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THEORY OF THIN ACCRETION DISCS

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I. INTRODUCTION

Accretion is the process by which a gravitating body accumulates matter from its surrounding. If the matter accreted has significant angular momentun a disc around the gravitating body will be formed. The viscosity present in the disc gas may convert gravitational potential energy into radiation in an efficient manner. Thus, accretion through a disc may be a powerful energy source in astrophysics.

The accretion discs around some of the planets are easily seen, even schoolchildren know Saturn's beautiful rings. The another, and particularly important, case in which the accretion discs can be actually observed is in some binary star systems such as the cataclysmic variables (Robinson,1976). For example, there is strong evidence for the existence of the accretion discs in some U-Gem systems (Smak 1971, Glasby 1970 and Bath 1972). Accretion discs may also play a major role in quasars, galactic nuclei, binary X-ray sources, star formations, solar system formation, and the like (White et al. 1983). (Recently, Malkan (1983) presented observational evidence for the massive accretion discs in some luminous quasars.) As a specially interesting subject, the only source confirming the existence of black holes is by means of the study of accretion discs. Therefore, the study of disc structure as well as disc radiation is obviously of great importance.

The study of dynamics of rotating gas masses started at as far back as an early stage in the consideration of the formation and evolution of the solar nebula with particular regard to Laplace's nebular hypothesis. The general properties of the evolution of rotating gas discs was well understood by the 1920s (Jeffreys 1924). In 1940s, Peek(1942) and von Weizsäcker

(1943) pointed out the importance of the effect of turbulent viscosity on the evolution of the solar nebula. von Weizsäcker (1948) derived the equations of motion of disc gas. The general solutions for time-dependent, viscous discs were given by Lüst (1952) and Lynden-Bell(1960).

In the 1960s interest switched to the radiation the rotating gas might emit owing to the discoveries of X-ray stars and quasars. Salpeter(1964) and Zel'dovich(1964) were the first to suggest that quasi-stellar objects could be massive black holes accreting the interstellar medium. Hayakawa and Matsuoko (1964) proposed firstly that X-rays might be produced by accretion of gas between two normal stars in close binary systems. Zel'dovich and Guseynov (1966), Novikov and Zel'dovich (1966) pointsd out that gas accretion onto compact components in binary systems should produce X-rays and thus could be the explanation of newly discoved X-ray sources. Shklovsky (1967) made the explanation of Sco X-1 as accretion onto a neutron star. Cameron and Mock (1967) considered that gas accretion onto white dwarfs could also produce X-rays. (The history of these early ideas was reviewed by Burbidge 1972.) The essential role of the angular momentum of the gas in binary accretion was first emphasized by Prendergast (Prendergast and Burbidge 1968). He built a model for disc-type accretion onto white dwarfs in binary systems. The similarity between cataclysmic variables and binary X-ray sources became apparent and in fact from then on the theory of accretion discs has been finding its main applications in both fields. On the larger scale, Lynden-Bell (1969) argued that galaxies might have supermassive black holes at their centres and analyzed disc -type accretion onto such holes.

At the biginning of the seventies the theory was established in a more complete form by several authors. Shvartsman (1971) studied discs around isolated stars. Lynden-Bell and Rees (1971) extended the work on supermassive holes hidden at galaxies' centres. Pringle and Rees (1972) made the application of the theory to the X-ray source Cyg X-1 which supposely contains a black hole. Shakura (1972), Shakura and Sunyaev (1973) wrote out the formulations which are now followed by most authors in the field. Lynden-Bell and Pringle (1974) gave also an elegant description of the problem. Lightman (1974) constructed a time-depedent model of accretion discs. The Roche-lobe viscous accretion -- one of the two basic accretion types -- was studied by Prendergast and Taam (1974), while Illarionov and Sunyaev (1975) and Shapiro and Lightman (1976) considered the another basic type -- stellar wind accretion. All of these models were Newtorian. The general relativistic calculations were done by Thorne (1973), Cunningham (1973), Novikov and Thorne (1973) and Page and Thorne (1974).

From then on the field has extented in many ramifications, including the hot disc model (Thorne and. Price 1975), the stability problem (Pringle, Rees and Pacholczyk 1973, Lightman and Eardley 1974, Shakura and Sunyaev 1976), the existence of thick discs (Paczynski and Wiita 1980, Abramowicz, Calvani and Nobili 1980), the relativistic effects on disc structure and spectrum (Bardeen and Petterson 1975, Cunningham 1975), and etc. The rate of publication of papers has become so high that an accurate review is very hard.

In this thesis I shall firstly give a brief description and discussion of so-called classical model of thin accretion discs which was recently reviewed by Pringle (1981), Treves (1981) and Verbunt (1982), and then I would like to propose a new possible method for studying thin accretion discs.

II. CLASSICAL MODEL OF THIN ACCRETION DISCS

1) General Picture

If a particle moves in a bound orbit around a central gravitating body (refer to Misner, Thorne and Wheeler 1973 and Bardeen 1973), and if there is a process which can cause the particle to lose its energy but retain its angular momentum, the particle's orbit will be changed gradually until to be a circular one, since a circular orbit is that of least energy for a given angular momentum. futher, if there is a process that can extract angular momentum as well as energy from the particle, it will spiral inwards, since the smaller particle's angular momentum is, the smaller radius of orbit of least energy is. Theoretically, the total amuont of energy that can be extracted by such a process is equal to the bindnig energy of the innermost stable circular orbit. for a normal star, the theoretical innermost stable circular orbit is in its deep interior. Therefore, a particle spiraling inward towards a normal star will impact star's surface far before its reaching the last stable circular orbit. But, for a sufficiently compact object, a particle surrounded will possibly approach or eventually reach the last circular orbit and thus release a considerable fraction of its rest mass energy during the journey of falling. For example, of order 10 percent of the rest mass energy can be obtained from orbits around a neutron star and up to 40 percent for orbits aruond a black hole. By comparing the fact that only less than 1 percent of rest mass energy can be released by means of nuclear reactions, it is noticeable that the accretion can be an extremely efficient mechanism of energy production. The problem now is to set up the process that can extract the energy and angular momentum.

Turn attention from the theoretical analysis of particle's motion to the practical phenomena in astrophysics. It is well known that more than 50 percent of stars are in binary systems (Batten 1973) and that about half of binary systems are close binaries (Paczynski 1971). Mass exchanges between components of close binary systems are quite common (Trimble 1983). There are two means by which a star in a close binary system can lose mass onto its companion. The stellar wind -- the outflow of matter from a star's surface in all directions -- is one of the main properties of stars (Cassinelli 1979). The rate of mass loss by stellar wind depends upon the type of star and varies from $\sim 10^{-14} M_{\odot}$/year for the Sun up to $10^{-5} M_{\odot}$/year for the Wolf-Rayet stars, MI supergiants and O-stars of the main sequence (Pottasch 1970). If the surface of the mass-losing star is inside its Roche lobe, then the only way it can dump gas onto its companion is by means of a stellar wind and only that gas blown toward the companion can be captured. But, if the surface of the mass-losing star fills its Roche lobe (Kopal 1972) -- at a definite stage of evolution, for example after leaving the main sequence, the star begins to increase in size and eventually fills its Roche volume -- then there will be an additional intensive outflow of matter from its surface, mostly through the inner Lagrangian point, toward the companion which will be called hereafter as mass-accreting star and might be in practice a black hole or a neutron star or a white dwarf or even a main-sequence star (Fig.1). Combine now the phenomena of mass exchange happening in close binary systems with the theory of particle's motion described above. Some fraction of the matter flowing out from the mass-losing star must fall into the sphere of influence of the gravitational field of the mass-accreting star and thus is captured by the latter. If the mass loss is associated only with stellar wind, the captured matter has small specific angular momentum and thus will fall onto the mass-accreting star in a narrow cone or in the form of spherically symmetric accretion which will not be discussed in this thesis (refer to Sunyaev 1978 and Lightman, Shapiro and Rees 1978) (Fig.2). But, in the case of Roche-lobe overflow, the matter has sufficiently large angular momentum, due to the revolution of the binary system and the rotation of the mass-losing star, which prevents free radial infall. Therefore, the captured matter will rotate around the mass-accreting star. The specific angular momentum of the transferred material is

$$\tilde{L} \sim a^2/P \tag{2.1}$$

where a and P are the separation of the two stars and the orbital period, respectively. Gas accreting onto a mass-accreting star with mass M will rotate at roughly the radius r_0 where its specific angular momentum is equal to the Keplerian value

$$\tilde{L} \sim (GMr_0)^{1/2} \tag{2.2}$$

where G is the gravitational constant. For Roche-lobe overflow,

$\tilde{L}$ is so great that r_0 is larger or much larger than the radius of mass-accreting star (The numerical calculations were done by Flannery 1975 and Lin and Pringle 1975).
The motion at $r \simeq r_0$ will be significantly noncircular. It is reasonable to assume that, for the rotating gas, both the dynamical (orbital) and radiative time scales are much shorter than the time scale for any viscous process to occur. The viscous processes are that can redistribute angular momentum among gas elements in order to let some of them spiral inward toward the mass-accreting star. Thus, energy and angular momentum can be extracted from those of gas elements by means of such processes. Under this assumption each element of gas will lose as much energy as it can by colliding with other gas elements and efficiently radiativing, while retaining its angular momentum. Eventually, all gas elements will settle down to moving on circular orbits with different radii according to their angular momenta. On the other hand, in the direction normal to the orbital plane, the gravity component will compress the gas. Thus a disc around the mass-accreting star will be formed (Fig.3 gave a nice picture of the accretion disc in Cyg. X-1). From then on, the viscosity will play an important role.
The gas incoming continuously from the mass-losing star interacts viscously with the gas of the disc. Some of the incoming gas gets deposited into the disc while the other is fed angular momentum from the disc by means of viscous processes and thereby gets ejected out of the disc region and back onto the mass-losing star or through the outer Lagrangian point into interstellar space (Fig.4).

It is disc itself, particularly its main body, that we are interested in here. In general, the gas of disc rotates differentially rather than rigidly (gas elements move on different orbits with different angular velocities). Viscosity present in the gas gradually remove angular momentum from gas elements. The angular momentum removed is transported by the viscous stresses from the inner part of the disc to the outer part, and is then carried away by ejected gas. At the same time, viscosity damps out shearing orbital motion of the gas, the energy of the shearing motion is dissipated in the gas as heat ("frictional heating"). Much more heat is generated than the gas can store, so most of the heat gets radiated away from the surfaces of the disc. Thus, viscosity causes the gas to lose energy. Since the only energy source is the gravitational potential, it is viscosity that converts gravitational potential energy into radiation.

The total amount of energy radiated by a unit mass of gas during its passage inward through the disc must equal approximately the gravitational binding energy of the unit mass when it eventually reaches the inner edge of the disc. For a main-sequence star or white dwarf or neutron star the inner edge of the disc is close to the star's surface (when the accreting star is a strongly magnetized neutron star or white dwarf, the disc extends inward only as far as the Alférn radius, refer to Lamb and Pethick 1974), for a black hole the inner edge of disc is at the last stable circular orbit, so the binding energy of the unit mass is

$$
\begin{aligned}
E_{bind} &\simeq \frac{1}{2} GM/r_I \\
&\simeq 10^{-6} C^2 && \text{for the Sun} \\
&\simeq 10^{-4} C^2 && \text{for white dwarf} \\
&\simeq 0.05C^2 && \text{for neutron star} \\
&\simeq 0.057C^2 && \text{for Schwarzschild hole} \\
&\simeq 0.42C^2 && \text{for extremely Kerr hole}
\end{aligned}
\tag{2.3}
$$

where r_I is disc's inner edge and C is the light speed. Thus, in order of magnitude the total luminosity of disc must be

$$
\begin{aligned}
L &\sim 10^{-6}\dot{M}C^2 \sim 10^{32}\,\text{ergs/sec}\left(\frac{\dot{M}}{10^{-9} M_\odot/yr}\right) && \text{for main-sequence} \\
L &\sim 10^{-4}\dot{M}C^2 \sim 10^{34}\,\text{ergs/sec}\left(\frac{\dot{M}}{10^{-9} M_\odot/yr}\right) && \text{for white dwarf} \\
L &\sim 10^{-1}\dot{M}C^2 \sim 10^{37}\,\text{ergs/sec}\left(\frac{\dot{M}}{10^{-9} M_\odot/yr}\right) && \text{for neutron star or black hole}
\end{aligned}
\tag{2.4}
$$

here $\dot{M}$ is the accretion rate. For typical observed X-ray binary systems, e.g., Cyg X-1 which supposely contains a black hole (Gursky 1973), the observations and models suggest that the mass-losing star is dumping gas onto its companion at a rate of $\dot{M} \sim 10^{-9} M_\odot/yr$, and this gas radiates a luminosity of $\sim 10^{37}$ ergs/sec. Thus it can be seen that accretion through a disc can be a extremely powerful energy source in astrophysics.

The general picture about binary accretion described above is essentially suitable for accretion in galactic nuclei. When one builds models for quasars and for active galactic nuclei, one typically is forced to invoke a strong energy source confined to a small region. Therefore, it is attractive to suppose that a supermassive black hole resides at the centre of a quasar or of a galaxy (Lynden-Bell 1969 and Gold, Axford and Ray 1965). In the case of a quasar the hole might have a mass $\sim 10^{10}$ to 10^{11} $M_\odot$. In the case of a normal galaxy such as our own, it might have a mass of $\sim 10^7$ to 10^8 $M_\odot$. Such a large black hole will accrete gas from its surroundings. The accreted gas, like the galaxy itself, will have significant angular momentum. Consequently, the gas will form an orbiting disc around the hole similar to that in the close binary systems. But the accretion rate and the resulting luminosity for a supermassive hole at the centre of a galaxy will much larger than those for binary accretion. For the most violent of quasars, $L \simeq 10^{47}$ ergs/sec, if all of that power is supplied by accretion onto a hole with ~ 10 per cent efficiency for converting rest mass energy into radiation, then $\dot{M} \lesssim 10$ $M_\odot$/yr. Similarly, for the nucleus of our own Galaxy, $L \simeq 10^{42}$ ergs/sec, so $\dot{M} \lesssim 10^{-4}$ $M_\odot$/yr.

Accertion discs might be divided into several classes, according to their geometrical thickness and optical depth. Which kind of disc is actually realized is dependent on the accretion rate and the boundary condition (see Table 1 and refer to Hoshi 1981 for the details). If the accretion rate is sufficiently low, an optically thin disc is formed.

As the accretion rate is increased, the disc evolves into either an optically thin, geometrically thick state or an optically thick, geometrically thin one, depending on the outer boundary condition. For the latter, when the accretion rate is increased to the extent of the critical value

$$\dot{M}_{cr} = 8\pi C r_I m_p / \sigma_T \tag{2.5}$$

where m_p is the mass of proton and σ_T is the Thompson cross-section, accordingly the total luminosity emitted by the disc approaches the "Eddington limit" (Eddington 1926)

$$L_{edd} = 4\pi CGM m_p / \sigma_T \simeq 1 \times 10^{38} ergs/sec \left(\frac{M}{M_\odot}\right) \tag{2.6}$$

the thickness of the disc will become of the order of distance to the centre of the mass-accreting star and the general nature of the accretion will be quite different from that of the geometrical thin one. We shall not consider here such thick supercritical accretion discs but concentrate on the subcritical accretion, $L \ll L_{edd}$, with a thin disc. The subcritical accretion means that the accretion rate

$$\dot{M} \ll \dot{M}_{cr} \sim \begin{cases} 10^{-3} M_\odot/yr\left(\frac{M}{M_\odot}\right) & \text{for main-sequence star} \\ 10^{-5} M_\odot/yr\left(\frac{M}{M_\odot}\right) & \text{for white dwarf} \\ 10^{-8} M_\odot/yr\left(\frac{M}{M_\odot}\right) & \text{for neutron star or black hole} \end{cases} \tag{2.7}$$

2) Newtonian Theory of Thin discs

i. Underlying Assumptions

In calculating the disc structure, the following assumptions and approximations are usually made (Novikov and Thorne 1973 and Lightman 1974):

a) The central plane of the disc coincides with the equatorial plane of the mass-accreting star.

b) The gravitational pull of the mass-losing star and the gas streaming off the mass-losing star in the binary system have negligible influence on the disc structure. Apparently, this assumption is valid in the inner part of the disc, not at the outer edge.

c) The self-gravitation of the disc is negligibly small.

d) The disc is thin, i.e., its half- thickness H satisfies everywhere

$$H_{(r)} \ll r \qquad (2.8)$$

where r is the distance from the centre of the mass-accreting star.

e) The gravitational force of the mass-accreting star is much greater than internal stress and pressure gradients inside the disc, thus the gas of the disc moves in nearly circular Keplerian orbits, i.e., the central gravitational force is balanced by the centrifugal force. Adoping cylindrical coordinates (r, ϕ, z) the circular velocity V_ϕ is everywhere

$$V_\phi \equiv \Omega r = (GM/r)^{1/2} \qquad (2.9)$$

where Ω is the angular velocity.
A very small radial flow produced by viscous stresses is superimposed on this orbital motion,

$$V_r << V_\phi \tag{2.10}$$

where V_r is the radial velocity of the gas.
In the direction perpendicular to the disc plane, there should be an even smaller flow described by a vertical velocity V_z. But V_z will disappear in calculation owing to introducing a new quantity, surface density Σ (see below). In an equivalent and simpler way, it is usually assumed that, in the vertical direction, hydrostatic equilibrium exists, there is no macroscopic motion,

$$V_z = 0 \tag{2.11}$$

(Lately, however, Urpin 1983 calculated a non-zero vertical velocity V_z).

ii. Equations of Disc Structure

Define firstly the surface density of the disc as

$$\Sigma_{(r,t)} = 2\int_0^H \rho_{(r,z,t)}\, dz$$

$$= \bar{\rho}_{(r,t)} \cdot 2H \qquad (2.12)$$

where $\rho_{(r,z,t)}$ is the local density of the gas and $\bar{\rho}_{(r,t)}$ is the vertically averaged density.

