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GEOMETRICAL ASPECTS OF GAUGE DIFFERENTIAL FORM THEORIES

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INTRODUCTION

It is now well recognised in mathematical phisics that in order to have a good description of gauge theories we need the language of fibre bundles and in particular classical gauge fields must be seen as connections on principal bundles and classical matter fields as sections of fibre bundles.

Recently many mathematicians and mathematical physicists have initiated in this framework a serious study of classical Yang Mills theory and many rigorous results are obtained in this way. (1)

The aim of this work is to give a systematic (although not complete in all implications and mathematical details) treatment of the proposed ways to make a geometrical formulation for gauge differential forms theories in the framework of fibre bundle as preliminary and partial step for a better understanding of these theories on similar lines.

Many results contained in this paper are new or unpublished, especially in the forth chapter, and I think the most interesting are Theorem 3.9 (with Corollary 3.10 and 3.11) and Theorem 4.12, 4.13

Let me first define which is the meaning of "gauge differential form".

Let M be a m-dimensional manifold and $E_k(M)$ deno te the module of differential forms on M of rank k.

Consider a $k+1$ form A locally defined on M and assume A be the dynamical variable of a field theory which possess the local inv ariance $A \longrightarrow A + d\Lambda$

Λ being a k form.

For k = 0 is the usual electromagnetic case and A can be seen as the local representative of a connection in a $\mathbb{R}$ or $U(1)$ bundle over M.

This simplest case admits two generalisations: on one hand we can substitute the gauge abelian groups $\mathbb{R}$ or $U(1)$ with another non abelian and we get the Yang Mills theories, on the other hand one can change the rank of A still retaining the abelian nature of the group. This is the case of "gauge differential forms" theories.

The third case of non abelian $k+1$ forms seems to fall in hard problems and certainly can not follow the usual shape of gauge theories because from the beginning we find the very strong difficulty of defining the curvature .

In fact this would be

$$F = dA + \frac{1}{2} A \wedge A$$

but if A $\in E_{k+1}(M)$, $dA \in E_{k+2}(M)$, $A \wedge A \in E_{2(k+1)}(M)$ and the rank can be equal only for $k = 0$.

Attempts to define never the less a theory which involves non abelian gauge differential forms necessarily implies therefore the use of more forms of different rank 2), so we leave this case and concentrate only in abelian gauge differential forms.

The main problem in extending the formalism of principal bundle with $k=0$ to the case $K > 0$ is immediately seen: the would be local representative of the connection $A \in E_k(M)$ is not a 1-form, but rather $k+1$ form and the would be gauge function Λ is not a zero form but a k-form.

The ways to overcome this difficulty are essentially two: one makes a slight generalisation of a $\mathbb{R}$-principal bundle which naturally leaves to a $k+1$ connection form and k gauge form 3), the other introduces as base space of the bundle the infinite dimensional k times iterated loop space $\Omega^k M$ 4) and makes use of the fact that a $k+i$ form on M induces naturally a i-form on $\Omega^k M$.

Although the relation with the naive point of view is closer in the first approach, the second is much more flexible and indeed whereas in the first case all bundles are globally trivializable (Th. 3.8) 5), the second takes account also of non globally trivializable bundles and of closed, non exact curvatures (Th. 4.16).

Let me finish this section making some remarks on physical applications of gauge differential forms in quantum field theory. 6)

Cronologically the first was, at least at my knowledge, a model introduced by Kalb and Ramond 7) to generalize electromagnetic interaction between pointlike charges to extended objects.
As starting point they take the Electrodynamic action with a point like source, i.e.

$$S = \frac{1}{2}\int_M dA \wedge {}^*dA + e\int_{\Sigma_1} A + \mu \int_{\Sigma_1} ds \qquad (1.1)$$

where Σ_1 is the curve in the space time M performed by the source and e and μ are the charge and the mass of the source. Consider now a k-dimensional extended object without boundary, this describes in M a $k+1$ dimensional surface without bundary and an obvious generalization of (1.1) to this case is

$$S = \frac{1}{2}\int_M dA \wedge {}^*dA + e\int_{\Sigma_{k+1}} A + \mu \int_{\Sigma_{k+1}} d\sigma \qquad (1.2)$$

where $d\sigma$ is the $k+1$ surface element.

The action (1.2) is invariant under the transformation

$$A \rightarrow A + d\Lambda$$

and thas is a gauge differential form theory which describes the interaction between k-dimensional objects without boundary.

This kind of theory was successively applied to discuss a bag model for the confinement in Q.C.D. taking k = 2 and m = 4 8).
we have an analogy with what happens for two pointlike charges (considered as boundary of a string) for k = 0 and m = 2 where the potential is linear in the distance of the charges, in such way that they are confined. In fact for k = 2 and m = 4 the potential is linear in the volume of the bag and tends to enforce it to collapse thus leading to a phenomenological model for confinement ;
free quarks inside the bag are requested to give the internal pressure to stop the collapse, but due to the potential growing with the distance they can never escape.
Somewhat related to this is an approach which tries to make a simpler model of the topological charge in Q.C.D. with a gauge differential form k = 2 identifying

$$A = Tr \left\{ B \wedge dB + \frac{1}{3} B \wedge B \wedge B \right\}$$

where B is a Yang Mills connection and A is a three form, considered as a k= 2 gauge differential form 9) under the gauge transformation induced by

$$B \rightarrow g^{-1} B g + g^{-1} d g$$

This kind of approach seems to be useful in the clarification of the U(1) problem in Q.C.D., leading also to an effective lagrangian for low energy 10), which in the U(1) sector can be written as

$$\mathcal{L} = dA \wedge {}^{*}dA + \theta\, dA +$$

I want to make a last remark.
There seems to be two kind of theories which involve k gauge differential forms: one is related with extended k-dimentional objects (e.g. Kalb and Ramond theory and bag model), the other only to pointlike structures. So the configuration space for the first case seems to be based on $\Omega^k M$ in the second case on M.
The third chapter is related to the second kind of theory although it takes care only of trivial bundles, the forth chapter is instead strongly connected with the first

kind of theory. However it is still not completly clear to me which are the links between the two.

MATHEMATICAL INTRODUCTION

In this chapter I will introduce the main mathematical objects which are used in the following. This introduction is not to be intended self consistent, but rather as a way to give essential definitions and theorems to proceed in the lecture, assuming not all theoreticians know the so - phisticated mathematical machinery of sheaf theory and Allendoerfer Eells theory.
For a necessary completion one can see e.g. Hirzebruch: Topological Methods in algebraic geometry 11) which I followed very close, or 12) for § 1-3 and 13), 14) for § 4.
Due to his aim, this chapter is as short as possible and does not contain any comment.

§ 1 Sheaves

Definition 2.1: A sheaf $\underset{\sim}{\mathcal{S}}$ of abelian group over X is a triple $\underset{\sim}{\mathcal{S}} = (S, \pi, X)$ such that

a) S and X are topological spaces and $\pi : S \longrightarrow X$ is an onto continous map;

b) every point $\alpha \in S$ has an open neighborhood N in S such that $\pi|_N$ is a homeomorphism between N and an open neighbourhood of $\pi(\alpha)$ in X

c) every $\pi^{-1}(x) \equiv S_x$, $x \in X$, called stalk, has the structure of and abelian group. The group operation associates to points $\alpha, \beta \in S_x$, the sum $\alpha + \beta \in S_x$ and the difference $\alpha - \beta \in S_x$. The difference depends continously on α and β.

Definition 2.2 : A map $h : \underset{\sim}{\mathcal{S}} = (S, \pi, X) \longrightarrow \underset{\sim}{\mathcal{S}}' = (S', \pi', X)$ is a homomorphism of sheaves if

a) h is a continous map from S to S'

b) $\pi = \pi' h$

c) $\forall x \in X$

$$h_x : S_x \longrightarrow S'_x$$

is a homomorphism of abelian groups.

Definition 2.3 : A presheaf over X consists of an abelian group S_U for each openset U of X and a homomorphism $r^U_V : S_U \longrightarrow S_V$ for each pair of open sets U, V of X with $V \subseteq U$ such that

a) if $U = \phi \qquad S_U = 0$

b) $r^U_U = Id_U$

c) if $W \subset V \subset U$ then $\quad r^U_W = r^V_W \, r^U_V$

Definition 2.4 : Let $\Sigma = \{ S_U , r^U_V \}$ and $\Sigma' = \{ S_U' , r'^U_V \}$

be presheaves over X. A homomorphism $h : \Sigma \to \Sigma'$ is a system $\{ h_U \}$ of homomorphisms $h_U : S_U \longrightarrow S_U'$ such that

$$r'^U_V \, h_U = h_V \, r^U_V$$

Every presheaf over X determines a sheaf over X by the following construction :

a) let $f \in S_U ,\ x \in U$; consider U running on all open sets containing x and put the equivalence relation

$f \in S_U ,\ g \in S_V \qquad f \simeq g$ if there exists an open neighbourhood W of x , $W \subseteq V$, $W \subseteq U$ such that

$$r^U_W f = r^V_W g$$

the equivalence class denoted by f_x is

called germ of f at x and let for each point , the direct limit of the abelian groups with respect to the homomorphisms

b) let $S = \bigcup_{x \in X} S_x$ and $\quad \pi : S \longrightarrow X$

$S_x \longrightarrow x$

c) let $\quad f_U = \bigcup_{y \in U} f_y$; the set f_U as U runs over all open sets of X and f over all elements of S_U form a basis for the topology of S.

Conversely given a sheaf there is a canonical presheaf associated to it by the following construction.

A section of a sheaf $\underset{\sim}{\mathcal{S}} = (S, \pi, X)$ over an open set U is a continous map $s: U \to S$ for which $\pi \circ s = Id_U$.

Denote by $\Gamma(U, \underset{\sim}{\mathcal{S}})$ the abelian group of all section of U.

If $V \subset U$ let $r^U_V : \Gamma(U, \underset{\sim}{\mathcal{S}}) \longrightarrow \Gamma(V, \underset{\sim}{\mathcal{S}})$ be the homomorphism which associates to each section of $\underset{\sim}{\mathcal{S}}$ over U its restriction to V.

The presheaf $\Sigma = \{\Gamma(U, \underset{\sim}{\mathcal{S}}), r^U_V\}$ is called the canonical presheaf of the sheaf $\underset{\sim}{\mathcal{S}}$ and proceeding by previous contruction to get a sheaf from Σ we will obtain $\underset{\sim}{\mathcal{S}}$.

Definition 2.5 : $\underset{\sim}{\mathcal{S}}' = (S', \pi', X)$ is a subsheaf of $\underset{\sim}{\mathcal{S}} = (S, \pi, X)$ if

a) S' is an open set of S

b) π' is the restriction of π toS' and maps S' onto X

c) $\pi'^{-1}(x) = S' \cap \pi^{-1}(x)$ is a subgroup of $\pi^{-1}(x)$

Theorem 2.6 Let $\underset{\sim}{\mathcal{S}}$ be a sheaf over a topological space X and $\underset{\sim}{\mathcal{S}}'$ a subsheaf of $\underset{\sim}{\mathcal{S}}$ with embedding $i: \underset{\sim}{\mathcal{S}}' \longrightarrow \underset{\sim}{\mathcal{S}}$

There exists a sheaf $\underset{\sim}{\mathcal{S}}''$ over X unique upto isomorphism, for which there is an exact sequence

$$0 \longrightarrow \underset{\sim}{\mathcal{S}}' \xrightarrow{i} \underset{\sim}{\mathcal{S}} \xrightarrow{h} \underset{\sim}{\mathcal{S}}'' \longrightarrow 0$$

$\forall x \in X$ the homomorphism h_x gives an isomorphism between the quotient group S_x / S'_x and S''_x .

2. Cohomology with coefficientsin a sheaf

Let $\Sigma = \{S_U, r_V^U\}$ be a presheaf over X and $\mathcal{U} = \{U_i\}_{i\in I}$ an open covering of X. A q-cochain is a function f which associates to each (q+1)-ple $(i_0 i_1 \dots i_q)$ of indices in I an element $f(i_0 \dots i_q) \in S_{(U_{i_0} \cap \dots \cap U_{i_q})}$

The q-cochains form a group $C^q(\mathcal{U}, \Sigma)$.

Define the coboundary homomorphism

$$\delta^q : C^q(\mathcal{U}, \Sigma) \longrightarrow C^{q+1}(\mathcal{U}, \Sigma)$$

$$(\delta^q f)(i_0, \dots i_q) = \sum_{k=0}^{q+1} (-1)^k \, r_W^{W_k} f(i_0, \dots \hat{i}_k, \dots i_{q+1})$$

where $\wedge$ means omitted symbol and

$$W = U_{i_0} \cap \dots \cap U_{i_{q+1}} \qquad W_k = U_{i_0} \cap \dots \cap \hat{U}_{i_k} \cap \dots \cap U_{i_{q+1}}$$

The cohomology group $H^q(\mathcal{U}, \Sigma)$ is defined by

$$H^q(\mathcal{U}, \Sigma) = \operatorname{Ker} \delta^q / \operatorname{im} \delta^{q-1}$$

The cohomology group $H^q(\mathcal{U}, \mathcal{S})$ is then defined as the cohomology group with coefficient in the canonical presheaf of $\mathcal{S}$.

Let now $\mathcal{B} = \{V_j\}_{j\in J}$ be a refinement of the open covering $\mathcal{U} = \{U_i\}_{i\in I}$. Choose a map $\tau : J \longrightarrow I$ so that $V_j \subset U_{\tau(j)}$ $\forall j \in J$ and define the homomorphism

$$\tau^* : C^q(\mathcal{U}, \Sigma) \longrightarrow C^q(\mathcal{B}, \Sigma)$$

$$(\tau^* f)(j_0 \dots j_q) = r_{W'}^{W} f(\tau j_0, \dots \tau j_q))$$

$$W = V_{j_0} \cap \dots \cap V_{j_q} \qquad W' = U_{\tau(j_0)} \cap \dots \cap U_{\tau(j_q)} \qquad W \subset W'$$

One can proove.

