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PERIODIC SOLUTIONS OF DIFFERENTIAL SYSTEMS USING PROJECTIONS ONTO SUBSPACES OF LOWER DIMENSION

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CONTENTS

Chapter 1

INTRODUCTION

The problem of the existence of periodic solution for differentiable systems

$$\frac{dx}{dt} = f(t,x) \tag{1.1}$$

in $\mathbb{R}^n$ has been investigated widely in the literature because of its intrinsic interest as a source of models to which employ the methods of nonlinear analysis.

Among the various useful tools introduced in the qualitative analysis of the equation (1.1) the Poincaré-Bendixson Theorem and Massera's Theorem play an important role for the study of the autonomous and non-autonomous systems, respectively. However such results have both a strict limitation in the applicability since they work in two dimensional spaces, this limitation prevents the possibility of good application to higher order differential equations of higher order without placing more restrictions on the class of the equations.

In 1961, D'Heedene R. N. [7] published a counterexample showing that no simple generalization of Poincaré-Bendixson Theorem to $n \geq 3$ case is possible. His example consist of a third-order system

$$\begin{aligned}\frac{dx}{dt} &= y\\ \frac{dy}{dt} &= z - x\\ \frac{dz}{dt} &= Z(x,y,z)\end{aligned}$$

and he proved that this system has almost periodic solutions but not periodic solutions. In 1949 Massera constructed an example that without adding more restrictive general assumptions the existence of periodic solutions cannot occur although the hypothesis of his second Theorem are satisfied, see Massera [14], Theorem 8.

In recent years many authors (see Bingxi Li [2] and Reissig R., Sansone G. Conti R. [16]) have tryed to overcome such difficulty and have proposed partial extensions of Poincaré-Bendixson and Massera's theorems to higher dimensional spaces.

From 1979 R. A. Smith in a series of papers [21, 22, 23, 24, 25, 26,] proposed a general method for extending to differential systems in $\mathbb{R}^n$ various results which are known for the two dimensional case. His idea mainly consist in the construction of a projection to a plane determined by the choice of a quadratic form which behaves like Liapunov function when evaluated along two solutions of equation (1.1). However the use of such quadratic form does not permit to have a sharp applications to some first order differential systems of Lotka-Volterra systems type. For such system, K. Gopalsamy [10] have obtained some results of the existence and uniqueness of periodic solution by using different class of Liapunov function which are not differentiable (given for instance, by the sum of the absolute value of the coordinates) but to calculate the derivate along the solution Gopalsamy used Dini's derivative.

Last year, analogue idea have been used too by H. Smith and Mallet-Paret [13]. They study a particular class of systems known like cooperation-competition and the selection of the projection is doing over two components and it is justified by the particular structure of the system.

The aim of this thesis is to describe R. A. Smith approach with a suitable generality as to obtain good application Lotka-Volterra system as well.

Chapter 2 is devoted to Massera's Theorem. In section 2.1, we give, following along the line of R. A. Smith [24] the Massera's convergence Theorem. In section 2.2 we analyse a special type of solution , amenable solution, which permit to get a generalization of Massera's second Theorem and later to speak about orbital stability.

Chapter 3 contains the generalization of Poincaré-Bendixson Theorem, like the previous chapter, in the same line of R. A. Smith [22] we get a Theorem for existence and uniqueness of periodic orbits of an autonomous system, this Poincaré-Bendixson Theorem does not give any result about stability. In section 3.2 we give some sufficient condition for the orbital stability.

Chapter 4 is devoted to the applications. In section 4.1 we consider equations of Lotka-Volterra system type and we prove the existence of solution which lying in a compact set S_0. In the section 4.2 we study equations in which it is natural to work in a domain with holes.

The main application shows how the general method developed along the thesis permits to induce in this extended approach some results by Gopalsamy which are not contained in the original development of R. A. Smith.

In this work we will consider the following notation:

The real space $\mathbb{R}^n$ is treated with the usual inner product $<,>$ and equivalent norm defined by

$$\|x\|_p^p = \sum_{i=1}^{n} | < x, v_i > |^p$$

where $\|\cdot\|_p^p$ is the l^p-norm in $\mathbb{R}^n$ and $\{v_i, \quad i = 1, ..., n\}$ is any basis of $\mathbb{R}^n$. Let us consider the following set

$$H = \{z \in \mathbb{R}^n : z = \sum_{i \in A} < z, v_i > v_i\}.$$

Now we can rewrite any $x \in \mathbb{R}^n$ as

$$x = \sum_{i \in A} < x, v_i > v_i + \sum_{i \in B} < x, v_i > v_i = \chi_A + \chi_B$$

and we will denote it by $x = (\chi_A, \chi_B)$, with $A = \{i = 1, ..., j\}$ and $B = \{i = j+1, ..., n\}$. Since note that is possible to write

$$\|x\|_p^p = \sum_{i \in A} | < x, v_i > |^p + \sum_{i \in B} | < x, v_i > |^p.$$

Let $K : \mathbb{R} \longrightarrow \mathbb{R}$ the following function

$$K(< x, v_i >) = \begin{cases} 1, & < x, v_i >> 0 \ \ or \ \ < x, v_i >= 0 \ \ and \ \ \frac{d}{dt} < x, v_i >> 0, \\ 0, & < x, v_i > = 0 \ and \ \frac{d}{dt} < x, v_i > = 0, \\ -1, & < x, v_i > < 0 \ \ or \ \ < x, v_i >= 0 \ \ and \ \ \frac{d}{dt} < x, v_i > < 0, \end{cases}$$

and it depends of the solution $x(t)$ of (1.1).

Chapter 2

MASSERA'S THEOREM

2.1. Massera's Convergence Theorem.

Let consider the non-autonomous ordinary differential equation

$$\frac{dx}{dt} = f(t,x) \tag{2.1}$$

in which $f(t,x) = f : \mathbb{R} \times S \longrightarrow \mathbb{R}^n$ is a continuous function such that for some open subset $S \subseteq \mathbb{R}^n$, f satisfies the local Lipschitz condition in x (uniformly in t). We assume the following restriction for the parameter t.

[H_1] *There is a constant* $T > 0$ *such that* $f(t,x) = f(t+T,x)$ *for all* $t \in \mathbb{R}$, *and all* $x \in S$.

If we consider the particular case when $S_0 \subseteq S$ is a compact set, the Lipschitz condition permits us to guarantee the global Lipschitz condition over the compact set $[0,T] \times S_0$, that is, there is a non-negative constant $L(S_0)$ such that the following inequality

$$|f(t,x_1) - f(t,x_2)| \leq L(S_0)|x_1 - x_2| \tag{2.2}$$

holds for all $t \in \mathbb{R}$ and $x_1,\ x_2 \in S_0$. From Gronwall's inequality we can ensure that if $x_1(t),\ x_2(t)$ are solutions of (2.1) such that $x_1(t),\ x_2(t) \in S_0$ for $\theta \leq t \leq \tau$ then (2.2) gives

$$\begin{aligned} |x_1(\theta) - x_2(\theta)|e^{-L(S_0)(\tau-\theta)} &\leq |x_1(\tau) - x_2(\tau)| \leq \\ &\leq |x_1(\theta) - x_2(\theta)|e^{L(S_0)(\tau-\theta)}. \end{aligned} \tag{2.3}$$

Massera made a discussion about the sufficient conditions which imply the existence of at least one T−periodic solution for systems of differential equations of the type (2.1). His first Theorem known like Massera's convergence Theorem (Massera [14], Yoshizawa [30]) is referred for scalar equations, i.e. $n = 1$ and can be stated as follows:

THEOREM 2.1. *(Massera) Assume* $[\mathbf{H}_1]$ *and suppose* $S = \mathbb{R}$ *and* $n = 1$. *If* $y(t)$ *is a solution of (2.1) which is bounded in an interval* $(t_0, +\infty)$ *then* $y(t)$ *must converge to a* T*-periodic solution* $u(t)$ *of (2.1) as* $t \to +\infty$.

At this point the uniqueness of the T-periodic solution is not assumed. We can observe that this theorem gives a relation between $y(t)$, a bounded solution of (2.1), and $u(t)$ T-periodic solution of (2.1), which is the limit. Trying to extend this result to the case $n = 2$, Massera observed that the conditions of the Theorem 2.1 are not enough to ensure the existence of T-periodic solutions of (2.1). In fact, Massera gave an example which shows that for the case $n = 2$, the boundedness of a solution does not imply the existence of a T-periodic solution.

EXAMPLE 2.1.

Consider the system in $\mathbb{R}^2$ (see Yoshizawa [30])

$$\begin{aligned} \frac{dx}{dt} &= f(u,v)\cos^2 \pi t - g(u,v)\sin \pi t \cos \pi t - \pi y \\ \frac{dy}{dt} &= g(u,v)\cos^2 \pi t + f(u,v)\sin \pi t \cos \pi t + \pi x \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} u &= x \cos \pi t + y \sin \pi t, \\ v &= y \cos \pi t - x \sin \pi t. \end{aligned} \tag{2.5}$$

The functions f and g are supposed to satisfy the following assumptions:

(a) $f,\ g$ have continuous first partial derivatives,

(b) $f(-u,-v) = f(u,v), \quad g(-u,-v) = g(u,v),$

(c) $f(1,0) = g(1,0) = 0,\ f(0,v) = 0,\ g(0,v) > 0$ *for all* $v,$

(d) $\int_{-\infty}^{\infty} \frac{1}{g(0,v)} dv < 2\pi.$

The condition *(a)* implies the local lipschitzianity and we can easily see that (2.4) is periodic of period 1 in t, because of assumption *(b)*.

For example, it is possible choice of f and g is given by $f = uv,\ g = (1-u)^2(1+v)^2 c$ where c is suitable constant, for instance $\frac{\pi^2}{2}$.

In the variables (u,v) system (2.4) becomes

$$\frac{du}{dt} = f(u,v)\, cos\, \pi t,$$
$$\frac{dv}{dt} = g(u,v)\, cos\, \pi t \tag{2.6}$$

It is clear that $u(t) = 0,\ v = v(t)$ is a solution of (2.6) because of the uniqueness we conclude that the first component, u(t) of any solution of (2.6) has constant sign. Moreover, since by *(c)* $u = \pm 1,\ v = 0$ are solutions of (2.6), we obtain that

$$(x = cos\, \pi t\ ,\ y = sin\ \pi t) \quad \text{and}$$
$$(x = -cos\, \pi t,\ y = -sin\ \pi t)$$

are two periodic solutions of (2.4), of period 2.

On the other hand there are not solution $(x(t), y(t))$ of (2.4) of period 1 because if $(x(t), y(t))$ is a solution of period 1, then

$$\begin{aligned} u(t+1) = &\ x(t+1) cos(\pi t + \pi) + y(t+1) sin(\pi t + \pi) = \\ &= -x(t) cos \pi t - y(t) sin \pi t, \end{aligned}$$

which shows that u must change signs and $u = 0$.

Hence, suppose that $u = 0,\ v = v(t)$ is a certain solution of (2.6). Then $v(t)$ will be given by

$$\int_{v_0}^{v} \frac{dv}{g(0,v)} = \int_{t_0}^{t} cos\, \pi t dt = \frac{1}{\pi}(sin \pi t - sin \pi t_0).$$

Finally, integrating if t could increase from $-\frac{1}{2}$ to $+\frac{1}{2}$, we have

$$\int_{v\left(-\frac{1}{2}\right)}^{v\left(\frac{1}{2}\right)} \frac{1}{g(0,v)} dv = \frac{2}{\pi}$$

which contradicts *(d)*. Hence $v(t)$ cannot be defined for all t and the corresponding (x, y) solution cannot be periodic (cannot exist in the future).

This example shows that if $n = 2$ the system (2.1) may have periodic solutions of period 2 without having periodic solutions of period 1. It is even possible to select f and g in such way that every solution is of period 2 except for a one-parameter family of solutions which do not exist for all values of t.

The present section is devoted to give an analogue of Massera's convergence Theorem in higher dimension which will be called as such generalization. We say analogue because if we consider Theorem 2.3 for $n = 1$ we do not get exactly Massera's convergence Theorem, since there is an extra hypothesis, $[\mathbf{H_3}]$ for $n = 1$, which is like monotonicity condition. Now we consider the following restrictions:

$[\mathbf{H_2}]$ *The coefficients*

$$a_i = \begin{cases} -1, & \text{if } i \in A; \\ 1, & \text{if } i \in B. \end{cases}$$

for all $i = 1, ..., n$.

$[\mathbf{H_3}]$ *There exist constants* $\lambda \geq 0$, $\varepsilon > 0$ *such that for all* $t \in \mathbb{R}$

$$\begin{aligned} &\lambda \sum_{i=1}^{n} a_i | < x_1 - x_2, v_i > |^p + \\ &+p \sum_{i=1}^{n} a_i K(< x_1 - x_2, v_i >) | < x_1 - x_2, v_i > |^{p-1} < f(t, x_1) - f(t, x_2), v_i > \leq \\ &\leq -\varepsilon \sum_{i=1}^{n} | < x_1 - x_2, v_i > |^p \end{aligned} \tag{2.8}$$

for all solutions $x_1(t)$, $x_2(t)$ *of (2.1) and* $1 \leq p < +\infty$, with K a function defined in Chapter 1.

Let us consider the scalar function $V(x(t)) = \sum_{i=1}^{n} a_i| < x(t), x_i > |^p$ with $x(t)$ a solution of (2.1). Let us consider $z(t)$ any differentiable function and we observe that for any function $|z(t)|$, the Dini's derivative is

$$D^+|z(t)| = K(z(t))\frac{dz}{dt}$$

(the definition of the function K depends only of the function $z(t)$). For $x_1(t), x_2(t)$ solutions of (2.1)

$$D^+\left[e^{\lambda t}V\left(x_1(t) - x_2(t)\right)\right] = e^{\lambda t}\left[\lambda V\left(x_1 - x_2\right) + \mathrm{D}^+V\left(x_1 - x_2\right)\right] =$$

$$= e^{\lambda t}\left[\lambda \sum_{i=1}^{n} a_i| < x_1 - x_2, v_i > |^p + \right.$$

$$\left. +p\sum_{i=1}^{n} a_i K\left(< x_1 - x_2, v_i >\right)| < x_1 - x_2, v_i > |^{p-1} < f(t, x_1) - f(t, x_2), v_i > \right] \leq$$

$$\leq -\varepsilon e^{\lambda t}\sum_{i=1}^{n} | < x_1 - x_2, v_i > |^p.$$

Then we can conclude that

$$D^+\left[e^{\lambda t}V\left(x_1(t) - x_2(t)\right)\right] \leq -\varepsilon e^{\lambda t}\|x_1(t) - x_2(t)\|_p^p \tag{2.9}$$

and this holds for all t such that $x_1(t)$, $x_2(t) \in S$. For the details related to Dini's derivative see Rouche, Habet's, Laloy [17] appendix I and Lakshmikanthan, Leela [12] pag. 7.

REMARK 2.1.

1) It is easy to verify from the definition of $D^+|z(t)|$ that

$$D^+|z(t)| = K(z(t))\frac{dz}{dt} = \frac{z(t)}{|z(t)|}\frac{dz}{dt}.$$

2) If $p > 1$ the term

$$\sum_{i=1}^{n} a_i K(< x_1(t)-x_2(t), v_i >)| < x_1(t)-x_2(t), v_i > |^{p-1} < f(t,x_1(t))-f(t,x_2(t)), v_i >$$

can be simplified without using K function. From (1) we can write this term

$$\sum_{i=1}^{n} a_i| < x_1(t) - x_2(t), v_i > |^{p-2} < x_1(t) - x_2(t), v_i >< f(t,x_1(t)) - f(t,x_2(t)), v_i > .$$

Note that if we consider the particular case for $p = 2$ with this replacement we get in $[\mathbf{H_3}]$ the same restriction obtained by R. A. Smith in [22, 24].