Consider now the motion of an annulus of gas with inner radius r and with radial extent Δr. The mass of the annulus is $2\pi r \cdot \Delta r \cdot \Sigma$ and its angular momentum is $2\pi r \cdot \Delta r \cdot \Sigma \cdot r^2 \Omega$.

a) Conservation of mass

The rate of change of the mass of the annulus is equal to the net flow of matter into it from neighboring annuli due to radial motions, that is

$$\frac{\partial}{\partial t}(2\pi r \cdot \Delta r \cdot \Sigma) = \bar{V}_{r(r,t)} \cdot 2\pi r \cdot \Sigma_{(r,t)} - \bar{V}_{r(r+\Delta r,t)} \cdot 2\pi (r+\Delta r) \cdot \Sigma_{(r+\Delta r,t)} \qquad (2.13)$$

where $\bar{V}_r$ is the vertically averaged radial velocity. Taking the the limit for small Δr, (2.13) becomes

$$r\frac{\partial \Sigma}{\partial t} + \frac{\partial}{\partial r}(r\Sigma\bar{V}_r) = 0 \qquad (2.14)$$

which is

$$\frac{\partial \Sigma}{\partial t} = -\frac{1}{r}\frac{\partial}{\partial r}(r\Sigma\bar{V}_r)$$

$$= \frac{1}{2\pi r} \cdot \frac{\partial \dot{M}}{\partial r} \qquad (2.15)$$

where

$$\dot{M}_{(r,t)} = -2\pi r \Sigma \bar{V}_r \tag{2.16}$$

is the mass crossing a cylindrical surface in unit time (the local accretion rate).

b) Conservation of angular momentum

The variation in the angular momentum of the annulus is given by the net flow of angular momentum into it from neighboring annuli due to radial motions, plus the net torque due to viscous forces acting on it,

$$\begin{aligned} &\frac{\partial}{\partial t}(2\pi r\cdot\Delta r\cdot\Sigma\cdot r^2\Omega) \\ &= \bar{V}_{r(r,t)}\cdot 2\pi r\cdot \Sigma_{(r,t)}\, r^2\Omega_{(r,t)} \\ &\quad -\bar{V}_{r(r+\Delta r,t)}\cdot 2\pi(r+\Delta r)\cdot\Sigma_{(r+\Delta r,t)}\cdot(r+\Delta r)^2\Omega_{(r+\Delta r,t)} \\ &\quad +\left[G_{(r+\Delta r,t)} - G_{(r,t)}\right] \end{aligned} \tag{2.17}$$

where $G_{(r,t)}$ is the torque due to viscous forces acting between one layer of matter and another,

$$G_{(r,t)} = 2\pi r\cdot T_{r\phi}\cdot r \tag{2.18}$$

where $T_{r\phi}$ is the viscous force per unit length around the circumference of the annulus,

$$T_{r\phi(r,t)} = 2\int_0^H \tau_{r\phi(r,z,t)}\, dz \tag{2.19}$$

where $\tau_{r\phi}$ is the only important coordinate component of viscous stress tensor, all other components are neglected (Landau and Lifshitz 1959),

$$\tau_{r\phi(r,z,t)} = \eta_{(r,z,t)} \cdot r \frac{d\Omega}{dr} \tag{2.20}$$

$r\frac{d\Omega}{dr}$ is called the rate of shearing, $\eta_{(r,z,t)}$ is the dynamic viscosity.

Putting (2.20) into (2.19), get

$$\begin{aligned} T_{r\phi} &= 2H \cdot \bar{\eta}_{(r,t)} \cdot r\frac{d\Omega}{dr} \\ &= \Sigma \bar{\nu}_{(r,t)} r \frac{d\Omega}{dr} \end{aligned} \tag{2.21}$$

where $\bar{\eta}_{(r,t)}$ is the vertically averaged dynamic viscosity while $\bar{\nu}_{(r,t)}$ is the vreticallly averaged kinematic viscosity, and

$$\bar{\eta}_{(r,t)} = \bar{\rho}_{(r,t)} \bar{\nu}(r,t) \tag{2.22}$$

Substituting (2.18) and (2.21) into (2.17) and taking the limit for small Δr, (2.17) becomes

$$\begin{aligned} &\frac{\partial}{\partial t}(\Sigma r^2 \Omega) + r^{-1}\frac{\partial}{\partial r}(\Sigma r^3 \Omega \bar{V}_r) \\ &= r^{-1}\frac{\partial}{\partial r}(\bar{\nu}\Sigma r^3 \frac{d\Omega}{dr}) \end{aligned} \tag{2.23}$$

this is the equation of angular momentum conservation. Equation (2.14) and (2.23) may be used to eliminate $\bar{V}_r$, yielding a single equation

$$\frac{\partial \Sigma}{\partial t} = 3r^{-1}\frac{\partial}{\partial r}\left[r^{1/2}\frac{\partial}{\partial r}(\bar{\nu}\Sigma r^{1/2})\right] \tag{2.24}$$

this is the equation for the evolution of the surface density.

c) Conservation of energy

The variation of the total kinetic and potential energies of the annulus

$$\frac{\partial}{\partial t}\left[2\pi r \cdot \Delta r \cdot \Sigma(\frac{\Omega^2 r^2}{2} - \frac{GM}{r})\right]$$

is determined by the difference between the rates of flow of these energies into and out of the annulus

$$\Delta r \frac{\partial}{\partial r}\left[2\pi r \cdot \Sigma \cdot \bar{V}_r \cdot \left(\frac{\Omega^2 r^2}{2} - \frac{GM}{r}\right)\right]$$

and the difference in work being done by viscous stress at the sides of the annulus

$$\Delta r \frac{\partial}{\partial r}(2\pi r \cdot T_{r\phi} \cdot \Omega r)$$

and irreversible energy losses into heat due to friction

$$\Delta r \cdot 2\pi r \cdot T_{r\phi} \cdot r\frac{d\Omega}{dr} = \Delta r \cdot 2\pi r \cdot \bar{\nu}\Sigma\left(r\frac{d\Omega}{dr}\right)^2 .$$

Putting everything together, obtain the equation of energy conservation

$$\frac{\partial}{\partial t}\left[\Sigma r\left(\frac{\Omega^2 r^2}{2} - \frac{GM}{r}\right)\right] = -\frac{\partial}{\partial r}\left[\Sigma \bar{V}_r r\left(\frac{\Omega^2 r^2}{2} - \frac{GM}{r}\right)\right] - \frac{\partial}{\partial r}\left(\Sigma \bar{\nu} r^3 \Omega \frac{d\Omega}{dr}\right) + \Sigma \bar{\nu} r^3 \left(\frac{d\Omega}{dr}\right)^2 \tag{2.25}$$

(2.25) can be derived by multiplying (2.23) by Ω.
It is the rate per unit area per unit time of energy dissipation into heat by viscous pro^ce^sses that we are interested in, according to standard fluid dynamics, the rate is

$$D_{(r,t)} = T_{r\phi} \cdot r\frac{d\Omega}{dr} = \bar{\nu}\Sigma\left(r\frac{d\Omega}{dr}\right)^2 \tag{2.26}$$

d) Energy transport

The heat generated by viscosity is transported predominantly in vertical direction to the surface of the disc because of disc thinness. If the energy transport mechanism is radiative transport (whether the turbulent energy transfer should be taken into account is still an open problem, see Urpin 1983 and Duschl 1983, and convection might be important in some regions of the disc, see Tayler 1980), and if the disc is optically thick because of electron scattering or true absorption, the energy transport is described by the radiative diffusion equation

$$F_{z(r,z,t)} = -\frac{4}{3}\frac{acT^3}{K\rho}\frac{\partial T}{\partial z} \qquad (2.27)$$

where F_z is the energy flux in vertical direction, T is the temperature which is a function of z and t, not of r (there is no radial flux, by assumption), K is the opacity and a is Stefan's constant.

Thus, the rate of heat loss per unit time per unit area on the disc's surface is approximately given by

$$Q^-_{(r,t)} = \frac{1}{3}\frac{ac}{\lambda}T_c^4 \qquad (2.28)$$

where T_c is the central temperature of the disc and λ is the optical depth,

$$\lambda_{(r,t)} = \bar{\rho} H \bar{k} \qquad (2.29)$$

where $\bar{k}$ is the vertically averaged opacity.

e) Vertical hydrostatic equilibrium

In the vertical direction, there exists hydrostatic equilibrium.

The pressure gradient is balanced by the component of the gravitational attraction normal to the disc plane (tidal force),

$$\frac{\partial P}{\partial z} = \rho \frac{\partial}{\partial z}\left[\frac{GM}{(r^2+z^2)^{1/2}}\right] \qquad (2.30)$$

which for small disc thickness becomes approximately

$$\frac{\partial P}{\partial z} = -\rho \frac{GMz}{r^3} \qquad (2.31)$$

where P is the pressure which is a function of z and t, not of r (radial pressure gradient is negligibly small, see assumption e).
Integrating (2.31) (using z-averaged density $\bar{\rho}$ instead of local density ρ), get

$$P_{(z,t)} = P_c\left[1-(z/H)^2\right] \qquad (2.32)$$

where

$$P_c = \frac{1}{2}\bar{\rho}\Omega^2 H^2 \qquad (2.33)$$

is the central pressure of the disc. The averaged pressure is then

$$\begin{aligned}\bar{P} &= \frac{1}{H}\int_0^H P_{(z)}dz \\ &= \frac{2}{3}P_c \\ &= \frac{1}{3}\bar{\rho}\Omega^2 H^2 \qquad (2.34)\end{aligned}$$

Introduce an averaged sound speed $\overline{V_s}$

$$\overline{V}_s = \frac{1}{\sqrt{3}} \Omega H \qquad (2.35)$$

by using the relation

$$\overline{V}_s^2 = \overline{P}/\overline{\rho} \qquad (2.36)$$

It can be seen that the disc half-thickness H is given by

$$H/r = \sqrt{3}\,\overline{V}_s / V_\phi \qquad (2.37)$$

which means that the assumption of disc thinness ($H/r \ll 1$) requires the gravitational energy of the disc gas ($\sim V_\phi^2$) to be much greater than its internal energy ($\sim \overline{V}_s^2$).

f) Viscosity

Viscosity governs the local structure of the disc (see 2.26) as well as the disc's evolution (see 2.24). Given a viscosity and a radiative process one can build an accretion disc model. But, unfortunately, the main uncertainty in disc theory is just the viscosity -- its nature and magnitude. The only reasonable certainty is that ordinary molecular viscosity is too small to lead to disc accretion. It is generally thought that turbulence in the gas and chaotic magnetic fields are probably the dominant sources of viscosity. The most fruitful approach to the problem is still the α-scheme proposed by Shakura and Sunyaev (1973) which has been followed by most authers in the field for ten years. Although other viscosity laws (Lynden-Bell and Pringle 1974, Pringle and Rees 1972, Eardlley and Lightman 1975, Pacheko and Steiner 1976, Stewart 1975 and Ichimaru 1976) and various viscous mechanisms (Vila 1978, Liang 1977, Paczynski and Jaroszynski 1978, Paczynski 1978 and etc.) have suggested, the fact that there has not been any substantive progress in discovering the properties of disc viscosity reflects the difficult nature of the problem.

The kinematic viscosity associated with a turbulent process is

$$\nu \simeq V_t l_t \tag{2.38}$$

where V_t is the turnover velocity and l_t is the characteristic size of the largest turbulent cells. The turbulent scale is limited by the disc thickness, $l_t \lesssim H$, and the turbulence is probably subsonic -- if the turbulent speed ever exceeds the sound speed, then shocks develop and quickly convert the turbulent energy into heat -- $V_t \lesssim V_s$. Thus, the vertically averaged shear stress due to turbulence may be written as

$$\begin{aligned} \left|\overline{\tau}_{r\phi}^{(turb.)}\right| &= \left|\overline{\nu \rho} \, r \frac{d\Omega}{dr}\right| \\ &\lesssim \overline{V}_s H \overline{\rho} \Omega \\ &\simeq \overline{V}_s^2 \overline{\rho} \\ &= \overline{P} \end{aligned} \tag{2.39}$$

(Note that $r\frac{d\Omega}{dr} = -\frac{3}{2}\Omega$ and (2.37) and (2.36) were used to obtain the above express).

On the other hand, for a chaotic magnetic field, because shearing tends to string the field out along the ϕ-direction, the magnetic shear stress will be somewhat smaller than the magnetic pressure

$$\left|\overline{\tau}_{r\phi}^{(mag.)}\right| \lesssim \overline{P}^{(mag.)} = B^2/8\pi \tag{2.40}$$

where B is the mean field strength. The magnetic pressure cannot exceed the thermal pressure

$$\overline{P}^{(mag.)} \lesssim \overline{P}^{(ther.)} = \overline{\rho}\,\overline{V}_s^2 \tag{2.41}$$

otherwise the field lines would bulge out of the disc, reconnect, and escape. Thus, the magnetic viscous stress will satisfy

$$\left| \overline{\tau}_{r\phi}^{(mag.)} \right| \lesssim \overline{P} = \overline{\rho}\,\overline{V}_s^2 \tag{2.42}$$

Combining (2.39) and (2.42), one can write

$$-\overline{\tau}_{r\phi} = \alpha\,\overline{P} \tag{2.43}$$

which means that one can condense lack of knowledge about the magnitude of the viscosity into a single dimensionless parameter α , and

$$\alpha \lesssim 1 \tag{2.44}$$

Clearly, excluding values of $\alpha > 1$ is based on stability considerations, it doesn't mean that $\alpha > 1$ is in practice impossible.

Using (2.19) and (2.42), $T_{r\phi}$ can be expressed in a simple form

$$-T_{r\phi} = 2\alpha H\overline{P} \tag{2.45}$$

g) Equation of state and opacity law

In the approximation that the interior of the disc is completely thermalized, so that the particles and photons are described by the same temperature, the pressure is given by

$$\begin{aligned} P &= P_{gas} + P_{rad.} \\ &= \frac{N_0 k}{\mu}\rho T + \frac{1}{3}aT^4 \end{aligned} \tag{2.46}$$

where N_o is Avogadro's number, k is Boltzmann's constant and μ is the mean molecular weight.

