ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

ATTESTATO DI RICERCA

"MAGISTER PHILOSOPHIAE"

ABOUT A NON HAMILTONIAN QUANTUM EVOLUTION MODEL

CANDIDATO:

dott. Fabio Benatti

RELATORI:

Prof. G.C. GHIRARDI

Prof. T. WEBER

Anno Accademico 1985/86

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

TRIESTE

Strada Costiera 11

ABOUT NON-HAMILTONIAN QUANTUM EVOLUTION MODEL

Attestato di Ricerca "Magister Philosophiae"

Candidato: dott. Fabio Benatti

Relatori: Prof. C.G. Ghirardi e Prof. T. Weber

Settore: Particelle Elementari

sotto settore: Fisica Matematica

Anno Acaademico 1985/86

LIST OF SYMBOLS

$B(H)$: vector space of bounded linear operators on the Hilbert space H

$B(H)_1$: vector space of trace class operators on H

$B(H)_p$: vector spaces of compact operators $\hat{A}$ for which $[Tr\,|\hat{A}|^p]^{1/p} < +\infty$, $0 < p \leq +\infty$

$B(H)^+_{(\cdot)}$: positive elements of $B(H)_{(\cdot)}$.

$B(H)^R_{(\cdot)}$: self adjoint elements of $B(H)_{(\cdot)}$.

$C_0(H)$: vector space of finite range operators.

$K(H)$: convex set of density matrices on H (state space).

$\|\cdot\|$: usual norm on B(H).

$\|\cdot\|_1$: trace norm.

$\|\cdot\|_p$: $[Tr\,|\hat{A}|^p]^{1/p}$ (norms for p>1, p=1)

$<.,.>$: duality product between linear spaces.

$[.,.]$: semi-inner product in a Banach space (s.i.p.).

$T[.]$: operation on B(H) .

$T^*[.]$: dual operation on B(H).

$\Sigma = \{\Sigma_t\}_{t\geq 0}$: semigroup on $B(H)_1$.

$\Sigma^* = \{\Sigma^*_t\}_{t\geq 0}$: dual semigroup on B(H).

INDEX

INTRODUCTION

The purpose of this work is to study some of the mathematical aspects and some of the phisical consequences of the dynamical model proposed by G.C.GHIRARDI, A.RIMINI, T WEBER in ref. [1].

It consists of a modification of the usual Hamiltonian dynamics for a physical system.

Quite generally we can look at it as at a special kind of interaction between a microsystem and an unspecified reservoir.

This interaction produces a reduced dynamics for the microsystem which is typical of the open systems [2].

The model possesses the far reaching property of providing the same type of modified dynamics for the centre of mass of the composite systems.

It is this nice feature that allows to get rid of the linear superpositions of macrostates and to obtain a classical motion for the centre of mass of, let us say, a crystal or an almost rigid body[1].

By a suitable choice of the parameters in the dissipative term modifying the Hamiltonian dynamics, the authors [1] can indeed show that quantum interference effects between far away localized macrostates are forbidden.

On the other hand, with the same choice, it is also shown that the usual evolution for microsystems is affected only after an enormous amount of time.

These features of the model allowed the authors to set up a program of unification of macro and micro-dynamics, once a common quantum ground for the two is accepted.

Following these lines it has also been provided a model of quantum measurement process which exhibits all the required physical properties but the unpleasant presence of linear superpositions of the apparatus-pointer positions.

It is also immediate to derive the well-known Von Neumann formula for the wave-packet reduction.

The modified dynamics is given by :

$$\frac{d\hat{\rho}_t}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}_t] - \lambda\hat{\rho}_t + \lambda T[\hat{\rho}_t] \qquad \langle I.1\rangle$$

in the Schroedinger picture and from now on it will be indicated as :

QUANTUM MECHANICS with SPONTANEOUS LOCALIZATIONS (Q.M.S.L.).

In <I.1> $\hat{H}$ is the quantum mechanical Hamiltonian and $\hat{\rho}_t$ is the evolved of the density matrix $\hat{\rho}$ describing the physical system at time t=0.

$T[\hat{\rho}_t]$ is an instantaneous process affecting the system with mean frequency λ, given by:

$$T[\hat{\rho}_t] = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{+\infty} dx \; e^{-\frac{\alpha}{2}(\hat{q}-x)^2}\,\hat{\rho}_t\, e^{-\frac{\alpha}{2}(\hat{q}-x)^2} \qquad \langle I.2\rangle$$

where $\hat{q}$ is the usual position operator in one dimension.

The spontaneous processes $T[\cdot]$ could be considered as the result of a coupling between the microsystem and a suitable surrounding after reduction to the microphysical degrees of freedom only.

What is important here is the fact that, whatever the coupling and the reservoir configuration be, a sort of localization in position for the microsystem in the state $\hat{\rho}_t$ is derived.

The parameters which have to be chosen, are the mean frequency λ and the precision $\alpha^{-1/2}$ of the localizations.

The Q.M.S.L. <I.1> has been solved in the free particle case [1], having as a solution $\Sigma_t \hat{\rho}$ which in the q-representation is given by:

$$\langle q | \Sigma_t \hat{\rho} | q' \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} d\mu \int_{-\infty}^{+\infty} dy \; e^{-i\mu y/\hbar} \; F(\lambda, \mu, q-q', t) \, \langle q+y | \hat{\rho}_s(t) | q'+y \rangle \qquad \text{<I.3>}$$

where:

$$\hat{\rho}_s(t) = e^{-\frac{i\hat{p}^2 t}{2m\hbar}} \; \hat{\rho} \; e^{\frac{i\hat{p}^2 t}{2m\hbar}} \qquad \text{<I.4>}$$

and

$$F(\lambda, \mu, q-q', t) = e^{-\lambda t} \; e^{\lambda \int_0^t e^{-\frac{\alpha}{4}\left[\frac{\mu\tau}{4} - q + q'\right]^2} d\tau} \qquad \text{<I.5>}$$

Putting $q-q'=\nu$, $F(\lambda,\mu,\nu,t)$ satisfies;

$$F(\lambda, \mu, \nu, t) < e^{-\lambda\beta t} \qquad \text{<I.6>}$$

$$\beta = 1 - \frac{\sqrt{\pi}}{\sqrt{\alpha}\,\nu/2} \operatorname{erf}\left[\frac{\sqrt{\alpha}}{2}\nu\right] \qquad \text{<I.7>}$$

$$1 - F(\lambda, \mu, 0, t) \leq \frac{\alpha\lambda\mu^2}{4m^2} t^3 \qquad \text{<I.8>}$$

Using these three properties the authors [1], were able to show that:

(a) the matrix elements of $\Sigma_t \hat{\rho}$ between two eigenvectors of $\hat{q}$: $|q'\rangle, |q''\rangle$" such that $|q'-q''| < \frac{1}{\sqrt{\alpha}}$ or q'=q" behave as in the pure Schroedinger case, for t<<T , T depending on the parameters λ, α and the mass of the system;

(b) the matrix elements of $\Sigma_t \hat{\rho}$, when $|q'-q''| > \frac{1}{\sqrt{\alpha}}$ are suppressed in time with a mean life $\tau = \frac{1}{\alpha\beta}$.

Within this amount of time the state of the system goes into a statistical mixture of localized states with precision not less than $1/\sqrt{\alpha}$.

At this point it is of great importance to investigate how the Q.M.S.L. extends to systems with a high number of particles, assuming that each of them is subjected to localizations processes occuring with frequency λ_i for the i-th particle.

The main result is that, when a composite system is such that the internal and the centre of mass dynamics decouple ($\hat{H} = \hat{H}_Q + \hat{H}_r$), then:

$$\frac{d\hat{\rho}_t^Q}{dt} = -\frac{i}{\hbar}\left[\hat{H}_Q, \hat{\rho}_t^Q\right] - \left(\sum_i^N \lambda_i\right)\left\{\hat{\rho}_t^Q - T_Q\left[\hat{\rho}_t^Q\right]\right\} \qquad \text{<I.9>}$$

where N is the number of constituents.

Moreover, if the structure of the composite system is rigid, then:

$$\frac{d\hat{\rho}_t^r}{dt} = -\frac{i}{\hbar}\left[\hat{H}_r, \hat{\rho}_t^r\right] \qquad \text{<I.10>}$$

$\hat{\rho}_t^Q = Tr^{(r)}\{\Sigma_t \hat{\rho}\}$:partial trace w.r.t. internal coordinates;

$\hat{\rho}_t^r = Tr^{(Q)}\{\Sigma_t \hat{\rho}\}$:partial trace w.r.t. barycentric coordinates;

Equation <I.9> holds because of the following property of T ;

$$T_Q\left[\hat{\rho}_t^Q\right] = Tr^{(r)}\left[T_i\left[\hat{\rho}_t\right]\right] \qquad \text{<I.11a>}$$

If the macrosystem is almost rigid then the result is stronger:

$$T_Q\left[\hat{\rho}_t\right] = T_i\left[\hat{\rho}_t\right] \qquad \text{<I.11b>}$$

Here $T_i[\cdot]$ is the spontaneous localization process affecting the i-th constituent.

<I.11b> tells that every process occurring with frequency λ_i is a process on the centre of mass which in turn is thus affected with frequency $\sum_i^N \lambda_i$.

If we choose the parameters in an appropriate way, for example: $\forall i$, $\lambda_i = 10^{-6}$ sec.(nearly one localization process on the single constituent every 10^{8} years), $\alpha = 10^{10}\,\mathrm{cm}^{-2}$, one obtains for a macroscopic body ($N \sim 10^{23}$) having mass m=1gr.:

$$\lambda = N\lambda_i = 10^{-7}\,\mathrm{sec}^{-1}\ ,\ \tau = 10^{-6}\,\mathrm{sec} \quad \text{for} \quad |q' - q''| = \nu = 4\cdot 10^{-5}\,\mathrm{cm}$$

The main result is that the linear superpositions of macrostates separated by a distance more than 10^{-4} cm are turned into statistical matrices in a time of the order of 10^{-6} sec.

Furthermore, the choice of λ_i and α makes the microdynamics almost unaffected by the localization processes.

These features of Q.M.S.L allow a nice application to the measurement problem [3].

The next obvious step is to study the interaction of a microsystem with a macroscopic measuring apparatus.

Let us use a simple model and consider a system whose Hamiltonian is given by:

$$\hat{H} = \frac{\hat{P}^2}{2M} + \frac{d\beta(t)}{dt} f(\hat{e})\hat{P} \qquad \text{<I.12>}$$

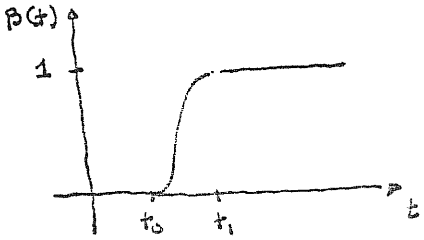

fig. 1

H describes a coupling between a microsystem and an apparatus characterized by the function in fig. 1, in the reference frame in which the microsystem is at rest, the apparatus being used to measure the observable $\hat{e}$ on it and schematized through its pointer centre of mass position.[4].

This Hamiltonian is able to account for the correlations between the position of the pointer, whose barycentric coordinates are Q,P, and the values $\{e_n\}$ of $\hat{e}$, set up by the interaction.

Unpleasantly if we solve the Schroedinger equation for the Hamiltonian given by <I.12> we get the interference effects between different macrostates of the pointer localized around different positions $f(e_n)$ and $f(e_m)$ on the scale, with dispersions much less than the space separation of any couple of them.

According with the new scheme, we can think of the pointer as affected by localization processes with a very high frequency λ , and, disregarding the ineffective localization processes on the microsystem, we get:

$$\frac{d\Sigma_t\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\frac{\hat{P}^2}{2M}, \Sigma_t\hat{\rho}\right] - \frac{i}{\hbar}\frac{d\beta(t)}{dt}\left[f(\hat{e})\hat{P}, \Sigma_t\hat{\rho}\right] - \lambda\Sigma_t\hat{\rho} + \lambda T[\Sigma_t\hat{\rho}] \qquad \text{<I.13>}$$

$\hat{\rho}$ now represents the preparation state of the system particle plus apparatus (pointer) before the starting of the interaction.

The model has been solved [3]; denoting by $|\Psi^S_m\rangle$ the eigenstates of $\hat{e}$ and by $|Q\rangle$ those of the centre of mass position of the pointer, we have:

$$\langle Q, \Psi^S_n | \Sigma_t\hat{\rho} | Q', \Psi^S_m\rangle =$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} d\mu \int_{-\infty}^{+\infty} dy\, e^{-i\mu y/\hbar} F_{n,m}(\lambda, \mu, Q-Q', t)\, \rho^{\lambda=0}_{n,m}(Q+y, Q'+y, t) \qquad \text{<I.14>}$$

$$F_{n,m}(\lambda, \mu, \nu, t) = \exp\left\{-\lambda t + \lambda\int_0^t d\tau \exp\left[\frac{\mu\tau}{M} - \nu + [f(e_n) - f(e_m)][\beta(t) - \beta(t-\tau)]\right]\right\} \qquad \text{<I.15>}$$

$$\rho^{\lambda=0}_{n,m}(Q+y, Q'+y, t) = \langle \Psi^S_n, Q+y | \exp\left\{-i\frac{\hat{H}t}{\hbar}\right\} \hat{\rho} \exp\left\{i\frac{\hat{H}t}{\hbar}\right\} | \Psi^S_m, Q'+y\rangle \qquad \text{<I.16>}$$

If the initial state of the composite system is:

$$|\Psi^S\rangle \otimes |\Phi^A_0\rangle = \left(\sum_n c_n |\Psi^S_n\rangle\right) \otimes |\Phi^A_0\rangle \quad \text{in} \quad H^S \otimes H^A \qquad \text{<I.17>}$$

where $|\Phi^A_0\rangle$ is a well-localized state of the center of mass of the pointer around the position zero on the scale with dispersion $\Delta Q \ll |f(e_{k+1}) - f(e_k)| \;\; \forall k$, then the purely Hamiltonian evolution gives the state:

$$|\Psi(t)\rangle = \sum_n c_n |\Psi^S_n\rangle \otimes |\Phi^A_n(t)\rangle \qquad \text{<I.18>}$$

after the interaction took place, where:

$|\phi^A_n(t)\rangle$ is a pointer macrostate localized around $f(\ell_n)$ with dispersion $\overline{\Delta Q} \simeq \Delta Q$ (due to its macroscopic features).

The corresponding density matrix exhibits off-diagonalities :

$$|\psi(t)\rangle\langle\psi(t)| = \sum_{n,m} c_n c^*_m \, |\psi^S_n\rangle\langle\psi^S_m| \otimes |\phi^A_n(t)\rangle\langle\phi^A_m(t)| \qquad \text{<I.19>}$$

The modified dynamics, on the contrary, because of the properties of $F_{m,n}(\alpha,\mu,\nu,t)$ gives:

$$\Sigma_t \hat{\rho} = \sum_n |c_n|^2 \, |\psi^S_n\rangle\langle\psi^S_n| \otimes |\phi^A_n(t)\rangle\langle\phi^A_n(t)| \qquad t > t_1 \quad (\text{see fig. 1}) \qquad \text{<I.20>}$$

where:

$$\hat{\rho} = \Sigma_0 \hat{\rho} = \sum_{n,m} c_n c^*_m \, |\psi^S_n\rangle\langle\psi^S_m| \otimes |\phi^A_0\rangle\langle\phi^A_0| \qquad \text{<I.21>}$$

Performing the trace over the pointer degrees of freedom in <I.21> , we see that the state of the microsystem $\hat{\sigma} = Tr^{Q} \hat{\rho}$ transforms as:

$$\hat{\sigma} \rightarrow \hat{\sigma}_t = \sum_n P_n \, \hat{\sigma} \, P_n \qquad \text{where:} \quad P_n = |\psi^S_n\rangle\langle\psi^S_n| \qquad \text{<I.22>}$$

corresponding to the wave-packet reduction of the usual quantum mechanics.

In this scheme the reduction is thus given by a dynamical effect, contained in the evolution model.

The modified dynamics as a general kind of quantum evolution does not require any ultimate observer for cutting the Von Neumann chain and, by physical arguments, indicates how to overcome the quantum measurement problems.

CHAPTER 1

Paragraph 1

This chapter is devoted to give an outline of the mathematical and phisical background of the Q.M.S.L.model.

The instantaneous processes which are supposed to affect the physical systems during their dynamics, are known as "covariant instruments" in the literature[2].

This concept is closely related to that of "generalized observable"[2], and both can be thought of as generalization of what is understood as "reduction of the wave packets" or "measurement process".

The simplest way of discussing it is to reduce the problem to the measurement process of a, so called, "yes-no "observable described by a projection operator, $\hat{P}$, on some Hilbert space.

In this case, after the interaction with the measuring apparatus, the initial state $\hat{W}_i$ of a statistical ensemble of N identically prepared physical systems, has changed into a final state $\hat{W}_f^{(\cdot)}$ where:

$$\hat{W}_f^{(1)} = \hat{P}\hat{W}_i\hat{P} / \mathrm{Tr}\{\hat{P}\hat{W}_i\} \qquad \langle 1.1.1\rangle$$

$$\hat{W}_f^{(2)} = (\hat{1}-\hat{P})\hat{W}_i(\hat{1}-\hat{P}) / \mathrm{Tr}\{(\hat{1}-\hat{P})\hat{W}_i\} \qquad \langle 1.1.2\rangle$$

$$\hat{W}_f^{(3)} = \hat{P}\hat{W}_i\hat{P} + (\hat{1}-\hat{P})\hat{W}_i(\hat{1}-\hat{P}) \qquad \langle 1.1.3\rangle$$

The first two final states describe the selected ensemble of those systems which did or did not trigger the apparatus, the third one the case in which no selection has been made.

Two assumptions lie behind this simple and well known scheme.

The first assumption is about the existence of an apparatus being able to discriminate, with infinite efficiency, between the eigenvalue 1 or 0 of , through the interaction with the systems of the statistical ensemble.

The second one is that this scheme forces ourselves in considering only the class of ideal measurements, for which if $\hat{W}_i = \hat{P}$ then $\hat{W}_f = \hat{P}$ without distorsion.

There are many critizisms [2], [5] against these restrictions and in general all the theory of "operations and effects" [6], [7], [8], [9] tries to give a more phisically and mathematically consistent description of what happens during the measurement process.

As it has been pointed out in ref.[10], an "operation" causes a state change on the physical system generating a new positive, linear, functional on the algebra of the observables.

Generally this algebra is the self-adjoint part of a C*-Algebra and the normalized, positive, linear functionals defined on it are the states of the system.

In [10] the so called "pure operations" are in one-to-one correspondence with those elements $\hat{E}$ of $\mathcal{A}$ such that:

$$\|\hat{E}\| \leq 1 \qquad \text{<1.1.4>}$$

If ϕ is a state, the state change due to the operation T_E , provides us with the new state ϕ_E given by:

$$\phi_E(\hat{A}) = \frac{\phi(\hat{E}^{+}\hat{A}\hat{E})}{\phi(\hat{E}^{+}\hat{E})} \qquad \forall\ \hat{A} \in \mathcal{A} \qquad \text{<1.1.5>}$$

where $\phi(\hat{E}^{+}\hat{E})$ is a normalization factor and represents the transition probability between the two states.

Paragraph 2

The connection between the algebraic approach and the usual Hilbert space formulation of Quantum Mechanics is given by the following:

THEOREM 1.2.1 [11]

Let ϕ be a state on a Von Neumann Algebra $\mathcal{M}$ acting on the Hilbert space H.