Suppose that $x_1(t),\ x_2(t) \in S$ for $t \in [\theta, \tau]$. The relation (2.9) shows us that the function $e^{\lambda t} V\big(x_1(t) - x_2(t)\big)$ is monotonic decreasing in $[\theta, \tau]$ and strictly decreasing for $x_1(t) \neq x_2(t)$ for all $t \in [\theta, \tau]$. By integrating (2.9) over the interval $[\theta, \tau]$ we get

$$\varepsilon \int_{\theta}^{\tau} e^{\lambda t} \|x_1(t) - x_2(t)\|_p^p \, dt \leq e^{\lambda\theta} V\big(x_1(\theta) - x_2(\theta)\big) - e^{\lambda\tau} V\big(x_1(\tau) - x_2(\tau)\big). \tag{2.10}$$

In fact, by following the Smith's in [24], along the same line, there is two possibilities. The first one is the case $\lambda = 0$. In this situation a comparison with Massera's Theorem, shows that the new result not only preserves the convergence to a $T-$periodic solution $u(t)$, but it also ensures that $u(t)$ is the only $T-$periodic solution in S.

THEOREM 2.2. *Suppose that (2.1) satisfies* $[\mathbf{H_1}]$, $[\mathbf{H_2}]$ *and* $[\mathbf{H_3}]$, *for* $\lambda = 0$. *If (2.1) has a solution* $y(t) \in S_0$, *a compact set in* S, *for all* $t \in [t_0, +\infty)$ *then (2.1) has a T-periodic solution* $u(t)$ *such that* $y(t) - u(t) \to 0$ *for* $t \to +\infty$. *Furthermore,* $u(t)$ *is the only one T-periodic solution that lies in* S *for all* t.

Proof. Let us consider $x(t)$, $y(t)$ solutions of (2.1) in S_0 a compact set, for all $t \in [t_0, +\infty)$. There exist a non-negative constant M such that $|V\big(x(t) - y(t)\big)| \leq M$ holds, for all $t \geq t_0$.

By placing $\lambda = 0$ in (2.10) we get

$$\varepsilon \int_{\theta}^{\tau} \|x(t) - y(t)\|_p^p \, dt \le V\big(x(\theta) - y(\theta)\big) - V\big(x(\tau) - y(\tau)\big) \le M$$

for all $\tau \ge \theta$. Then $\int_{\theta}^{+\infty} \|x(t) - y(t)\|_p^p \, dt$ converges. From the left-hand inequality in (2.3) we get

$$\|x(\theta) - y(\theta)\|_p^p \int_{\theta}^{+\infty} e^{-pL(S_0)(t-\theta)} dt \le \int_{\theta}^{+\infty} \|x(t) - y(t)\|_p^p \, dt \le M$$

That is,

$$\|x(\theta) - y(\theta)\|_p^p \frac{1}{pL(S_0)} \le \int_{\theta}^{+\infty} \|x(t) - y(t)\|_p^p \, dt \quad \text{for all } \theta \ge t_0.$$

Then we conclude that

$$x(\theta) - y(\theta) \longrightarrow 0, \qquad for\ \theta \to +\infty. \tag{2.7}$$

Condition (2.7) holds in particular when $x(t) = y(t+T)$ as $y(t+T)$ is a solution in S_0 for all $t \ge t_0$. Since S_0 is a compact set by Weierstrass's Theorem. There exist a strictly increasing sequence of positive integers $m(1),\ m(2),\ m(3), \dots,$ and $c \in S$ such that $y\big(t_0 + m(h)T\big) \longrightarrow c,$ as h $\to \infty$. Let $u(t)$ be the solution of (2.1) such that $u(t_0) = c$ in fact, we can ensure that $u(t)$ is in S for all $t_0 \le t < \alpha$. Then $y\big(t_0 + m(h)T\big) \longrightarrow u(t)$ as $h \to +\infty$ point wise for $t_0 \le t < \alpha$. We observe that the solution $u(t)$ cannot leave the compact set S_0 in $[t_0, \alpha)$. In fact, if it does so there is a neighborhood in which $y\big(t_0 + m(h)T\big)$ does it too for all $h \ge h_0$ and this is absurd, because we supposed that $y(t) \in S_0$ for all $t \ge t_0$. For this we can ensure that $u(t) \in S_0$ for all $[t_0, +\infty)$ and $y\big(t_0 + m(h)T\big) \longrightarrow u(t)$ as $h \to \infty$ point wise for $t_0 \le t < \alpha$ holds.

Finally $u(t)$ is a T-periodic solution for all t because

$$u\big(t_0 + T\big) = \lim_{h \to +\infty} y\big(t_0 + T + m(h)T\big) = \lim_{h \to +\infty} y\big(t_0 + m(h)T\big) = c = u(t_0),$$

and this equality holds for (2.7), i.e. $y(t+T) - y(t) \to 0$ as $t \to +\infty$.

For the uniqueness, let us consider $\tilde{u}(t)$ another T-periodic solution and we select $S_0 \subset S$ the bigger compact set which includes $u(t)$ and $\tilde{u}(t)$ for all t. Then for (2.7) we conclude that $u(t) - \tilde{u}(t) \longrightarrow 0$ as $t \to +\infty$. This happens only if $u(t),\ \tilde{u}(t)$ coincide. This establishes Theorem 2.2.

From this Theorem we can get the following Corollary, whose proof is omitted because is completely similar to the previous one.

COROLLARY 2.2. *Suppose that (2.1) satisfies* $[\mathbf{H}_1]$, $[\mathbf{H}_2]$ *and* $[\mathbf{H}_3]$, *for* $\lambda = 0$. *If (2.1) has a solution* $y(t) \in S_0$, *a compact set in* S, *for all* $t \in (-\infty, t_0]$ *then (2.1) has a* T*-periodic solution* $u(t)$ *such that* $y(t) - u(t) \to 0$ *for* $t \to -\infty$. *Furthermore,* $u(t)$ *is the only one* T*-periodic solution who lies in* S *for all* t.

For the second possibility λ, we consider

$$V(x) = \sum_{i=1}^{n} a_i | < x, v_i > |^p$$

a continuous function and $\mathcal{P} : \mathbb{R}^n \longrightarrow H \subset \mathbb{R}^n$ a linear map defined by

$$\mathcal{P}x = \sum_{i \in A} < x, v_i > v_i, \quad \text{for every } \mathrm{x} \in \mathbb{R}^n.$$

We assume in the sequel that $dim\ A = 1$ and $dim\ B = n-1$. In this case the image of this map $\mathcal{P}$ is a straight line in $\mathbb{R}^n$.

Since

$$\|x\|_p^p = \sum_{i=1}^{n} | < x, v_i > |^p = V(x) \ + \ 2\|\mathcal{P}x\|_p^p .$$

We have

$$V(x) \ + \ 2\|\mathcal{P}x\|_p^p \geq \ \|\mathcal{P}x\|_p^p \tag{2.11}$$

for every $x \in \mathbb{R}^n$.

THEOREM 2.3 . *Suppose that (2.1) satisfies* $[\mathbf{H}_1]$, $[\mathbf{H}_2]$ *and* $[\mathbf{H}_3]$, *for* $\lambda > 0$ *and* $j = 1$. *If (2.1) has a solution* $y(t) \in S_0$, *a compact set in* S, *for all* $t \in [t_0, +\infty)$ *then (2.1) has a* T*-periodic solution* $u(t)$ *such that* $y(t) - u(t) \to 0$ *for* $t \to +\infty$.

Let consider the following example given for R. A. Smith [24] to explain the analogy between Theorem 2.2 and Theorem 2.3.

EXAMPLE 2.2.

Consider the following bidimensional system,

$$\begin{aligned} \frac{d\psi}{dt} &= \phi(t,\psi) \\ \frac{d\eta}{dt} &= -\mu\eta, \end{aligned} \tag{2.12}$$

with μ a positive constant . It is clear that this system is of the form (2.1). If we suppose that

(a) $a_1 = -1,\ a_2 = 1,$
(b) $\lambda = \mu - \varepsilon,$
(c) $S = \mathbb{R}^2,$
(d) $v_1 = (1,0); \quad v_2 = (0,1).$

The system (2.12) verify hypothesis $[\mathbf{H_3}]$, provided that with the consideration did in Remark 2.1.

$$-[\mu - 2\varepsilon]|\psi_1 - \psi_2|^2 \leq |\psi_1 - \psi_2||\phi(t,\psi_1) - \phi(t,\psi_2)|, \tag{2.13}$$

for $\phi_1,\ \phi_2 \in \mathbb{R}$.

In the special case when the partial derivative $\phi_\varepsilon(t,\psi)$ exist and satisfies

$$-\mu < \inf_t \phi_\varepsilon(t,\psi)$$

the condition (2.13) holds for $\varepsilon \to 0$. Then the system (2.12) satisfies $[\mathbf{H_3}]$ with $\lambda > 0$, $j = 1,\ S = \mathbb{R}^2$. Theorem 2.2, gives us a version of Massera's Theorem for the scalar equation $\frac{d\psi}{dt} = \phi(t,\psi)$, with the extra restriction $-\mu < \inf_t \phi_\varepsilon(t,\psi)$, this is satisfied in the special case when

$$\phi(t,\psi) = \frac{1}{2}\mu sin\,\psi,$$

for which the periodic solution of (2.12) are the constants solutions $\psi = h\pi,\ \eta = 0$, with integer h. The set S may contain many different periodic solutions when the conditions of the Theorem 2.3 holds. In This case, Theorem 2.3 is more relaxed than Theorem 2.2. And we can observe that Theorem 2.3 is not a really generalization of Massera's Theorem because for the linear equation (2.12) we need to suppose in addition that $-\mu < \inf_t \phi_\varepsilon(t, \psi)$.

Proof. (Theorem 2.2) We suppose the case for $j = 1$, then the linear application $\mathcal{P}x$ can be understood as a real function, if we restrict to the hyperplane H, all the real line properties, $\left(\mathcal{P} : \mathbb{R}^n \longrightarrow H\right)$. Let us consider $y(t+T)$ and y(t) solutions of (2.1) $t \geq t_0$, then from (2.9) we know that the function $e^{\lambda t} V\left(x_1(t) - x_2(t)\right)$ is monotonic decreasing in $t \in [t_0, +\infty)$.

The proof of this Theorem will be divided in two cases, the first we consider $V\left(y(\tilde{t}) - y(\tilde{t}+T)\right) < 0$ for some $t \geq \tilde{t}$. From (2.11) follows that

$$\begin{aligned}
&\sum_{i=1}^{n} | < y(t) - y(t+T), v_i > |^p = \\
&= V\left(y(t) - y(t+T)\right) + \ 2\|\mathcal{P}\left(y(t) - y(t+T)\right)\|_p^p < \\
&< 2\|\mathcal{P}\left(y(t) - y(t+T)\right)\|_p^p
\end{aligned} \tag{2.14}$$

for all $t \geq \tilde{t}$. For this reason the real function (referred to H) $\mathcal{P}\left(y(t) - y(t+T)\right)$ has a constant sign in $[\tilde{t}, \infty)$. Then, the sequence $\left\{\mathcal{P}\left(y(\tilde{t}+hT) - y(\tilde{t}+hT+T)\right)\right\}_{h\geq 1}$ has a constant sign, consequently the sequence $\left\{\mathcal{P}y(\tilde{t}+hT)\right\}_{h\geq 1}$ is monotonic and, it is bounded because $\left\{y(\tilde{t}+hT)\right\}_{h\geq 1}$ lies in S_0 a compact set. For this reason the series converges and then

$$\sum_{h=1}^{\infty} \|\mathcal{P}\left(y(t) - y(t+T)\right)\|_p^p$$

converges. Then for the relation (2.14) we show that the series

$$\sum_{h=1}^{\infty} \|\left(y(t) - y(t+T)\right)\|_p^p$$

converges for all $1 < p < \infty$. This permits us to affirm that

$$\lim_{h\to+\infty} \|\left(y(t) - y(t+T)\right)\|_p^p = 0,$$

therefore $\{y(\tilde{t}+hT)\}_{h\geq 1}$ is a Cauchy sequence in $\mathbb{R}^n$. Then there is $c \in S_0$ such that $\lim_{h\to+\infty} y(\tilde{t}+hT) = c$.

Let us consider $u(t)$ a solution of (2.1) such that its initial condition $u(t_1) \in S$ for all $\tilde{t} \leq \alpha$, therefore the sequence $\{y(t+hT)\}$ converges to $u(t)$ point wise for all $\tilde{t} \leq \alpha$, when $h \to \infty$. This function $u(t)$ cannot leave the set S_0 because if this happens in some neighborhood, the solution $y(t+hT)$ does it too and this is an absurd because $y(t) \in S_0$ for all $t \geq t_0$. For this $u(t) \in S_0$ for all $t \geq t_0$

$$u(\tilde{t}+T) = \lim_{h\to+\infty} y(\tilde{t}+T+hT) = \lim_{h\to+\infty} y(\tilde{t}+hT) = c = u(\tilde{t}),$$

moreover $y(t) - u(t) \to 0$ for $t \to +\infty$, because

$$y(\tilde{t}+hT) - u(\tilde{t}+hT) = y(\tilde{t}+hT) - u(\tilde{t}) \longrightarrow 0$$

when $h \to +\infty$. Then Theorem 2.3 is established for $V\big(y(\tilde{t}) - y(\tilde{t}+T)\big) < 0$ for some $\tilde{t} \geq t_0$.

Now let us consider the another case when $V\big(y(\tilde{t}) - y(\tilde{t}+T)\big) > 0$ for all $t \geq t_0$. In fact, if we $x(t) = y(t+T)$ in (2.10), then

$$\varepsilon \int_{\theta}^{\tau} e^{\lambda t} \|y(t+T) - y(t)\|_p^p \, dt \leq e^{\lambda\theta}\big(y(\theta+T) - y(\theta)\big) - e^{\lambda\tau}\big(y(\tau+T) - y(\tau)\big).$$

for all $\tau \geq \theta \geq t_0$. From this inequality and Cauchy-Schwarz inequality

$$\left[\int_{\theta}^{\tau} \|y(t+T) - y(t)\|_p \, dt\right]^p \leq M \int_{\theta}^{\tau} \left[e^{\frac{\lambda}{p}t} \|y(t+T) - y(t)\|_p\right]^p dt \leq \frac{M}{\varepsilon}\left[e^{\lambda\theta}\|y(\theta+T) - y(\theta)\|_p^p\right]$$

holds for all $\theta \leq \tau$ and

$$M = \left[\int_{\theta}^{\tau} e^{-\frac{\lambda}{p}tq}\right]^{\frac{p}{q}}.$$

such that $\frac{1}{p} + \frac{1}{q} = 1$.

We get an upper bound which does not depend on of τ. For $\tau \to +\infty$ we get

$$\int_{\theta}^{+\infty} e^{\lambda t} \|y(t+T) - y(t)\|_{p} \, dt \quad \text{converges for all } \theta \geq t_0 \tag{2.15}$$

If we replace in the inequality (2.3) $x(t),\ \tau,\ \theta$ by respectively $y(t+T),\ t,\ \theta + hT$ respectively

$$\sum_{i=1}^{n} |< y(\theta + hT) - y(\theta + T + hT), v_i > |^p e^{-pL(S_0)(\tau - \theta)} \leq \sum_{i=1}^{n} |< y(\tau) - y(\tau + T), v_i > |^p$$

and if we integrate this inequality in $(\theta + hT, \theta + T + hT)$ with respect to the variable t then:

$$\int_{\theta+hT}^{\theta+T+hT} e^{-pL(S_0)(t-\theta)} \|y(\theta + hT) - y(\theta + T + hT)\|_p^p \, dt \leq \int_{\theta+hT}^{\theta+T+hT} \|y(t+T) - y(t)\|_p^p \, dt$$

that is, if we consider the constant $M = \frac{1}{pL(S_0)}\left(1 - e^{-pL(S_0)T}\right)$ then

$$M\|y(\theta + hT) - y(\theta + T + hT)\|_p \leq \int_{\theta+hT}^{\theta+T+hT} \|y(\theta + hT) - y(\theta + T + hT)\|_p^p \, dt$$

for all $\theta \geq t_0$. With this relation we can conclude that

$$M \sum_{h=1}^{\infty} \|y(\theta + hT) - y(\theta + T + hT)\|_p^p \leq \int_{\theta+T}^{\infty} \|y(t) - y(t+T)\|_p^p$$

this is for all $\theta \geq t_0$, by using (2.15) we can ensure the convergence of the series

$$\sum_{h=1}^{\infty} \|y(\theta + hT) - y(\theta + T + hT)\|_p^p$$

for all $\theta \geq t_0$. Therefore $\{y(\theta+hT)\}_{h\geq 1}$ is a Cauchy sequence in $\mathbb{R}^n$. From this follows, like in the other case, that $y(t)$ converges to $u(t)$ a $T-$periodic solution in S_0. This establishes Theorem 2.2.

