



**ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES**

"MAGISTER PHILOSOPHIAE" THESIS

Alessandro Fonda

**SOME EXISTENCE RESULTS FOR
NON - CONVEX VALUED
DIFFERENTIAL INCLUSIONS**

Supervisor : Professor Arrigo Cellina

Trieste, October 1986

TRIESTE

Scuola Internazionale Superiore di Studi Avanzati
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1. INTRODUCTION.

We will study the problem of the existence of solutions to the differential inclusion

$$x'(t) \in F(t, x(t)) \quad (1)$$

where $x' = dx/dt$ and F is a multivalued function defined on a subset of \mathbb{R}^{n+1} , taking as values closed subsets of \mathbb{R}^n .

Such type of equations arise rather naturally in any kind of experimental processes in which the parameters are known only within a certain degree of approximation ; in Control Theory one often faces equations with a parameter which are particular cases of equation (1). One can also represent an implicit differential equation of the form

$$f(t, x(t), x'(t)) = 0$$

as a differential inclusion of the type (1), with

$$F(t, x) = \{ v : f(t, x, v) = 0 \}.$$

Differential inclusions can also be employed in the study of differential equations with non-continuous right hand side (see e.g. [3]).

In order to prove the existence of solutions to the differential inclusion (1) we will need some assumptions on F

which will be essentially of two different kinds : continuity (in some sense to be specified) , and regularity of the values (like e.g. convexity and compactness) .

Early results were obtained essentially by Marchaud [27] and Zaremba [36] in the thirties by assuming the continuity of F and the convexity of its values . Later, equations of the type (1) were studied by many authors including Filippov [12] and Wazewski [35] . However most results were obtained under the assumption that F had convex values . Filippov [13] was the first to avoid this assumption and he finally proved the existence of a solution to (1) by asking the continuity of F (in the sense of the Hausdorff topology) and only compactness for the values of F .

In the following years, the pioneering work of Filippov has been improved in various directions. Kaczynski and Olech [21] and Antosiewicz and Cellina [1] proved the existence of solutions by assuming Caratheodory conditions on F : the technique used by the formers was a refinement of Filippov's, while the latters used a completely new fixed point approach based on a continuous selection argument . This new approach gave the impulse to the study of the existence of continuous selections for multivalued maps taking as values decomposable subsets of L^1 . Fryszkowski [14] proved the existence of a continuous selection from a lower semicontinuous map with decomposable values, while Cellina, Colombo and Fonda [9] obtained continuous approximate selections for an upper semicontinuous map . These results have been later improved by

Bressan and Colombo [5] by assuming the domain of the multifunction to be only paracompact .

A generalization to the previous existence results for (1) was given by Olech [29], while Lojasiewicz [26] and Himmelberg and Van Vleck [20] weakened the compactness condition on the values of F .

In 1980 two papers by Bressan [4] and Lojasiewicz [25] appeared , in which the lower semicontinuous case was analyzed : Bressan's approach followed that of Antosiewicz - Cellina 's while Lojasiewicz's followed the one of Kaczynski - Olech .

In this paper we shall give a survey on the most remarkable results obtained in the study of the existence of solutions to equations of the type (1) , concentrating on the case in which F does not have convex values . We shall try to emphasize the different techniques and the main difficulties involved in each of them . Some technical proofs will be omitted, but the main ideas will be sketched.

Section 2 contains the main notations and basic definitions which will be used throughout the paper; some interrelations between the various definitions of continuity for a multivalued map are also given.

Section 3 is a brief survey on some classical results which can be obtained when the multivalued map F is convex valued. We also present here the Michael Selection Theorem for lower semicontinuous multivalued maps and the Cellina Approximate Selection Theorem for upper semicontinuous

maps.

Section 4 is the central part of this paper ; we analyze here the case when the multivalued map F involved in the differential inclusion (1) has non-convex values . At first we prove Filippov's theorem ; then the case in which F satisfies Caratheodory type conditions is studied in detail by presenting both the different approaches of Kaczynski - Olech and of Antosiewicz - Cellina . An abstract theorem of Fryszkowski is presented and we show how it can be applied in order to obtain further generalizations of the above results. We also present the result in some sense complementary obtained by Cellina, Colombo and Fonda , together with the improvements obtained by Bressan and Colombo . Finally we survey the papers by Himmelberg - Van Vleck and Lojasiewicz by showing how the compactness assumption on the values of F can be remarkably weakened.

Section 5 is concentrated on the problem of the existence of solutions to the differential inclusion

$$x'(t) \in Ax(t) + F(t,x(t))$$

where A is a maximal monotone operator and F is a compact - valued multivalued map . We present the result of Colombo, Fonda and Ornelas [11] which is a generalization of a previous paper by Cellina and Marchi [10] .

2. NOTATIONS AND BASIC DEFINITIONS.

In a metric space X with metric d , we will denote by $h(.,.)$ the Hausdorff pseudometric defined on the space of nonempty, closed subsets of X :

$$h(A,B) = \max \left\{ \sup_{x \in A} \inf_{y \in B} d(x,y), \sup_{y \in B} \inf_{x \in A} d(x,y) \right\}$$

As usual we set $B(a,r)$ $[B[a,r]]$ to be the open $[$ closed $]$ ball of radius r about a point a . Analogously $B(A,r)$ $[B[A,r]]$ will be the open $[$ closed $]$ r -neighborhood of a set A .

Let X, Y be metric spaces and F be a multifunction from X to the set of subsets of Y . We will use the following definitions.

F is said to have closed graph at $x_0 \in X$ iff, for each sequence (x_n) converging to x_0 and each sequence (y_n) with $y_n \in F(x_n)$, $y_n \rightarrow y_0$ implies $y_0 \in F(x_0)$.

F is lower semicontinuous (l.s.c.) at $x_0 \in X$ iff, for each open set V such that $V \cap F(x_0) \neq \emptyset$, there exists a neighborhood $U(x_0)$ of x_0 such that $V \cap F(x) \neq \emptyset$ for each $x \in U(x_0)$.

F is upper semicontinuous (u.s.c.) at $x_0 \in X$ iff, for each open set V such that $F(x_0) \subset V$ there exists a neighborhood $U(x_0)$ of x_0 such that $F(x) \subset V$ for any $x \in U(x_0)$.

F is ε - upper semicontinuous at $x_0 \in X$ iff, for each $\varepsilon > 0$ there exists a neighborhood $U(x_0)$ of x_0 such that $F(x) \subset B(F(x_0), \varepsilon)$ for each $x \in U(x_0)$.

F is continuous at x_0 iff it is continuous at x_0 with respect to the Hausdorff pseudometric h on the space of nonempty subsets of Y .

We will say that one of the above properties holds on X iff the property holds for all $x \in X$.

Remark . We will use sometimes the following equivalent formulations .

F is l.s.c. on X iff for every closed subset C of Y , the set $\{x: F(x) \subset C\}$ is closed .

F is u.s.c. on X iff for every closed subset C of Y , the set $\{x: F(x) \cap C \neq \emptyset\}$ is closed .

Let us now summarize in a schematic way the main relationships between the above concepts (for the proofs, see [3]).

(a) If $F(x_0)$ is closed, then the following holds at x_0 :

$$\begin{array}{lcl} \text{u.s.c.} & \Rightarrow & \varepsilon\text{-u.s.c.} \Rightarrow \text{closed graph} \\ \text{continuity} & \Rightarrow & \varepsilon\text{-u.s.c.} \ \& \ \text{l.s.c.} \end{array}$$

(b) If $F(x_0)$ is compact, then at x_0 we have :

$$\begin{aligned} \text{u.s.c.} &\Leftrightarrow \varepsilon - \text{u.s.c.} \Rightarrow \text{closed graph} \\ \text{continuity} &\Leftrightarrow \text{u.s.c.} \ \& \ \text{l.s.c.} \end{aligned}$$

(c) If Y is compact and $F(x_0)$ is closed, then at x_0 we have :

$$\begin{aligned} \text{u.s.c.} &\Leftrightarrow \varepsilon - \text{u.s.c.} \Leftrightarrow \text{closed graph} \\ \text{continuity} &\Leftrightarrow \text{u.s.c.} \ \& \ \text{l.s.c.} \end{aligned}$$

Finally, F will be said to be measurable iff the set $\{x: F(x) \cap C \neq \emptyset\}$ is measurable for any closed subset C of Y . A rather complete treatment of measurable multifunctions can be found in [19] or in [34].

In sections 4 and 5 we will deal with subsets of L^1 having the following property.

Definition 2.1 . A subset K of $L^1(T, E)$ is said to be *decomposable* iff, taken $u, v \in K$ and any measurable subset A of T , we have

$$u \cdot \chi[A] + v \cdot \chi[T \setminus A] \in K$$

where $\chi[A]$ and $\chi[T \setminus A]$ stand for the characteristic functions of A and $T \setminus A$ respectively .

3. DIFFERENTIAL INCLUSIONS WITH CONVEX VALUES.

In this section we will show some classical results concerning differential inclusions of the type

$$x'(t) \in F(t, x(t)) \quad , \quad x(0) = x_0 \quad (1)$$

where F is a multivalued map defined on an open subset Ω of \mathbb{R}^{n+1} taking values into the set of nonempty, closed and convex subsets of \mathbb{R}^n . The existence results of this section can be easily proved by using classical continuous selection results (cf. [3]).