Lemma 2.7 τ^* induces a homomorphism

$$t^{\mathcal{U}}_{\mathcal{B}} : H^q(\mathcal{U}, \Sigma) \longrightarrow H^q(\mathcal{B}, \Sigma)$$

indipendent of the choice of the refining map τ, and $\{H^q(\mathcal{U}, \Sigma), t^{\mathcal{U}}_{\mathcal{B}}\}$ is a directed inductive family.

The cohomology group $H^q(X, \Sigma)$ is the direct limit on the groups $H^q(\mathcal{U}, \Sigma)$ with respect to the homomorphism $t^{\mathcal{U}}_{\mathcal{B}}$ where $\mathcal{U}$ runs over all covering of X.

The cohomology group $H^q(X, \underset{\sim}{\mathcal{S}})$ is the cohomology group relative to the canonical presheaf of $\underset{\sim}{\mathcal{S}}$.

Theorem 2.8 $\quad H^0(\mathcal{U}, \underset{\sim}{\mathcal{S}}) \simeq H^0(X, \underset{\sim}{\mathcal{S}}) \simeq \Gamma(X, \underset{\sim}{\mathcal{S}})$

Theorem 2.9 Let $0 \longrightarrow \Sigma' \longrightarrow \Sigma \longrightarrow \Sigma'' \longrightarrow 0$ an exact sequence of presheaves, where $S''_U \simeq S_U/S'_U$, over a topological space X; then there is for each open covering $\mathcal{U}$ of X an exact cohomology sequence

$$0 \longrightarrow H^0(\mathcal{U}, \Sigma') \longrightarrow H^0(\mathcal{U}, \Sigma) \longrightarrow H^0(\mathcal{U}, \Sigma'') \longrightarrow \cdots$$
$$\cdots \longrightarrow H^q(\mathcal{U}, \Sigma') \longrightarrow H^q(\mathcal{U}, \Sigma) \longrightarrow H^q(\mathcal{U}, \Sigma'') \longrightarrow \cdots$$

Moreover the inductive limit over all open coverings $\mathcal{U}$ of X gives rise to the exact cohomology sequence

$$0 \longrightarrow H^0(X, \Sigma') \longrightarrow H^0(X, \Sigma) \longrightarrow H^0(X, \Sigma'') \longrightarrow \cdots$$
$$\cdots \longrightarrow H^q(X, \Sigma') \longrightarrow H^q(X, \Sigma) \longrightarrow H^q(X, \Sigma'') \longrightarrow \cdots$$

Theorem 2.10 Let Σ be a presheaf over a paracompact space X and let $\underset{\sim}{\mathcal{S}}$ be the corresponding sheaf. Then there is an isomorphism :

$$h_* : H^q(X, \Sigma) \longrightarrow H^q(X, \underset{\sim}{\mathcal{S}})$$

Theorem 2.11 An exact sequence

$$0 \longrightarrow \underset{\sim}{\mathcal{S}}' \longrightarrow \underset{\sim}{\mathcal{S}} \longrightarrow \underset{\sim}{\mathcal{S}}'' \longrightarrow 0$$

of sheaves over a paracompact space X gives an exact cohomology sequence

$$0 \longrightarrow H^0(X, \underset{\sim}{\mathcal{S}}') \longrightarrow H^0(X, \underset{\sim}{\mathcal{S}}) \longrightarrow H^0(X, \underset{\sim}{\mathcal{S}}'') \longrightarrow \cdots$$

$$\cdots \longrightarrow H^q(X, \underset{\sim}{\mathcal{S}}') \longrightarrow H^q(X, \underset{\sim}{\mathcal{S}}) \longrightarrow H^q(X, \underset{\sim}{\mathcal{S}}'') \longrightarrow \cdots$$

Theorem 2.12 Let $\underset{\sim}{\mathcal{S}}$ be a sheaf over a topological space X, let $\mathcal{U}$ be an open covering of X such that

$$U_{i_0} \cap \cdots \cap U_{i_q} \equiv U_{\Sigma_q} \qquad \forall q \geq 1$$

satisfies

$$H^i(U_{\Sigma_q}, \underset{\sim}{\mathcal{S}}) = 0 \qquad i \geq 1$$

then

$$H^p(\mathcal{U}, \underset{\sim}{\mathcal{S}}) \simeq H^p(X, \underset{\sim}{\mathcal{S}})$$

3 Fibre bundles

Let X be a topological space. A sheaf $\underset{\sim}{\mathcal{S}} = (S, \pi, X)$ of multiplicative generally non abelian group over X is defined if a) and b) of Definition 2.1 hold and

c) every stalk has the structure of a group, the group operations associates to points $\alpha, \beta \in S_x$ the element $\alpha\beta, \alpha\beta^{-1} \in S_x$; $\alpha\beta^{-1}$ depends continously on α and β.

Definitions of presheaf, canonical presheaf, etc. carry over with similar modifications.

Cohomology groups $H^q(X, \underset{\sim}{\mathcal{S}})$ cannot be defined in the non abelian case, however for q=1 one can define a cohomology set $H^1(X, \underset{\sim}{\mathcal{S}})$ with distinguished element $\mathbb{1}$ by the following construction.

Let $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of X. A $\mathcal{U}$-cocycle is a function $f: i,j \in I \longrightarrow f_{ij} \in \Gamma(U_i \cap U_j, \underset{\sim}{\mathcal{S}})$

such that

$$f_{ij} f_{jk} = f_{ik} \qquad \text{on} \qquad U_i \cap U_j \cap U_k$$

Denote $Z^1(\mathcal{U}, \underset{\sim}{\mathcal{S}})$ the set of $\mathcal{U}$-cocycles.

$f, f' \in Z^1(\mathcal{U}, \underset{\sim}{\mathcal{S}})$ are said to be cohomologous if $\forall i \in I$ there exists an $g_i \in \Gamma(U_i, \underset{\sim}{\mathcal{S}})$ such that

$$f'_{ij} = g_i^{-1} f_{ij} g_j \qquad \text{on } U_i \cap U_j \qquad \forall i,j \in I$$

Definition 2.13 The cohomology set $H^1(\mathcal{U}, \underset{\sim}{\mathcal{S}})$ is the set of equivalence classes of $\mathcal{U}$-cocycles.

Let now $\mathcal{B} = \{V_j\}_{j\in J}$ be a refinement of the open covering $\mathcal{U} = \{U_i\}_{i\in I}$. Choose a map $\tau : J \longrightarrow I$ so that $V_j \subset U_{\tau(j)}$ $\forall j \in J$ and define a map

$$\tau^* : Z^1(\mathcal{U}, \underset{\sim}{\mathcal{S}}) \longrightarrow Z^1(\mathcal{B}, \underset{\sim}{\mathcal{S}})$$

$$(\tau^* f)_{ij} = f_{\tau(i),\tau(j)} \quad \text{on } V_i \cap V_j \quad \forall i,j \in J$$

Lemma 2.14 τ^* induces a homomorphism

$$t^{\mathcal{U}}_{\mathcal{B}} : H^1(\mathcal{U}, \underset{\sim}{\mathcal{S}}) \longrightarrow H^1(\mathcal{B}, \underset{\sim}{\mathcal{S}})$$

indipendent of the choice of the refining map τ, and $\{t^{\mathcal{U}}_{\mathcal{B}}, H^1(\mathcal{U}, \underset{\sim}{\mathcal{S}})\}$ is a directed inductive family.

Definition 2.15 The cohomology set $H^1(X, \underset{\sim}{\mathcal{S}})$ is the direct limit of the set $H^1(\mathcal{U}, \underset{\sim}{\mathcal{S}})$ with respect to the maps $t^{\mathcal{U}}_{\mathcal{B}}$, as $\mathcal{U}$ runs over all coverings of X.

One can prouve that $H^1(X, \underset{\sim}{\mathcal{S}})$ can also be defined as the union of all sets $H^1(\mathcal{U}, \underset{\sim}{\mathcal{S}})$ as $\mathcal{U}$ runs over all coverings of X; the distinguished element $1 \in H^1(X, \underset{\sim}{\mathcal{S}})$, is represented for any open covering $\mathcal{U} = \{U_i\}_{i\in I}$ by the cocycle $f_{ij} = 1 \in \Gamma(U_i \cap U_j, \underset{\sim}{\mathcal{S}})$

Let X be a topological space, F a topological space, G a topological group with identity e and $\underset{\sim}{G}$ the sheaf whose canonical presheaf is $\{\Gamma(U, \underset{\sim}{G}), r^U_V\}$ where $\Gamma(U, \underset{\sim}{G})$ is the group of continous functions from U to G.

Definition 2.16 A topological space W, together with a map $\pi : W \to X$ is called a fibre bundle over X with structure group G and typical fibre F if

a) there is a continous effective action of G on F, i.e. a continous map $G \times F \to F$

satisfying
$$g_1(g_2 f) = (g_1 g_2) f \qquad g_1, g_2 \in G$$
$$ef = f \qquad \forall f \in F$$
$$gf = f \ \forall f \in F \Rightarrow g = e$$

b) an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of X and homeo - morphisms
$$h_i : \pi^{-1}(U_i) \to U_i \times F$$
$$\pi^{-1}(x) \to (x, F)$$

c) elements $g_{ij} \in \Gamma(U_i \cap U_j, \mathcal{G})$ $\forall i, j \in I$, called transition functions, such that
$$(h_i h_j^{-1})(x, f) = (x, g_{ij}(x) f) \qquad \forall x \in U_i \cap U_j,\ f \in F$$

Remark :

a) defines an action of G on W : $G \times W \to W$ which on U_i can be represented by
$$g h_i^{-1}(x, f) = h_i^{-1}(x, gf)$$

Two systems $\{(U_i, h_i)\}_{i\in I}\ \{U'_j, h'_j\}_{j\in J}$ over X define the same fibre bundle if $\forall i \in I, j\in J$ exists $g_{ij} \in \Gamma(U_i \cap U'_j, \underset{\sim}{G})$ such that

$$(h_i\, h_j'^{-1})(x,f) = (x, g_{ij}(x)f) \qquad \forall x \in U_i \cap U'_j,\ f \in F$$

Definition 2.17 Let (W, π, X) and (W', π', X) be fibre bundles over X with structure group G and fibre F. An isomorphism $K: (W, \pi, X) \to (W', \pi', X)$ is a homeomorphism $K: W \to W'$ such that

a) $\pi = \pi' \circ K$

b) $\forall x \in X$ there is an open neighbourhood U of x, an element $g_U \in \Gamma(U, \underset{\sim}{G})$ and homeomorphisms

$$h_U : \pi^{-1}(U) \to U \times F$$
$$h'_U : \pi'^{-1}(U) \to U \times F$$

such that

$$(h'_U \circ K \circ h_U^{-1})(x,f) = (x, g_U(x)f) \qquad \forall x \in U,\ f \in F$$

Theorem 2.18 The isomorphism classes of fibre bundles over X with structure group G and fibre F are in a natural one-one corrispondence with the elements of the cohomology set $H^1(X, \underset{\sim}{G})$. The trivial fibre bundle $W = X \times F$ corresponds to the distinguished element $1\!\!1 \in H^1(X, \underset{\sim}{G})$

Definition 2.19 If F=G and the action of G on itself is left translation, then the fibre bundles is called a principal bundle with structure group G.

4 Connections and gauge transformations.

Let us now consider the case in which a fibre bundle is a differentiable principal bundle; this means that W and X are now manifolds, G is a Lie group and $\Gamma(U, \underset{\sim}{G})$ is the group of all C^∞ maps from U to G, π and h_i are now C^∞ maps.

Let τ be an element of the lie algebra $\mathcal{G}$ of G and $g_\tau(t)$ the one parameter subgroup generated by τ.

Denote with $f_g : \begin{array}{l} W \longrightarrow W \\ w \longrightarrow gw \end{array}$ $\qquad w \in W \quad g \in G$

Definition 2.20 A fondamental vector field $Y(\tau)$ is the vector field on W induced by a one parameter group of transformation $f_{g_\tau(t)}$

Definition 2.21 A connection form α is a $\mathcal{G}$-valued one form on W such that

a) $\alpha\,(Y(\tau)) = \tau$

b) $f_g^{*}\,\alpha = Ad\, g^{-1}\alpha$

Let $\mathcal{U}=\{U_i\}$ be an open covering of X and $\sigma_i : U_i \longrightarrow W$ a section (i.e. $\pi\circ\sigma_i = Id_{U_i}$)
satisfying $\sigma_i(x) = h_i^{-1}(x,e)$ and define the $\mathcal{G}$ valued one form on U_i by

$$A_i = \sigma_i^{*}\alpha$$

This is the usual (local) gauge potential.

Theorem 2.21 The local forms A_i; verify in $U_i \cap U_j$ the compatibility conditions

$$A_j = Ad_{g_{ij}^{-1}}\, A_i + g_{ij}^{-1}\, dg_{ij}$$

where g_{ij} are the transition functions of the bundle.
We have also the converse

Theorem 2.22 Given a collection of local forms A_i on X verifying the avove compatibility conditions, there exists a unique form α on W such that $A_i = \sigma_i^{*}\alpha$

Definition 2.23 The $\mathcal{G}$ -valued two form on W

$$\beta = d\alpha + \frac{1}{2}\,\alpha\wedge\alpha$$

is called curvature form.
The $\mathcal{G}$-valued two form on U_i $\sigma_i^{*}\beta = F_i$ is the (local) field strength.
Let us finally see how fits the gauge transformation in this scheme.