As before we consider the following corollary of this Theorem, we should observe that the demonstration of this corollary fails if we consider t by $-t$.

COROLLARY 2.4. *Suppose that (2.1) satisfies* $[\mathbf{H_1}]$, $[\mathbf{H_2}]$ *and* $[\mathbf{H_3}]$, *for* $\lambda \geq 0$ *and* $j = 1$. *If (2.1) has a solution* $\tilde{y}(t) \in S_0$, *a compact set in* S, *for all* $t \in (-\infty, t_0]$ *then (2.1) has a* T*-periodic solution* $\tilde{u}(t)$*in* S_0 *such that* $\tilde{y}(t) - \tilde{u}(t) \to 0$ *for* $t \to -\infty$.

Proof. S_0 is a compact set, $V\big(\tilde{y}(t) - \tilde{y}(t-T)\big)$ is bounded for all $-\infty < t \leq t_0$ and $e^{\lambda t}V\big(\tilde{y}(t) - \tilde{y}(t-T)\big) \to 0$ as $t \to -\infty$. It follows that $e^{\lambda t}V\big(\tilde{y}(t) - \tilde{y}(t-T)\big) \leq 0$ for all t in $(-\infty, t_0]$ because this function is monotonic decreasing by (2.9). If $\tilde{y}(t)$ is not $T-$periodic, it is strictly decreasing, then $V\big(\tilde{y}(t) - \tilde{y}(t-T)\big) \leq 0$ for all $-\infty < t \leq t_0$. From this relation and (2.11) we get

$$2\|\mathcal{P}\tilde{y}(t)-\tilde{y}(t-T)\|_p^p \geq V(\tilde{y}(t)-\tilde{y}(t-T)) + \|\mathcal{P}\tilde{y}(t)-\tilde{y}(t-T)\|_p^p = \|\tilde{y}(t)-\tilde{y}(t-T)\|_p^p \quad (2.16)$$

for all $t \leq t_0$. The function $\mathcal{P}\big(\tilde{y}(t) - \tilde{y}(t-T)\big)$ has a constant sign in $(-\infty, t_0]$. Then $\big\{\mathcal{P}\big(\tilde{y}(t_0 - hT)\big)\big\}_{h\geq 1}$ is a monotonic sequence. S_0 is a compact set, then the sequence is also bounded and the series

$$\sum_{h=1}^{\infty} |\mathcal{P}\tilde{y}(t_0 - hT) - \mathcal{P}\tilde{y}(t_0 - hT - T)|^p \quad \text{converges.}$$

By using (2.16) the series

$$\sum_{h=1}^{\infty} |\tilde{y}(t_0 - hT) - \tilde{y}(t_0 - hT - T)|^p \quad \text{also converges.}$$

Then $\big\{\mathcal{P}\big(\tilde{y}(t_0 - hT)\big)\big\}_{h\geq 1}$ is a Cauchy sequence in $\mathbb{R}^n$. From this, like in proof of Theorem 2.3, the equation (2.1) has a $T-$periodic solution $\tilde{u}(t)$ such that $\tilde{y}(t) - \tilde{u}(t) \to 0$ for $t \to -\infty$. This establishes Corollary 2.4.

2.2. Amenable Stability.

In this section we consider a characterization of the amenable solution of (2.1). By arguing like Smith in [24, 25] the notion of amenable solution will be useful for the generalization of Massera's second Theorem in this chapter and for the orbital stability in the next chapter. In this section we will not consider the restriction $[\mathbf{H}_1]$.

Definition 2.5. *A solution $x(t)$ of (2.1) is said to be amenable if there exists an η in $\mathbb{R}^n$ such that $x(t) \in S$ for all $t \in (-\infty, \eta]$ and*

$$\sum_{i=1}^{n} \int_{-\infty}^{\eta} e^{p\lambda t} | < x(t), v_i > |^p dt \quad \text{converges for } 1 \leq p < \infty.$$

Note that every solution of (2.1) in S which is bounded in $t \in (-\infty, \eta]$ is obviously amenable. In particular, every periodic solution in S is amenable.

Furthermore instead of to consider solutions x_1, x_2 of (2.1) we consider amenable solutions of (2.1), then not only (2.9) holds, but if we integrate it for some interval (θ, η) we get an analogue relation to (2.10) but for amenable solutions,

$$e^{\lambda\eta} V\big(x_1(\eta) - x_2(\eta)\big) \leq e^{\lambda\theta} V\big(x_1(\theta) - x_2(\theta)\big) - \int_{\theta}^{\eta} \varepsilon e^{\lambda t} \|x_1(t) - x_2(t)\|_p^p \, dt$$

for x_1, $x_2 \in S$ and for all $\theta \leq t \leq \eta$.

If we consider this relation and in addition we suppose that $e^{\lambda t}| < x_1(t), v_i > |$ and $e^{\lambda t}| < x_2(t), v_i > |$ are in $L^p(-\infty, \eta)$ then $e^{\lambda t}| < x_1(t) - x_2(t), v_i > |$ is in $L^p(-\infty, \eta)$. Therefore

$$e^{\lambda\eta} V\big(x_1(\eta) - x_2(\eta)\big) \leq - \int_{\theta}^{\eta} \varepsilon e^{\lambda t} \|x_1(t) - x_2(t)\|_p^p \, dt \leq 0$$

for each pair of amenable solutions of (2.1), because

$$e^{\lambda\theta_h} | < x_1(\theta_h) - x_2(\theta_h), v_i > |^p \longrightarrow 0$$

for some sequence $\{\theta_h\}_{h\geq 1}$ such that $\theta_h \longrightarrow -\infty$ as $h \to \infty$.

with this previous analysis we can conclude the following result

LEMMA 2.6. *Suppose that $x(t)$, $y(t)$ are solution of (2.1) in S.*
a) if $x(t)$, $y(t)$ are amenable solutions then $V\big(x(t) - y(t)\big) < 0$ for all t,
b) if $y(t)$ is an amenable solution and $V\big(x(t) - y(t)\big) < 0$ for all t then the solution $x(t)$ is also amenable.

From Lemma 2.6 and (2.11) follows that for all amenable solutions $x(t)$, $y(t)$ following holds:

$$\begin{aligned} 2\|\mathcal{P}x(t) - \mathcal{P}y(t)\|_p^p &\geq \\ \geq V(x(t) - y(t)) \; + \; 2\|\mathcal{P}x(t) - \mathcal{P}y(t)\|_p^p &= \\ = \sum_{i=1}^{n} | < x(t) - y(t), v_i > |^p \geq \|\mathcal{P}x(t) - \mathcal{P}y(t)\|_p^p . \end{aligned} \tag{2.17}$$

This relation shows that if $\mathcal{P}x(t) = \mathcal{P}y(t)$ for some value of $t \leq \eta$ then $x(t) = y(t)$ for all values $t \leq \eta$. If h, k are positive constants, the amenable solution $x(t-h)$, $y(t-k)$ lie in S throughout $-\infty < t \leq \eta$ and can therefore replace $x(t)$, $y(t)$ in (2.17). This implies that if Γ, Γ_0 are the amenable orbits described by $x(t)$, $y(t)$ respectively, then

$$\begin{aligned} V(p - q) \; + \; 2\|\mathcal{P}p - \mathcal{P}q\|_p^p &= \\ = \sum_{i=1}^{n} | < p - q, v_i > |^p \geq \|\mathcal{P}p - \mathcal{P}q\|_p^p \end{aligned} \tag{2.18}$$

for all $p \in \Gamma$ and all $q \in \Gamma_0$.

Note that if the curves $\mathcal{P}\Gamma, \mathcal{P}\Gamma_0$ intersect then $\mathcal{P}p = \mathcal{P}q$ for some $p,\ q$. From (2.18) follows that $p = q$ and therefore $\Gamma = \Gamma_0$

Consider $r \in \mathbb{R}$. We denote $\Lambda_r \subseteq S$ the set

$$\Lambda_r = \left\{x(r) \in S :\ x(t) \textit{ is an amenable solution of } (2.1) \textit{ in } S \textit{ for all } t\ \in (-\infty, r]\right\}$$

which will be known as a *amenable set* of (2.1) in S.

When (2.1) satisfies $[\mathbf{H_3}]$ note that Λ_r is the union of all amenable orbits of (2.1) in S. Then (2.18) shows that $\mathcal{P} : \Lambda_r \longrightarrow \mathcal{P}\Lambda_r$ is a linear one-one and bicontinuous map, that is Λ_r is homeomorphic $\mathcal{P}\Lambda_r \subset H$.

If $\phi : \mathcal{P}\Lambda_r \longrightarrow \Lambda_r$ is the inverse mapping of this homeomorphism then (2.18) shows that

$$\begin{aligned} &2\|\varsigma_1 - \varsigma_2\|_p^p \geq \\ &\geq V(\phi(\varsigma_1) - \phi(\varsigma_2)) \ +\ 2\|\mathcal{P}\phi(\varsigma_1) - \phi(\varsigma_2)\|_p^p = \\ &= \sum_{i=1}^{n} \ | < \phi(\varsigma_1) - \phi(\varsigma_2), v_i > |^p \geq \ \|\varsigma_1 - \varsigma_2\|_p^p . \end{aligned}$$

for all $\varsigma_1,\ \varsigma_2$ in $\mathcal{P}\Lambda_r$

THEOREM 2.7. *Suppose that (3.1) satisfies* $[\mathbf{H_2}]$, $[\mathbf{H_3}]$. *If* Γ_0 *is a closed trajectory in* S *and it is amenable stable then* Γ_0 *is orbitally stable.*

Proof. By following the idea of Smith [25] with the opportune changes, it is possible to prove this Theorem.

If $x(t)$ satisfies (2.1) and $\varsigma(t) = \mathcal{P}x(t)$ then

$$\frac{d\varsigma}{dt} = \mathcal{P}\frac{dx}{dt}$$

because $\mathcal{P} : \mathbb{R}^n \longrightarrow H \subset \mathbb{R}^n$ is linear. Some results which are related with amenable solution of (2.1) will be treated in the next section because they are useful to show the generalization of Massera's second Theorem.

2.3. Massera's Second Theorem

In this section, not only will be proved the Massera's second Theorem, but we will give some relation between the amenable solutions of (2.1) and the j-dimensional equation (2.22). The proof of Massera's second theorem will be realized with the help of Corollary 2.11. In fact, to guarantee the existence of a T-periodic solution of (2.1), Smith in [24] ensured the existence of T-periodic solution $u(t)$ for the system (2.22) (in our case over the hyperplane H) with the help of the following theorem and its allow to find the candidate to be T-periodic solution of (2.1) by using the map ϕ ($:\mathcal{P}\Lambda_r \longrightarrow \Lambda_r$ the inverse mapping of this homeomorphism).

Let us consider the following theorem, for dimension 2.

THEOREM .. *Let consider (2.1) with a $T-$periodic solution function on $S = \mathbb{R}^2$ and $n = 2$. Suppose that all solutions of (2.1) are defined in an interval of the form (θ, ∞) and if once of them, $y(t)$, is bounded in some interval $[t_0, \infty)$ then there exist at least one $T-$periodic solution $u(t)$ of (2.1).*

This theorem is knowed like Massera's Second Theorem and it does not give any relation $y(t)$ and $u(t)$ like in Theorem 2.1. This theorem is an extension of the convergence theorem and we can observe that Massera needed to suppose that all solution should be defined in an interval (θ, ∞).

Our interest is to give such generalization of this Theorem for this in this section we will consider the definition of (2.1) holds for $S = \mathbb{R}^n$. Suppose the following restriction

[$\mathbf{H_4}$] *Each solution of (2.1) is defined in an interval of the form (θ, ∞).*

The following Theorem will be refereed to as the generalization of Massera's second Theorem.

THEOREM 2.8. *Suppose that (2.1) satisfies* $[\mathbf{H_1}]$, $[\mathbf{H_2}]$, $[\mathbf{H_3}]$ *and* $[\mathbf{H_4}]$ *for* $S = \mathbb{R}^n$, $\lambda \geq 0$ *and* $j = 2$. *If (2.1) has a solution* $y(t)$ *which is bounded for all interval* $[t_0, \infty)$ *then (2.1) has at least one* $T-$*periodic solution.*

Under the same assumptions of this theorem we will study the following results related with amenable solutions.

LEMMA 2.9. *Suppose that* $y(t)$ *is an amenable solution of (2.1). If* $\varsigma \in H$ *and* θ, $r \in \mathbb{R}$ *with* $\theta < r$ *then there exist a solution* $z_\theta(t)$ *of (2.1) defined on* $\theta \leq t < \infty$ *such that* $\varsigma = \mathcal{P} z_\theta(r)$ *and* $V\big(z_\theta(r) - y(t)\big) \leq 0$ *for all* $t \in [\theta, \infty)$.

Proof. Let $\mathcal{P}x = \chi_A$ in H for any $x \in \mathbb{R}^n$ and we denote by $x(t, \chi_A, \theta)$ any solution of (2.1) such that $x(\theta) = y(\theta) + (\chi_A, 0)$ (see Chapter 1) with $y(\theta)$, the value of $t = \theta$ in $y(t)$ an amenable solution. The solution $x(t, \chi_A, \theta)$ there exist for all $\theta \leq t < \infty$ for $[\mathbf{H_4}]$.

If $\chi_A = 0$, we have $x(t, 0, \theta) = y(t)$ in this situation

$$x(\theta, \chi_A^1, \theta) - x(\theta, \chi_A^2, \theta) = (\chi_A^1 - \chi_A^2, \theta)$$

and when we replace by x in the definition of the function V, we get

$$\begin{aligned} V\big(\chi_A^1 - \chi_A^2, 0\big) &= -\sum_{i \in A} | < x(\theta, \chi_A^1, \theta) - x(\theta, \chi_A^2, \theta), v_i > |^p = \\ &= -\sum_{i \in A} | < x_1(\theta) - x_2(\theta), v_i > |^p. \end{aligned}$$

From (2.5) we know that the function $e^{\lambda t} V\big(x(t, \chi_A^1, \theta) - x(t, \chi_A^2, \theta)\big)$ is decreasing in $[\theta, \infty)$ wherefore the following relation

$$\begin{aligned} e^{\lambda\theta} V\big(x(\theta, \chi_A^1, \theta) - x(\theta, \chi_A^2, \theta)\big) &= -e^{\lambda\theta} \sum_{i \in A} | < x_1(\theta) - x_2(\theta), v_i > |^p \geq \\ &\geq \; e^{\lambda t} V\big(x(t, \chi_A^1, \theta) - x(t, \chi_A^2, \theta)\big), \end{aligned} \tag{2.15}$$

hold for all $t \geq \theta$. If we consider (2.8) result

$$2\|\mathcal{P}\left(x(t,\chi_A^1,\theta) - x(t,\chi_A^2,\theta)\right)\|_p^p \geq$$
$$\geq V\left(x(t,\chi_A^1,\theta) - x(t,\chi_A^2,\theta)\right) + 2\|\mathcal{P}\left(x(t,\chi_A^1,\theta) - x(t,\chi_A^2,\theta)\right)\|_p^p, \tag{2.16}$$

this for all $\theta \leq t$. That is

$$-V\left(x(t,\chi_A^1,\theta) - x(t,\chi_A^2,\theta)\right) \leq \|\mathcal{P}\left(x(t,\chi_A^1,\theta) - x(t,\chi_A^2,\theta)\right)\|_p^p.$$

From this and (2.15)

$$e^{\lambda\theta}\sum_{i\in A} | < x_1(\theta) - x_2(\theta), v_i > |^p \leq e^{\lambda t}\|\mathcal{P}\left(x(t,\chi_A^1,\theta) - x(t,\chi_A^2,\theta)\right)\|_p^p, \tag{2.17}$$

for all $t \geq \theta$.