The following conditions are equivalent:

(1) ϕ is normal;

(2) ϕ is ultra-weakly continuous;

(3) there exists a density matrix $\hat{\rho}$ such that $\phi(\hat{A}) = Tr\{\hat{\rho}\hat{A}\}$

end

(see appendix A.1 for the definitions and theorems of this paragraph)

Translating <1.1.5> into the usual language we have:

$$Tr\{\hat{\rho}_{\phi_E}\hat{A}\} = Tr\{\hat{\rho}_\phi \hat{E}^\dagger \hat{A}\hat{E}\} / Tr\{\hat{\rho}_\phi \hat{E}^\dagger \hat{E}\} \qquad \text{<1.2.2>}$$

Using the properties of the trace, from <1.2.2> we obtain:

$$Tr\{\hat{\rho}_{\phi_E}\hat{A}\} = Tr\{\hat{E}\hat{\rho}_\phi \hat{E}^\dagger \cdot \hat{A}\} / Tr\{\hat{E}\hat{\rho}_\phi \hat{E}^\dagger\} \quad \forall \hat{A} \in \mathcal{M} \qquad \text{<1.2.3>}$$

Considering $\hat{A} = |\psi\rangle\langle\psi| \quad \psi \in H, \ \|\psi\| = 1$ we can set:

$$\hat{\rho}_{\phi_E} = \frac{\hat{E}\hat{\rho}_\phi \hat{E}^\dagger}{Tr\{\hat{E}\hat{\rho}_\phi \hat{E}^\dagger\}} \qquad \text{<1.2.4>}$$

<1.2.4> enables us to introduce the "operation" as a map from $B(H)_1$ into $B(H)_1$. Disregarding the subscripts and considering the transformation:

$$T[\hat\rho] = [\mathrm{Tr}\{\hat E^\dagger \hat E \; \hat\rho\}] \cdot \hat\rho = [\mathrm{Tr}\{\hat F \cdot \hat\rho\}] \cdot \hat\rho$$

<1.2.5>

we see that:

T is linear <1.2.6a>

T is positive <1.2.6b>

$\hat F = \hat E^\dagger \hat E$ is an observable and $\|\hat F\| \le 1$ <1.2.6c>

if $\hat E^\dagger = \hat E = \hat E^2$ then T can be associated with the selection of those systems which gave the answer yes in the measurement of $\hat E$ <1.2.6d>

since $\mathrm{Tr}\{\hat F \hat\rho\} \le 1 = \mathrm{Tr}\,\hat\rho \quad \forall\, \hat\rho \in K(H)$ we have $\hat F \le \hat{1}$ <1.2.6e>

Thus we are led to consider, as buildings blocks of general measurements processes, not only the projections but also the more general $\hat F$'s called "effects" [9].

The set of all possible effects is convex and its extremal points are the projections [2], which are weakly dense [2].

Since $T[\hat\rho] = [\mathrm{Tr}\{\hat F \hat\rho\}] \cdot \hat\rho$ and $\hat\rho$ is of trace 1, we have:

$$T[\hat\rho] = \mathrm{Tr}\{T[\hat\rho]\} \cdot \hat\rho \quad , \quad \hat\rho \in K(H)$$

<1.2.7>

Hence we see that there is a correspondence between the selective operation T and the effect $\hat F$.

We shall see that $\hat F$ is uniquely determined by T and that this correspondence is not one-to-one.

In order to prove that, it is necessary to extend T to all of $B(H)_1$.

Up to now it has been considered as a map from $K(H)$ into $B(H)_1$.

This can be easily done obtaining a positive, complex linear map which is continuos on $B(H)_1$ with respect to the trace-norm topology (see appendix A.2).

It then follows that:

$$|\mathrm{Tr}\{\hat{X}\cdot T[\hat{A}]\}| \leq \|\hat{X}\| \, \|\hat{A}\|_1 \qquad \forall \hat{X} \in B(H) \;, \forall \hat{A} \in B(H)_1 \tag{1.2.8}$$

From the duality between $B(H)_1$, $B(H)$ given by the duality product:

$$\langle \hat{A}, \hat{X} \rangle = \mathrm{Tr}\{\hat{A}\hat{X}\} \qquad \hat{A} \in B(H)_1 \;, \; \hat{X} \in B(H) \tag{1.2.9}$$

we have a unique $\hat{X}^* = T^*[\hat{X}]$ such that:

$$\mathrm{Tr}\{\hat{X}\cdot T[\hat{A}]\} = \mathrm{Tr}\{T^*[\hat{X}]\cdot \hat{A}\} \qquad [21] \tag{1.2.10}$$

T^* is the adjoint map of T and it is complex linear, positive and ultraweakly continuos on $B(H)$ (see appendix A.2).

From <1.2.7> we have that:

$$\mathrm{Tr}\{T[\hat{\rho}]\} = \mathrm{Tr}\{T^*[\hat{1}]\cdot\hat{\rho}\} = \mathrm{Tr}\{\hat{F}\hat{\rho}\} \qquad \forall \hat{\rho} \in K(H) \tag{1.2.11}$$

Hence: $$\hat{F} = T^*[\hat{1}] \tag{1.2.12}$$

Paragraph 3

It seems reasonable to require a more general kind of positivity for the map T and its adjoint T* . In this paragraph it will be discussed the request of complete positivity [8], [9], [12] .

For more technical details the interested reader is refferred to [13].

Given two Hilbert spaces, H and H_N , H_N finite dimensional, the C*-Algebra of bounded operators on $H \otimes H_N$ is isomorphic to the algebra of NxN matrices with entries in $B(H)$:

$$B(H \otimes H_N) \simeq B(H) \otimes M_N \qquad \text{<1.3.1>}$$

$$\underline{\hat{X}} \in B(H \otimes H_N) \quad \text{is the matrix:} \quad [\hat{X}_{ij}] \quad i,j = 1,\dots,N \qquad \text{<1.3.2>}$$

in the representation:

$$\pi(\underline{\hat{X}})\,|\psi\rangle \otimes |e_i\rangle = \sum_{1}^{N}{}_j\, [\hat{X}_{ji}\,|\psi\rangle] \otimes |e_j\rangle \qquad \text{<1.3.3>}$$

$|\psi\rangle \in H$, $|e_j\rangle$ j-th N- unit vector

Hence: <1.3.4>

$$\pi(\underline{\hat{X}})\,|\psi\rangle \otimes |e_i\rangle = \begin{pmatrix} \hat{X}_{11} & \dots & \hat{X}_{1N} \\ \vdots & & \\ \hat{X}_{i1} & \dots & \hat{X}_{iN} \\ \vdots & & \\ \hat{X}_{N1} & \dots & \hat{X}_{NN} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ |\psi\rangle \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{X}_{1i}\,|\psi\rangle \\ \vdots \\ \hat{X}_{ii}\,|\psi\rangle \\ \vdots \\ \hat{X}_{Ni}\,|\psi\rangle \end{pmatrix}$$

Given a positive linear map T on $B(H)$ we can extend it to a linear map on $B(H \otimes H_N)$ in the following way:

$$\underline{T}[\underline{\hat{X}}] = [T\hat{X}_{ij}] \qquad \text{<1.3.5>}$$

If $\underline{T}$ is positive for any N, then it is called <u>completely positive</u>.

It is easy to show that if T is N-positive it is also positive, being sufficient to observe that:

$$\underline{\hat{X}} = \begin{pmatrix} \hat{X} & 0 \dots 0 \\ 0 & \\ \vdots & \underline{O} \\ 0 & \end{pmatrix} \geq 0 \quad \Longrightarrow \quad \underline{T}[\underline{\hat{X}}] = \begin{pmatrix} T[\hat{X}] & 0 \dots 0 \\ 0 & \\ \vdots & \underline{O} \\ 0 & \end{pmatrix} \geq 0$$

Indeed

$$\langle \psi \otimes e_1 | \underline{T} \, \underline{X} | \psi \otimes e_1 \rangle \geq 0 \qquad \forall |\psi\rangle \in H$$

means:

$$(\langle\psi|, 0, \ldots, 0) \begin{pmatrix} T[\hat{X}] & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & \underline{0} & \\ 0 & & & \end{pmatrix} \begin{pmatrix} |\psi\rangle \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \langle\psi| T[\hat{X}] |\psi\rangle \geq 0$$

What it is not trivial is the N-positivity of T if it is only positive.

An example of a positive map which is not 2-positive is given in ref.[14].

The physical reasonableness of this property is shown by two arguments.

The first one is typical of the operational approach to quantum mechanics.

Let us consider two non interacting systems, one of them being an N-level system and described in a Hilbert space .

Let us suppose the other is affected by a selective operation which does not interfere with the N-level system.

Phisycally it should be possible to extend T to an operation $\underline{T}$ on $\mathcal{B}(H \otimes H_N)$ in such a way that:

$$\underline{T}(\hat{W} \otimes \hat{W}_N) = (T\hat{W}) \otimes \hat{W}_N \qquad \langle 1.3.6 \rangle$$

where $\hat{W}_N \in K(H_N)$ is the state of the N-level system and $\hat{W} \in K(H)$ the state of the other one.

Clearly, as an operation, $\underline{T}$ has to be positive, and, by <1.3.6>, N-positive.

Since this is independent of N, T has to be completely positive.

To be extremely rigorous one would have to prove that $\underline{T}$ is an operation on $\mathcal{B}(H \otimes H_N)$ once it is on $\mathcal{B}(H)_1$.

That is no difficult.

More interesting is the following:

THEOREM 1.3.7 [8], [9]

T is a completely positive map on $\mathcal{B}(\mathcal{H})_1$ iff its adjoint T* is a completely positive map on $\mathcal{B}(\mathcal{H})$.

end

The second argument is strictly connected with the interpretation of the Q.M.S.L model as an evolution typical of the open systems once it has been proven it comes out disregarding the degrees of freedom of some "reservoir".

It is worthwhile reproducing the argument of ref. [6], [7] ,in order to compare that with the similar situation in the derivation of a quantum dynamical semigroup evolution [14], [15], [16], [17].

Let us consider a closed system compound by a microsystem M and an apparatus A.

N copies of it are prepared in an initial state $\underline{\hat{W}}_i = \hat{W}_M \otimes \hat{W}_A \in K(\mathcal{H}_M \otimes \mathcal{H}_A)$, $\hat{W}_M$ and $\hat{W}_A$ being the normalized states of M and A respectively.

The interaction from which the measurements process arises, can be studied as a diffusion process governed by a scattering S-matrix $\underline{S}$ on $\mathcal{H}_M \otimes \mathcal{H}_A$.

After the interaction being over, the new state of the closed system is given by:

$$\underline{\tilde{W}} = \underline{S}\, \hat{W}_i\, \underline{S}^{+} \qquad \langle 1.3.8\rangle$$

If the measurement of a yes-no observable $\hat{Q}_A$ is performed on the apparatus, the resulting state, according to Von Neumann, is:

$$\underline{\hat{W}}_f = (\hat{1}_M \otimes \hat{Q}_A)\, \underline{\tilde{W}}\, (\hat{1}_M \otimes \hat{Q}_A) / \mathrm{Tr}\{(\hat{1}_M \otimes \hat{Q}_A)\, \underline{\tilde{W}}\} \qquad \langle 1.3.9\rangle$$

This happens if we select those systems giving yes answer.

The $\hat{W}_f$ state of the microsystems is obtained by tracing over the degrees of freedom of the apparatus:

$$\hat{W}_f = Tr^{(A)} \underline{\hat{W}}_f \qquad \text{<1.3.10>}$$

We recognize the selective operation T on $\hat{W}_M$ as given by:

$$T[\hat{W}_M] = Tr^{(A)}[(\hat{1}_M \otimes \hat{Q}_A)\underline{\tilde{W}}(\hat{1}_M \otimes \hat{Q}_A)] = Tr^{(A)}[(\hat{1}_M \otimes \hat{Q}_A)\underline{S}\,\underline{\hat{W}}_i\,\underline{S}^\dagger] \qquad \text{<1.3.11>}$$

Let us consider now the spectral resolution of $\hat{W}_A$:

$$\hat{W}_A = \Sigma_j\, \rho_j^A\, |g_j^A\rangle\langle g_j^A| \qquad \text{<1.3.12>}$$

with $\Sigma_j\, \rho_j^A = 1$ <1.3.13>

where, for sake of semplicity, although very roughly, we consider non degenerate the spectrum of a possible macroobject.

Taking the trace we use an orthonormal basis in the range of $\hat{Q}_A$: Hence, for any $|\phi\rangle$ and $|\psi\rangle$ belonging to H_M , we have:

$$(\psi, T[\hat{W}_M]\phi) = \Sigma_i \Sigma_j\, \rho_j^A\, (\psi \otimes f_i^A, (\hat{1}_M \otimes \hat{Q}_A)\underline{S}\,\hat{W}_M \otimes |g_j^A\rangle\langle g_j^A|\,\underline{S}^\dagger f_i^A \otimes \phi)$$

$$= \Sigma_i \Sigma_j\, \rho_j^A\, (\psi \otimes f_i^A, \underline{S}\,\hat{W}_M \otimes |g_j^A\rangle\langle g_j^A|\,\underline{S}^\dagger f_i^A \otimes \phi)$$

<1.3.13>

Calling : $\hat{A}_{ij} = (f_i^A, \underline{S}\, g_j^A)$ <1.3.14>

and consequently: $\hat{A}^\dagger_{ij} = (g_j^A, \underline{S}^\dagger f_i^A)$ <1.3.15>

we have :

$$(\psi, T[\hat{W}_M]\phi) = \Sigma_i \Sigma_j\, \rho_j^A\, (\psi, \hat{A}_{ij}\hat{W}_M \hat{A}^\dagger_{ij}\phi) \qquad \text{<1.3.16>}$$

$$T[\hat{W}_M] = \Sigma_i \Sigma_j \; \rho^A_j \; \hat{A}_{ij} \; \hat{W}_M \; \hat{A}^\dagger_{ij} \qquad \text{<1.3.17>}$$

From <1.3.11> it follows that:

$$Tr^{(M)}\{T[\hat{W}_M]\} = Tr^{(A+M)}\{(\hat{1}_M \otimes \hat{G}_A)\underline{\tilde{W}}\} \leq 1 \qquad \text{<1.3.18>}$$

therefore:

$$\forall \; \hat{W}_M \in K(H_M) \qquad Tr^{(M)}\{[\Sigma_{i,j} \; \rho^A_j \; \hat{A}^\dagger_{ij} \; \hat{A}_{ij}] \cdot \hat{W}_M\} \leq 1 \qquad \text{<1.3.19>}$$

Hence:

$$\Sigma_{i,j} \; \rho^A_j \; \hat{A}^\dagger_{ij} \; \hat{A}_{ij} \leq \hat{1} \qquad \text{<1.3.20>}$$

We recognize in the sum <1.3.20> an effect $\hat{F} = \sum_{i,j} \rho^A_j \; \hat{A}^\dagger_{ij} \; \hat{A}_{ij}$ <1.3.21>

All the convergence problems are taken care of by the following theorem where a better formulation of the above results is given.

THEOREM 1.3.22 [9]

For an arbitrary operation T, there exist operators $\hat{A}_k$, $k \in K$ (a finite or countably infinite index set) on the Hilbert space H , satisfying:

$$\sum_{k \in K_0} \hat{A}^\dagger_k \hat{A}_k \leq \hat{1} \qquad \text{for all finite subsets} \quad K_0 \subseteq K \qquad \text{<1.3.22a>}$$

such that, with arbitrary $\hat{S} \in B(H)_1$ and $\hat{X} \in B(H)$, the maps T and T* are given by:

$$T[\hat{S}] = \sum_{k \in K} \hat{A}_k \hat{S} \hat{A}_k^\dagger$$ <1.3.22b>

and

$$T^*[\hat{X}] = \sum_{k \in K} \hat{A}_k^\dagger \hat{X} \hat{A}_k$$ <1.3.22c>

respectively.

In particular, the effect $\hat{F}$ coresponding to T is given by:

$$\hat{F} = \sum_{k \in K} \hat{A}_k^\dagger \hat{A}_k = T^*[\hat{1}]$$ <1.3.22d>

If the index set is infinite, <1.3.22b> implies that, independently of the ordering of K , the infinite sum in <1.3.22b> converges in the trace norm topology, while those in <1.3.22c> and <1.3.22d> converge ultraweakly.

Viceversa, given any countably or even uncountably infinite set of operators $\hat{A}_k$ on H, $k \in K$, satisfying condition <1.3.22a>, then <1.3.22b> defines an operation T , whose adjoint T^* and the corresponding effect are given by <1.3.22c> and <1.3.22d> respectively.

end

We can now proof that any map of the type <1.3.22c> is completely positive,once <1.3.22a> is satisfied, [8], [9], and therefore, by THEOREM 1.3.7, <1.3.22b> too.

Let us, indeed, consider the sum: $\sum_{k \in K_0} (\hat{A}_k^\dagger \otimes \hat{1}_N)(\hat{A}_k \otimes \hat{1}_N) \quad \forall K_0 \subseteq K$ <1.3.23>

where $\hat{1}_N$ is the identity operator on an N-dimensional Hilbert space H_N .

By <1.3.22a> the operator in <1.3.23> is bounded by the identity operator on $H \otimes H_N$.

Hence, by the fact that $\underline{\hat{A}}_k = \hat{A}_k \otimes \hat{1}_N$, we can construct the adjoint of an operation on $B(H \otimes H_N)$, :

$$\underline{T}^*[\underline{X}] = \sum_{k \in K} (\hat{A}_k^\dagger \otimes \hat{1}_N) \, \underline{X} \, (\hat{A}_k \otimes \hat{1}_N) \qquad \langle 1.3.24\rangle$$

Since $\underline{X} = \hat{X} \otimes \hat{A}_N$, T* satisfies the conditions for the N-positivity.

In fact: $\underline{T}[\underline{X}] = \left(\sum_{k \in K} \hat{A}_k^\dagger \hat{X} \hat{A}_k\right) \otimes \hat{A}_N = \left[T[X](A_N)_{ij}\right]$ <1.3.25>

Complete positivity follows from being N any natural number. It finds its physical justification when we derive the state change produced by considering a microsystem as an open system in interaction with some apparatus, like during a measurement process, or with some surroundings, like a thermal bath.

It should be stressed that the operators $\hat{A}_k$ in THEOREM 1.3.22, or the $\sqrt{\varsigma_j^A}\,\hat{A}_{ij}$ in the Kraus'argument [8] contain all the features of what has been called apparatus (its efficiency, for example).

In the last above mentioned case, we deal with a system S in interaction with a reservoir R : R+S evolves unitarily as a closed system.

Let $\hat{\varrho}$ be the initial state of R and $\hat{\sigma}$ that of S , the evolution of S, disregarding R , is given by:

$$T_r^{(S)}[\Sigma_t \hat{\sigma} \cdot \hat{A}_S] = T_r^{(S+R)}[\hat{\varrho} \otimes \hat{\sigma} \;\; \underline{U}^\dagger \hat{1}_R \otimes \hat{A}_S \underline{U}] \qquad \langle 1.3.26\rangle$$

where $\underline{U}$ is the Hamiltonian evolution operator on the C*-Algebra $B(H_R \otimes H_S)$ and $\hat{A}_S \in B(H_S)$.

Concluding this paragraph we show that the correspondence between operations and effects is not one-to-one.

Let us consider two index sets $K_1=\{1\}$ and $K_2=\{1,2\}$ with $\hat{A}_1^1=\sqrt{\hat{F}}$; $\hat{A}_1^2=\sqrt{\frac{\hat{F}}{2}}$, $\hat{A}_2^2=\hat{U}\sqrt{\frac{\hat{F}}{2}}$, $\hat{U}$ being unitary.

$\hat{F}_1=\hat{F}$ and $\hat{F}_2=\frac{\hat{F}}{2}+\sqrt{\frac{\hat{F}}{2}}\hat{U}^{+}\hat{U}\sqrt{\frac{\hat{F}}{2}}=\hat{F}=\hat{F}_1$ means that two different operations can be associated to the same effect.

This is not other but the fact that two different devices can be used to measure the same observable.

Note that this is impossible if the operation transforms pure states into pure states [6], [10].

In the above mentioned case that happens if and only if $\hat{U}=e^{i\alpha}\hat{\mathbb{1}}$.

Indeed:

$$T_1\left[|\psi\rangle\langle\psi|\right]=\sqrt{\hat{F}}\,|\psi\rangle\langle\psi|\sqrt{\hat{F}}=|\phi\rangle\langle\phi| \qquad \text{<1.3.27>}$$

$$T_2\left[|\psi\rangle\langle\psi|\right]=\frac{1}{2}|\phi\rangle\langle\phi|+\frac{1}{2}\hat{U}|\phi\rangle\langle\phi|\hat{U} \qquad \text{<1.3.28>}$$

Paragraph 4

What briefly discussed at the end of the last paragraph can be better explained by using the concept of generalized observable [2] (see also Appendix A.3).

First of all it is convenient to stress that the effect : $\hat{F}=\sum_{k\in K}\hat{A}_k^{+}\hat{A}_k$ is connected with the operation $T[\hat{W}]=\sum_{k\in K}\hat{A}_k\hat{W}\hat{A}_k^{+}$ which describes how a phisical state changes due to the interaction with an instruments "f".

Different instrumentss can measure the same effect and therefore, although belonging to the same equivalence class , they determine different operations.

The same apparatus can be used for selecting those systems by which it was not tirggered.

This corresponds to an operation complementary to T : T' .

Both together realize a non selective operation $\tilde{T}$.

Such a situation is easy analized in terms of what we already know.