The principal sources of opacity are free-free and electron scattering (bound-free and bound-bound processes are negligible in the case of a fully ionized plasma). The Rosseland mean opacity is then

$$k = K_{ff} + k_{es} \tag{2.47}$$

and

$$k_{es} = 0.40 \ \mathrm{cm}^2 \mathrm{g}^{-1} \tag{2.48}$$

$$k_{ff} = 0.64 \times 10^{23} (\rho / \mathrm{g\ cm}^{-3}) (T / {}^{\circ}\mathrm{K})^{-7/2} \mathrm{cm}^2 \mathrm{g}^{-1} \tag{2.49}$$

(Novikov and Thorne 1973).

iii. Stationary Solution

In the stationary case ($\frac{\partial}{\partial t} = 0$), it can be seen from (2.15) that the accretion rate

$$\dot{M} = -2\pi r \Sigma \bar{V}_r \qquad (2.50)$$

is a constant. This means that a constant inward mass flux exists in a steady disc. Note that the radial velocity $\bar{V}_r < 0$.

The angular momentum equation (2.23) may also be integrated to give

$$\dot{M}\Omega r^2 = -2\pi T_{r\phi} r^2 + C \qquad (2.51)$$

where C is the constant of integration which is determined by the condition that $T_{r\phi} = 0$ at the inner edge of the disc, r_I,

$$C = \dot{M}(GMr_I)^{1/2} \qquad (2.52)$$

this is the rate of angular momentum deposited in the mass-accreting star. Thus, (2.51) becomes

$$r^2\dot{M}\Omega\left[1 - \left(\frac{r_I}{r}\right)^{1/2}\right] = -2\pi T_{r\phi} r^2 \qquad (2.53)$$

This is just the expression of the rate at which viscous stresses carry angular momentum out of radius r.

Using (2.50), (2.21) and (2.53), get

$$\bar{V}_r \approx -\frac{3}{2}\frac{\bar{\nu}}{r} \qquad \text{for } r \gg r_I \qquad (2.54)$$

This result gives a magnitude estimate for radial velocity. Using (2.53) and (2.26), the energy dissipation rate per unit area in the stationary case is

$$D = \frac{3GM\dot{M}}{4\pi r^3}\left[1 - \left(\frac{r_I}{r}\right)^{1/2}\right] \qquad (2.55)$$

Note that the energy dissipation rate is independent of viscosity. The major uncertainty of the theory has vanished in the stationary case, but this is at the expense of the assumption that the viscosity can just provide a constant mass flux $\dot{M}$. Of course, the other local disc properties, for example the surface density, the optical thickness, etc., do depend on the magnitude of the viscosity.

The heat generated between radii r_a and r_b is

$$\begin{aligned}
&\int_{r_a}^{r_b} D \cdot 2\pi r \, dr \\
&= \frac{3}{2} GM\dot{M}\left\{\frac{1}{r_a}\left[1-\frac{2}{3}\left(\frac{r_I}{r_a}\right)^{1/2}\right]-\frac{1}{r_b}\left[1-\frac{2}{3}\left(\frac{r_I}{r_b}\right)^{1/2}\right]\right\} \\
&\simeq \frac{3}{2} GM\dot{M}\left(\frac{1}{r_a}-\frac{1}{r_b}\right) \qquad \text{for } r_b > r_a \gg r_I \qquad (2.56)
\end{aligned}$$

Notice that gravitational potential energy is released at a rate $GM\dot{M}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$, and only half of this energy can go into heat, the other half must go into orbital kinetic energy (Virial theorem). So, the rate at which gravitational energy gets converted into heat is $\frac{1}{2}GM\dot{M}\left(\frac{1}{r_a}-\frac{1}{r_b}\right)$, this is only one-third of the heat generated between radii r_a and r_b for a region $r \gg r_I$. The remaining two-thirds is transported from the inner region by viscous stresses. Thus, the viscous stresses transport outward not only angular momentum, but also energy.

From (2.56) it is easy to find that the total disc luminosity is

$$L = \int_{r_I}^{\infty} D 2\pi r dr = \frac{1}{2} GM\dot{M}/r_I \qquad (2.57)$$

This is only one half of the total gravitational potential energy released by matter moving inward from infinity to the inner edge of the disc. The matter just outside the inner boundary of the disc retains still another half of the potential energy it has lost as kinetic energy.

The heat generated by viscosity must be emitted from the surfaces of the disc, the energy emission rate per unit time per unit area on the disc's surface is therefore (the disc has two surfaces)

$$Q^+ = \frac{1}{2} D = \frac{34M\dot{M}}{8\pi r^3}\left[1 - \left(\frac{r_I}{r}\right)^{1/2}\right] \qquad (2.58)$$

Fig. 5 illustrates the luminosity of the unit surface of the disc, Q^+, as a function of the radius r. The maximum of Q^+ takes place at $r = 1.36 r_I$. The figure shows also the contribution of different regions of the disc to the total disc luminosity. The main contribution comes from the region $r < 10 r_I$.

Q^+ is equal to the radiation flux Q^- on the surface

$$Q^+ = Q^- = \frac{1}{3} \frac{ac}{\lambda} T_c^4 \qquad (2.59)$$

To summarize, the set of equqtions governing the structure of the stationary disc is as follows.

Kepler's law (2.9) $\Omega = (GM/r^3)^{1/2}$

Conservation of mass (2.50) $\dot{M} = -2\pi r \Sigma \bar{V}_r$

Conservation of angular momenyum (2.53) (using 2.45)

$$\dot{M}\Omega\left[1 - \left(\frac{r_I}{r}\right)^{1/2}\right] = 4\pi\alpha H\bar{P}$$

Vertical equilibrium (2.37) $H = \sqrt{3}\,\bar{V}_s/\Omega$

Thermal energy balance (2.59)

$$\frac{3GM\dot{M}}{8\pi r^3}\left[1 - \left(\frac{r_I}{r}\right)^{1/2}\right] = \frac{1}{3}\frac{ac}{\lambda}T_c^4$$

Definition of sound speed (2.36) $\bar{V}_s^2 = \bar{P}/\bar{\rho}$

Definition of surface density (2.12) $\Sigma = 2H\bar{\rho}$

Equation of state (2.46) $\bar{P} = \frac{N_0 k}{\mu}\bar{\rho}\,T_c + \frac{1}{3}aT_c^4$

These equqtions enable us to solve for $\bar{P}$, $\bar{\rho}$, T_c, $\bar{V}_s$, H, Σ and $\bar{V}_r$ as funtions of M, $\dot{M}$, r and the unknown viscosity parameter α.

Following Shakura and Sunyaev (1973), take $r_I = 3r_g$ for the case of that the accreting star is a Schwarzschild black hole where r_g is the Schwarzschild gravitational radius

$$r_g = 2GM/C^2 \tag{2.60}$$

and introduce nondimensional variables

$$
\begin{aligned}
\tilde{M} &= M/M_{\odot} \\
\tilde{\dot{M}} &= \dot{M}/\dot{M}_{crit} \qquad\qquad (2.61)\\
\tilde{r} &= r/3r_g
\end{aligned}
$$

where $\dot{M}_{crit} = 3 \times 10^{-8}(M/M_{\odot})M_{\odot}/yr$ is the accretion rate which yields the Eddington luminosity for an efficiency of rest mass energy conversion $\simeq 6\%$.

The explicit form of solution is particularly simple if one characterized the regions of the disc by their pressures and opacities.

a) The inner region in which radiation pressure dominates over gas pressure, $P_r \gg P_g$, and opacity is predominantly due to electron scattering, $k_{es} \gg k_{ff}$, one has

$$
\begin{aligned}
H &= 3.2 \times 10^{6}(1-\tilde{r}^{-1/2})\tilde{M}\,\tilde{\dot{M}} \text{ cm} \\
\Sigma &= 4.6\,\alpha^{-1}\tilde{r}^{3/2}(1-\tilde{r}^{-1/2})^{-1}\tilde{\dot{M}}^{-1} \text{ g/cm}^2 \qquad\qquad (2.62)\\
\bar{V}_r &= 7.7 \times 10^{10}\alpha\,\tilde{r}^{-5/2}(1-\tilde{r}^{-1/2})\tilde{\dot{M}}^{2} \text{ cm/sec} \\
T_c &= 2.3 \times 10^{7}\,\tilde{r}^{-3/4}(\alpha\tilde{M})^{-1/4}\ {}^{\circ}\text{K}
\end{aligned}
$$

b) The intermediate region in which election scattering opacity is still predominant, $k_{es} \gg k_{ff}$, but gas pressure dominates over radiation pressure, $P_g \gg P_r$, the solution is

$$
\begin{aligned}
H &= 1.2 \times 10^{4}\alpha^{-1/10}\tilde{r}^{21/20}(1-\tilde{r}^{-1/2})^{1/5}\tilde{M}^{9/10}\tilde{\dot{M}}^{1/5} \text{ cm} \\
\Sigma &= 1.7 \times 10^{5}\alpha^{-4/5}\tilde{r}^{-3/5}(1-\tilde{r}^{-1/2})^{3/5}\tilde{M}^{1/5}\tilde{\dot{M}}^{3/5} \text{ g/cm}^2 \qquad\qquad (2.63)\\
\bar{V}_r &= 2 \times 10^{6}\alpha^{4/5}\tilde{r}^{-2/5}(1-\tilde{r}^{-1/2})^{-3/5}\tilde{M}^{-1/5}\tilde{\dot{M}}^{2/5} \text{ cm/sec} \\
T_c &= 3.1 \times 10^{8}\alpha^{-1/5}\tilde{r}^{-9/10}(1-\tilde{r}^{-1/2})^{2/5}\tilde{M}^{-1/5}\tilde{\dot{M}}^{2/5}\ {}^{\circ}\text{K}
\end{aligned}
$$

c) The outer region in which gas pressure is still dominant, $P_g \gg P_r$, but the opacity is determined by free-free absorption, $k_{ff} \gg k_{es}$, the result is

$$
\begin{aligned}
H &= 6.1 \times 10^{3} \alpha^{-1/10} \tilde{r}^{9/8} (1-\tilde{r}^{-1/2})^{3/20} \tilde{M}^{9/10} \tilde{\dot{M}}^{3/20} \ \mathrm{cm} \\
\Sigma &= 6.1 \times 10^{5} \alpha^{-4/5} \tilde{r}^{-3/4} (1-\tilde{r}^{-1/2})^{7/10} \tilde{M}^{1/5} \tilde{\dot{M}}^{7/10} \ \mathrm{g/cm^2} \\
\bar{V}_r &= 5.8 \times 10^{5} \alpha^{4/5} \tilde{r}^{-1/4} (1-\tilde{r}^{-1/2})^{-7/10} \tilde{M}^{-1/5} \tilde{\dot{M}}^{3/10} \ \mathrm{cm/sec} \\
T_c &= 8.6 \times 10^{7} \alpha^{-1/5} \tilde{r}^{-3/4} (1-\tilde{r}^{-1/2})^{3/10} \tilde{M}^{-1/5} \tilde{\dot{M}}^{3/10} \ {}^{\circ}\mathrm{K}
\end{aligned}
\qquad (2.64)
$$

iv. Radiation Spectrum

The disc is divided into rings and each ring is supposed to emit independently. So we must determine firstly the spectrum emitted locally at each part on the disc surface and then integrate over the whole disc surface.
The local spectrum depends mainly on the opacity law. In the outer region the free-free opacity is dominant, and the disc is optically thick. Thus, the radiation is of blackbody,

$$I_\nu = B_\nu = \frac{2h}{c^2}\nu^3\left[\exp(h\nu/kT_S)-1\right]^{-1} \tag{2.65}$$

where I_ν is the radiation intensity, B_ν is the Planck function, h is Planck's constant, ν is frequency, T_S is the disc surface temperature. The radiation flux crossing outward through the disc surface is

$$Q^- = \pi\int I_\nu d\nu = \sigma T_S^4 \tag{2.66}$$

where σ is Stefan-Boltzman constant. Using (2.59) and (2.58) get

$$T_S = \left\{\frac{3GM\dot{M}}{8\pi r^3\sigma}\left[1-(r_I/r)\right]^{1/2}\right\}^{1/4} \tag{2.67}$$

for $r \gg r_I$ which is approximately

$$T_S = T_I(r/r_I)^{-3/4} \tag{2.68}$$

where

$$T_I = (3GM\dot{M}/8\pi r_I^3\sigma)^{1/4} \qquad (2.69)$$

The radiation spectrum given by the whole disc is obtained by integrating the local spectrum

$$S_\nu = \int_{r_I}^{r_{out}} \pi I_\nu \cdot 2\pi r \, dr \qquad (2.70)$$

where r_{out} is the outer boundary of the disc. For $\nu \ll kT_I/h$, the radiation comes predominantly from $r \gg r_I$, obtain

$$S_\nu \propto \nu^{1/3} \qquad (2.71)$$

This is sometimes regarded as a characteristic accretion disc spectrum and was first noted by Lynden-Bell (1969). In the intermediate region and the inner region electron scattering opacity modifies the emitted spectrum so it is no longer of blackbody. In the innermost region, the temperature is so high that comptonization effect strongly the shape of the emitted spectrum, the radiation spectrum has a Wien distribution,

$$S_\nu \propto e^{-h\nu/kT_s} \qquad (2.72)$$

Fig.6 gives four characteristic local spectra of radiation formed in the corresponding regions of the disc.
Fig.7 illustrates the total radiation spectrum of the disc around a nonrotating black hole.
Fig.8 shows the distribution of the surface temperature along the radius of the disc for different accretion rates and viscosity parameters.

From fig.5 it is known that the strongest radiation flux, thus the hottest surface temperature, occur at radius $r = (49/36)r_I$. From (2.67) the maximum black-body temperature of the disc around a black hole or neutron star is

$$T_{b.b.} \sim 5 \times 10^6 \,{}^\circ K \, (M/M_\odot)^{-1/2} \left(\frac{\dot{M}}{10^{-9} M_\odot/yr}\right)^{1/4} \tag{2.73}$$

Because the black body spectrum peaks at a photon energy of (Wien's displacement law)

$$h\nu_{max} \simeq (2.44 \times 10^{-4} ev)(T_s/{}^\circ K) \tag{2.74}$$

the spectrum of the total radiation from the disc should peak at

$$h\nu_{max} \simeq (1 \text{ kev}) \left(\frac{\dot{M}}{10^{-9} M_\odot/yr}\right)^{1/4} (M/M_\odot)^{-1/2} \tag{2.75}$$

for the case that the accreting star is a neutron star or black hole. Thus, for reasonable accretion rates the disc around a neutron star or a black hole can emit strongly in the X-ray region ($\sim$1 to 10 kev). In the intermediate and inner regions of the disc, the electron-scattering modifies the blackbody spectrum, and causes the spectrum to peak at energies higher than that of (2.75).

3) Relativistic Form of Disc Equations

Following Novikov and Thorne (1973), in the extention of disc equations from Newtonian form to general relativistic one, assume that the spacetime geometry outside the mass-accreting star (a black hole here) is that of Kerr, and that the disc lies in the equatorial plane of the Kerr metric. Geometrized units ($G = C = \hbar = 1$) is used in this section. The form of Kerr metric in and near equatorial plane ($|\theta - \pi/2| \ll 1$) is

$$dS^2 = -\frac{r^2\Delta}{A}dt^2 + \frac{A}{r^2}(d\phi - \omega dt)^2 + \frac{r^2}{\Delta}dr^2 + dz^2 \qquad (2.76)$$

where

$$\Delta = r^2 - 2Mr + a^2$$
$$A = r^4 + r^2a^2 + 2Mra^2$$
$$\omega = 2Mar/A$$

where M and a are the mass and specific angular momentum of the black hole, respectively.