For any θ, r, with $\theta < r$ let us consider $G_\theta : H \longrightarrow H$ a continuous map such that $G_\theta = \mathcal{P}x(r,\chi_A,\theta)$ for all $\chi_A \in H$. If we replace $t = r$, results from (2.17)

$$e^{\lambda\theta}\sum_{i\in A} | < x_1(\theta) - x_2(\theta), v_i > |^p \leq e^{\lambda t}\|G_\theta(\chi_A^1) - G_\theta(\chi_A^2)\|_p^p \tag{2.18}$$

for χ_A^1, $\chi_A^2 \in H$

The relation (2.18), shows that G_θ is a one-one map and by using Brouwer's Theorem on invariance of domain the set $G_\theta(H)$ is an open subset in H. Now we will prove that $G_\theta(H) = H$ which is useful to show that $\varsigma = \mathcal{P}z_\theta(r)$.

In fact, suppose, by contradiction, that $G_\theta(H)$ is not the whole H, then there exist $b \in \partial\left(G_\theta(H)\right)$ because $G_\theta(H)$ is an open set. Then there is $\left\{G_\theta\left(\chi_A^h\right)\right\}_{h\geq 1} \subset H$ such that $lim\ G_\theta\left(\chi_A^h\right) = b$. This means $\left\{G_\theta\left(\chi_A^h\right)\right\}_{h\geq 1}$ is a Cauchy sequence in H and for (2.18) follows that $\left\{\left(\chi_A^h\right)\right\}_{h\geq 1}$ converges in H. Then there exist $a \in H$ such that $\chi_A^h \to a$. Hence $lim\ G_\theta\left(\chi_A^h\right) = G_\theta(a) = b$, that is, $b \in int\left(G_\theta(H)\right)$. If we give $\varsigma \in H$, we can find $b(\theta)$ in H such that $\varsigma = G_\theta\left(b_(\theta)\right) = \mathcal{P}x(r, b(\theta), \theta)$.

If we denote by $z_\theta(t) = x(t, b(\theta), \theta)$ a solution of (2.1) in $[\theta, \infty)$ then $\varsigma = \mathcal{P}z_\theta(r)$. For $y(t) = x(t, 0, \theta)$, if we consider $\chi_A^1 = b(\theta)$, $\chi_A^2 = 0$ from (2.15) we deduce $V\left(z_\theta(t) - y(t)\right) \leq 0$ for all $\theta \leq t$. This establish Lemma 2.9.

THEOREM 2.10. *Suppose that (2.1) satisfies* $[\mathbf{H}_2]$, $[\mathbf{H}_3]$ *and* $[\mathbf{H}_4]$ *for* $S = \mathbb{R}^n$, $\lambda \geq 0$ *and* $j \geq 1$. *If (2.1) has at least one amenable solution then* $\mathcal{P}\Lambda_r = H$ *for all* $r \in \mathbb{R}$ *and the restriction* $\mathcal{P} : \Lambda_r \longrightarrow H$ *is an homeomorphism.*

Proof. It is sufficient to show for every $\varsigma \in H$, there is an amenable solution $u(t)$ of (2.1) such that $\varsigma = \mathcal{P}u(t)$ to prove $H \subset \mathcal{P}\Lambda_r$. The solution $u(t)$ will be find like a limit of some sequence of the solutions $z_\theta(t) = x(t, b(\theta), \theta)$ which we have seen before in the previous Lemma 2.9.

Let us consider $\varsigma = \mathcal{P}z_\theta(r) = \mathcal{P}x(r, b(\theta), \theta)$, $x(t, 0, \theta) = y(t)$ and by putting $\chi_A^1 = b(\theta)$, $\chi_A^2 = 0$, $t = r$ in (2.16) results

$$2\|\varsigma - \mathcal{P}y(r)\|_p^p \geq V\left(z_\theta(r) - y(r)\right) + 2\|\varsigma - \mathcal{P}y(r)\|_p^p, \quad \text{for all} \theta \leq r \qquad (2.19)$$

If we consider $x(t) = x(t, b(\theta), \theta)$, $\tau = r$ in (2.6) and later by using (2.15) we get,

$$-e^{\lambda r} V\left(x(r, b(\theta), \theta) - y(\theta)\right) \geq \varepsilon \int_\theta^r e^{\lambda t} \|x(t, b(\theta), \theta) - y(t)\|_p^p \, dt$$

This relation and (2.19) give

$$\int_\theta^r e^{\lambda t} \|x(t, b(\theta), \theta) - y(t)\|_p^p \, dt \leq \frac{2}{\varepsilon} e^{\lambda r} \|\varsigma - \mathcal{P}y(r)\|_p^p \qquad (2.20),$$

for all $\theta \leq r$. The boundness of $\|x(r, b(\theta), \theta)\|_p^p$ holds for (2.19), this for all $\theta \leq r$. We are interested to find the solution $u(t)$, for this let us consider the sequence of real numbers $\{\theta_h\}_{h \geq 1}$ such that $x(r, b(\theta_h), \theta_h) \to q$ and $\theta_h \to -\infty$, as $h \to \infty$, for $q \in H$. Let $u(t)$ be a solution of (2.1) such that $u(r) = q$, from the hypothesis $[\mathbf{H}_4]$ follows that $u(t)$ exist in $[r, \infty)$. Furthermore $\mathcal{P}u(r) = \varsigma$ because $\varsigma = \mathcal{P}x(r, b(\theta_h), \theta_h)$ for all h.

Now we will show that $u(t)$ exist for $(-\infty, r]$. A sufficient condition is to show that $u(t)$ exist in an interval $[\beta, r]$ for all $\beta < r$. If we take h large, $\theta_h \leq \beta - 1$ and (2.20), we get

$$\int_{\beta-1}^{\beta} e^{\lambda t} \|x(t, b(\theta_h), \theta_h) - y(t)\|_p^p \, dt \leq \frac{2}{\varepsilon} e^{\lambda r} \|\varsigma - \mathcal{P}y(r)\|_p^p$$

Applying the mean value Theorem to this integral there exist $t_h \in [\beta-1,\beta]$ such that

$$e^{\lambda(\beta-1)}\|x(t_h,b(\theta_h),\theta_h)-y(t_h)\|_p^p \leq \frac{2}{\varepsilon}e^{\lambda r}\|\varsigma - \mathcal{P}y(r)\|_p^p.$$

We can observe that, when β is fixed, follows that $\|x(t_h,b(\theta_h),\theta)\|_p^p$ and t_h are bounded for all large h. In consequence of the Weierstrass subsequence Theorem we can suppose that there exist a sequence $t_h \to l$ such that $x(t_h,b(\theta_h),\theta_h) \longrightarrow p$ as $h \to \infty$, for $l \in [\beta-1,\beta]$ and $p \in \mathbb{R}^n$.

Now if we denote by $w(t)$ the solution of (2.1) such that $w(l)=p$, by $[\mathbf{H_4}]$ the solution $w(t)$ exist in $[l,\infty)$. Since the solutions vary continuously with their initial values, we can conclude, in analogous way like before that $x(t,b(\theta_h),\theta) \to w(t)$ point wise in $[l,\infty)$ when $h \to \infty$. In particular

$$w(r) = \lim_{h\to\infty} x(r,b(\theta_h),\theta_h) = u(r)$$

and then $w(t)$ is an extension of $u(t)$ in $[l,r]$. Hence $l \leq \beta$, there exist $u(t)$ in $[\beta,\infty)$ and this for all $\beta \leq r$. For this $u(t)$ exist in all $(-\infty,\infty)$.

The last thing which we need to prove is $u(t)$ is an amenable solution. For $t \geq \theta_h$, (2.15) gives $0 \geq V\big(x(t,b(\theta_h),\theta_h)-y(t)\big)$. When $h\to\infty$, this gives $0 \geq V\big(y(t)-w(t)\big)$ for all $t \geq l$. Since $u(t)$ coincides with $w(t)$ this follows that $0 \geq V\big(y(t)-u(t)\big)$ in $[\beta,\infty)$ for all $\beta \leq r$. That is, $0 \geq V\big(y(t)-u(t)\big)$ for all $(-\infty.\infty)$. Since $y(t)$ is an amenable solution from Lemma 2.6 results that $u(t)$ is an amenable solution too. Then $u(r) \in \Lambda_r$. We proved above that $\varsigma = \mathcal{P}u(r)$ and therefore $\varsigma \in \mathcal{P}\Lambda_r$ for each $\varsigma \in H$. That is $H = \mathcal{P}\Lambda_r$. In (2.14) we showed that the map $\mathcal{P} : H \longrightarrow \mathcal{P}H$ is an homeomorphism, this establishes the proof of the Theorem 2.10.

An application directly of this theorem is the following

COROLLARY 2.11. *Suppose that (2.1) satisfies* $[\mathbf{H_2}]$, $[\mathbf{H_3}]$ *and* $[\mathbf{H_4}]$ *for* $S = \mathbb{R}^n$, $\lambda \geq 0$ *and* $j \geq 1$. *If (2.1) has at least one amenable solution then there is a continuous function* $\phi(t,\varsigma) : \mathbb{R} \times H \longrightarrow \mathbb{R}^n$ *such that the relations*

$$\begin{aligned} \varsigma(t) &= \mathcal{P}x(t) \\ x(t) &= \phi(t,\varsigma(t)) \end{aligned} \tag{2.21}$$

give a correspondence between amenable solution $x(t)$ *and the solutions* $\varsigma(t)$ *of the* $j-$*dimensional equation*

$$\frac{d\varsigma}{dt} = \mathcal{P}f(t,\phi(t,\varsigma)). \tag{2.22}$$

Proof. If suppose that $(t,\varsigma) \in \mathbb{R} \times H$, the Theorem 2.10 give us the existence of only one point $\phi(t,\varsigma)$ in Λ_t such that $\varsigma = \mathcal{P}\phi(t,\varsigma)$. It is possible to define a function $\phi : \mathbb{R} \times H \longrightarrow \mathbb{R}^n$ which verifies $x = \phi(t, \mathcal{P}x)$ for all $x \in \Lambda_t$ because $\Lambda \subseteq \mathbb{R}^n$. Furthermore (2.14) gives the following

$$\begin{aligned} 2\|\mathcal{P}x_1(t) - x_2(t)\|_p^p \geq \|\phi(t,\varsigma_1) - \phi(t,\varsigma_2)\|_p^p \geq \\ \geq \|\mathcal{P}x_1(t) - x_2(t)\|_p^p. \end{aligned} \tag{2.23}$$

Now we will show that $\phi(t,\varsigma)$ is a continuous function for all point $(r,\varsigma_0) \in \mathbb{R} \times H$. From the definition of Λ_r there exist an amenable solution $x_0(t)$ of (2.1) such that $\phi(r,\varsigma_0) \in \Lambda_r$ and $x_0(r) = \phi(r,\varsigma_0)$. Then $\varsigma_0 = \mathcal{P}x_0(r)$. Furthermore for $t \in \mathbb{R}$ fixed, $x_0(t) \in \Lambda_t$ then hold $x_0(t) = \phi(t, \mathcal{P}x_0(t))$ for all t. For this and (2.23)

$$\|\phi(t,\varsigma) - \phi(r,\varsigma_0)\|_p^p \leq 2\|\varsigma - \mathcal{P}x_0(t)\|_p^p + \|x_0(t) - x_0(r)\|_p^p.$$

From (2.8) follows

$$\|x_0(r) - x_0(t)\|_p^p \geq \|\varsigma_0 - \mathcal{P}x_0(t)\|_p^p.$$

Then ϕ is a continuous function at the point (r,ς_0) because

$$\|\phi(t,\varsigma) - \phi(r,\varsigma_0)\|_p^p \leq 3\|x_0(t) - x_0(r)\|_p^p.$$

Since the map $\mathcal{P} : \mathbb{R}^n \to H \subset \mathbb{R}^n$ does not depend an explicit way of the variable t and is a linear map, this allows us to write

$$\frac{d}{dt}\big(\mathcal{P}x(t)\big) = \mathcal{P}\big(\frac{dx}{dt}\big) = \mathcal{P}f(t, x(t)) \tag{2.24}$$

for all solution $x(t)$ of (2.1).

Moreover if $x(t)$ is an amenable solution then $x(t) \in \Lambda_t$ and then $x(t) = \phi(t, \mathcal{P}x(t))$. Then $\frac{d}{dt}(\mathcal{P}x(t)) = \mathcal{P}f(t, \phi(t, \mathcal{P}x(t)))$ and for $\phi(t) = \mathcal{P}x(t)$ (2.21) and (2.22) hold. The last part which we need to prove is whether each solution of (2.22) has the same form. It is clear from (2.22) that the continuity and local Lipschitz condition for f is verified in each point $(r, \varsigma) \in \mathbb{R} \times H$. By Picard's Theorem there is only one solution $\varsigma(t)$ of (2.22). Hence every solution of $\varsigma(t)$ of (2.22)is of the form $\mathcal{P}x(t)$, where $x(t)$ is amenable solution of (2.1). This establishes the Corollary 2.11.

Proof of Theorem 2.8. We suppose that (2.1) should have a solution $y(t)$ such that it is bounded in $[t_0, \infty)$, this means that there is a non negative constant M such that $|y(t)| \leq M$ for all $t \in [t_0, \infty)$. There exist a sequence of integer positives such that $y(m_h T) \to a$ as $h \to \infty$, with $a \in \mathbb{R}^n$. Suppose that the solution $x(a, t)$ with initial condition $x(0, a) = a$ exist for all t such that $r \leq t \leq s$, with s, $r \in \mathbb{R}$ and $r < 0 < s$. From $[\mathbf{H}_1]$ holds that $y(t + m_h T)$ is a solution of (2.1) defined in $[t_0 - m_h T, \infty)$ and then the sequence $y(t + m_h T) \to x(a, t)$ converges point wise in $[r, s]$, for $h \to \infty$. For h sufficiently large $|y(t + m_h T)| \leq M$ for all $r \leq t \leq s$ and then $|x(t, a)| \leq M$ for $r \leq t \leq s$. This shows that there exist a point $x(t, a) \in \mathbb{R}^n$ which can never meet the boundary of the set N,

$$N = \big\{x \in \mathbb{R}^n : |x| \leq 1 + M\big\}.$$

The solution $x(t, a)$ exists and lies in N, for all t such that $-\infty < t < \infty$. Hence (2.1) has a amenable solution which satisfies the conditions of the Corollary 2.8. For $j = 2$ the $T-$periodic differential equation (2.22) is bidimensional. There exist all the solutions for all $t \in (-\infty, \infty)$, because 2.1 verifies $[\mathbf{H}_6]$. Since it has the bounded solution $\mathcal{P}x(t, a)$ Massera's second Theorem shows that (2.22) has at least one $T-$periodic solution $\varsigma(t)$. Then Corollary 2.10 shows that $\phi(t, \varsigma(t))$ is a $T-$periodic solution of (2.1) because $\phi(t + T, \varsigma) = \phi(t, \varsigma)$ for all (t, ς) in $\mathbb{R} \times \mathbb{R}^j$. This establishes the Theorem 2.8.

Chapter 3

POINCARÉ - BENDIXSON THEOREM

3.1. Existence and Uniqueness Theorems.

Let us consider the autonomous ordinary differential equation

$$\frac{dx}{dt} = f(x) \tag{3.1}$$

in which $f : S \longrightarrow \mathbb{R}^n$ is a continuous function such that for some closed subset $S \subseteq \mathbb{R}^n$, f satisfies the local Lipschitz condition in S, such that $f(x)$ satisfy the Lipschitz condition in $B(x)$.