Given an index set J and operators $\hat{A}_k$ such that :

$$\sum_J{}_j \hat{A}_j^{\dagger} \hat{A}_j = \hat{1} \qquad \text{<1.4.1>}$$

and two complementary subsets K, K' : K U K' = J ; we have:

$$T[\hat{W}] = \sum_{k \in K} \hat{A}_k \hat{W} \hat{A}_k^{\dagger} \;, \; \forall \; \hat{W} \in K(H) \qquad \text{<1.4.2>}$$

$$T'[\hat{W}] = \sum_{k \in K'} \hat{A}_k \hat{W} \hat{A}_k^{\dagger} \;, \; \forall \; \hat{W} \in K(H) \qquad \text{<1.4.3>}$$

$$\tilde{T}[\hat{W}] = \sum_J{}_j \hat{A}_j \hat{W} \hat{A}_j^{\dagger} \;, \forall \; \hat{W} \in K(H) \qquad \text{<1.4.4>}$$

Note that:

$$\mathrm{Tr}\{T[\hat{W}]\} = \mathrm{Tr}\{\hat{F}\hat{W}\} \leq 1 \;; \quad \mathrm{Tr}\{T'[\hat{W}]\} = \mathrm{Tr}\{\hat{F}'\hat{W}\} \leq 1$$

where: $$\hat{F}' = (T')^{*}[\hat{1}] = \sum_{K'}{}_k \hat{A}_k^{\dagger} \hat{A}_k \qquad \text{<1.4.5>}$$

But:

$$\mathrm{Tr}\{\tilde{T}[\hat{W}]\} = \mathrm{Tr}\{\sum_{j \in J} \hat{A}_j \hat{W} \hat{A}_j^{\dagger}\} = \mathrm{Tr}\{\sum_{j \in J} \hat{A}_j^{\dagger} \hat{A}_j \cdot \hat{W}\} = \mathrm{Tr}\,\hat{W} = 1 \qquad \text{<1.4.6>}$$

for all $\hat{W} \in K(H)$.

Therefore $\tilde{T}$ is a trace and probability preserving map.

Let us consider a positive, normalized, integrable function on the real line: $f: \mathbb{R} \to \mathbb{R}^{+} \;; \quad \int_{-\infty}^{+\infty} f(x)\, dx = 1$ <1.4.7>

We construct a POSITIVE OPERATOR VALUED (P.O.V.) measure (see appendix A.3) as follows:

$$\hat{F}(E) = \int_{-\infty}^{+\infty} (f * \chi_E)(q)\, d\hat{P}_q(q) \qquad \text{<1.4.8>}$$

E belongs to the Borel σ-algebra of the real line.

$d\hat{P}_q(q) = |q\rangle\langle q|\, dq$ is the spectral measure of the usual position operator $\hat{q}$, and $(f * \chi_E)(q) = \int_{-\infty}^{+\infty} f(q-x)\, \chi_E(x)\, dx$

By this way <1.4.8> becomes;

$$\hat{F}(E) = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dx\, f(q-x)\, \chi_E(x)\, d\hat{P}_q(q) = \int_E dx\, f(\hat{q}-x) \qquad \text{<1.4.9>}$$

just considering the usual notion of function of an operator.

Since $\hat{F}(\mathbb{R}) = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dq\, f(q-x)\, |q\rangle\langle q| = \int_{-\infty}^{+\infty} |q\rangle\langle q|\, dq = \hat{\mathbb{1}}$

$\hat{F}(\cdot)$ is a P.O.V. measure on the real line.

Let us choose $f(x) = \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha x^2}$ so that :

$$\hat{F}_\alpha(E) = \sqrt{\frac{\alpha}{\pi}} \int_E e^{-\alpha(\hat{q}-x)^2}\, dx \qquad \text{<1.4.10>}$$

Let us compute the weak limit of $\hat{F}_\alpha(E)$ as $\alpha \to 0$: $\forall\, |\phi\rangle, |\psi\rangle$ we have :

$$\lim_{\alpha\to 0} (\phi, \hat{F}_\alpha(E)\psi) = \lim_{\alpha\to 0} \sqrt{\frac{\alpha}{\pi}} \int_E dx \int_{-\infty}^{+\infty} dq\, e^{-\alpha(q-x)^2}\, \psi(q)\, \overline{\phi(q)} =$$

$$= \int_E dx \int_{-\infty}^{+\infty} dq\, \delta(q-x)\, \psi(q)\, \overline{\phi(q)} = \int_E \psi(x)\, \overline{\phi(x)}\, dx = (\phi, \hat{P}_q(E)\psi) \qquad \text{<1.4.11>}$$

In the limit of zero dispersion we get the usual projection-valued measure on the real line associated with the position operator $\hat{q}$.

We can thus think of $\hat{F}_\alpha(E)$ as of the effect relative to the measurement of the presence of a particle within the open set E , being this one performed by an instrument of finite, non zero dispersion $\sigma = (\sqrt{2\alpha})^{-1}$.

We can go further and show that this "generalized position observable" follows from an "instrument" that is from a "POSITIVE MAP VALUED" (P.M.V.) measure (see appendix A.3).

This one is a generalization of the concept of "operation".

In the case of the usual spectral measure of $\hat{q}$, once a partition $\{E_i\}$ of the real line has been chosen, the state change is given by:

$$T[\hat{W}] = \sum_i \hat{P}_q(E_i)\, \hat{W}\, \hat{P}_q(E_i) \qquad \text{<1.4.12>}$$

in the non selective case.

This map is partition dependent and as such not covariant.

Let us now introduce a P.M.V measure in the following way:
for any $E \in \mathcal{A}$ =Borel σ-algebra on the real line, for any $\hat{W} \in K(H)$:

$$\mathcal{E}_\alpha(E)[\hat{W}] = \int_E \sqrt{\frac{\alpha}{\pi}}\, e^{-\frac{\alpha}{2}(\hat{q}-x)^2}\, \hat{W}\, e^{-\frac{\alpha}{2}(\hat{q}-x)^2}\, dx \qquad \text{<1.4.13>}$$

<A.3.10> and <A.3.11> are satisfied and:

$$\mathrm{Tr}\{\mathcal{E}_\alpha(\mathbb{R})[\hat{W}]\} = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dx \sqrt{\frac{\alpha}{\pi}} \langle q|\, e^{-\frac{\alpha}{2}(\hat{q}-x)^2}\, \hat{W}\, e^{-\frac{\alpha}{2}(\hat{q}-x)^2}$$

$$= \int_{-\infty}^{+\infty} dq\, \langle q|\hat{W}|q\rangle \int_{-\infty}^{+\infty} dx \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha(q-x)^2} = \mathrm{Tr}\,\hat{W} = 1 \qquad \text{<1.4.14>}$$

Hence $\mathcal{E}_\alpha(\cdot)$ is an "instrument".

Moreover:

$$\mathrm{Tr}\{\mathcal{E}_\alpha(E)[|\psi\rangle\langle\psi|]\} = \int_{-\infty}^{+\infty} dq \int_E dx \sqrt{\frac{\alpha}{\pi}}\, e^{-\alpha(q-x)^2}\, |\psi(q)|^2 \qquad \text{<1.4.15>}$$

Compare with:

$$\mathrm{Tr}\left\{ \hat{F}_\alpha(E)\left[|\psi\rangle\langle\psi|\right]\right\} = \int_{-\infty}^{+\infty} dq \int_E dx \sqrt{\frac{\alpha}{\pi}} \langle q| e^{-\alpha(\hat{q}-x)^2} |\psi\rangle\langle\psi|q\rangle$$

$$= \int_{-\infty}^{+\infty} dq \int_E dx \sqrt{\frac{\alpha}{\pi}} e^{-\alpha(q-x)^2} |\psi(q)|^2 \qquad \text{<1.4.16>}$$

This is true for all $|\psi\rangle$ in H , hence for all $\hat{W} \in K(H)$.

Therefore $\hat{F}_\alpha(E)$ is the effect associated to the operation $\mathcal{E}_\alpha(E)$.

It is covariant too:

if E=(a,b) then: $\mathcal{E}(E)[\hat{W}] = \sqrt{\frac{\alpha}{\pi}} \int_a^b dx\, e^{-\frac{\alpha}{2}(\hat{q}-x)^2} \hat{W} e^{-\frac{\alpha}{2}(\hat{q}-x)^2} =$

$$= \sqrt{\frac{\alpha}{\pi}} \int_a^b dx\, e^{-i\hat{p}x/\hbar} e^{-\frac{\alpha}{2}\hat{q}^2} e^{i\hat{p}x/\hbar} \hat{W} e^{-i\hat{p}x/\hbar} e^{-\frac{\alpha}{2}\hat{q}^2} e^{i\hat{p}x/\hbar}$$

$$\mathcal{E}(E+c)[\hat{W}] = \sqrt{\frac{\alpha}{\pi}} \int_{a+c}^{b+c} dx\, e^{-\frac{\alpha}{2}(\hat{q}-x)^2} \hat{W} e^{-\frac{\alpha}{2}(\hat{q}-x)^2} =$$

$$= \sqrt{\frac{\alpha}{\pi}} \int_a^b dx\, e^{-i\hat{p}c/\hbar} e^{-i\hat{p}x/\hbar} e^{-\frac{\alpha}{2}\hat{q}^2} e^{i\hat{p}c/\hbar} e^{i\hat{p}x/\hbar} \hat{W} e^{-i\hat{p}x/\hbar} e^{-i\hat{p}c/\hbar} e^{-\frac{\alpha}{2}\hat{q}^2} e^{i\hat{p}c/\hbar} e^{i\hat{p}x/\hbar}$$

$$= e^{-i\hat{p}c/\hbar}\left[\sqrt{\frac{\alpha}{\pi}} \int_a^b dx\, e^{-i\hat{p}x/\hbar} e^{-\frac{\alpha}{2}\hat{q}^2} e^{i\hat{p}x/\hbar} \hat{W}_c e^{-i\hat{p}x/\hbar} e^{-\frac{\alpha}{2}\hat{q}^2} e^{i\hat{p}x/\hbar}\right] e^{i\hat{p}c/\hbar}$$

$$\hat{W}_c = e^{i\hat{p}c/\hbar} \hat{W} e^{-i\hat{p}c/\hbar}$$

$$\mathcal{E}(E+c)[\hat{W}] = e^{-i\hat{p}c/\hbar}\left\{\mathcal{E}(E)\left[e^{i\hat{p}c/\hbar}\hat{W} e^{-i\hat{p}c/\hbar}\right]\right\} e^{i\hat{p}c/\hbar}$$

Concluding:

we identify the instantaneous processes in the Q.M.S.L. model as non selective operations given as P.M.V. measure on the real line.

Each of them $\mathcal{E}_\alpha(\cdot)$ is connected with a covariant "instrument" measuring the "generalized observable" $\hat{F}_\alpha(\cdot)$ as P.O.V. measure, or, analagously the usual position observable with finite efficiency $\sqrt{\alpha}$.

$$\mathcal{T}[\hat{W}] = \mathcal{E}_\alpha(\mathbb{R})[\hat{W}] = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{+\infty} dx \; e^{-\frac{\alpha}{2}(\hat{q}-x)^2}\,\hat{W}\,e^{-\frac{\alpha}{2}(\hat{q}-x)^2} \qquad \langle 1.4.17\rangle$$

$$\hat{F}_\alpha(E) = \sqrt{\frac{\alpha}{\pi}}\int_E dx \; e^{-\frac{\alpha}{2}(\hat{q}-x)^2} \qquad \langle 1.4.18\rangle$$

The non selective character of the process is expressed by:

$$\mathrm{Tr}\{\mathcal{T}[\hat{w}]\} = 1 \qquad \langle 1.4.19\rangle$$

the selection being obtained when we have, instead of $\mathcal{T}$:

$$\mathcal{T}_E[\hat{W}] = \mathcal{E}_\alpha(E)[\hat{W}] \qquad \langle 1.4.20\rangle$$

We also stress that:

$$\hat{F}_\alpha(\mathbb{R}) = \sqrt{\frac{\alpha}{\pi}}\int_{\mathbb{R}} dx \; e^{-\frac{\alpha}{2}(\hat{q}-x)^2} = \hat{\mathbb{1}} \qquad \langle 1.4.21\rangle$$

accordingly with <1.4.4> and <1.4.5>.

CHAPTER 2

Paragraph 1

Here we will deal with the Q.M.S.L.model:

$$\frac{d\,\Sigma_t\hat\rho}{dt} = -\frac{i}{\hbar}\,[\hat H, \Sigma_t\hat\rho] - \lambda\,\Sigma_t\hat\rho + \lambda\, T[\Sigma_t\hat\rho] \qquad \langle 2.1.1\rangle$$

where H is the free quantum Hamiltonian for a particle of mass m, and:

$$T[\hat\rho] = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{+\infty} dx\; e^{-\frac{\alpha}{2}(\hat q - x)^2}\,\hat\rho\; e^{-\frac{\alpha}{2}(\hat q - x)^2} \;;\quad \hat\rho \in B(H)_1^h \qquad \langle 2.1.2\rangle$$

From now on the r.h.s. of <2.1.1> will be indicated as: $L[\Sigma_t\hat\rho]$

It will be shown that L is the generator of a strongly continuous, positivity preserving semigroup of contraction operators on $B(H)_1^h$.

$B(H)_1^h$ is the real Banach space of self-adjoint trace class operators.

Among them the physical states are represented by the positive ones with trace 1.

It is sufficient to restrict ourselves to $B(H)_1^h$ instead of working with since, by linearity and by the fact that: $\hat\rho = \frac{1}{2}[\hat\rho + \hat\rho^\dagger] + \frac{i}{2}[\hat\rho - \hat\rho^\dagger] \quad \forall \hat\rho \in B(H)_1$, we can extend all the conclusions to $B(H)_1$.

In order to get a more familiar form for the linear positive map we use the following argument [3].

Let us consider the position representation of $T[\hat\rho]$:

$$\langle q|\,T[\hat\rho]\,|q'\rangle = \sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{+\infty} dx\; e^{-\frac{\alpha}{2}(q-x)^2}\,\langle q|\hat\rho|q'\rangle\; e^{-\frac{\alpha}{2}(q'-x)^2}$$

$$= e^{-\frac{\alpha}{4}(q-q')^2}\,\langle q|\hat\rho|q'\rangle \qquad \langle 2.1.3\rangle$$

We can obtain formula <2.1.3> by considering:

$$T[\hat{\rho}] = \sum_{0}^{\infty}{}_m \; \hat{A}_m \, \hat{\rho} \, \hat{A}_m \qquad \langle 2.1.4\rangle$$

with

$$\hat{A}_m = \sqrt{\left(\frac{\alpha}{2}\right)^m \frac{1}{m!}} \; \hat{q}^m \, e^{-\frac{\alpha}{4}\hat{q}^2} \qquad \langle 2.1.5\rangle$$

Indeed:

$$\langle q|\left(\sum_{0}^{\infty}{}_m \; \hat{A}_m \, \hat{\rho} \, \hat{A}_m\right)|q'\rangle = \sum_{0}^{\infty}{}_m \left(\frac{\alpha}{2}\right)^m \frac{1}{m!} (qq')^m \, e^{-\frac{\alpha}{4}(q^2+q'^2)} \langle q|\hat{\rho}|q'\rangle$$

$$= \langle q|\hat{\rho}|q'\rangle \, e^{-\frac{\alpha}{4}(q^2+q'^2)} \, e^{+\frac{\alpha}{2}qq'}$$

Noting that:

$$\sum_{0}^{\infty}{}_m \; \hat{A}_m \hat{A}_m = \hat{1} \qquad \langle 2.1.6\rangle$$

we recognize <2.1.4> as a non selective operation on $B(H)_1$.

It then follows that the sum converges in the trace norm and that there exists the adjoint map T*, given by: $T^*[\hat{X}] = \sum_{0}^{\infty}{}_m \; \hat{A}_m \, \hat{X} \, \hat{A}_m \; , \quad \hat{X} \in B(H)$ where the sum is ultraweakly convergent (see appendix A.2).

We stress here two important consequences of <2.1.4>.

Writing:

$$L[\hat{\rho}] = L_0[\hat{\rho}] + L_\lambda[\hat{\rho}] \qquad \langle 2.1.7\rangle$$

where:

$$L_0[\hat{\rho}] = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] \qquad \langle 2.1.8\rangle$$

$$L_\lambda[\hat{\rho}] = -\lambda\hat{\rho} + \lambda T[\hat{\rho}] \qquad \langle 2.1.9\rangle$$

$$T[\hat{\rho}] = \sum_{0}^{\infty}{}_m \; \hat{A}_m \hat{\rho} \, \hat{A}_m \qquad \langle 2.1.10\rangle$$

it follows:

a) $\| T[\hat{\rho}] \|_1 \leq \| \hat{\rho} \|_1 \qquad \forall \hat{\rho} \in B(H)_1^h$ <2.1.11>

Indeed: if $\hat{\rho} \in B(H)_1^h$, then $\hat{\rho} = \hat{\rho}_+ - \hat{\rho}_-$ with:
$\hat{\rho}_+ = \dfrac{\hat{\rho} + |\hat{\rho}|}{2}$ and $\hat{\rho}_- = \dfrac{|\hat{\rho}| - \hat{\rho}}{2}$ the positive parts of $\hat{\rho}$.
Since $T[\hat{\rho}] = T[\hat{\rho}_+] - T[\hat{\rho}_-]$ and $T[\hat{\rho}_+]$, $T[\hat{\rho}_-]$ are positive trace class operators it follows that:
$$\| T[\hat{\rho}_\pm] \|_1 = \mathrm{Tr}\, | T[\hat{\rho}_\pm] | = \mathrm{Tr} \{ T[\hat{\rho}_\pm] \} = \mathrm{Tr}\, \hat{\rho}_\pm = \| \hat{\rho}_\pm \|_1 \qquad \text{<2.1.12>}$$

due to <2.1.6> and to the cyclicity of the trace.
Hence, being $\| \ \|_1$ a norm, we get <2.1.11>.

b) Defining the norm of T as: $\| T \| = \sup_{\hat{\rho} \in B(H)_1^h} \| T[\hat{\rho}] \|_1 / \| \hat{\rho} \|_1$
we have: $\| T \| \leq 1$ and, since $\| T[\hat{\rho}] \|_1 = \| \hat{\rho} \|_1$ if $\hat{\rho} \in B(H)_1^+$, it follows that: $\| T \| = 1$

c) From a) and b) we obtain that L_d is a bounded operator on $B(H)_1^h$ with norm: $\| L_d \| = \| - d\, \mathbb{1} + d\, T \| \leq 2\, d$ <2.1.13>

d) We can use c) and the duality between $B(H)_1^h$ and $B(H)^h$ given by the trace, in order to construct the adjoint L_d^* of L_d on $B(H)^h$, which turns out to be bounded and such that: $\| L_d^* \| = \| L_d \|$ [21] <2.1.14>

We want to proof the following:

STATEMENT 1

$L = L_0 + L_d$ is the generator of a strongly continuous semigroup of contractions on $B(H)_1^h$: $\Sigma = \{ \Sigma_t \}_{t \geq 0}$
Moreover: Σ preserves positivity.

Since the Hamiltonian H in L_0 is unbounded, we cannot use the results in ref.[12] about the general form of a completely positive dynamical semigroups.

We will face this problem introducing the concept of dissipativeness of an operator on a Banach space [23], and studying the operator L_d in <2.1.9> as a linear perturbation of L_0 [11], [26].

In the following we will deal with general unbounded self adjoint Hamiltonian H.

The first step consists in defining a semi-inner product (s.i.p) on the Banach space $B(H)_1^h$, as follows: $(\hat{\sigma},\hat{\rho}) \longrightarrow [\hat{\sigma},\hat{\rho}]$; $\hat{\sigma},\hat{\rho} \in B(H)_1^h$

where:

$$[\hat{\sigma}, \hat{\rho}_1+\hat{\rho}_2] = [\hat{\sigma},\hat{\rho}_1]+[\hat{\sigma},\hat{\rho}_2] \qquad \hat{\sigma}, \hat{\rho}_1, \hat{\rho}_2 \in B(H)_1^h \tag{2.1.15}$$

$$[\hat{\sigma}, \alpha\hat{\rho}] = \alpha[\hat{\sigma},\hat{\rho}] \qquad \hat{\sigma},\hat{\rho} \in B(H)_1^h \ ; \ \alpha \in \mathbb{R} \tag{2.1.16}$$

$$[\hat{\sigma},\hat{\sigma}] > 0 \quad \text{for} \quad \hat{\sigma} \in B(H)_1^h \ , \ \hat{\sigma} \neq \hat{0} \tag{2.1.17}$$

$$|[\hat{\sigma},\hat{\rho}]| \leq \sqrt{[\hat{\sigma},\hat{\sigma}]}\sqrt{[\hat{\rho},\hat{\rho}]} \qquad \hat{\sigma},\hat{\rho} \in B(H)_1^h \tag{2.1.18}$$

It is always possible to equip a Banach space X with a (s.i.p) compatible with the norm $\|\cdot\|_X$ in the sense that: $\|A\|_X^2 = [A,A]$, $A \in X$

In the case in which $X = B(H)_1^h$ the construction of such a (s.i.p) is the following.