We begin by stating the Michael's continuous selection theorem.

THEOREM 3.1 (Michael). *Let X be a metric space, Y a Banach space. Let F from X into the closed convex subsets of Y be lower semicontinuous. Then there exists $f: X \rightarrow Y$, a continuous selection from F .*

As an easy consequence we have the following existence result.

THEOREM 3.2. *Let F be l.s.c. from some open region Ω of \mathbb{R}^{n+1} into the nonempty, closed and convex subsets of \mathbb{R}^n . Let*

$(0, x_0) \in \Omega$. Then there exists a solution to (1) defined on a neighborhood of 0 .

A similar approach can be used in the upper semicontinuous case; we will obtain an existence result by applying the following Approximate Selection Theorem of Cellina (see [3]).

THEOREM 3.3 (Cellina) . Let X be a metric space, Y a Banach space, F a map from X into the convex subsets of Y , upper semicontinuous. Then for every $\varepsilon > 0$ there exists a locally Lipschitzian map f_ε from X to Y such that

$$\text{Graph}(f_\varepsilon) \subset \text{Graph}(F) + \varepsilon B$$

By applying Theorem 3.3 we obtain the following existence result.

THEOREM 3.4 . Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set containing $(0, x_0)$, and let F be an u.s.c. map from Ω into the nonempty closed convex subsets of \mathbb{R}^n . We assume moreover the map $(t, x) \rightarrow m(F(t, x))$ to be locally compact . Then there exists an interval $[0, T]$, $T > 0$ over which a solution to (1) is defined .

4. DIFFERENTIAL INCLUSIONS WITH NON - CONVEX VALUES.

4.1 Introduction. A method frequently employed in the study of the existence of solutions to a differential inclusion

$$x'(t) \in F(t, x(t)) \quad , \quad x(0) = x_0 \quad (1)$$

- or to an ordinary differential equation as well - consists in constructing a sequence $x_n(\cdot)$ of " approximate solutions " , i.e. such that there exists a sequence $\varepsilon_n(\cdot)$ tending (uniformly) to zero and such that

$$d(x'_n(t) , F(t, x_n(t) - \varepsilon_n(t))) \rightarrow 0 \quad . \quad (2)$$

Then one tries to prove that the sequence $x_n(\cdot)$ contains a subsequence uniformly converging to a certain limit $x_0(\cdot)$, and that this limit is a solution to the equation in question . Unfortunately , when dealing with differential inclusions, there is a main difficulty in trying to make the derivatives $x'_n(\cdot)$ converge . In fact, the convergence of $x_n(\cdot)$ to $x_0(\cdot)$ can be often obtained as a strong convergence in the space of absolutely continuous functions , and this only implies $x'_n(\cdot) \rightarrow x'_0(\cdot)$ weakly in L^1 , while we should try to have at least a pointwise

convergence . As an immediate consequence of the fact that closed convex sets are also weakly closed , and of the uniform convergence of $x_n(.)$ and $\varepsilon_n(.)$, from (2) we obtain:

$$x'_0(t) \in \bigcap_{\varepsilon > 0} \text{cl co} \bigcup_{|x - x_0(t)| < \varepsilon} F(t, x) \quad \text{a.e. in } [0, 1] . \quad (3)$$

where cl co stands for the closed convex hull . If the right hand side of (3) is equal to $F(t, x)$, which is equivalent to say that $F(t, x)$ is convex and $F(t, .)$ is upper semicontinuous for each fixed t , then (3) means that $x_0(t)$ is indeed a solution to (1) . In general, when this is not the case, we will have to construct approximate selections in some accurate way in order to have a subsequence converging to a solution.

First of all we will prove an existence result by Filippov [13]. Then the generalization to Caratheodory type conditions by Kaczynski and Olech [21] and Antosiewicz and Cellina [1] will be examined analyzing their different methods of proof. The Theorem of Fryszkowski [14] , generalizing the proof in [1] , and the complementary one of Cellina, Colombo and Fonda [9], together with the improvements of Bressan and Colombo [5] are surveyed in section 4.3 . Finally the results contained in Bressan [4] , Lojasiewicz [25] , and the ones in Himmelberg - Van Vleck [20] and again in Lojasiewicz [26] are presented with some comments and remarks .

4.2 Filippov's result . We will follow here the presentation in [3] .

THEOREM 4.2.1. (Filippov) . *Let Ω be an open subset of \mathbb{R}^{n+1} containing $(0, x_0)$, and let F be a continuous map from Ω to the nonempty compact subsets of \mathbb{R}^n . Then there exists $T > 0$ and an absolutely continuous function $x(\cdot)$ defined on $[0, T]$, a solution to the differential inclusion*

$$x'(t) \in F(t, x(t)) \quad , \quad x(0) = x_0 \quad .$$

Moreover, the derivative $x'(\cdot)$ is regulated .

Proof . We will construct piecewise linear approximate solutions. In order to make the derivatives converge, we will need the following compactness Lemma on the space $\mathcal{B}(I, \mathbb{R}^n)$ of bounded functions from an interval I of \mathbb{R} to \mathbb{R}^n .

Lemma . *Assume that a subset $\mathcal{H} \subset \mathcal{B}(I, \mathbb{R}^n)$ of bounded functions satisfies*

- (α) $\forall t \in I$, $\mathcal{H}(t) \equiv \{x(t) \mid x(\cdot) \in \mathcal{H}\}$ is precompact ;
- (β) \mathcal{H} is equioscillating, i.e. $\forall \varepsilon > 0 \exists$ a finite partition of I into subintervals J_k ($k = 1, \dots, r$) such that, $\forall x(\cdot) \in \mathcal{H}, \forall k \in \{1, \dots, r\}$,

$$\sup_{t_1, t_2 \in J_k} |x(t_1) - x(t_2)| < \varepsilon .$$

Then \mathcal{H} is precompact in $\mathcal{B}(I, \mathbb{R}^n)$.

Let $a, b > 0$ be such that $[-a, a] \times B[x_0, b] \subset \Omega$; we will consider F restricted to $[-a, a] \times B[x_0, b]$, which is uniformly continuous and takes values into a certain ball $B[0, M]$. Set $T = \min \{a, b/M\}$.

Once we set $\eta_k = 2^{-k}$, by the uniform continuity there exists δ_k such that $|t - t'| \leq \delta_k$ and $|x - x'| \leq M \cdot \delta_k$ imply $h(F(t, x), F(t', x')) \leq \eta_k$.

We will now define the partitions of the interval $I = [0, T]$. Choose a number h_1 smaller than δ_1 and such that T/h_1 is an integer; define the first partition of I by the intervals $[ih_1, (i+1)h_1[$. In general, the m -th partition is obtained by subdividing each interval of the preceding partition into a finite number of right-open intervals each of equal length h_m , with $h_m < \delta_m$. The points ih_m are the nodal points of the m -th partition. When τ is a nodal point, we denote by $O(\tau)$ the order of the partition where τ first appears as a nodal point. When $O(\tau) > 1$, we set $s(\tau)$ to be the initial point of the interval of the $(O(\tau) - 1)$ -th partition to which τ belongs.

We will now define the η_k -approximate solution x_k

as an absolutely continuous function verifying $x_k(0) = x_0$, whose derivative is constant on the intervals of the k -th partition of I . We will use an induction argument on the nodal points of the k -th partition. If $\tau = 0$, then x_k is obviously defined on $[0, \tau[$. Assume by induction that x_k is defined on $[0, \tau^*[$, and moreover that :

(a) at each nodal point $\tau \in [0, \tau^*[$, $x'_k(\tau) \in F(\tau, x_k(\tau))$;

(b) at each nodal point $\tau \in [0, \tau^*[$, $d(x'_k(\tau), x'_k(s(\tau))) \leq \eta_{O(\tau)-1}$

Since

$$|\tau^* - s(\tau^*)| < h_{O(\tau^*)-1}, \quad d(x_k(\tau^*), x_k(s(\tau^*))) \leq M h_{O(\tau^*)-1},$$

we have

$$h(F(\tau^*, x_k(\tau^*)), F(s(\tau^*), x_k(s(\tau^*)))) \leq \eta_{O(\tau^*)-1}$$

By (a), $x_k(s(\tau^*)) \in F(s(\tau^*), x_k(s(\tau^*)))$ and so there exists $v \in F(\tau^*, x_k(\tau^*))$ such that $d(v, x_k(s(\tau^*))) \leq \eta_{O(\tau^*)-1}$. Set

$$x_k(t) = x_k(\tau^*) + (t - \tau^*)v, \quad t \text{ on }]\tau^*, \tau^* + h_k].$$

Then the inductive hypotheses are satisfied up to $\tau^* + h_k$. So the existence of x_k is proved on the whole $[0, T]$.