We call x_0 a critical point (equilibrium point or singular point) of (3.1) if $f(x_0) = 0$, that is, if x_0 is a constant solution of (3.1). If we consider $x(t)$ a solution of (3.1), then the locus of $x(t)$ for $t_0 \leq t < +\infty$ Γ is called a complete orbit of (3.1). For $x(t)$ a periodic solution of (3.1), its complete orbit or periodic orbit is a closed curve in $\mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is called ω-limit point of Γ if there exist a sequence $\{t_h\}_{h\geq 1} \subset [t_0, +\infty)$ such that $t_h \rightarrow \infty$ and $x(t_h) \rightarrow p$. We denote by $\Omega(\Gamma)$ the set of all ω-limit points of Γ. $\Omega(\Gamma)$ has the property to be an invariant set, that is, if $x(t)$ is a solution of (3.1) such that $x(t_0) \in \Omega(\Gamma)$ for some t_0 in $\mathbb{R}$ then, the solution $x(t)$ exists throughout $(-\infty, +\infty)$ and $x(t) \in \Omega(\Gamma)$ for all t. We could observe that Ω is a union of complete orbits of (3.1) if Γ is contained in a compact set, $\Omega(\Gamma)$ is bounded, non empty, closed, invariant set. See Bathia-Szego [1], Cronin J. [6].

In this chapter we will give a sufficient condition for the set $\Omega(\Gamma)$ to include a periodic orbit of (3.1).

There are a lot of methods used to show the existence of periodic orbits of autonomous ordinary differential equations Bingxi Li [2]. One of them is the following:

THEOREM 3.1. *(Poincaré-Bendixson) If $S \subset \mathbb{R}^2$includes a bounded semi-orbit Γ of (3.1) and the set $\Omega(\Gamma)$ includes no critical points (3.1) then $\Omega(\Gamma)$ consist of a periodic orbit of (3.1).*

It is well known that the full extension of the Poincaré-Bendixson Theorem to the case $n > 2$ is not possible without adding further hypothesis. Recently, R. A. Smith gave assumptions which are easier than other methods and following his results we obtain some restrictions which permits us to apply the Poincaré -Bendixson Generalized Theorem in an easy way for a wider class of ordinary differential equations.

Now, let us have the following hypothesis:

[$\mathbf{H_1}$] *There is a continuous function $U : \mathbb{R}^n \longrightarrow \mathbb{R}$*

$$U(x) = \sum_{i=1}^{n} a_i | < x, v_i > |^p, \tag{3.2}$$

such that $U(x_1 - x_2) \leq 0$ in $(-\infty, +\infty)$ for every pair of bounded solutions x_1, x_2 of (3.1) which lies in S throughout $(-\infty, +\infty)$.

Moreover we will suppose

[$\mathbf{H_2}$] *The coefficients*

$$a_i = \begin{cases} -1, & \text{if } i \in A; \\ 1, & \text{if } i \in B. \end{cases}$$

for all $i = 1, ..., n$.

In this chapter, we assume that *dim* $A = 2$ and *dim* $B = n - 2$. Let us consider the following hyperplane in $\mathbb{R}^n$

$$H = \{z \in \mathbb{R}^n : z = \sum_{i \in A} < z, v_i > v_i\}$$

and let $\mathcal{P} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear map defined by

$$\mathcal{P}x = 2^{\frac{1}{p}} \sum_{i \in A} < x, v_i > v_i, \quad \text{for every } x \in \mathbb{R}^n, \quad \text{and} \quad 1 \leq p < +\infty.$$

We can observe that

$$\|x\|_p^p = \sum_{i=1}^{n} | < x, v_i > |^p = U(x) + \|\mathcal{P}x\|_p^p,$$

we have

$$2\|x\|_p^p \geq \|\mathcal{P}x\|_p^p = \|x\|_p^p - U(x) \tag{3.3}$$

for every $x \in \mathbb{R}^n$.

Before giving the Generalization of Poincaré-Bendixson Theorem we should observe the following properties of $\mathcal{P}$ a linear map.

LEMMA 3.3. *If* $x_1(t)$, $x_2(t)$ *are bounded solutions of (3.1) in* $(-\infty, +\infty)$ *such that* $U(x_1(t) - x_2(t)) \leq 0$ *then , the following relations hold:*

(*i*) $\|\mathcal{P}x_1(t) - \mathcal{P}x_2(t)\|_p \geq \|x_1(t) - x_2(t)\|_p$,
(*ii*) $\|\frac{d}{dt}(\mathcal{P}x_1(t))\|_p \geq \|\frac{dx_1}{dt}(t)\|_p$, *for all real* t .

Proof. Let us consider $x_1(t)$, $x_2(t)$ bounded solutions of (3.1) which lie in S throughout $(-\infty, +\infty)$. If we replace $x(t)$ by $x_1(t) - x_2(t)$ in (3.1) and from hypothesis we get

$$2^{\frac{1}{p}}\|x_1(t) - x_2(t)\|_p \geq \|\mathcal{P}x_1(t) - \mathcal{P}x_2(t)\|_p \geq \|x_1(t) - x_2(t)\|_p \quad \textit{for all } t. \tag{3.4}$$

Since $\mathcal{P}$ is a linear map then $\mathcal{P}x_1(t)$ is differentiable and the following equality

$$\frac{d}{dt}\left(\mathcal{P}x_1(t)\right) = \mathcal{P}\left(\frac{dx_1}{dt}(t)\right)$$

holds. If we replace $x_2(t)$ in (3.4) by the solution $x_1(t+h)$ for any constant h then

$$\|\mathcal{P}x_1(t+h) - \mathcal{P}x_1(t)\|_p \geq \|x_1(t+h) - x_1(t)\|_p$$

for every t and h real numbers. Dividing by $|h|$ and letting $|h| \to 0$ we get

$$\left\|\frac{d}{dt}\left(\mathcal{P}x_1(t)\right)\right\|_p \geq \left\|\frac{dx_1}{dt}(t)\right\|_p \tag{3.5}$$

this establishes Lemma 3.3.

Remark 3.4. We have observed that Ω is a union of complete orbits of (3.1) because $\Omega(\Gamma)$ do not include critical points. Since any two points y_1, y_2 of $\Omega(\Gamma)$ can be written as $x_1(0)$, $x_2(0)$ for suitable solutions $x_1(t)$, $x_2(t)$ of (3.1), it then follows from (3.4) that

$$2^{\frac{1}{p}}\|y_1 - y_2\|_p \geq \|\mathcal{P}y_1 - \mathcal{P}y_2\|_p \geq \|y_1 - y_2\|_p$$

for every $y_1, y_2 \in \Omega(\Gamma)$. This relation shows that $\mathcal{P} : \Omega \longrightarrow \mathcal{P}\Omega$ gives a bicontinuous and one-one mapping. Then

a) disjoint complete orbits in $\Omega(\Gamma)$ are mapped into disjoint plane curves,
b) periodic orbits in $\Omega(\Gamma)$ are mapped into simple closed curves,
c) if the complete orbit $\Gamma_1 \subset \Omega(\Gamma)$ is not periodic then the plane curve $\mathcal{P}\Gamma_1$ does not intersect itself,
d) if $x(t)$ a solution of (3.1) describes a complete orbit Γ_1 in $\Omega(\Gamma)$ then the tangent vector to the plane curve $\mathcal{P}\Gamma_1$ at the point $\mathcal{P}x_1(t)$ is the vector $\frac{d}{dt}\left(\mathcal{P}x_1(t)\right)$ which is nonzero by (3.5),
e) $\Gamma_1 \subset \Omega(\Gamma)$ because $\Omega(\Gamma)$ is a closed set.

THEOREM 3.3. *Suppose that (3.1) satisfies* $[\mathbf{H_1}]$ *and* $[\mathbf{H_2}]$. *If (3.1) has a bounded semi-orbit* $\Gamma \subset S$ *and* $\Omega(\Gamma)$ *contains no critical points of (3.1), then* $\Omega(\Gamma)$ *includes at least one periodic orbit of (3.1).*

Proof. This demonstration is based on Theorem 3.1. Let $x_1(t)$, $x_2(t)$, $x_3(t)$ be solutions of (3.1) such that the complete orbits Γ_1, Γ_2, Γ_3 verify respectively the following assumptions: $\Gamma_1 \subset \Omega(\Gamma)$, $\Gamma_2 \subset \Omega(\Gamma_1)$ and $\Gamma_3 \subset \Omega(\Gamma_2)$ then $\Omega(\Gamma_2) \subset \Omega(\Gamma_1) \subset \Omega(\Gamma)$ because each one of them is a closed set. Suppose $x_3(0) = q$ is a point of Γ_3 and $v = \mathcal{P}\frac{dq}{dt}$ be the nonzero vector which is tangential to the plane curve $\mathcal{P}\Gamma_3$ at the point $\mathcal{P}q$. Let us consider an open disc D included in H with center $\mathcal{P}q$ and radius (small) $\delta > 0$.

Since $q \in \Gamma_3$, q is $w-$limit point of Γ_1 and Γ_2 then the arcs of the plane curves $\mathcal{P}\Gamma_1$ and $\mathcal{P}\Gamma_2$ must pass through the disc D an infinite number of times for $t \to +\infty$. Then (3.4) shows that if

$$\mathcal{P}x_1(t) \to \mathcal{P}q, \quad x_1(t) \to q.$$

From (3.1) it follows that if

$$\frac{dx_1}{dt}(t) \to \frac{dq}{dt}, \qquad \text{then}$$
$$\mathcal{P}\frac{dx_1}{dt}(t) \to \mathcal{P}\frac{dq}{dt} = v.$$

Hence, if the radius δ is small then along all of arcs $\mathcal{P}\Gamma_1$ in D the tangent vector is approximately equal to v. These arcs $\mathcal{P}\Gamma_1$ are close to the straight line segments parallel to v. For $\mathcal{P}\Gamma_2$ the same argument is also true.

We define the transversal R to be the diameter of D such that it is perpendicular to v. Then each arc $\mathcal{P}\Gamma_1$ and $\mathcal{P}\Gamma_2$ cuts R at most once while remaining in D. We will prove that either Γ_1 or Γ_2 is a periodic orbit. We suppose that neither of them are periodic orbits. From Remark 3.3, c), the plane curves $\mathcal{P}\Gamma_1$, $\mathcal{P}\Gamma_2$ do not intersect themselves respectively.

Each time that $\mathcal{P}\Gamma_2$ intersects the transversal R it does so in different points. Let us consider α, $\beta \in \Gamma_2$ such that $\mathcal{P}\alpha$ and $\mathcal{P}\beta$ are successive intersections of $\mathcal{P}\Gamma_2$ with R then $\mathcal{P}\alpha \neq \mathcal{P}\beta$. α and β are $w-$limits of Γ_1 because $\Gamma_2 \subset \Omega(\Gamma)$. The orientation of the plane curve $\mathcal{P}\Gamma$ will be selected in the same direction of the vector v, that is the curve $\mathcal{P}\Gamma$ intersect the transversal R in the direction of v.

From the Fig. 3.1, it is clear that, $\mathcal{P}\Gamma_1$ intersects R close to $\mathcal{P}\alpha$ and later intersects R close to $\mathcal{P}\beta$. Γ_1 cannot intersect R again close to $\mathcal{P}\alpha$ because $\mathcal{P}\Gamma_1$ cannot cross itself. Then this is a contradiction because we supposed that α is an $w-$limit point of the orbits Γ_1 then at least one of the orbits Γ_1, Γ_2 is periodic. This establishes Theorem 3.3.

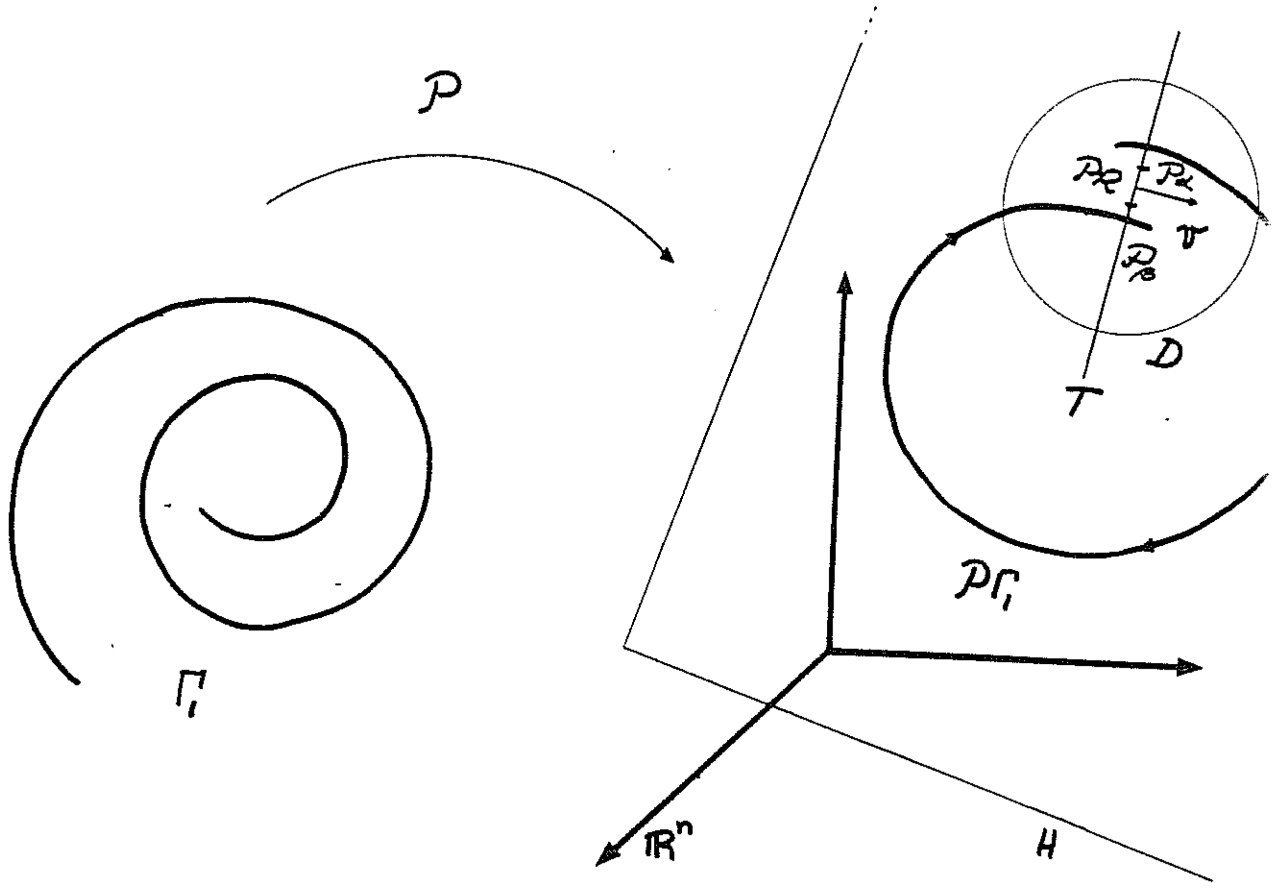

Fig. 3.1

We should observe that Theorem 3.2 is not a generalization of Poincaré - Bendixson Theorem because it allows the possibility that $\Omega(\Gamma)$ includes more than one periodic orbit. For this reason we will suppose the following restrictions,

[$\mathbf{H}_8$] *There exist constants* $\lambda \geq 0$, $\varepsilon_1 > 0$ *and* $V(x) = \sum_{i=1}^{n} a_i | < x, v_i > |^p$ *such that*

$$\lambda \sum_{i=1}^{n} a_i | < x_1 - x_2, v_i > |^p + p \sum_{i=1}^{n} a_i K\big(< x_1 - x_2, v_i >\big) | < x_1 - x_2, v_i > |^{p-1} \cdot$$

$$\cdot < f(x_1) - f(x_2), v_i > \leq \tag{3.6}$$

$$\leq -\varepsilon_1 \sum_{i=1}^{n} | < x_1 - x_2, v_i > |^p$$

for all $t \in \mathbb{R}^n$, *for all solutions* $x_1(t)$, $x_2(t)$ *of (3.1) and* $p \geq 1$.