For simplicity in splitting formulas into Newtorian limits plus relativistic corrections, introduce the following functions with value unity far from the hole

$$\mathcal{A} \equiv 1 + a_*^2/r_*^2 + 2a_*^2/r_*^3$$
$$\mathcal{B} \equiv 1 + a_*/r_*^{3/2}$$
$$\mathcal{C} \equiv 1 - 3/r_* + 2a_*/r_*^{3/2}$$
$$\mathcal{D} \equiv 1 - 2/r_* + a_*^2/r_*^2$$
$$\mathcal{E} \equiv 1 + 4a_*^2/r_*^2 - 4a_*^2/r_*^3 + 3a_*^4/r_*^4$$

$$\mathcal{F} \equiv 1-2a_*/r_*^{3/2}+a_*^2/r_*^2 \qquad (2.77)$$
$$\mathcal{G} \equiv 1-2/r_* +a_*/r_*^{3/2}$$
$$g \equiv \exp\left[-\frac{3}{2}\int_{r_*}^{\infty}\mathcal{B}^{-1}\mathcal{C}^{-1}\mathcal{F}r_*^{-2}dr_*\right]$$
$$\mathcal{L} \equiv \frac{\tilde{L}-\tilde{L}_{ms}}{(Mr)^{1/2}} = \frac{\mathcal{F}}{\mathcal{C}^{1/2}} - \frac{\tilde{L}_{ms}}{(Mr)^{1/2}}$$
$$\mathcal{Q} \equiv \mathcal{L}-\frac{3}{2r_*^{1/2}}\,g\int_{r_{ms*}}^{r_*}\frac{\mathcal{F}\mathcal{L}}{\mathcal{B}\mathcal{C}g}\,r_*^{-3/2}\,dr_*$$

where r_* and a_* are dimensionless measures of radius r and a

$$r_* \equiv r/M, \qquad a_* \equiv a/M \qquad (2.78)$$

$\tilde{L}$ is the angular momentum per unit mass for circular orbit

$$\tilde{L} = M^{1/2}r^{1/2}\mathcal{F}/\mathcal{C}^{1/2} \qquad (2.79)$$

$\tilde{L}_{ms}$ is the angular momentum per unit mass for the last stable circular orbit

$$\tilde{L}_{ms} = \frac{2M}{\sqrt{3X}}(3x-2a), \qquad x = M^{1/2}r_{ms}^{1/2} \qquad (2.80)$$

where r_{ms} is the minimum radius of stable circular orbits which is determined by the following equation

$$r_{ms}^2 -6Mr_{ms} +8aM^{1/2}r_{ms}^{1/2} -3a^2 = 0 \qquad (2.81)$$

The relativistic equations governing disc structure are:

a) Rest-mass conservation

$$\dot{M}_o = -2\pi r \Sigma V^{-\hat{r}} \mathcal{D}^{1/2} \tag{2.82}$$

where $\dot{M}_o$ is the rest-mass accretion rate, Σ is the rest-mass surface density and $V^{-\hat{r}}$ is the mass-averaged radial velocity of the gas,

$$V^{-\hat{r}} \equiv \frac{1}{\Sigma} \int_{-H}^{+H} V^{\hat{r}} \rho_o \, dz \tag{2.83}$$

ρ_o is the rest-mass density.

b) Angular momentum conservation

$$T_{r\phi} = 2\int_0^H \tau_{\hat{r}\hat{\phi}} dz = \frac{\dot{M}_o}{2\pi} (M/r^3)^{1/2} \frac{\mathcal{C}^{1/2} \mathcal{Q}}{\mathcal{B} \mathcal{D}} \tag{2.84}$$

c) Energy conservation

$$Q^- = \frac{3\dot{M}_o}{8\pi r^2} \frac{M}{r} \frac{\mathcal{Q}}{\mathcal{B} \mathcal{C}^{1/2}} \tag{2.85}$$

d) Vertical equilibrium

$$H = (P/\rho_o)^{1/2} (r^3/M)^{1/2} \mathcal{A} \mathcal{B}^{-1} \mathcal{C}^{1/2} \mathcal{D}^{-1/2} \mathcal{E}^{-1/2} \tag{2.86}$$

e) Energy transport

$$aT_c^4 = k \Sigma Q^- \tag{2.87}$$

f) Viscosity expression

$$\tau_{\hat{r}\hat{\phi}} = \alpha P \tag{2.88}$$

g) Equation of state and opacity law

$$P = P^{(rad.)} + P^{(gas)}$$
$$k = k_{ff} + k_{es} \qquad (2.89)$$

Novikov and Thorne (1973) also solved the above set of equations by dividing the disc into three regions which are same with that of Shakura and Sunyaev (1973). Therefore, their solution is that of Shakura and Sunyaev (1973) plus relativistic corrections.

4) Critical Discussion on Classical Model

Obviously, the classical model of thin accretion discs was an outstanding one. It appeared to be very simple, but was able to describe a clear picture of disc accretion process and was flexible enough to accommodate much observational data. Precisely because of this, the classical model provided a frame within which most of authors in the field have worked for last ten years. But, I wouldn't intend to detail the virtues of the classical model here, rather, evidently what would be crucial is to point out the defects of the model in order to find a way to improve the theory of thin accretion discs.

The main failing of accretion disc theory is that it generally has no predictive power. This is due to the lack of our knowledge about the nature and magnitude of the disc viscosity. As Wheeler (1981) commented on, the study of accretion discs is still in its infancy, analogous to the early days of Eddington when stars were modelled using elementary scaling laws without benefit of knowledge of the nuclear processes powering the stars. Similarly, most of the current literature on accretion discs is couched in terms of parametrized scaling laws, our knowledge of viscous processes powering the accretion discs is only at the crudest level. Accretion discs are a new kind of star -- flat two-dimensional-configuration stars. A breakthrough in study of viscous processes in accretion discs, analogous to that in study of nuclear processes in ordinary spherical stars in 1940s, is to be expected.

Probably one should invoke an observational approach to discovering the properties of disc viscosity rather than the purely theoritical one. The studies in this direction done by Webbink (1976), Lynden-Bell and Pringle (1974) and Bath and Pringle (1981),(1982) are meaningful.
The obvious defects of the classical model itself are its approximation and rough. All physical quantities describing the disc properties lost, at least partially, their local characteristic, they were replaced either by their values on the z=0 plane or by the z-direction-averaged values.
The temperature T appeared in the disc structure equations was only that on the central plane, while the other basic disc parameters, the radial velocity V_r , the pressure P, the density ρ (thus sound speed V_s) and energy flux Q^-, were averaged in vertical direction. Although the disc is by assumption thin, and the properties of the central plane may be most important in the disc region, a disc is not a plane, nor homogeneous, it has a definite structure in vertical direction, all quantities must vary with z-values from the central plane to the surfaces, and the dependences of them on z might be very important and is just what we want to know. Therefore, using the averaged values or that on the central plane instead of the local ones everywhere was rather crude. Besides, the assumption that V_ϕ is Keplerian everywhere without changing with z-values is also doubtful.
By virtue of introductions of some vertically integrated quantities (the surface density Σ and the integrated shear stress $T_{r\phi}$), the basic equations (the mass conservation, the angular momentum conservation and the thermal energy balance) were averaged vertically by integrating them over

the thickness of the disc. Clearly, such integrations could not be performed precisely because of uncertainty in vertical averaging and, hence, the derived relations and all results obtained from these relations were approximate. This was the reason why the results of the classical model had a character of order-of-magnitude estimations. Furthermore, since the dependences of disc parameters on z-coordinate might be very important (for example, the dependences of ρ and T on z determines the radiation spectrum), more accurate calculations of the parameters might probably not only give some quantitative corrections on the results of the classical model, but also even change qualitatively some of its general pictures about accretion process.

Besides the omissions of the radial pressure gradient, the radial energy flux and all components of shear stress tensor except $\tau_{r\phi}$, the vertical velocity V_z was killed completely by the introduction of the surface density Σ, or by the assumption of vertically hydrostatic equilibrium. Although V_z seems to be very small in comparisons with V_ϕ and even V_r, it might probably play an important role in the disc structure. For example, in the classical model, the equations and thus the solution of the disc structure were splitted into a radial part and a vertical one. Probably, the possibility of the splitting is partially based on the omission of V_z. If V_z -- a typically vertical parameter -- is not neglected and appears in a radial equation, such a splitting will no longer possible, and some connection and influence between the radial structure and the vertical one will be found -- this picture looks to be more reasonable.

III. A NEW POSSIBLE METHOD FOR STUDYING THIN DISCS

1) Basic Equations

Let us first write out the basic equations which describe the structrue and evolution of electromagnetically neutral accretion discs in a general coordinate system $\{t, x^1, x^2, x^3\}$ where t is the time coordinate and x^1, x^2, x^3 are three spatial coordinates. And then we shall specify the formulae to the spherical coordinates (r, θ, ϕ).
We shall use the covariant derivative operator ∇_i which is defined as

$$\nabla_i X = \frac{\partial X}{\partial x^i} \qquad \text{for a scalar function } X$$

$$\nabla_i X^k = \frac{\partial X^k}{\partial x^i} + \Gamma^k_{ij} X^j \qquad \text{for a vector function } X^k \tag{3.1}$$

$$\nabla_i X^{jk} = \frac{\partial X^{jk}}{\partial x^i} + \Gamma^k_{is} X^{js} + \Gamma^j_{is} X^{sk} \qquad \text{for a tensor function } X^{jk}$$

where the indices i,j,k run the values 1,2,3 and the Einstein summation convention is used. The Christoffel symbol Γ^k_{ji} is defined by

$$\Gamma^k_{ji} = \frac{1}{2} g^{sk} \left\{ \frac{\partial g_{is}}{\partial x^j} + \frac{\partial g_{js}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^s} \right\} \tag{3.2}$$

where g_{is} is the metric tensor and g^{sk} is the matrix reverse to g_{is} in the sense that

$$g_{is} g^{sk} = \delta^k_i = (\text{ Kronecker's delta }) \tag{3.3}$$

From this definition it follows that

$$\nabla_i X^i = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi^i} \left\{ \sqrt{g}\, X^i \right\}$$

$$\nabla_i X^{ij} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial \chi^i} \left\{ \sqrt{g}\, X^{ij} \right\} + \Gamma^j_{ik} X^{ik} \qquad (3.4)$$

where $g = \sqrt{\det. \| g_{ij} \|}$.

In the spherical coordinates we have

$$g_{rr} = 1, \qquad g_{\theta\theta} = r^2, \qquad g_{\phi\phi} = r^2 \mathrm{Sin}^2\theta \qquad (3.5)$$

all other components of g_{ij} are equal to zero. Thus

$$g = r^2 \mathrm{Sin}\,\theta \qquad (3.6)$$

The non-zero components of the matrix g^{ij} and the Christoffel symbol Γ^i_{jk} are

$$g^{rr} = \frac{1}{g_{rr}} = 1, \quad g^{\theta\theta} = \frac{1}{g_{\theta\theta}} = \frac{1}{r^2}, \quad g^{\phi\phi} = \frac{1}{g_{\phi\phi}} = \frac{1}{r^2 \mathrm{Sin}^2\theta} .$$

$$\Gamma^r_{\theta\theta} = -r, \qquad \Gamma^r_{\phi\phi} = -r\mathrm{Sin}^2\theta, \qquad (3.7)$$

$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = \frac{1}{r}, \qquad \Gamma^\theta_{\phi\phi} = -\mathrm{Sin}\theta\,\mathrm{Cos}\theta,$$

$$\Gamma^\phi_{\phi r} = \Gamma^\phi_{r\phi} = \frac{1}{r}, \qquad \Gamma^\phi_{\phi\theta} = \Gamma^\phi_{\theta\phi} = \mathrm{Ctg}\,\theta .$$

The following physical quantities will be used to describe the properties of the disc gas:

ρ = Density

P = Pressure

T = Temperature

S = Specific entropy

V^i = Generalized velocity

F^i = Energy flux

χ = Transport coefficient

ν = Kinematic viscosity

ν_R = Radiative viscosity

ν_B = Bulk viscosity

Φ = Gravitational potential

where V^i is the generalized velocity,

$$V^i = \frac{d\chi^i}{dt} \tag{3.8}$$

which differs from the physical velocity $V_{(i)}$ by

$$V_{(i)} = e^k_{(i)} V_k = e^k_{(i)} g_{jk} V^j \tag{3.9}$$

where $e^k_{(i)}$ is the unit vector in the direction of the coordinate x^k. In the case of spherical coordinates the connections between the generalized and physical velocities are

$$\begin{aligned} V_{(r)} &= V^r \\ V_{(\theta)} &= rV^\theta \\ V_{(\phi)} &= r\mathrm{Sin}\theta\ V^\phi \end{aligned} \tag{3.10}$$

Note that $V^\phi \equiv \Omega$ = (angular velocity of rotation).

WE shall also use the viscous stress tensor τ^{ij} and the heat dissipation function (heat generated by viscous friction per unit volume and unit time) Q, thy are defined in our notation as (refer to Tassoul 1978)

$$\tau^{ij} = \rho(\nu+\nu_R)(g^{ik}\nabla_k V^j + g^{jk}\nabla_k V^i - \frac{2}{3}g^{ij}\nabla_k V^k) + \rho(\nu_B + \frac{5}{3}\nu_R)g^{ij}\nabla_k V^k \tag{3.11}$$

$$Q = \frac{1}{2}\rho(\nu+\nu_R)(g^{ik}\nabla_k V^j + g^{jk}\nabla_k V^i - \frac{2}{3}g^{ij}\nabla_k V^k)\cdot(g_{il}\nabla_j V^l + g_{jl}\nabla_i V^l - \frac{2}{3}g_{ij}\nabla_k V^k) + \rho(\nu_B + \frac{5}{3}\nu_R)(\nabla_k V^k)^2 \tag{3.12}$$

With the help of these quantities the equations which describe the structure and evolution of discs can be written as (refer to Tassoul 1978)

a) Conservation of mass

$$\frac{\partial\rho}{\partial t} + \nabla_i(\rho V^i) = 0 \tag{3.13a}$$

b) Conservation of momentum

$$\frac{\partial V^i}{\partial t} + V^j\nabla_j V^i = -g^{ij}\nabla_j\Phi - \frac{1}{\rho}g^{ij}\nabla_j P + \frac{1}{\rho}\nabla_j\tau^{ij} \tag{3.13b}$$

c) Conservation of thermal energy

(if there is no nuclear reaction)

$$\rho T(\frac{\partial S}{\partial t} + V^i\nabla_i S) = Q - \nabla_i F^i \tag{3.13c}$$

d) Energy transport (if there is no convection)

$$F^i = -\chi g^{ij}\nabla_j T \tag{3.13d}$$

e) Equation of state

$$P = P(\rho, T)$$
$$S = S(\rho, T) \tag{3.13e}$$

In principle, these ten equations enable us to solve for ten unknown quantities (ρ, P, T, S, V^i, F^i) as functions of t and x^i, if the gravitational potential Φ, the viscosities ν, ν_R, ν_B and the transport coefficient χ are known.
In the case of spherical coordinates, from the consideration of axis symmetry, all quantities are functions of r, θ and t, not of ϕ, i.e., $\frac{\partial}{\partial\phi} = 0$ for any quantity, thus we have nine unknown quantities ρ, P, T, S, V^r, V^θ, V^ϕ, F^r, F^θ (F^ϕ=0), and the basic equations become (next page)

i) $$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho V^r)+\frac{1}{\sin\theta}\frac{\partial}{\partial \theta}(\sin\theta\,\rho V^\theta)=0 \tag{3.14a}$$

ii) $$\frac{\partial V^r}{\partial t}+V^r\frac{\partial V^r}{\partial r}+V^\theta\frac{\partial V^r}{\partial \theta}-rV^\theta V^\theta-r\sin^2\theta\,V^\phi V^\phi$$
$$=-\frac{\partial \Phi}{\partial r}-\frac{1}{\rho}\frac{\partial p}{\partial r}+\frac{1}{\rho}\left\{\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2\tau^{rr})\right.\right.$$
$$\left.\left.+r^2\frac{\partial}{\partial \theta}(\sin\theta\,\tau^{r\theta})\right]-r\tau^{\theta\theta}-r\sin^2\theta\,\tau^{\phi\phi}\right\} \tag{3.14b}$$

iii) $$\frac{\partial V^\theta}{\partial t}+V^r\frac{\partial V^\theta}{\partial r}+\frac{2}{r}V^rV^\theta+V^\theta\frac{\partial V^\theta}{\partial \theta}-\sin\theta\cos\theta\,V^\phi V^\phi$$
$$=-\frac{1}{r^2}\frac{\partial \Phi}{\partial \theta}-\frac{1}{r^2\rho}\frac{\partial p}{\partial \theta}+\frac{1}{\rho}\left\{\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2\tau^{\theta r})\right.\right.$$
$$\left.\left.+r^2\frac{\partial}{\partial \theta}(\sin\theta\,\tau^{\theta\theta})\right]+\frac{2}{r}\tau^{r\theta}-\sin\theta\cos\theta\,\tau^{\phi\phi}\right\} \tag{3.14c}$$