Let us consider the linear functional $F_{\hat{\rho}}$ in $[B(H)_1^h]^*$ given by:

$$F_{\hat{\rho}}[\hat{\sigma}] = \|\hat{\rho}\|_1 \, \mathrm{Tr}\{\hat{W}^\dagger \hat{\sigma}\} \qquad \hat{\rho},\hat{\sigma} \in B(H)_1^h \tag{2.1.19}$$

where $\hat{W}$ is the partial isometry appearing in the polar decomposition of $\hat{\rho}$:

$|\hat{\rho}| = \sqrt{\hat{\rho}^\dagger\hat{\rho}} = \hat{W}^\dagger\hat{\rho}$ [27]

Then:

$$F_{\hat{\rho}}[\hat{\rho}] = \|\hat{\rho}\|_1 \, \mathrm{Tr}\{\hat{W}^\dagger\hat{\rho}\} = \|\hat{\rho}\|_1 \, \mathrm{Tr}\,|\hat{\rho}| = \|\hat{\rho}\|_1^2$$

Hence: $[\hat{\rho},\hat{\sigma}] = F_{\hat{\rho}}[\hat{\sigma}]$ is the required compatible (s.i.p).

Indeed: $[\hat{\rho},\hat{\rho}] = F_{\hat{\rho}}[\hat{\rho}] = \|\hat{\rho}\|_1^2$

and: $|[\hat{\rho},\hat{\sigma}]| = \|\hat{\rho}\|_1 \, |Tr\{\hat{W}^{+}\hat{\sigma}\}| \leq \|\hat{\rho}\|_1 \|\hat{\sigma}\|_1 = \sqrt{[\hat{\rho},\hat{\rho}]}\,\sqrt{[\hat{\sigma},\hat{\sigma}]}$

the <2.1.15> and <2.1.16> being trivially verified by <2.1.19>.

$F_{\hat{\rho}}[\cdot]$ is also called tangent functional at $\hat{\rho}$.

DEFINITION 2.1.20.[23]

A linear operator A on a Banach space X is dissipative if:

$Re\,[x, Ax] \leq 0$ for at least a (s.i.p.), $\forall x \in X$.

We will need the following:

THEOREM 2.1.21.[23]

A necessary and sufficient condition for a linear operator A, with dense domain in a Banach space X, to generate a stongly continuous semigroup of contractions is that A be dissipative and that Ran(I-A)=X .

STATEMENT 2

L_0 generates a strongly continuous group of isometries on $B(H)_1^h$.

<u>proof</u>

Being H unbounded, L_0 generates a group on B(H) which is continuous in the strong operators-topology.

This means that it is also continuous in the weak operators-topology on B(H).

If $\{T_t\}_{t\in\mathbb{R}}$ is the group generated by L_0 , it then follows that $T_t\hat{\rho}$ is weakly convergent to $\hat{\rho}$ as t goes to zero for every $\hat{\rho}$ in K(H).

Since every $\hat{\sigma} \in B(H)_1^h$ can be written as: $\hat{\sigma} = c_+\hat{\rho}_+ - c_-\hat{\rho}_-$ where:

$$\hat{\rho}_+ = \frac{\hat{\sigma}_+}{Tr\,\hat{\sigma}_+} \;,\; \hat{\rho}_- = \frac{\hat{\sigma}_-}{Tr\,\hat{\sigma}_-} \;;\; c_+ = Tr\,\hat{\sigma}_+ \;,\; c_- = Tr\,\hat{\sigma}_-$$

by linearity $T_t\hat{\sigma}$ converges weakly to $\hat{\sigma}$ as t goes to zero.

By a theorem of Wehrl[28] we have that: if a sequence of density matrices $\{\hat{\rho}_n\}$ converges weakly to a density matrix $\hat{\rho}$, then it also converges in trace norm to the same $\hat{\rho}$.

Then: $\|T_t\hat{\rho} - \hat{\rho}\|_1 \longrightarrow 0$ as $t \to 0$ $\quad \forall\, \hat{\rho} \in B(H)_1^h$

Hence: $\|T_t\hat{\sigma} - \hat{\sigma}\|_1 \leq c_+\|T_t\hat{\rho}_+ - \hat{\rho}_+\|_1 + c_-\|T_t\hat{\rho}_- - \hat{\rho}_-\|_1 \longrightarrow 0$ as $t \to 0$

Moreover: $\|T_t\hat{\sigma}\|_1 \leq \|T_t\hat{\sigma}_+\|_1 + \|T_t\hat{\sigma}_-\|_1 = \|\hat{\sigma}_+\|_1 + \|\hat{\sigma}_-\|_1 = \|\hat{\sigma}\|_1$

since T_t is positivity preserving and $Tr\left\{e^{i\hat{H}t/\hbar}\,\hat{\sigma}_\pm\, e^{-i\hat{H}t/\hbar}\right\} = Tr\,\hat{\sigma}_\pm$.

Therefore T_t is a strongly continuous group of isometries on $B(H)_1^h$.

end

By Theorem 2.1.21. and Statement 2 it follows that L_0 and $-L_0$ are strong-densely defined and dissipative with respect to every (s.i.p.) on $B(H)_1^h$.

STATEMENT 3

L_d is dissipative.

<u>proof</u>

We equip $B(H)_1^h$ with a (s.i.p.) compatible with the trace norm.

Hence:

$$\begin{aligned} Re\,[\hat{\sigma}, L_d\hat{\sigma}] &= Re\,[\hat{\sigma}, -d\,\hat{\sigma}] + Re\,[\hat{\sigma}, d\,T[\hat{\sigma}]] \\ &\leq -d\,\|\hat{\sigma}\|_1^2 + d\,\|\hat{\sigma}\|_1\,\|T[\hat{\sigma}]\|_1 \leq 0 \qquad \forall\, \hat{\sigma} \in B(H)_1^h \end{aligned}$$

by <2.1.11>.

Since L_d is bounded and dissipative and L_0 generates a group of isometries which is strong-continuous on $B(H)_1^h$, the following theorem allows us to show that $L = L_0 + L_d$ generates a strong-continuous semigroup of contractions on $B(H)_1^h$.

THEOREM 2.1.22.[11], [26]

Let S be the generator of a strong-continuous semigroup of contractions, on the Banach space X , and P a dissipative operator with $D(S) \subseteq D(P)$ and $\|PA\| \leq a\|A\| + b\|SA\|$ for all $A \in D(S)$, for some $a \geq 0$ and $b < 1$.

It follows that S+P generates a strong-continuous semigroup of contractions on X .

Concluding: the semigroup $\{\Sigma_t\}_{t \geq 0}$ generated by L is such that:

(i) $\frac{d\Sigma_t\hat{\sigma}}{dt} = L[\Sigma_t\hat{\sigma}]$ in the strong sense $\forall\, \hat{\sigma} \in B(H)_1^h \cap D(L)$ <2.1.23>

(ii) $\lim_{t \to 0+} \|\Sigma_t\hat{\sigma} - \hat{\sigma}\|_1 = 0 \qquad \forall\, \hat{\sigma} \in B(H)_1^h$ <2.1.24>

(iii) $\|\Sigma_t\hat{\sigma}\|_1 \leq \|\hat{\sigma}\|_1 \qquad \forall\, \hat{\sigma} \in B(H)_1^h$ <2.1.25>

Moreover, if $\hat{\rho} \in K(H)$ and $\hat{\rho} \in D(L)$ we have:

$$\mathrm{Tr}\left\{\frac{d\Sigma_t\hat{\rho}}{dt}\Big|_{t=0}\right\} = \frac{d\,\mathrm{Tr}\,\Sigma_t\hat{\rho}}{dt}\Big|_{t=0} = \mathrm{Tr}\{L[\hat{\rho}]\} = 0$$

Therefore: $\mathrm{Tr}\{\Sigma_t\hat{\rho}\} = \mathrm{Tr}\,\hat{\rho} = 1$ and: $1 = \mathrm{Tr}\{\Sigma_t\hat{\rho}\} \leq \|\Sigma_t\hat{\rho}\|_1 \leq \|\hat{\rho}\|_1 = 1$

Hence $\|\Sigma_t\hat{\rho}\|_1 = \|\hat{\rho}\|_1$ for $\hat{\rho} \in K(H)$ and $\|\Sigma_t\| = 1$;thus:

(iv) L generates a semigroup of contractions , norm preserving on $K(H)$. <2.1.26>

We know that the strong continuity of the semigroup $\{\Sigma_t\}_{t\geq 0}$ coincides with the weak continuity (see ref.[11], Chap.3, par.3.1.1.).

That is $\{\Sigma_t\}_{t\geq 0}$ is continuous in the trace norm topology of $B(H)_1^h$ and also in the topology induced by the functionals in $[B(H)_1^h]^* = B(H)^h$.

This topology is obviously weaker than the previous one; a typical neighbourhood of some $\hat{\rho} \in B(H)_1^h$ is given by: $\{\hat{\sigma} \in B(H)_1^h : |\mathrm{Tr}\{\hat{X}[\hat{\rho} - \hat{\sigma}]\}| < \varepsilon\ ;\ \varepsilon > 0,\ \hat{X} \in B(H)^h\}$

It will be indicated by $\sigma(X, X^*)$, where $X = B(H)_1^h$.

Weak continuity of $\{\Sigma_t\}_{t\geq 0}$ means continuity in t of every function given by: $F_{\hat{X}}[\Sigma_t \hat{\rho}] = Tr\{\hat{X} \cdot \Sigma_t \hat{\rho}\}$; $\hat{X} \in B(H)^h$, $\hat{\rho} \in B(H)_1^h$ and $\sigma(X, X^*) - \sigma(X, X^*)$ continuity of Σ_t in the sense that, if $F_{\hat{X}} \in B(H)^h$ then $F_{\hat{X}} \circ \Sigma_t$ also belongs to $B(H)^h$.

This very fact allows the definition of the adjoint of Σ_t as a $\sigma(X^*, X)$ -continuous semigroup $\Sigma^* = \{\Sigma_t^*\}_{t\geq 0}$ on $X^* = B(H)^h$.

$$\Sigma_t^* F_{\hat{X}}[\hat{\rho}] = F_{\hat{X}}[\Sigma_t \hat{\rho}] \qquad \text{<2.1.27>}$$

that is:

$$Tr\{\Sigma_t^* \hat{X} \cdot \hat{\rho}\} = Tr\{\hat{X} \cdot \Sigma_t \hat{\rho}\}; \quad \hat{X} \in B(H)^h, \ \hat{\rho} \in B(H)_1^h \qquad \text{<2.1.28>}$$

Σ^* turns out to be a semigroup of contractions, continuous in the topology induced on $B(H)^h$ by the elements of $B(H)_1^h$ thought of as linear functionals on $B(H)^h$.

This topology will be indicated by $\sigma(X, X_*)$, where $X = B(H)^h$, due to the fact that $B(H)_1^h$ is the predual of $B(H)^h$.

We already know who $\sigma(X, X_*)$ is in this case: it is the ultraweak topology on $B(H)^h$ (see appendix A.2.).

Therefore: $\Sigma^* = \{\Sigma_t^*\}_{t\geq 0}$ is an ultraweak continuous semigroup of contractions on $B(H)^h$, generated by the dual L* of L and satisfying:

(i) $$L^*[\cdot] = \frac{i}{\hbar}[\hat{H}, \cdot] - \alpha 1\!\!1 + \alpha T[\cdot] \qquad \text{<2.1.29>}$$

(ii) $$\frac{d \Sigma_t^* \hat{X}}{dt} = L^*[\Sigma_t^* \hat{X}] \quad \text{for } \hat{X} \in D(L^*) \cap B(H)^h \qquad \text{<2.1.30>}$$

in the ultraweak sense.

(iii) $$\lim_{t \to 0^+} |Tr\{\hat{\rho}[\Sigma_t^* \hat{X} - \hat{X}]\}| = 0 \quad \forall \hat{\rho} \in B(H)_1^h \qquad \text{<2.1.31>}$$

(iv) $\| \Sigma_t^* \| = 1$ <2.1.32>

(v) $\mathrm{Tr}\{\hat{\rho}[\Sigma_t^* \hat{X}]\}$ is a continuous linear functional on $B(H)^h$ equipped with the ultraweak topology. <2.1.33>

We observe now that $\hat{X} = \hat{\mathbb{1}}$ implies:

$L^*[\hat{\mathbb{1}}] = 0$ and $\Sigma_t^* \hat{\mathbb{1}} = \hat{\mathbb{1}}$ <2.1.34>

In order to conlude the proof of Statement 1 , we need demonstrate the last point:

STATEMENT 4

$\forall t \geq 0 \quad \Sigma' = \{\Sigma_t\}_{t \geq 0}$ is positivity preserving.

<u>proof</u>

We shall use the following:

THEOREM 2.2.35.[21]

Let U be a C*-algebra and ω a linear functional on it.

If ω is continuous and $\|\omega\| = \lim_\alpha \omega(E_\alpha^2)$ for some approximate identity of U, then ω is a state, hence positive.

Now the situation is the following:

$\Sigma_t^* : B(H)^h \longrightarrow B(H)^h$ <2.1.36>

$\| \Sigma_t^* \| = 1$ <2.1.37>

$\Sigma_t^* \hat{\mathbb{1}} = \hat{\mathbb{1}}$ <2.1.38>

Σ_t^* , by linearity, can be extended to a bounded map from the C*-algebra B(H) into itself.

Let us consider a state $\omega_{\hat{\varrho}}(\cdot) = \mathrm{Tr}\{\hat{\varrho}\cdot[\cdot]\}$ on B(H), with $\hat{\varrho} \in K(H)$ and hence $\|\omega_{\hat{\varrho}}\| = 1$.

$\Omega[\cdot] = (\omega_{\hat{\varrho}} \circ \Sigma_t^*)[\cdot]$ is a continuous linear functional on B(H) such that:

$$\|\Omega\| \leq \|\omega_{\hat{\varrho}}\| \, \|\Sigma_t^*\| = 1 \tag{<2.1.39>}$$

$$\Omega(\mathbb{1}) = \omega_{\hat{\varrho}}(\mathbb{1}) = 1 \tag{<2.1.40>}$$

By the above Theorem Ω is positive, wathever $\hat{\varrho}$ be.

Thus: $\forall\ \hat{\varrho} \in K(H)$, $\hat{X} \in B(H)^+$ implies $\mathrm{Tr}\{\hat{\varrho}\cdot\Sigma_t^*\hat{X}\} \geqslant 0$.

It then follows that $\Sigma_t^*\hat{X} \geqslant 0$ and, by duality, that $\{\Sigma_t\}_{t\geqslant 0}$ is also positivity preserving.

Paragraph 2

In this paragraph it will be shown that the Von Neumann entropy of a state $\hat{\rho}$ increases in time under the dynamics with spontaneous localizations.

STATEMENT 1

$$\| \Sigma_t \hat{\rho} \|_1 = \| \hat{\rho} \|_1 \qquad \langle 2.2.1\rangle$$

$$\| \Sigma_t \hat{\rho} \|_2 \leq \| \hat{\rho} \|_2 \qquad \langle 2.2.2\rangle$$

for $\hat{\rho} \in$ K(H).

<u>proof</u>

Since $\{\Sigma_t\}_{t\geq 0}$ preserves the positivity and $\hat{\rho} \geq 0$ we have:

$$\begin{aligned} \| \Sigma_t \hat{\rho} \|_1 &= \mathrm{Tr}\{\Sigma_t \hat{\rho}\} = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} d\mu \ \frac{e^{-i\mu y/\hbar}}{\sqrt{2\pi\hbar}} F(\alpha,\mu,0,t) \langle q+y | \hat{\rho}_S(t) | q+y\rangle \\ &= F(\alpha,0,0,t) \int_{-\infty}^{+\infty} dq \ \langle q | \hat{\rho}_S(t) | q\rangle \\ &= \mathrm{Tr}\{\hat{\rho}_S(t)\} = \mathrm{Tr}\{\hat{\rho}\} = \| \hat{\rho} \|_1 \end{aligned} \qquad \langle 2.2.3\rangle$$

Since $\| \Sigma_t \hat{\rho} \|_2^2 = \mathrm{Tr}\{[\Sigma_t \hat{\rho}]^\dagger [\Sigma_t \hat{\rho}]\} = \mathrm{Tr}\{[\Sigma_t \hat{\rho}]^2\}$, if $\mathrm{Tr}\{[\Sigma_t \hat{\rho}]^2\}$ decreases in time, also does $\| \Sigma_t \hat{\rho} \|_2$.

Indeed: $\quad 0 \geq \dfrac{d \| \Sigma_t \hat{\rho} \|_2^2}{dt} = 2 \| \Sigma_t \hat{\rho} \|_2 \dfrac{d \| \Sigma_t \hat{\rho} \|_2}{dt}$

implies:

$$\frac{d \| \Sigma_t \hat{\rho} \|_2}{dt} \leq 0 \quad ; \quad \| \Sigma_t \hat{\rho} \|_2 \leq \| \Sigma_0 \hat{\rho} \|_2 = \| \hat{\rho} \|_2$$

But this is really the case, since:

$$\begin{aligned} \frac{d \,\mathrm{Tr}\{[\Sigma_t \hat{\rho}]^2\}}{dt} &= 2 \,\mathrm{Tr}\left\{ \Sigma_t \hat{\rho} \cdot \frac{d \Sigma_t \hat{\rho}}{dt} \right\} = 2 \,\mathrm{Tr}\left\{ -\alpha [\Sigma_t \hat{\rho}]^2 \right\} + \\ &+ \mathrm{Tr}\left\{ \alpha \Sigma_t \hat{\rho} \cdot T[\Sigma_t \hat{\rho}] - \frac{i}{\hbar} \Sigma_t \hat{\rho} \cdot [\hat{H}, \Sigma_t \hat{\rho}] \right\} \leq 0 \end{aligned}$$

where the third term is zero by cyclicity when $\Sigma_t \hat{\rho}$ belongs to D(L).

Indeed:

$$\begin{aligned} \mathrm{Tr}\{\Sigma_t \hat{\rho} \cdot T[\Sigma_t \hat{\rho}]\} &= \int_{-\infty}^{+\infty} dq \ \langle q | \Sigma_t \hat{\rho} \cdot T[\Sigma_t \hat{\rho}] | q \rangle = \\ &= \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \ \langle q | \Sigma_t \hat{\rho} | q' \rangle \langle q' | T[\Sigma_t \hat{\rho}] | q \rangle \end{aligned} \qquad \langle 2.2.4\rangle$$

where: $\langle q'|\, T[\Sigma_t \hat{\rho}]\, |q\rangle = \sqrt{\frac{\alpha}{\pi}} \int_{-\infty}^{+\infty} dx \; e^{-\frac{\alpha}{2}(q'-x)^2} \langle q'|\, \Sigma_t[\hat{\rho}]\, |q\rangle \, e^{-\frac{\alpha}{2}(q-x)^2} =$

$$= \sqrt{\frac{\alpha}{\pi}} \quad e^{-\frac{\alpha}{4}(q-q')^2} \quad \langle q'|\, \Sigma_t \hat{\rho}\, |q\rangle$$

Hence <2.2.4> becomes: $\displaystyle\int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \; |\langle q'|\, \Sigma_t \hat{\rho}\, |q\rangle|^2 \; e^{-\frac{\alpha}{4}(q-q')^2}$

$$\leq \int_{-\infty}^{+\infty} dq \int_{-\infty}^{+\infty} dq' \; |\langle q'|\, \Sigma_t \hat{\rho}\, |q\rangle|^2 = \mathrm{Tr}\{[\Sigma_t \hat{\rho}]^2\}$$

thus:

$$\frac{d \;\mathrm{Tr}\{[\Sigma_t \hat{\rho}]^2\}}{dt} \leq 0$$

end

Note that $\Sigma_t \hat{\rho}$ is Hilbert-Schmidt, that is $\|\Sigma_t \hat{\rho}\|_2 < +\infty$ since $\Sigma_t \hat{\rho}$ is trace class if $\hat{\rho}$ is.

DEFINITION 2.2.5

We shall indicate by $B(H)_p$ those subensembles of bounded operators for which: $\|\hat{A}\|_p = [\mathrm{Tr}\, |\hat{A}|^p]^{1/p} < +\infty \qquad 0 < p \leq +\infty$

REMARK

$B(H)_1$ is the ensemble of trace class operators.

$B(H)_2$ that of Hilbert-Schmidt's.

What we know about $B(H)_1$ and $B(H)_2$ extends to any $B(H)_p$ with $1 \leq p \leq +\infty$

THEOREM 2.2.6.[20]

For $p \geq 1$, $\|\cdot\|_p$ is a norm and with this norm $B(H)_p$ is a Banach space.

THEOREM 2.2.7.[20]

The finite range operators are dense in any $B(H)_p$ in the respective norm topologies.

REMARK

Density is also true for $0<p<1$, but the $\|\cdot\|_p$'s are not norms in this case.

THEOREM 2.2.8.[20]

If $\hat{A}$ belongs to $\mathcal{B}(H)_p$ and $\hat{B}$ to $\mathcal{B}(H)_q$, then: $\|\hat{A}\hat{B}\|_1 \leq \|\hat{A}\|_p \|\hat{B}\|_q$ with $\frac{1}{p}+\frac{1}{q}=1 \ ; \ p,q \geq 1$.

