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Area of Mathematics

Minimality and stability results for a class of free-discontinuity and nonlocal isoperimetric problems

Ph.D. Thesis

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Academic Year 2012/2013

Il presente lavoro costituisce la tesi presentata da Marco Bonacini, sotto la direzione del Prof. Massimiliano Morini, al fine di ottenere l'attestato di ricerca post-universitaria *Doctor Philosophiæ* presso la SISSA, Curriculum in Matematica Applicata, Area di Matematica. Ai sensi dell'art. 1, comma 4, dello Statuto della Sissa pubblicato sulla G.U. no. 36 del 13.02.2012, il predetto attestato è equipollente al titolo di *Dottore di Ricerca in Matematica.*

Trieste, Anno Accademico 2012-2013.

Ringraziamenti

È prassi, al termine di un lavoro di tesi, dare la precedenza nei ringraziamenti al proprio relatore; in questo caso Massimiliano Morini merita a buon diritto la prima posizione. La sua costante e meticolosa guida mi ha permesso di portare a compimento questo percorso; dei suoi preziosi insegnamenti saprò fare tesoro anche nel futuro. La pazienza, infine, è virtù di cui in questi anni non ha fatto risparmio.

La SISSA, *in primis* nelle persone dei suoi docenti, è stato luogo di crescita e maturazione professionale senza eguali: averlo frequentato è motivo di orgoglio e gratitudine.

Un'esperienza significativa come quella triestina è preziosa soprattutto per le persone che la hanno accompagnata. Un collaboratore fenomenale come Riccardo non si può rimpiazzare: l'augurio è che con lo stesso entusiasmo di questi giorni si possa continuare a ragionare insieme di matematica. Se la cultura cinese mi è ora più familiare, e non cessa di affascinarmi, lo devo a Lei. Ricordo anche i tanti momenti in cui la presenza di Alberto, Maurizio e Mauro ha portato grande allegria (con un giusto tocco demenziale).

Per Elisa, Flaviana e Simone ogni parola correrebbe il rischio di essere inadeguata.

Trieste, 25 Ottobre 2013

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Introduction

In this thesis we study some relevant *free-discontinuity* and *nonlocal* energies arising in several physical systems modeled by non-convex variational problems. The *trait d'union* between the different functionals that we consider lies in the fact that they share a strong geometrical content: indeed, they can all be regarded as *nonlocal variants of the perimeter functional*, where the non-locality may be given, depending on the context, by an elastic term or by a long-range interaction of Coulombic type.

An interesting and mathematically challenging peculiarity of the physical systems modeled by this class of energies is the emergence of complex patterns in the observed configurations, as well as the formation of morphological instabilities of interfaces between elastic phases. These phenomena are understood as the result of the competition between different forms of interaction: indeed, the common feature of the functionals that we study is the presence of two competing forms of energy, namely a local geometric surface energy of perimeter type and a bulk nonlocal contribution. While the first one favors phase separation along sharp interfaces, the latter drives the system towards scattered or oscillating configurations: as a result, the interaction between the two competing terms makes the minimization of such energies highly nontrivial, and complex, interesting behaviors emerge.

The main focus of this thesis is to establish local and global minimality results for some paradigmatic examples encoding the main features of energies of this kind. In particular, we will consider the following three different models:

- the Mumford-Shah functional, which is the prototype of free-discontinuity problems, introduced in the context of image segmentation and appearing also in the variational formulation of fracture mechanics;
- a model functional related to the epitaxial growth of elastic films over a flat substrate in presence of a mismatch strain, which provides an example of the appearing of morphological instabilities and can be regarded as an instance of the so-called *stress-driven rearrangement instabilities*;
- a nonlocal variant of the standard perimeter, with the addition of a repulsive long-range interaction, which has recently received attention for its connection with some relevant physical models (like diblock copolymers).

The unifying technical approach to the study of these problems is based on the investigation of second order necessary and sufficient minimality conditions, which should rigorously determine the theoretical connection between the notions of *stability* and *minimality*. In particular, we aim at establishing a minimality sufficiency criterion stating, roughly speaking, that *strictly stable regular critical configurations are local minimizers*. With such a condition in hand, we can address the stability issue for critical configurations in order to provide a picture of the energy landscape of the functionals under consideration; in some cases, also global minimality results are established.

Besides providing an analytical tool which has proven effective in different situations, the previous criterion has an independent theoretical interest, as it can be regarded as the analog, in this free-discontinuity framework, of the classical minimality sufficient condition based on the positivity of the second variation. Indeed, although the question whether strict stability implies local minimality is very classical for the standard functionals of the Calculus of Variations, its investigation in the context of free-discontinuity problems seems to have been started only in recent years, by F. Cagnetti, M.G. Mora and M. Morini in [19], and by N. Fusco and M. Morini in [45]. Following their approach, similar results have been obtained for different energy functionals: see [1, 12, 14, 15, 20, 56].

The general strategy leading to the proof of such a minimality criterion is close in spirit to the one devised in [45] and consists mainly of two fundamental steps. In the first part, one shows that the strict positivity of the second variation of the total energy guarantees a minimality property weaker than the desired one (usually, we prove minimality with respect to small $W^{2,p}$-perturbations of the interface or of the discontinuity set). This is accomplished by a careful analysis of the continuity properties of the second variation, which usually requires delicate regularity estimates for elliptic PDEs combined with geometric arguments. Such a minimality property plays the role, in this framework, of the classical notion of weak minimizer for the standard functionals of the Calculus of Variations.

The second step of the outline consists in showing that weak local minimizers are in fact local minimizers with respect to the desired stronger topology. This is achieved through a contradiction argument, with an appeal to the regularity theory of quasi-minimizers of the area functional and of the Mumford-Shah functional (see Sections 1.2 and 1.3). Clearly, the implementation of this strategy is different according to the different problems and contexts; below we sketch the main steps of this second part of the proof in the particular case of the Mumford-Shah functional, in order to give a flavor of this type of arguments. We remark that similar ideas have been used also in [1] for a nonlocal isoperimetric problem related to the modeling of diblock copolymers, and in [27], where the appeal to the regularity of quasi-minimizers appears for the first time in the context of isoperimetric inequalities and leads to an alternative proof of the quantitative isoperimetric inequality.

We now turn to the description in deeper details of the three different variational models whose analysis forms the core of this thesis.

The Mumford-Shah functional. In Chapter 2 (which contains the results of [15]) we undertake the study of second order minimality conditions for the *Mumford-Shah functional*. Such a functional represents the prototype of the so-called "free-discontinuity problems", according to the expression coined by E. De Giorgi in [32] to denote a class of variational problems whose common feature is the simultaneous presence of volume energies, concentrated on N-dimensional sets, and surface terms, concentrated on $(N-1)$-dimensional sets: in this context the unknown of the problem is typically a pair (K,u), where K is a closed set on which the surface energy is supported (which is not *a priori* fixed), and u is a function defined in the complement of K.

The minimization of the Mumford-Shah functional was proposed in the seminal papers [68, 69] in the context of image segmentation, and plays an important role also in variational models for fracture mechanics. Its homogeneous version in a bounded open set $\Omega\subset\mathbb{R}^2$ is defined over admissible pairs (Γ,u), with Γ closed subset of $\overline{\Omega}$ and $u\in H^1(\Omega\setminus\Gamma)$, as

$$\mathcal{MS}(\Gamma,u) := \int_{\Omega\setminus\Gamma} |\nabla u|^2\,\mathrm{d}x + \mathcal{H}^1(\Gamma\cap\Omega), \tag{1}$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure. Since its introduction, several results concerning the existence and regularity of minimizers, as well as the structure of the optimal set, have been obtained (see, *e.g.*, [8] for a detailed account on this topic). We shall mention that an existence theory passes through a suitable weak formulation of the problem, proposed by E. De Giorgi and L. Ambrosio in [33], where the relaxed version of the functional is defined on the space $SBV(\Omega)$ of *special functions with bounded variation* as

$$\overline{\mathcal{MS}}(u) := \int_\Omega |\nabla u|^2 \, dx + \mathcal{H}^1(S_u),$$

S_u denoting the jump set of $u \in SBV(\Omega)$.

Due to the deep lack of convexity of the functional (1), one cannot expect in general that a critical point is also a minimizer; nonetheless, it was shown in [65] by means of a calibration method that critical points are in fact global minimizers, with respect to their own boundary conditions, in sufficiently small domains. Such a smallness assumption was afterwards interpreted as a second order condition by F. Cagnetti, M.G. Mora and M. Morini in [19]. Here a proper notion of second variation was introduced by considering one-parameter families of perturbations of the regular part of the discontinuity set: given a vector field X and the associated flow Φ_t, the second variation of $\mathcal{MS}$ at (Γ, u) along the flow Φ_t was defined as $\frac{d^2}{dt^2}\mathcal{MS}(\Phi_t(\Gamma), u_{\Phi_t})|_{t=0}$, where u_{Φ_t} is the minimizer in $H^1(\Omega \setminus \Phi_t(\Gamma))$ of the first term of the functional, under the same Dirichlet conditions as u.

In [19] it was also shown that a critical point (Γ, u) with positive definite second variation minimizes the functional with respect to pairs of the form $(\Phi(\Gamma), v)$, where Φ is any diffeomorphism sufficiently close to the identity in the C^2-norm, with $\Phi - Id$ compactly supported in Ω, and $v \in H^1(\Omega \setminus \Phi(\Gamma))$ satisfies $v = u$ on $\partial\Omega$.

In the main result of Chapter 2 (Theorem 2.7) we strongly improve the aforementioned result, by showing that in fact the positive definiteness of the second variation implies strict local minimality with respect to the weakest topology which is natural for this problem, namely the L^1-topology. To be more precise, we prove that if (Γ, u) is a critical point with positive second variation, then there exists $\delta > 0$ such that

$$\mathcal{MS}(\Gamma, u) < \mathcal{MS}(K, v)$$

for all admissible pairs (K, v), provided that v attains the same boundary conditions as u and $0 < \|u - v\|_{L^1(\Omega)} < \delta$. We mention that for technical reasons the boundary conditions imposed here are slightly different from those considered in [19], as we prescribe the Dirichlet condition only on a portion $\partial_D\Omega \subset \partial\Omega$ away from the intersection of the discontinuity set Γ with $\partial\Omega$.