Definition 2.24 A gauge transformation in (W, π, X, G) is a map $f: W \longrightarrow W$ such that

a) $\pi \circ f = \pi$

b) $\forall i \in I \qquad (h_i \circ f \circ h_i^{-1})(x, a) = (x, g_i(x) a)$

$a \in G \quad g_i \in \Gamma(U_i \mathcal{G})$

where g_i satisfy on $U_i \cap U_j$ the compatibility condition

$$g_j(x) = f_{ij}^{-1}(x) \quad g_i(x) \; f_{ij}(x)$$

Clearly a gauge transofrmation can be also seen on U_i as a change of section through

$$\sigma_i'(x) = g_i(x) \, \sigma_i(x)$$

Lemma 2.25 Under this change the (local) gauge potential changes according to

$$A_i' = \sigma_i'^{*} \alpha = g_i^{-1} A_i g_i + g_i^{-1} dg_i$$

and the(local) field strength according to

$$F_i' = \sigma_i'^{*} \beta = g_i^{-1} F_i g_i$$

If G is abelian F can be globally defined on X and is gauge invariant.

Remark : Condition b) in Definition 2.24 can be substituted by

$$f(gp) = g f(p) \qquad \begin{matrix} g \in G \\ p \in W \end{matrix}$$

5 Allendoerfer-Eels cohomology theory.

Let $E_K(X)$ denote the additive group of smooth k-forms on a manifold X and let d^K be the exterior differential acting on k-forms.

Definition 2.26 The cohomology group

$$\operatorname{Ker} d^K / \operatorname{im} d^{K-1} \equiv H^K_{deR}(X)$$

is called the k-th de Rham cohomology group of X.
Let now $\mathcal{H}_{deR}(X)$ be its derived cohomology algebra.
For each $\theta \in E_K(X)$ we define the smooth singular cochain $h\theta$ by the formula

$$h\theta \cdot c = \int_c \theta$$

for smooth real chain c. It follows from Sto.ke's theorem that h induces a homomorphism h^* on cohomology classes.

Theorem 2.27 h^* is an algebra isomorphism of $\mathcal{H}_{deR}(X)$ onto the singular cohomology älgebra $H(X,\mathbb{R})$ of X.

Remark: $H(X,\mathbb{R})$ is isomorphic to the cohomology of the sheaf of germs od constant $\mathbb{R}$-valued fuctions.

The purpose of Allendoerfer-Eels theory 15) is to find an analogous theorem for $H(X,\mathbb{Z})$.

Definition 2.27 An $(\mathbb{Z}, r)$ pair (θ, ω) of forms on X of dimention n is a couple of forms θ and ω such that

a) ω is a $r-1$ smooth form defined on X except perhaps for a closed rare set $e(\omega)$ lieing on a smooth locally finite polyhedra of dimension not exceeding $n-r$ (if $r=0$, ω is zero function on X).

b) θ is an extension of $d\omega$ to $X-e(\theta)$ where $e(\theta)$ is a closed subset of $e(\omega)$ lieing on a smooth locally finite polyhedron of dimension not exceeding $n-r-1$

c) for any chain admissible to (θ, ω), i.e. such that

$$\partial c \cap e(\omega) = \phi \qquad c \cap e(\theta) = \phi$$

one as

$$\int_c \theta - \int_{\partial c} \omega \in \mathbb{Z}$$

Theorem 2.28 Let c_t $(0 \le t \le 1)$ be a smooth deformation of the chain c_0 on X such that

$$\partial c_t \cap e(\omega) = \phi \qquad c_t \cap e(\theta) = \phi \qquad \forall t$$

then

$$\int_{c_0} \theta - \int_{\partial c_0} \omega = \int_{c_1} \theta - \int_{\partial c_1} \omega$$

Definition 2.29 Two $(\theta, \omega), (\theta', \omega')$ $(\mathbb{Z}, r)$ pairs are defined to be equivalent if for any chain which is admissible for both pairs

$$\int_c \theta - \int_{\partial c} \omega = \int_c \theta' - \int_{\partial c} \omega'$$

The equivalence classes are denoted by $[\theta, \omega]$ and form a $\mathbb{Z}$ module denoted by $\mathcal{C}^{n}(X, \mathbb{Z})$.

One can define the exterior differential

$$d^{n}: \mathcal{C}^{n}(X,\mathbb{Z}) \longrightarrow \mathcal{C}^{n+1}(X,\mathbb{Z})$$
$$[\theta,\omega] \longrightarrow d^{n}[\theta,\omega] \equiv [0,\theta]$$

Then d is an $\mathbb{Z}$ homomorphism such that $d \circ d = 0$

Theorem 2.30 Denote $\zeta^{0}(X,\mathbb{Z})$ = kernel of d^{0} then the natural map $\zeta^{0}(X,\mathbb{Z}) \longrightarrow \mathbb{Z}$ is an isomorphism. (X connected)

Theorem 2.31 There is a canonical isomorphism of the cohomology algebra $\mathcal{H}_{A.E.}(X,\mathbb{Z})$ of the $\mathbb{Z}$ pair forms onto the cohomology algebra $H(X,\mathbb{Z})$.

Given any smooth integer chain c for each $(\mathbb{Z}, r)$ pair $[\theta, \omega]$ define the smooth singular cochain $h[\theta,\omega]$ by

$$h[\theta,\omega] \cdot c = \int_{c} \theta - \int_{\partial c} \omega$$

h commutes with d and thus defines a homomorphism on cohomology classes. Theorem 2.31 asserts that this is an isomorphism.

GENERALIZED PRINCIPAL BUNDLES

1 The bundle

In this chapter we consider an extension, due to Tulczyjew 3), of the $\mathbb{R}$-principal bundle to the case of a gauge group of k ránk forms.

The first step to discuss the $k > 0$ case of gauge differen tial forms, is the definition of a suitable trivial bundle which generalize the bundle $(M \times \mathbb{R}, \pi, M, \mathbb{R})$, M being a manifold and π a C^∞ map.

Sections of this bundle can be identified (through the graph relation) with the zero form in M, $E_0(M)$.

This suggests the generalization to a bundle such that his sections are $E_k(M)$, the additive group of k-forms on M, and this is nothing but the k-skewsymmetric cotangent bundle $P^k \equiv (\Lambda^k T^* M, \pi^k, M)$ where π^k is the canonical progection.

On this bundle acts naturally and effectively the infinite dimensional abelian group of k-forms by pointwise addition:

$$\gamma^k ; E_k(M) \times P^k \longrightarrow P^k$$
$$(s, p) \longrightarrow p + s(\pi^k(p))$$

or on each fibre

$$E_k(M) \times \Lambda^k T_x^* M \longrightarrow \Lambda^k T_x^* M$$
$$(s, p_x) \longrightarrow p_x + s(x)$$

Clearly $E_k(M)$ can be considered as a group of the sections of the sheaf of germs of k-forms on M, denoted by $\underset{\sim}{\Lambda}^k$, i.e. $E_k(M) \equiv \Gamma(M, \underset{\sim}{\Lambda}^k)$.

The action γ^k allow us to define the fundamental vector field on P^k as follows.

Define

$$\gamma_s^k : P^k \longrightarrow P^k$$
$$p \longrightarrow \gamma^k(s, p)$$

and let $c : I \longrightarrow E_k(M)$ be a C^1 curve with $c(0) = s$; denote by $\gamma^k_{c(t)}$ the induced flow on P^k ; then the fundamental vector field $W^k(s)$ is defined by

$$W^k(s)\Big|_p = \frac{d}{dt}\gamma^k_{c(t)}(p)\Big|_{t=0}$$

B ng $\Lambda^k T^* M$ the total spaces of P^k, they admit naturally the canonical k-forms $\theta^k \in E_k(P^k)$ and $\omega^k = d\theta \in E_{k+1}(P^k)$ defined by

$$(\theta^k, v_1 \ldots v_k)\Big|_p = (p \mid T\pi^k(v_1) \ldots T\pi^k(v_k))\Big|_{\pi(p)}$$

where $v_i \in T_p(\Lambda^k T^* M)$ for $k>0$ and

$$\vartheta^0 = pr : \mathbb{R} \times M \longrightarrow \mathbb{R}$$

One immediately see that the following equations hold

$$i_{W^k(s)}\vartheta^k = 0$$

$$\gamma_s^* \vartheta^k = \vartheta^k + \pi^{k*} s \qquad \mathcal{L}_{W^k(s)} \vartheta^k = \pi^{k*} s$$

$$\iota_{W^k(s)}\,\omega^k = \pi^{k*} s$$

$$\gamma_s^*\,\omega^k = \omega^k + \pi^{k*} ds \qquad \mathcal{L}_{W^k(s)}\,\omega^k = \pi^{k*} ds$$

If $\{x^\mu\}$ are the coordinates in $U \subset M$ and $\{x^\mu, p_{\mu_1 \ldots \mu_k}\}$ are the coordinates in $\Lambda^k T^* U$, we have

$$\vartheta^k\Big|_{\Lambda^k T^* U} = \frac{1}{k!}\, p_{\mu_1 \ldots \mu_k}\, dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_k}$$

Let me now give in analogy with 2.16, 2.19 the definition of a generalized principal bundle.

Let M be a manifold and $\underset{\sim}{\Lambda}^k$ denote the sheaf of germs of differential forms of rank k in M.

Definition 3.1 A manifold P together with a C^∞ map $\pi : P \to M$ is called a generalized principal bundle of type k if

a) there is a differentiable effective action γ of $E_k(M)$ on P

b) an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ of M and differmorphisms

$$h_i : \pi^{-1}(U_i) \longrightarrow (\pi^k)^{-1}(U_i)$$
$$\pi^{-1}(x) \longrightarrow (\pi^k)^{-1}(x)$$

satisfying

$$h_i(\gamma(s, p)) = h_i(p) + s(\pi(p))$$

c) elements $g_{ij} \in \Gamma(U_i \cap U_j, \underset{\sim}{\Lambda}^k) \quad \forall i,j \in I,$

called transition forms, such that

$$(h_i\, h_j^{-1})\, p^k = p^k + g_{ij}(\pi^k(p^k)) \qquad \forall x \in U_i \cap U_j, \; p^k \in (\pi^k)^{-1}(U_i \cap U_j)$$

Two systems $\{(U_i h_i)\}_{i \in I}$, $\{(U'_j, h'_j)\}_{j \in J}$ define the same generalized principal bundle of type k if $\forall i \in I, j \in J$ there exists $f_{ij} \in \Gamma(U_i \cap U'_j, \underset{\sim}{\Lambda}^k)$ such that

$$(h_i\, h'_j)^{-1}(p) = p + f_{ij}(\pi(p)) \cdot \quad \forall x \in U_i \cap U'_j$$

The main differences with a principal bundle are the following :

a) the generalized principal bundle is not locally isomorphic to a trivial bundle but to $\Lambda^k T^* M$

b) the group depends on the base space

c) the action is effective but not free

d) the fibre is an affine space associated to the vector space $\Lambda^K T^*_x M$

2 Connections and gauge transformations

In analogy with γ^k on P^k, the action of $E_k(M)$ on P allow us to define a fundamental vector field

$$W(s)|_p = \frac{d}{dt} \gamma_{c(t)}(p) \Big|_{t=0}$$

On U_i we clearly have

$$W(s) = T h_i^{-1} W^k(s)$$

Definition 3.2 A k+1 form $\alpha \in E_{k+1}(P)$ will be called a connection form for a generalized principal bundle of type k if

$$\begin{aligned} i_{W(s)} \alpha &= \pi^* s \\ \mathcal{L}_{W(s)} \alpha &= \pi^* ds \end{aligned} \qquad \forall s \in E_k(M)$$

The connection α has the transformation property

$$\gamma_s^* \alpha = \alpha + \pi^* ds$$

Let $\mathcal{U}=\{U_i\}_{i\in I}$ be an open covering of M and $\sigma_i : U_i \longrightarrow P$ a section satisfying $\sigma_i(x)=h_i^{-1}(0)$, 0 being the zeroth 0 form; define the k+1 form on U_i by

$$A_i = \sigma_i^* \alpha \qquad (= A_{\mu_1 \cdots \mu_{k+1}} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_{k+1}})$$

This is the usual (local) gauge potential for gauge differential forms of rank k+1.

Theorem 3.3 The local forms A_i verify in $U_i \cap U_j$ the compatibility conditions

$$A_i = A_j + dg_{ij} \qquad g_{ij} \in \Gamma(U_i \cap U_j, \Lambda^k)$$

where g_{ij} are the transition forms of the bundle.

Proof. Define $\phi_i \in E_{k+1}(U_i)$ by

$$\phi_i = \pi^* A_i + h_i^* \omega^k$$

Suppose now $X \in T_p P$ with $p = \sigma_i(x)$, then it can be decomposed in a unique way as a sum $X = Y + W(s)$ where Y is tangent at p to the section σ , and W is vertical.

We have $\quad Y = T\sigma_i \, T\pi(Y) \qquad T\pi(W) = 0$

Thus

$$i_X \phi_i = i_{T\pi(Y)} A_i + i_W h_i^* \omega^k =$$

$$= i_Y \alpha + \pi^* s = i_Y \alpha + i_W \alpha = i_X \alpha$$

Moreover ϕ_i and $\alpha|_{\pi^{-1}(U_i)}$ have the same transformation pro-

perty , thus
$$\phi_i = \alpha|_{\pi^{-1}(U_i)}$$

therefore on $\pi^{-1}(U_i \cap U_j)$ we have
$$\pi^* A_i + h_i^* \omega^k = \pi^* A_j + h_j^* \omega^k$$

i.e.
$$\pi^* A_j = \pi^* A_i + (h_j h_i^{-1})^* \omega^k = \pi^* A_j + \pi^* dg_{ij}$$

Then $A_i = A_j + dg_{ij}$ on $U_i \cap U_j$

Theorem 3.4 Given a collection of local k+1 forms A_i on M satisfying the above compatibility conditions, there exists a unique form α on P such that
$$A_i = \sigma_i^* \alpha$$

Proof. Define $\phi_i = \pi^* A_i + h_i^* \omega^k$; by compatibility condition on $\pi^{-1}(U_i \cap U_j)$, $\phi_i = \phi_j$ thus one can extend ϕ to the whole P; moreover
$$i_{W(s)} \phi_i = \pi^* s$$
$$\mathcal{L}_{W(s)} \phi_i = \pi^* ds$$

thus ϕ define a connection and
$$\sigma_i^* \phi_i = A_i$$

Definition 3.5 $\beta = d\alpha \in E_{k+2}(P)$ is called curvature form.