THEOREM 2.2.9.[22], [25].

If T is a linear map from $\mathcal{B}(H)_{q_1}$ into $\mathcal{B}(H)_{p_1}$ and
from $\mathcal{B}(H)_{q_2}$ into $\mathcal{B}(H)_{p_2}$

defined on the finite range operators, satisfying:

(i) $\|T\hat{A}\|_{p_1} \leq M_1 \|\hat{A}\|_{q_1}$

(ii) $\|T\hat{A}\|_{p_2} \leq M_2 \|\hat{A}\|_{q_2}$

then, for t in the interval [0,1] and p, q such that:

(iii) $1/p = (1-t)/p_1 + t/p_2$

(iv) $1/q = (1-t)/q_1 + t/q_2$

we have:

$$\|T\hat{A}\|_p \leq M_1^{1-t} M_2^{t} \|\hat{A}\|_q$$

The proof of this theorem is given in appendix A.5.

DEFINITION 2.2.10.[25]

We shall call α -entropies the following maps from $K(H)$ into $\mathbb{R}^+$:

$$S_\alpha(\hat{\rho}) = \frac{1}{1-\alpha} \ln\{\mathrm{Tr}[\hat{\rho}^\alpha]\} \quad , \alpha \in \mathbb{R}^+ - \{1\} \ , \ \hat{\rho} \in K(H)$$

THEOREM 2.2.11

$$\lim_{\alpha\to 1} S_\alpha(\hat{\rho}) = S(\hat{\rho}) = -\mathrm{Tr}\{\hat{\rho}\ln\hat{\rho}\}$$

proof

$$\hat{\rho}^\alpha = \Sigma_j \rho_j^\alpha \hat{P}_j \quad ; \quad \Sigma_j \rho_j = 1 \quad ; \quad \rho_j \geq 0 \quad \forall j$$

where ρ_j is an eigenvalue of $\hat{\rho}$ and they all are repeated according to the multiplicity (finite), and $\hat{P}_j$ is the corresponding projection.

$$\lim_{\alpha\to 1} \frac{1}{1-\alpha} \ln\left[\Sigma_j \rho_j^{\alpha}\right] = -\lim_{\alpha\to 1} \frac{d}{d\alpha}\left\{\ln\left[\Sigma_j \rho_j^{\alpha}\right]\right\} =$$

$$= -\lim_{\alpha\to 1} \frac{1}{\Sigma_j \rho_j^{\alpha}} \Sigma_j \frac{d e^{\alpha \ln \rho_j}}{d\alpha} = -\frac{1}{Tr\,\hat{\rho}} \Sigma_j \rho_j \ln \rho_j = -\Sigma_j \rho_j \ln \rho_j$$

end

In the present situation the linear map Σ_t satisfies the requirements of theorem 2.1.9. with:

$M_1 = 1 \;;\; M_2 = 1$

$p_1 = q_1 = 2 \;;\; p_2 = q_2 = 1$

Then: $\|\Sigma_t \hat{\rho}\|_p \le \|\hat{\rho}\|_p$ for $p = \left[\frac{1-t}{2} + t\right]^{-1} = \frac{2}{1+t}$ <2.2.12>

That is <2.2.12> holds for any p belonging to the interval [1, 2].

Hence: $\left[Tr\left\{|\Sigma_t \hat{\rho}|^p\right\}\right]^{1/p} \le \left[Tr\left\{|\hat{\rho}|^p\right\}\right]^{1/p} \quad \forall\, p \in [1,2]$

Consequently: $Tr\left\{|\Sigma_t \hat{\rho}|^p\right\} \le Tr\left\{|\hat{\rho}|^p\right\} \quad \forall\, p \in [1,2]$

But $\hat{\rho}$ is a density matrix and $\Sigma_t \hat{\rho}$ too, that is: $|\hat{\rho}| = \hat{\rho} \;;\; |\Sigma_t \hat{\rho}| = \Sigma_t \hat{\rho}$

Thus:

$$Tr\left\{[\Sigma_t \hat{\rho}]^p\right\} \le Tr\left\{\hat{\rho}^p\right\}$$

$$\ln\left[Tr\left\{[\Sigma_t \hat{\rho}]^p\right\}\right] \le \ln\left[Tr\left\{\hat{\rho}^p\right\}\right]$$

$$\frac{1}{1-p} \ln\left[Tr\left\{[\Sigma_t \hat{\rho}]^p\right\}\right] \ge \frac{1}{1-p} \ln\left[Tr\left\{\hat{\rho}^p\right\}\right] \quad \forall\, p \in [1,2]$$

$$S_p(\Sigma_t \hat{\rho}) \ge S_p(\hat{\rho}) \quad \forall\, p \in [1,2]$$

and: $$S(\Sigma_t \hat{\rho}) \ge S(\hat{\rho})$$

by theorem 2.5.11.

end

APPENDIX A.1 [11], [13], [18], [19], [20]

It is appropriate to give here all the definitions and properties concerning the mathematical technicalities in Chapter 1.

DEFINITION A.1.1

A C*-Algebra is a normed algebra $\mathcal{A}$ such that:

$\mathcal{A}$ is complete in the norm $\|\cdot\|$ (Banach algebra) <i>

$\mathcal{A}$ possesses an involution $*: \mathcal{A} \to \mathcal{A}$;

$*(AB)=B^*A^*$; $\|A^*\| = \|A\|$; $*(\alpha A + \beta B) = \bar{\alpha} A^* + \bar{\beta} B^*$; $A, B \in \mathcal{A}$; $\alpha, \beta \in \mathbb{C}$ <ii>

$\|A^*A\| = \|A^*\| \, \|A\| = \|A\|^2$ <iii>

For example B(H) is a C*-Algebra under the norm: $\|\hat{A}\| = \sup\limits_{\|\psi\|=1} \|\hat{A}\psi\|$; $\hat{A} \in B(H)$, $\psi \in H$

DEFINITION A.1.2

A *-Subalgebra $\mathcal{B}$ of $\mathcal{A} = B(H)$ such that it coincides with its bicommutant is a Von Neumann Algebra.

REMARK A.1.3.

By the Von Neumann bicommutant theorem $\mathcal{B}$ coincides with its strong and weak closure.

We can now examine some subclasses of B(H) and their features.

DEFINITION A.1.4.

$\hat{A} \in B(H)$ is compact if and only if it transforms weakly convergent sequences of vectors in H into strong convergent ones.

THEOREM A.1.5.

The set of compact operators is a linear space which is closed in the norm $\|\cdot\|$.
It is a two sided ideal in B(H) and coincides with the norm closure of the linear space of finite range-operators.

THEOREM A.1.6.

If $\hat{A} \in B(H)$ then there exists $|\hat{A}| = \sqrt{\hat{A}^{\dagger}\hat{A}}$ and an isometry W such that $\hat{A} = W|\hat{A}|$.

DEFINITION A.1.7

The set of all operators $\hat{A}$ in B(H) such that $\mathrm{Tr}|\hat{A}| < +\infty$ is the set of the trace class operators: $B(H)_1$.

DEFINITION A.1.8.

The set of all operators $\hat{A}$ in B(H) such that $\mathrm{Tr}\,|\hat{A}|^2 < +\infty$ is the set of Hilbert-Schmidt operators: $B(H)_2$.

THEOREM A.1 .9.

$\hat{A} \in B(H)_1$ is of the form $\hat{A}=\hat{C}\hat{B}$ with $\hat{C} \in B(H)_2$, $\hat{B} \in B(H)_2$ and for any $\hat{X} \in B(H)$, $\mathrm{Tr}\{\hat{A}\hat{X}\} = \mathrm{Tr}\{\hat{X}\hat{A}\}$ holds.

THEOREM A.1.10

The following statements are equivalent:

$\hat{A} \in B(H)_1$ <i>

$|\hat{A}| \in B(H)_1$ <ii>

$\sqrt{|\hat{A}|} \in B(H)_2$ <iii>

$Tr\, |\hat{A}| < +\infty$ <iv>

DEFINITION A.1.11.

The sub-classes of compact operators such that:

$[\, Tr\, |\hat{A}|^p \,]^{1/p} < +\infty \;, \quad 0 < p \leq +\infty \;, \quad \hat{A} \in B(H)$ <i>

or

$[\, \Sigma_j \; |\mu_j(\hat{A})|^p \,]^{1/p} < +\infty$ <ii>

where: $\{\mu_j(\hat{A})\}$

are the eigenvalues of $|\hat{A}|$, will be indicated by $B(H)_p$. <iii>

THEOREM A.1.12.[20]

If $\{\hat{T}_n\} \in B(H)_p$ is a sequence of operators such that $\|\hat{T}_n - \hat{T}_m\|_p \to 0$ as $m, n \to \infty$ then there exists a compact operator $\hat{T}$ such that $\hat{T}_n \to \hat{T}$ (in the topology of $B(H)_p$) as $n \to \infty$.

THEOREM A.1.13.[20]

Let $\hat{T}$ be compact.

Then there exists a sequence $\{\hat{T}_n\}$ of compact operators, having finite dimensional ranges, such that:

$\hat{T}_n \to \hat{T}$ in the uniform topology as <i>

$\|\hat{T}_n - \hat{T}\|_p \to 0$ as $n \to \infty$ if $\hat{T} \in B(H)_p$ <ii>

$\|\hat{T}_n\|_p \to \|\hat{T}\|_p$ as $n \to \infty$ if $\hat{T} \in B(H)_p$ <iii>

THEOREM A.1.14.[20]

We have $\mathcal{B}(H)_p \subseteq \mathcal{B}(H)_{p'}$ if $p \le p'$; $\|\cdot\|_p$ decreases as p increases. <i>

If $\hat{T}_1, \hat{T}_2$ are in $\mathcal{B}(H)_p$, then $\hat{T}_1 + \hat{T}_2$ is in $\mathcal{B}(H)_p$ <ii>

and: <iii>

$$\|\hat{T}_1 + \hat{T}_2\|_p \le 2^{1/p} \{ \|\hat{T}_1\|_p + \|\hat{T}_2\|_p \} \qquad p \ge 1$$

$$\|\hat{T}_1 + \hat{T}_2\|_p^p \le 2 \{ \|\hat{T}_1\|_p^p + \|\hat{T}_2\|_p^p \} \qquad 0 < p \le 1$$ <iv>

If $\hat{T}_1$ is in $\mathcal{B}(H)_{r_1}$ and $\hat{T}_2$ is in $\mathcal{B}(H)_{r_2}$, then $\hat{T}_1 \hat{T}_2$ is in $\mathcal{B}(H)_r$

where $1/r_1 + 1/r_2 = 1/r$ <v>

Moreover: $\|\hat{T}_1 \hat{T}_2\|_r \le 2^{1/r} \|\hat{T}_1\|_{r_1} \|\hat{T}_2\|_{r_2}$ <vi>

If $\hat{T}$ is in $\mathcal{B}(H)_r$ and $\hat{A}$ is bounded, then $\hat{A}\hat{T}$ and $\hat{T}\hat{A}$ are in $\mathcal{B}(H)_r$ <vii>

Moreover: $\|\hat{A}\hat{T}\|_r \le \|\hat{A}\| \|\hat{T}\|_r$; $\|\hat{T}\hat{A}\|_r \le \|\hat{T}\|_r \|\hat{A}\|$ <viii>

THEOREM A.1.15.[20]

Let $1 \le p \le +\infty$ and let $q^{-1} + p^{-1} = 1$.

Let C_0 denote the set of non zero operators with finite dimensional ranges.

Then if $\hat{A}$ is in $\mathcal{B}(H)_p$:

$$\|\hat{A}\|_p = \sup_{\hat{B} \in C_0} \frac{|\mathrm{Tr}\{\hat{A}\hat{B}\}|}{\|\hat{B}\|_q}$$ <i>

Let p and q be as above, let $\hat{A}$ be in $\mathcal{B}(H)_p$ and $\hat{B}$ in $\mathcal{B}(H)_q$.

Then:

$$\mathrm{Tr}\{\hat{A}\hat{B}\} = \mathrm{Tr}\{\hat{B}\hat{A}\}$$ <ii>

and

$$|\mathrm{Tr}\{\hat{A}\hat{B}\}| \le \|\hat{A}\|_p \|\hat{B}\|_q$$ <iii>

Let p, q, $\hat{A}$, $\hat{B}$ be as above.

Then:

$$\|\hat{A}\hat{B}\|_1 \leq \|\hat{A}\|_p \|\hat{B}\|_q \qquad \text{<iv>}$$

Let p be as above and $\hat{A}$, $\hat{A}_1$ be in $\mathcal{B}(H)_p$.

Then:

$$\|\hat{A}+\hat{A}_1\|_p \leq \|\hat{A}\|_p + \|\hat{A}_1\|_p \qquad \text{<v>}$$

We can assign different topologies of locally convex space to B(H). These are determined by different families of seminorms which fix what sets are open.

We usually work with:

the norm topology given by: $\|\hat{A}\| = \sup_{\|\psi\|=1} \|\hat{A}\psi\|$; $\psi \in H$

the strong topology given by the following family of seminorms: $p_\psi(A) = \|A\psi\|$; $\psi \in H$

the weak topology given by the family: $p_{\psi,\phi}(A) = |(\psi, A\phi)|$; $\psi, \phi \in H$

We introduce here a further topology: the ultraweak one.

It is determined by the following family of seminorms: $\{p_{\underline{\xi},\underline{\eta}}\}$

$$p_{\underline{\xi},\underline{\eta}}(\hat{A}) = \sum_n |(\xi_n, \hat{A}\eta_n)|$$

where $\underline{\xi} = \{\xi_n\}$ and $\underline{\eta} = \{\eta_n\}$ are sequences of vectors in H such that $\sum_n \|\xi_n\|^2, \sum_n \|\eta_n\|^2 < +\infty$

A generating family of neighbourhoods is given by sets in B(H) of the form:

$$\left\{ \hat{A} \in B(H) : p_{\underline{\xi}^i,\underline{\eta}^i}(\hat{A}) < \varepsilon_i \ ; \forall i = 1, \dots, n \quad \varepsilon_i > 0 \quad \sum_n \|\xi_n^i\|^2 < +\infty \ , \sum_n \|\eta_n^i\|^2 < +\infty \right\}$$

THEOREM A.1.16 [11]

The ultraweak topology is finer than the weak one and the two coincide on very bounded set in B(H).

REMARK A.1.17.

Given a vector space E and its algebraic dual E*, the initial topology on E is the one making all the linear functionals in E* continuous.

Considering E** we can assign to E* two different topologies: the initial one and the weak*topology.

The last is obtained as follows:

for any $x \in E$, Jx is an element of E'', the topological dual of E' with the sup norm, where $J : E \to E''$ is such that $\langle Jx, f \rangle = \langle f, x \rangle$

The weak*-topology on E' is the one making continuous all the functionals on E'of the type Jx.

A generating family of neighbourhoods is given by sets of the form:

$$\{ f \in E' : |\langle Jx_i, f \rangle| = |\langle f, x_i \rangle| < \varepsilon_i \quad \forall \varepsilon_i > 0 \quad i = 1, \dots, n \}$$

Since in general $E \subseteq E''$ the weak*-topology is less fine than the weak (initial) one generated by the following neighbourhoods:

$$\{ f \in E' : |\langle F_i, f \rangle| < \varepsilon_i \quad \forall \varepsilon_i > 0 \quad F_i \in E'' \quad i = 1, \dots, n \}$$

THEOREM A.1.18 [11]

In the duality given by:

$$Tr : (\hat{x}, \hat{T}) \to Tr\{\hat{x}\hat{T}\} \qquad \hat{x} \in B(H), \; \hat{T} \in B(H)_1$$

B(H) is the topological dual of $B(H)_1$.

The ultraweak topology coincide with the weak*-topology.

REMARK A.1.19

The space of the ultraweakly continuous linear functionals on B(H) is in one to one correspondence with the elements of $B(H)_1$.

Indeed the seminorms defining that topology are of the form:

$$p_T(\hat{x}) = |Tr\{\hat{x}\hat{T}\}| \quad ; \quad \hat{x} \in B(H), \; \hat{T} \in B(H)_1$$

REMARK A.1.20.

Since $\|\hat{T}\|_1 = \sup_{\|\hat{x}\| = 1} |Tr\{\hat{x}\hat{T}\}|$, from the Hahn-Banach theorem and its corollaries it follows that: $\|\hat{x}\| = \sup_{\|\hat{T}\|_1 = 1} |Tr\{\hat{x}\hat{T}\}|$

APPENDIX A.2. [6], [7], [8], [9]

In this appendix we want to discuss some of the properties of the "operations" : $T : B(H)_1 \longrightarrow B(H)_1$

introduced in Chapter 1.

It will be useful to see how the formalism of the perevious appendix works.

The first aspect we face is the continuity of T with respect to the trace norm topology given by: $\| \hat{A} \|_1 = Tr\, |\hat{A}|$

By linear convexity T, which is initially defined only on the set of density matrices K(H), can be extended to a positive, linear map on $B(H)_1$.

Let us consider the action of T on $B(H)_1^h$ which is the real Banach-subspace of the self-adjoint operators in $B(H)_1$.

Every $\hat{A}$ in $B(H)_1^h$ can be decomposed as follows:

$$\hat{A} = \hat{A}_+ - \hat{A}_- \qquad \text{<A.2.1>}$$

where $\hat{A}_+$ and $\hat{A}_-$ are positive self-adjoint operators given by:

$$\hat{A}_+ = \frac{\hat{A} + |\hat{A}|}{2} \; ; \quad \hat{A}_- = \frac{|\hat{A}| - \hat{A}}{2} \qquad \text{with } |\hat{A}| = \sqrt{\hat{A}^\dagger \hat{A}} = \sqrt{\hat{A}^2} \qquad \text{<A.2.2>}$$

We have:

$$\|\hat{A}\|_1 = tr\,|\hat{A}| = tr\,\hat{A}_+ + tr\,\hat{A}_- \qquad \text{<A.2.3>}$$

and

$$|Tr\,\hat{A}| = |Tr\,\hat{A}_+ - Tr\,\hat{A}_-| \leq Tr\,\hat{A}_+ + Tr\,\hat{A}_- = \|\hat{A}\|_1 \qquad \text{<A.2.4>}$$

WE construct $\hat{W}_+ = \dfrac{\hat{A}_+}{Tr\,\hat{A}_+}$ and $\hat{W}_- = \dfrac{\hat{A}_-}{Tr\,\hat{A}_-}$:they are elements of K(H).

Hence:

$$\| T[\hat{A}] \|_1 = \sup_{\|\hat{X}\|=1} | Tr \{ \hat{X} \cdot T[\hat{A}] \} | = \sup_{\|\hat{X}\|=1} | Tr \{ \hat{X} [Tr \hat{A}_+ \cdot T[\hat{W}_+] - Tr \hat{A}_- \cdot T[\hat{W}_-]] \} |$$

$$\leq \sup_{\|\hat{X}\|=1} [Tr \hat{A}_+ \cdot | Tr \hat{X} T[\hat{W}_+] | + Tr \hat{A}_- \cdot | Tr \hat{X} . T[\hat{W}_-] |]$$

$$\leq \sup_{\|\hat{X}\|=1} \| \hat{X} \| \{ Tr \hat{A}_+ \| T[\hat{W}_+] \|_1 + Tr \hat{A}_- \| T[\hat{W}_-] \|_1 \} \leq C_1 \| \hat{A} \|_1 \qquad \text{<A.2.5>}$$

where we used THEOREM A.1.15 and REMARK A.1.20 , setting: $C_1 = \sup_{\hat{W} \in K(H)} \| T[\hat{W}] \|_1 \leq 1$

Therefore $Tr \{ \hat{X} \cdot T [\hat{A}] \}$ is a linear continuous functional on $B(H)_1^h$.

Indeed:

$$| Tr \{ \hat{X} \cdot T[\hat{A}] \} | \leq \| \hat{X} \| \, \| T[\hat{A}] \|_1 \leq \| \hat{X} \| \, \| \hat{A} \|_1 \qquad \text{<A.2.6>}$$

By the duality $B(H) = [B(H)_1]^*$ (see A.1.), which also holds for $B(H)^h$, $B(H)_1^h$ we have:

$$[Tr \{ \hat{X} \cdot T[\hat{A}] \}] = Tr \{ \hat{X}^* . \hat{A} \} \qquad \text{<A.2.7>}$$

with X uniquely determined by X* [21].

We write: $\hat{X}^* = T^*[\hat{X}]$ <A.2.8>

thus defining T* as the linear map dual of T in the above defined duality.

T* is real linear and can be extended to B(H).