It is not difficult to see that the sequence (x'_k) is indeed equioscillating (see [3]). This is a consequence of the fact that our functions were constructed inductively satisfying property (b) above. The oscillation of each x_k on the intervals of the r -th partition comes out to be at most

$$\sum_{j=0}^{\infty} \eta_j = 1/2^{O(r)-2}$$

thus not depending on k . Applying the Lemma, we have that both $x_k(\cdot)$ and $x'_k(\cdot)$ converge uniformly to an absolutely continuous function $x^*(\cdot)$ and to a regulated function $v^*(\cdot)$, respectively. Indeed, v^* is the derivative of x^* , $x^*(0) = x_0$, and it can be shown that

$$d(v^*(t), F(t, x^*(t))) = 0$$

which implies

$$x^{*'}(t) = v^*(t) \in F(t, x^*(t)) \quad . \quad \blacksquare$$

4.3 Existence under Caratheodory type conditions . The result of Filippov has been improved by Kaczynski and Olech [21] and Antosiewicz and Cellina [1] , by assuming on F Caratheodory type conditions . Their proofs follow completely different approaches . Kaczynski and Olech use the "approximate solutions" approach , generalizing Filippov's construction , while Antosiewicz and Cellina use a completely new - for non-convex valued differential inclusions - fixed point approach, based on the construction of a continuous selection for a certain "integral" map. Since all the generalizations which were given in the following years were in some way inspired on these two approaches, we believe useful to report here both of them. Here is the result.

THEOREM 4.3.1 . Assume $F(.,.)$ is a multivalued map from $I \times \mathbb{R}^n$ ($I = [0,1]$) into the compact subsets of \mathbb{R}^n satisfying the following conditions:

- (i) for each $x \in \mathbb{R}^n$, $t \rightarrow F(t,x)$ is measurable in I ;
- (ii) for each $t \in I$, $x \rightarrow F(t,x)$ is continuous in \mathbb{R}^n ;
- (iii) F is integrably bounded , i.e. there exists $m \in L^1(I)$ such that, for each t and x , $F(t,x) \subset B(0,m(t))$.

Then there exists an absolutely continuous function $x(.)$ of I into \mathbb{R}^n such that, for almost every $t \in I$,

$$x'(t) \in F(t,x(t)) \quad , \quad x(0) = x_0 \quad . \quad (1)$$

Proof by Kaczynski and Olech. We can suppose without loosing generality that $x_0 = 0$. The first thing we are going to do is to construct an L^1 - compact family of measurable functions which will be used for the choice of the derivatives of the approximate solutions. Let K be the closed ball in \mathbb{R}^n about the origin, with radius $M = \int_I m(t) dt$. By (iii), any solution of (1) assumes values in K . Let us introduce a modulus of continuity for F

$$\eta(t,r) = \max \{h(F(t,y), F(t,x)) \mid x,y \in K, |x - y| < r\}$$

It can be checked that η is integrable in t for each fixed r , and continuous non increasing in r for each fixed t . Take (r_i) , a sequence of real numbers converging to zero and such that $r_{i+1} < r_i/4$ and moreover

$$\int_I \sum_{i \geq 1} \eta(t,r_i) dt < +\infty$$

Let A_i be a finite $r_{i+1}/2$ - net for K . For a_1, \dots, a_n such that $a_i \in A_i$ ($i = 1, \dots, n$) and $|a_i - a_{i-1}| < r_i$ ($i = 2, \dots, n$), choose an integrable function $u(a_1, \dots, a_n)(.) : I \rightarrow \mathbb{R}^n$ satisfying

$$u(a_1, \dots, a_n)(t) \in F(t, a_n) \quad \forall t$$

and, if $n > 1$,

$$|u(a_1, \dots, a_n)(t) - u(a_1, \dots, a_{n-1})(t)| \leq \eta(t, r_{n-1}).$$

(Such a function exists as a measurable selection of the proper measurable map

$$F(t, a_n) \cap \{x: |x - u(a_1, \dots, a_{n-1})(t)| \leq \eta(t, r_{n-1})\} \quad)$$

Let h_i be chosen in such a way that $1/h_i$ and h_i/h_{i+1} are integers greater than one and moreover

$$\int_t^{t+h_i} m(s) ds < r_i/4, \quad i = 1, 2, \dots$$

Given a sequence $a_i^n(\cdot) : I \rightarrow A_i$ of maps which are constant on the intervals $[kh_i, (k+1)h_i)$ ($k = 0, \dots, 1/h_i - 1$), it is not difficult to show that the set

$$\{u(a_1^n(\cdot), \dots, a_i^n(\cdot))(\cdot), i = 1, 2, \dots; n \geq i\}$$

is a compact set in L^1 . It is in this set that we are going to define the derivatives of our approximate solutions. In fact one can define sequences $a_i^n(\cdot)$ of the above kind and $x_n(\cdot)$ such that

$$x'_n(t) = u(a_1^n(t), \dots, a_n^n(t))(t)$$

and

$$|x_n(t) - a_i^n(t)| < r_i$$

by using an induction argument (see [21]). As a consequence, the functions $x_n(\cdot)$ are indeed approximate solutions, equicontinuous and uniformly bounded; hence there exists a subsequence uniformly converging to a certain $x_0(\cdot)$, and because of the above construction we can choose a subsequence whose derivatives converge pointwisely to x'_0 . Finally x_0 is easily proved to be a solution to (1) . ■

Proof by Antosiewicz and Cellina. The proof will be based on the following continuous selection result.

Proposition. *Let F be as in the Theorem. Define on the space \mathcal{X} of the absolutely continuous mappings $u : I \rightarrow \mathbb{R}^n$ such that $u(0) = x_0$ and $\|u'(t)\| \leq M \equiv \int_I m(t)dt$ the map G :*

$$G(u)(t) = F(t, u(t)) \tag{4}$$

There exists a continuous mapping $g : \mathcal{X} \rightarrow L^1$ such that, for every $u \in \mathcal{X}$, $g(u)(t) \in G(u)(t)$ at almost every $t \in I$.

Proof. Let us consider the restriction of F to $I \times B[0, M]$, and

assume first that F is continuous - hence uniformly continuous.

First of all we will prove the existence of an approximate selection, i.e. for every $\varepsilon > 0$ there exists a continuous mapping g_ε such that

$$d(g_\varepsilon(u)(t), G(u)(t)) < \varepsilon \quad (5)$$

at almost every $t \in I$. Let $\varepsilon > 0$ be given. By the uniform continuity of F there exists $\delta > 0$ such that

$$|t - s| < \delta, \quad \|x - y\| < \delta \quad \Rightarrow \quad h(F(t, x), F(s, y)) < \varepsilon$$

Since \mathcal{K} is a compact subset of the set of continuous functions, we can select a finite covering $\{B(u_i, \delta/2) : i = 1, \dots, m\}$ of \mathcal{K} , and, since each $G(u_i)$ is measurable, we can choose $v_i \in L^1(I)$ ($i = 1, \dots, m$) such that $v_i(t) \in G(u_i)(t)$ at almost every $t \in I$. Let $(p_i)_{i=1, \dots, m}$ be a continuous partition of unity associate to the above covering, and set, for each $u \in \mathcal{K}$,

$$\tau_0(u) = 0, \quad \tau_i(u) = \tau_{i-1}(u) + p_i(u) \quad (i = 1, \dots, m).$$

Define the interval $J_i(u) = [\tau_{i-1}(u), \tau_i(u)[$ for every i , and denote

by $\chi[J_i(u)]$ the characteristic function of $J_i(u)$. Finally we define

$$g_\varepsilon(u) = \sum \chi[J_i(u)] v_i \quad \text{on } [0,1[$$

and $g_\varepsilon(u)(1) = v_j(1)$, where $j = \min \{i \geq 1 : \tau_i(u) = 1\}$. It can be easily checked that the map g_ε has the required property (5).

By using a procedure similar to the one above, one can construct a sequence of continuous mappings $g^n : \mathcal{K} \rightarrow L^1(I)$ with the properties that, for each $u \in \mathcal{K}$,

$$d(g^n(u)(t), G(u)(t)) < 2^{-n-1} \quad (6)$$

at almost every $t \in I$, and

$$\mu(\{t \in I : |g^n(u)(t) - g^{n-1}(u)(t)|\}) < 2^{-n}.$$

As a result there will exist for each $u \in \mathcal{K}$ a measurable mapping $g(u)$ of I into \mathbb{R}^n such that the sequence $(g^n(u))$ converges to $g(u)$ a.e. in measure and that a subsequence of $(g^n(u))$ converges to $g(u)$ a.e. in I . Thus, by (6) we will have that, for each $u \in \mathcal{K}$, $g(u)(t) \in G(u)(t)$ at almost every $t \in I$.

For the general case of F satisfying the hypothesis in Theorem 4.3.1, it can be proved that for each u ,

$G(u) : t \rightarrow F(t, u(t))$ remains measurable in I ; moreover the following holds true:

for each $\varepsilon > 0$ there exists a closed subset E of I with $\mu(I \setminus E) \leq \varepsilon$ such that the family $(F(t, \cdot))_{t \in E}$ is uniformly equicontinuous.