iv) $$\frac{\partial V^\phi}{\partial t}+V^r\frac{\partial V^\phi}{\partial r}+V^\theta\frac{\partial V^\phi}{\partial \theta}+\frac{2}{r}V^rV^\phi+2\,\mathrm{ctg}\,\theta\,V^\theta V^\phi$$
$$=\frac{1}{\rho}\left\{\frac{1}{r^2\sin\theta}\left[\sin\theta\frac{\partial}{\partial r}(r^2\tau^{\phi r})+r^2\frac{\partial}{\partial \theta}(\sin\theta\,\tau^{\phi\theta})\right]\right.$$
$$\left.+\frac{2}{r}\tau^{r\phi}+2\,\mathrm{ctg}\,\theta\,\tau^{\theta\phi}\right. \tag{3.14d}$$

v> $\rho T\left[\frac{\partial s}{\partial t}+v^r\frac{\partial s}{\partial r}+v^\theta\frac{\partial s}{\partial \theta}\right]$

$$= \rho(\nu+\nu_R)\Big[2\left(\frac{\partial v^r}{\partial r}\right)^2+\frac{1}{r^2}\left(\frac{\partial v^r}{\partial \theta}\right)^2+r^2\left(\frac{\partial v^\theta}{\partial r}\right)^2$$
$$+2\left(\frac{\partial v^\theta}{\partial \theta}\right)^2+r^2\sin^2\theta\left(\frac{\partial v^\phi}{\partial r}\right)^2+\sin^2\theta\left(\frac{\partial v^\phi}{\partial \theta}\right)^2$$
$$+2\left(\frac{\partial v^r}{\partial \theta}\right)\left(\frac{\partial v^\theta}{\partial r}\right)+\frac{4}{r}v^r\frac{\partial v^\theta}{\partial \theta}+\frac{4}{r^2}(v^r)^2$$
$$+2\,\mathrm{ctg}^2\theta\,(v^\theta)^2+\frac{4}{r}\mathrm{ctg}\theta\,v^r v^\theta\Big]$$
$$+\rho\left(\nu_B+\nu_R-\frac{2}{3}\nu\right)\left[\frac{\partial v^r}{\partial r}+\frac{\partial v^\theta}{\partial \theta}+\frac{2}{r}v^r+\mathrm{ctg}\theta\,v^\theta\right]^2$$
$$-\frac{\partial F^r}{\partial r}-\frac{\partial F^\theta}{\partial \theta}-\frac{2}{r}F^r-\mathrm{ctg}\theta\,F^\theta \qquad (3.14\,e)$$

vi> $$F^r = -\chi\frac{\partial T}{\partial r} \qquad (3.14\,f)$$

vii> $$F^\theta = -\chi\frac{1}{r^2}\frac{\partial T}{\partial \theta} \qquad (3.14\,g)$$

viii> $$P = P(\rho, T) \qquad (3.14\,h)$$

ix> $$S = S(\rho, T) \qquad (3.14\,i)$$

The system is also closed.

The components of τ^{ij} appeared in (3.14) are:

$$\tau^{rr} = 2\rho(\nu+\nu_R)\left(\frac{\partial V^r}{\partial r}\right) + \rho\left(\nu_B+\nu_R-\frac{2}{3}\nu\right)\cdot$$
$$\left(\frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta} + \frac{2}{r}V^r + \mathrm{ctg}\theta\, V^\theta\right)$$

$$\tau^{\theta\theta} = 2\rho(\nu+\nu_R)\frac{1}{r^2}\left(\frac{\partial V^\theta}{\partial \theta} + \frac{1}{r}V^r\right)$$
$$+ \rho\left(\nu_B+\nu_R-\frac{2}{3}\nu\right)\frac{1}{r^2}\left(\frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta} + \frac{2}{r}V^r + \mathrm{ctg}\theta\, V^\theta\right)$$

$$\tau^{\phi\phi} = 2\rho(\nu+\nu_R)\frac{1}{r^2\sin^2\theta}\left(\frac{1}{r}V^r + \mathrm{ctg}\theta\, V^\theta\right)$$
$$+\rho\left(\nu_B+\nu_R-\frac{2}{3}\nu\right)\frac{1}{r^2\sin^2\theta}\left(\frac{\partial V^r}{\partial r} + \frac{\partial V^\theta}{\partial \theta} + \frac{2}{r}V^r + \mathrm{ctg}\theta\, V^\theta\right)$$

$$\tau^{r\theta} = \tau^{\theta r} = \rho(\nu+\nu_R)\left(\frac{1}{r^2}\frac{\partial V^r}{\partial \theta} + \frac{\partial V^\theta}{\partial r}\right)$$

$$\tau^{\theta\phi} = \tau^{\phi\theta} = \rho(\nu+\nu_R)\frac{1}{r^2}\frac{\partial V^\phi}{\partial \theta}$$

$$\tau^{\phi r} = \tau^{r\phi} = \rho(\nu+\nu_R)\frac{\partial V^\phi}{\partial r}.$$

(3.15)

Substituting (3.15) into (3.14), we get the explicit form of basic equations of disc in spherical coordinates which are given in the appendix.

2) Method and Result

All quantities present in equations (3.14) are functions of r, θ and t. (Generally, the gravitational potential Φ is also a function of r, θ and t, but if we neglect the gravitational pull of the mass-losing star and the self-gravitation of the disc matter, i.e., the potential is that due to a central point mass, Φ will be a function of r only). We must find a way to separate two spatial variables r and θ in order to solve the equations. Let us first emphasize the following two assumptions:

a) The disc is thin, i.e.,

$$|\mathrm{Cos}\,\theta| \leq |\mathrm{Cos}\,\theta_S| << 1 \qquad (3.16)$$

where θ_S is the value of θ on the surface of the disc.

b) All quantities are symmetric about the central plane of the disc, i.e., at any two points which possess the same r-value but values of θ and $\theta+\frac{\pi}{2}$, respectively ($\theta=\frac{\pi}{2}$ is the central plane of the disc), the values of anyone of quantities ρ, P, T, S, V^r, V^ϕ, F^r, ν, ν_R, ν_B, χ are exactly same, but, V^θ or F^θ at such two points will only have the same absolute values but with opposite signs.

From these two assumptions it is followed that we may expend every quantity as a power series of $\mathrm{Cos}\,\theta$

$$\rho_{(r,\theta,t)} = \rho_{0}(r,t) + \rho_{2}(r,t) \cos^2\theta + \cdots$$

$$p_{(r,\theta,t)} = p_{0}(r,t) + p_{2}(r,t) \cos^2\theta + \cdots$$

$$T_{(r,\theta,t)} = T_{0}(r,t) + T_{2}(r,t) \cos^2\theta + \cdots$$

$$S_{(r,\theta,t)} = S_{0}(r,t) + S_{2}(r,t) \cos^2\theta + \cdots$$

$$V^r_{(r,\theta,t)} = V^r_0(r,t) + V^r_2(r,t) \cos^2\theta + \cdots$$

$$V^\phi_{(r,\theta,t)} = V^\phi_0(r,t) + V^\phi_2(r,t) \cos^2\theta + \cdots$$

$$F^r_{(r,\theta,t)} = F^r_0(r,t) + F^r_2(r,t) \cos^2\theta + \cdots \qquad (3.17)$$

$$\nu_{(r,\theta,t)} = \nu_0(r,t) + \nu_2(r,t) \cos^2\theta + \cdots$$

$$\nu_R(r,\theta,t) = \nu_{R0}(r,t) + \nu_{R2}(r,t) \cos^2\theta + \cdots$$

$$\nu_B(r,\theta,t) = \nu_{B0}(r,t) + \nu_{B2}(r,t) \cos^2\theta + \cdots$$

$$\chi(r,\theta,t) = \chi_0(r,t) + \chi_2(r,t) \cos^2\theta + \cdots$$

$$V^\theta_{(r,\theta,t)} = \cos\theta \left[V^\theta_0(r,t) + V^\theta_2(r,t) \cos^2\theta + \cdots \right]$$

$$F^\theta_{(r,\theta,t)} = \cos\theta \left[F^\theta_0(r,t) + F^\theta_2(r,t) \cos^2\theta + \cdots \right]$$

By substituting (3.17) into (3.14), each equation of (3.14) may be written as a power series of Cosθ which is equal to zero. Since all terms in such a series are independent each other, we may put each term equal to zero and thus obtain in different orders new equations in which all quantities will be functions of r and t only. We shall do this up to the order of $\cos^2\theta$. Accordingly, we shall keep all quantities, expect V^θ and F^θ, up to $\cos^2\theta$ order, V^θ will be kept to $\cos\theta$ oeder but F^θ will be kept to $\cos^3\theta$ order since it would be more important than F^r.

Using this method into mass conservation equation (3.14a), we obtain on the order of $\cos^0\theta$

$$\frac{\partial \rho_0}{\partial t} + \rho_0 \frac{\partial V_0^r}{\partial r} + V_0^r \frac{\partial \rho_0}{\partial r} + \frac{2}{r} \rho_0 V_0^r - \rho_0 V_0^\theta = 0 \tag{3.18}$$

and on the order of $\cos^2\theta$ (3.14a) reads

$$\frac{\partial \rho_2}{\partial t} + \rho_0 \frac{\partial V_2^r}{\partial r} + \rho_2 \frac{\partial V_0^r}{\partial r} + V_0^r \frac{\partial \rho_2}{\partial r} + V_2^r \frac{\partial \rho_0}{\partial r} + \frac{2}{r} \rho_0 V_2^r + \frac{2}{r} \rho_2 V_0^r - 3V_0^\theta \rho_2 + \frac{3}{2} \rho_0 V_0^\theta = 0 \tag{3.19}$$

(3.14b) -- (3.14e) are much more complicated than (3.14a), therefore, it would be appropriate to make some estimate of magnitudes of the quantities in these equations before applying our method to them.

(2.36) and (2.37) tell us that the sound velocity satisfies

$$\begin{aligned} V_s^2 &= P/\rho \\ &\sim (rV^\phi)^2 (H/r)^2 \\ &= (rV^\phi)^2 \cos^2\theta_s \\ &\ll (rV^\phi)^2 \end{aligned} \tag{3.20}$$

and we know that the viscosity

$$\nu \lesssim HV_S << r(rV^{\phi}) \qquad (3.21)$$

According to (2.54), V^r is of the order of $\frac{\nu}{r}$ for the region far away from the disc's inner edge

$$V^r \sim \frac{\nu}{r} << rV^{\phi} \qquad (3.22)$$

and (rV^{θ}) is even smaller than V^r.

On the basis of these estimates of magnitudes of the quantities, we make the following regulations in using our method to equations (3.14b -- e):

i) Neglect all terms which are

a) One of ν, ν_R, ν_θ or their derivatives times one of V^r, V^{θ} or their derivatives, e.g., νV^r, $\nu_R \frac{\partial^2 V^r}{\partial r^2}$, $\nu_\theta \frac{\partial V^{\theta}}{\partial \theta}$, etc.

b) V^r, V^{θ} or thier derivatives times each other, e.g., $V^r \frac{\partial V^r}{\partial r}$, $(V^{\theta})^2$, etc.

ii) When there is V_0^{ϕ} only in an equation, neglect all terms containing V^r or V^{θ}; when V_2^{ϕ} appears, retain the terms containing V_0^r or V_0^{θ} but neglect that containing V_2^r.

iii) On the order of $\mathrm{Cos}^0\theta$, neglect pressure gradient.

Thus we have from (3.14b)

on $\mathrm{Cos}^0\theta$ order $$(V_0^{\phi})^2 = GM/r^3 \qquad (3.23)$$

on $\mathrm{Cos}^2\theta$ order $$\rho_2 \frac{\partial V_0^r}{\partial t} - r\rho_2 (V_0^{\phi})^2 - 2r\rho_0 V_0^{\phi} V_2^{\phi} + r\rho_0 (V_0^{\phi})^2 = -\rho_2 \frac{GM}{r^2} - \frac{\partial P_2}{\partial r} \qquad (3.24)$$

from (3.14c)

on $\mathrm{Cos}\,\theta$ order $$-r^2 \rho_0 (V_0^{\phi})^2 = 2P_2 \qquad (3.25)$$

from (3.14d)

on $\cos^0\theta$ order

$$\frac{\partial V_0^\phi}{\partial t} + V_0^r \frac{\partial V_0^\phi}{\partial r} + \frac{2}{r} V_0^r V_0^\phi$$

$$= (\nu_0 + \nu_{R0}) \frac{\partial^2 V_0^\phi}{\partial r^2} + \frac{2}{r^2}(\nu_0 + \nu_{R0}) V_2^\phi$$

$$+ \frac{4}{r}(\nu_0 + \nu_{R0}) \frac{\partial V_0^\phi}{\partial r} + \frac{\partial(\nu_0 + \nu_{R0})}{\partial r} \frac{\partial V_0^\phi}{\partial r}$$

$$+ \frac{1}{\rho_0}(\nu_0 + \nu_{R0}) \frac{\partial \rho_0}{\partial r} \frac{\partial V_0^\phi}{\partial r} \qquad (3.26)$$

on $\cos^2\theta$ order

$$\rho_0 \frac{\partial V_2^\phi}{\partial t} + \rho_2 \frac{\partial V_0^\phi}{\partial t} + \rho_0 V_0^r \frac{\partial V_2^\phi}{\partial r} + \rho_2 V_0^r \frac{\partial V_0^\phi}{\partial r}$$

$$- 2 \rho_0 V_0^\theta V_2^\phi + \frac{2}{r} \rho_0 V_0^r V_2^\phi + \frac{2}{r} \rho_2 V_0^r V_0^\phi + 2 \rho_0 V_0^\theta V_0^\phi$$

$$= \rho_0 (\nu_0 + \nu_{R0}) \frac{\partial^2 V_2^\phi}{\partial r^2} + \rho_0 (\nu_2 + \nu_{R2}) \frac{\partial^2 V_0^\phi}{\partial r^2} + \rho_2 (\nu_0 + \nu_{R0}) \frac{\partial^2 V_0^\phi}{\partial r^2}$$

$$- \frac{4}{r^2} \rho_0 (\nu_0 + \nu_{R0}) V_2^\phi + \frac{2}{r^2} \rho_0 (\nu_2 + \nu_{R2}) V_2^\phi + \frac{2}{r^2} \rho_2 (\nu_0 + \nu_{R0}) V_2^\phi$$

$$+ \frac{4}{r} \rho_0 (\nu_0 + \nu_{R0}) \frac{\partial V_2^\phi}{\partial r} + \frac{4}{r} \rho_0 (\nu_2 + \nu_{R2}) \frac{\partial V_0^\phi}{\partial r} + \frac{4}{r} \rho_2 (\nu_0 + \nu_{R0}) \frac{\partial V_0^\phi}{\partial r}$$

$$- \frac{6}{r^2}(\nu_0 + \nu_{R0}) \rho_0 V_2^\phi + \rho_0 \frac{\partial(\nu_0 + \nu_{R0})}{\partial r} \frac{\partial V_2^\phi}{\partial r} + (\nu_0 + \nu_{R0}) \frac{\partial \rho_0}{\partial r} \frac{\partial V_2^\phi}{\partial r}$$

$$+ \rho_0 \frac{\partial(\nu_2 + \nu_{R2})}{\partial r} \frac{\partial V_0^\phi}{\partial r} + \rho_2 \frac{\partial(\nu_0 + \nu_{R0})}{\partial r} \frac{\partial V_0^\phi}{\partial r} + (\nu_2 + \nu_{R2}) \frac{\partial \rho_0}{\partial r} \frac{\partial V_0^\phi}{\partial r}$$

$$+ (\nu_0 + \nu_{R0}) \frac{\partial \rho_2}{\partial r} \frac{\partial V_0^\phi}{\partial r} + \frac{4}{r^2} V_2^\phi \rho_0 (\nu_2 + \nu_{R2}) + \frac{4}{r^2} V_2^\phi \rho_2 (\nu_0 + \nu_{R0})$$

$$(3.27)$$

from (3.14e)

on $\cos^0\theta$ order $\quad \rho_0(\nu_0+\nu_{R0})r^2\left(\frac{\partial V_0^\phi}{\partial r}\right)^2 - \frac{\partial F_0^r}{\partial r} + F_0^\theta - \frac{2}{r}F_0^r$

$$= \rho_0 T_0\left(\frac{\partial S_0}{\partial t} + V_0^r\frac{\partial S_0}{\partial r}\right) \tag{3.28}$$

on $\cos^2\theta$ order $\quad \rho_0(\nu_2+\nu_{R2})r^2\left(\frac{\partial V_0^\phi}{\partial r}\right)^2 + \rho_2(\nu_0+\nu_{R0})r^2\left(\frac{\partial V_0^\phi}{\partial r}\right)^2$

$$+ 2\rho_0(\nu_0+\nu_{R0})r^2\left(\frac{\partial V_0^\phi}{\partial r}\right)\left(\frac{\partial V_2^\phi}{\partial r}\right) + 4\rho_0(\nu_0+\nu_{R0})(V_2^\phi)^2$$

$$- \rho_0(\nu_0+\nu_{R0})r^2\left(\frac{\partial V_0^\phi}{\partial r}\right)^2 - \frac{\partial F_2^r}{\partial r} - \frac{2}{r}F_2^r + 3F_2^\theta - \frac{3}{2}F_0^\theta$$

$$= \rho_0 T_0\frac{\partial S_2}{\partial t} + \rho_0 T_2\frac{\partial S_0}{\partial t} + \rho_2 T_0\frac{\partial S_0}{\partial t} + \rho_0 T_0 V_0^r\frac{\partial S_2}{\partial r}$$

$$\rho_0 T_2 V_0^r\frac{\partial S_0}{\partial r} + \rho_2 T_0 V_0^r\frac{\partial S_0}{\partial r} - 2\rho_0 T_0 V_0^\theta S_2 \tag{3.29}$$