It is easy to prove that

$$i_{W(s)}\beta = 0 \qquad\qquad\qquad\qquad (3.6)$$
$$\mathcal{L}_{W(s)}\beta = 0 \qquad\qquad \gamma_s^{*}\beta = \beta$$

From (3.6) it follows that there exists a $F \in E_{k+2}(M)$ such that

$$\beta = \pi^{*}F \qquad\qquad\qquad\qquad (3.7)$$

F is the usual field strength of gauge differential forms of rank k+1 and is defined on the whole M.

Let us finally see how fit gauge transofrmations for gauge differential forms in this scheme.

Definition 3.8 A gauge transformation on a generalized principal bundle of type k $(P, \pi, M, E_k(M))$ is a map $f: P \longrightarrow P$ such that

a) $\pi \circ f = \pi$

b) $\exists\, g_i \in \Gamma(M, \underset{\sim}{\Lambda}^{k})$ such that if $\mathcal{U} = \{U_i\}_{i\in I}$ is a covering of M,

$$(h_i \circ f \circ h_i^{-1})(p^k) = p^k + g_i(\pi^k(p^k))$$
$$\forall p^k \in (\pi^k)^{-1}(U_i)$$

Remark: Condition a), b) can be substituted by

a) $\pi \circ f = \pi$

b) $f(\gamma(s,p)) = \gamma(s, f(p))$

Clearly a gauge transformation can be also seen on U_i as a change of section or trivialization through

$$\sigma_i'(x) = \gamma_{g(x)} \circ \sigma_i(x) \qquad (h_i')^{-1}(p^k) = \gamma_{g(\pi(p))} \circ h_i^{-1}(p^k)$$

Under this change, $A_i = \sigma_i^* \alpha$ behaves as follows :

$$\pi^* A_i - \pi^* A_i' = h_i^* \omega^k - h_i'^* \omega^k = \pi^* dg$$

thus

$$A_i' = A_i + dg \qquad g \in \Gamma(M, \underset{\sim}{\Lambda}^k) \text{ on } U_i \tag{3.7}$$

The curvature is clearly gauge invariant $F = F'$.

These are the usual gauge transformation properties of gauge potential for k+1 gauge differential forms.

3 Isomorphism classes.

Definition 3.8 Let $\mathcal{G} = (P, \pi, M, E_k(M))$ and $\mathcal{G}' = (P', \pi', M, E_k(M))$ be generalized principal bundles of type k. An isomorphism $\rho : \mathcal{G} \to \mathcal{G}'$ is a differmorphism $\rho : P \to P'$ such that

a) $\pi = \pi' \circ \rho$

b) $\forall x \in M$ there is an open neighbourhood U of x and element $g_U \in \Gamma(U, \underset{\sim}{\Lambda}^k)$ and differmorphisms

$$h_U : \pi^{-1}(U) \longrightarrow (\pi^k)^{-1}(U)$$
$$h'_U : (\pi')^{-1}(U) \longrightarrow (\pi^k)^{-1}(U)$$

such that

$$(h'_U \circ p \circ h_U^{-1})(p^k) = p^k + g_U(\pi^k(p^k)) \quad \forall p^k \in (\pi^k)^{-1}(U)$$

Theorem 3.9 5) The isomorphism classes of generalized principal bundles of type k over M are in a natural one-one correspondence with the elements of the cohomology group

$$H^1(M, \underset{\sim}{\Lambda}^k)$$

Proof. Let $\mathcal{U} = \{U_i\}_{i \in I}$ an open covering of M and $f = \{f_{ij}\} \in Z^1(\mathcal{U}, \underset{\sim}{\Lambda}^k)$, then we can construct a generalized principal bundle of type k over M taking the disjoint union of $(\pi^k)^{-1}(U_i)$, and identifying $p^k \in (\pi^k)^{-1}(U_i)$ and $p^k + f_{ij}(\pi^k(p^k)) \in (\pi^k)^{-1}(U_j)$ for every $p^k \in (\pi^k)^{-1}(U_i \cap U_j)$.
If $f \in Z^1(\mathcal{U}, \underset{\sim}{\Lambda}^k)$ and $g \in Z^1(\mathcal{B}, \underset{\sim}{\Lambda}^k)$ then the corresponding generalized principal bundles obtained by the above construction are isomorphic if and only if g and h represent the same element of the cohomology group $H^1(M, \underset{\sim}{\Lambda}^k)$.
In fact take a covering $\mathcal{W} = \{W_\ell\}_{\ell \in L}$ which is a refinement of both $\mathcal{U} = \{U_i\}_{i \in I}$ and $\mathcal{B} = \{B_j\}_{j \in J}$, then there are maps

$\tau_J : L \longrightarrow J$ such that $W_\ell \supset B_{\tau_J(\ell)}$.
$\tau_I : L \longrightarrow I$ such that $W_\ell \supset B_{\tau_I(\ell)}$

Given a $\mathcal{U}$-cocycle f and a $\mathcal{B}$-cocycle g one defines a cocycle in $\mathcal{W}$ by

$$(\tau_J^* g)_{\ell\ell'} = g_{\tau_J(\ell)\,\tau_J(\ell')} \qquad \text{in } W_\ell \cap W_{\ell'}$$
$$(\tau_I^* f)_{\ell\ell'} = f_{\tau_I(\ell)\,\tau_I(\ell')} \qquad \text{in } W_\ell \cap W_{\ell'}$$

The maps

$$\tau_I^* : H^1(\mathcal{U}, \underset{\sim}{\Lambda}^k) \longrightarrow H^1(\mathcal{W}, \underset{\sim}{\Lambda}^k)$$
$$\tau_J^* : H^1(\mathcal{B}, \underset{\sim}{\Lambda}^k) \longrightarrow H^1(\mathcal{W}, \underset{\sim}{\Lambda}^k)$$

are injections.

By definition 3.1 the two bundles constructed from f and g are isomorphic if $(\tau_J^* g)$ and $(\tau_I^* f)$ are cohomologous, this means that they represent the same cocycle in $H^1(M, \underset{\sim}{\Lambda}^k)$

Conversely given an element $\xi \in H^1(M, \underset{\sim}{\Lambda}^k)$ we shall represent it by a cocycle of some covering $\mathcal{U} = \{ U_i \}_{i \in I}$ and construct the bundle for which that cocycle is the cocycle of transition forms.

Corollary 3.10 All generalized principal bundles of type k over M are isomorphic.

Proof Using the partition of unity one can prove in a standard way that

$$H^1(M, \underset{\sim}{\Lambda}^k) = 0$$

(the sheaf $\underset{\sim}{\Lambda}^k$ is fine).

Corollary 3.11 The field strength F is an exact form, i.e. the gauge potential A_i can be extended to the whole M .

Proof. The only element of $H^1(M, \Lambda^k)$ is the O cocycle, thus for every covering $\mathcal{U}$ one construct a bundle whose transition forms are O, thus on $U_i \cap U_j$

$$A_i = A_j$$

and one can define A over all M and $(h_i h_j^{-1}) p^k = p^k$, i.e. one can extend h_i to the whole M. Thus

$$\alpha = \pi^* A + h^* \omega^k$$
$$d\alpha = \pi^* dA = \pi^* F \qquad A \in E_{k+1}(M)$$

4 "Matter" like gauge differential forms.

Let us end up this chapter discussing the "matter" like Kalb-Ramond fields 6).

The lagrangian which defines this dynamical theory is constructed in terms of k and k+1 form $A \in E_{k+1}(M)$, $\phi \in E_k(M)$ and has the following form

$$\mathcal{L} = \frac{1}{g^2} dA \wedge {}^* dA + A \wedge {}^* A + d\phi \wedge {}^* A + d\phi \wedge {}^* d\phi$$

It possess the gauge invariance

$$A \rightarrow A + ds$$
$$\phi \rightarrow \phi - s$$

We want now show that the "matter" like gauge transformation of ϕ naturally fits in the above scheme.

In fact let $(P, \pi, M, E_K(M))$ a generalized principal bundle of type k over M and α the connection form whose gauge potential is $A \in E_{k+1}(M)$.

A "matter" like k-form ϕ is defined as follow: let σ be a global cross section of P which exists because of 3.10, and h a trivialisation; define $\phi \in E_k(M)$ by

$$\phi \equiv h \circ \sigma \tag{3.12}$$

We already know that a change of trivialisation relative to a gauge transformation represented by $s \in \Gamma(M, \underset{\sim}{\Lambda}^k)$ induces on A the transformation

$$A \to A' = A + ds$$

but by definition 3.12 and by

$$(h')^{-1} = \gamma_s \, h^{-1}$$

it follows that

$$\phi \to \phi' = \phi - s$$

as required.

PRINCIPAL $\mathcal{G}^k$ BUNDLE OVER LOOP SPACES

1 Motivation

As we have seen in the preceding section although Tulczyjew's generalized principal bundles give a coordinate gauge indipendent description of gauge differential forms, they are necessarily trivial, i.e. the local representative of the connection k+1 form A can be extended to the whole manifold M and thus the (de Rham) cohomology class of the curvature $F=dA$ is necessarily the trivial one.
So we don't have the usual possibility of $U(1)$-principal bundle (k=0) to have non trivial insomorphism classes of bundles and non trivial (integer) cohomology classes for the curvature $F \in E_2(M)$.
In order to obtain something similar for gauge differential forms, we have to leave the manifold M where the action of gauge differential forms is defined pointwise, giving rise to fibres which are isomorphic to $\mathbb{R}^n$ (for some n) and thus leading to trivial bundle.

The main observation to get up this difficulty was already made by Nambu 16) in 1976 and it goes as follows.
Let $\Omega^k M$ be the space of all C^∞ maps from S^k to M (for the moment I leave out all details); a k-form $\Lambda \in E_k(M)$ defines naturally a $\mathbb{R}$ or $U(1)$ valued function over $\Omega^k M$ by

$$\Omega^k M \ni \omega \longrightarrow \int_{|\omega|} \Lambda \in \mathbb{R}$$

$$\Omega^k M \ni \omega \longrightarrow \exp 2\pi i \int_{|\omega|} \Lambda \in U(1)$$

where $|\omega|$ is the image of ω in M.
We thus have a natural interpretation of gauge transformation for gauge differential forms and in particular the possibility of defining $U(1)$ transformations in such a way to come closer to the $U(1)$ -principal bundle (k=0).
Starting from this naive suggestion essentially two temptatives are made, at least at my knowledge, to give a geometrical interpretation of gauge differential form theories on loop spaces: one is due again to Tulczyjew 3) one to Freund and Nepomechie 17).
Although both constructions are not worked out in all details (e.g. we do not know the topology of loop space in the first, the domain of gauge forms in the second), with reasonable assumptions we could find again that we have

only trivial isomorphism classes of bundles.
Being the aim of bundles over loop spaces to find non trivial isomorphism classes, I shall discuss a modification of the above attempts, worked out with R. Percacci 5) which seems to have the desired properties.

2 The loop space structure.

Let $\Omega^k M$ the set of all C^∞ maps $\omega : (S^k, s_0) \longrightarrow (M, *)$ where $s_0 \in S^k$ and $* \in M$, assuming M to be a riemannian manifold.

We can introduce a metric on $\Omega^k M$ as follows, 18).

Denote I^k the k-cube parametrised by $(s_1, \dots s_k)$ then $\omega \in \Omega^k M$ can be written as

$$\omega : (I^k, \partial I^k) \longrightarrow (M, *)$$

Let $\omega, \omega' \in \Omega^k M$; $(\bar{s}_1 \dots \bar{s}_k) \in I^k$, PM the space of all C^∞ path in M, i.e. maps $[0,1] \longrightarrow M$,then $\omega(\bar{s}_1 \dots \bar{s}_k)$ $\omega'(\bar{s}_1 \dots \bar{s}_k)$ are two points in M; define the distance

$$d(\omega(\bar{s}_1 \dots \bar{s}_k), \omega'(\bar{s}_1 \dots \bar{s}_k)) =$$
$$= \inf_{C \in PM} \{ \text{length of } C \mid C(0) = \omega(\bar{s}_1 \dots \bar{s}_k), C(1) = \omega'(\bar{s}_1 - \bar{s}_k) \}$$

This induces a metric in $\Omega^k M$ given by

$$d(\omega, \omega') = \sup_{(\bar{s}_1 \dots \bar{s}_k) \in I^k} d(\omega(\bar{s}_1 \dots \bar{s}_k), \omega'(\bar{s}_1 \dots \bar{s}_k))$$

Theorem 4.4 The space $\Omega^k M$ is a smooth manifold 18).

Remark:

1) Eventually we can also take a submanifold of $\Omega^k M$ to avoid patologies in the following.

2) For simplicity in the following we often identify ω with its image in M, denoting them with the same symbol.

On $\Omega^k M$ we introduce now a class of function and forms induced by forms in M.

Let $\Lambda \in E_K(M)$,as we have seen before, Λ defines a smoth function

$$\hat{\Lambda} : \omega \longrightarrow \int_\omega \Lambda \tag{4.5}$$

Now if $\Lambda \in E_{k+i}(M)$ one defines the i-form on $\Omega^k M$, $\hat{\Lambda}$ by

$$\hat{\Lambda}(\tilde{v}_1, \dots \tilde{v}_i)\Big|_\omega = \int_\omega i_{v_1} \dots i_{v_i} \Lambda \tag{4.6}$$

Where in the left hand side $\tilde{v}$ is a vector on $\Omega^k M$, i.e. a section of TM such that $\pi \circ \tilde{v} = \omega$ and in the right hand side v is the value of $\tilde{v}$ at the point p, and thus is considered as vector in TM.