[$\mathbf{H}_4$] *There exist constants* $\mu \geq 0,\ \varepsilon_2 > 0$ *and there exist* $W(x) = \sum_{i=1}^{n} b_i| < x, v_i > |^p$ *such that*

$$-\mu \sum_{i=1}^{n} b_i| < x_1 - x_2, v_i > |^p + p\sum_{i=1}^{n} b_i K\big(< x_1 - x_2, v_i >\big)| < x_1 - x_2, v_i > |^{p-1}\cdot$$

$$\cdot < f(x_1) - f(x_2), v_i > \leq \tag{3.7}$$

$$\leq\ -\varepsilon_2 \sum_{i=1}^{n} | < x_1 - x_2, v_i > |^p$$

for all $t \in \mathbb{R}^n$, *for all solutions* $x_1(t)$, $x_2(t)$ *of (3.1) and* $p \geq 1$.

[$\mathbf{H}_5$] *The coefficients*

$$a_i - b_i = \begin{cases} -1, & \text{if } i \in A; \\ 1, & \text{if } i \in B. \end{cases}$$

for all $i = 1, ..., n$.

THEOREM 3.4. *Suppose that (3.1) satisfies* [$\mathbf{H}_1$], [$\mathbf{H}_3$] ,[$\mathbf{H}_4$] *If (3.1) has a bounded semi-orbit* $\Gamma \subset S$ *and* $\Omega(\Gamma)$ *contains no critical points of (3.1), then* $\Omega(\Gamma)$ *consist of a single periodic orbit of (3.1).*

Proof. Theorem 3.2 is the principal tool to ensure the existence of periodic orbits of (3.1), from this we need to consider $S \subset \mathbb{R}^n$ the closed set like before and, the closure of Γ, $\overline{\Gamma} = S_0 \subset S$. The relations (3.6) and (3.7) can be written respectively as

$$D^+[e^{\lambda t}V(x_1(t) - x_2(t))]\ \leq\ -\varepsilon_1 e^{\lambda t}\|x_1(t) - x_2(t)\|_p^p, \tag{3.8}$$

$$D^+[e^{-\mu t}V(x_1(t) - x_2(t))]\ \leq\ -\varepsilon_2 e^{-\mu t}\|x_1(t) - x_2(t)\|_p^p. \tag{3.9}$$

If the solutions $x_1(t)$, $x_2(t)$ of (3.1) lies in S_0 for all $t \in (-\infty, +\infty)$ the boundness of $x_1(t)$, $x_2(t)$ allows us to say

$$\lim_{x\to-\infty} e^{\lambda t}V\big(x_1(t)-x_2(t)\big) = \lim_{x\to+\infty} e^{-\mu t}W\big(x_1(t)-x_2(t)\big) = 0.$$

From (3.8) and (3.9) we can deduce that the functions

$$e^{\lambda t}V\big(x_1(t)-x_2(t)\big) \text{ and } e^{-\mu t}W\big(x_1(t)-x_2(t)\big)$$

are monotonic decreasing in $(-\infty,+\infty)$ and therefore satisfy the relation

$$e^{\lambda t}V\big(x_1(t)-x_2(t)\big) \le 0 \le e^{-\mu t}W\big(x_1(t)-x_2(t)\big)$$

for all t. Then $V(x)\le 0$ and $W(x)\ge 0$ for all t. Let $U(x)=V(x)-W(x)$, i.e.,

$$U(x)=\sum_{i=1}^{n}(a_i-b_i)|<x_1(t)-x_2(t),v_i>|^p.$$

This implies that $U\big(x_1(t)-x_2(t)\big)\le 0$ in $(-\infty,+\infty)$. Then all the hypothesis of Theorem 3.2 holds if we replace S *by* S_0. Then $\Omega(\Gamma)$ includes at least one periodic orbit Γ_0.

It remains to prove that this periodic orbit γ_0 is the whole set $\Omega(\Gamma)$. Let us consider x_1, x_2 solutions of (3.1) such that describing Γ,Γ_0 respectively. then we have the conditions of the lemma 1.2 (see Appendix). Suppose that (i) of this lemma holds, then $x_1-x_2\to 0$ as $t\to\infty$, we can ensure that $\Omega(\Gamma)=\Omega(\Gamma_0)=\Gamma_0$. If (i) fails, then (ii) holds for all $t\ge\tau(t)$ and all $0\le h\le k$. We consider the case when k is chosen to be the period T of x_2, (ii) shows that the arc of $\mathcal{P}\Gamma_0$ with $t\ge\tau(T)$ could intersect $\mathcal{P}\Gamma_0$ only if Γ intersects Γ_0, then $\Gamma=\Gamma_0$ and $\Omega(\Gamma)=\Omega(\Gamma_0)=\Gamma_0$. Let us therefore delete from Γ the arc on which $t<\tau(k)$ and suppose henceforth that $\mathcal{P}\Gamma$ does not intersect $\mathcal{P}\Gamma_0$. This implies that Γ is not a periodic orbit. This implies that Γ is not a periodic orbit. Also (23) shows that $\mathcal{P}x_1(t)$ is close to $\mathcal{P}\Gamma_0$ only if $x_1(t)$ is close to Γ_0. In Lemma 1.2 let us now take $x_2(t)=x_1(t+\frac{1}{2}T)$, where $x_1(t)$ is the solution which describes Γ. If (1.2.(i)) holds then (1.2.(ii)) must hold. That is

$$|x_1(t)-x_1(t+h+\frac{1}{2}T)| \le |x\mathcal{P}x_1(t)-\mathcal{P}x_1(t+h+\frac{1}{2})|,$$

for all $t\ge\tau(k)$ and $0\le h\le k$. Since Γ is not periodic this shows that

$$\mathcal{P}x_1(t)\neq\mathcal{P}x_1(t+h+\frac{1}{2}T) \qquad (3.10)$$

when $t\ge\tau_1(T)$, $0<h\le t$.

Here we have chosen $k=T$ where T is the period o Γ_0. We know prove that the simple closed curve $\mathcal{P}\Gamma_0$ is approached spirally by $\mathcal{P}\Gamma$ as $t\to\infty$. Choose any point $\tilde{R}$ an Γ on Γ_0 and let $\tilde{T}$ be a transversal with center at $\mathcal{P}\tilde{R}$ perpendicular to the nonzero tangent vector of $\mathcal{P}\Gamma_0$ at $\mathcal{P}\tilde{R}$.

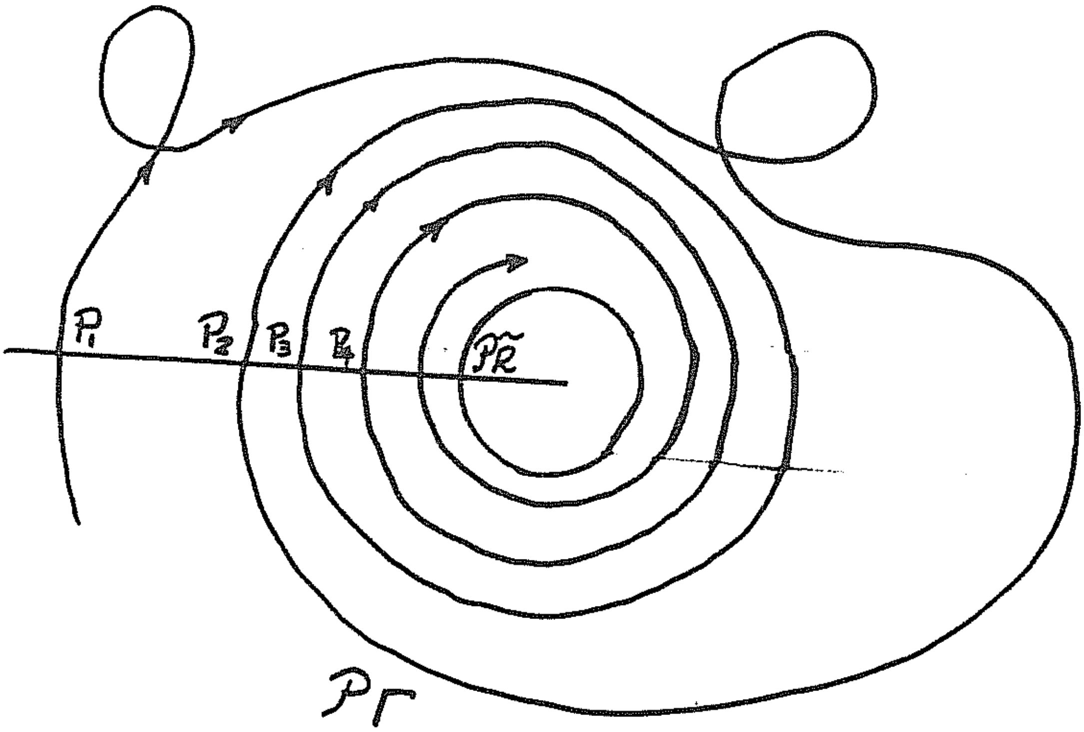

Fig. 3.2

Since $\tilde{R} \in \Omega(\Gamma)$, there exist points of intersection of $\tilde{T}$ and $\mathcal{P}\Gamma$ as near as we please to $\mathcal{P}\tilde{R}$. We can therefore choose intersection points $P_1 = \mathcal{P}x_1(t_1)$, $P_2 = \mathcal{P}x_2(t_2)$ of $\mathcal{P}\Gamma$ with $\tilde{T}$ so that $\tau(T) < t_1 < t_2$ and P_2 lies between P_1 and $\mathcal{P}\tilde{R}$. We can suppose that P_2 is so close to $\mathcal{P}\tilde{R}$ that $\mathcal{P}x_1(t)$remains close to $\mathcal{P}\Gamma_0$ throughout $t_2 \leq t \leq t_2 + 3T$.

Then there are at least two more intersections of $\mathcal{P}\Gamma$ with $\tilde{T}$ in the interval $t_2 \leq t \leq t_2+3T$, namely P_3, P_4 close to the points $\mathcal{P}x_1(t_2+T)$, $\mathcal{P}x_1(t_2+2T)$, respectively, (see Fig 2.2). If P_3 did not lie between P_2 and P_4 then the arc P_2P_3 of $\mathcal{P}\Gamma$would intersect the arc P_3P_4 at a point where $\mathcal{P}x_1(t) = \mathcal{P}x_1(t+d)$ with d approximately equal to T. This would contradict (3.10). Hence, P_3 must lie between P_2 and P_4. If we assume first that P_3 lies between P_2 and $\mathcal{P}\tilde{R}$ then P_4 lies between P_3 and $\mathcal{P}\tilde{R}$. By repeat use of (3.10) it follows that successive arcs P_3P_4, P_4P_5, $P_5,P_6, \ldots$, of $\mathcal{P}\Gamma$ encircle $\mathcal{P}\Gamma_0$ without intersecting each other and meet $\tilde{T}$ in the sequence of the points P_3, P_4, $P_5, \ldots$, which tend monotonically to $\mathcal{P}\tilde{R}$. That is $\mathcal{P}\Gamma$ approaches $\mathcal{P}\Gamma_0$ spirally as $t \to \infty$. We now consider the possibility that P_3 might not lie between P_2 and $\mathcal{P}\tilde{R}$ and show that this leads to a contradiction. In this case, P_2 would lie between P_3 and $\mathcal{P}\tilde{R}$. Then (3.10) would show that if $\mathcal{P}\Gamma$is followed backwards from P_2, successive arcs of it encircle $\mathcal{P}\Gamma_0$ and cut $\tilde{T}$ at points nearer to $\mathcal{P}\tilde{R}$ on each occasion until the point P_1 on $\mathcal{P}\Gamma$ is reached. This implies that P_1 lies between $\mathcal{P}\tilde{R}$ and P_2, contradicting the way in which P_1, P_2 where chosen. This contradiction proves that P_3 must lie between P_2 and $\mathcal{P}\tilde{R}$ and therefore $\mathcal{P}\Gamma_0$is approached spirally by $\mathcal{P}\Gamma$ as $t \to \infty$. Hence, $\mathcal{P}\Gamma_0 = \mathcal{P}\Omega(\Gamma)$ and (32) gives $\Gamma_0 = \Omega(\Gamma)$. This establishes Theorem 3.4.

3.2. Orbital Stability.

Up to now we discussed about the existence of periodic orbits for autonomous systems but this results do not ensure that orbitally stable closed orbit exist. This is a big disadvantage of this theory if we are interested to apply in practice, for example in a physical or biological systems. Erle [9] made a deeper study of Poincaré-Bendixson Theorem (n=2) in the case when some closed trajectories are non isolated. Smith in [25, 26] showed how this theory can be used to prove the existence of a stable trajectory of autonomous ordinary differential equations in $\mathbb{R}^n$. Let S an open subset of $\mathbb{R}^n$ considered at the beginning of this chapter. Let $x(t)$ a non-constant solution of (3.1) and Γ_0 its correspondent orbit. If the system (3.1) verify all the conditions of the generalization of Poincaré-Bendixson Theorem, we can ensure that the existence of one orbit which coincides with $\Omega(\Gamma)$.

We denote

$$N(\Gamma_0, \delta) = \left\{ x(t) \in R^n : \quad d(x(t), \Gamma) < \delta \right\}$$

a neighborhood of Γ_0.

We say that Γ_0 is orbitally stable if for all $\varepsilon >$ 0there exist $\delta(\varepsilon) > 0$ such that

$$x(t) \in N(\delta(\varepsilon), \Gamma_0), \quad x(t) \in S \cap N(\Gamma, \varepsilon) \quad \text{throughout} \quad t_0 \leq t < \infty.$$

Moreover if

$$\lim_{t \to \infty} d(x(t), \Gamma_0) = 0$$

then Γ_0 is asymptotic orbitally stable. Note that if we consider $x(t)$ an amenable solution $x(t)$ of (3.1) in the previous definition of orbital stability we get the definition for amenable stability. We can observe that in most of the cases the is easier to verify amenable stability than orbital stability but this does not means that amenable stability is weaker than orbital stability, in fact, by Theorem 2.7

In addition Γ_0 is also an isolated periodic orbit. The converse of this proposition is the following

THEOREM 3.5. *Suppose that (3.1) verify* $[\mathbf{H}_2]$, $[\mathbf{H}_3]$. *If (3.1) has a closed trajectory* $\Gamma_0 \subset S$ *which is isolated and orbitally stable, then* Γ_0*is asymptotically stable.*

Proof. See Smith [25].

Now we will consider the equations which satisfies the following condition

$[\mathbf{H}_6]$ *There exist an open bounded set* $C \subset S \subset \mathbb{R}^n$ *with closure* $\overline{C} \subset S$ *such that its boundary* ∂C *is crossed inward by any solution of (3.1) with meet it.*

This restriction means that if any solution with $x(t_0) \in \partial C$ for some t_0 then $x(t) \in \overline{C}$ for all $t > t_0$ and there exist $t_1 > t_0$ such that $x(t) \in C$ for all $t > t_1$, i.e. there is not critical points of (3.1)on ∂C.

In the particular case when $f(x)$ is an analytic function in S, that is, if f may be written as a multiple power series in the coordinates of $x - x_0$, there exist precisely results about the stability.

THEOREM 3.6. *Suppose that (3.1) verify* $[\mathbf{H}_2]$, $[\mathbf{H}_3]$ *in* C. *If (3.1) does not have critical points then each semiorbit in* C *converges to a closed trajectory as* $t \to \infty$ *and* C *contains one closed trajectory which is orbitally stable. Moreover in the case that* $f(x)$ *is an analytic function in* S, C *contains only a finite number or closed trajectories and at least one of them is asymptotically orbitally stable.*

We can note that there exist cases for the set C such that it contains at least one critical point and these cases are not possible to be treated by this Theorem, for example, when C is an spherical ball which satisfies $[\mathbf{H}_6]$, the Brouwer fixed point Theorem shows that C has at least one critical point. For this reason we have the more general result

THEOREM 3.7 Suppose that (3.1) verify $[\mathbf{H}_2]$, $[\mathbf{H}_3]$, $[\mathbf{H}_6]$ and has only one critical point

k in C. *If $f(x)$ is a continuously differentiable function in some neighborhood of k such that for all eigenvalues of it $n \times n$-jacobian matrix at k verify*

$$re\, z_1 \geq re\, z_2 > 0 > re\, z_3 \geq, ..., \geq re\, z_n$$

then each semiorbit in C converges to k or to a closed trajectory as $t \to \infty$ and C contains at least one closed trajectory which is orbitally stable. Moreover if $f(x)$ is analytic in S then C contains a finite number of closed trajectories and at least one of these is asymptotically orbitally stable.