It is easy to use our expansion scheme to equations (3.14f -- i), the result is

$$F_0^r = -\chi_0\frac{\partial T_0}{\partial r} \tag{3.30}$$

$$F_2^r = -\chi_0\frac{\partial T_2}{\partial r} - \chi_2\frac{\partial T_0}{\partial r} \tag{3.31}$$

$$F_0^\theta = \frac{2}{r^2}\chi_0 T_2 \tag{3.32}$$

$$P_0 = P_0(\rho_0, T_0) \tag{3.33}$$

$$P_2 = P_2(\rho_0, \rho_2, T_0, T_2) \tag{3.34}$$

$$S_0 = S_0(\rho_0, T_0) \tag{3.35}$$

$$S_2 = S_2(\rho_0, \rho_2, T_0, T_2) \tag{3.36}$$

Up to now we have obtained 16 equations which are (3.18), (3.19) and (3.23) -- (3.36), but there are 17 unknown quantities in these equations, namely,

$$\rho_0,\ P_0,\ T_0,\ S_0,\ F_0^r,\ F_0^\theta,\ V_0^\phi,\ V_0^r,\ V_0^\theta,$$
$$\rho_2,\ P_2,\ T_2,\ S_2,\ F_2^r,\ F_2^\theta,\ V_2^\phi,\ V_2^r.$$

Therefore, we must find another equation in order to make the system closed.

It is reasonable and necessary to find some relations among the disc quantities from the boundary conditions, since the above basic equations do not tell us what happens on the boundary. First, we define the disc surface as on that pressure is zero. This definition means that there is no pressure gradient along the disc surface, the pressure gradient vector must be normal to the surface. Second, we assume that there is no matter flow across the boundary, this is to say that the normal component of velocity of disc matter on the disc surface is zero, the matter can move only along the surface. Combining the above two boundary conditions, we know that the pressure gradient vector and velocity vector on the disc surface must be perpendicular to each other, that is,

$$\left[(\nabla_i P)V^i\right]_{P=0} = 0 \qquad (3.37)$$

Applying our expansion scheme into (3.37) upto $\mathrm{Cos}^2\theta$ order ($\mathrm{Cos}^2\theta$ become $\mathrm{Cos}^2\theta_s$ on the surface) and we have

$$\cos^2\theta_S = -P_c/P_2 \tag{3.38}$$

from

$$P = P_c + P_2\cos^2\theta_S = 0 \tag{3.39}$$

Substituting (3.38) into the expansion of (3.37), we get the following relation

$$P_2 V_c^r \frac{\partial P_c}{\partial r} - P_c V_2^r \frac{\partial P_c}{\partial r} - P_c V_c^r \frac{\partial P_2}{\partial r} + 2P_c P_2 V_c^\theta = 0 \tag{3.40}$$

Although this relation is obtained from the boundary conditions, it is held everywhere in the disc since all quantities in it are functions of r and t only. Thus, we have found the last necessary equation and have had a closed set of equations describing the structure and evolution of the disc.

3) Discussion

By comparing our equations with that of the classical model, it can be seen that there are following differences between them:

i) All quantities in our equations are local, thy are neither vertically averaged ones, nor replaced by the values on the central plane, thus they are more accurate than that in the classical model in describing the disc properties.

ii) There is V^{θ} -- the vertical velocity -- in our equations which was absent in the classical model. It is interesting that V_c^{θ} appears in (3.18) -- the zero-order equation of mass conservation which is rewritten here

$$\frac{\partial \rho_c}{\partial t} + \rho_c \frac{\partial V_c^r}{\partial r} + V_c^r \frac{\partial \rho_c}{\partial r} + \frac{2}{r} \rho_c V_c^r - \rho_c V_c^{\theta} = 0 \qquad (3.18)$$

Note that ρ_c and V_c^r are values of ρ and V^r on the central plane of the disc, respectively, and V^{θ} is zero on the central plane (see 3.17). If there were no the term of ($-\rho_c V_c^{\theta}$) in (3.18), it would be a equation which describe purely the structure of the central plane of the disc. But, V_c^{θ} does appear in (3.18). Since V^{θ} is a typically vertical parameter, this means that there exists connection and influence between the horizontal and vertical structures of the disc. Splitting the disc equations into a horizontal part and a vertical one -- this was done in the classical model -- is rather unreasonable.

iii) In our equations, V^{ϕ} is no longer everywhere Keplerian, only V_0^{ϕ} -- the zero-order value of V^{ϕ} -- is Keplerian, there is some non-Keplerian correction on the circular velocity which is represented by V_2^{ϕ} -- this is again different from the classical model where V^{ϕ} is Keplerian everywhere in the disc.

iv) Unlike the classical model in which the horizontal pressure gradient and energy flux were neglected totally, there are radial pressure gradient and energy flux in our equations.

v) From our thermal energy conservation equation it can be seen that the heat is generated not only by the frictions between different rings with different radii (this is the same with that in the classical model), but also by the frictions between different layers with different θ-values (this was absent in the classical model).

Since our equations have the above advantages, some new results obtained in solving them which can help us to know better the propertities of thin accretion discs are to be expected.

$$
\begin{aligned}
\text{i)}\ & \frac{\partial \rho}{\partial t} + \rho \frac{\partial v^r}{\partial r} + v^r \frac{\partial \rho}{\partial r} + \frac{2}{r} \rho v^r + \rho \frac{\partial v^\theta}{\partial \theta} \\
& \qquad + v^\theta \frac{\partial \rho}{\partial \theta} + \rho v^\theta \mathrm{ctg}\theta = 0
\end{aligned}
$$

$$
\begin{aligned}
\text{ii)}\ & \rho \frac{\partial v^r}{\partial t} + \rho v^r \frac{\partial v^r}{\partial r} + \rho v^\theta \frac{\partial v^r}{\partial \theta} - r\rho v^\theta v^\theta - r\rho \sin^2\theta\, v^\phi v^\phi \\
= & -\rho \frac{\partial \Phi}{\partial r} - \frac{\partial p}{\partial r} + \frac{4}{3}\rho\nu \frac{\partial^2 v^r}{\partial r^2} + 3\rho\nu_R \frac{\partial^2 v^r}{\partial r^2} + \rho\nu_B \frac{\partial^2 v^r}{\partial r^2} \\
& + \frac{1}{3}\rho\nu \frac{\partial^2 v^\theta}{\partial r \partial \theta} + \frac{1}{r^2}\rho\nu \frac{\partial^2 v^r}{\partial \theta^2} \\
& + \rho\nu_B \frac{\partial^2 v^\theta}{\partial r \partial \theta} + 2\rho\nu_R \frac{\partial^2 v^\theta}{\partial r \partial \theta} \\
& + \frac{1}{r^2}\rho\nu_R \frac{\partial^2 v^r}{\partial \theta^2} + \frac{8}{3r}\rho\nu \frac{\partial v^r}{\partial r} + \frac{6}{r}\rho\nu_R \frac{\partial v^r}{\partial r} + \frac{2}{r}\rho\nu_B \frac{\partial v^r}{\partial r} \\
& + \frac{1}{r^2}\rho\nu\, \mathrm{ctg}\theta \frac{\partial v^r}{\partial \theta} + \frac{1}{r^2}\rho\nu_R\, \mathrm{ctg}\theta \frac{\partial v^r}{\partial \theta} + \rho\nu_B\, \mathrm{ctg}\theta \frac{\partial v^\theta}{\partial r} \\
& + 2\rho\nu_R\, \mathrm{ctg}\theta \frac{\partial v^\theta}{\partial r} + \frac{1}{3}\rho\nu\, \mathrm{ctg}\theta \frac{\partial v^\theta}{\partial r} - \frac{2}{r^2}\rho\nu_B v^r - \frac{6}{r^2}\rho\nu_R v^r \\
& + \frac{8}{3r^2}\rho\nu v^r - \frac{2}{r}\rho\nu \frac{\partial v^\theta}{\partial \theta} - \frac{2}{r}\rho\nu_R \frac{\partial v^\theta}{\partial \theta} - \frac{2}{r}\rho\nu\, \mathrm{ctg}\theta\, v^\theta \\
& - \frac{2}{r}\rho\nu_R\, \mathrm{ctg}\theta\, v^\theta + \rho \frac{\partial v^r}{\partial r}\frac{\partial \nu_B}{\partial r} + \nu_B \frac{\partial v^r}{\partial r}\frac{\partial \rho}{\partial r} + \frac{2}{r}\rho v^r \frac{\partial \nu_B}{\partial r} \\
& + \frac{2}{r}\nu^B v^r \frac{\partial \rho}{\partial r} + \rho \frac{\partial v^\theta}{\partial \theta}\frac{\partial \nu_B}{\partial r} + \nu_B \frac{\partial v^\theta}{\partial \theta}\frac{\partial \rho}{\partial r} + \rho v^\theta \mathrm{ctg}\theta \frac{\partial \nu_B}{\partial r} \\
& + \nu_B\, \mathrm{ctg}\theta\, v^\theta \frac{\partial \rho}{\partial r} + 3\rho \frac{\partial v^r}{\partial r}\frac{\partial \nu_R}{\partial r} + 3\nu_R \frac{\partial v^r}{\partial r}\frac{\partial \rho}{\partial r} + \frac{2}{r}\rho v^r \frac{\partial \nu_R}{\partial r} \\
& + \frac{2}{r} v^r \nu_R \frac{\partial \rho}{\partial r} + \rho \frac{\partial v^\theta}{\partial \theta}\frac{\partial \nu_R}{\partial r} + \nu_R \frac{\partial v^\theta}{\partial \theta}\frac{\partial \rho}{\partial r} + \rho v^\theta \mathrm{ctg}\theta \frac{\partial \nu_R}{\partial r} \\
& + v^\theta \mathrm{ctg}\theta\, \nu_R \frac{\partial \rho}{\partial r} + \frac{4}{3}\rho \frac{\partial v^r}{\partial r}\frac{\partial \nu}{\partial r} + \frac{4}{3}\nu \frac{\partial v^r}{\partial r}\frac{\partial \rho}{\partial r} - \frac{4}{3r}\rho v^r \frac{\partial \nu}{\partial r} \\
& - \frac{4}{3r} v^r \nu \frac{\partial \rho}{\partial r} - \frac{2}{3}\rho \frac{\partial v^\theta}{\partial \theta}\frac{\partial \nu}{\partial r} - \frac{2}{3}\nu \frac{\partial v^\theta}{\partial \theta}\frac{\partial \rho}{\partial r} - \frac{2}{3}\rho v^\theta \mathrm{ctg}\theta \frac{\partial \nu}{\partial r} \\
& - \frac{2}{3} v^\theta \mathrm{ctg}\theta\, \nu \frac{\partial \rho}{\partial r} + \frac{1}{r^2}\rho \frac{\partial v^r}{\partial \theta}\frac{\partial \nu}{\partial \theta} + \frac{1}{r^2}\nu \frac{\partial v^r}{\partial \theta}\frac{\partial \rho}{\partial \theta} + \rho \frac{\partial v^\theta}{\partial r}\frac{\partial \nu}{\partial \theta} \\
& + \nu \frac{\partial v^\theta}{\partial r}\frac{\partial \rho}{\partial \theta} + \frac{1}{r^2}\rho \frac{\partial v^r}{\partial \theta}\frac{\partial \nu_R}{\partial \theta} + \frac{1}{r^2}\nu_R \frac{\partial v^r}{\partial \theta}\frac{\partial \rho}{\partial \theta} + \rho \frac{\partial v^\theta}{\partial r}\frac{\partial \nu_R}{\partial \theta} \\
& + \nu_R \frac{\partial v^\theta}{\partial r}\frac{\partial \rho}{\partial \theta}
\end{aligned}
$$

iii) $r^2\rho\frac{\partial v^\theta}{\partial t} + r^2\rho v^r\frac{\partial v^\theta}{\partial r} + 2r\rho v^r v^\theta + r^2\rho v^\theta\frac{\partial v^\theta}{\partial\theta} - r^2\rho\sin\theta\cos\theta\, v^\phi v^\phi$

$$= -\rho\frac{\partial\Phi}{\partial\theta} - \frac{\partial p}{\partial\theta} + \frac{1}{3}\rho\nu\frac{\partial^2 v^r}{\partial r\partial\theta} + 2\rho\nu_R\frac{\partial^2 v^r}{\partial r\partial\theta} + \rho\nu_B\frac{\partial^2 v^r}{\partial r\partial\theta}$$

$$+ r^2\rho\nu\frac{\partial^2 v^\theta}{\partial r^2} + r^2\rho\nu_R\frac{\partial^2 v^\theta}{\partial r^2} + \frac{4}{3}\rho\nu\frac{\partial^2 v^\theta}{\partial\theta^2} + \rho\nu_B\frac{\partial^2 v^\theta}{\partial\theta^2}$$

$$+ 3\rho\nu_R\frac{\partial^2 v^\theta}{\partial\theta^2} + \frac{8}{3r}\rho\nu\frac{\partial v^r}{\partial\theta} + \frac{6}{r}\rho\nu_R\frac{\partial v^r}{\partial\theta} + 4r^2\rho\nu\frac{\partial v^\theta}{\partial r}$$

$$+ 4r\rho\nu_R\frac{\partial v^\theta}{\partial r} + \frac{4}{3}\rho\nu\,\mathrm{ctg}\theta\frac{\partial v^\theta}{\partial\theta} + 3\rho\nu_R\,\mathrm{ctg}\theta\frac{\partial v^\theta}{\partial\theta}$$

$$+ \rho\nu_B\,\mathrm{ctg}\theta\frac{\partial v^\theta}{\partial\theta} + \frac{2}{r}\rho\nu_B\frac{\partial v^r}{\partial\theta} + \frac{2}{3}\rho\nu\left(\frac{1}{\sin^2\theta} - 3\,\mathrm{ctg}^2\theta\right)v^\theta$$

$$- \rho\nu_R\left(\frac{1}{\sin^2\theta} + 2\,\mathrm{ctg}^2\theta\right)v^\theta - \frac{1}{\sin^2\theta}\rho\nu_B v^\theta + \rho\frac{\partial v^r}{\partial\theta}\frac{\partial\nu}{\partial r}$$

$$+ \nu\frac{\partial v^r}{\partial\theta}\frac{\partial\rho}{\partial r} + \rho\frac{\partial v^r}{\partial\theta}\frac{\partial\nu_R}{\partial r} + \nu_R\frac{\partial v^r}{\partial\theta}\frac{\partial\rho}{\partial r} + r^2\rho\frac{\partial v^\theta}{\partial r}\frac{\partial\nu}{\partial r}$$

$$+ r^2\nu\frac{\partial v^\theta}{\partial r}\frac{\partial\rho}{\partial r} + r^2\rho\frac{\partial v^\theta}{\partial r}\frac{\partial\nu_R}{\partial r} + r^2\nu_R\frac{\partial v^\theta}{\partial r}\frac{\partial\rho}{\partial r} + \rho\frac{\partial v^r}{\partial r}\frac{\partial\nu_B}{\partial\theta}$$