Let ψ denote the chart in $\Omega^K M$ and ω be in his domain. One can easily prove the following property of the differential in $\Omega^K M$:

Lemma 4.7 $$d\hat{\Lambda} = \widehat{d\Lambda}$$

Proof. Let $\tilde{v} \in T_\omega \Omega^K M$; the diretional derivative of $\hat{\Lambda}$ at ω in the direction $\tilde{v}$ is given by

$$\nabla\hat{\Lambda}(\omega,\tilde{v}) = \lim_{h\to 0} \frac{1}{h}\Big[\int_{\psi^{-1}(\psi(\omega)+h\tilde{v})} \Lambda - \int_\omega \Lambda\Big] =$$

$$= \lim_{h\to 0} \int_{\frac{1}{h}[\psi^{-1}(\psi(\omega)+h\tilde{v})-\psi^{-1}(\psi(\omega))]} \Lambda = \int_\omega i_v d\Lambda$$

therefore $d\hat{\Lambda} : \tilde{v} \longrightarrow \nabla\hat{\Lambda}(\omega,\tilde{v})$ is given by $\widehat{d\Lambda}$.

One consider now the $\hat{\Lambda}$ forms as cochains and they have in this context one other interesting property.

Every (cellular) ℓ-chain in $\Omega^K M$ gives rise to a ℓ+k chain in M by the following procedure.

Let c_ℓ be a ℓ-cell in $\Omega^K M$ defined by

$$c_\ell : B^\ell \longrightarrow \Omega^K M$$
$$(t_1 \dots t_\ell) \longrightarrow \omega^K(s_1 \dots s_K, t_1 \dots t_\ell)$$

We define a ℓ+k cellular chain in M denoted by $hc_\ell \in C_{\ell+K}(M)$ through

$$hc_\ell : S^K \times B^\ell \longrightarrow M$$
$$(s_1 \dots s_K, t_1 \dots t_\ell) \longrightarrow \omega^K(s_1 \dots s_K, t_1 \dots t_\ell)$$

It is easy to see that the homomorphism

$$h : C_\ell(\Omega^K M) \longrightarrow C_{\ell+K}(M)$$

commutes with the boundary operator

$$h\,\partial_{\Omega^k M} = \partial_M h$$

One can now state one other property of the $\hat{\Lambda}$ forms

Theorem 17) Let $\Lambda \in E_{k+\ell}(M)$, $c_\ell \in C_\ell(\Omega^k M)$

then

$$\int_{c_\ell} \hat{\Lambda} = \int_{h c_\ell} \Lambda$$

3 Principal bundle construction.

We first identify in analogy with the k=0 case of the $U(1)$ princiapl bundle, the gauge group which we are intere- sted in. From Definition 2.24 we know that in the abe - lian case the gauge transformations are given by the (global) sections of the trivial bundle through the graph relation.

To obtain all such sections with value in $U(1)$ starting from those $\mathbb{R}$-valued, for the k=0 case, we can proced through sheaf theory. Being needed in the following we also give a presheaf treatment of all arguments involved.

Let $\underset{\sim}{\mathbb{Z}}$ (resp $\underset{\sim}{\mathbb{R}}$) be the sheaf of germs of smooth $\mathbb{Z}$ (resp $\mathbb{R}$) valued functions and $Pr\,\underset{\sim}{\mathbb{Z}}$ (resp $Pr\,\underset{\sim}{\mathbb{R}}$). denote the canonical presheaf.

There is an obvious injection.

$$\underset{\sim}{\mathbb{Z}} \xrightarrow{i} \underset{\sim}{\mathbb{R}}$$

$$Pr\,\underset{\sim}{\mathbb{Z}} \xrightarrow{i} Pr\,\underset{\sim}{\mathbb{R}}$$

By general sheaf theory one can define a unique quotient sheaf $\underset{\sim}{U(1)}$ requiring the following exact sequence hold

$$0 \to \underset{\sim}{\mathbb{Z}} \xrightarrow{i} \underset{\sim}{\mathbb{R}} \xrightarrow{\pi} \underset{\sim}{U(1)} \to 0 \qquad (4.8a)$$

Where
$$\pi : f_x \in \underset{\sim}{\mathbb{R}} \longrightarrow \exp 2\pi i f_x \in \underset{\sim}{U(1)}$$

The wanted $U(1)$ gauge transformation group is now $\Gamma(M, \underset{\sim}{U(1)})$
Otherwise we can define the quotient presheaf $Pr\,\underset{\sim}{\mathbb{R}} / Pr\,\underset{\sim}{\mathbb{Z}}$
and the exact sequence of presheaves holds

$$0 \to Pr\,\underset{\sim}{\mathbb{Z}} \to Pr\,\underset{\sim}{\mathbb{R}} \xrightarrow{P} Pr\,\underset{\sim}{\mathbb{R}}/Pr\,\underset{\sim}{\mathbb{Z}} \to 0 \qquad (4.8b)$$

We can also take as group of gauge transformations
$H^0(M, Pr\,\underset{\sim}{\mathbb{R}}/Pr\,\underset{\sim}{\mathbb{Z}})$; by theorem 2.10 being M para compact $H^0(M, Pr\,\underset{\sim}{\mathbb{R}}/Pr\,\underset{\sim}{\mathbb{Z}}) \simeq \Gamma(M, \underset{\sim}{U(1)})$
Notice that being $Pr\,\underset{\sim}{\mathbb{Z}}$ and $Pr\,\underset{\sim}{\mathbb{R}}$ the canonical pre-sheaf of $\underset{\sim}{\mathbb{Z}}$ and $\underset{\sim}{\mathbb{R}}$ we also have

$$H^0(M, Pr\,\underset{\sim}{\mathbb{Z}}) \equiv \Gamma(M, \underset{\sim}{\mathbb{Z}})$$
$$H^0(M, Pr\,\underset{\sim}{\mathbb{R}}) \equiv \Gamma(M, \underset{\sim}{\mathbb{R}})$$

We can remark that taking the naive attitude of dividing directly the $\mathbb{R}$ valued sections $\Gamma(M, \underset{\sim}{\mathbb{R}})$ by the $\mathbb{Z}$ -valued sections $\Gamma(M, \underset{\sim}{\mathbb{Z}})$ we do not get in general the whole set of $U(1)$ valued sections.
In fact the exact sheaf sequence (4.8a) gives rise to the cohomology exact sequence on the paracompact space M

$$0 \to \Gamma(M, \underset{\sim}{\mathbb{Z}}) \to \Gamma(M, \underset{\sim}{\mathbb{R}}) \to \Gamma(M, \underset{\sim}{U(1)}) \to$$
$$\to H^1(M, \underset{\sim}{\mathbb{Z}}) \to \dots$$

and this means that if $0 \neq H^1(M, \mathbb{Z})$ (which is isomorphic to $H^1(M, \mathbb{Z})$ if M is connected), $\Gamma(M, U(1))$ is not the quotient of $\Gamma(M, \mathbb{R})$ by $\Gamma(M, \mathbb{Z})$. (example $M = S^1$, $H^1(S^1, \mathbb{Z}) = \mathbb{Z}$).

One can immediately see that both these procedures cannot be applied to Tulczyjew's generalized principal bundles(k > 0); in fact being the action defined pointwise we cannot found neither an analog of $\mathbb{Z}$ neither an analog of π . (or p).
This is due to the fact that we regard in such approach differential forms as sections of the bundle $\Lambda^k T^*M$.
However as we have seen in § 4.2 using integration theory one can define through the differential k-forms a class of functions over $\Omega^k M$.
So we have now to find an analog of $\mathbb{Z}$ on $\Omega^k M$ obtained utilizing some class of differential forms on M.
This is provided by Allendoerfer Eels theory of smooth forms with singularity (see chapter 2.), and will suggest how to proceed in the construction of gauge transformations which are needed in our case.
We already know that in the Allendoerfer Eels complex in M we have the canonical isomorphism

$$\zeta^0(M, \mathbb{Z}) \rightarrow \mathbb{Z}$$

we thus want to construct an induced A-E complex on $\Omega^k M$,

One proceed as follows: an r+k A-E pair (θ , φ)

in M induces an A-E r pair in $\Omega^k M$ by

$$(\theta, \varphi) \longrightarrow (\hat{\theta}, \hat{\varphi})$$

$$e(\theta) \longrightarrow e(\hat{\theta}) \equiv \{ \omega \in \Omega^k M \mid e(\theta) \cap |\omega| \neq \phi \}$$

$$e(\varphi) \longrightarrow e(\hat{\varphi}) \equiv \{ \omega \in \Omega^k M \mid e(\varphi) \cap |\omega| \neq \phi \}$$

In fact by 4.2 we know that

$$\int_{c_e} \hat{\theta} - \int_{\partial c_e} \hat{\varphi} = \int_{hc_e} \theta - \int_{h\partial c_e = \partial h c_e} \varphi$$

We denote the corresponding complex by $\hat{\theta}$.

This suggests to take as analog of $\mathbb{Z}$, the sheaf of germs of $\hat{\mathcal{C}}^k$ functions: $\hat{\mathcal{Z}}$, whose canonical presheaf I denote by $Pr\hat{\mathcal{Z}}$.

Let us first make same comment on $\hat{\mathcal{C}}^k$. Being $h\omega \in C_k(M)$ a cycle, $\partial h\omega = 0$, a k-pair (θ, φ) in M define a O-pair in $\Omega^k M$ simply by taking $(\hat{\theta}, 0)$.

Two O-pairs in $\Omega^k M$ $(\hat{\theta}, 0)$ $(\hat{\theta}', 0)$, are equivalent if and only if $\hat{\theta}(\omega) = \hat{\theta}'(\omega)$ for all ω in their common demain; in particular if $(\hat{\theta}, 0)$ is a representative of the class $[\hat{\theta}, 0]$ in $\hat{\mathcal{Z}}^k$ one has

$$\forall c_1 \in C_1(\Omega^k M) \mid \partial c_1 \cap e(\hat{\theta}) = \phi$$

$$\int_{\partial c_1} \hat{\theta} = \int_{\partial h c_1} \theta = 0$$

and this impies that in his domain $\hat{\theta}$ has a constant $\mathbb{Z}$-value, so that the analogy of $\hat{\mathfrak{Z}}^k$ in $\Omega^k M$ and $\mathbb{Z}$ in M is strongly supported.

We now need an analog of $\mathbb{R}$, which I shall call $\hat{\mathcal{L}}^k$ whose canonical presheave I denote by $Pr\,\hat{\mathcal{L}}^k$, such that there is inclusion

$$\hat{\mathfrak{Z}}^k \longrightarrow \hat{\mathcal{L}}^k \qquad Pr\,\hat{\mathfrak{Z}}^k \longrightarrow Pr\,\hat{\mathcal{L}}^k$$

This can be provided by the following construction.

Let λ a smooth k-form defined on M exept perhaps for a closed rare set $e(\lambda)$ which lies on a smooth local - ly finite polyhedron of dimension m-k-1.

As before one can define $\hat{\lambda}$ and $e(\hat{\lambda})$; we finally divide this set of smooth singular formsby the following equivalence relation

$$\hat{\lambda} \simeq \hat{\lambda}' \iff \hat{\lambda}(\omega) = \hat{\lambda}'(\omega) \quad \forall \omega \text{ in common domain}$$

We denote this quotient set by $\hat{\rho}^k$ and the sheaf of germs of $\hat{\rho}^k$ functions in $\Omega^k M$ is denoted, as usual, by $\hat{\mathcal{L}}^k$.

So we are now in position to construct the analog of the exact short sequence(4.8a) (4.8b), denoting by $\mathcal{G}^k$ the quotient sheaf and $\mathcal{G}^k$ the quotient presheaf

$$0 \longrightarrow \hat{\mathfrak{Z}}^k \longrightarrow \hat{\mathcal{L}}^k \longrightarrow \mathcal{G}^k \longrightarrow 0 \qquad (4.9a)$$

$$0 \longrightarrow Pr\,\hat{\mathfrak{Z}}^k \longrightarrow Pr\,\hat{\mathcal{L}}^k \longrightarrow \mathcal{G}^k \longrightarrow 0 \qquad (4.9b)$$

The wanted $U(1)$ gauge transformations for analogy with (4.8) would be now sections of $\mathcal{G}^k$.

However there are two remarks to be made that suggest a more restrictive definition of the gauge group.

The first is that we want somehow that coverings used in $\Omega^k M$ for the definition of the cohomology group $H^p(\mathcal{U}, \mathcal{G}^k)$ or $\check{H}^p(\mathcal{U}, \mathcal{G}^k)$ are determined from covering in M in such way as to avoid the case in which open sets of loops with the same image on M have different sections associated of $\mathcal{G}^k$.

Therefore one can limit to the subgroup of all covering of $\Omega^k M$ which contains all coverings $\{\Omega^k U_i\}_{i\in I}$ such that $\{U_i\}_{i\in I}$ is a covering of M.

So our gauge group would be either

$$\hat{H}^0(\Omega^k M, \mathcal{G}^k) \equiv \underset{\Omega^k\mathcal{U} \text{ covering of } \Omega^k M}{\operatorname{ind\,lim}} H^0(\Omega^k\mathcal{U}, \mathcal{G}^k) = \Gamma(\Omega^k M, \mathcal{G}^k)$$

(the last equality follows by theory 2.8) either

$$\hat{\check{H}}^0(\Omega^k M, \mathcal{G}^k) \equiv \underset{\Omega^k\mathcal{U} \text{ covering of } \Omega^k M}{\operatorname{ind\,lim}} \check{H}^0(\Omega^k\mathcal{U}, \mathcal{G}^k)$$

These two does not coincide because we cannot apply theorem 2.10. Whereas the first seems more attractive I prefer to choose the second one because being $Pr\mathcal{G}^k$ not the quotient presheaf of $Pr\hat{\mathcal{L}}^k$ with respect to $Pr\hat{\mathcal{Z}}^k$ we don't have a cohomology presheaf exact sequence, and more over it makes no general relation with the intuitive picture that an element of $Pr\mathcal{G}^k$ should have locally the form $\exp 2\pi i\, \hat{\Lambda}$.