These facts permit to construct the desired function g by choosing an invading family of sets E_n having the above property with $\varepsilon = 1/n$, and defining g separately on the sets $E_{n+1} \setminus E_n$ (see [1] for details) . ■

End of the proof of Theorem 4.3.1 . By the above Proposition, there exists a continuous mapping $g : \mathcal{K} \rightarrow L^1(I)$ such that, for each $u \in \mathcal{K}$, $g(u)(t) \in F(t, u(t))$ at almost every $t \in I$. Let $h(u)$, for each $u \in \mathcal{K}$, be the mapping of I into \mathbb{R}^n defined by setting

$$h(u)(t) = \int_0^t g(u)(s) ds$$

for every $t \in I$. It is immediately seen that h is a continuous function mapping \mathcal{K} into itself. Hence, by Schauder's Theorem , there exists a point $x \in \mathcal{K}$ for which $x = h(x)$, i.e., $x(t) = h(x)(t)$ at every $t \in I$. This implies that $x(0) = 0$ and $x'(t) = g(x)(t) \in F(t, x(t))$ at almost every $t \in I$. ■

4.4 Multivalued maps with decomposable values . As we have shown in section 4.3 , most of the effort in the proof of Antosiewicz and Cellina has been done in constructing a continuous selection from the map G . This was accomplished by "piecing" together several given L^1 - maps . The subsets of L^1 which are closed under this operation of "piecing" are said to be "decomposable" (see Definition 2.1) . The paper by Antosiewicz and Cellina initiated the problem of whether decomposability could be used as a substitute for convexity (see [18], [31]) . Fryszkowski [14] proved an analogue of Michael's Continuous Selection Theorem with decomposability instead of convexity, and Cellina, Colombo and Fonda [9] proved the analogue of Cellina's Approximate Selection Theorem . These results were proved essentially under the assumption of compactness of the domain of the multivalued map under consideration . However, Bressan and Colombo [5] succeeded in eliminating this assumption, too. Here we state the analogues of Theorems 3.1 and 3.3 in the case of multivalued maps with decomposable values.

THEOREM 4.4.1. *Let X be a separable metric space , T a measure space with a nonatomic measure μ_0 , and E a Banach space . Let F be a lower semicontinuous multivalued map from X to the closed decomposable subsets of $L^1(T,E)$. Then F has a continuous selection .*

THEOREM 4.4.2 . *Let X , T , and E be as above , and F be an ε -*

upper semicontinuous multivalued map defined on X , taking as values decomposable subsets of $L^1(T,E)$. Then for every $\varepsilon > 0$ there exists a continuous map $f_\varepsilon : X \rightarrow L^1(T,E)$ such that

$$\text{Graph}(f_\varepsilon) \subset B(\text{Graph}(F), \varepsilon)$$

Without entering the technical details in the proofs of Theorems 4.4.1 and 4.4.2 above, we will only give here the proof of Theorem 4.4.2 in the simplified case when the metric space X is supposed to be compact (see [9]).

Proof. We show that Theorem 4.4.2 is true when X is compact. Fix $\varepsilon > 0$; by the upper semicontinuity of F , for each $s \in S$ there is a $\delta(s) > 0$ such that $F(s') \subset B(F(s), \varepsilon/3)$ whenever $s' \in B(s, \delta(s))$. We can choose $\delta(s) < \varepsilon/3$. Since X is compact and $\{B(s, \delta(s)/2) : s \in X\}$ is an open covering of X , there exist $s_1, \dots, s_n \in X$ such that, setting $\delta_i = \delta(s_i)/2$, the balls $B(s_i, \delta_i)$ ($i=1, \dots, n$) form a finite subcovering of X . Let $\{p_i : i=1, \dots, n\}$ be a continuous partition of unity subordinate to it, and choose arbitrarily $u_i \in F(s_i)$ ($i=1, \dots, n$). We shall construct $f_\varepsilon(s)$ as an appropriate decomposition of these maps.

Let us choose $v_{ij} \in F(s_j)$ ($i, j = 1, \dots, n$) such that

$$\begin{aligned}
d_1(u_i, v_{ij}) &= \int_T |u_i(t) - v_{ij}(t)| d\mu_0 \\
&\leq \inf \int_T |u_i(t) - v(t)| d\mu_0 + \varepsilon/3 = d_1(u_i, F(s_j)) + \varepsilon/3
\end{aligned}$$

and define the set functions

$$\mu_{ij}(E) = \int_E |u_i(t) - v_{ij}(t)| d\mu_0 \quad (i, j = 1, \dots, n)$$

for every measurable subset E of T . It is easy to see that for each $i, j \in \{1, \dots, n\}$, μ_{ij} is a finite non atomic measure on T . Following an idea of Fryszkowski [14] we apply here a consequence of Lyapunov's theorem (cf.[16]) : there exists a family $(A_\alpha)_{\alpha \in [0,1]}$ of measurable subsets of T such that:

$$P_1) A_\alpha \subset A_\beta \quad \text{if } \alpha \leq \beta$$

$$P_2) \mu_{ij}(A_\alpha) = \alpha \mu_{ij}(T) \quad (i, j = 1, \dots, n)$$

$$P_3) \mu_0(A_\alpha) = \alpha \mu_0(T)$$

Set $\alpha^0 \equiv 0$, $\alpha^i(s) = p_1(s) + \dots + p_i(s)$ and define the approximate selection as

$$f_\varepsilon(s) = \sum_{i=1}^n u_i \chi_{[A_{\alpha^i}(s) \setminus A_{\alpha^{i-1}}(s)]}$$

We claim that f_ε has the required properties. First of all f_ε is

continuous. In fact, fix s^0 ; then

$$\begin{aligned}
\|f_\varepsilon(s) - f_\varepsilon(s^0)\|_1 &= \left\| \sum u_i (\chi[A_{\alpha^i}(s) \setminus A_{\alpha^{i-1}}(s)] - \chi[A_{\alpha^i}(s^0) \setminus A_{\alpha^{i-1}}(s^0)]) \right\| \\
&\leq \sum \int_T |u_i(t)| \cdot |\chi[A_{\alpha^i}(s) \setminus A_{\alpha^{i-1}}(s)](t) - \chi[A_{\alpha^i}(s^0) \setminus A_{\alpha^{i-1}}(s^0)](t)| d\mu_0 \\
&\leq \sum \int_T |u_i| \{ |\chi[A_{\alpha^i}(s)] - \chi[A_{\alpha^i}(s^0)]| + |\chi[A_{\alpha^{i-1}}(s)] - \chi[A_{\alpha^{i-1}}(s^0)]| \} d\mu_0 \\
&= \sum \left\{ \int_{A_{\alpha^i}(s) \Delta A_{\alpha^i}(s^0)} |u_i| d\mu_0 + \int_{A_{\alpha^{i-1}}(s) \Delta A_{\alpha^{i-1}}(s^0)} |u_i| d\mu_0 \right\}
\end{aligned}$$

and, by the integrability of u_i and (P_3) , the continuity of f_ε follows.

It is clear that if $F(S)$ is decomposable then $f_\varepsilon(S) \subset F(S)$.

It remains to verify that f_ε is an ε -approximate selection of F .

For this purpose, fix $s \in S$ and let $I(s) = \{i \in \{1, \dots, n\} : p_i(s) > 0\}$

and $j \in I(s)$ such that $\delta_j = \max \{\delta_i : i \in I(s)\}$.

Then, for every $i \in I(s)$, we have:

$$s_i \in B(s_j, 2\delta_j)$$

so that

$$F(s_i) \subset B(F(s_j), \varepsilon/3)$$

and

$$\mu_{ij}(T) \leq d_1(u_i, F(s_j)) + \varepsilon/3 \leq (2/3)\varepsilon.$$

Moreover, since $F(s_j)$ is decomposable, we have that

$$\omega_j \equiv \sum v_{ij} \chi[A_{\alpha^i(s)} \setminus A_{\alpha^{i-1}(s)}] \in F(s_j)$$

Finally,

$$\begin{aligned} d((s, f_\varepsilon(s)), (s_j, \omega_j)) &\leq d(s, s_j) + \left\| \sum (u_i - v_{ij}) \chi[A_{\alpha^i(s)} \setminus A_{\alpha^{i-1}(s)}] \right\|_1 \\ &\leq \delta(s) + \sum \|(u_i - v_{ij}) \chi[A_{\alpha^i(s)} \setminus A_{\alpha^{i-1}(s)}]\|_1 \\ &\leq \varepsilon/3 + \sum \mu_{ij}(A_{\alpha^i(s)} \setminus A_{\alpha^{i-1}(s)}) \\ &= \varepsilon/3 + \sum [\alpha^i(s) - \alpha^{i-1}(s)] \mu_{ij}(T) \\ &\leq \varepsilon/3 + (2/3)\varepsilon \sum p_i(s) \\ &= \varepsilon \end{aligned}$$

■

4.5 . Further existence results . First of all we will show how Theorem 4.4.1 can be applied in order to prove the following result by Bressan [4] and Lojasiewicz [25] .

THEOREM 4.5.1 . *Let $F(.,.)$ be a lower semicontinuous multivalued map defined on a compact subset K of \mathbb{R}^{n+1} , taking compact values in a bounded region of \mathbb{R}^n . Then there exists a solution to the problem*

$$x'(t) \in F(t, x(t)) \quad , \quad x(0) = x_0 \quad . \quad (1)$$

In order to prove Theorem 4.5.1 , we need the following

Proposition . *The map G , defined on the set of absolutely continuous functions by*

$$G(u) = \{v \in L^1 : v(t) \in F(t, u(t)) \text{ for a.e. } t \}$$

is lower semicontinuous .