$$+ \nu_B\frac{\partial v^r}{\partial r}\frac{\partial\rho}{\partial\theta} + \frac{2}{r}\rho v^r\frac{\partial\nu_B}{\partial\theta} + \frac{2}{r}\nu_B v^r\frac{\partial\rho}{\partial\theta} + \rho\frac{\partial v^\theta}{\partial\theta}\frac{\partial\nu_B}{\partial\theta}$$

$$+ \nu_B\frac{\partial v^\theta}{\partial\theta}\frac{\partial\rho}{\partial\theta} + \rho v^\theta\,\mathrm{ctg}\theta\frac{\partial\nu_B}{\partial\theta} + v^\theta\,\mathrm{ctg}\theta\,\nu_B\frac{\partial\rho}{\partial\theta} + \rho\frac{\partial v^r}{\partial r}\frac{\partial\nu_R}{\partial\theta}$$

$$+ \nu_R\frac{\partial v^r}{\partial r}\frac{\partial\rho}{\partial\theta} + \frac{4}{r}\rho v^r\frac{\partial\nu_R}{\partial\theta} + \frac{4}{r}\nu_R v^r\frac{\partial\rho}{\partial\theta} + 3\rho\frac{\partial v^\theta}{\partial\theta}\frac{\partial\nu_R}{\partial\theta}$$

$$+ 3\nu_R\frac{\partial v^\theta}{\partial\theta}\frac{\partial\rho}{\partial\theta} + \rho v^\theta\,\mathrm{ctg}\theta\frac{\partial\nu_R}{\partial\theta} + v^\theta\,\mathrm{ctg}\theta\,\nu_R\frac{\partial\rho}{\partial\theta}$$

$$- \frac{2}{3}\rho\frac{\partial v^r}{\partial r}\frac{\partial\nu}{\partial\theta} - \frac{2}{3}\nu\frac{\partial v^r}{\partial r}\frac{\partial\rho}{\partial\theta} + \frac{2}{3r}\rho v^r\frac{\partial\nu}{\partial\theta} + \frac{2}{3r}\nu v^r\frac{\partial\rho}{\partial\theta}$$

$$+ \frac{4}{3}\rho\frac{\partial v^\theta}{\partial\theta}\frac{\partial\nu}{\partial\theta} + \frac{4}{3}\nu\frac{\partial v^\theta}{\partial\theta}\frac{\partial\rho}{\partial\theta} - \frac{2}{3}\rho v^\theta\,\mathrm{ctg}\theta\frac{\partial\nu}{\partial\theta} - \frac{2}{3}\nu v^\theta\,\mathrm{ctg}\theta\frac{\partial\rho}{\partial\theta}$$

$$\text{iv)}\quad \rho\frac{\partial v^\phi}{\partial t}+\rho v^r\frac{\partial v^\phi}{\partial r}+\rho v^\theta\frac{\partial v^\phi}{\partial \theta}+\frac{2}{r}\rho v^r v^\phi+2\rho\,\mathrm{ctg}\,\theta\, v^\theta v^\phi$$

$$=\rho\nu\frac{\partial^2 v^\phi}{\partial r^2}+\rho\nu_R\frac{\partial^2 v^\phi}{\partial r^2}+\frac{1}{r^2}\rho\nu\frac{\partial^2 v^\phi}{\partial \theta^2}+\frac{1}{r^2}\rho\nu_R\frac{\partial^2 v^\phi}{\partial \theta^2}$$

$$+\frac{4}{r}\rho\nu\frac{\partial v^\phi}{\partial r}+\frac{4}{r}\rho\nu_R\frac{\partial v^\phi}{\partial r}+\frac{3}{r^2}\rho\nu\,\mathrm{ctg}\,\theta\frac{\partial v^\phi}{\partial \theta}$$

$$+\frac{3}{r^2}\rho\nu_R\,\mathrm{ctg}\,\theta\frac{\partial v^\phi}{\partial \theta}+\rho\frac{\partial v^\phi}{\partial r}\frac{\partial \nu}{\partial r}+\nu\frac{\partial v^\phi}{\partial r}\frac{\partial \rho}{\partial r}$$

$$+\rho\frac{\partial v^\phi}{\partial r}\frac{\partial \nu_R}{\partial r}+\nu_R\frac{\partial v^\phi}{\partial r}\frac{\partial \rho}{\partial r}+\frac{1}{r^2}\rho\frac{\partial v^\phi}{\partial \theta}\frac{\partial \nu}{\partial \theta}$$

$$+\frac{1}{r^2}\nu\frac{\partial v^\phi}{\partial \theta}\frac{\partial \rho}{\partial \theta}+\frac{1}{r^2}\rho\frac{\partial v^\phi}{\partial \theta}\frac{\partial \nu_R}{\partial \theta}+\frac{1}{r^2}\nu_R\frac{\partial v^\phi}{\partial \theta}\frac{\partial \rho}{\partial \theta}$$

$$\text{v)}\quad \rho T\left[\frac{\partial s}{\partial t}+v^r\frac{\partial s}{\partial r}+v^\theta\frac{\partial s}{\partial \theta}\right]=\frac{4}{3}\rho\nu\left(\frac{\partial v^r}{\partial r}\right)^2$$

$$+3\rho\nu_R\left(\frac{\partial v^r}{\partial r}\right)^2+\frac{4}{3}\rho\nu\left(\frac{\partial v^\theta}{\partial \theta}\right)^2+3\rho\nu_R\left(\frac{\partial v^\theta}{\partial \theta}\right)^2+\rho\nu r^2\left(\frac{\partial v^\theta}{\partial r}\right)^2$$

$$+\rho\nu_R r^2\left(\frac{\partial v^\theta}{\partial r}\right)^2+\rho\nu r^2\sin^2\theta\left(\frac{\partial v^\phi}{\partial r}\right)^2+\rho\nu_R r^2\sin^2\theta\left(\frac{\partial v^\phi}{\partial r}\right)^2$$

$$+\frac{1}{r^2}\rho\nu\left(\frac{\partial v^r}{\partial \theta}\right)^2+\frac{1}{r^2}\rho\nu_R\left(\frac{\partial v^r}{\partial \theta}\right)^2+\rho\nu\sin^2\theta\left(\frac{\partial v^\phi}{\partial \theta}\right)^2$$

$$+\rho\nu_R\sin^2\theta\left(\frac{\partial v^\phi}{\partial \theta}\right)^2+2\rho\nu\left(\frac{\partial v^\theta}{\partial r}\right)\left(\frac{\partial v^r}{\partial \theta}\right)+2\rho\nu_R\left(\frac{\partial v^\theta}{\partial r}\right)\left(\frac{\partial v^r}{\partial \theta}\right)$$

$$+\frac{4}{3r}\rho\nu v^r\frac{\partial v^\theta}{\partial \theta}+\frac{8}{r}\rho\nu_R v^r\frac{\partial v^\theta}{\partial \theta}+\frac{4}{3r^2}\rho\nu(v^r)^2+\frac{8}{r^2}\rho\nu_R(v^r)^2$$

$$+\frac{4}{3}\rho\nu\,\mathrm{ctg}^2\theta\,(v^\theta)^2+3\rho\nu_R\,\mathrm{ctg}^2\theta\,(v^\theta)^2+\frac{4}{3r}\rho\nu\,\mathrm{ctg}\,\theta\, v^r v^\theta$$

$$+\frac{8}{r}\rho\nu_R\,\mathrm{ctg}\,\theta\, v^r v^\theta+\rho\nu_B\left(\frac{\partial v^r}{\partial r}\right)^2+\frac{4}{r^2}\rho\nu_B(v^r)^2+\rho\nu_B\left(\frac{\partial v^\theta}{\partial \theta}\right)^2$$

$$+\rho\nu_B\,\mathrm{ctg}^2\theta\,(v^\theta)^2+\frac{4}{r}\rho\nu_B v^r\frac{\partial v^r}{\partial r}+\frac{4}{r}\rho\nu_R v^r\frac{\partial v^r}{\partial r}-\frac{8}{3r}\rho\nu v^r\frac{\partial v}{\partial r}$$

$$+2\rho\nu_B\frac{\partial v^r}{\partial r}\frac{\partial v^\theta}{\partial \theta}+2\rho\nu_R\frac{\partial v^r}{\partial r}\frac{\partial v^\theta}{\partial \theta}-\frac{4}{3}\rho\nu\frac{\partial v^r}{\partial r}\frac{\partial v^\theta}{\partial \theta}+2\rho\nu_B\,\mathrm{ctg}\,\theta\, v^\theta\frac{\partial v}{\partial r}$$

$$+2\rho\nu_R \operatorname{ctg}\theta\, v^\theta \frac{\partial v^r}{\partial r} - \frac{4}{3}\rho\nu \operatorname{ctg}\theta\, v^\theta \frac{\partial v^r}{\partial r} + \frac{4}{r}\rho\nu_B v^r \frac{\partial v^\theta}{\partial \theta}$$

$$+\frac{4}{r}\rho\nu_B \operatorname{ctg}\theta\, v^r v^\theta + 2\rho\nu_B v^\theta \operatorname{ctg}\theta \frac{\partial v^\theta}{\partial \theta} + 2\rho\nu_R \operatorname{ctg}\theta\, v^\theta \frac{\partial v^\theta}{\partial \theta}$$

$$-\frac{4}{3}\rho\nu \operatorname{ctg}\theta\, v^\theta \frac{\partial v^\theta}{\partial \theta} - \frac{\partial F^r}{\partial r} - \frac{\partial F^\theta}{\partial \theta} - \frac{2}{r}F^r - \operatorname{ctg}\theta\, F^\theta$$

vi) $F^r = -\varkappa \frac{\partial T}{\partial r}$

vii) $F^\theta = -\varkappa \frac{1}{r^2} \frac{\partial T}{\partial \theta}$

viii) $P = P(\rho, T)$

ix) $S = S(\rho, T)$

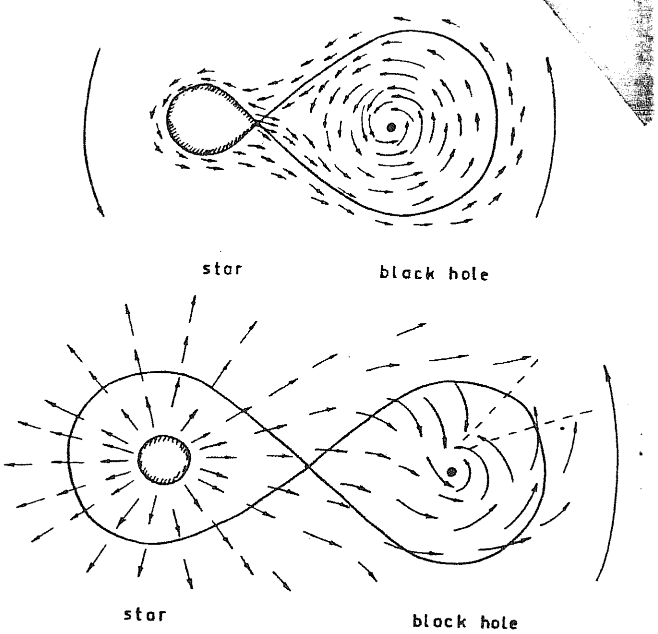


Fig. 1 Two regimes of matter capture by a collapsar: a) a normal companion fills up its Roche lobe, and the outflow goes, in the main, through the inner lagrangian point; b) the companion's size is much less than Roche lobe the outflow is connected with a stellar wind. The matter loses part of its kinetic energy in the shock wave and thereafter, gravitational capture of accreting matter becomes possible

(From Shakura and Sunyaev 1973)

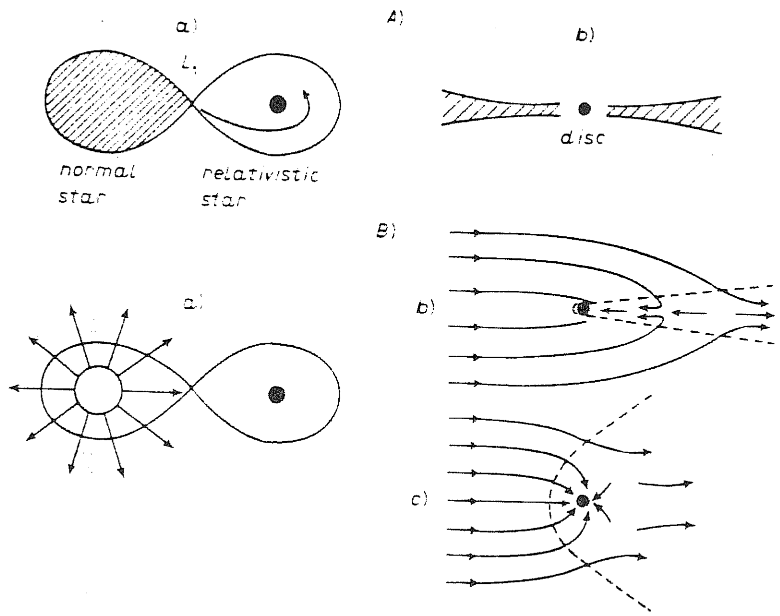


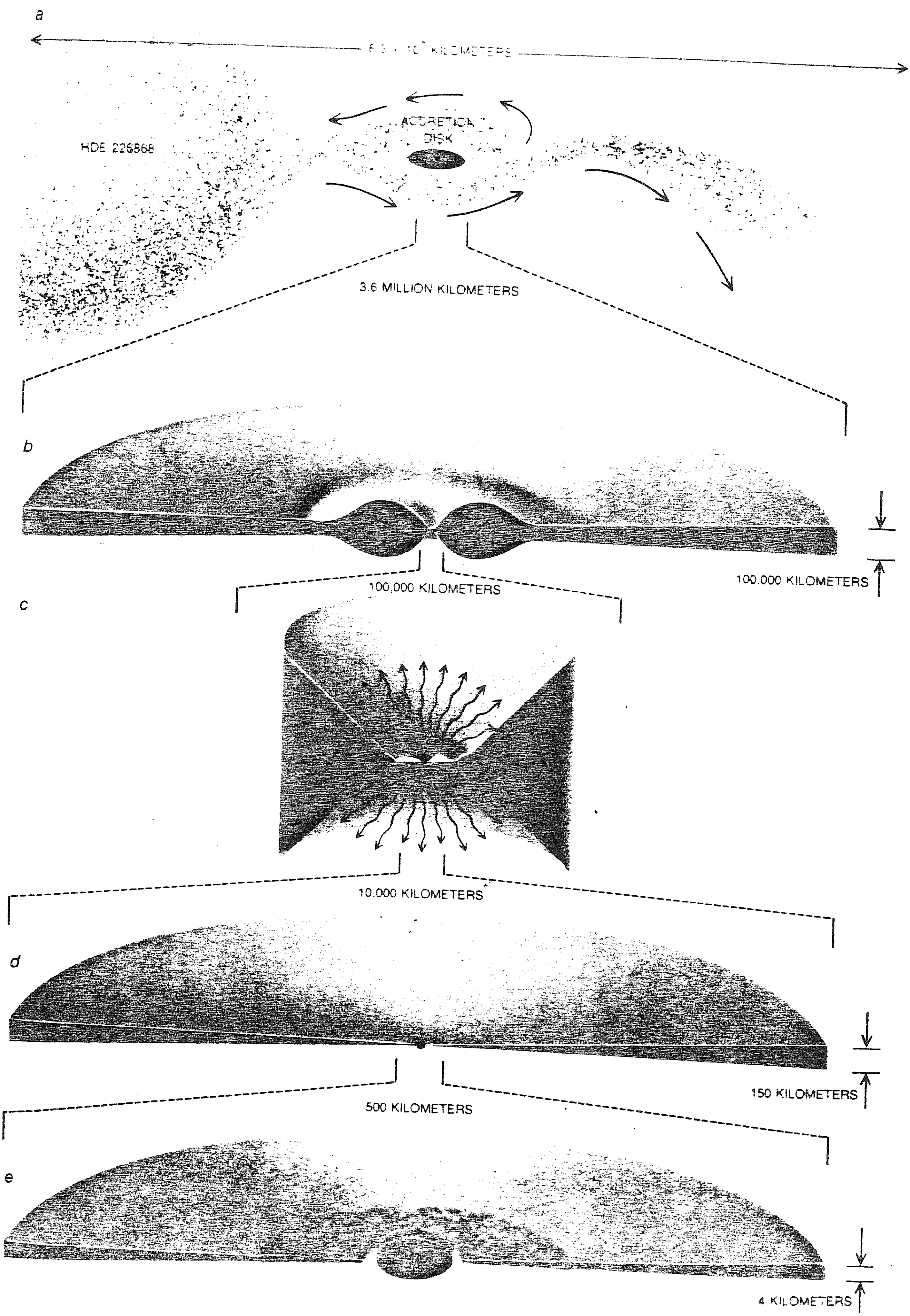
Fig. 2. Three types of the accretion picture on the relativistic star in a binary system [31]: *A*) semi-detached system: *a*) the normal component fills up is critical Roche lobe, the outflow of matter takes place through the inner Lagrangian point; *b*) the accreting matter has great angular momentum and forms a disc around the relativistic object which radiates X-rays. *B*) Detached system: *a*) the size of the normal star is small compared with the critical Roche lobe, the mass loss is associated with stellar wind; *b*) great rate of accretion: the matter behind the shock wave loses its thermal energy due to free-free radiation; the main part of captured matter has small angular momentum and falls onto the relativistic star in the narrow cone and forms the dense stream with relatively low temperature; *c*) small rate of accretion: the collisionless and emissionless bow shock wave is formed. The fall of matter obeys the laws of the spherically symmetric accretion.