This instead is garanteed by the second definition because all elements of $\hat{H}^0(\Omega^k U, \mathcal{G}^k)$ are locally exactly of that form which also garantee the possibility of using an exact cohomology presheaf sequence.

Thus we assume as gauge group

$$\mathcal{G}^k \equiv \hat{H}^0(\Omega^k M, \mathcal{G}^k)$$

whose elements are locally in $\Omega^k M$ represented by

$$\exp 2\pi i \hat{\Lambda} : \omega \longrightarrow \exp 2\pi i\, \hat{\Lambda}(\omega)$$

where $\Lambda \in E_k(M)$ (or more precisely by $\exp 2\pi i \hat{\lambda}$, $\lambda \in \rho_k(M)$)

Let me finally make the convention to call

$$\hat{H}^0(\Omega^k U, \mathcal{G}^k) \equiv \Gamma(\Omega^k U, \mathcal{G}^k)$$

A bundle is now fixed giving the transitions function with values in $\mathcal{G}^k$.

We again require that transition function in our case are determined by forms in M, therefore we assume the covering of $\Omega^k M$ relative to the bundle is of a particular kind.

Definition 4.10 A topological space P^k together with a continous map $\pi : P^k \to \Omega^k M$ is called a $\mathcal{G}^k$ bundle over $\Omega^k M$ if there exist :

a) an open covering of M, $\mathcal{U} = \{U_i\}_{i\in I}$ such that $\{\Omega^k U_i\}_{i\in I}$ is a covering of $\Omega^k M$, and homeomorhisms

$$\varphi_i : \pi^{-1}(\Omega^k U_i) \longrightarrow \Omega^k U_i \times U(1)$$
$$\pi^{-1}(\omega) \longrightarrow \omega \times U(1)$$

b) elements $f_{ij} \in \Gamma(\Omega^k U_i \cap \Omega^k U_j, \mathcal{G}^k)\ \forall i,j \in I$ such that

$$(\varphi_i \circ \varphi_j^{-1})(\omega, a) = (\omega, f_{ij}(\omega)a) \qquad \begin{array}{l}\forall a \in U(1)\\ \omega \in \Omega^k(U_i \cap U_j)\end{array}$$

Two systems $\{(\Omega^k U_i, \varphi_i)\}_{i\in I}$ $\{(\Omega^k U'_j), \varphi'_j\}_{j\in J}$ give rise to the same $\mathcal{G}^k$ bundle if $\forall i \in I, j \in J$ there exists $g_{ij} \in \Gamma(\Omega^k U_i \cap \Omega^k U'_j, \mathcal{G}^k)$ such that

$$(\varphi_i \circ \varphi_j'^{-1})(\omega, a) = (\omega, g_{ij}(\omega)a) \qquad \begin{array}{l}\forall a \in U(1)\\ \omega \in \Omega^k U_i \cap \Omega^k U'_j\end{array}$$

Remark: forms in φ^k which differs for exact forms give rise to the same element in $\mathcal{G}^k$, remembering this we can see that the action of $\mathcal{G}^k$ on $U(1)$ is effective.

We now show how in suitable case the bundle over $\Omega^k M$ can be determined from assignement of k-forms on M.

In fact let $\{U_i\}_{i\in I}$ be a covering of M such that $\{\Omega^k U_i\}_{i\in I}$ is a covering of $\Omega^k M$, and $(U_i \cap U_j)$ is homeomorphic to $\overset{S^n \ n\le k}{V_i}$ then given a set of $\Lambda_{ij} \in \Gamma(U_i \cap U_j, \underline{\mathcal{R}}^k)$ these determine the transition functions of a $\mathcal{G}^k$ bundle through

$$\Gamma(\Omega^k U_i \cap \Omega^k U_j, \mathcal{G}^k) \ni g_{ij} \equiv \exp 2\pi i \hat{\Lambda}_{ij}$$

and all possible transition function are of this form, because $0 = H^1(\Omega^k U_i \cap \Omega^k U_j, \underline{\mathcal{R}}^k)$ as we shall prove later, and from the cohomology exact sequence derived from (4.9b) in this case we get $\Gamma(\Omega^k U_i \cap U_j, \mathcal{G}^k) \simeq \dfrac{\Gamma(\Omega^k U_i \cap U_j, \underline{\mathcal{R}}^k)}{\Gamma(\Omega^k U_i \cap U_j, \mathcal{Z}^k)}$

We now define what we mean by an isomorphism between two $\mathcal{G}^k$ bundles and we will find as usual a cohomological classification for the classes of isomorphic $\mathcal{G}^k$-bundles.

Let P^k and P'^k be two total spaces of $\mathcal{G}^k$ bundles $\mathcal{B}, \mathcal{B}'$ over the same manifold $\Omega^k M$.

Definition 4.11 An isomorphism K form $\mathcal{B}$ to $\mathcal{B}'$. is given by a homeomorphism $K : P^k \longrightarrow P'^k$ such that $\forall \omega \in \Omega^k M$

a) $\pi^{-1}(\omega)$ maps onto $(\pi')^{-1}(\omega)$

b) $\exists$ an open enighbouhood of ω, of the kind $\Omega^k U$ where U is a neighbouhood of $|\omega|$, an element $g \in \Gamma(\Omega^k U, \mathcal{G}^k)$ and

$$\varphi : \pi^{-1}(\Omega^k U) \longrightarrow \Omega^k U \times U(1)$$
$$\varphi' : \pi'^{-1}(\Omega^k U) \longrightarrow \Omega^k U \times U(1)$$

such that

$$(\varphi' \circ K \circ \varphi^{-1})(\omega, a) = (\omega, g(\omega)a) \quad \forall \omega \in \Omega^k U, \ a \in U(1)$$

Now we have the following

Theorem 4.12 The isomorphism classes of $\mathcal{G}^k$ fibre bundles over $\Omega^k M$ are in a natural one-one correspondence with the elements of the cohomology set

$$\hat{H}^1(\Omega^k M, \mathcal{G}^k) \equiv \underset{\Omega^k \mathcal{U} \text{ coverings of } \Omega^k M}{\mathrm{ind.\,lim}} H^1(\Omega^k \mathcal{U}, \mathcal{G}^k)$$

Proof. Let $\{\Omega^k V_i\}_{i \in I}$ a covering of $\Omega^k M$ and $f = \{f_{ij}\} \in Z^1(\Omega^k \mathcal{U}, \mathcal{G}^k)$, then we can construct a $\mathcal{G}^k$ bundle over $\Omega^k M$ taking the disjoint union of the cartesian product $\Omega^k V_i \times U(1)$ and identifying $(\omega, a) \in \Omega^k V_i \times U(1)$ and $(\omega, f_{ij}(\omega)\, a) \in \Omega^k V_j \times U(1)$ for every $\omega \in \Omega^k V_i \cap \Omega^k V_j$.

If $f \in Z^1(\Omega^k \mathcal{U}, \mathcal{G}^k)$ and $g \in Z^1(\Omega^k \mathcal{B}, \mathcal{G}^k)$ then the corresponding bundles obtained by the above construction are isomorphic if and only if g and h represent the same element of the cohomology set $\hat{H}^1(\Omega^k M, \mathcal{G}^k)$.

Then the proof goes in similar way to 3.9.

Let me now prove that

Theorem 4.13 $\hat{H}^1(\Omega^k M, \mathcal{G}^k) \simeq H^{k+2}(M, \mathbb{Z})$

Proof:

From the presheaf exact sequence (4.9b) we have for every covering $\Omega^k \mathcal{U}$, the exact cohomology sequence

$$0 \to \Gamma(\Omega^k M, \hat{\underset{\sim}{\mathcal{Z}}}^k) \to \Gamma(\Omega^k M, \hat{\mathcal{F}}^k) \to$$
$$\to H^0(\Omega^k \mathcal{U}, \mathcal{G}^k) \to H^1(\Omega^k \mathcal{U}, \hat{\underset{\sim}{\mathcal{Z}}}^k) \to$$
$$\to H^1(\Omega^k \mathcal{U}, \hat{\mathcal{F}}^k) \to H^1(\Omega^k \mathcal{U}, \mathcal{G}^k) \to$$
$$\to H^2(\Omega^k \mathcal{U}, \hat{\mathcal{Z}}^k) \to H^2(\Omega^k \mathcal{U}, \hat{\mathcal{F}}^k) \to \dots$$

Now $H^p(\Omega^k\mathcal{U}, \hat{\mathcal{L}}^k) = 0 \quad \forall p$ and $\forall \Omega^k\mathcal{U}$ in fact let me consider for simplicity the p=1 case.

Then a 1-cocycle $f \in Z^1(\Omega^k\mathcal{U}, \hat{\mathcal{L}}^k)$ is given by $f_{ij} \in \Gamma(\Omega^k(U_i \cap U_j), \hat{\mathcal{L}}^k)$ such that

$$f_{ij} + f_{jk} + f_{ki} = 0$$

One can now find an element $\lambda_{ij} \in \Gamma(U_i \cap U_j, \mathcal{L}^k)$ such that $f_{ij} = \hat{\lambda}_{ij}$ and

$$\lambda_{ij} + \lambda_{jk} + \lambda_{ki} = 0$$

However

$$H^p(M, \mathcal{L}^k) = 0$$

i.e. $\forall$ open covering $\mathcal{U}$ of M $\quad H^p(\mathcal{U}, \mathcal{L}^k) = 0$

Therefore there exist $\mu_i \in \Gamma(U_i, \mathcal{L}^k)$ such that

$$\lambda_{ij} = \mu_i - \mu_j|_{U_i \cap U_j}$$

and hence

$$f_{ij} = \hat{\lambda}_{ij} = \hat{\mu}_i - \hat{\mu}_j|_{\Omega^k(U_i \cap U_j)} = g_i - g_j|_{\Omega^k(U_i \cap U_j)}$$

where $g_i \in \Gamma(\Omega^k U_i, \hat{\mathcal{L}}^k)$, i.e. $\quad H^1(\Omega^k\mathcal{U}, \hat{\mathcal{L}}^k) = 0$

We have thus the isomorphisms

$$H^1(\Omega^k\mathcal{U}, \mathcal{G}^k) \simeq H^2(\Omega^k\mathcal{U}, \underset{\sim}{\xi}^k)$$

Taking the direct limit we get

$$\hat{H}^1(\Omega^k M, \mathcal{G}^k) \simeq \hat{H}^2(\Omega^k M, \underset{\sim}{\xi}^k)$$

Let me now consider the presheaf exact sequence

$$0 \to Pr\, \underset{\sim}{\xi}^k \to Pr\, \underset{\sim}{\mathcal{C}}^k \xrightarrow{\hat{d}^0} Pr\, \widehat{\underset{\sim}{d\mathcal{C}}^k} \xrightarrow{\hat{d}^1} 0$$

We get the cohomology exact sequence

$$\cdots \longrightarrow H^1(\Omega^k\mathcal{U}, \overset{\vee}{\underset{\sim}{\xi}}{}^k) \longrightarrow H^1(\Omega^k\mathcal{U}, \hat{\underset{\sim}{\mathscr{C}}}{}^k) \longrightarrow H^1(\Omega^k\mathcal{U}, d\hat{\underset{\sim}{\mathscr{C}}}{}^k) \longrightarrow$$
$$\longrightarrow H^2(\Omega^k\mathcal{U}, \overset{\vee}{\underset{\sim}{\xi}}{}^k) \longrightarrow H^2(\Omega^k\mathcal{U}, \hat{\underset{\sim}{\xi}}{}^k) \longrightarrow$$

With an argoment similar to the previous one one can prove from $H^p(M, \underset{\sim}{\mathscr{C}}{}^k) = 0$ that $\forall\, \Omega^k\mathcal{U}$ $H^p(\Omega^k\mathcal{U}, \hat{\underset{\sim}{\mathscr{C}}}{}^k) = 0$ and thus we have the isomorphism, passing to the direct limit

$$\hat{H}^1(\Omega^k M, d\hat{\underset{\sim}{\mathscr{C}}}{}^k) \simeq \hat{H}^2(\Omega^k M, \overset{\vee}{\underset{\sim}{\xi}}{}^k)$$

Now given any $\Omega^k\mathcal{U}$ covering of $\Omega^k M$ there exists a refinement $\Omega^k\tilde{\mathcal{U}}$ such that

$$H^1(\Omega^k\tilde{\mathcal{U}}, d\hat{\underset{\sim}{\mathscr{C}}}{}^k) \simeq H^1(\Omega^k\tilde{\mathcal{U}}, \overset{\vee}{\underset{\sim}{\xi}}{}^{k+1}).$$

In fact let $\omega \in \Omega^k U_i$; then consider a tubular neighbourhood N_ω of the image of ω in M contained in U_i with convex section.

Then $\Omega^k N_\omega$ is contractible to $|\omega|$ and $\Omega^k N_\omega \cap \Omega^k N_{\omega'}$ is such that his component connected to the zero loop is again contractable to S^n $n \leq k$.

Therefore if $\xi \in \Gamma(N_\omega \cap N_{\omega'}, \overset{\vee}{\underset{\sim}{\xi}}{}^{k+1})$ there exists a $\mathscr{C} \in \Gamma(N_\omega \cap N_{\omega'}, \underset{\sim}{\mathscr{C}}{}^k)$ such that $\xi = d\mathscr{C}$.