Proof . Let M be such that $F(t, x) \subset B[0, M]$ for every t, x , and let $K = I \times B[0, b]$. Fix $u_0(.)$ and let $\varepsilon > 0$ be given . According to classical results of Scorza - Dragoni type (see [4] , [26]), we can find a compact set $E \subset I$ and a $\rho > 0$ such that $\mu(I \setminus E) < \varepsilon/2M$ and

$$|u(.) - u_0(.)| < \rho \Rightarrow F(t, u_0(t)) \subset B[F(t, u(t)), \varepsilon/2T]$$

for every $t \in E$. Take now any $f_0 \in G(u_0)$ and any $u(.)$ with $|u(.) - u_0(.)| < \rho$. What we have to show is that there exists an $f \in G(u)$ such that $|f - f_0|_1 < \varepsilon$. In order to do this, define the following multivalued map:

$$\begin{aligned} H(t) &= F(t, u(t)) \cap B[\{f_0(t)\}, \varepsilon/2T] & \text{if } t \in E \\ &= F(t, u(t)) & \text{if } t \in I \setminus E \end{aligned}$$

Since the map H is measurable, we can select a measurable selection $f : I \rightarrow \mathbb{R}^n$; clearly, $f \in G(u)$ and moreover

$$\begin{aligned} |f - f_0|_1 &= \int_E \|f(t) - f_0(t)\| dt + \int_{I \setminus E} \|f(t) - f_0(t)\| dt \\ &< T \cdot \varepsilon/2T + 2M \cdot \varepsilon/2M = \varepsilon. \end{aligned}$$

■

Proof of Theorem 4.5.1. The proposition above together with Theorem 4.4.1 permits us to define a continuous function g from the space of absolutely continuous functions to $L^1(I, \mathbb{R}^n)$ such that

$$g(u)(t) \in F(t, u(t))$$

for every $u(.)$ and almost every $t \in I$. Define now the map h :

$$h(u)(t) = \int_0^t g(u)(s) ds$$

If we define \mathcal{K} to be the set of absolutely continuous functions u such that $u(0) = x_0$ and $\|u'(t)\| \leq M$ at almost every t in I , then it is easily seen that \mathcal{K} is a nonempty, compact and convex subset of the set of continuous functions, and that h is a continuous function mapping \mathcal{K} into itself. By Schauder's Theorem, h has a fixed point $x(\cdot)$, which is the solution we are looking for. ■

We are now going to present some recent results which generalize the theorem presented in section 4.4 with Caratheodory type conditions on the map F . Himmelberg and Van Vleck [20] and Lojasiewicz [26] succeeded in weakening considerably condition (iii) in Theorem 4.3.1 (which yields in particular the compactness of the values of F), by asking F to be weakly integrably bounded, i.e.

DEFINITION . a map F is said to be *weakly integrably bounded* iff there exists $m \in L^1(I)$ such that for each t and x , $F(t,x) \cap B(0,m(t)) \neq \emptyset$; we will say that F is *locally weakly integrably bounded* iff for each $\rho > 0$, F is weakly integrably bounded for $|x| < \rho$ by a function $m_\rho \in L^1(I)$.

Condition (ii) is also generalized (see also Olech [29]) in order to include classical results obtained for convex valued maps F . Here are their main results:

THEOREM 4.5.2 . ([20]) . Assume F satisfies the following conditions:

- (i') F is measurable in t for each x ;
- (ii') for each t , $F(t, \cdot)$ has closed graph and, at each point x for which $F(t, x)$ is not convex, $F(t, \cdot)$ is lower semicontinuous ;
- (iii') F is weakly integrably bounded [locally weakly integrably bounded].

Then there exists a global [local] solution to

$$x'(t) \in F(t, x(t)) \quad , \quad x(0) = x_0 . \quad (1)$$

THEOREM 4.5.3 . ([26]) . Assume F satisfies the following conditions:

- (i'') F is measurable in (t, x) ;

- (ii'') for a.e. t , for each point x , either $F(t, \cdot)$ has closed graph at x and $F(t, x)$ is convex, or $F(t, \cdot)$ restricted to some neighborhood of x is lower semicontinuous ;
- (iii'') F is locally weakly integrably bounded .

Then (1) has a local solution .

Remark . Comparing conditions (ii') and (ii'') above, one can see that (ii') requires stronger upper semicontinuity conditions, while (ii'') requires more lower semicontinuity .

We will give here the proof of Theorem 4.5.2 due to Himmelberg and Van Vleck. The proof will be carried out by following an "approximate solutions" approach, refining the previous approaches by Filippov [13] , Kaczynski and Olech [21] , and Olech [29].

Proof of Theorem 4.5.2 .In order to reconduct ourselves to a differential inclusion with an integrably bounded multivalued map, we will have to consider the following auxiliary multifunction

$$\tilde{F}(t, x) = (F(t, x) \cap B[0, 2m(t)]) \cup \partial B[0, 2m(t)]$$

If F satisfies the hypotheses of Theorem 4.5.2, it can be shown that the map \tilde{F} , besides being integrably bounded with compact values, has the properties that for each t , $\tilde{F}(t, \cdot)$ has closed graph and $\tilde{F}(t, \cdot)$ is lower semicontinuous whenever $F(t, \cdot)$ is lower semicontinuous.

For technical reasons to be explained later, we also need to introduce the map

$$F^*(t, x) = \bigcap_{\varepsilon > 0} \text{cl} \bigcup_{\substack{z \in Z \\ |z - x| < \varepsilon}} \tilde{F}(t, z)$$

where Z is an appropriate countable dense subset of \mathbb{R}^n . It can be shown (see [20]) that F^* has the following properties:

- (a) $F^*(t, x) \subset \tilde{F}(t, x)$ for each t, x ;
- (b) $(F^*)^*(t, x) = F^*(t, x)$ for each t, x ;
- (c) F^* is weakly integrably bounded by m ;
- (d) F^* is integrably bounded by $2m$;
- (e) F^* satisfies (i') ;
- (f) F^* has closed graph;
- (g) $F^*(t, \cdot)$ is lower semicontinuous at every x for which $F(t, x)$ is not convex .

We will now introduce an appropriate modulus of continuity. Let $M = \int_1 2m(t) dt$. Let $H^*(t)$ be the set of values

x at which $F^*(t, \cdot)$ is continuous ; for $s > 0$ set

$$K^*(t, s) = \{x \in B[0, 2M] : B(x, 2s) \subset H^*(t)\}$$

Define the functions α^* and η^* as follows:

$$\alpha^*(t, r, x) = \sup \{h(F^*(t, x), F^*(t, y)) : y \in \mathbb{R}^n, |x - y| < r\}$$

$$\begin{aligned} \eta^*(t, r, s) &= \sup \{\alpha^*(t, r, x) : x \in K^*(t, s)\} && \text{if } K^*(t, s) \neq \emptyset \\ &= 0 && \text{otherwise,} \end{aligned}$$

where $r > 0$, $s > 0$ are given.

We will not prove here the measurability of the function $\eta^*(\cdot, r, s)$. This is a consequence of property (b) of the map F^* , and this is the reason why we have to use the auxiliary map F^* . The proof of this fact is due to Olech [30] and the details can be found in [20].

As in the proof of Theorem 4.4.1 due to Kaczynski and Olech, we are going to construct an L^1 -compact family of functions which will be used for the definition of the derivatives of our approximate solutions.

Fix a sequence (s_i) that decreases to zero. Then it is possible to choose a sequence (r_i) , also decreasing to zero, with the following properties:

$$r_1 \leq 1 \quad \text{and} \quad r_{i+1} < r_i/2 \quad \text{for } i = 1, 2, \dots$$

$$\int_I \left[\sum_{i \geq 1} \eta^*(t, r_i, s_i) \right] dt < \int_I m(t) dt$$

$$\mu(E_i) \equiv \mu(\{t : \eta^*(t, r_i, s_i) > m(t)/2^i\}) < 1/2^i \quad \text{for } i = 1, 2, \dots$$

Let A_i be a finite $r_{i+1}/2$ net for $B[0, 2M]$. For a_1, \dots, a_n such that $a_i \in A_i$ ($i = 1, \dots, n$) and $|a_i - a_{i-1}| \leq r_i$ ($i = 1, \dots, n$), choose an integrable function $u(a_1, \dots, a_n)(\cdot) : I \rightarrow \mathbb{R}^n$ satisfying

$$u(a_1, \dots, a_n)(t) \in F^*(t, a_n) \cap B[0, (1 + 1/2 + \dots + 1/2^{n-1}) m(t)] \quad (7)$$

and, for $n > 1$,

$$|u(a_1, \dots, a_n)(t) - u(a_1, \dots, a_{n-1})(t)| \leq \alpha^*(t, r_{n-1}, a_{n-1}) \quad (8)$$

$$\text{for } t \in (I \cap E_{n-1}) \cap \{t : a_{n-1} \in K^*(t, s_{n-1})\}$$

In order to do this, set, for any $a_1 \in A_1$, $u(a_1)(\cdot)$ to be a measurable selector for the measurable multifunction $F^*(\cdot, a_1) \cap B[0, m(\cdot)]$, which is nonempty by (c) above. Once $u(a_1, \dots, a_{n-1})(\cdot)$ has been defined, taking values in $F^*(\cdot, a_{n-1})$, and $a_n \in A_n$ with

$|a_n - a_{n-1}| \leq r_n$ we define $u(a_1, \dots, a_n)(\cdot)$ as follows. For $t \in (\mathbb{I} \setminus E_{n-1}) \cap \{t : a_{n-1} \in K^*(t, s_{n-1})\}$, choose a measurable function $u^*(\cdot)$ such that

$$u^*(t) \in F^*(t, a_n)$$

and

$$|u^*(t) - u(a_1, \dots, a_n)(t)| \leq \alpha^*(t, r_{n-1}, a_{n-1})$$

This can be done by choosing a measurable selection from the measurable multivalued map

$$F^*(\cdot, a_n) \cap B[u(a_1, \dots, a_{n-1})(\cdot), \alpha^*(\cdot, r_{n-1}, a_{n-1})]$$

which has nonempty values since $|a_{n-1} - a_n| \leq r_n < r_{n-1}$ implies $h(F^*(t, a_{n-1}), F^*(t, a_n)) \leq \alpha^*(t, r_{n-1}, a_{n-1})$ and since $u(a_1, \dots, a_{n-1})(t) \in F^*(t, a_{n-1})$.

