left[(k^{2}/\omega)\left(\tfrac{1}{2}+s_{2}^{0}+s_{4}^{0}\right) - (m^{2}a^{2}/2\omega)(u_{2}^{0}+u_{4}^{0})\right]$$

$$\langle 0|T_{ij}^{sub,\ell}|0\rangle = \langle 0|T_{ij}^{sub,tr}|0\rangle + (6\pi^2 a^2)^{-2}\delta_{ij}\int_0^\infty dk\, k^2 [\omega(s_4^Q + u_2 + u_4) - (m^2 a^2/2\omega)(2s_4^Q + u_2^Q + u_4^Q) + c(r_1 + r_3)/2 - c^2(1+u_2+2s_2)/4\omega] \qquad (9.8)$$

In order to deal with the integrals showing ultraviolet divergences, we shall use a non-covariant dimensional regularization method due to Bunch (1980). This amounts to replacing the factor $dk\, k^2$ appearing in the integrals (9.8) by $dk\, k^{N-2}$; quantities formally depending on N will then result. In fact one can make the replacement

$$\int_0^\infty dk\, k^s \omega^{-r} \longrightarrow \lim_{N\to 4} (ma)^{1+s-r} \frac{\Gamma[(s+1)/2]\,\Gamma[(-N-s+r+3)/2]}{2\Gamma(r/2)} \qquad (9.9)$$

and use expansion (4.8) for the diverging $\Gamma(N-4)$.

All divergences are therefore $\propto (N-4)^{-1}$. Any time (4.8) will be used in (9.8) only the pole residuals will be kept, factorizing $(N-4)^{-1}$.

The result of these calculations is expressed by the formula

$$\langle 0|T_{\mu\nu}|0\rangle - \langle 0|T_{\mu\nu}^{reg}|0\rangle = [3m^4/32\pi^2(N-4)]\, g_{\mu\nu} - (m^2/8\pi^2)[1/4 + 1/(N-4)](R_{\mu\nu} - g_{\mu\nu}R/2) - (960\pi^2)^{-1}\cdot {}^3H_{\mu\nu} + (576\pi^2)^{-1}[1/10 - 1/(N-4)]\, {}^1H_{\mu\nu} +$$

$$+ \mathcal{N}_{\mu\nu} \tag{9.10}$$

where $\mathcal{N}_{\mu\nu}$ is a non-tensorial quantity coming from the non-covariance of the method which handled at a different level non-diverging and diverging quantities in $\langle 0|T^{sub}_{\mu\nu}|0\rangle$.

The appearance of a finite term proportional to ${}^{3}H_{\mu\nu}$ has the same origin of the trace anomaly as discussed in (Bunch, 1979).

According to (9.1) one has the following corrections to the bare values of $\Lambda_0$, $G_0$ and $b_0$

$$\Delta \Lambda_0 = -3\, G_0 m^4 / 4\pi (N-4)$$

$$\Delta G_0^{-1} = -m^2 / 2\pi (N-4)$$

$$\Delta {}'b_0 = -G_0 / 72\pi (N-4)\,. \tag{9.11}$$

Renormalization amounts to assuming that from the beginning $\Lambda_0 + \Delta\Lambda_0 = \Lambda_{finite}$; only $\Lambda_{finite}$, $G_{finite}$ and ${}'b_{finite}$ will be physically relevant.

No simple factors connect the corrections (9.11) with the ones of the scalar case (see Bunch, 1980); this is not surprising as the divergences here result from the anomalous longitudinal part.

Our approach leads to a pole singularity in the trace of $\langle 0|T_{\mu\nu}|0\rangle$ which is of the form (see (3.2))

$$\Box R/(N-4) \qquad (3.12)$$

and lasts also in the $m=0$ case. This is a confirmation that the photon case cannot be simply dealt by studying the $m=0$ limit of this theory, since the photon trace anomaly must be finite.

Finally let us notice that Grib and Nesteruk (1982) do not find the singularity (3.12) since they only study the case of a radiation dominated universe ($a(\eta) \propto \eta$) and there $R$ is precisely zero.

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