We regard this result as a first step of a more general study of second order minimality conditions for free-discontinuity problems. Besides considering more general functionals, it would be very interesting to extend our local minimality criterion to the case of discontinuity sets with singular points, like the so-called "triple junction", where three lines meet forming equal angles of $2\pi/3$, and the "crack-tip", where a line terminates at some point.

As anticipated in the first part of this Introduction, the general strategy of the proof consists of two fundamental steps. First, one shows that strict stability is sufficient to guarantee local minimality with respect to perturbations of the discontinuity set which are close to the identity in the $W^{2,\infty}$-norm (see Theorem 2.27). This amounts to adapting to our slightly different context the techniques developed in [19], with the main new technical difficulties stemming from allowing also boundary variations of the discontinuity set.

The second step of the outline consists in showing that the above local $W^{2,\infty}$-minimality in fact implies the desired local L^1-minimality. This is obtained by showing firstly, as an intermediate result, that the local $W^{2,\infty}$-minimality implies minimality with respect to small $C^{1,\alpha}$-perturbations of the discontinuity set. This is perhaps the most technical part of the proof. The main idea is to restrict the functional to the class of pairs (Γ, v) such that $\|v - u\|_{W^{1,\infty}(\Omega\setminus\Gamma)} \leq 1$, so that the Dirichlet energy behaves like a volume term, and $\mathcal{MS}$ can be regarded as a volume perturbation of the area functional. This allows to use the regularity theory for quasi-minimizers of the area functional (see Section 1.2) to deduce the local $C^{1,\alpha}$-minimality through a suitable contradiction argument.

A contradiction argument is also finally used to establish the sought L^1-minimality. To give a flavor of this type of reasoning, we sketch here the main steps of this last part of the proof. One assumes by contradiction the existence of admissible pairs (Γ_n, u_n) with u_n converging to u in $L^1(\Omega)$, such that the minimality inequality fails along the sequence:

$$\mathcal{MS}(\Gamma_n, u_n) \leq \mathcal{MS}(\Gamma, u) \tag{2}$$

for every n. By an easy truncation argument, we may also assume that $\|u_n\|_\infty \leq \|u\|_\infty$, so that $u_n \to u$ in $L^p(\Omega)$ for every $p \geq 1$. Then we replace each (Γ_n, u_n) by a new pair (K_n, v_n) chosen as solution to a suitable penalization problem, namely

$$\min\Big\{\mathcal{MS}(K,w)+\beta\Big(\sqrt{(\|w-u\|^2_{L^2(\Omega)}-\varepsilon_n)^2+\varepsilon_n^2}-\varepsilon_n\Big)\,:\,(K,w)\text{ admissible, }w=u\text{ on }\partial_D\Omega\Big\}$$

with $\varepsilon_n := \|u_n - u\|^2_{L^2(\Omega)} \to 0$, and $\beta > 0$ large enough. Note that, by (2) and by minimality, we have

$$\mathcal{MS}(K_n, v_n) \leq \mathcal{MS}(\Gamma_n, u_n) \leq \mathcal{MS}(\Gamma, u). \tag{3}$$

The advantage is now that the pairs (K_n, v_n) satisfy a uniform *quasi-minimality* property (see Definition 1.10). It is easy to show that, up to subsequences, the sequence (K_n, v_n) converges to a minimizer of the limiting problem

$$\min\Big\{\mathcal{MS}(K,w)+\beta\|w-u\|^2_{L^2(\Omega)}\,:\,(K,w)\text{ admissible, }w=u\text{ on }\partial_D\Omega\Big\}. \tag{4}$$

Now a calibration argument developed in [67] implies that we may choose β so large that (Γ, u) is the unique global minimizer of (4). With this choice of β we have in particular that $v_n \to u$ in $L^1(\Omega)$, and in turn, by exploiting the regularity properties of quasi-minimizers of the Mumford-Shah functional, we infer that the corresponding discontinuity sets K_n are locally $C^{1,\alpha}$-graphs and converge in the $C^{1,\alpha}$-sense to Γ. Recalling (3), we have reached a contradiction to the $C^{1,\alpha}$-minimality of (Γ, u).

A variational model in epitaxial films theory. A paradigmatic example of the occurrence of morphological instabilities of interfaces is given by the mechanism of epitaxial deposition of an elastic film on a relatively thick substrate, in presence of a lattice mismatch at the interface between film and substrate. A threshold effect, known as the Asaro-Grinfeld-Tiller (AGT) instability, characterizes the observed configurations: after reaching a critical value of the thickness, a flat layer becomes morphologically unstable, and typically the free surface starts to develop irregularities (see, for instance, [51]). This phenomenon is understood to be governed by the competition between two opposing forms of energy, the bulk elastic energy and the surface energy, and is the subject of investigation of Chapter 3 (which is based on the results contained in [13] and [12]).

A proper variational formulation of the problem is proposed in [16]: here the film is modeled as a linear elastic solid grown on a flat substrate in a two-dimensional framework (corresponding to three-dimensional configurations with planar symmetry); equilibrium configurations correspond to *volume-constrained* minimizers of the total energy

$$F(h,u) := \int_{\Omega_h} Q(E(u))\,\mathrm{d}z + \mathcal{H}^1(\Gamma_h)\,, \tag{5}$$

where h is a periodic, non-negative function whose subgraph $\Omega_h = \{(x,y) : 0 < y < h(x)\}$ represents the reference configuration of the film, and whose graph Γ_h describes its free profile. The stored elastic energy is assumed to be the integral over the reference configuration of a linear function Q of the symmetrized gradient $E(u)$ of the displacement u. A Dirichlet boundary condition is imposed at the interface between the film and the flat substrate, forcing the film to be elastically strained. Since we are imposing a volume constraint, we may interpret the minimization of the functional (5) as a variant of the isoperimetric problem, where an elastic term is introduced.

Existence of minimizers is established in [16] through the relaxation of the functional, while in [42] a regularity theory for local minimizers is developed for a slightly different model (see also [36]): it is shown that the profile of a volume constrained local minimizer may exhibit at most a finite number of cusp points (possibly leading to vertical cracks in the material), being analytic away from these singularities. A rigorous justification of the zero-contact-angle condition between film and substrate, in the wetting regime, is also attained.

In [45] qualitative properties of equilibrium configurations are studied by means of a new local minimality criterion based on the positivity of the second variation of the total energy: in particular, the authors provide a detailed description of the energy landscape of the functional and determine analytically the critical threshold for the local minimality of the flat configuration. The second variation of the functional (5) at a regular critical pair (h,u), along the direction of a given variation ϕ with zero mean value, is defined as

$$\frac{d^2}{dt^2}F(h+t\phi,u_t)|_{t=0}\,, \tag{6}$$

where u_t is the minimizer of the elastic energy in $\Omega_{h+t\phi}$ under the prescribed boundary conditions. Then it is shown that the strict positivity of the associated quadratic form implies minimality of (h,u) with respect to competitors whose free profile is in a L^∞-neighborhood of the graph of h.

We now come to the description of the results contained in Chapter 3. In the first part of the chapter (Section 3.1) we investigate how the presence of *surface anisotropy* affects the resulting equilibrium configurations: the length of the free profile of the film in (5) is replaced by a term depending also on the orientation of the normal vector to Γ_h, of the form

$$\int_{\Gamma_h}\psi(\nu)\,\mathrm{d}\mathcal{H}^1\,, \tag{7}$$

where ψ is a convex, positively 1-homogeneous function of the normal ν to the surface of the film. The main information about the anisotropy is carried by the *Wulff shape* associated with ψ, which is the set that minimizes (7) under a volume constraint. We consider first the case of "weak" anisotropies, in which the surface density ψ satisfies a strong convexity condition (see (3.1)) and the corresponding Wulff shape is a regular set: after having observed that the derivation of the relaxed energy follows from the same arguments as in [16, 42], we show that the threshold effect that describes the stability of flat morphologies in the isotropic case remains valid; correspondingly we analytically determine the volume threshold of local minimality of the flat configuration (Theorem 3.18). These results are obtained by extending to this anisotropic framework the local minimality criterion established in [45].

An interesting new phenomenon occurs when considering "crystalline" anisotropies (meaning that the boundary of the Wulff shape associated with ψ contains a horizontal facet intersecting the y-axis): in this case *the AGT instability is suppressed*, that is the flat configuration is *always* a local minimizer, no matter how thick the film is (Theorem 3.53).

Starting from Section 3.2, we undertake the task of extending the sufficiency minimality criterion introduced in [45] to the physically relevant three-dimensional case and to a larger class of nonlinear elastic energies, which appear in the context of Finite Elasticity. In this setting the functional representing the energy of the system takes the form

$$F(h,u) := \int_{\Omega_h} W(Du)\,\mathrm{d}z + \int_{\Gamma_h} \psi(\nu)\,\mathrm{d}\mathcal{H}^{N-1}, \tag{8}$$

where W is now a regular function defined on an open subset of the space of $N\times N$ matrices with positive determinant. In this context, by *strong local minimizer* we mean a pair (h,u) which minimizes (8) among all configurations (g,v) such that g is in a small L^∞-neighborhood of h and satisfies the volume constraint $|\Omega_g| = |\Omega_h|$, and the gradients of the deformations Du, Dv are close in L^∞. Necessary conditions for local minimality are the first order conditions

$$\begin{cases} \mathrm{div}(DW(Du)) = 0 & \text{in } \Omega_h, \\ DW(Du)[\nu] = 0 & \text{on } \Gamma_h, \\ W(Du) + H^\psi = \text{const} & \text{on } \Gamma_h, \end{cases} \tag{9}$$

where H^ψ denotes the anisotropic mean curvature of Γ_h (see (1.6)).