Moreover:

$$\| T^*[\hat{X}] \| = \sup_{\|\hat{B}\|_1=1} | Tr \{ T^*[\hat{X}] \hat{B} \} | = \sup_{\|\hat{B}\|_1=1} | Tr \{ \hat{X} \cdot T[\hat{B}] \} | \leq \| \hat{X} \| \qquad \text{<A.2.9>}$$

Hence T* is a continuos linear map on B(H).

For any $\hat{B} \in B(H)_1^h$ it then follows:

$$\| T[\hat{B}] \|_1 = \sup_{\|X\|=1} | Tr \{ \hat{X} \cdot T[\hat{B}] \} | = \sup_{\|\hat{X}\|=1} | Tr \{ T^*[\hat{X}] \cdot \hat{B} \} | \qquad \text{<A.2.10>}$$

that is the continuity of T on all of $B(H)_1^h$.

T^* is also normal, that is ultraweakly continuous on B(H).

Indeed: given a sequence $\{\hat{X}_n\}$ converging to $\hat{X}$ in $B(H)^h$, we have:

$$|\mathrm{Tr}\{T^*[\hat{X}-\hat{X}_n]\cdot\hat{A}\}| = |\mathrm{Tr}\{[\hat{X}-\hat{X}_n]\cdot T[\hat{A}]\}| \leq \|\hat{X}-\hat{X}_n\|_1 \cdot \|T\hat{A}\|_1 \qquad \langle A.2.11\rangle$$

which goes to zero as n goes to infinity for any $\hat{A}\in B(H)_1^h$.

Let us now consider:

$$T[\hat{A}] = \sum_K{}_k \hat{T}_k \hat{A} \hat{T}_k^\dagger \qquad \langle A.2.12\rangle$$

$$T^*[\hat{X}] = \sum_K{}_k \hat{T}_k^\dagger \hat{X} \hat{T}_k \qquad \langle A.2.13\rangle$$

$$\hat{F} = \sum_K{}_k \hat{T}_k^\dagger \hat{T}_k \leq \hat{1} \qquad \langle A.2.14\rangle$$

where $\hat{A}\in B(H)_1$, $\hat{X}\in B(H)$.

The set K is at most countable by condition <A.2.14> and $\hat{F}_n = \sum_1^m{}_k \hat{T}_k^\dagger \hat{T}_k$ is a bounded, increasing sequence of positive operators in B(H).

As such it is weakly and ultraweakly convergent in B(H) to some operator $\hat{F} = \sum_K{}_k \hat{T}_k^\dagger \hat{T}_k$.

Let us consider:

$$\hat{X}_n = \sum_1^m{}_k \hat{T}_k^\dagger \hat{X} \hat{T}_k \quad , \quad \hat{X}\in B(H)^+ \qquad \langle A.2.15\rangle$$

$\{\hat{X}_n\}$ is an increasing bounded sequence of positive operators in B(H).

Indeed: $0 \leq \hat{X}_n \leq \|\hat{X}\|\,\hat{F}_n \leq \|\hat{X}\|\cdot\hat{1} \quad \forall n\in\mathbb{N}$

Hence it is weakly and ultraweakly convergent to some $T^*[\hat{X}] = \sum_k \hat{T}_k^\dagger \hat{X} \hat{T}_k$

By linearity the argument extends to all of B(H), independently of the order of summation.

Written: $\hat{A}_n = \sum_1^m{}_k \hat{T}_k \hat{A} \hat{T}_k^\dagger \quad , \quad \hat{A}\in B(H)_1^+ \qquad \langle A.2.16\rangle$

since $B(H)_1$ is a two sided ideal in B(H), $\{\hat{A}_n\}$ is a sequence in $B(H)_1^+$ which is a Cauchy sequence of positive operators.

Thence:

$$\|\hat{A}_n - \hat{A}_m\|_1 = \mathrm{Tr}\left\{\sum_{m+1}^{n}{}_k \hat{T}_k \hat{A} \hat{T}_k^\dagger\right\} = \mathrm{Tr}\left\{\sum_{m+1}^{n}{}_k \hat{T}_k^\dagger \hat{T}_k \cdot \hat{A}\right\}$$
$$= \mathrm{Tr}\left\{[\hat{F}_n - \hat{F}_m]\cdot \hat{A}\right\} \longrightarrow 0 \quad \text{as } n, m \longrightarrow \infty \qquad \text{<A.2.17>}$$

by ultraweakly convergence of $\{\hat{F}_n\}$ in B(H).

Hence $\{\hat{A}_n\}$ is trace norm convergent to a positive operator $T[\hat{A}] = \sum_k \hat{T}_k \hat{A} \hat{T}_k^\dagger$ in $B(H)_1$.

By linearity this is true for any $\hat{A} \in B(H)_1$, independently of the order of summation.

We have now the ultraweakly convergence of the second and third series in <A.2.13> and <A.2.14> and the trace norm convergence of that in <A.2.12>.

It remains to be proved the duality between T and T* .

We have to show that:

$$\mathrm{Tr}\left\{\hat{X}\cdot T[\hat{A}]\right\} = \mathrm{Tr}\left\{T^*[\hat{X}]\cdot \hat{A}\right\} \quad \forall \hat{X} \in B(H),\ \forall \hat{A} \in B(H)_1 \qquad \text{<A.2.18>}$$

By ultraweakly convergence: $\mathrm{Tr}\{\hat{X}_n \hat{A}\} \xrightarrow[n\to\infty]{} \mathrm{Tr}\{\hat{X}\cdot\hat{A}\}$ <A.2.19>

But:

$$\mathrm{Tr}\{\hat{X}_n \hat{A}\} = \mathrm{Tr}\left\{\sum_1^n{}_k \hat{T}_k^\dagger \hat{X} \hat{T}_k \cdot \hat{A}\right\} = \mathrm{Tr}\left\{\hat{X}\cdot \sum_1^n{}_k \hat{T}_k \hat{A} \hat{T}_k^\dagger\right\}$$
$$= \mathrm{Tr}\{\hat{X}\hat{A}_n\}$$

and: $\left|\mathrm{Tr}\left\{\hat{X}\cdot[T[\hat{A}] - \hat{A}_n]\right\}\right| \leq \|\hat{X}\|\, \|T[\hat{A}] - \hat{A}_n\|_1 \xrightarrow[n\to\infty]{} 0$

by trace norm convergence of $\{\hat{A}_n\}$.

Hence:

$$\mathrm{Tr}\left\{T^*[\hat{X}]\cdot\hat{A}\right\} = \lim_{n\to\infty}\left[\mathrm{Tr}\{\hat{X}_n\hat{A}\}\right] = \lim_n\left[\mathrm{Tr}\{\hat{X}\cdot\hat{A}_n\}\right] = \mathrm{Tr}\left\{\hat{X}\cdot T[\hat{A}]\right\}$$

A self adjoint operator A is usually expressed in terms of its spectral resolution.

Let A be such that $\hat{A} = \hat{A}^{\dagger}$ and:

$$\hat{A} = \int_{S_p[\hat{A}]} x \, d\hat{P}_A(x) \tag{A.3.1}$$

where $S_p[\hat{A}]$ is the bounded or unbounded spectrum of $\hat{A}$, $d\hat{P}_A(x)$ is the "projection valued measure" on $S_p[\hat{A}]$.

Generally $\hat{P}_A(x)$ is a map from the Borelian σ-algebra of a measure set Ω into the positive operators on a Hilbert space H such that:

$$\hat{P}_A(E) \geq \hat{P}_A(\phi) = 0 \qquad \forall \; E \in \mathcal{E} \tag{A.3.2}$$

$$\hat{P}_A(E)^{\dagger} = \hat{P}_A(E) \qquad \forall \; E \in \mathcal{E} \tag{A.3.3}$$

$$\sum_{i}^{\infty} \hat{P}_A(E_i) = \hat{P}_A\left[\bigcup_{i=1}^{\infty} E_i\right] \qquad \forall \; \{E_i\} \in \mathcal{E} \; : \; E_i \cap E_j = \phi \tag{A.3.4}$$

$$\hat{P}_A(E)\,\hat{P}_A(F) = \hat{P}_A(E) \qquad \forall \; E, F \; : \; E \subseteq F \tag{A.3.5}$$

$$\hat{P}_A(\Omega) = \hat{\mathbb{1}} \tag{A.3.6}$$

This map: $\hat{P}_A(\cdot) \; : \; \mathcal{E} \longrightarrow B(H)^{+}$

is uniquely determined by the observable (self adjoint operator) $\hat{A}$.

Since we stressed the opportunity of using more general effects than the projections, we introduce the following:

POSITIVE OPERATOR VALUED MEASURE (P.O.V.): $\hat{F}(\cdot) \; : \; \mathcal{E} \longrightarrow B(H)^{+}$

It satisfies:

$$\hat{F}(E) \geq \hat{F}(\phi) = 0 \qquad \forall\, E \in \mathcal{A} \tag{A.3.7}$$

$$\sum_i^{\infty} \hat{F}(E_i) = \hat{F}\Big[\bigcup_{i=1}^{\infty} E_i\Big] \qquad \forall\, \{E_i\} \in \mathcal{A} : E_i \cap E_j = \phi \tag{A.3.8}$$

$$\hat{F}(\Omega) = \hat{1} \tag{A.3.9}$$

This $\hat{F}(\cdot)$ is called "generalized observable".

Given a state $\hat{W}$, the probability of finding a value of the generalized observable $\hat{F}(\cdot)$ within a set E of $\mathcal{A}$ is given by:

$$\mathrm{Tr}\{\hat{W} \cdot \hat{F}(E)\} = \int_E \mathrm{Tr}\,[\hat{W} \cdot d\hat{F}(x)] \tag{A.3.10}$$

and the mean value in the state $\hat{W}$ by:

$$\langle \hat{F} \rangle_{\hat{W}} = \int_{\Omega} x \; \mathrm{Tr}\{\hat{W} \cdot d\hat{F}(x)\} \tag{A.3.11}$$

The serie in <A.3.8> and the above integral, if convergent, are to be understood as strong or ultraweakly limits in B(H).

In such a case $\hat{F}(\cdot)$ determines a self adjoint operator:

$$\hat{B} = \int_{\Omega} x \, d\hat{F}(x) \tag{A.3.12}$$

The correspondence is not one to one.

Since $\hat{B} = \hat{B}^{\dagger}$, $\hat{B}$ it is also determined by its unique projection valued measure.

We find here the mathematical expression of the fact that an observable is specified by the measurements we can perform on it, using a whole equivalence class of instruments [9], each of them providing a different operation on the physical states.

The set of the "operations" on the state space K(H) is a convex set within the linear space of the bounded, linear transformations on K(H):

It can be topologized in two ways: by the norm given by:

$$\| T \| = \sup_{\hat{W} \in K(H)} \| T[W] \|_1 \qquad \text{<A.3.13>}$$

or by the seminorms $\{ P_{\hat{\rho}} \}$ where:

$$P_{\hat{\rho}}(T) = \| T \hat{\rho} \|_1 \qquad \text{<A.3.14>}$$

The last one is called the strong topology on $\mathcal{L}(K(H))$

We are now able to introduce the following:

POSITIVE MAP VALUED MEASURE (P.M.V): $\mathcal{E} : \mathcal{A} \longrightarrow \mathcal{L}^+(K(H))$

$\mathcal{L}^+(K(H))$ is the cone of all positive linear maps on $\mathcal{L}(K(H))$;

which satisfies:

$$\mathcal{E}(E) \geq \mathcal{E}(\phi) = 0 \qquad \forall E \in \mathcal{A} \qquad \text{<A.3.15>}$$

$$\mathcal{E}\Big[\bigcup_{i=1}^{\infty} E_i\Big] = \sum_{1}^{\infty}{}_i \, \mathcal{E}(E_i) \qquad \forall \{E_i\} \in \mathcal{A} : E_i \cap E_j = \phi \qquad \text{<A.3.16>}$$

$$Tr\{ \mathcal{E}(\Omega) \, E[\hat{\rho}] \} = 1 \qquad \forall \hat{\rho} \in K(H) \qquad \text{<A.3.17>}$$

This P.M.V. will be called an "instruments".

The sum in <A.3.16> converges in the strong topology.

This is a natural generalization of what has been called "operation" in Chapter 1.

In that case the set Ω is countable and $\hat{\mathcal{E}}(\cdot)$ is a pure point P.M.V. measure.

As for the discrete case we can show that a P.M.V. measure uniquely determines a P.O.V. measure on the same set and therefore a unique generalized observable.

The rule is obviously given by the correspondence:

$$\mathrm{Tr}\{\mathcal{E}(E)[\hat{W}]\} = \mathrm{Tr}\{\hat{F}(E)\cdot\hat{W}\} \quad , \forall\ \hat{W} \in K(H) \qquad \text{<A.3.18>}$$

The proof is similar to that presented in A.2.: it has only to be proved that $\hat{F}(\cdot)$ is a P.O.V., but this easily follows from the properties of and the strong convergence of the sum in <A.3.16>.

APPENDIX A.4.

We want now to investigate the solution of the modified dynamics in the free case, whose formal properties have been pointed out in chapter 1.

The solution, as discussed in the introduction, is:

$$\langle q|\Sigma_t \hat{\rho}|q'\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} d\mu \int_{-\infty}^{+\infty} d\gamma \; e^{-i\mu\gamma/\hbar} \, F(\alpha,\mu,q-q',t)\langle q+\gamma|\hat{\rho}_S(t)|q'+\gamma\rangle \qquad \langle A.4.1\rangle$$

$$F(\alpha,\mu,q-q',t) = e^{-\alpha t} \; e^{\alpha\int_0^t e^{-\frac{\alpha}{4}\left[\frac{\mu\tau}{m}-(q-q')\right]^2} d\tau} \qquad \langle A.4.2\rangle$$

$$\hat{\rho}_S(t) = e^{-i\frac{\hat{p}^2}{2m\hbar}t} \; \hat{\rho} \; e^{i\frac{\hat{p}^2}{2m\hbar}t} \qquad \langle A.4.3\rangle$$

Let us formally manipulate the formula <A.4.1> as follows:

we write $F(\alpha,\mu,q-q',t)$ as:

$$F(\alpha,\mu,q-q',t) = \int_{-\infty}^{+\infty} d\nu \; \delta(\nu-q+q')\, F(\alpha,\mu,\nu,t)$$

$$= \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} d\nu \int_{-\infty}^{+\infty} d\lambda \; e^{-i\lambda(\nu-q+q')/\hbar} \, F(\alpha,\mu,\nu,t) \qquad \langle A.4.4\rangle$$

Hence <A.4.1> becomes:

$$\langle q|\Sigma_t\hat{\rho}|q'\rangle = \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty} d\mu \int_{-\infty}^{+\infty} d\nu \int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} d\gamma \; e^{-i\nu\lambda/\hbar} \, e^{-i\mu\gamma/\hbar} \, F(\alpha,\mu,\nu,t) \times$$

$$\times \exp\left\{\frac{i}{\hbar}\lambda[q-q']\right\} \langle q+\gamma|\hat{\rho}_S(t)|q'+\gamma\rangle$$

Remembering that: $\exp\{-i\hat{p}\gamma/\hbar\}|q\rangle = |q+\gamma\rangle$

we obtain:

$$\langle q|\Sigma_t\hat{\rho}|q'\rangle = \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty} d\mu \int_{-\infty}^{+\infty} d\nu \int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} d\gamma \; e^{-i\nu\lambda/\hbar} \, e^{-i\mu\gamma/\hbar} \times$$

$$\times F(\alpha,\mu,\nu,t) \; \langle q| \exp\left\{\frac{i\hat{q}\lambda}{\hbar}\right\} \exp\left\{i\frac{\hat{p}\gamma}{\hbar} - \frac{i\hat{p}^2 t}{2m\hbar}\right\} \hat{\rho} \, \exp\left\{\frac{i\hat{p}^2 t}{2m\hbar} - \frac{i\hat{p}\gamma}{\hbar}\right\} \exp\left\{-\frac{i\hat{q}\lambda}{\hbar}\right\}|q'\rangle;$$

thus:

$$\Sigma_t\hat{\rho} = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} d\lambda \int_{-\infty}^{+\infty} d\gamma \; \hat{F}(\alpha,\gamma,\lambda,t) \, W^{\dagger}(\lambda,\gamma) \, U_t \hat{\rho} \, U_t^{\dagger} \, W(\lambda,\gamma) \qquad \langle A.4.5\rangle$$

$$\hat{F}(\alpha,\gamma,x,t) = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}d\mu\int_{-\infty}^{+\infty}d\nu\; e^{-i\mu\gamma/\hbar}\, e^{-i\nu x/\hbar}\, F(\alpha,\mu,\nu,t)$$ <A.4.6>

$$W(x,\gamma) = e^{-i\hat{p}\gamma/\hbar - i\hat{q}x/\hbar} \quad , \quad U_t = e^{-i\frac{\hat{p}^2 t}{2m\hbar}}$$ <A.4.7>

Since $F(\alpha,\mu,\nu,t)$ is only locally integrable, being bounded and going as a constant at infinity, it can be interpreted as a regular, tempered distribution on the Schwartz space $\mathcal{S}$, for any t>0.

The linear functional $F_t[\cdot]$ on $\mathcal{S}$ the function $F(\alpha,\mu,\nu,t)$ gives rise is positive for any t>0.

Its Fourier transform $\hat{F}_t[\varphi] = F_t[\hat{\varphi}]$, $\varphi \in \mathcal{S}$, gives a meaning to formula <A.4.6>.

Formula <A.4.5> is meaningful in the weak convergence of the integral.

Indeed, compactness and self-adjointness of $\hat{\rho} \in B(H)_1^+$ allows to restrict ourselves to the simpler case:

$$\Sigma_t[|\psi\rangle\langle\psi|] = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}d\gamma\; \hat{F}(\alpha,\gamma,x,t)\; W^{+}(x,\gamma)\,U_t|\psi\rangle\langle\psi|U_t^{+}\,W(x,\gamma)$$ <A.4.8>

with: $|\psi\rangle \in H = \mathcal{L}^2(\mathbb{R})$

For any $|\phi\rangle, |\psi\rangle \in H$, $\langle\phi|\,\Sigma_t[|\psi\rangle\langle\psi|]\,|\phi\rangle$ is well defined:

$$\langle\phi|\,\Sigma_t[|\psi\rangle\langle\psi|]|\phi\rangle = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}d\gamma\; \hat{F}(\alpha,\gamma,x,t)\,|f_t(x,\gamma)|^2 = \hat{F}_t[\,f_t^{*}f_t]$$

f_t and $f_t^{*}f_t$ being square-integrable on $\mathbb{R}^2$.

This result comes out from:

$$\|f_t\|^2 = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}d\gamma\; |f_t(x,\gamma)|^2 = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}d\gamma\; |\langle\phi|W^{+}(x,\gamma)\,U_t|\psi\rangle|^2$$ <A.4.9>

$$\langle\phi|W^{+}(x,\gamma)\,U_t|\psi\rangle = \int_{-\infty}^{+\infty}dq\; \overline{\phi(q)}\;\langle q|\,e^{i\hat{q}x/\hbar}\,e^{i\hat{p}\gamma/\hbar}\,|\psi_t\rangle\, e^{i\gamma(x,\gamma)}$$ <A.4.10>

Since: $e^{i\hat{q}x/\hbar + i\hat{p}\gamma/\hbar} = e^{i\hat{q}x/\hbar}\,e^{i\hat{p}\gamma/\hbar}\,e^{\frac{i}{2\hbar}x\gamma}$; $\gamma(x,\gamma) = \frac{x\gamma}{2\hbar}$

and we put: $|\psi_t\rangle = U_t|\psi\rangle$

Thus:

$$\langle A.4.10\rangle = \int_{-\infty}^{+\infty} dq\ \overline{\phi(q)}\ e^{iqx/\hbar}\ \psi_t(q-y)\ e^{ixy/2\hbar}$$

$$\langle A.4.9\rangle = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dq' \int_{-\infty}^{+\infty} dq\ e^{i(q-q')x/\hbar}\ \overline{\phi(q)}\ \phi(q')\ \psi_t(q-y)\ \overline{\psi_t(q'-y)}$$

$$= 2\pi\hbar \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dq\ |\phi(q)|^2\ |\psi_t(q-y)|^2 < +\infty$$

being $|\phi\rangle$ and $|\psi\rangle$ square-integrable on the real line.

Note that this result is independent of t since, for any t, $U(t)$ maps $\mathcal{L}^2(\mathbb{R})$ into $\mathcal{L}^2(\mathbb{R})$.

Due to the fact that: $0 < F(\alpha,\mu,\nu,t) \leq 1$, the expression <A.4.8> has a meaning uniformly in t .

Therefore we can perform the weak limit:

$$\underset{t\to 0^+}{\text{w-lim}}\ \Sigma_t\hat{\rho} = \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} d\mu \int_{-\infty}^{+\infty} d\nu\ e^{-i\mu y/\hbar}\ e^{-i\nu x/\hbar}\ F(\alpha,\mu,\nu,0)\ W^\dagger(x,y)\,\hat{\rho}\,W(x,y)$$

$$= \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx\ \delta(y)\ \delta(x)\ W^\dagger(x,y)\,\hat{\rho}\ W(x,y) = \hat{\rho}$$

We can obviously state more by the results of Chapter 2, paragraph 1.