One also has $u^*(t) \in B[0, (1 + 1/2 + \dots + 1/2^{n-1})m(t)]$ since $u(a_1, \dots, a_{n-1})(t) \in B[0, (1 + 1/2 + \dots + 1/2^{n-2})m(t)]$ and $\eta^*(t, r_{n-1}, s_{n-1}) \leq m(t)/2^{n-1}$ (for $t \notin E_{n-1}$).

Finally, set

$$\begin{aligned} u(a_1, \dots, a_n)(t) &= u^*(t) & \text{if } t \in (\mathbb{I} \setminus E_{n-1}) \cap \{t : a_{n-1} \in K^*(t, s_{n-1})\} \\ &= \tilde{u}(t) & \text{elsewhere,} \end{aligned}$$

where $\tilde{u}(\cdot)$ is taken to be a measurable selection from $F^*(t, a_n) \cap B[0, m(t)]$. Then $u(a_1, \dots, a_n)(\cdot)$ satisfies the required properties (7) and (8). Moreover $u(a_1, \dots, a_n)(t) \in F(t, a_n)$ since $F^*(t, a_n) \subset \tilde{F}(t, a_n)$, by property (a), and $\tilde{F}(t, a_n) \cap B[0, \lambda m(t)] = F(t, a_n) \cap B[0, \lambda m(t)]$ for any λ with $0 \leq \lambda < 2$.

Let us now define the partition of our interval I . Let $h_i > 0$ be such that $1/h_i, h_i/h_{i+1}$ are integers and

$$\int_t^{t+h_i} 2m(t) dt < r_{i+1}/4.$$

The interval I is subdivided into subintervals of the form $[kh_i, (k+1)h_i]$.

We are going to define approximate solutions $x_n(\cdot)$ of (1); the derivatives $x'_n(\cdot)$ will be appropriately chosen among the maps we have constructed above.

Let us construct sequences of maps $a_i^n(\cdot) : I \rightarrow A_i$, constant on the intervals $[kh_i, (k+1)h_i]$ ($k = 0, \dots, 1/h_i - 1$), and of absolutely continuous maps $x_n(\cdot)$ in such a way that

$$|a_i^n(t) - a_{i-1}^n(t)| \leq r_i \quad (i = 2, \dots, n),$$

$$x_n(0) = 0, \quad |x_n(t) - a_i^n(t)| \leq r_i \quad \text{for } t \in I \quad (9)$$

and

$$|x_n(kh_i) - a_i^n(kh_i)| \leq r_{i+1}/2, \quad (k = 0, \dots, 1/h_i - 1)$$

for $i = 1, \dots, n$.

We will use an induction argument to define the above functions on the intervals $[kh_n, (k+1)h_n)$ ($k = 0, \dots, 1/h_n - 1$). On the first interval $[0, h_n)$ we set $a_i^n(t) = a_i$, where a_1, \dots, a_n are chosen so that $|a_i - x_0| \leq r_{i+1}/2$. Moreover we define

$$x_n(t) = \int_0^t u(a_1^n(s), \dots, a_n^n(s))(s) ds \quad (t \in [0, h_n]).$$

We have

$$|x_n(t) - a_i^n(t)| = |x_n(t) - a_i| \leq \left| \int_0^t u(a_1^n(s), \dots, a_n^n(s))(s) ds \right| + r_{i+1}/2$$

$$\leq \int_0^{h_n} 2m(s) ds + r_{i+1}/2 \leq r_{n+1}/4 + r_{i+1}/2 \leq r_i.$$

for $t \in [0, h_n]$ and $i = 1, \dots, n$.

Now assume by induction that x_n is defined on $[0, kh_n]$ and the corresponding a_i^n 's are defined at least on

$[0, kh_n)$. Let us first see how to define the a_i^n 's on $[kh_n, (k+1)h_n)$. There exists an integer $j = j(k)$ such that kh_n/h_i is an integer for $i = j, \dots, n$ and is not an integer for $i = 1, \dots, j-1$.

For $i = 1, \dots, j-1$, set $m(i)$ to be an integer such that $m(i)h_i < kh_n < (m(i)+1)h_i$, and put $a_i^n(t) = a_i^n(m(i)h_i)$ for $t \in [kh_n, (k+1)h_n)$.

For $i = j, \dots, n$, choose $a_i \in A_i$ so that $|a_i - x_n(kh_n)| \leq r_{i+1}/2$ and set $a_i^n(t) = a_i$ for $t \in [kh_n, (k+1)h_n)$.

The functions $a_i^n(\cdot)$ defined above are clearly constant on the intervals $[kh_i, (k+1)h_i)$.

Define finally x_n as

$$x_n(t) = \int_0^t u(a_1^n(s), \dots, a_n^n(s)) ds$$

Now we check that the above properties indeed hold true for $a_i^n(\cdot)$ and $x_n(\cdot)$ in such a way defined. On $[kh_n, (k+1)h_n)$ we have:

If $i \geq j$, then

$$\begin{aligned} |x_n(t) - a_i^n(t)| &\leq |x_n(t) - x_n(kh_n)| + |x_n(kh_n) - a_i^n(t)| \\ &\leq r_{n+1}/4 + r_{i+1}/2 < r_i . \end{aligned}$$

If $i < j$, then

$$\begin{aligned} |x_n(t) - a_i^n(t)| &\leq |x_n(t) - x_n(m(i)h_i)| + |x_n(m(i)h_i) - a_i^n(t)| \\ &\leq r_{i+1}^{1/4} + r_{i+1}^{1/2} < r_i. \end{aligned}$$

So (9) holds true; the remaining properties can be proved by a similar procedure.

The constructed sequence (x_n) is a sequence of approximate solutions, i.e.

$$x'_n(t) = u(a_1^n(t), \dots, a_n^n(t))(t) \in F(t, a_n^n(t)) = F(t, x_n(t) - \varepsilon_n(t))$$

where $\varepsilon_n(t) = x_n(t) - a_n^n(t)$ tends uniformly to zero as $n \rightarrow \infty$.

Now it remains to show that the sequence $(x_n(\cdot))$ constructed above possesses a subsequence uniformly converging to a certain $x_0(\cdot)$ and that x_0 is indeed a solution to problem (1). Since $x'_n(t)$ is contained in $B[0, 2m(t)]$, it is in fact true that there exists a subsequence - again denoted by (x_n) - uniformly converging to x_0 , and such that $x'_n \rightarrow x'_0$ weakly in $L^1(I)$. Since $x'_n(t) \in F(t, a_n^n(t))$, we have, by standard arguments,

$$x'_0(t) \in \text{co } F(t, x_0(t)) \quad \text{a.e. in } I.$$

Let $G(t) = \{x : F(t, x) \text{ is not convex}\}$, $T = \{t : x_0(t) \in G(t)\}$. We want to show that $x'_0(t) \in F(t, x_0(t))$ holds for almost all $t \in T$, since it clearly holds already a.e. on $I \setminus T$. Set $T_s^* = \{t : B(x_0(t), s) \subset H^*(t)\}$. One can prove that $G(t)$ is an open set contained in $H^*(t)$ and hence $\bigcup_{s>0} T_s^* \supseteq T$. Therefore we need only show that the sequence (x'_n) is L^1 -compact on T_s^* , for each fixed s . In order to do this we will prove, for any $\varepsilon > 0$, the existence of an ε -net for the set $\{x'_n(\cdot)\}$.