In our main result we provide a sufficient condition for a *critical pair* (that is, a pair (h,u) satisfying (9)) to locally minimize the total energy: precisely, we show that *any regular critical configuration with strictly positive second variation is a strong local minimizer for F*, according to the previous definition (Theorem 3.45). We also prove a stronger result in the case of linear elasticity (see Theorem 3.46), namely we replace the L^∞-closeness of the deformation gradients appearing in the definition of local minimizer by a uniform bound on the Lipschitz constant of the deformations.

As before, this minimality criterion can be applied to the study of the local minimality of flat morphologies, when the amount of material deposited is small. Moreover, also in this case the presence of a flat horizontal facet in the Wulff shape associated with the anisotropy ψ eliminates the AGT instability.

We also mention that our result could be useful to deal with the three-dimensional version of the elastic film evolution by surface diffusion with curvature regularization, studied in [43] in the two-dimensional case. In particular, it is a natural question in this context to ask whether the strict positivity of the second variation guarantees the Lyapunov stability with respect to this evolution; we think that our criterion could be instrumental in establishing such a result.

One of the crucial difficulties that arise when treating the three-dimensional case is the lack of a regularity theory for minimizers of (8), which prevents us to fully extend the results of [45]. In fact, we remark that the minimality property that we are able to prove is weaker than the one considered in [45], as it requires the L^∞-closeness of the deformation gradients (or a bound on the Lipschitz constant of the deformation in the linear elastic case). While this constraint seems to be not too restrictive in the nonlinear case, we expect that in the linearized framework the local minimality should hold without such a condition; however, our strategy to improve the result in this direction needs a regularity theory which is not yet available in three dimensions.

A few comments on the strategy of the proof are in order. The following crucial observation is a consequence of the Implicit Function Theorem: starting from a regular pair (h,u) and assuming that the elastic second variation at u is uniformly positive in Ω_h (see condition (3.52)), it is possible to find a critical point u_g for the elastic energy in Ω_g (that is, a deformation satisfying the first two conditions of (9) in Ω_g), provided that g is sufficiently close to h in the $W^{2,p}$-topology (see Proposition 3.29). This allows us to define the second variation of the functional (8) at the critical pair (h,u) similarly to (6).

As we pointed out before, the proof of the minimality criterion is inspired by the two-steps strategy devised in [45]. Firstly, we show that the strict positivity of the second variation is sufficient, in dimension $N = 2, 3$, for a weaker notion of local minimality, namely with respect to competitors (g, v) with $\|g - h\|_{W^{2,p}}$ sufficiently small. Since the expression of the second variation involves the trace of the gradient of $W(Du)$ on Γ_h, a crucial point in the proof of this result consists in controlling this term in a proper Sobolev space of negative fractional order. We overcome this difficulty by proving careful new estimates for the elliptic system associated with the first variation of the elastic energy in Lemma 3.42, which provides a highly non-trivial generalization to the three-dimensional and nonlinear cases of the estimates proved in [45, Lemma 4.1].

The second part of the proof consists in showing that, in any dimension, the aforementioned weaker notion of minimality implies the desired strong local minimality. This is obtained by a contradiction argument, similar to the one previously described in the case of the Mumford-Shah functional: assuming the existence of a sequence (g_n, v_n) converging to (h, u) and violating the minimality of (h, u), one replaces (g_n, v_n) by a new pair (k_n, w_n) selected as solution to a suitable penalized minimum problem, whose energy is still below the energy of (h, u). Due to minimality, the pairs (k_n, w_n) enjoy better regularity properties: since the L^∞-bound on the deformation gradients allows us to regard the elastic energy as a volume perturbation of the surface area, we may appeal to the regularity theory for quasi-minimizers of the area functional to deduce the $C^{1,\alpha}$-convergence of k_n to h. In turn, with the aid of the Euler-Lagrange equations for the minimum problem solved by (k_n, w_n) we obtain the $W^{2,p}$-convergence of k_n to h, and we reach a contradiction to the local minimality of (h, u) with respect to $W^{2,p}$-perturbations established in the first step of the proof.

A nonlocal isoperimetric problem. The last problem that we consider is the following nonlocal variant of the isoperimetric problem:

$$\text{minimize} \qquad \mathcal{F}(E) := \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \,\mathrm{d}x\mathrm{d}y\,, \qquad |E| = m, \tag{10}$$

with $\alpha \in (0, N-1)$ and $\gamma > 0$, where $\mathcal{P}(E)$ denotes the standard perimeter of E in $\mathbb{R}^N$ and χ_E its characteristic function. In Chapter 4 (which contains the results of [14]) we provide an accurate description of the energy landscape of the family of functionals (10).

The energy (10) appears in the modeling of different physical problems. For instance, when $N = 3$ and $\alpha = 1$ it corresponds to the celebrated Gamow's water-drop model for the constitution of the atomic nucleus (see [46]), and it is also related, by Γ-convergence, to the Ohta-Kawasaki model for diblock copolymers (see [72]). For a more specific account on the physical background of this kind of problems, we refer to [70].

From a mathematical point of view, functionals of the form (10) recently drew the attention of many authors. The issue of existence and non-existence of global minimizers is investigated in [55, 57, 58, 62], and there is a growing literature on asymptotic regimes in bounded or periodic domains (see [24, 25, 28, 49, 50, 71, 77]). We mention also the paper [1], dealing with local minimizers in a periodic setting, whose results inspired also our analysis.

Once again, the main feature of the energy (10) is the presence of two competing terms, the sharp short-range interface energy, given by the standard perimeter, and the long-range repulsive interaction, represented by the double integral. Indeed, while the first term is minimized by the ball (by the isoperimetric inequality), the nonlocal term is in fact maximized by the ball, as a consequence of the Riesz's rearrangement inequality (see [61, Theorem 3.7]), and favors scattered or oscillating configurations.

We remark that, by scaling, minimizing (10) under the volume constraint $|E| = m$ is equivalent to the minimization of the functional

$$\mathcal{P}(E) + \gamma \left(\frac{m}{|B_1|}\right)^{\frac{N-\alpha+1}{N}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y$$

under the constraint $|E| = |B_1|$, where B_1 is the unit ball in $\mathbb{R}^N$. It is clear from this expression that, for small masses, the perimeter is the leading term and this suggests that in this case the ball should be the solution to the minimization problem; on the other hand, for large masses the nonlocal term becomes dominant and causes the existence of a solution to fail.

In fact it was proved, although not in full generality, that the functional (10) is uniquely minimized (up to translations) by the ball for every value of the volume below a critical threshold: in [58] for the planar case $N = 2$, in [57] for $3 \leq N \leq 7$, and in [55] for any dimension N but for $\alpha = N - 2$. Moreover, the existence of a critical mass above which the minimum problem does not admit a solution was established in [58] in dimension $N = 2$, in [57] for every N and for $\alpha \in (0, 2)$, and in [62] in the physical interesting case $N = 3$, $\alpha = 1$.

Here we aim at providing a more detailed picture of the energy landscape of the functional (10) by a totally different approach, based on the positivity of the second variation. The main findings of our analysis are the following. First, we confirm and strengthen some of the above results, proving in full generality that *the ball is the unique global minimizer for small masses*, without restrictions on the parameters N and α (Theorem 4.10).

Moreover, for α *small* we also provide a complete characterization of the ground state, showing that *the ball is the unique global minimizer, as long as a minimizer exists* (Theorem 4.11). Precisely, we show that there exists $m_1 > 0$ such that for $m \in (0, m_1)$ the ball is the unique global minimizer under the volume constraint $|E| = m$, while for $m > m_1$ a solution to the minimization problem fails to exist. More in general, in this regime we can write $(0,\infty) = \cup_{k=0}^{\infty}(m_k, m_{k+1}]$, with $m_0 = 0$, $m_{k+1} > m_k$, in such a way that for $m \in [m_{k-1}, m_k]$ a minimizing sequence for the functional is given by a configuration of at most k disjoint balls with diverging mutual distance (Theorem 4.12). The results stated in Theorem 4.11 and Theorem 4.12 are completely new (the first one was only known in the special case $N = 2$: see [58]).

Finally, we also investigate for the first time in this context the issue of *local minimizers*, that is, sets which minimize the energy with respect to competitors sufficiently close in the L^1-sense (where we measure the distance between two sets by the quantity (4.7), which takes into account the translation invariance of the functional). For any N and α we show the existence of a volume threshold below which the ball is also an isolated local minimizer, determining it explicitly in the three dimensional case with a Newtonian potential (Theorem 4.9).

One of the main tools in proving the aforementioned results is represented by Theorem 4.8, where we show that the strict positivity of the quadratic form associated with the second variation of $\mathcal{F}$ at a regular critical set E is a sufficient condition for local minimality with respect to L^1-perturbations. The general strategy to establish this theorem is mainly inspired, besides [45], by [1], which deals with energies in the form (10) in a periodic setting. Here we have to tackle the nontrivial technical difficulties coming from working with a more general nonlocal term (the exponent α is allowed to range in the whole interval $(0, N-1)$) and from the lack of compactness of the ambient space $\mathbb{R}^N$. Then we treat the global minimality issues described above, which require additional arguments and nontrivial refinements of the previous ideas.

An issue which remains unsolved concerns the structure of the set of masses for which the problem does not have a solution: is it always true that it has the form $(m, +\infty)$ for all admissible values of α and N? Notice that we provide a positive answer to this question in the case of α small. Another interesting question asks if there are other global (or local) minimizers different from the ball. Finally, our analysis leaves open the case of $\alpha \in [N-1, N)$, which seems to require different techniques.

Organization of the thesis. In a preliminary chapter (Chapter 1) we fix the main notation and we give an account of one of the main tools needed throughout the thesis, namely the regularity theory for quasi-minimizers of the area functional (Section 1.2) and of the Mumford-Shah functional (Section 1.3). In the final section of this preliminary chapter we also collect some definitions and properties of Sobolev spaces of fractional order (Section 1.4). The core of the thesis is made up of Chapter 2, Chapter 3 and Chapter 4, where the three different variational models described above are studied in details. In Appendix A we provide the proof of a density lower bound for the discontinuity set of quasi-minimizers of the Mumford-Shah functional in presence of mixed Dirichlet-Neumann boundary conditions: this is required in Chapter 2. Finally, Appendix B contains an auxiliary technical result needed in Section 3.6.