For what concern us here, it is enough to give a mathematical meaning to the following formal calculations.

We want to proof that $\Sigma_t\hat{\rho}$ exhibits the semigroup property:

$$\Sigma_{t_1}\circ\Sigma_{t_2} = \Sigma_{t_1+t_2} \qquad \forall\ t_1, t_2 \geq 0$$

Let I be:

$$\hat{I} = \Sigma_{t_1}[\Sigma_{t_2}\hat{\rho}] = \int_{-\infty}^{+\infty} d\mu_1 \int_{-\infty}^{+\infty} d\mu_2 \int_{-\infty}^{+\infty} d\nu_1 \int_{-\infty}^{+\infty} d\nu_2 \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \int_{-\infty}^{+\infty} dy_1 \int_{-\infty}^{+\infty} dy_2 \times \frac{1}{(2\pi\hbar)^4}\times$$

$$\times F(\alpha,\mu_1,\nu_1,t_1)\ F(\alpha,\mu_2,\nu_2,t_2)\ \exp\left\{-\frac{i}{\hbar}[\mu_1 y_1 + \mu_2 y_2 + \nu_1 x_1 + \nu_2 x_2]\right\}\times$$

$$\times\ W^\dagger(x_1,y_1)\,U_{t_1}\,W^\dagger(x_2,y_2)\,U_{t_2}\,\hat{\rho}\ U^\dagger_{t_2}\,W(x_2,y_2)\ U^\dagger_{t_1}\,W(x_1,y_1)$$

We shall call $\hat{\Pi}$ the product of operators in the integral:

$$\hat{\Pi} = \hat{A}\,\hat{\rho}\,\hat{A}^\dagger \qquad \text{where: } \hat{A} = e^{i\hat{q}x_1/\hbar}\ e^{i\hat{p}y_1/\hbar}\ e^{-i\hat{p}^2 t_1/2m\hbar}\ e^{i\hat{q}x_2/\hbar}\ e^{i\hat{p}y_2/\hbar}\ e^{-\frac{i\hat{p}^2 t_2}{2m\hbar}}$$

$$\text{and: } \exp\left\{-i\frac{\hat{p}^2 t_1}{2m\hbar}\right\}\exp\left\{i\frac{\hat{q}x_2}{\hbar}\right\} = \exp\left\{\frac{i}{\hbar}\left[\hat{q}-\frac{\hat{p}}{m}t_1\right]x_2\right\}\exp\left\{-\frac{i\hat{p}^2 t_1}{2m\hbar}\right\}$$

Hence: $$\hat{A} = e^{i\hat{q}x_1/\hbar}\, e^{i\hat{p}y_1/\hbar}\, e^{i(\hat{q} - \frac{\hat{p}}{m}t_1)\frac{x_2}{\hbar}}\, e^{-\frac{i\hat{p}^2}{2m\hbar}(t_1+t_2)}$$

$$e^{\frac{i}{\hbar}(\hat{q}-\frac{\hat{p}}{m}t_1)x_2} = e^{i\hat{q}x_2/\hbar}\, e^{-i\hat{p}tx_2/\hbar m}\, e^{-\frac{itx_2^2}{2m\hbar}}$$

$$e^{i\hat{p}y_1/\hbar}\, e^{i\hat{q}x_2/\hbar} = e^{\frac{i}{\hbar}\hat{q}x_2}\, e^{i\hat{p}y_1/\hbar}\, e^{ix_2y_1/\hbar}$$

Thus: $$\hat{A} = e^{i/\hbar\,(y_1x_2 - \frac{tx_2^2}{2m})}\, e^{i\hat{q}(x_1+x_2)/\hbar}\, e^{i\hat{p}(y_1+y_2-\frac{t_1x_2}{m})}\, e^{-\frac{i\hat{p}^2}{2m\hbar}(t_1+t_2)}$$

and: $$\hat{\Pi} = W^\dagger(x_1+x_2, y_1+y_2-\tfrac{t_1x_2}{m})\, U_{t_1+t_2}\, \hat{\rho}\, U^\dagger_{t_1+t_2}\, W(x_1+x_2, y_1+y_2-\tfrac{t_1x_2}{m})$$

First change of variables: $\begin{cases} y_1+y_2 = w \\ y_1 = u \end{cases}$ $\quad J = \left|\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\right| = 1$

$$I = \int_{-\infty}^{+\infty} d\mu_1 \int_{-\infty}^{+\infty} d\mu_2 \int_{-\infty}^{+\infty} d\nu_1 \int_{-\infty}^{+\infty} d\nu_2 \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \int_{-\infty}^{+\infty} dw \int_{-\infty}^{+\infty} du \times \frac{1}{(2\pi\hbar)^4} \times$$

$$\times\ F(d,\mu_1,\nu_1,t_1)\, F(d,\mu_2,\nu_2,t_2)\ \exp\left\{-\frac{i}{\hbar}\left[(\mu_1-\mu_2)u + \mu_2 w + \nu_1x_1 + \nu_2x_2\right]\right\} \times$$

$$\times\ W^\dagger(x_1+x_2, w-\tfrac{t_1x_2}{m})\, U_{t_1+t_2}\, \hat{\rho}\, U_{t_1+t_2}\, W(x_1+x_2, w-\tfrac{t_1x_2}{m})$$

Integrating in u and μ_2 we obtain:

$$I = \int_{-\infty}^{+\infty} d\mu_1 \int_{-\infty}^{+\infty} d\nu_1 \int_{-\infty}^{+\infty} d\nu_2 \int_{-\infty}^{+\infty} dx_1 \int_{-\infty}^{+\infty} dx_2 \int_{-\infty}^{+\infty} dw\ \frac{1}{(2\pi\hbar)^3}\, F(d,\mu_1,\nu_1,t_1)\, F(d,\mu_1,\nu_2,t_2) \times$$

$$\times \exp\left\{-\frac{i}{\hbar}\left[\nu_1x_1 + \nu_2x_2 + \mu_1 w\right]\right\}\ \hat{\Pi}(x_1+x_2, w - t_1x_2/m, t_1+t_2)$$

Second change of variables: $\begin{cases} x = x_1+x_2 \\ x_2 = v \end{cases}$ $\quad J = \left|\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}\right| = 1$

$$I = \int_{-\infty}^{+\infty} d\mu_1 \int_{-\infty}^{+\infty} d\nu_1 \int_{-\infty}^{+\infty} d\nu_2 \int_{-\infty}^{+\infty} dw \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dv\ \frac{1}{(2\pi\hbar)^3}\, F(d,\mu_1,\nu_1,t_1)\, F(d,\mu_1,\nu_2,t_2) \times$$

$$\times \exp\left\{-\frac{i}{\hbar}\left[(\nu_2-\nu_1)v + \nu_1 x + \mu_1 w\right]\right\}\ \hat{\Pi}(x, w - t_1v/m, t_1+t_2)$$

Third change of variables: $\begin{cases} y = w - t_1v/m \\ z = v \end{cases}$ $\quad J = \left|\begin{pmatrix} 1 & \frac{t_1}{m} \\ 0 & 1 \end{pmatrix}\right| = 1$

$$I = \int_{-\infty}^{+\infty} d\mu_1 \int_{-\infty}^{+\infty} d\nu_1 \int_{-\infty}^{+\infty} d\nu_2 \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dz \times \frac{1}{(2\pi\hbar)^3}\, F(d,\mu_1,\nu_1,t_1)\, F(d,\mu_1,\nu_2,t_2) \times$$

$$\times \exp\left\{-\frac{i}{\hbar}\left[(\nu_2-\nu_1)z + \nu_1 x + \mu_1 y + \mu_1\frac{t_1 z}{m}\right]\right\} \times\ \hat{\Pi}(x, y, t_1+t_2)$$

Integrating in z and ν_2 we obtain: $I = \frac{1}{(2\pi\hbar)^2} \int_{-\infty}^{+\infty} d\mu_1 \int_{-\infty}^{+\infty} d\nu_1 \int_{-\infty}^{+\infty} dy \int_{-\infty}^{+\infty} dx\ \times$

$$\times\ F(d,\mu_1,\nu_1,t_1)\, F(d,\mu_1,\nu_1 - \frac{\mu_1 t_1}{m}, t_2) \times \exp\left\{-\frac{i}{\hbar}\left[\nu_1 x + \mu_1 y\right]\right\} \times \hat{\Pi}(x,y,t_1+t_2)$$

Now: $$F(\alpha,\mu,\nu,t) = e^{-\alpha t}\, e^{\alpha \int_0^t e^{-\frac{\alpha}{4}\left[\frac{\mu\tau}{m}-\nu\right]^2} d\tau} = e^{-\alpha t} \cdot \exp\left\{ \frac{2m\alpha}{\sqrt{\alpha}\,\mu} \int_{-\frac{\sqrt{\alpha}}{2}\nu}^{-\frac{\sqrt{\alpha}}{2}\nu + \frac{\sqrt{\alpha}}{2}\frac{\mu t}{m}} e^{-\tau^2} d\tau \right\}$$

Hence:

$$F(\alpha,\mu_1,\nu_1,t_1)\, F\left(\alpha,\mu_1,\nu_1-\frac{\mu_1 t_1}{m},t_2\right) = e^{-\alpha(t_1+t_2)} \exp\left\{ \frac{2m\alpha}{\sqrt{\alpha}\,\mu_1} \int_{-\frac{\sqrt{\alpha}\nu_1}{2}}^{-\frac{\sqrt{\alpha}}{2}\nu_1 + \frac{\sqrt{\alpha}}{2}\frac{\mu_1 t_1}{m}} e^{-\tau^2} d\tau \right\} \times$$
$$\times \exp\left\{ \frac{2m\alpha}{\sqrt{\alpha}\,\mu_1} \int_{-\frac{\sqrt{\alpha}\nu_1}{2} + \frac{\sqrt{\alpha}}{2}\frac{\mu_1 t_1}{m}}^{-\frac{\sqrt{\alpha}}{2}\nu_1 + \frac{\sqrt{\alpha}}{2}\frac{\mu_1(t_1+t_2)}{m}} e^{-\tau^2} d\tau \right\} = F(\alpha,\mu_1,\nu_1,t_1+t_2)$$

Finally:

$$\Sigma_{t_1}[\Sigma_{t_2}\hat{\rho}] = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, \hat{F}(\alpha,y,x,t_1+t_2)\, W^{\dagger}(x,y)\, U_{t_1+t_2}\,\hat{\rho}\, U^{\dagger}_{t_1+t_2} W(x,y) = \Sigma_{t_1+t_2}\hat{\rho}$$

We want to stress here the following fact.

The above property is valid for all $t\in\mathbb{R}$, not only for positive times.

Nonetheless $\{\Sigma_t\}$ is a semigroup and not a group.

Infact the modified dynamics does not remain unaltered by time-reversing:

$$\frac{d\Sigma_t\hat{\rho}}{dt} = -\frac{i}{\hbar}[H,\Sigma_t\hat{\rho}] + \alpha\,\Sigma_t\hat{\rho} - \alpha\, T[\Sigma_t\hat{\rho}] \quad \text{is the Q.M.S.L. for } t\to -t.$$

The semigroup property and the consequent lack of reversibility comes in from the very beginning, that is in the construction of the dynamical model.

The formal expression in <A.4.5> allows an easy derivation of the semigropup dual to $\{\Sigma_t\}_{t\geq 0}$.

Indeed:

$$\mathrm{Tr}\,[\hat{X}\,\Sigma_t\hat{\rho}] = \mathrm{Tr}\,[\Sigma_t^{*}\hat{X}\cdot\hat{\rho}] \qquad \hat{X}\in B(H)^h \ , \ \hat{\rho}\in B(H)^h_1$$

and :

$$\Sigma_t^{*}\hat{X} = \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy\, \hat{F}(\alpha,y,x,t)\, U_t^{\dagger}\, W(x,y)\,\hat{X}\, W^{\dagger}(x,y)\, U_t$$

We want to use the expression:

$$\Sigma_t^{*}\hat{X} = \frac{1}{2\pi\hbar}\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\ \hat{F}(\alpha,y,x,t)\ U_t^{\dagger}\,W(x,y)\,\hat{X}\,W^{\dagger}(x,y)\,U_t \qquad \text{<A.4.11>}$$

in order to derive some interesting results and to compare them with those obtained in [1].

In the following some well known relations will be used:

$$\hat{q}\,\exp\left\{\frac{i\hat{p}y}{\hbar}\right\} = \exp\left\{\frac{i\hat{p}y}{\hbar}\right\}[\hat{q}-\hat{\mathbb{1}}y] \qquad \text{<A.4.12>}$$

$$\hat{q}^2\exp\left\{\frac{i\hat{p}y}{\hbar}\right\} = \exp\left\{\frac{i\hat{p}y}{\hbar}\right\}[\hat{q}^2-2y\hat{q}+y^2\hat{\mathbb{1}}] \qquad \text{<A.4.13>}$$

$$\hat{p}\,\exp\left\{\frac{i\hat{q}x}{\hbar}\right\} = \exp\left\{\frac{i\hat{q}x}{\hbar}\right\}[\hat{p}+\hat{\mathbb{1}}x] \qquad \text{<A.4.14>}$$

$$\hat{p}^2\exp\left\{\frac{i\hat{q}x}{\hbar}\right\} = \exp\left\{\frac{i\hat{q}x}{\hbar}\right\}[\hat{p}^2+2x\hat{p}+x^2\hat{\mathbb{1}}] \qquad \text{<A.4.15>}$$

$$\exp\left\{\frac{i\hat{p}y}{\hbar}\right\}\exp\left\{\frac{i\hat{q}x}{\hbar}\right\}\exp\left\{-\frac{i\hat{p}y}{\hbar}\right\} = \exp\left\{\frac{i\hat{q}x}{\hbar}+\frac{ixy}{\hbar}\right\} \qquad \text{<A.4.16>}$$

Writing: $\quad W(x,y) = e^{-\frac{i}{\hbar}(\hat{q}x+\hat{p}y)} = e^{-\frac{i}{\hbar}\hat{p}y}\,e^{-\frac{i}{\hbar}\hat{q}x}\,e^{-\frac{i}{\hbar}xy}$

we will evaluate:

$$U_t^{\dagger}\exp\left\{-\frac{i}{\hbar}\hat{p}y\right\}\exp\left\{-\frac{i}{\hbar}\hat{q}x\right\}\hat{X}\exp\left\{+\frac{i}{\hbar}\hat{q}x\right\}\exp\left\{\frac{i}{\hbar}\hat{p}y\right\} \qquad \text{<A.4.17>}$$

in the cases:

$\hat{X} = \hat{q}$ <A.4.18> $\qquad \hat{X} = \hat{p}$ <A.4.19>

$\hat{X} = \hat{q}^2$ <A.4.20> $\qquad \hat{X} = \hat{p}^2$ <A.4.21> $\qquad \hat{X} = \hat{q}\hat{p}$ <A.4.22>

These are not bounded operators of course, nonetheless it is significant to study how they evolve in time due to the fact that we can always study how their mean values evolve in an opportune state.

$$U_t^{\dagger}\, e^{-i\hat{p}\gamma/\hbar}\, e^{-i\hat{q}x/\hbar}\, \hat{q}\, e^{i\hat{q}x/\hbar}\, e^{+i\hat{p}\gamma/\hbar}\, U_t = \hat{q}_t - \gamma\hat{\mathbb{1}}$$ <A.4.18>

by<A.4.12>

($\hat{x}_t = U_t^{\dagger}\hat{x}\,U_t$:that is the Heisenberg evolved of $\hat{x}$)

$$U_t^{\dagger}\, e^{-i\hat{p}\gamma/\hbar}\, e^{-i\hat{q}x/\hbar}\, \hat{p}\, e^{i\hat{q}x/\hbar}\, e^{i\hat{p}\gamma/\hbar}\, U_t = \hat{p}_t + x\hat{\mathbb{1}}$$ <A.4.19>

by<A.4.14>

$$U_t^{\dagger}\, e^{-i\hat{p}\gamma/\hbar}\, e^{-i\hat{q}x/\hbar}\, \hat{q}^2\, e^{i\hat{q}x/\hbar}\, e^{i\hat{p}\gamma/\hbar}\, U_t = \hat{q}_t^2 - 2\gamma\hat{q}_t + \gamma^2\hat{\mathbb{1}}$$ <A.4.20>

by<A.4.13>

$$U_t^{\dagger}\, e^{-i\hat{p}\gamma/\hbar}\, e^{-i\hat{q}x/\hbar}\, \hat{p}^2\, e^{i\hat{q}x/\hbar}\, e^{i\hat{p}\gamma/\hbar}\, U_t = \hat{p}_t^2 + 2x\hat{p}_t + x^2\hat{\mathbb{1}}$$ <A.4.21>

by<A.4.15>

$$U_t^{\dagger}\, e^{-i\hat{p}\gamma/\hbar}\, e^{-i\hat{q}x/\hbar}\, \hat{q}\hat{p}\, e^{i\hat{q}x/\hbar}\, e^{i\hat{p}\gamma/\hbar}\, U_t = \hat{q}_t\hat{p}_t - \gamma\hat{p}_t + x\hat{q}_t - x\gamma\hat{\mathbb{1}}$$ <A.4.22>

Now:

$$\Sigma_t^{*}[\hat{q}] = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} d\gamma\, \frac{\hat{F}(\alpha,\gamma,x,t)}{2\pi\hbar}\, [\hat{q}_t - \hat{\mathbb{1}}\gamma]$$ <A.4.23>

$$\Sigma_t^{*}[\hat{p}] = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} d\gamma\, \frac{\hat{F}(\alpha,\gamma,x,t)}{2\pi\hbar}\, [\hat{p}_t + \hat{\mathbb{1}}x]$$ <A.4.24>

$$\Sigma_t^{*}[\hat{q}^2] = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} d\gamma\, \frac{\hat{F}(\alpha,\gamma,x,t)}{2\pi\hbar}\, [\hat{q}_t^2 - 2\gamma\hat{q}_t + \gamma^2\hat{\mathbb{1}}]$$ <A.4.25>

$$\Sigma_t^{*}[\hat{p}^2] = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} d\gamma\, \frac{\hat{F}(\alpha,\gamma,x,t)}{2\pi\hbar}\, [\hat{p}_t^2 + 2x\hat{p}_t + x^2\hat{\mathbb{1}}]$$ <A.4.26>

$$\Sigma_t^{*}[\hat{q}\hat{p}] = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} d\gamma\, \frac{\hat{F}(\alpha,\gamma,x,t)}{2\pi\hbar}\, [\hat{q}_t\hat{p}_t - \gamma\hat{p}_t + x\hat{q}_t - x\gamma\hat{\mathbb{1}}]$$ <A.4.27>

We shall use the following properties of the function $F(\alpha,\mu,\nu,t)$ [1]:

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} d\gamma\, \frac{\hat{F}(\alpha,\gamma,x,t)}{2\pi\hbar} = F(\alpha,0,0,t) = 1$$ <A.4.28>

$$\partial_\mu F(d,\mu,\nu,t)\Big|_{\mu=\nu=0} = 0 \qquad \text{<A.4.29>}$$

$$\partial_\nu F(d,\mu,\nu,t)\Big|_{\mu=\nu=0} = 0 \qquad \text{<A.4.30>}$$

$$\partial^2_\mu F(d,\mu,\nu,t)\Big|_{\mu=\nu=0} = -\frac{\alpha d}{6m^2}\,t^3 \qquad \text{<A.4.31>}$$

$$\partial^2_\nu F(d,\mu,\nu,t)\Big|_{\mu=\nu=0} = -\frac{\alpha d}{4m}\,t^2 \qquad \text{<A.4.32>}$$

$$\partial^2_{\mu\nu} F(d,\mu,\nu,t)\Big|_{\mu=\nu=0} = \frac{\alpha d}{2}\,t \qquad \text{<A.4.33>}$$

Thus:

$$\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\; y\,\frac{\hat{F}(d,y,x,t)}{2\pi\hbar} = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}d\nu\int_{-\infty}^{+\infty}d\mu\;\frac{e^{-i\nu x/\hbar}}{2\pi\hbar}\,i\hbar\,F(d,\mu,\nu,t)\,\partial_\mu\left[\frac{e^{-i\mu y/\hbar}}{2\pi\hbar}\right]$$

$$= i\hbar\int_{-\infty}^{+\infty}d\mu\;\partial_\mu[\delta(\mu)]\,F(d,\mu,0,t)$$

$$= -i\hbar\,\partial_\mu F(d,0,0,t) = 0 \qquad \text{<A.4.34>}$$

by <A.4.29>

$$\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\; y^2\,\frac{\hat{F}(d,y,x,t)}{2\pi\hbar} = \int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\int_{-\infty}^{+\infty}d\nu\int_{-\infty}^{+\infty}d\mu\;\frac{e^{-i\nu x/\hbar}}{2\pi\hbar}\,F(d,\mu,\nu,t)\left\{-\hbar^2\partial^2_\mu\left[\frac{e^{-i\mu y/\hbar}}{2\pi\hbar}\right]\right\}$$

$$= -\hbar^2\int_{-\infty}^{+\infty}d\mu\;\partial^2_\mu[\delta(\mu)]\,F(d,\mu,0,t)$$

$$= -\hbar^2\,\partial^2_\mu F(d,0,0,t) = \frac{\hbar^2\alpha d}{6m^2}\,t^3 \qquad \text{<A.4.35>}$$

by <A.4.31>

By the same kind of calculations:

$$\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\; x\,\hat{F}(d,y,x,t) = 0 \qquad \text{<A.4.36>}$$

by<A.4.30>

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; x^2 \, \frac{\hat{F}(\alpha, y, x, t)}{2\pi\hbar} = \frac{\hbar^2 \alpha d}{4m} t^2$$ <A.4.37>

by <A.4.32>

$$\int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \; xy \frac{\hat{F}(\alpha, y, x, t)}{2\pi\hbar} = -\hbar^2 \int_{-\infty}^{+\infty} d\mu \int_{-\infty}^{+\infty} d\nu \; [\partial_\mu \delta(\mu)] \, [\partial_\nu \delta(\nu)] \, F(\alpha, \mu, \nu, t)$$

$$= -\hbar^2 \partial^2_{\mu\nu} F(\alpha, 0, 0, t) = -\frac{\alpha d \hbar^2}{2} t$$ <A.4.38>

by <A.4.33>.