Moreover $\{\Omega^k N_\omega\}_{\omega \in \Omega^k M} \equiv \Omega^k\tilde{\mathcal{U}}$ is a refinement of $\Omega^k\mathcal{U}$

Thus in the direct limit we have

$$\hat{H}^1(\Omega^k M, d\underset{\sim}{\mathscr{C}}{}^k) \simeq \hat{H}^1(\Omega^k M, \overset{\vee}{\underset{\sim}{\xi}}{}^{k+1})$$

From the presheaf exact sequence

$$0 \to Pr\,\underset{\sim}{\mathcal{J}}^{k+1} \to Pr\,\underset{\sim}{\hat{\mathcal{E}}}^{k+1} \xrightarrow{\hat{d}^1} Pr\,d\underset{\sim}{\hat{\mathcal{E}}}^{k+1} \xrightarrow{\hat{d}^2} 0$$

we also get

$$0 \to \Gamma(\Omega^k M, \underset{\sim}{\hat{\mathcal{J}}}^{k+1}) \to \Gamma(\Omega^k M, \underset{\sim}{\hat{\mathcal{E}}}^{k+1}) \xrightarrow{(\hat{d}^1)^*}$$

$$\xrightarrow{(\hat{d}^1)^*} \Gamma(\Omega^k M, d\underset{\sim}{\hat{\mathcal{E}}}^{k+1}) \to \hat{H}^1(\Omega^k M, \underset{\sim}{\hat{\mathcal{J}}}^{k+1})$$

Therefore

$$\hat{H}^1(\Omega^k M, \underset{\sim}{\mathcal{J}}^{k+1}) \simeq \frac{\Gamma(\Omega^k M, d\hat{\mathcal{E}}^{k+1})}{im\,(\hat{d}^1)^*} \simeq \frac{\Gamma(\Omega^k M, \hat{\mathcal{J}}^{k+2})}{im\,(\hat{d}^1)^*}$$

But from the presheaf exact sequence

$$0 \to Pr\,\underset{\sim}{\hat{\mathcal{J}}}^{k+2} \to Pr\,\underset{\sim}{\hat{\mathcal{E}}}^{k+2} \xrightarrow{\hat{d}^2} Pr\,d\underset{\sim}{\hat{\mathcal{E}}}^{k+2} \xrightarrow{\hat{d}^3} 0$$

we also get

$$0 \to \Gamma(\Omega^k M, \underset{\sim}{\hat{\mathcal{J}}}^{k+2}) \to \Gamma(\Omega^k M, \underset{\sim}{\hat{\mathcal{E}}}^{k+2}) \xrightarrow{\hat{d}_2^*}$$

$$\xrightarrow{(\hat{d}^2)^*} \Gamma(\Omega^k M, d\underset{\sim}{\hat{\mathcal{E}}}^{k+2}) \to \hat{H}^1(\Omega^k M, \underset{\sim}{\hat{\mathcal{J}}}^{k+2}) \to 0$$

and hence

$$\Gamma(\Omega^k M, \hat{\mathcal{J}}^{k+2}) = Ker\,(\hat{d}^2)^*$$

so

$$\hat{H}^2(\Omega^k M, \underset{\sim}{\hat{\mathcal{J}}}^k) = \frac{Ker\,(\hat{d}^2)^*}{im\,(\hat{d}^1)^*}$$

But it is not difficult to see that this is nothing but the image under the isomorphism Λ of $H^{k+2}_{AE}(M) \simeq H^{k+2}(M, \underset{\sim}{\mathbb{Z}})$.

We thus have the following result :

if $H^{k+2}(M, \underset{\sim}{\mathbb{Z}}) \neq 0$, then there exists non trivial $\mathcal{G}^k$ bundles over $\Omega^k M$ and they are classified by $H^{k+2}(M, \mathbb{Z})$ (example $M = S^{k+2}$, $H^{k+2}(M, \mathbb{Z}) = \mathbb{Z}$)

Remark: by a way similar to the one followed in this proof, one can obtain

$$\hat{H}^1(\Omega^k M, \underset{\sim}{\hat{\mathcal{J}}}^k) \simeq H^{k+1}(M, \mathbb{Z})$$

Thus $\hat{H}^1(\Omega^k S^n, \underset{\sim}{\hat{\mathcal{J}}}^k) = 0 \quad n \leq k$.

4. Connection and gauge transformation.

We want now define a connection on a $\mathcal{G}^k$ bundle $\mathcal{B}$.

Let
$$I \longrightarrow \Gamma(M, \mathcal{G}^k)$$
$$t \longrightarrow g(t)$$

be a one parameter subgroup of the gauge group,

consider the induced transformation of the total space P^k of a $\mathcal{G}^k$ bundle

$$f_{g(t)} : P^k \longrightarrow P^k$$
$$p \longrightarrow f(p, g(t))$$

We now define as usual the fundamental vector

$$W(\gamma) = \frac{d}{dt} f(p, g(t))\Big|_{t=0}$$

where γ is the element of the algebra of $\mathcal{G}^k$ which generates $g(t)$, i.e.

$$\gamma = \frac{d}{dt} g(t)\Big|_{t=0}$$

Let σ_i be a preferred section of P^k, i.e. a map

$$\sigma_i : \Omega U_i \longrightarrow P^k$$
$$\pi \circ \sigma_i = Id_{\Omega \wedge U_i} \qquad \sigma_i(\omega) = \varphi_i(\omega, 1)$$

Definition 4.14 A connection form α on $\mathcal{P}$ is a one form on P^k such that

a) $\alpha(W(\gamma))\big|_p = \gamma \circ \pi(p) \qquad p \in P^k$

b) $f_g^* \cdot \alpha = \alpha \qquad g \in \mathcal{G}^k$

c) for every section σ_i,

$$\sigma_i^* \alpha = \hat{A}_i \qquad A_i \in E_{k+1}(U_i)$$

Theorem 4.15 The local forms $\hat{A}_i = \sigma_i^* \alpha$ verify in $\Omega^k(U_i \cap U_j)$ the compatibility condition

$$\hat{A}_j = \hat{A}_i + f_{ij}^{-1} d f_{ij}$$

where f_{ij} is the transition function.

proof. Define on $\pi^{-1}(\Omega^k U_i)$ the 1-form

$$\phi = \pi^* \hat{A}_i + \varphi_i^{-1} d\varphi_i$$

where d is the exterior derivative in P^k; $\phi = \alpha|_{\pi^{-1}(\Omega^k U_i)}$

But then on $\pi^{-1}(\Omega^k(U_i \cap U_j))$

$$\pi^* \hat{A}_i + \varphi_i^{-1} d\varphi_i = \pi^* \hat{A}_j + \varphi_j^{-1} d\varphi_j$$

i.e.

$$\pi^* \hat{A}_j = \pi^* \hat{A}_i + \pi^* (f_{ij} d f_{ij})$$

Remark: If the open set $U_i \cap U_j$ is homotopic to S^n $n \leq k$ then $\hat{H}^1(\Omega^k(U_i \cap U_j), \mathcal{Z}^u) = 0$

therefore:

$$f_{ij}^{-1} d f_{ij} = d\hat{\lambda}_{ij} \qquad f_{ij} = \exp 2\pi i \hat{\lambda}_{ij}$$

Theorem 4.16: Given a collection of k+1 forms A_i on U_i satisfying

$$\hat{A}_j = \hat{A}_i + f_{ij}^{-1} d f_{ij}$$

(or if $U_i \cap U_j$ is homotopic to S^n $n \leq k$, $A_j = A_i + d\lambda_{ij}$)

there exists a unique connection form α on σ such that $\hat{A}_i = \sigma_i^* \alpha$

Proof. Define in $\pi^{-1}(\Omega^k U_i)$ the form

$$\alpha_i = \pi^* \hat{A}_i + \varphi_i^{-1} d\varphi_i$$

this behaves as a connection on $\pi^{-1}(\Omega^k U_i)$ and by the compatibility condition on $\pi^{-1}(\Omega^k U_i \cap U_j))$

$$\alpha_i = \alpha_j$$

i.e. there exists a connection form α defined on the whole P^k.

Let us now see how a gauge transformation fits in this scheme.

Definition 4.17 A gauge transformation in $\mathcal{P}$ is a map

$$f : P^k \longrightarrow P^k$$

such that

a) $\pi \circ f\,(p) = \pi(p)$ $\qquad \forall p \in P^k$

b) $f(pg) = f(p)g$ $\qquad g \in \mathcal{G}^k$

We immediately see that gauge transformations are in one to one correspondence with elements of $\mathcal{G}^k$ as written before, in fact being the action effective one can write

$$f(p) = p\; g(\pi(p))$$

for some $g \in \mathcal{G}^k$

Clearly a gauge transformation on $\Omega^k U_i$ can also be seen as a change of section through

$$\sigma_i'(w) = \sigma_i(w)\, g(w)|_{\Omega^k U_i} \qquad g \in \mathcal{G}^k$$

From theorem 4.15 easy follows that

$$\hat{A}'_i = \hat{A}_i + g^{-1}dg$$

Thus if U_i is homeomorphic to S^n $n \leq k$

$$g|_{U_i} = \exp 2\pi i \hat{\lambda}_i \qquad \lambda_i \in \Gamma(U_i, R^k)$$

And we have

$$\hat{A}'_i = \hat{A}_i + d\hat{\lambda}_i$$

Viceversa given a gauge transformation of a k+1 form A_i defined in a open set homeomorphic to S^n $n \leq k$

$$A'_i = A_i + d\lambda_i$$

one can interpret this as a gauge transformation induced on a connection α in a $\mathcal{G}^k$ bundle over $\Omega^k M$

Finally let us define a curvature

Definition Let α be a connection in a $\mathcal{G}^k$ bundle over $\Omega^k M$, then $\beta = d\alpha$ is called the curvature form.

Remark: In $\Omega U_i \cap \Omega U_j$

$$\sigma_i^* \beta = \sigma_j^* \beta$$

i.e.

$$\beta = \pi^* \hat{F}$$

F being a closed k+2 form on M.

Thus a curvature of a $\mathcal{G}^k$ -bundle give rise naturally to a closed k+2 form on M.

Now using a covering $\Omega^k \mathcal{U}$ of $\Omega^k M$, such that U_i, $U_i \cap U_j$ $\forall i,j \in I$ are homeomorphic to S^n $n \leq k$ we have

$$\hat{F}|_{\Omega^k U_i} = d\hat{A}_i$$

$$\hat{A}_i - \hat{A}_j|_{\Omega^k(U_i \cap U_j)} = d\hat{\lambda}_{ij}$$

But we already know that

$$\exp 2\pi i\,\hat{\lambda}_{ij} \cdot \exp 2\pi i\,\hat{\lambda}_{jk} \cdot \exp 2\pi i\,\hat{\lambda}_{ki}\Big|_{\pi^{\kappa}(U_i \cap U_j \cap U_k)} = 1$$

thus

$$\hat{\lambda}_{ij} + \hat{\lambda}_{jk} + \hat{\lambda}_{ki}\Big|_{\pi^{\kappa}(U_i \cap U_j \cap U_k)} \in \mathbb{Z}$$

and the cohomology class of $\hat{F}$, and thus of F, is integral (19).

5 Relation with Yang Mills theories.

There is a relation between Yang Mills theory in a 4--dimensional manifold M and $\mathcal{G}^2$-bundle whose link is given by the second Chern number which is more usual between physicists to call topological charge.

In fact let $P(M, SU(N))$ be a (Yang-Mills) $SU(N)$-principal bundle over M (4-dimensional) and α a connection on it with curvature $\beta = d\alpha + \frac{1}{2}\alpha \wedge \alpha$

Given the form $Tr\, \beta \wedge \beta$, is a well known result in Chern-Weil theory 12) 14) that there exists a unique closed form $Q \in E_4(M)$ such that

$$Tr\, \beta \wedge \beta = \pi^* Q$$

π being the projection of $P(M, SU(N))$.

Moreover if F is the local representative of the curvature on U

$$Q|_U = Tr\, F \wedge F|_U$$

The second Chern number is given by

$$C_2 = \int_M Tr\, F \wedge F$$

Let now assume U is contractible then there exists a 3-form on U, $K \in E_3(U)$ such that

$$Q|_U = dK$$

and K can be written in terms of the local representative of the connection in U, A as

$$K = \mathrm{Tr}\; dA\wedge A + \frac{1}{3} A\wedge A\wedge A$$

If A undergoes a gauge transformation

$$A \longrightarrow g^{-1}Ag + g^{-1}dg$$

then K transforms as

$$K \longrightarrow K + \mathrm{Tr}\,(g^{-1}dg\wedge g^{-1}dg\wedge g^{-1}dg) + $$
$$+ d\;\mathrm{Tr}\;(\tfrac{1}{3}\; dg\wedge g^{-1}A)$$

Thus if given $P(M, SU(N))$ there exists a $\mathcal{U} = \{U_i\}_{i\in I}$ covering of M compatible with P such that $\Omega^2\mathcal{U} = \{\Omega^2 U_i\}_{i\in I}$ is a covering of $\Omega^2 M$ and $U_i\cap U_j$ is homeomorphic to $S^n\; n\leq 2$, one can define a $\mathcal{G}^2$-bundle over $\Omega^2 M$ by the following construction.