Bibliographic note. The results of Chapter 2 have been obtained in collaboration with M. Morini and appear in the paper [15]. The content of Chapter 3 corresponds to the two articles [13] and [12]. Finally, Chapter 4 describes the results of a joint work with R. Cristoferi (see [14]).

CHAPTER 1

Preliminaries

In this chapter we fix the main notation and we collect some preliminary results that we shall need in the following. In particular, in Section 1.1 we recall the definition of some tangential differential operators and related identities. A fundamental tool required throughout the thesis is the regularity theory for quasi-minimizers of the area functional and of the Mumford-Shah functional: a brief account on this topic, with particular emphasis on the properties that we need for our purposes, is given in Sections 1.2 and 1.3. Finally, in Section 1.4 we collect some definitions and results concerning fractional Sobolev spaces.

Main notation. Throughout the thesis, the scalar product of two vectors $x, y \in \mathbb{R}^N$ is denoted by $x \cdot y$, or equivalently by $\langle x, y \rangle$, and the associated norm by $|\cdot|$. The canonical basis of $\mathbb{R}^N$ is usually denoted by $(e_1, \ldots, e_N)$. The space $\mathbb{M}^N$ of $N \times N$ real matrices is endowed with the Euclidean scalar product $A : B := \mathrm{trace}\,(A^T B)$, where A^T is the transpose of A, and with the corresponding norm $|A|$. We denote by $\mathbb{M}^N_+$ the subset of matrices with positive determinant. The symbol I stands for the identity matrix, while $Id : \mathbb{R}^N \to \mathbb{R}^N$ indicates the identity map. If $a, b \in \mathbb{R}$, the maximum and the minimum of $\{a, b\}$ are usually denoted by $a \vee b$ and $a \wedge b$, respectively.

The symbols $\mathcal{L}^N(E)$ and $\mathcal{H}^k(E)$ stand for the Lebesgue measure and the k-dimensional Hausdorff measure of a set $E \subset \mathbb{R}^N$, respectively. We will often write $|E|$ in place of $\mathcal{L}^N(E)$. The characteristic function of E is denoted by χ_E, and the symmetric difference of two sets $E, F \subset \mathbb{R}^N$ is given by $E \triangle F := (E \setminus F) \cup (F \setminus E)$.

We denote the ball centered at a point $x \in \mathbb{R}^N$ with radius $\rho > 0$ by $B_\rho(x)$, writing for simplicity $B_\rho := B_\rho(0)$, and the unit sphere in $\mathbb{R}^N$ by $\mathbb{S}^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. The volume of the unit ball in $\mathbb{R}^N$ is usually denoted by $\omega_N := |B_1|$. Given $\nu \in \mathbb{S}^{N-1}$ and $\rho > 0$, we shall denote by $C_{\nu,\rho}$ the cylinder

$$C_{\nu,\rho} := \left\{ x \in \mathbb{R}^N \,:\, |x - (x \cdot \nu)\nu| < \rho,\, |x \cdot \nu| < \rho \right\}.$$

For $g : B^{N-1}_\rho \to (-\rho, \rho)$, where B^{N-1}_ρ is the ball in $\mathbb{R}^{N-1}$ centered at the origin with radius ρ, we define the *graph* of g (with respect to the direction ν) to be the set

$$\mathrm{gr}_\nu(g) := \left\{ x \in C_{\nu,\rho} \,:\, x \cdot \nu = g(x - (x \cdot \nu)\nu) \right\}$$

(here we have identified B^{N-1}_ρ with the set $\{x \in \mathbb{R}^N : |x| < \rho,\, x \cdot \nu = 0\}$, with an abuse of notation).

1.1. Geometric preliminaries

Given a smooth orientable $(N-1)$-dimensional manifold $\Gamma \subset \mathbb{R}^N$, we indicate the tangent space and the normal space to Γ at $x \in \Gamma$ by $T_x\Gamma$ and $N_x\Gamma$, respectively. By $\nu : \mathcal{U} \to \mathbb{S}^{N-1}$ we denote a smooth vector field defined in a tubular neighborhood $\mathcal{U}$ of Γ and normal to Γ on Γ (we can take, for instance, the gradient of the signed distance function to Γ).

We now recall the definition of some tangential differential operators, referring to [75, Chapter 2, Section 7] for more details. If $g : \mathcal{U} \to \mathbb{R}^d$ is a C^1-function, we denote by $D_\Gamma g(x)$

(or by $\nabla_\Gamma g(x)$, if $d=1$) the *tangential differential* of g at $x \in \Gamma$, that is, the linear operator from $\mathbb{R}^N$ into $\mathbb{R}^d$ given by $dg(x) \circ \pi_x$, where $dg(x)$ is the usual differential of g at x and π_x is the orthogonal projection on the tangent space to Γ at x. We will usually identify $D_\Gamma g(x)$ with a $d \times N$ matrix. Notice that

$$(D_\Gamma g(x))^T[h] \cdot \nu(x) = h \cdot D_\Gamma g(x)[\nu(x)] = 0 \quad \text{for every } h \in \mathbb{R}^d,$$

that is, $(D_\Gamma g(x))^T$ maps $\mathbb{R}^d$ into $T_x\Gamma$. We remark also that

$$D\nu = D_\Gamma \nu \qquad \text{on } \Gamma. \tag{1.1}$$

If $d = N$, we define also the *tangential divergence* of g by

$$\mathrm{div}_\Gamma g := \sum_{j=1}^{N} e_j \cdot \nabla_\Gamma g_j = \sum_{j=1}^{N-1} \tau_j \cdot \partial_{\tau_j} g$$

where $g_i := g \cdot e_i$ is the i-th component of g with respect to the canonical basis of $\mathbb{R}^N$, $\tau_1, \ldots, \tau_{N-1}$ is any orthonormal basis for $T_x\Gamma$, and for every $v \in S^{N-1}$ the symbol ∂_v denotes the derivative in the direction v. We have, by definition,

$$\mathrm{div} g = \mathrm{div}_\Gamma g + \nu \cdot \partial_\nu g,$$

from which follows in particular, as $\partial_\nu \nu = 0$ by (1.1), that $\mathrm{div}\,\nu = \mathrm{div}_\Gamma \nu$ on Γ. We will make repeated use of the following identities:

$$\mathrm{div}_\Gamma(\varphi g_1) = \nabla_\Gamma \varphi \cdot g_1 + \varphi\, \mathrm{div}_\Gamma g_1,$$
$$\nabla_\Gamma(g_1 \cdot g_2) = (D_\Gamma g_1)^T[g_2] + (D_\Gamma g_2)^T[g_1],$$

for $\varphi \in C^1(\mathcal{U})$ and $g_1, g_2 \in C^1(\mathcal{U}; \mathbb{R}^N)$.

Let $S : \mathcal{U} \to \mathbb{M}^N$ be of class C^1. We recall that the divergence of S is defined as the unique vector function $\mathrm{div} S : \mathcal{U} \to \mathbb{R}^N$ such that

$$a \cdot \mathrm{div} S = \mathrm{div}(S^T a) \quad \text{for every } a \in \mathbb{R}^N.$$

Under identification of S with the matrix associated in the orthonormal basis $(e_1, \ldots e_N)$, the divergence $\mathrm{div} S$ is the vector function whose components in the orthonormal basis $(e_1, \ldots e_N)$ are the divergences of the rows:

$$\mathrm{div} S = (\mathrm{div} S_1, \ldots, \mathrm{div} S_N),$$

where S_i is the i-th row of S. Analogously, we define the *tangential divergence* $\mathrm{div}_\Gamma S : \mathcal{U} \to \mathbb{R}^N$ as the unique vector function such that

$$a \cdot \mathrm{div}_\Gamma S = \mathrm{div}_\Gamma(S^T a) \quad \text{for every } a \in \mathbb{R}^N,$$

and, as before, the tangential divergence $\mathrm{div}_\Gamma S$ is the vector function whose components in the orthonormal basis $(e_1, \ldots e_N)$ are the tangential divergences of the rows:

$$\mathrm{div}_\Gamma S = (\mathrm{div}_\Gamma S_1, \ldots, \mathrm{div}_\Gamma S_N).$$

We remark that all the tangential differential operators introduced so far have an intrinsic meaning, since they only depend on the restriction of g to Γ. The above definitions can be also extended to the case where Γ is a countably $\mathcal{H}^{N-1}$-rectifiable set (see [8, Remark 7.30]).

The following *divergence formula* is stated in [75, equation 7.6]: if the closure of Γ is a compact C^2-manifold with smooth $(N-2)$-dimensional boundary $\partial\Gamma$, then for every vector field $g : \mathcal{U} \to \mathbb{R}^N$ of class C^1 holds

$$\int_\Gamma \mathrm{div}_\Gamma g \, \mathrm{d}\mathcal{H}^{N-1} = \int_\Gamma H(g \cdot \nu) \, \mathrm{d}\mathcal{H}^{N-1} + \int_{\partial\Gamma} g \cdot \eta \, \mathrm{d}\mathcal{H}^{N-2}, \tag{1.2}$$

where H is the scalar mean curvature of Γ with respect to ν (see equation (1.5) below), and η is the outward pointing unit co-normal of $\partial\Gamma$ (that is, η is a unit vector normal to $\partial\Gamma$, tangent to Γ, and points out of Γ at each point of $\partial\Gamma$). Notice that (1.2) allows to extend to tangential operators the usual integration by parts formula: indeed,

$$\int_\Gamma \varphi\, \mathrm{div}_\Gamma g \,\mathrm{d}\mathcal{H}^{N-1} = -\int_\Gamma \nabla_\Gamma \varphi \cdot g \,\mathrm{d}\mathcal{H}^{N-1} \tag{1.3}$$

for every $g \in C^1(\mathcal{U};\mathbb{R}^N)$ such that $g(x) \in T_x\Gamma$ for $x \in \Gamma$, and for every $\varphi \in C^1(\mathcal{U})$ with $\mathrm{supp}\,\varphi \cap \Gamma \subset\subset \Gamma$.