All these manipulations with Dirac deltas and its derivatives are justified in the sense of tempered distributions by the properties of the function $F(\alpha, \mu, \nu, t)$.

Hence:

<A.4.23> = $\Sigma_t^* [\hat{q}] = \hat{q}_t$

<A.4.24> = $\Sigma_t^* [\hat{p}] = \hat{p}_t$

<A.4.25> = $\Sigma_t^* [\hat{q}^2] = \hat{q}_t^2 + \frac{\alpha d \hbar^2}{6 m^2} t^3 \hat{\mathbb{1}}$

<A.4.26> = $\Sigma_t^* [\hat{p}^2] = \hat{p}_t^2 + \frac{\alpha d \hbar^2}{4m} t^2 \hat{\mathbb{1}}$

<A.4.27> = $\Sigma_t^* [\hat{q}\hat{p}] = \hat{q}_t \hat{p}_t + \frac{\alpha d \hbar^2}{2} \hat{\mathbb{1}}$

Computing the mean values of these evolved unbounded operators in an opportune state $\hat{\rho}$ we obtain exactly the results in [1].

More rigorously we can study how the Weyl Algebra evolves under the semigroup $\{\Sigma_t\}_{t \geq 0}$.

We evaluate:

$$\Sigma_t^*\left[e^{i\hat{q}s/\hbar}\right] \quad , \quad s \in \mathbb{R}$$

<A.4.39>

$$\Sigma_t^*\left[e^{i\hat{p}r/\hbar}\right] \quad , \quad r \in \mathbb{R}$$

<A.4.40>

Using:

$$U_t^\dagger e^{-i\hat{p}y/\hbar} e^{-i\hat{q}x/\hbar} e^{i\hat{q}s/\hbar} e^{i\hat{q}x/\hbar} e^{+i\hat{p}y/\hbar} U_t = e^{-iys/\hbar} e^{i\hat{q}_t s/\hbar}$$

<A.4.41>

by<A.4.16>

$$U_t^\dagger e^{-i\hat{p}y/\hbar} e^{-i\hat{q}x/\hbar} e^{i\hat{p}r/\hbar} e^{+i\hat{q}x/\hbar} e^{i\hat{p}y/\hbar} U_t = e^{ixr/\hbar} e^{i\hat{p}r/\hbar}$$

<A.4.42>

by<A.4.16>

we obtain:

$$\langle A.4.39\rangle = \Sigma_t^*\left[e^{i\hat{q}s/\hbar}\right] = e^{i\hat{q}_t s/\hbar}\left\{\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\, \frac{e^{-iys/\hbar}}{2\pi\hbar}\, \hat{F}(d,y,x,t)\right\}$$

$$= e^{i\hat{q}_t s/\hbar}\left\{\int_{-\infty}^{+\infty}d\mu\, \delta(\mu+s)\int_{-\infty}^{+\infty}d\nu\, \delta(\nu)\, F(d,\mu,\nu,t)\right\}$$

$$= F(d,-s,0,t)\, e^{i\hat{q}_t s/\hbar} = F(d,s,0,t)\, e^{i\hat{q}_t s/\hbar}$$

$$\langle A.4.40\rangle = \Sigma_t^*\left[e^{i\hat{p}r/\hbar}\right] = e^{i\hat{p}r/\hbar}\left\{\int_{-\infty}^{+\infty}dx\int_{-\infty}^{+\infty}dy\, \frac{e^{ixr/\hbar}}{2\pi\hbar}\, \hat{F}(d,y,x,t)\right\}$$

$$= e^{i\hat{p}r/\hbar}\, F(d,0,r,t)$$

Since:

$$\Sigma_t^*[e^{i\hat{p}\pi/\hbar}] \cdot \Sigma_t^*[e^{i\hat{q}s/\hbar}] \cdot \Sigma_t^*[e^{-i\hat{p}\pi/\hbar}] =$$

$$= F(d,0,\pi,t)\, F(d,0,-\pi,t)\, F(d,s,0,t) \quad e^{i\hat{p}\pi/\hbar} \quad e^{i\hat{q}_t s/\hbar} \quad e^{-i\hat{p}\pi/\hbar}$$

$$= [F(d,0,\pi,t)]^2 \; F(d,s,0,t) \quad e^{is\pi/\hbar} \quad e^{i\hat{q}_t s/\hbar}$$

and:

$$\Sigma_t^*[e^{i(\hat{q}+\pi)s/\hbar}] = F(d,s,0,t)\; e^{is\pi/\hbar} \; e^{i\hat{q}_t s/\hbar}$$

with:

$$F(d,0,\pi,t) \neq 1 \qquad \text{if} \quad \pi \neq 0$$

we see that the Weyl relations are not preserved in time by the map Σ_t^* which is thus not an autom-orphism of the algebra.

But it is not surprising since if it were such an automorphism, we would have:

$$\frac{d\, \Sigma_t^*[\hat{A}\hat{B}]}{dt} = \frac{d\, \Sigma_t^*[\hat{A}]}{dt}\, \Sigma_t^*[\hat{B}] + \Sigma_t^*[\hat{A}]\, \frac{d\, \Sigma_t^*[\hat{B}]}{dt}$$

That is: $\quad L^*[\hat{A}\hat{B}] = L^*[\hat{A}]\,\hat{B} + \hat{A}\, L^*[\hat{B}]$

where $\quad \hat{A}, \hat{B} \in D(L^*)$

This is forbidden by the dissipative part L_d^* of L^* :

$$L_d^*[\hat{A}\hat{B}] = -d\hat{A}\hat{B} + d\sum_m \hat{A}_m[\hat{A}\hat{B}]\hat{A}_m$$

$$L_d^*[\hat{A}]\cdot\hat{B} + \hat{A}\, L_d^*[\hat{B}] = -2d\hat{A}\hat{B} + d\sum_m \hat{A}_m\hat{A}\hat{A}_m\hat{B} + d\sum_m \hat{A}\hat{A}_m\hat{B}\hat{A}_m =$$

$$= -d\hat{A}\hat{B} + d\sum_m \hat{A}_m\hat{A}\hat{B}\hat{A}_m + d\sum_m \hat{A}_m\hat{A}[\hat{A}_m,\hat{B}]$$

$$-d\hat{A}\hat{B} + d\sum_m \hat{A}\hat{A}_m\hat{A}_m\hat{B} + d\sum_m \hat{A}\hat{A}_m[\hat{B},\hat{A}_m]$$

$$= L_d^*[\hat{A}\hat{B}] + d\sum_m [\hat{A},\hat{A}_m][\hat{B},\hat{B}_m]$$

APPENDIX A.5 [22], [24]

We want to present here a proof of theorem 2.2.9 which is an adaptation of that given in [22] for $\mathcal{L}_p$ spaces.

THEOREM 2.3.4

If T is a linear map, defined on the finite range operators, such that:

$$\|T\hat{A}\|_{p_1} \leq M_1 \|\hat{A}\|_{q_1} \qquad \text{<i>}$$

$$\|T\hat{A}\|_{p_2} \leq M_2 \|\hat{A}\|_{q_2} \qquad \text{<ii>}$$

with $\hat{A} \in C_0(H)$

then, for any $t \in [0,1]$ and p, q satisfying:

$$\frac{1}{p} = \frac{1-t}{p_1} + \frac{t}{p_2} \qquad \text{<iii>}$$

$$\frac{1}{q} = \frac{1-t}{q_1} + \frac{1}{q_2} \qquad \text{<iv>}$$

we have: $\|T\hat{A}\|_p \leq M_1^{1-t} M_2^{t} \|\hat{A}\|_q$ <v>

and by continuity T extends to the whole of $B(H)_q$ with the inequality <v> preserved.

proof

We introduce the following complex quantities:

$$\begin{cases} \dfrac{1}{p(z)} = \dfrac{1-z}{p_1} + \dfrac{z}{p_2} \\[2ex] \dfrac{1}{q(z)} = \dfrac{1-z}{q_1} + \dfrac{z}{q_2} \end{cases} \qquad z \in \mathbb{C}$$

A general operator in $C_0(H)$ has the form: $\hat{F} = \sum_{1}^{N} {}_j \, \alpha_j \, |\psi_j\rangle\langle\phi_j|$

From the Hoelder's inequality we know that: $\|\hat{A}\hat{B}\|_1 \leq \|\hat{A}\|_2 \|\hat{B}\|_S$

$\frac{1}{r} + \frac{1}{s} = 1$, $\hat{A} \in B(H)_r$, $\hat{B} \in B(H)_s$.

This is certainly true for $\hat{A}$ and $\hat{B}$ in $C_0(H)$, since it is dense in any $B(H)_p$.

By THEOREM A.1.15 it follows:

$$\|\hat{A}\|_r = \sup_{\hat{B}\in C_0} \frac{|Tr\{\hat{A}\hat{B}\}|}{\|\hat{B}\|_s} \quad , \quad \frac{1}{r}+\frac{1}{s}=1$$

We choose a value for t in [0,1], thus fixing 1/p and 1/q too.

Let us suppose p>1 and q>0 .

Setting $p^*=1/(1-1/p)$, we consider:

$$\|T\hat{F}\|_p = \sup_{\|\hat{E}\|_{p^*}=1} |Tr\{T\hat{F}\cdot\hat{E}\}|$$

$$\hat{F} = \sum_1^N {}_j\, |\mu_j|\, e^{iu_j}\, |\psi_j\rangle\langle\phi_j| \quad ; \quad \hat{E} = \sum_1^M {}_k\, |d_k|\, e^{iv_k}\, |\pi_k\rangle\langle\gamma_k|$$

$\{|\psi_j\rangle\}_{1\le j\le N}$, $\{|\phi_j\rangle\}_{1\le j\le N}$, $\{|\pi_k\rangle\}_{1\le k\le M}$, $\{|\gamma_k\rangle\}_{1\le k\le M}$ belonging to H.

We can always choose $\hat{F}$ and $\hat{E}$ such that:

$$\|\hat{F}\|_q = 1 \qquad \text{<iii>}$$

and

$$\|\hat{E}\|_{p^*} = 1 \qquad \text{<iv>}$$

We set : $\phi(t) = Tr\{T\hat{F}\cdot\hat{E}\}$

Let us introduce:

$$\hat{F}_z = \sum_1^N {}_j\, |\mu_j|^{q/q(z)}\, e^{iu_j}\, |\psi_j\rangle\langle\phi_j|$$

$$\hat{E}_z = \sum_1^M {}_k\, |d_k|^{p^*(1-\frac{1}{p(z)})}\, e^{iv_k}\, |\pi_k\rangle\langle\gamma_k|$$

$$\phi(z) = Tr\{T\hat{F}_z\cdot\hat{E}_z\}$$

Note that:

$$\mathrm{Re}\left\{\frac{q}{q(z)}\right\} = q\ \mathrm{Re}\left[\frac{1-iy}{q_1} + \frac{iy}{q_2}\right] = \frac{q}{q_1} \quad : \quad z = iy$$

$$\mathrm{Re}\left\{p^*\left(1-\frac{1}{p(z)}\right)\right\} = p^*\ \mathrm{Re}\left[1 - \frac{1-iy}{p_1} - \frac{iy}{p_2}\right] = \frac{p^*}{p_1^*}$$

$$\mathrm{Re}\left\{\frac{q}{q(z)}\right\} = q\ \mathrm{Re}\left[\frac{1-1-iy}{q_1} + \frac{1+iy}{q_2}\right] = \frac{q}{q_2} \quad : \quad z = 1+iy$$

$$\mathrm{Re}\left\{p^*\left(1-\frac{1}{p(z)}\right)\right\} = p^*\ \mathrm{Re}\left[1 - \frac{1-1-iy}{p_1} - \frac{1+iy}{p_2}\right] = \frac{p^*}{p_2^*}$$

and that

$$\phi(z) = \mathrm{Tr}\left\{T\hat{F}_z \cdot \hat{E}_z\right\} = \sum_{1}^{N}{}_j \sum_{1}^{M}{}_k |\mu_j|^{q/q(z)} |d_k|^{p^*\left(1-\frac{1}{p(z)}\right)} e^{i(u_j+v_k)} \mathrm{Tr}\left\{\left[T[\psi_j\rangle\langle\phi_j|\right] |\pi_k\rangle\langle\gamma_k|\right\}$$

is an holomorphic function of z, due to the fact that $|\mu_j|$ and $|d_k|$ are positive numbers and:

$$\frac{1}{q(z)} = \frac{1-z}{q_1} + \frac{z}{q_2}$$

$$1 - \frac{1}{p(z)} = 1 - \frac{1-z}{p_1} - \frac{z}{p_2}$$

Using Hoelder's inequality and the properties <i>, <ii> we obtain:

$$|\phi(iy)| \le \|T\hat{F}_{iy}\|_{p_1} \|\hat{E}_{iy}\|_{p_1^*} \le M_1 \|\hat{F}_{iy}\|_{q_1} \|\hat{E}_{iy}\|_{p_1^*} \qquad \text{<A.5.1>}$$

$$|\phi(1+iy)| \le \|T\hat{F}_{1+iy}\|_{p_2} \|\hat{E}_{1+iy}\|_{p_2^*} \le M_2 \|\hat{F}_{1+iy}\|_{q_2} \|\hat{E}_{1+iy}\|_{p_2^*} \qquad \text{<A.5.2>}$$

Now by <iii> and <iv> we have:

$$\| \hat{F}_{i\gamma} \|_{q_1}^{q_1} = \sum_{1}^{N}{}_{j} \left| |\mu_j|^{\frac{q}{q(i\gamma)}} \right|^{q_1} = \sum_{1}^{N}{}_{j} |\mu_j|^{q} = \| \hat{F} \|_q^q = 1$$

$$\| \hat{E}_{i\gamma} \|_{p_1^*}^{p_1^*} = \sum_{1}^{M}{}_{k} \left| |d_k|^{p^*\left(1-\frac{1}{p(i\gamma)}\right)} \right|^{p_1^*} = \sum_{1}^{M}{}_{k} |d_k|^{p^*} = \| \hat{E} \|_{p^*}^{p^*} = 1$$

$$\| \hat{F}_{1+i\gamma} \|_{q_2}^{q_2} = \sum_{1}^{N}{}_{j} \left| |\mu_j|^{\frac{q}{q(1+i\gamma)}} \right|^{q_2} = \sum_{1}^{N}{}_{j} |\mu_j|^{q} = \| \hat{F} \|_q^q = 1$$

$$\| \hat{E}_{1+i\gamma} \|_{p_2^*}^{p_2^*} = \sum_{1}^{M}{}_{k} \left| |d_k|^{p^*\left(1-\frac{1}{p(1+i\gamma)}\right)} \right|^{p_2^*} = \sum_{1}^{M}{}_{k} |d_k|^{p^*} = \| \hat{E} \|_{p^*}^{p^*} = 1$$

Hence <A.5.1> and <A.5.2> become:

$$|\phi(i\gamma)| \le M_1 \quad ; \quad |\phi(1+i\gamma)| \le M_2$$

In order to conclude the argument we need the following:

LEMMA A.5.1

Suppose that $f(z)$, continuous and bounded in the strip S and regular in the interior of S , satisfies the conditions:

$$|f(\alpha+i\gamma)| \le M_1 \quad ; \quad |f(\beta+i\gamma)| \le M_2 \quad \forall \gamma$$

Then , if L(t) is a linear function taking the values 1 and o for t=a and t=b respectively, we have: $|f(x_0+iy_0)| \le M_1^{L(x_0)} M_2^{1-L(x_0)} \quad \forall y_0$

end

In the present case we choose L(t)=1-t.

Thus:

$$|\phi(t+iy_0)| \leq M_1^{1-t} M_2^{t} \qquad \forall y_0$$

Hence:

$$\|\hat{F}\|_p = \sup_{\|\hat{E}\|_{p^*}=1} |\phi(t)| \leq M_1^{1-t} M_2^{t}$$

Choosing: $$\hat{F} = \hat{G} / \|\hat{G}\|_q$$

we arrive at the desired result:

$$\|T\hat{G}\|_p \leq M_1^{1-t} M_2^{t} \|\hat{G}\|_q$$

The extension to all of $B(H)_q$ is possible because of the continuity of T on $C_0(H)$ and the density of it in any $B(H)_q$.

end

REFERENCES

[1] Ghirardi G.C., Rimini A., Weber T.:Phys.Rew.D 34, 470(1986)

[2] Davies E.B.:Quantum Theory of Open Systems(Academic Press, 1976)

[3] Benatti F., Ghirardi G.C., Rimini A., Weber T.:Quantum Mechanics with Spontaneous Localization and the Quantum Theory of Measurement, to be published

[4] D'Espagnat B.:Conceptual Foundations of Quantum Mechanics(Benjamin, Reading, Mass., 1976)

[5] Ghirardi G.C., Miglietta F., Rimini A., Weber T.:Phys.Rew.D 24, 347(1981)

[6] Hellwig K.E., Kraus K.:Comm.Math.Phys.11,214(1969)

[7] Hellwig K.E., Kraus K.:Comm.Math.Phys.16,142(1970)

[8] Kraus K.:Ann.Phys.(N.Y.), 311(1971)

[9] Kraus K.:Lecture Notes in Physics 190(Springer, Berlin, 1983)

[10] Haag R., Kastler D.:J.Math.Phys.5, 848(1964)

[11] Bratteli O., Robinson D.W.:Operator Algebras and Quantum Statistical Mechanics I(Springer, N.Y., 1979)

[12] Lindblad G.:Comm.Math.Phys.48,119(1976)

[13] Takesaki M.:Theory of Operator Algebras I(Springer, N.Y., 1979)

[14] Gorini V., Kossakowski A. and Sudarshan E.C.G.:J.Math.Phys.17, 821(1976)

[15] Gorini V., Frigerio A., Verri M., Kossakowski A. and Sudarshan E.C.G.: Rep.Math.Phys.13, 149(1978)

[16] Gorini V., Frigerio A.:J.Math.Phys.17,2123(1976)

[17] Gorini V., Kossakowski A.:J.Math.Phys.17, 1298(1976)

[18] Dixmier J.:Les Algebres d'Operateurs dans l'Espace Hilbertienne(Gauthier-Villars, Paris, 1957)

[19] Schatten R.:Norm Ideals of Completely Continuous Operators(Springer, Berlin, 1960)

[20] Dunford N., Schwartz J.T.:Linear Operators II(Interscience Publishers, 1963)

[21] Yosida K.:Functional Analysis(Springer, Berlin, 1971, 3-rd ed.)

[22] Calderon A.P., Zygmund A.:Contributions to Fourier Analysis, Ann. of Math. Studies, 166, No 25(Princeton University Press, 1950)

[23] Lumer G., Phillips R.S.:Pac.J.Math.11, 679(1961)

[24] Kunze K.A.:Trans.Amer.Math.Soc.89, 519(1958)

[25] Thirring W.:A Course in Mathematical Physics IV(Springer, N.Y., 1979-81)

[26] Reed M., Simon B.:Fourier Analysis-Self Adjointness(Academic Press, N.Y.)

[27] Reed M., Simon B.:Functional Analysis(Academid Press, N.Y.)

[28] Wehrl A.:Rep.Math.Phys.10, 159(1976)

[29] Lindblad G.:Comm.Math.Phys.33, 305(1973)