Proof. See Appendix, Lemma 1.7 establish proof of theorems 3.6 and 3.7.

Chapter 4

APPLICATIONS

In this chapter we will apply the results of the previous chapters. More precisely, in section 4.1 we prove the existence of positives solution of a first order Lotka-Volterra type and for this end we will apply the result from section 2.1. In section 4.2 we do the same than the previous section for autonomous systems, with the evaluation in which is natural to choose the set W as a domain with holes , and for it we use theorems of the section 3.1 with the help of Wazewski's principle, Conley [4,5], Srzednicky [27, 28].

4.1. Non-Autonomous Case.

The following problem was treated by Gopalsamy [10] and he showed, by using a Liapunov function $V(x)$ and Brouwer fixed point Theorem, the existence of periodic solutions. In our case we will treat it with generalization of Massera's Convergence Theorem and in this particular case not only we can give some sufficient conditions but uniqueness for the $T-$periodic solution.

Let us consider the Lotka-Volterra equations of competition species type and consider the following n-dimensional system

$$\frac{dx_i}{dt} = x_i(t)\left[\alpha_i(t) - \sum_{j=1}^{n} \beta_{ij}(t)x_j(t)\right] \tag{4.1}$$

for $t > 0$, $n \in \mathbb{N}$ and $i = 1, ..., n$. Suppose that a_i, $b_{ij} : \mathbb{R} \longrightarrow \mathbb{R}$ are continuous, $T-$periodic (for some $T > 0$) functions, for all $i, j = 1, ..., n$.

We denote

$$\mathbb{R}^n_+ = \left\{x = (x_1, ..., x_n) \in \mathbb{R}^n \;\; such\ that \;\; x_i \geq 0, i = 1, ..., n\right\},$$

$$[g_i]_U = \max_{t\in[0,T]} g_i(t) = \sup_{t\in\mathbb{R}} g_i(t),$$

$$[g_i]_L = \min_{t\in[0,T]} g_i(t) = \inf_{t\in\mathbb{R}} g_i(t),$$

$$[g_i]^+ = max\{g_i, 0\},$$

$$[g_i]^- = min\{g_i, 0\}.$$

To analyse this particular system we need to define some compact set $S_0 \subset S$ in $\mathbb{R}^n$ such that $[\mathbf{H_1}]$, $[\mathbf{H_2}]$ and $[\mathbf{H_3}]$ hold. We will denote by

$$f_i(t,x) = x_i(t)\big[\alpha_i(t) - \sum_{j=1}^{n} \beta_{ij}(t)x_j(t)\big] \tag{4.2}$$

for all $i = 1, ..., n$ for all t.

Any values of the population densities x_i, with $i = 1, ..., n$ can be represented as points on the space $\mathbb{R}^n$. Since the densities x_i cannot be negative because we are interested that the species compete, then we consider

$$S = \{x = (x_1, ..., x_n) \in \mathbb{R}^n : \quad x_i(t) \ > \ 0\}$$

As a first step, we build a positively invariant compact set $S_0 \subset S$. The form of S_0 will be the following

$$S_0 \ = \ [\varepsilon, R_1] \times ... \times [\varepsilon, R_n]$$

with $\varepsilon > 0$ sufficiently small.

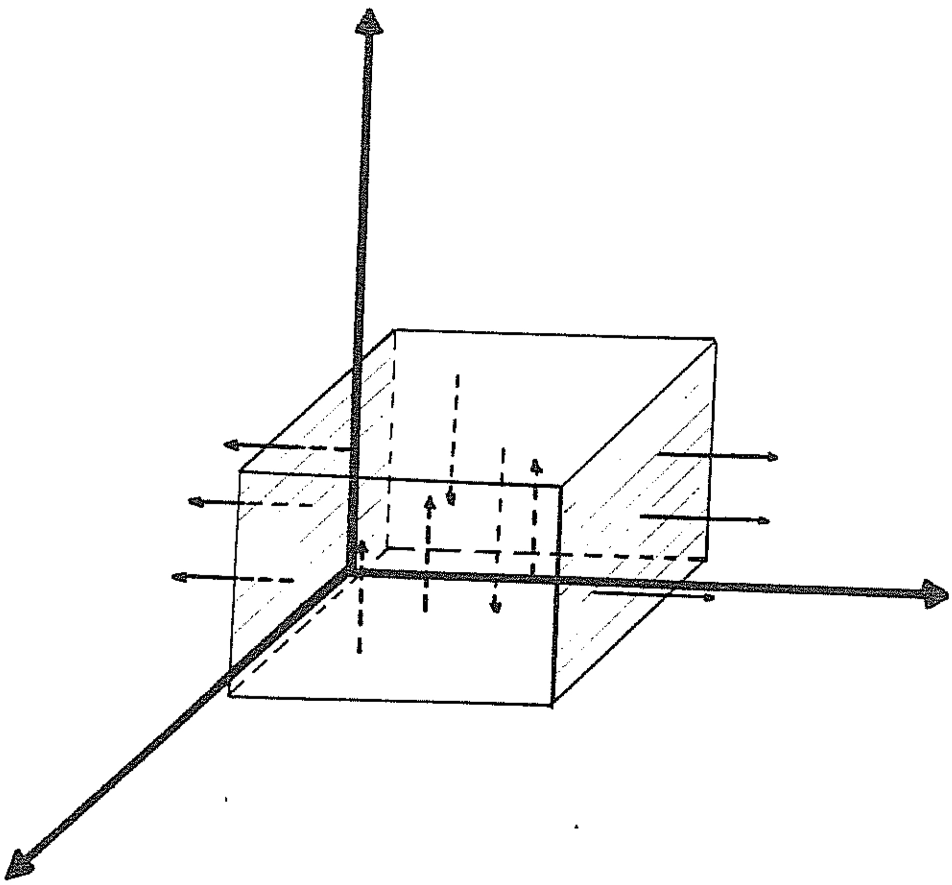

Fig 4.1

To build this compact set S_0 we will get some condition from the coefficients α_i, β_{ij} of (4.2) in such way the flow $f_i(t,x)$ can verify the following assumptions,

(1) $f_i(t,x) > 0$ such that $x_i = \varepsilon$ with $\varepsilon > 0$ a small number for some $i = 1, ..., n$ and $\varepsilon \le x_i \le R_j$, with $R_j > 0$ some real number, for $j = 1, ..., n$ and $j \ne i$, for all t.

To get

$$\alpha_i(t) - \varepsilon\beta_{ii}(t) - \Big[\sum_{\substack{j=1\\ j\neq i}}^{n} \beta_{ij}(t)\ x_j\Big] > 0$$

is sufficient to consider

$$[\alpha_i]_L - \varepsilon[\beta_{ii}]_U^+ - \Big[\sum_{\substack{j=1\\ j\neq i}}^{n} [\beta_{ij}]_U^+\ x_j\Big] > 0$$

if $\varepsilon \to 0$ we obtain

$$[\alpha_i]_L > \Big[\sum_{\substack{j=1\\ j\neq i}}^{n} [\beta_{ij}]_U^+ \, R_j\Big] \tag{4.3}$$

in analogous way we can do for the another case,

(2) $f_i(t,x) < 0$ such that $x_i = R_i$ with $R_i > 0$ a some real number for some $i = 1,...,n$ and $\varepsilon \leq x_i \leq R_j$, with $\varepsilon > 0$ (small) real number, for $j = 1,...,n$ and $j \neq i$, for all t.

To get

$$\alpha_i(t) - R_i\beta_{ii}(t) - \Big[\sum_{\substack{j=1\\ j\neq i}}^{n} \beta_{ij}(t)\; x_j\Big] < 0$$

is sufficient to consider

$$[\alpha_i]_U - R_i[\beta_{ii}]_L^- - \Big[\sum_{\substack{j=1\\ j\neq i}}^{n} [\beta_{ij}]_L^+ \; x_j\Big] < 0$$

If we suppose

$$[\alpha_i]_U - R_i[\beta_{ii}]_L^- < \Big[\sum_{\substack{j=1\\ j\neq i}}^{n} [\beta_{ij}]_L^- \; R_j\Big] \tag{4.4}$$

we get the conditions for (2)

Under restrictions (4.3) and (4.4) we built the compact set

$$S_0 = \Big\{ x = (x_1,...,x_n) \in \mathbb{R}_+^n \quad \textit{such that} \quad \varepsilon \leq x_i(t) \leq R_i, \quad i = 1,...,n \Big\}$$

for all t. Furthermore we can observe that this set is invariant with respect to (4.1), i.e. if any solution $x(t)$ of (4.1) which has initial condition $x(0)$ in S_0, then $x(t) \in S_0$ for all t. From the previous analysis we have obtained the following

PROPOSITION 4.1. *Let consider the system (4.1). Suppose that $\alpha_i(t) > 0$ and $\beta_{ii} > 0$ for all t, for $i, j = 1, ..., n$. If there exist $R_1, ..., R_n$ real constants such that the inequalities (4.3) and (4.4) hold, then there exist an $\varepsilon > 0$ such that the $n-$dimensional rectangle S_0 is an invariant set.*

From the Generalization of Massera's Theorem we will check that the following hypothesis holds (see $[\mathbf{H_3}]$ with $p = 1$)

There exits constants $\lambda \geq 0$, $\varepsilon > 0$ such that for all $t \in \mathbb{R}$

$$\lambda \sum_{i=1}^{n} a_i | < x_1 - x_2, v_i > | + \sum_{i=1}^{n} a_i K(< x_1 - x_2, v_i >) | < f(t, x_1) - f(t, x_2), v_i > \leq$$
$$\leq -\varepsilon \sum_{i=1}^{n} | < x_1 - x_2, v_i > |$$

for $x(t)$, $y(t)$ solution of (4.1) in S. With $a_j = 1$ if $j = 1, ..., n$ and K like in Chapter 1.

Let consider

$$X_i = log\, x_i, \qquad \text{and} \qquad Y_i = log\, y_i \tag{4.5}$$

with x_i, y_i solutions of (4.1) defined in S_0 (a sufficient condition is to require $x(0)$, $y(0) \in S_0$ for any solutions of (4.1)).

From (4.1) and (4.5) that

$$\frac{d}{dt}\Big(X_i - Y_i\Big) = -\beta_{ii}(t)\left[e^{X_i(t)} - e^{Y_i(t)}\right] - \sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_{ij}(t)\left[e^{X_j(t)} - e^{Y_j(t)}\right] \tag{4.6}$$

for $i = 1, ..., n$. We know that $\mathrm{D}^+|z(t)| = K\big(z(t)\big)\frac{dz}{dt}$. For the scalar function $V(x) = \sum_{i=1}^{n} |x_i(t)|$ (with $a_i = 1$) and for x, y solutions of (4.1) we can obtain

$$D^+\Big[e^{\lambda t}V(x-y)\Big] = \lambda e^{\lambda t}V(x-y) + e^{\lambda t}\mathrm{D}^+V(x-y) =$$
$$= \lambda e^{\lambda t}\sum_{i=1}^{n}|X_i - Y_i| +$$
$$+ e^{\lambda t} \sum_{i=1}^{n} K(X_i - Y_i) \sum_{i=1}^{n} \beta_{ij}(e^{X_i} - e^{Y_i}) =$$

$$\leq \lambda e^{\lambda t}\sum_{i=1}^{n}|X_i - Y_i| -$$

$$- e^{\lambda t} \quad \sum_{i=1}^{n}\left[\beta_{ii}(t)(e^{X_i} - e^{Y_i}) - e^{\lambda t}\sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}(t)|e^{X_i} - e^{Y_i}|\right] =$$

$$= \lambda e^{\lambda t}\sum_{i=1}^{n}|X_i - Y_i| -$$

$$- e^{\lambda t} \quad \sum_{i=1}^{n}\left[\beta_{ii}(t) - |e^{X_i} - e^{Y_i}|e^{\lambda t}\sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}(t)|e^{X_i} - e^{Y_i}|\right] =$$

$$= \lambda e^{\lambda t}\sum_{i=1}^{n}|X_i - Y_i| - \sum_{i=1}^{n}\left[\beta_{ii}(t) - \sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}(t)\right]|e^{X_i} - e^{Y_i}|$$

Then if we consider $\lambda = 0$ and

$$0 < \varepsilon \leq \beta_{ii}(t) - \sum_{\substack{j=1\\ j\neq i}}^{n}\beta_{ij}(t) \qquad (4.7),$$

we can conclude that

$$\frac{d}{dt}V(x-y) \leq -\varepsilon\sum_{i=1}^{n}|e^{X_i} - e^{Y_i}| = -\varepsilon\sum_{i=1}^{n}|x_i(t) - y_i(t)|$$

Now we have all the sufficient conditions to apply the generalization of Massera's Theorem and we can ensure that if (4.1) has a solution $y(t) \in S_0$ such that the initial condition $y(0) \in S_0$ not only there exist $u(t)$ a $T-$periodic orbit in S, such that $y(t) - u(t) \longrightarrow 0$ for $t \to \infty$ more over, this $u(t)$ is unique in S_0. Now if we consider the case studied by Gopalsamy [10] we can observe that it is a particular case of our analysis, in fact, if we replace the coefficients of (4.1) by the correspondents coefficients $\beta_{ij} > 0$, $bij^{+} = bij$, $bij^{-} = 0$ the relations (4.3) and (4.4) become

$$\sum_{\substack{j=1\\ j\neq i}}^{n}\left(\beta_{ij}(t)\right)_U^{+} R_j < \left(\alpha_i(t)\right)_L$$

and

$$\frac{\alpha_i(t)_U}{\beta_{ii}(t)} < R_i$$

That is, is possible to obtain the invariant compact set S_0 if we consider the following restriction for the coefficients of (4.1)

$$\sum_{\substack{j=1 \\ j\neq i}}^{n} (\beta_{ij}(t))_U^+ \frac{\alpha_i(t)_U}{\beta_{ii}(t)} < (\alpha_i(t))_L.$$

With the same Theorem 3.3, we can conclude that the periodic solution $u(t)$ of (4.1) is globally asymptotically stable in the sense gave by Gopalsamy in [10], that is, for every other solution $y(t)$ of (4.1) such that the initial condition $y_i(0) \in S_0$ and is defined by all $t > 0$,

$$\lim_{t\to\infty} |u_i(t) - y_i(t)| = 0$$

for all $i = 1, ..., n$.

4.2. Autonomous Case.

We pass now to the investigation of a class of autonomous systems of Lotka-Volterra type, we consider the equation

$$\frac{dx_i}{dt} = f(x) \tag{4.9}$$

where $f_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a locally lipschitzian function. Our purpose is to outline two different possible applications of the generalized Poincaré-Bendixson Theorem.

First of all, we observe that, in the situation described for the preceding example of non-autonomous equation, if we can find a $n-$th dimensional rectangle

$$S_0 = [\varepsilon, R_1] \times ... \times [\varepsilon, R_n],$$

which is positively invariant for the flow induced by (4.9), then there is at least one equilibrium point $x^* \in S_0$.

Indeed, this is a standard consequence of Brouwer fixed point Theorem and of a basic lemma from the theory of dynamical systems (fix for any $h \in \mathbb{N}$, the period $T_h = \frac{1}{k}$ and observe that, by Brouwer Theorem, there is $x_h^* \in S_0$, an initial point of a T_h-periodic solution of (4.9). Then passing to a subsequences $x_h^* \longrightarrow x^*$ with $f(x^*) = 0$). Such x^* lies in the interior of S of $\mathbb{R}_+^n$. Now the problem arises whether there are non trivial periodic solutions for equation (4.9) which are contained in S.

To this end, the generalized Massera's Theorem still can be used in order to get a negative answer.