For every $x \in \Gamma$ we set

$$\mathbf{B}(x) := D_\Gamma \nu(x) = D\nu(x). \tag{1.4}$$

The bilinear form associated with $\mathbf{B}(x)$ is symmetric and, when restricted to $T_x\Gamma \times T_x\Gamma$, it coincides with the *second fundamental form of* Γ *at* x. We consider also the function $H : \mathcal{U} \to \mathbb{R}$ defined by

$$H := \mathrm{div}\,\nu. \tag{1.5}$$

On Γ we have $H = \mathrm{div}\,\nu = \mathrm{div}_\Gamma \nu = \mathrm{trace}\,\mathbf{B}$, that is, for every $x \in \Gamma$ the value $H(x)$ coincides with the *scalar mean curvature of* Γ *at* x (with respect to ν).

If $\psi : \mathbb{R}^N \setminus \{0\} \to (0,+\infty)$ is a smooth, positively 1-homogeneous and convex function, we define the *anisotropic second fundamental form* $\mathbf{B}^\psi$ and the *anisotropic mean curvature* H^ψ of Γ (with respect to ν) by

$$\mathbf{B}^\psi := D(\nabla\psi \circ \nu) \qquad \text{and} \qquad H^\psi := \mathrm{trace}\,\mathbf{B}^\psi = \mathrm{div}\,(\nabla\psi \circ \nu) \tag{1.6}$$

respectively. Notice that, also in this case, we have $H^\psi = \mathrm{div}_\Gamma\,(\nabla\psi \circ \nu)$ on Γ.

Let $\Phi : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ be an orientation-preserving diffeomorphism of class C^1. We will usually denote by $\Gamma_\Phi := \Phi(\Gamma)$ the image of Γ through Φ. A possible choice for the unit normal to Γ_Φ is given by the vector field

$$\nu_\Phi = \frac{(D\Phi)^{-T}[\nu]}{|(D\Phi)^{-T}[\nu]|} \circ \Phi^{-1}. \tag{1.7}$$

Accordingly, we define the functions $\mathbf{B}_\Phi$, H_Φ, $\mathbf{B}^\psi_\Phi$ and H^ψ_Φ as in (1.4), (1.5) and (1.6), with Γ and ν replaced by Γ_Φ and ν_Φ, respectively. The following identity is a particular case of the so-called *generalized area formula* (see, *e.g.*, [75, Chapter 2, Section 8]): for every $\psi \in L^1(\Gamma_\Phi)$

$$\int_{\Gamma_\Phi} \psi \,\mathrm{d}\mathcal{H}^{N-1} = \int_\Gamma (\psi \circ \Phi) J_\Phi \,\mathrm{d}\mathcal{H}^{N-1}, \tag{1.8}$$

where $J_\Phi := |(D\Phi)^{-T}[\nu]| \det D\Phi$ is the $(N-1)$-dimensional Jacobian of Φ.

The two dimensional case. We now specialize some of the above definitions in the particular case where Γ is a smooth embedded curve in $\mathbb{R}^2$, since the results in Chapter 2 and in the first part of Chapter 3 are obtained in a two-dimensional framework. As before, we let $\nu : \mathcal{U} \to \mathbb{S}^1$ be a smooth vector field defined in a tubular neighborhood $\mathcal{U}$ of Γ and normal to Γ on Γ, and we let $\tau := \nu^\perp$ be the unit tangent vector to Γ (where $^\perp$ stands for the clockwise rotation by $\frac{\pi}{2}$). In this case the tangential divergence of a smooth vector field $g : \mathcal{U} \to \mathbb{R}^2$ has the simpler expression $\mathrm{div}_\Gamma g := \tau \cdot \partial_\tau g$, and the divergence formula in (1.2) becomes

$$\int_\Gamma \mathrm{div}_\Gamma g \,\mathrm{d}\mathcal{H}^1 = \int_\Gamma H(g\cdot\nu)\,\mathrm{d}\mathcal{H}^1 + \int_{\partial\Gamma} g \cdot \eta \,\mathrm{d}\mathcal{H}^0\,, \tag{1.9}$$

where the last integral reduces to a sum over the endpoints of Γ, and η is a unit vector tangent to Γ and pointing out of Γ at each point of $\partial\Gamma$. Notice that $H = \mathrm{div}_\Gamma \nu = D\nu[\tau,\tau]$ coincides, on Γ, with the *curvature* of Γ.

If $\Phi : \overline{\mathcal{U}} \to \overline{\mathcal{U}}$ is a smooth orientation-preserving diffeomorphism, then the divergence formula (1.9) holds on $\Gamma_\Phi = \Phi(\Gamma)$ with the vector η replaced by

$$\eta_\Phi = \frac{D\Phi[\eta]}{|D\Phi[\eta]|} \circ \Phi^{-1}. \tag{1.10}$$

1.2. Quasi-minimizers of the area functional

We recall that a measurable set $E \subset \mathbb{R}^N$ is said to be of *finite perimeter* in an open set $\Omega \subset \mathbb{R}^N$ if

$$\mathcal{P}(E;\Omega) := \sup\left\{ \int_E \mathrm{div} g \, \mathrm{d}x \,:\, g \in C^1_c(\Omega;\mathbb{R}^N),\ \|g\|_\infty \le 1 \right\} < +\infty,$$

or equivalently if the distributional derivative $D\chi_E$ of its characteristic function χ_E is a vector-valued bounded Radon measure in Ω. In this case $\mathcal{P}(E;\Omega) = |D\chi_E|(\Omega)$ is called the *perimeter* of E in Ω. We usually write $\mathcal{P}(E) := \mathcal{P}(E;\mathbb{R}^N)$. For self-contained presentations of the theory of sets of finite perimeter we refer the reader to the books [8, 64]. Throughout the thesis, for every set E of finite perimeter we denote by $\partial^* E$ its reduced boundary and by ν_E the generalized outer unit normal to E. By modifying E in a set of measure zero, we can always assume without loss of generality that (see [64, Proposition 12.19])

$$\mathrm{supp}\,(D\chi_E) = \left\{ x \in \mathbb{R}^N \,:\, 0 < |E \cap B_r(x)| < \omega_N r^N \text{ for all } r > 0 \right\} = \partial E.$$

We now recall the definition of *quasi-minimizers* of the area functional, which is a sort of generalization of the classical notion of local perimeter minimizer. The idea is to allow for the presence, in the minimality inequality, of higher-order perturbations, in order to provide a minimality condition which is satisfied by solutions of a larger class of geometric variational problems (for instance, in presence of volume constraints).

DEFINITION 1.1. A set of finite perimeter $E \subset \mathbb{R}^N$ is said to be an (ω, r_0)*-minimizer of the area functional*, with $\omega > 0$ and $r_0 > 0$, if for every ball $B_r(x)$ with $r \le r_0$ and for every set $F \subset \mathbb{R}^N$ of finite perimeter such that $E \triangle F \subset\subset B_r(x)$ we have

$$\mathcal{P}(E) \le \mathcal{P}(F) + \omega |E \triangle F|.$$

In the literature, sets satisfying the previous condition are sometimes referred to as *strong* ω*-minimizers*, while the expression *quasi-minimizer* designates a more general class for which the term $|E \triangle F|$ in the above definition is replaced by a power $r^{N-1+2\gamma}$, $\gamma \in (0,1)$, of the radius of the ball B_r.

For the purposes of this thesis, we are mainly interested in the regularity properties enjoyed by quasi-minimizers. A brief account of the development of this now classical theory necessarily starts from the first regularity results for minimal boundaries obtained by E. De Giorgi in [31] (see also [35]) in the setting of sets of finite perimeter. Then the partial regularity of sets quasi-minimizing the perimeter was proved by Tamanini in [79], while a regularity theory in the framework of rectifiable currents, for currents "almost-minimizing a parametric elliptic integrand", was established by several authors: Allard [3], Almgren [4, 5], Bombieri [11], Schoen and Simon [74].

We direct the attention of the interested reader to the books [6] by L. Ambrosio and [64] by F. Maggi, which set out a complete and clear presentation of the regularity theory for quasi-minimizers, and to which we refer for the proofs of the theorems below.

A first fundamental result provides uniform density estimates for quasi-minimizers in balls centered at points in ∂E (see [64, Theorem 21.11]).

Theorem 1.2 (density estimates). *There exists a positive constant $C(N)$, depending only on the dimension N, with the following property. If $E \subset \mathbb{R}^N$ is an (ω, r_0)-minimizer of the area functional with $\omega r_0 \leq 1$, then*

$$\frac{1}{4^N} \leq \frac{|E \cap B_r(x)|}{\omega_N r^N} \leq 1 - \frac{1}{4^N}, \quad C(N) \leq \frac{\mathcal{P}(E; B_r(x))}{r^{N-1}} \leq 3N\omega_N$$

for every $x \in \partial E$ and $r < r_0$.

We now introduce the key notion of *excess*, which measures the integral oscillation of the generalized normal vector to E in small balls centered at some point $x \in \partial E$:

$$\operatorname{Exc}(E, x, r) := \frac{1}{r^{N-1}} \min_{\nu \in \mathbb{S}^{N-1}} \int_{\partial^* E \cap B_r(x)} |\nu_E(y) - \nu|^2 \, \mathrm{d}\mathcal{H}^{N-1}(y) \,. \tag{1.11}$$

The main result in the regularity theory expresses the fact that the smallness of the previous quantity in some ball $B_\rho(x)$ forces the boundary ∂E to coincide, in a smaller ball, with the graph of a $C^{1,\gamma}$-function. This is the content of the following fundamental theorem, for which we refer to [64, Theorem 26.3] (in the statement, we follow the notation introduced at the beginning of this chapter).