(From Sunyaev 1978)

Fig.3 (Next page)

MODEL FOR A BLACK HOLE IN CYGNUS X-1 is a likely explanation for the observations made in the visible and X-ray regions of the spectrum. Gas is being pulled off the supergiant primary star HDE 226868 (*a*) by the gravitational attraction of the black hole. As the gas falls toward the black hole, the hole moves in its orbit out of the way, causing the gas to miss it. The gas nearest the black hole is whipped around it into a tight circular orbit, forming a thin accretion disk. The second illustration (*b*), at a scale of about 20 times smaller than the first, shows the expected shape of the accretion disk. The gravitational pull of the black hole compresses the disk, making it thin. At the same time thermal pressures in the gas react against the compression and try to thicken the disk. Only in the central bulge (*c*) are the pressures sufficient actually to thicken the disk. The large pressures in the bulge are caused by heat from X rays emitted near the black hole. In the core of the accretion disk (*d*) the thermal pressures are even higher than they are in the bulge; the gravity is so enormously strong, however, that it prevents the disk from thickening. The X rays observed from the earth are generated only in the innermost 200 kilometers of the core (*e*), which has the black hole itself at its center In the innermost 50 or 100 kilometers the disk becomes translucent, violently turbulent and much hotter than it is elsewhere. The disk terminates near the black hole, where the gravitational field becomes so strong that gas can no longer move in an orbit but is sucked directly in. The termination point and the structure of the inner disk are sensitive to the black hole's speed of rotation (*see illustration on page 37*). Here it is assumed that the black hole is rotating very rapidly; if the rotation is slow, the X-ray-emitting region may be 400 kilometers in radius rather than 200.

(From Thorne 1974)


a
ACCRETION
DISK
HDE 226868
3.6 MILLION KILOMETERS
b
100,000 KILOMETERS
100,000 KILOMETERS
c
10,000 KILOMETERS
d
150 KILOMETERS
500 KILOMETERS
e
4 KILOMETERS

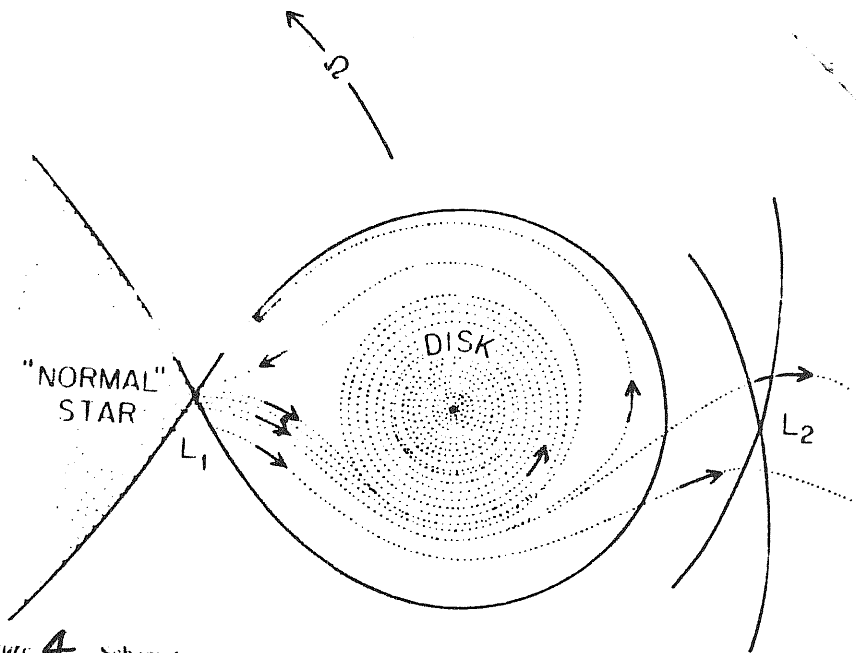


Figure **4.** Schematic representation of the flow of gas off a "normal" star that fills its Roche lobe, onto a compact star. The equipotentials Φ_{gc} = const. are drawn as solid curves; the flow lines of the gas are drawn as dotted curves

(From Novikov and Thorne 1973)

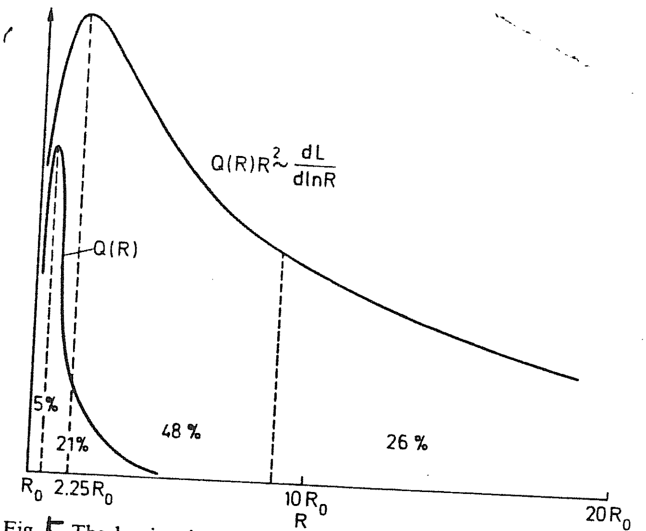


Fig. 5 The luminosity of the surface unit of the disk as a function of the radius. The function $Q(R)\,R^2$ is proportional to the luminosity of the ring with radius R and $\Delta R \sim R$. The numbers illustrate the contribution of the corresponding regions to the integral luminosity of the disk

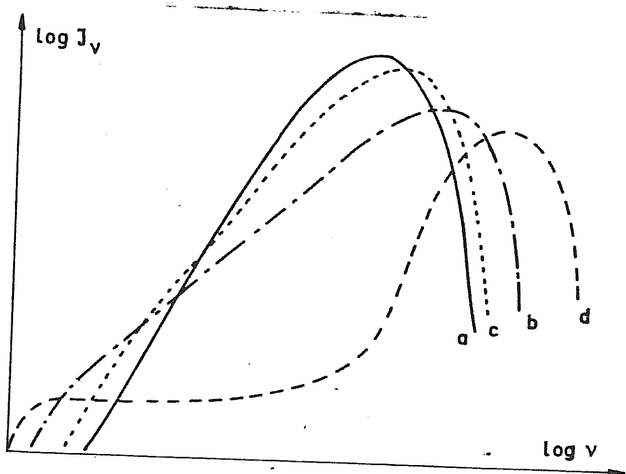


Fig. 6 Characteristic local spectra of radiation formed in the disk a) the black body spectrum $Q = bT^4$. b) the radiation spectrum of an isothermic, homogeneous medium where the main contribution to the opacity comes from scattering $Q = \text{const}\sqrt{n}\,T^{2.25}$. c) the same in an isothermal, exponential atmosphere: $Q = \text{const}\,T^{2.5}$. d) the spectrum formed as a result of comptonization $Q = \text{const}\,T^4$. The intensities are normalized so that the energy flux of radiation Q is the same in all four cases. The change of effective temperature of the radiation is clearly seen

(From Shakura and Sunyaev 1973)

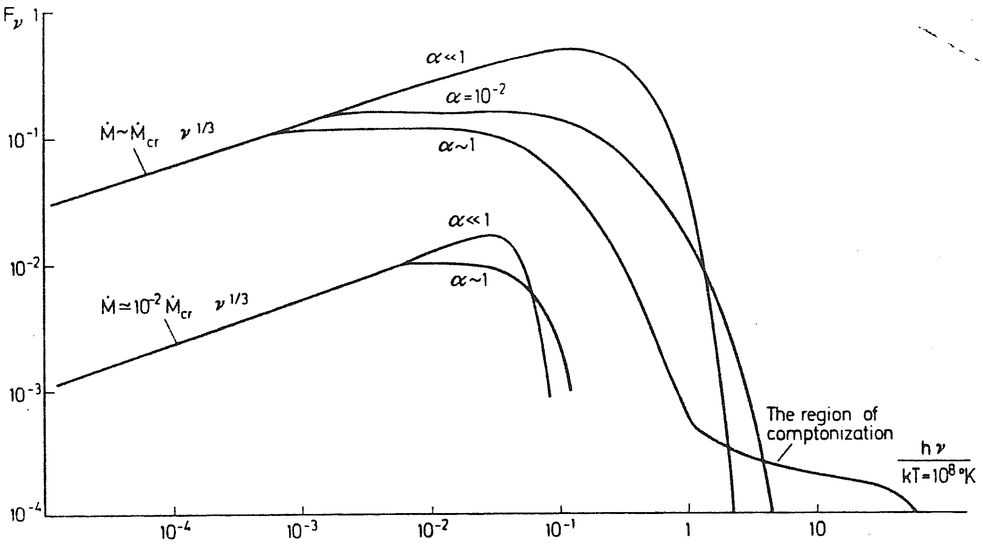


Fig. 7 The integral radiation spectrum of the disk, computed for different $\dot{M}$ and α

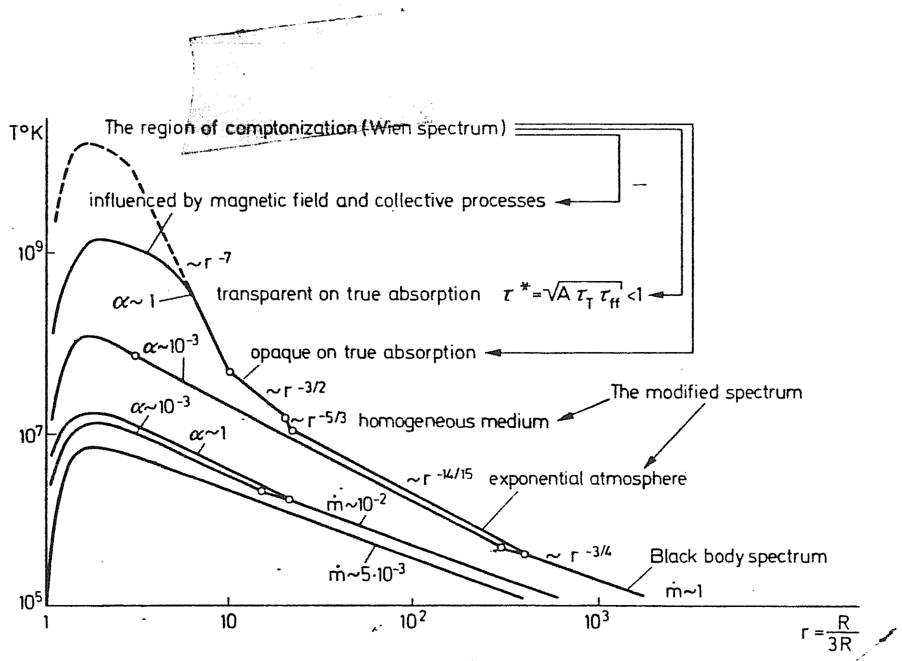
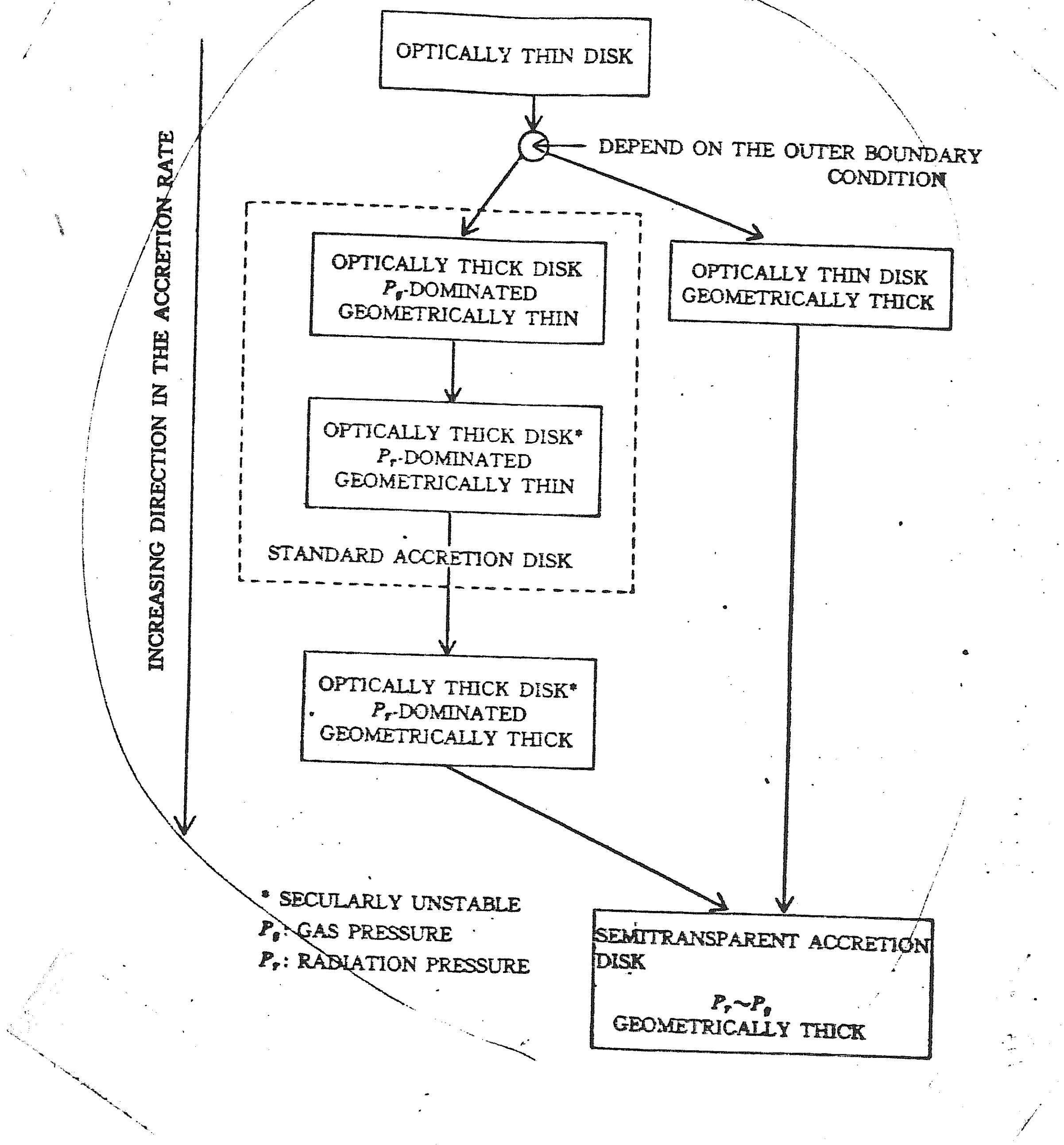


Fig. 8 Distribution of temperature along the radius of the disk if $\dot{M}$ and the parameter α are different

(From Shakura and Sunyaev 1973)

Table I. Classification of accretion disks.



(From Hoshi 1981)

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