For instance if we are in the situation of the case (4.1) with the autonomous equation

$$\frac{dx_i}{dt} = x_i(t)\left[\alpha_i - \sum_{j=1}^{n} \beta_{ij} x_j(t)\right] \tag{4.10}$$

with

$$\beta_{jj} > 0, \qquad \beta_{jj} > \sum_{\substack{j=1 \\ j \neq i}}^{n} \beta_{ij} \quad \text{and}$$

$$\alpha_i > \sum_{j=1}^{n} \beta_{ij} \frac{\alpha_i}{\beta_{jj}},$$

then we know that for each $T > 0$, there is a $T-$periodic solution $x_T(t)$ such that

$$\lim_{t \to \infty} |x_T(t) - y(t)| = 0$$

for every $y(t)$ solution of (4.10) such that $y(0) \in S_0$ (see section 4.1).

In this cases necessarily, we get that $x_T(t) = x_*$ =constant with respect to t and therefore x^* is the unique periodic solution of (4.10) which is contained in S_0.

With this remark, we get immediately that also the result by Gopalsamy for autonomous equations ([7, Theorem 4.1]) is a particular case of our analysis.

Now, we try to propose some other situations for which the existence of nontrivial periodic solutions is guaranteed. In order to obtain such results we can choose two different ways.

A first possibility consist into a modification of the structure of the equations so that the flow-invariance property of the rectangle S_0 is preserved, but the equilibrium point

x^* is no more asymptotically stable and, precisely, two eigenvalues of the Jacobian matrix $f\prime(x^*)$ have positive real part, while all the other eigenvalues have negative real part. This is the case considered by R. A. Smith in [22, 24]. Roughly speaking, in this situation, we find a two-dimensional unstable manifold U passing through x^* and we can try to use the projection technique in order to get a limit cycle of equation (4.9) whose projection on U is a closed curve. Of course, because of the preceding remark, we cannot hope to get such result for equations of the form (4.10) since in this case the conditions which are needed to ensure the flow-invariance property are also sufficient to guarantee the asymptotic stability of the internal equilibrium point. A single example for this approach is given in the next example 4.1.

A second possibility consists into a modification of the conditions for the flow of (4.1) at the boundary of S_0. In order to explain better the situation, we assume, for sake of simplicity, that the flow enters all the boundary of S_0 except that for two opposite faces. This situation can be easily achieved if we assume that there is $i \in \{1, ..., n\}$ such that

$$f_i(x_1, ..., x_{i-1}, \varepsilon, x_{i+1}, ..., x_n) < 0 < f_i(x_1, ..., x_{i-1}, R_i, x_{i+1}, ..., x_n)$$

for all $\varepsilon \leq x_i \leq R_j$, for $j \neq i$,

$$f_j(x_1, ..., x_{j-1}, \varepsilon, x_{j+1}, ..., x_n) > 0 > f_j(x_1, ..., x_{j-1}, R_j, x_{j+1}, ..., x_n)$$

for all $\varepsilon \leq x_k \leq R_k$, $k \neq j$.

Under the above conditions, the flow enters in S_0 at the faces $x_j = \varepsilon, \;\; x_j = R_j$ for $j \neq i$, while it is directed outward at the faces $x_i = \varepsilon, \;\; x_i = R_i$.

Next, assume that the equilibrium point $x^* \in S_0$ is repeller, that is all the eigenvalues of $f\prime(x^*)$ have positive real part. In this cases it is a standard from Conley index theory that it is possible to find a sufficiently small neighborhood $U(x^*)$ of x^* homeomorphic to a ball such that all the points of its boundary are egress points for the considered flow.

Finally, we define

$$W = S_0 \setminus U(x^*).$$

By the above required properties, it is possible to check that W is a Wazewski set (see R. Srzednicki [27])and its set of egress points is given by $W^- = F_1 \cup F_2$, where F_1 and F_2 are the opposite faces $x_i = \varepsilon, \;\; x_i = R_i$.

Such situation may be represented though the following figure

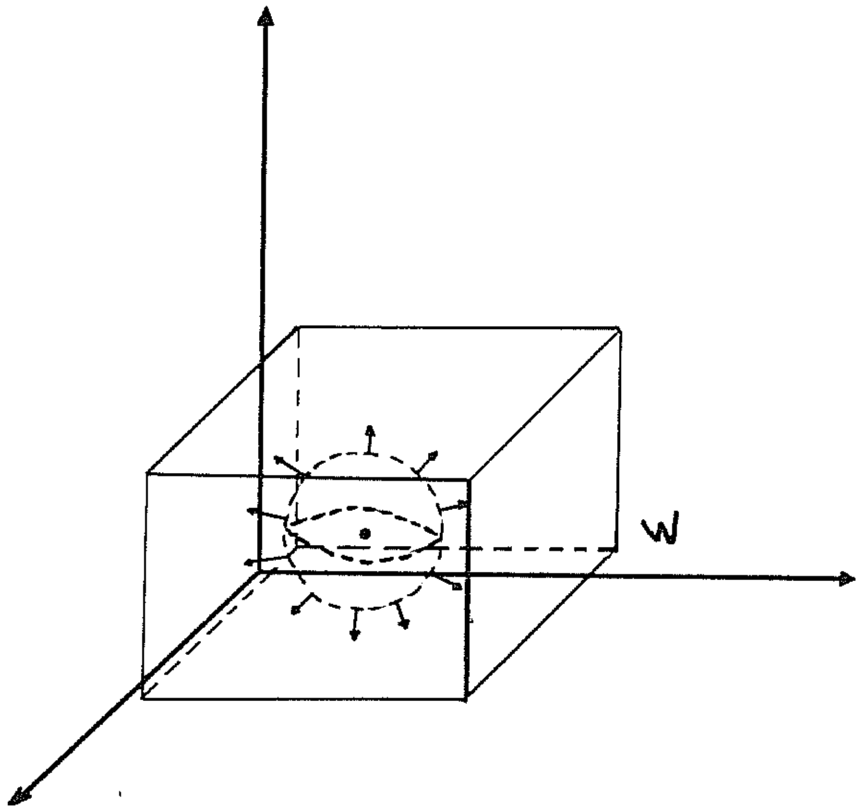

Fig 4.2

Assume, besides the above conditions, that the phase space $\mathbb{R}^n$ has odd dimension. In this case, we have that W is homeomorphic to the n−dimensional closed ball with one hole which, in turns has the same Euler Characteristic of the $(n-1)$−dimensional sphere. Since $n-1$ is even, then we get $\chi(W) = 2$.

On the other hand

$$\chi(W^-) = (F_1) + (F_2) - (F_1 \cap F_2) = 1 + 1 - 2 = 0$$

so that by R. Srzednicki formula we compute

$$d_B(f, int\, W, 0) = \chi(W) - \chi(W^-) = 2 - 2 = 0.$$

Then this fact confirm which we told before, that there is not critical points in $int\, W$, see Srzednicki [28]. Furthermore, W^- is not a strong deformation retract of W because W^- is not a connected set, while W is connected, then from Wazewski's criterion, see Conley [4,5] or Wazewski [29], there exist solutions which stay included in W, for all $t > 0$.

The fact that $d_B(f, int\,W, 0) = 0$, do not prevent the possibility of other equilibrium points in $int\,W$, but this condition at least guarantee that a function f without zeroes in $int\,W$, but with the prescribed behavior on the boundary can be found.

At this point, we are in the position to say that, if n is odd, then function f exists such that the flow induced by equation (4.9) is like the one depicted in *Fig.* 4.2 and $f(x) \neq 0$ for all $x \in W$. Then we can apply Wazewski retract Theorem which ensures the existence of at least one solution $x(t)$ of (4.9) such that $x(t) \in W$, for all $t \geq 0$. Then the w−limit of $x(t)$ is a subset of W which does not include critical points. Hence, if f satisfies suitable conditions for the projections technique we can produce the existence of a non trivial periodic solution contained in W.

Such second possibility, seems to be more suitable for producing examples than to prove the existence theorems. Nevertheless, our purpose was find that of outlining possible situations which were not considered by R. A. Smith, in order to avoid the presence of critical points in the limit sets. It is also clear that we easily can modify the above discussion varying the number of the exit faces and assuming, at the same time other conditions on the eigenvalues.

We end this section, showing a simple example in which the first possibility is exploited.

EXAMPLE 4.1.

Consider the following system in $\mathbb{R}^3$,

$$\frac{dx}{dt} = x[\ \ k + a\phi(x) - y + \psi(z)]$$
$$\frac{dy}{dt} = y[-k + b\phi(y) + x - \psi(z)]$$
$$\frac{dx}{dt} = z(k - z)\delta(x, y, z)$$

with ϕ, $\psi : \mathbb{R} \longrightarrow \mathbb{R}$, $\delta : \mathbb{R}^3 \longrightarrow \mathbb{R}$ are continuous and differentiable functions such that $\phi(k) = \psi(k) = 0$ for $k > 0$ and a, $b > 0$ real constants such that $ab > 1$.

We can observe that $K = (k, k, k)$ is the critical point of the previous system and the Jacobian matrix evaluated in K has two eigenvalues with positive real part and the third one with negative real part. If we restrict to the manifold which passing through K, more precisely, for $z = k$, and by using the projection technique we can find the limit cycle of the system, which lyes in the three-dimensional rectangle given by the assumptions which we will see later, whose projection is on this manifold.

Suppose the following assumption over the coefficients of the previous system.
If $x = \varepsilon$ for all $\varepsilon \leq y \leq R_2$ and $\varepsilon \leq z \leq R_3$

$$k + a\phi(\varepsilon) - R_2 + [\psi]_L > 0.$$

If $x = R_1$ for all $\varepsilon \leq y \leq R_2$ and $\varepsilon \leq z \leq R_3$

$$k + a\phi(R_2) + [\psi]_U < 0.$$

If $y = \varepsilon$ for all $\varepsilon \leq x \leq R_1$ and $\varepsilon \leq z \leq R_3$

$$-k + b\phi(\varepsilon) + [\psi]_L > 0.$$

If $y = R_2$ for all $\varepsilon \leq x \leq R_1$ and $\varepsilon \leq z \leq R_3$

$$-k + b\phi(R_2) + R_2 + [\psi]_U < 0.$$

Then we get the positive invariance of the set S_0.

APPENDIX

Appendix A

In this section we consider some lemmas which are needed in the proof of the uniqueness of the solution from (3.1) (Lemmas 1.1 and 1.2) and the sufficient conditions for the orbital stability (Lemmas 1.3 - 1.7), i.e. in the proof of the theorems 3.6 and 3.7 is needed the lemma 1.7 in such way needs the previous lemmas.

LEMMA 1.1. *If $f(x)$ satisfy the local the local Lipschitz condition in x, then for each compact S_0 of S there exist a constant $L(S_0) > 0$ such that*

$$\sum_{i=1}^{n} | < f(x) - f(y); v_i > |^p \leq L(S_0) \sum_{i=1}^{n} | < x(t) - y(t); v_i > |^p$$

for all $t \in \mathbb{R}^n$ and $x(t)$, $y(t) \in S_0$.

LEMMA 1.2. *Let us consider $x_1(t)$, $x_2(t)$ solutions of (2.1) in $[t_0, +\infty)$. If $x_1(t)$, $x_2(t)$ are in S for all $t \geq t_0$ then at least one of the following statements i), ii) is true:*

i) there exist constants $h \geq 0$, $c > 0$ such that

$$\|x_1(t) - x_2(t+h)\|_p \leq ce^{-\lambda t} \quad for\ all \quad t \geq t_0; \tag{1.1}$$

ii) for each $k > 0$ there exist $T(k) \geq 0$ such that

$$\|x_(t) - x_2(t+h)\|_p \leq \|\mathcal{P}x_1(t) - \mathcal{P}x_2(t+h)\|_p, \tag{1.2}$$

for all $t \geq T(k)$ and all h with $0 \leq h \leq k$.

To prove this lemma we consider S_0, the closure of the orbit Γ, $\mathcal{P}$ linear map replaced by Π linear map and $U(x) = W(x) - V(x)$ as in the proof of the Theorem 3.4 then by R. A. Smith in [22], we obtain such proof.

We need to prove the theorems related with the orbital stability for autonomous systems in the chapter 3.

Lemma 1.3. *If Γ_0 is both internally and externally stable then Γ_0 is amenably stable.*

Lemma 1.4. *Each chain Q and B has an upper bound Γ_u and a lower bound Γ_l. Furthermore we can suppose that either $\Gamma_l \in Q$ is an externally stable closed trajectory.*

Lemma 1.5. *B contains an externally stable closed trajectory.*

Lemma 1.6. *B contains an orbitally stable closed trajectory.*

For the proof of all the previous lemmas see R. A. Smith [25].

Lemma 1.7. *If the critical point $k \in \Omega(\Gamma)$ for some semi-orbit $\Gamma \subset C$ then $\Omega(\Gamma)$ contains no other points.*

Proof. See R. A. Smith in [21, 25] . The proofs of this lemmas are omitted because we can get them in Smith. Essentially Lemma 1.6 permit us to prove the theorems about the orbital stability, that is, Theorem 3.6 and 3.7.

Appendix B

Let $S_0 \subset \mathbb{R}^n$ such that is a compact ENR. From the definition of the Lefschetz number see Dold [8], Conley [4].

$$\lambda(f) = \sum_{p=1}^{\infty}(-1)^p tr(f_p)$$

with $f_p : H_p(\mathbb{R}^n) \longrightarrow H_p(\mathbb{R}^n)$, a continuous function defined over $H_p(\mathbb{R}^n)$ the $p-$dimensional homology group of $\mathbb{R}^n$ with coefficients which are rational numbers and the graded vector space $H_p(\mathbb{R}^n) = \{H_p(\mathbb{R}^n)\}_{p=0}^{\infty}$ of finite type.

The Euler Characteristic of S_0, $\chi(S_0)$, is the Lefschetz number of the identity map Id_{S_0} over S_0. We note that

$$\chi(S_0) = \sum_{p=1}^{\infty}(-1)^p dim\, H_p(S_0).$$

By definition of singular homology and from homotopy axioms, follow that the Euler Characteristic may be introduced by the following properties for compact ENR sets:

(1) Additivity:

For A, B compact ENR sets of $\mathbb{R}^n$

$$\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B).$$

(2) Normalization:

$$\chi(A) = 1 \quad \text{if A is a convex set,}$$
$$\chi(\emptyset) = 0.$$

(3) Homotopy Equivalence:

If A and B compact sets of $\mathbb{R}^n$ have the same homotopy type, then

$$\chi(A) = \chi(B).$$

Note that Euler characteristic does not depend from the connected decomposition. For example, let us consider the set A the $n-$dimensional ball with a "hole" and B the $n-$dimensional ball inside A (hole). The computation of Euler Characteristic for A follows from (1) and (2) that

$$\begin{aligned} & \chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B) \\ \text{and} \quad & \chi(A \cup B) = \chi(B) = 1 \\ \text{then} \quad & \chi(A \cap B) = \chi(A). \end{aligned}$$

Since

$$\chi(A \cap B) = \chi(A) = \chi(ball) = \begin{cases} 0, & \text{if n is even} \\ 2, & \text{if n is odd.} \end{cases}$$

The Euler characteristic can be used in order to apply topological methods in the research of periodic solutions to differential systems. Specifically it is used like the topological degree, in fact if $\chi(S_0) \neq 0$ in some cases implies the existence of periodic solutions of such differential system. See Conley C. [4,5]. For example in R. Srzednicki [28] consider for a system

$$\begin{aligned} \frac{dx}{dt} &= f(t, x) \\ x_0 &= x(t_0) \end{aligned}$$

the following result

THEOREM 2.1. *Assume that S is an open subset of $\mathbb{R}^n$ and $f : \mathbb{R}^n \times S \longrightarrow \mathbb{R}^n$ is continuous, differentiable and T-periodic in t. Let W a block of the type $(p-q)$. If W and W^- are compact ANR (absolute neighborhood retract) and $\chi(W) \neq \chi(W^-)$ then there exist a point $x_0 \in int\, W$ such that the solution $x(t)$ of the previous system is T−periodic in t. Moreover, $x(t) \in W$ for each $t \in \mathbb{R}^n$.*

For the opportune definitions like ANR, block of the type $(p-q)$, see C. Conley [4,5].

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