Fix $\varepsilon > 0$. There exists a $\delta > 0$ such that $m(E) < \delta$ implies $\int_E 2m(t) dt < \varepsilon/4$. Choose an integer m such that $1/2^m < \delta/2$, $\int_I \sum_{i \geq m} \eta^*(t, r_i, s_i) dt < \varepsilon/2$, and $r_i < s/4$, $s_i < s/4$ for $i \geq m$. For $t \in \bigcap_{i \geq m} [(I \setminus E_i) \cap \{t : a_i^{n+p}(t) \in K^*(t, s_i)\}]$, we have

$$|u(a_1, \dots, a_{n+p})(t) - u(a_1, \dots, a_n)(t)| \leq \sum_{i=n}^{n+p-1} \alpha^*(t, r_i, a_i)$$

and, since by construction we have

$$x'_{n+p}(t) = u(a_1^{n+p}(t), \dots, a_{n+p}^{n+p}(t))(t),$$

$$|x'_{n+p}(t) - u(a_1^{n+p}(t), \dots, a_n^{n+p}(t))(t)| \leq \sum_{i=n}^{n+p-1} \alpha^*(t, r_i, a_i^{n+p}(t))$$

By (9) , $|x_n(t) - a_i^n(t)| \leq r_i$, and hence there is an N such that if $n \geq N$, $i \geq n$, and $p \geq 0$, then

$$\begin{aligned} |a_i^{n+p}(t) - x_0(t)| &\leq |a_i^{n+p}(t) - x_{n+p}(t)| + |x_{n+p}(t) - x_0(t)| \\ &\leq r_i + s/4 < s/2 . \end{aligned}$$

(N can be chosen in order to be greater than m). Hence we have $B(a_i^{n+p}(t), 2s_i) \subset B(a_i^{n+p}(t), s/2) \subset B(x_0(t), s) \subset H^*(t)$ if $t \in T_s^*$ and $n \geq N$, $i \geq n$, $p \geq 0$. Thus, $a_i^{n+p}(t) \in K^*(t, s_i)$ and so

$$|x'_{n+p}(t) - u(a_1^{n+p}(t), \dots, a_n^{n+p}(t))(t)| \leq \sum_{i=n}^{n+p-1} \eta^*(t, r_i, s_i)$$

for $t \in \bigcap_{i=n}^{n+p-1} (I \setminus E_i) \cap T_s^*$. So, if we set $E = (\bigcup_{i \geq m} E_i) \cap T_s^*$, we obtain

$$\begin{aligned} &|x'_{n+p}(t) - u(a_1^{n+p}(t), \dots, a_n^{n+p}(t))(t)|_{L^1(T_s^*)} \\ &\leq \int_E |x_{n+p}(t) - u(a_1^{n+p}(t), \dots, a_n^{n+p}(t))(t)| dt + \int_{I \setminus E} \left[\sum_{i \geq m} \eta^*(t, r_i, s_i) \right] dt \\ &\leq 2 \int_E 2m(t) dt + \varepsilon/2 \\ &< \varepsilon . \end{aligned}$$

Remember now that the set $\{u(a_1^{n+p}(.), \dots, a_n^{n+p}(.))(.) : p = 0, 1, \dots\}$, n fixed, is finite. Hence the above inequalities tell us we have found the finite ε -net we were looking for. The set $\{x'_n(.) : n \geq 1\}$ is therefore L^1 -compact in $L^1(T_S^*)$.

We can conclude that (x'_n) converges strongly in $L^1(T_S^*)$ and pointwisely almost everywhere to x'_0 . By the facts that $|x_n(t) - a_n^n(t)| \rightarrow 0$ and $x'_n(t) \in F(t, a_n^n(t))$, recalling that $F(t, .)$ has closed graph, we can conclude that $x'_0(t) \in F(t, x_0(t))$ a.e. on T_S^* . Since $T \subset \bigcup T_S^*$, the proof is complete. ■

5. LOWER SEMICONTINUOUS PERTURBATIONS OF MAXIMAL MONOTONE
DIFFERENTIAL INCLUSIONS

5.1. Introduction.

In [10], Cellina and Marchi proved an existence result for differential inclusions of the form

$$(1) \quad \dot{x} \in -Ax + F(t, x) \quad ,$$

where A is a maximal monotone operator and F is a continuous map with compact (not necessarily convex) values which verifies a sublinear growth condition. The main tool used in their proof is a continuous selection theorem for the map

$$(2) \quad x \mapsto \{u \in L^1(I) : u(t) \in F(t, x(t)) \text{ a.e.}\}$$

defined on a compact subset of $L^1(I, \mathbb{R}^n)$. This approach goes back to a paper of Antosiewicz and Cellina [1], who considered the special case $A = 0$ with no convexity assumptions on the values of F . The results in [1] were generalized by Bressan [4] and Lojasiewicz [25] assuming the map F to be:

- (a) jointly measurable in (t, x)
- (b) lower semicontinuous in x .

In this paper we show that (1) still has a solution if A is a maximal monotone operator and F satisfies only (a) and (b) above and the same sublinear growth condition. Our proof follows the same fixed point argument of [10] and is based on a selection theorem of Fryszkowski [14], which contains the selection theorems used in [4], [1] and [25]. In fact, Fryszkowski's result permits a general treatment of operators of the type (2). The main part of this paper consists thus in proving that the operator (2) satisfies the assumptions of Fryszkowski's theorem.

5.2. Assumptions and statement of the main result.

In what follows, A is a maximal monotone operator in \mathbb{R}^n , i.e. a set-valued map from a subset $D(A)$ of \mathbb{R}^n into the subsets of \mathbb{R}^n , with the following two properties:

$$(A1) \quad \forall x_1, x_2 \in D(A), \quad \forall v_i \in Ax_i, \quad i = 1, 2, \\ \langle v_1 - v_2, x_1 - x_2 \rangle \geq 0 ;$$

$$(A2) \quad \text{the range of } I + A \text{ is all of } \mathbb{R}^n .$$

It is known that $\overline{D(A)}$ is convex, and that Ax is convex closed for any $x \in D(A)$ (see [6]) .

We will consider a map F from $[a, +\infty) \times \overline{D(A)}$ into the compact subsets of \mathbb{R}^n with the following properties:

$$(F1) \quad F(.,.) \text{ is } \mathcal{L} \otimes \mathcal{B} \text{ -measurable, i.e. for any closed set } C \subset \mathbb{R}^n \text{ the set}$$

$$F^-(C) := \{ (t, x) \in [a, +\infty) \times \overline{D(A)} : F(t, x) \cap C \neq \emptyset \}$$

belongs to the σ -algebra generated by the sets of the form $L \times B$, where L is a Lebesgue measurable subset of $[a, +\infty)$ and B is a Borel subset of $\overline{D(A)}$;

$$(F2) \quad \text{for each } t \geq a, F(t, .) \text{ is lower semicontinuous, i.e. for any closed set } C \subset \mathbb{R}^n \text{ the set}$$

$$F(t, .)^+(C) := \{ x \in \overline{D(A)} : F(t, x) \subset C \}$$

is closed in \mathbb{R}^n ;

$$(F3) \quad \text{there exist two non-negative locally integrable functions } a, b : [a, +\infty) \rightarrow \mathbb{R} \text{ such that, for every } (t, x) \in [a, +\infty) \times \overline{D(A)},$$

$$|F(t, x)| := \sup\{ |y| : y \in F(t, x) \} \leq a(t) |x| + b(t) .$$

In the present paper we study the existence of solutions to

the initial value problem

$$(P) \quad \dot{x} \in -Ax + F(t, x) \quad , \quad x(a) = x^0 \in \overline{D(A)} .$$

By a solution of (P) we mean a function $x \in C([a, +\infty), \mathbb{R}^n)$ which is absolutely continuous on every compact subset of $(a, +\infty)$ and is such that $x(a) = x^0$ and $x(t) \in D(A)$ for a.e. $t > a$ and, for some measurable selection $f(\cdot)$ from $F(\cdot, x(\cdot))$,

$$(P_f) \quad \dot{x} \in -Ax + f(t) \quad \text{for a.e. } t \geq a$$

(see [10] and [2]).

Our main result is the following:

THEOREM 5.2.1 If A is a maximal monotone operator and (F1) - (F3) hold, then problem (P) has a solution for any $x^0 \in \overline{D(A)}$.

5.3. Some known results .

In this section we state some known facts which will be used in the following. The first lemma illustrates the properties of a maximal monotone differential inclusion. For any compact interval I in $[a, +\infty)$, we denote by $|\cdot|_{i,I}$ the usual norm in $L^i(I) := L^i(I, \mathbb{R}^n)$, and we set $L_{loc}^i([a, +\infty)) := L_{loc}^i([a, +\infty), \mathbb{R}^n)$ ($i = 1$ or $i = \infty$).

LEMMA 5.3.1. ([2, Thm 1.2]) For any $f \in L_{loc}^1([a, +\infty))$ and any initial value $x^0 \in \overline{D(A)}$ there exists a unique solution u_f to (P_f) . For every $t \geq a$,

$$|u_f(t) - u_g(t)| \leq \int_a^t |f(s) - g(s)| \, ds$$

and, given any interval $I := [\tau, \tau + T]$, there exists a constant C

depending only on A such that

$$|\dot{u}_f|_{1,I} \leq C [(1 + T + |f|_{1,I}) \cdot (1 + |u_f|_{\infty,I}) + |u_f(\tau)|^2] .$$

As a straightforward consequence of Lemma 3.1 we have that the map $i : L^1_{loc}([a, +\infty)) \rightarrow L^1_{loc}([a, +\infty))$

$$f \mapsto u_f$$

is well - defined and continuous. The next lemma gives a kind of a priori estimate on the solutions of (P_f) . We denote by $u_0(\cdot)$ the solution of (P_f) with $f = 0$.

LEMMA 5.3.2 ([10, Lemma 2.1]) . Set

$$\psi(t) = \int_a^t (a(s) |u_0(s)| + b(s)) \cdot \exp\left(\int_s^t a(l) dl\right) ds .$$

Fix a function $w : [a, +\infty) \rightarrow \overline{D(A)}$ and let $f(\cdot)$ be a measurable selection from $F(\cdot, w(\cdot))$. The following holds:

$$\text{if } |w(t) - u_0(t)| \leq \psi(t) \quad \text{then also} \quad |u_f(t) - u_0(t)| \leq \psi(t) .$$

We now need the following

DEFINITION . A subset H of $L^1(I)$ is called decomposable if, whenever $u, v \in H$ and E is a measurable set in I , we have $u \chi_E + v \chi_{I \setminus E} \in H$. By $\text{Dec } L^1(I)$ we denote the set of all closed and decomposable subsets of $L^1(I)$.