Let f_{ij} be the transition function, A_i the local representatives of the YM connection in U_i, then

$$K_i \equiv \mathrm{Tr}\; dA_i\wedge A_i + \frac{1}{3} A_i\wedge A_i\wedge A_i$$

$$K_j\Big|_{U_i\cap U_j} \equiv K_i + \mathrm{Tr}\; f_{ij}^{-1}df_{ij}\wedge f_{ij}^{-1}df_{ij}\wedge f_{ij}^{-1}df_{ij}$$
$$+ d\;\mathrm{Tr}\;\frac{1}{3}\, df_{ij}\wedge f_{ij}^{-1}A_i$$

Now being $U_i\cap U_j \simeq S^n\; n\leq 2$, $H^3_{deR}(S^n) = 0$ thus there exists a 2-form ξ_{ij} such that

$$d\xi_{ij} = f_{ij}^{-1}df_{ij}\wedge f_{ij}^{-1}df_{ij}\wedge f_{ij}^{-1}df_{ij}$$

Take now by definition

$$g_{ij} = \exp 2\pi i\left(\mathrm{Tr}\,\frac{1}{3}\;\widehat{df_{ij}\wedge f_{ij}^{-1}A_i} + \hat{\xi}_{ij}\right)$$

and consider $g_{ij} \in \Gamma(\Omega^2 U_i \cap \Omega^2 U_j, \mathcal{G}^2)$
as the transition function of a $\mathcal{G}^2$-bundle over $\Omega^2 M$.
In particular in the case of $M = S^4$ we can proceed as follows 5) : let $U_x = S^4/\{x\}$ $x \in S^4$
then $U_x \simeq \mathbb{R}^4$ i.e. contractible therefore given $P(M, SU(N))$
$P|_U$ is trivializable.
Clearly $\{U_x\}_{x \in S^4}$ is a covering of S^4 and

$$\bigcup_{x \in S^4} \Omega^2 U_x = \Omega^2 S^4$$

Now $U_x \cap U_y \approx S^3$ thus we are not exactly in the situation required by the above construction. However let γ_{xy} a path connecting x to y , $U_x \cap U_y / \gamma_{xy} \simeq \mathbb{R}^4$, then

$$f_{ij}^{-1} d f_{ij} \wedge f_{ij}^{-1} d f_{ij} \wedge f_{ij}^{-1} d f_{ij} \Big|_{U_x \cap U_y / \gamma_{xy}} = d \xi(\gamma_{xy})$$

One now define

$$g'_{ij} = \exp 2\pi i \hat{\xi}(\gamma_{xy}).$$

and consider as transition functions g_{ij} the classess of all g'_{ij} which coincide on their demain in $\Omega^2(U_x \cap U_y)$
Clearly on $\Omega^2(U_x \cap U_y / \gamma_{xy})$ a representative is given by

$$g_{ij} = \exp 2\pi i \hat{\xi}(\gamma_{xy})$$

A connection in the bundle is defined by $\{\hat{K}_i\}_{i \in I}$
and $\hat{Q}$ definesa curvature.

If the bundle $P(M,SU(N))$ is not trivial, then the cohomology class of $\hat{Q}$ is not trivial.

6 Configuration space and θ-vacua.

In this final section I want to make some remarks on the configuration space of gauge differential forms theories in the context of $\mathcal{G}^k$ bundle, which proves the usefulness of this approach.

All what will be said here is very rough and these notes are intended to be only a preliminary work.

We assume that the space-time has a structure $M = \mathbb{R}(\text{time}) \times \Sigma$ Σ being a connected d-dimensional manifold.

Let $\mathcal{B}^k$ be a $\mathcal{G}^k$-bundle over $\Omega^k M$ with connection α and $\hat{F} \in E_2(\Omega^k M)$ $(F \in E_{k+2}(M))$ the representative in $\Omega^k M$ of the curvature form; thus for any k+2 dimensional submanifold N without boundary we have $\int_N F \in \mathbb{Z}$.

In order to define the canonical configuration space we have first to fix the gauge; we choose as usual to work in the temporal gauge which is defined as follow.

Let $\{\hat{A}_i\}_{i \in I}$ the local representatives of the connection α, being $M = \mathbb{R} \times \Sigma$ one can split $\hat{A}_i$ into

$$\hat{A}_i = \hat{A}_i^{(0)} \wedge dx^0 + \hat{A}_i^{(\Sigma)}$$

where x^0 is the canonical variable in $\mathbb{R}$, $\hat{A}_i^{(0)}$ and $\hat{A}_i^{(\Sigma)}$ are respectively k and a k+1 forms not containing dx^0.
The gauge is fixed putting $\hat{A}_i^{(0)} = 0$. It is easy to see that the residual freedom amounts to gauge transformations

$$\{\hat{A}_i^{(\Sigma)}\}_{i\in I} \longrightarrow \{\hat{A}_i^{(\Sigma)} + g^{-1(\Sigma)} \hat{d} g^{(\Sigma)}\}_{i\in I}$$

where A is a k+1 form on Σ and $g^{(\Sigma)} \in \Gamma(\Omega^k\Sigma, \mathcal{G}^k)$

Because of the integrality conditions on F, every $\{\hat{A}_i^{(\Sigma)}\}_{i\in I}$ defines a connection in a $\mathcal{G}^k$bundle over $\Omega^k\Sigma$; notice that in the case k=d-1 this bundle is necessarily trivial.
Let $\mathcal{B}^k$ now be resulting $\mathcal{G}^k$ bundle over $\Omega^k\Sigma$ and $\mathcal{A}^k$ the space of all C^∞ connection on $\mathcal{B}^k$ represented by the local representatives in $\Omega^k\Sigma$ $\hat{A}_i$ (I omit from here on the symbol $^{(\Sigma)}$ for simplicity).
Then if $\{\hat{A}_{i1}\}_{i\in I}, \{\hat{A}_{i2}\}_{i\in I} \in \mathcal{A}^k$; $\{\hat{A}_{i1} - \hat{A}_{i2}\}_{i\in I}$ defines a horizontal one form on $\mathcal{B}^k$, $\mathcal{T}$, such that

$$\sigma_i^* \mathcal{T} = \hat{\mathcal{T}} \qquad \mathcal{T} \in E_{k+1}(\Sigma)$$

Being the space of $\mathcal{T}$ forms a linear space, it can be made isomorphic to a Hilbert space, thus $\mathcal{A}^k$ is the affine space associated to the Hilbert (vector) space.

The group of gauge transformations $\mathcal{G}^k \equiv \Gamma(\Omega^k \Sigma\, \mathcal{G}^k)$ acts on $\mathcal{A}^k$ by

$$\mathcal{G}^k \times \mathcal{A}^k \longrightarrow \mathcal{A}^k$$
$$(g\ ,\ \{\hat{A}_i\}_{i\in I}) \longrightarrow \{\hat{A}_i + g^{-1}dg\}_{i\in I}$$

The canonical configuration space is now defined as

$$\mathcal{Q}^k = \mathcal{A}^k / \mathcal{G}^k$$

We do not analyze here the properties of $\mathcal{Q}^k$ except for the first homotopy group of $\mathcal{Q}^k$. In fact although it is not clear how to proceed to get a quantisation scheme of such theories, if the usual relation between $\mathcal{Q}^k$ and inequivalent quantisations, labelled by θ-vacua, holds, one has that they are classified 20) by elements of

$$\Theta = \mathrm{Hom}\,(\pi_1(\mathcal{Q}^k),\ U(1))$$

and we shall prove that the expected result for the case $M = S^n$ holds in the context of $\mathcal{G}^k$ bundles.

In fact from the exact sequence

$$0 \longrightarrow \mathcal{G}^k \longrightarrow \mathcal{A}^k \longrightarrow \mathcal{Q}^k \longrightarrow 0$$

one gets the homotopy exact sequence

$$\pi_0(\mathcal{Q}^k) \longrightarrow \pi_0(\mathcal{A}^k) \longrightarrow \pi_0(\mathcal{G}^k) \longrightarrow \pi_1(\mathcal{Q}^k) \longrightarrow$$
$$\longrightarrow \pi_1(\mathcal{A}^k) \longrightarrow \ldots$$

Being $\mathcal{A}^k$ an affine space $\pi_i(\mathcal{A}^k) = 0 \quad \forall i \geq 0$ and thus we have an isomorphism

$$\pi_1(\mathcal{Q}^k) \simeq \pi_0(\mathcal{G}^k)$$

Theorem 21 Let Σ be k-connected, then

$$\pi_0(\mathcal{G}^k) \simeq H^{k+1}(\Sigma, \mathbb{Z})$$

Remark: We first observe that from Serre exact sequence 22)

$$H^{k+1}(\Sigma, \mathbb{Z}) \simeq H^1(\Omega^k\Sigma, \mathbb{Z})$$

Then being $\Omega^k\Sigma$ simply connected by Hurewitz theorem

$$\pi_1(\Omega^k\Sigma) \simeq \pi_k(\Sigma) = 0$$

an integer cohomology class in $\Omega^k\Sigma$ can be uniquely represented by a closed form which have integral period over all integral cycles in $\Omega^k\Sigma$.

Proof: $\pi_0(\mathcal{G}^k) = \overline{[\Omega^k\Sigma, U(1)]}$, the homotopy classes of maps from $\Omega^k\Sigma$ to $U(1)$ which are locally of the form (see section 4.3)

$$\exp 2\pi i \hat{\Lambda} \qquad \Lambda \in E_k(U)$$

The set $[\Omega^k\Sigma, U(1)]$ of all homotopy classes of maps from $\Omega^k\Sigma$ to $U(1)$ has a natural abelian group structure and it is easy to see that $\pi_0(\mathcal{G}^k)$ is a subgroup of $[\Omega^k\Sigma, U(1)]$.

Let now α the fundamental closed 1-form on S^1, (which locally can be written $\alpha = \frac{1}{2\pi} d\varphi$) whose cohomology class generates $H^1_{deR}(S^1)$. Then there is a map

$$h: \quad U(1)^{\Omega^k\Sigma} \longrightarrow Z^1_{deR}(\Omega^k\Sigma)$$

$$(c : \Omega^k\Sigma \to U(1)) \longrightarrow c^*\alpha$$

Consider the restriction of h to $\mathcal{G}^k$. h maps homotopic maps to cohomologous cycles and hence there is an induced group homomorphism

$$\hat{h} : \overline{[\Omega^k\Sigma, U(1)]} \longrightarrow H^1_{deR}(\Omega^k\Sigma)$$

We first prove that the image of $\hat{h}$ is in $H^1(\Omega^k\Sigma, \mathbb{Z})$. In fact if $g \in \mathcal{G}^k$ and $\{\Omega^k U_i\}_{i\in I}$ is a contractible covering of $\Omega^k\Sigma$, one has

$$g|_{\Omega^k U_i} = p \circ f_i$$

Where $p : \mathbb{R} \to U(1)$ is the natural projection and $f_i = \hat{\lambda}_i$ for some $\lambda_i \in E_k(U_i)$. Then we have

$$g^*\alpha|_{\Omega^k U_i} = f_i^* p^* \alpha = df_i = d\hat{\lambda}_i$$

Since on $\Omega^k U_i \cap \Omega^k U_j$, $d\hat{\lambda}_i = d\hat{\lambda}_j$ the collection of 1-form $\{d\hat{\lambda}_i\}_{i\in I}$ represents a closed one form $\hat{\beta}$ on $\Omega^k\Sigma$.

Moreover since $\hat{\lambda}_i - \hat{\lambda}_j|_{\Omega^k U_i \cap \Omega^k U_j} \in \mathbb{Z}$ one has that $\hat{\beta}$ defines an integer cohomology class.

To prove $\hat{h}$ is onto, let β be a closed 1-form repre - senting an integral cohomology class of $\Omega^k\Sigma$, being $H^1(\Omega^k\Sigma, \mathbb{Z}) \simeq H^{k+1}(\Sigma, \mathbb{Z})$ we can always choose $B = \hat{\beta}$ $\beta \in E_{k+1}(\Sigma)$

If $\{\Omega^k U_i\}_{i\in I}$ is a contractible covering of $\Omega^k\Sigma$ then

$$\hat{\beta} = d\hat{\lambda}_i$$

and on $\Omega^k(U_i \cap U_j)$, $\hat{\lambda}_i - \hat{\lambda}_j \in \mathbb{Z}$.

Therefore defining $g_i = \exp 2\pi i f_i$, since $\Omega^k U_i$ is contractible, we have $g_i|_{\Omega^k(U_i \cap U_j)} = g_j|_{\Omega^k(U_i \cap U_j)}$ and hence a function $g \in \mathcal{G}^k$ is defined such that $h(g) = \beta$

To see that $\hat{h}$ is into, notice that if $[\hat{\beta}] = 0$, $\hat{\beta} = d\hat{\lambda}$ globally and the function $g = \exp(2\pi i\, \hat{\lambda})$ is nul-homotopic. Thus if g is not nul-homotopic, $\beta = g^{+}\alpha$ is not a cobaundary and hence $\operatorname{Ker} \hat{h} = 0$

Thus $\hat{h}$ is an isomorphism of $\overline{[\Omega^k \Sigma, U(1)]}$ onto $H^1(\Omega^k \Sigma, \mathbb{Z})$.

Let now consider the case when $\Sigma = S^n$; we immediately get

$$\Pi_1(\Omega^k) = \begin{cases} \mathbb{Z} & k = d-1 \\ 0 & \text{otherwise} \end{cases}$$

Thus θ -vacua occur only for k=d-1 in which case

$$\Theta = \operatorname{Hom}(\mathbb{Z}, U(1)) = U(1)$$

In particular for d=1 k=0 we recover the known θ -vacua of compact electrodynamic in 1+1 dimensions.

For k=d-1 the situation is similar to the usual Yang Mills instanton picture, rinforcing the link between Yang Mills theory on S^4 and $\mathcal{G}^2$ bundle over S^4. In fact our result could be understood heuristically in terms of "n vacua", labelled by $n = \int_\Sigma A \in \mathbb{Z}$ and tunneling. The analog of the instanton would be in this case the field for which F is an

integral multiple of the fundamental harmonic $d+1$ form on
For $k < d-1$ the cohomology class of F in Σ need not be trivial. This class is a topological invariant which is conserved in time evolution, much like monopole charge. The situation is similar to compact electrodynamic in $d+1$ dimensions, for $d>1$.

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