Theorem 1.3 (regularity of quasi-minimizers). *For every $\gamma \in (0, \frac{1}{2})$ there exist positive constants $\varepsilon_0(N,\gamma)$, $C_0(N,\gamma)$, depending only on the dimension N and on γ, with the following property. If $E \subset \mathbb{R}^N$ is an (ω, r_0)-minimizer of the area functional with $\omega r_0 \leq 1$, and*

$$\operatorname{Exc}(E, x_0, r) + \omega r \leq \varepsilon_0$$

for some $x_0 \in \partial E$ and $r < r_0$, then $x_0 \in \partial^* E$ and, setting $\nu := \nu_E(x_0)$, one has

$$(\partial E - x_0) \cap C_{\nu, \frac{r}{10}} = \operatorname{gr}_\nu(f)$$

for some function $f \in C^{1,\gamma}(B^{N-1}_{r/10})$ with

$$\|f\|_{C^1} \leq C_0, \qquad |\nabla f(z) - \nabla f(z')| \leq C_0 \Big(\frac{|z - z'|}{r} \Big)^\gamma$$

for every $z, z' \in B^{N-1}_{r/10}$.

By the previous result, one can actually deduce the *partial regularity* of ∂E, namely that if E is an (ω, r_0)-minimizer then the reduced boundary $\partial^* E$ is a $(N-1)$-dimensional surface of class $C^{1,\gamma}$ for every $\gamma \in (0, \frac{1}{2})$, relatively open in ∂E, and such that $\mathcal{H}^{N-1}(\partial E \setminus \partial^* E) = 0$.

As observed by White in [82], the uniform $C^{1,\gamma}$-estimates provided by Theorem 1.3, combined with the continuity properties of the excess, allow to deduce the following important result concerning uniform sequences of quasi-minimizers converging to a regular set. The proof is well-known to specialists and can be found, for instance, in [27, Lemma 3.6].

Theorem 1.4 (uniform sequences of quasi-minimizers). *Let $E_n \subset \mathbb{R}^N$ be a sequence of (ω, r_0)-minimizers of the area functional such that*

$$\sup_n \mathcal{P}(E_n) < +\infty \quad \text{and} \quad \chi_{E_n} \to \chi_E \text{ in } L^1(\mathbb{R}^N)$$

for some bounded set E of class C^2. Then, for n sufficiently large, E_n is a set of class $C^{1,\gamma}$ for every $\gamma \in (0, \frac{1}{2})$, and $\partial E_n \to \partial E$ in $C^{1,\gamma}$ in the sense that

$$\partial E_n = \{x + \varphi_n(x)\nu_E(x) \,:\, x \in \partial E\},$$

with $\varphi_n \to 0$ *in* $C^{1,\gamma}(\partial E)$ *for all* $\gamma \in (0, \frac{1}{2})$.

PROOF. From the bound on the perimeters of E_n and the L^1-convergence to E, we deduce that $D\chi_{E_n} \overset{*}{\rightharpoonup} D\chi_E$ and $|D\chi_{E_n}| \overset{*}{\rightharpoonup} |D\chi_E|$ weakly* in the sense of measures (see [6, Theorem 4.2.5]). It follows that each point $x \in \partial E$ is the limit of a sequence of points $x_n \in \partial E_n$: on the contrary, we could find a ball $B_\rho(x)$ such that $\mathrm{supp}\,(D\chi_{E_n}) \cap B_\rho(x) = \emptyset$ for infinite indices n (since we are assuming $\mathrm{supp}\,(D\chi_{E_n}) = \partial E_n$), from which it would follow that $\mathcal{P}(E; B_\rho(x)) = 0$.

Fix now any point $x_0 \in \partial E$ and let $x_n \in \partial E_n$, $x_n \to x_0$. By the regularity of E we can find $r \in (0, r_0)$ such that $\mathcal{H}^{N-1}(\partial E \cap \partial B_{2r}(x_0)) = 0$ and

$$\mathrm{Exc}(E, x_0, 2r) < \frac{\varepsilon_0}{2^N}\,, \tag{1.12}$$

where ε_0 is the constant provided by Theorem 1.3 corresponding to a fixed $\gamma \in (0, \frac{1}{2})$. Observe that for every unit vector $\nu \in \mathbb{S}^{N-1}$

$$\int_{\partial^* E_n \cap B_{2r}(x_0)} |\nu_{E_n}(y) - \nu|^2 \,\mathrm{d}\mathcal{H}^{N-1}(y) \to \int_{\partial E \cap B_{2r}(x_0)} |\nu_E(y) - \nu|^2 \,\mathrm{d}\mathcal{H}^{N-1}(y)\,.$$

This follows from the weak*-convergence of $D\chi_{E_n}$ and $|D\chi_{E_n}|$ to $D\chi_E$ and $|D\chi_E|$ respectively, from the assumption $\mathcal{H}^{N-1}(\partial E \cap \partial B_{2r}(x_0)) = 0$ and from the representation

$$\frac{1}{2}\int_{\partial^* E_n \cap B_{2r}(x_0)} |\nu_{E_n}(y) - \nu|^2 \,\mathrm{d}\mathcal{H}^{N-1}(y) = |D\chi_{E_n}|(B_{2r}(x_0)) - \langle \nu, D\chi_{E_n}(B_{2r}(x_0))\rangle\,.$$

Hence, if the minimum value defining $\mathrm{Exc}(E, x_0, 2r)$ is attained at some $\nu_0 \in \mathbb{S}^{N-1}$, by the previous convergence we deduce that

$$\begin{aligned}
\limsup_{n\to\infty} \mathrm{Exc}(E_n, x_n, r) &\le \limsup_{n\to\infty} \frac{1}{r^{N-1}} \int_{\partial^* E_n \cap B_r(x_n)} |\nu_{E_n}(y) - \nu_0|^2 \,\mathrm{d}\mathcal{H}^{N-1}(y) \\
&\le \limsup_{n\to\infty} \frac{1}{r^{N-1}} \int_{\partial^* E_n \cap B_{2r}(x_0)} |\nu_{E_n}(y) - \nu_0|^2 \,\mathrm{d}\mathcal{H}^{N-1}(y) \\
&= \frac{1}{r^{N-1}} \int_{\partial E \cap B_{2r}(x_0)} |\nu_E(y) - \nu_0|^2 \,\mathrm{d}\mathcal{H}^{N-1}(y) \\
&= 2^{N-1}\mathrm{Exc}(E, x_0, 2r)\,,
\end{aligned}$$

where we used the fact that $B_r(x_n) \subset B_{2r}(x_0)$ for n large enough. Hence by (1.12) we conclude that for n sufficiently large we have

$$\mathrm{Exc}(E_n, x_n, r) < \frac{\varepsilon_0}{2}\,.$$

By reducing r if necessary, we can also assume that $\omega r < \frac{\varepsilon_0}{2}$.

We are in position to apply Theorem 1.3 to E_n in the ball $B_r(x_n)$: setting $r_1 := \frac{r}{10}$, we have functions $g_n \in C^{1,\gamma}(B^{N-1}_{r_1})$ uniformly bounded in $C^{1,\gamma}$, with $g_n(0) = \nabla g_n(0) = 0$, such that

$$(\partial E_n - x_n) \cap C_{\nu_n, r_1} = \mathrm{gr}_{\nu_n}(g_n),$$

where ν_n is the exterior normal to ∂E_n at x_n.

By compactness, $\nu_n \to \bar\nu$ for some $\bar\nu \in \mathbb{S}^{N-1}$ (up to subsequences). Hence for n large enough $C_{\bar\nu, r_1/2} \subset C_{\nu_n, r_1} + x_n - x_0$, and there exist functions $f_n \in C^{1,\gamma}(B^{N-1}_{r_1/2})$, uniformly bounded in $C^{1,\gamma}$, satisfying

$$\mathrm{gr}_{\bar\nu}(f_n) \cap C_{\bar\nu, r_1/2} = (\mathrm{gr}_{\nu_n}(g_n) + x_n - x_0) \cap C_{\bar\nu, r_1/2},$$

so that

$$(\partial E_n - x_0) \cap C_{\bar{\nu}, r_1/2} = \mathrm{gr}_{\bar{\nu}}(f_n).$$

By Ascoli-Arzelà Theorem f_n converges to some function f in $C^{1,\beta}$ for every $\beta < \gamma$, with $f(0) = \nabla f(0) = 0$. Since $E_n \to E$ in L^1, it is easily seen that the limit function f satisfies

$$(\partial E - x_0) \cap C_{\bar{\nu}, r_1/2} = \mathrm{gr}_{\bar{\nu}}(f),$$

and moreover $\bar{\nu} = \nu_E(x_0)$. Now the conclusion of the theorem follows by a covering argument, by using the compactness of ∂E. □

REMARK 1.5. The previous results continue to hold when working in a bounded open set $\Omega \subset \mathbb{R}^N$. Precisely, we say that a set $E \subset \Omega$ of finite perimeter is an (ω, r_0)*-minimizer of the area functional in* Ω, with $\omega > 0$ and $r_0 > 0$, if for every ball $B_r(x)$ with $r \le r_0$ and for every set $F \subset \Omega$ of finite perimeter such that $E \triangle F \subset\subset B_r(x)$ we have

$$\mathcal{P}(E; \Omega) \le \mathcal{P}(F; \Omega) + \omega |E \triangle F|.$$

In this case the same argument used in the proof of Theorem 1.4, combined with the regularity of quasi-minimizers up to the boundary $\partial\Omega$ (which follows from a work by Grüter [54]), leads to the following conclusion (see [56, Theorem 3.3]): assume that Ω is smooth and $E_n \subset \Omega$ is a sequence of (ω, r_0)-minimizers of the area functional in Ω, such that

$$\sup_n \mathcal{P}(E_n; \Omega) < +\infty \quad \text{and} \quad \chi_{E_n} \to \chi_E \text{ in } L^1(\Omega)$$

for some set E whose boundary inside Ω is of class $C^{1,\gamma}$, $\gamma \in (0, \frac{1}{2})$, such that either $\overline{\partial E \cap \Omega} \cap \partial\Omega = \emptyset$ or $\overline{\partial E \cap \Omega}$ meets $\partial\Omega$ orthogonally. Then $\overline{\partial E_n \cap \Omega}$ is of class $C^{1,\gamma}$ for n sufficiently large, and $\partial E_n \to \partial E$ in $C^{1,\gamma}$. This means that we can find a sequence of diffeomorphisms $\Phi_n : \overline{\Omega} \to \overline{\Omega}$ of class $C^{1,\gamma}$ such that $\Phi_n(\partial E) = \partial E_n$ and $\|\Phi_n - Id\|_{C^{1,\gamma}} \to 0$.