The following proposition will play a central role in the proof of our result.

PROPOSITION 5.3.3 (Fryszkowski) . Let S be a compact metric space and $G : S \rightarrow \text{Dec } L^1(I)$ be a lower semicontinuous multivalued map. Then there exists $g : S \rightarrow L^1(I)$, a continuous selection from G .

For the proof, see [14] and [5, Thm 3] .

In the following $h^*(A,B)$ will denote the separation of a set A from a set B , i.e.

$$h^*(A,B) := \sup_{\alpha \in A} d(\alpha,B) .$$

5.4. Proof of the main result.

In order to apply the selection theorem of Fryszkowski, we need the following result .

PROPOSITION 5.4.1. Let F be as in Section 2 , I be a compact interval in $[a,+\infty)$ and let K be a compact subset of $L^1(I)$, bounded in $L^\infty(I)$. Then the operator

$$G : K \rightarrow \text{Dec } L^1(I) \\ x \mapsto \{u \in L^1(I) : u(t) \in F(t, x(t)) \text{ for a.e. } t \in I\}$$

is well - defined and lower semicontinuous.

Proof. It is easily seen that $G(x_1) = G(x_2)$ whenever $x_1(\cdot) = x_2(\cdot)$ a. e. . Moreover $G(x)$ clearly is decomposable, for any $x \in K$.

In order to prove the lower semicontinuity, let C be a closed subset of $L^1(I)$ and let (x_n) be a sequence in K converging in $L^1(I)$ to some x_0 in K and such that $G(x_n) \subset C$. We just need to prove that $G(x_0) \subset C$ or, since C is closed, that

$$(3) \quad h^*(G(x_0), G(x_n)) \rightarrow 0 \quad \text{as } n \rightarrow \infty .$$

Let \hat{x}_n (resp. \hat{x}_0) be Borel functions such that $\hat{x}_n = x_n$ a.e. (resp. $\hat{x}_0 = x_0$ a.e.) . We begin by proving the following

CLAIM :
$$\int_I h^*(F(t, \hat{x}_0(t)), F(t, \hat{x}_n(t))) dt \rightarrow 0$$

as $n \rightarrow \infty$.

Proof of the Claim . Set

$$h_n(t) = h^*(F(t, \hat{x}_0(t)), F(t, \hat{x}_n(t))) \quad , \quad \eta_n = \int_I h_n(t) dt \quad .$$

First of all we remark that the maps

$$t \mapsto F(t, \hat{x}_0(t)) \quad , \quad t \mapsto F(t, \hat{x}_n(t))$$

are measurable. Next we show that $h_n(\cdot)$ is measurable. By Theorem 3.5 e) in [19], the map

$$(t, z) \mapsto d(z, F(t, \hat{x}_n(t)))$$

is Carathéodory, and by Theorem 6.5 in the same paper the multivalued map given by

$$\Phi(t) = \{d(z, F(t, \hat{x}_n(t))) : z \in F(t, \hat{x}_0(t))\}$$

is weakly measurable. Hence Theorem 6.6 again in [19] gives the measurability of $h_n(\cdot)$.

Now we will prove that every subsequence (η_{n_k}) of (η_n) has a subsequence converging to 0. In fact $(x_{n_k}(\cdot))$ contains a subsequence (still denoted $(x_{n_k}(\cdot))$) converging to $x_0(\cdot)$ a.e.. Then the lower semicontinuity of $F(t, \cdot)$ together with the fact that the values of F are compact imply that $h_{n_k}(t) \rightarrow 0$ for a.e. $t \in I$. Moreover, by (F3),

$$\begin{aligned} h_{n_k}(t) &= h^*(F(t, \hat{x}_0(t)), F(t, \hat{x}_{n_k}(t))) \\ &\leq h^*(F(t, \hat{x}_0(t)), \{0\}) + h^*(\{0\}, F(t, \hat{x}_{n_k}(t))) \\ &\leq |F(t, \hat{x}_0(t))| + |F(t, \hat{x}_{n_k}(t))| \\ &\leq a(t) \{ |\hat{x}_0(t)| + |\hat{x}_{n_k}(t)| \} + 2b(t) \\ &\leq 2 \{M a(t) + b(t)\} \end{aligned}$$

for a suitable constant M . The Lebesgue dominated convergence theorem gives

$$\eta_{n_k} = \int_I h_{n_k}(t) dt \rightarrow 0 \quad \text{as } k \rightarrow \infty$$

and this proves the claim.

Finally, in order to prove (3), fix $u_0 \in G(x_0)$ and consider the multivalued function

$$\Gamma_n : t \mapsto \overline{B}(u_0(t), h_n(t) + 1/n) \cap F(t, \hat{x}_n(t)),$$

which clearly has closed nonempty values. To show that Γ_n is measurable, it is enough to prove the measurability of the map

$$\Psi : t \mapsto \overline{B}(u_0(t), h_n(t) + 1/n)$$

(see [19, Thm. 4.1]). But Ψ is the composition of the measurable map $t \mapsto (u_0(t), h_n(t) + 1/n)$ with the continuous map $(x, r) \mapsto \overline{B}(x, r)$, hence is measurable. Therefore we can choose a L^1 selection u_n from Γ_n . Clearly $u_n \in G(x_n)$ and we have

$$\|u_n - u_0\|_{1,I} \leq \int_I (h_n(t) + 1/n) dt.$$

By the above claim, the r.h.s. of this inequality converges to 0 as $n \rightarrow \infty$, uniformly in u_0 . Hence (3) is proved.

Proof of Theorem 5.2.1. We will follow essentially the proof of Thm. 2.2 in [10]. Define K as the closure in $L^1_{loc}([a, +\infty))$ of the set of those absolutely continuous functions v having the following properties:

- (i) $v(a) = x^0$ and $v(t) \in \overline{D(A)}$ ($t \geq a$);
- (ii) $|v(t) - u_0(t)| \leq \psi(t)$ ($t \geq a$);
- (iii) for every interval $I = [\tau, \tau + T]$ ($\tau \geq a$),

$$\|\dot{v}\|_{1,I} \leq C [(1 + T + N(I))(1 + M(I)) + r^2(\tau)],$$

where

$$\begin{aligned} M(I) &= \exp\left(\int_I a(t) dt\right) \cdot \int_I (a(t)|u_0(t)| + b(t)) dt + |u_0|_{\infty, I}, \\ N(I) &= M(I) \int_I a(t) dt + \int_I b(t) dt, \\ r(\tau) &= |u_0(\tau)| + 2 \int_a^\tau (a(t)|u_0(t)| + b(t)) \exp\left(2 \int_t^\tau a(s) ds\right) dt. \end{aligned}$$

It is easily seen (as in [10]) that K is nonempty, convex, compact in $L^1_{loc}([a, +\infty))$ and bounded in $L^\infty_{loc}([a, +\infty))$. Set, for $n = 1, 2, \dots$, $I_n := [a, a + n]$, $K_n := \{v|_{I_n} : v \in K\} \subset L^1(I_n)$.

We will construct recursively a sequence of continuous maps $g_n: K_n \rightarrow L^1(I_n)$ verifying, for each $x \in K_n$,

$$(4) \quad g_n(x)(t) \in F(t, x(t)) \quad \text{for a.e. } t \in I_n,$$

$$(5) \quad \text{if } n > 1, \quad g_n(x)(t) = g_{n-1}(x)(t) \quad \text{for a.e. } t \in I_{n-1}.$$

Define the operator G_1 in the same way as G with K_1 in place of K , and, for $n > 1$, assuming that g_{n-1} has already been defined, define the operator

$$G_n : K_n \rightarrow \text{Dec } L^1(I_n) \\ x \mapsto \{u|_{I_n} : u(t) \in F(t, x(t)) \text{ for a.e. } t \in I_n \text{ and} \\ u(t) = g_{n-1}(x)(t) \text{ for a.e. } t \in I_{n-1}\}.$$

By Proposition 5.4.1 and Proposition 5.3.3, the operator G_1 has a continuous selection g_1 . Therefore we can consider the operator G_2 , and it is not difficult to see, in view of Proposition 5.4.1, that it is lower semicontinuous. Applying again Proposition 5.3.3, we see that G_2 admits a continuous selection g_2 which by construction satisfies (4) and (5). Similarly, for any $n > 2$, we obtain g_n from g_{n-1} satisfying (4) and (5). Now we define

$$g : K \rightarrow L^1_{loc}([a, +\infty))$$

by setting $g(x)|_{I_n} = g_n(x)$, $n = 1, 2, \dots$.

Using (4) and (5), it is easy to see that g is well-defined and continuous and satisfies

$$g(x)(t) \in F(t, x(t)) \quad \text{for a.e. } t \geq a.$$

To conclude the proof we define, as in [10],

$$s : K \rightarrow L^1_{\text{loc}}([a, +\infty))$$

$$x \rightarrow i(g(x)) \quad .$$

The map s is continuous and, by Lemma 53.2 $s(K) \subset K$.

Since K is compact and convex, the theorem of Schauder - Tichonov yields a fixed point of s , which is a solution to (P) .

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