REMARK 1.6. Let $\psi : \mathbb{R}^N \to [0, +\infty)$ be a convex and positively 1-homogeneous function of class C^2 out of the origin, and assume that for every $v \in \mathbb{S}^{N-1}$

$$D^2\psi(v)[w, w] > \bar{c}\, |w|^2 \qquad \text{for all } w \perp v,$$

for some constant $\bar{c} > 0$. We say that E is an (ω, r_0)*-minimizer of the anisotropic perimeter* if for every ball $B_r(x)$ with $r \le r_0$ and for every set $F \subset \mathbb{R}^N$ of finite perimeter such that $E \triangle F \subset\subset B_r(x)$ we have

$$\int_{\partial^* E} \psi(\nu_E)\, \mathrm{d}\mathcal{H}^{N-1} \le \int_{\partial^* F} \psi(\nu_F)\, \mathrm{d}\mathcal{H}^{N-1} + \omega |E \triangle F|.$$

The result in Theorem 1.4 remains valid under the assumption that the sets E_n are quasi-minimizers of the anisotropic perimeter, according to the previous definition. This can be deduces by following the same strategy, using now the standard regularity theory for *almost-minimal currents*, and precisely using the result stated in [39, Theorem 15] (see also the proof of [39, Theorem 8]). Notice that the quasi-minimality property considered in [39], namely

$$\int_{\partial^* E} \psi(\nu_E)\, d\mathcal{H}^{N-1} \le \int_{\partial^* F} \psi(\nu_F)\, d\mathcal{H}^{N-1} + \omega r \mathcal{P}(E \triangle F)$$

whenever $E \triangle F$ is compactly contained in a ball of radius r, is clearly implied by our definition of quasi-minimality as a consequence of the isoperimetric inequality.

REMARK 1.7. We say that a set $E \subset \mathbb{R}^N$ is *periodic* if its characteristic function is one-periodic in the first $N-1$ coordinate directions. It is clear from the proof that Theorem 1.4, as well as its anisotropic version discussed in Remark 1.6, are still valid if we replace the assumption of boundedness of E by the request that E and E_n are periodic sets, with perimeters uniformly bounded in the cell of periodicity and $\chi_{E_n} \to \chi_E$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$.

By a standard first variation argument (see, *e.g.*, [8, Proposition 4.71]) we can show that the curvature of an (ω, r_0)-minimizer of class C^1 is uniformly bounded by the constant ω.

LEMMA 1.8. *Let $E \subset \mathbb{R}^N$ be an (ω, r_0)-minimizer of the area functional, and assume that ∂E is of class C^1. Then there exists a function $H \in L^\infty(\partial E)$, with $\|H\|_{L^\infty(\partial E)} \leq \omega$, such that the equation*

$$\mathrm{div}_{\partial E}\nu_E = H$$

holds weakly on ∂E, that is, for every $X \in C^1(\mathbb{R}^N;\mathbb{R}^N)$

$$\int_{\partial E} \mathrm{div}_{\partial E} X \,\mathrm{d}\mathcal{H}^{N-1} = \int_{\partial E} H(X\cdot\nu_E)\,\mathrm{d}\mathcal{H}^{N-1}\,.$$

PROOF. Let $X \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ be a vector field whose support is contained in a ball with radius smaller than r_0, and consider the associated flow

$$\frac{\partial}{\partial t}\Phi(t,x) = X(\Phi(t,x)), \qquad \Phi(0,x) = x.$$

It can be shown that

$$|E\triangle\Phi(t,E)| = |t|\int_{\partial E}|X\cdot\nu_E|\,\mathrm{d}\mathcal{H}^{N-1} + o(t)\,, \qquad \lim_{t\to 0}\frac{o(t)}{t} = 0\,.$$

Then by the quasi-minimality of E we deduce that for every t small enough

$$\mathcal{P}(E) \leq \mathcal{P}(\Phi(t,E)) + \omega\,|t|\int_{\partial E}|X\cdot\nu_E|\,\mathrm{d}\mathcal{H}^{N-1} + \omega\, o(t)\,.$$

Dividing by t and letting $t\to 0$, we obtain

$$\int_{\partial E}\mathrm{div}_{\partial E}X\,\mathrm{d}\mathcal{H}^{N-1} \leq \omega\int_{\partial E}|X\cdot\nu_E|\,\mathrm{d}\mathcal{H}^{N-1}\,. \tag{1.13}$$

The previous inequality holds for every vector field $X \in C^1(\mathbb{R}^N;\mathbb{R}^N)$, without the requirement that the support of X is contained in a small ball, by a simple argument which uses a partition of unity. Hence the left-hand side of (1.13) defines a continuous linear functional, whose norm is bounded by ω, and we conclude by the Riesz representation theorem. □

We conclude by recalling the following simple lemma from [1, Lemma 4.1], which shows that any regular set is in fact a quasi-minimizer, with a constant depending on the set itself.

LEMMA 1.9. *Let $E \subset \mathbb{R}^N$ be a bounded set of class C^2. Then there exists a constant $C_E > 0$, depending only on E, such that for every set of finite perimeter $F \subset \mathbb{R}^N$*

$$\mathcal{P}(E) \leq \mathcal{P}(F) + C_E|E\triangle F|.$$

PROOF. Let $\nu \in C^1(\mathbb{R}^N;\mathbb{R}^N)$ be any smooth vector field such that $\nu = \nu_E$ on ∂E and $\|\nu\|_\infty \leq 1$. Then

$$\begin{aligned}\mathcal{P}(F) - \mathcal{P}(E) &\geq \int_{\partial^* F}\nu\cdot\nu_F\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E}\nu\cdot\nu_E\,\mathrm{d}\mathcal{H}^{N-1}\\ &= \int_F \mathrm{div}\nu\,\mathrm{d}x - \int_E\mathrm{div}\nu\,\mathrm{d}x \geq -C_E|E\triangle F|\,,\end{aligned}$$

where $C_E := \|\mathrm{div}\nu\|_\infty$. □

1.3. Quasi-minimizers of the Mumford-Shah functional

Given an open set $\Omega \subset \mathbb{R}^N$, we recall that the space $SBV(\Omega)$ of special functions of bounded variation is defined as the set of all functions $u \in L^1(\Omega)$ whose distributional derivative Du is a bounded Radon measure of the form

$$Du = \nabla u\, \mathcal{L}^N + D^j u = \nabla u\, \mathcal{L}^N + (u^+ - u^-)\nu_u \mathcal{H}^{N-1} \llcorner S_u,$$

where $\nabla u \in L^1(\Omega; \mathbb{R}^N)$ is the approximate gradient of u, S_u is the jump set of u (which is countably $(\mathcal{H}^{N-1}, N-1)$-rectifiable), u^+ and u^- are the traces of u on S_u and ν_u is the approximate normal on S_u. We refer to [8] for a complete treatment of the space SBV and a precise definition of all the notions introduced above.

As observed in the Introduction, the space SBV is the proper space where to set and solve the minimum problem for the Mumford-Shah functional by the direct method of the Calculus of Variations. Here we are mainly interested in the regularity properties enjoyed by minimizers of the Mumford-Shah functional, and, more in general, by *quasi-minimizers*, according to the following definition.

Definition 1.10. We say that $u \in SBV(\Omega)$ is a *quasi-minimizer of the Mumford-Shah functional* if there exists $\omega > 0$ such that for every ball $B_\rho(x)$

$$\int_{\Omega \cap B_\rho(x)} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^{N-1}(S_u \cap B_\rho(x)) \leq \int_{\Omega \cap B_\rho(x)} |\nabla v|^2 \,\mathrm{d}x + \mathcal{H}^{N-1}(S_v \cap B_\rho(x)) + \omega \rho^N \tag{1.14}$$

for every $v \in SBV(\Omega)$ with $\{v \neq u\} \subset\subset B_\rho(x)$.

Similarly to the case of the area functional, treated in the previous section, a powerful regularity theory is also established for quasi-minimizers of the Mumford-Shah functional: the partial regularity of the discontinuity set of a quasi-minimizer u is proved in any number of dimensions in the papers [7, 9] (see also [8]), where it is shown that there exists a closed $\mathcal{H}^{N-1}$-negligible singular set $\Sigma \subset \overline{S}_u$ such that $\overline{S}_u \setminus \Sigma$ is locally a hypersurface of class $C^{1,1/4}$.

The proof is based on the decay properties of the quantity

$$E_u(x,r) := D_u(x,r) + r^{-2} A_u(x,r),$$

defined for $u \in SBV(\Omega)$, $x \in \Omega$ and $r > 0$, where

$$D_u(x,r) := \int_{B_r(x) \cap \Omega} |\nabla u|^2 \,\mathrm{d}y, \qquad A_u(x,r) := \min_{T \in \mathcal{A}} \int_{\overline{S}_u \cap B_r(x)} \operatorname{dist}^2(y,T) \,\mathrm{d}\mathcal{H}^{N-1}(y),$$

$\mathcal{A}$ denoting the set of affine $(N-1)$-planes in $\mathbb{R}^N$. The quantity $E_u(x,r)$ plays the role in this context of the excess (1.11), introduced by De Giorgi to study the regularity of minimal surfaces. The main result expresses the fact that the rate of decay of E_u in small balls determines the $C^{1,1/4}$-regularity of the jump set of u, provided that u satisfies the quasi-minimality property (1.14):

Theorem 1.11. *Let $u \in SBV(\Omega)$ be a quasi-minimizer of the Mumford-Shah functional, according to Definition 1.10, for some constant $\omega > 0$. There exist $R_0 > 0$, $\varepsilon_0 > 0$ (depending only on ω and on the dimension N) such that if*

$$E_u(x,r) < \varepsilon_0 r^{N-1}$$

for some $x \in \overline{S}_u \cap \Omega$ and $r < R := R_0 \wedge \operatorname{dist}(x, \partial\Omega)$, then there exist a smaller radius $r' \in (0,r)$ (depending only on ω, R and r) and a function $f \in C^{1,1/4}(B^{N-1}_{r'})$ with $f(0) = \nabla f(0) = 0$ such that

$$(\overline{S}_u - x) \cap C_{\nu,r'} = \mathrm{gr}_\nu(f),$$

where ν denotes the normal to S_u at x. Moreover, $\|f\|_{C^{1,1/4}} \leq C$ for some constant C depending only on ω.

The previous result is a consequence of [8, Theorem 8.2 and Theorem 8.3]: the only missing part is the uniform bound in $C^{1,1/4}$, which is not explicitly stated but can be deduced by checking that the constants appearing in the proof depend only on ω. Notice that the theorem provides the regularity of S_u in balls well contained in Ω; concerning the regularity of the discontinuity set at the intersection with the boundary of Ω, under Neumann conditions, we have the following result, which is essentially contained in the book [30] (see, in particular, [30, Remark 79.42]; see also [63]), in the two-dimensional case.

THEOREM 1.12. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open set with boundary of class C^1, and let $u \in SBV(\Omega)$ satisfy (1.14) for some $\omega > 0$. Then there exist $b \in (0,1)$ and $\tau > 0$ (depending only on ω and on Ω) such that, setting*

$$\Omega(\tau) := \{x \in \Omega : \mathrm{dist}\,(x, \partial\Omega) < \tau\},$$

the intersection $\overline{S}_u \cap \Omega(\tau)$ is a finite disjoint union of curves of class $C^{1,b}$ intersecting $\partial\Omega$ orthogonally, with $C^{1,b}$-norm uniformly bounded by a constant depending only on ω and Ω.

We conclude this section by recalling a well known property of quasi-minimizers of the Mumford-Shah functional, namely a lower bound on the $\mathcal{H}^{N-1}$-dimensional density of the jump set in balls centered at any point of its closure. The estimate was proved in [34] in balls entirely contained in the domain Ω (see also [8, Theorem 7.21]); we refer also, when a Dirichlet condition is imposed at the boundary of the domain, to [21] for balls centered at $\partial\Omega$, and to [10] for balls possibly intersecting $\partial\Omega$ but not necessarily centered at $\partial\Omega$. Finally, we refer to [30, Section 77] in the case of balls intersecting $\partial\Omega$ when a Neumann condition is imposed.

In fact, we will need to consider also, in the two-dimensional case, the mixed situation where we impose a Dirichlet condition on a part $\partial_D\Omega$ of the boundary and a Neumann condition on the remaining part $\partial_N\Omega$. The result is still valid for balls centered at the intersection between the Dirichlet and the Neumann part of the boundary, under the additional assumption that $\partial_D\Omega$ and $\partial_N\Omega$ meet orthogonally. We are not aware of any result of this kind in the existing literature, but the proof, which we postpone to Appendix A, can be obtained by following closely the strategy of the original proof in [34], combined also with some new ideas contained in [10]. The precise statement is the following.

THEOREM 1.13. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open set, let $\partial_D\Omega \subset \partial\Omega$ be relatively open and of class C^1, $\partial_N\Omega := \partial\Omega \setminus \overline{\partial_D\Omega}$ of class C^1, and assume that $\partial_D\Omega$ meets $\partial_N\Omega$ orthogonally. Let $\Omega' \subset \mathbb{R}^2$ be a bounded, open set of class C^1 such that $\Omega \subset \Omega'$ and $\partial\Omega \cap \Omega' = \partial_D\Omega$. Let $u \in SBV(\Omega')$ be such that $\overline{S}_u \cap \overline{\partial_D\Omega} = \emptyset$ and $u \in W^{1,\infty}(\Omega' \setminus \overline{S}_u)$.*

Let $w \in SBV(\Omega')$, with $w = u$ in $\Omega' \setminus \Omega$, satisfy for every $x \in \overline{\Omega}$ and for every $\rho > 0$

$$\int_{\Omega' \cap B_\rho(x)} |\nabla w|^2 \,\mathrm{d}x + \mathcal{H}^1(S_w \cap B_\rho(x)) \leq \int_{\Omega' \cap B_\rho(x)} |\nabla v|^2 \,\mathrm{d}x + \mathcal{H}^1(S_v \cap B_\rho(x)) + \omega\rho^2$$

for every $v \in SBV(\Omega')$ such that $v = u$ in $\Omega' \setminus \Omega$ and $\{v \neq w\} \subset\subset B_\rho(x)$. Then there exist $\rho_0 > 0$ and $\theta_0 > 0$ (depending only on ω, u and Ω) such that

$$\mathcal{H}^1(S_w \cap B_\rho(x)) \geq \theta_0 \rho$$

for every $\rho \leq \rho_0$ and $x \in \overline{S}_w$.

1.4. Fractional Sobolev spaces

We collect in this section some definition and properties of fractional Sobolev spaces. Given a smooth, compact, embedded $(N-1)$-dimensional submanifold $\Gamma \subset \mathbb{R}^N$, we recall that the Gagliardo seminorm of a function ϑ on Γ is defined as

$$[\vartheta]_{s,p,\Gamma} := \Big(\int_\Gamma \int_\Gamma \frac{|\vartheta(z) - \vartheta(w)|^p}{|z-w|^{N-1+sp}} \,\mathrm{d}\mathcal{H}^{N-1}(z)\,\mathrm{d}\mathcal{H}^{N-1}(w) \Big)^{\frac{1}{p}}$$

for $0 < s < 1$ and $1 < p < \infty$, and that $\vartheta \in W^{s,p}(\Gamma)$ if

$$\|\vartheta\|_{W^{s,p}(\Gamma)} := \|\vartheta\|_{L^p(\Gamma)} + [\vartheta]_{s,p,\Gamma} < \infty.$$

When $p = 2$ we switch to the equivalent notation $H^s(\Gamma)$ for $W^{s,2}(\Gamma)$. We start with the following simple lemma, which ensures that the product of an H^1-function by a Hölder-continuous function belongs to the space $H^{\frac{1}{2}}(\Gamma)$.

LEMMA 1.14. *Let $N \le 3$ and $\alpha > \frac{1}{2}$. If $\varphi \in H^1(\Gamma)$ and $u \in C^{0,\alpha}(\Gamma)$, then*

$$\|\varphi u\|_{H^{\frac{1}{2}}(\Gamma)} \le C \|\varphi\|_{H^1(\Gamma)} \|u\|_{C^{0,\alpha}(\Gamma)}$$

for some constant C depending only on α and on Γ.

PROOF. We can bound the Gagliardo $H^{\frac{1}{2}}$-seminorm of φu as follows: choosing $q > 2$ such that $(2\alpha - 1)q > 2N - 2$, adding and subtracting the term $\varphi(z)u(w)$ and using Hölder inequality, we have

$$\begin{aligned}
[\varphi u]^2_{\frac{1}{2},2,\Gamma} &\le 2 \int_\Gamma \int_\Gamma \frac{|\varphi(z) - \varphi(w)|^2 |u(w)|^2}{|z-w|^N} \,\mathrm{d}\mathcal{H}^{N-1}(z)\,\mathrm{d}\mathcal{H}^{N-1}(w) \\
&\quad + 2 \int_\Gamma \int_\Gamma \frac{|\varphi(z)|^2 |u(z) - u(w)|^2}{|z-w|^N} \,\mathrm{d}\mathcal{H}^{N-1}(z)\,\mathrm{d}\mathcal{H}^{N-1}(w) \\
&\le 2\,\|u\|^2_\infty \|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma)} + 2\|u\|^2_{C^{0,\alpha}} \int_\Gamma \int_\Gamma |\varphi(z)|^2 |z-w|^{2\alpha - N} \,\mathrm{d}\mathcal{H}^{N-1}(z)\,\mathrm{d}\mathcal{H}^{N-1}(w) \\
&\le 2\,\|u\|^2_{C^{0,\alpha}(\Gamma)} \Big[\|\varphi\|^2_{H^{\frac{1}{2}}(\Gamma)} \\
&\quad + \|\varphi\|^2_{L^q(\Gamma)} \big(\mathcal{H}^{N-1}(\Gamma)\big)^{\frac{2}{q}} \Big(\int_\Gamma \int_\Gamma |z-w|^{\frac{q(2\alpha - N)}{q-2}} \,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y) \Big)^{\frac{q-2}{q}} \Big].
\end{aligned}$$

Now the last integral is finite by the choice of q, and the conclusion follows since $H^1(\Gamma)$ is continuously imbedded in $L^q(\Gamma)$ for every q. □

We now consider the particular case of fractional Sobolev spaces defined on graphs of regular functions, listing a series of properties that will be used several times in Section 3.6. We follow the presentation contained in [45, Section 8.1], rephrasing the statements to consider also the case of dimension $N = 3$. Given a positive, C^1-function $h : \mathbb{R}^{N-1} \to (0, +\infty)$, 1-periodic in the coordinate directions, we denote its graph and its subgraph over the unit square $Q := (0,1)^{N-1}$ by

$$\Gamma_h := \big\{ (x, h(x)) \in \mathbb{R}^N : x \in Q \big\}, \quad \Omega_h := \big\{ (x,y) \in \mathbb{R}^N : x \in Q,\ 0 < y < h(x) \big\},$$

respectively, and by $\Gamma^\#_h$ and $\Omega^\#_h$ their periodic extensions. We also denote by c_0 a positive constant such that $\min_{\overline{Q}} h \ge c_0$.

Let $W^{s,p}_{\#}(\Gamma_h)$ be the subspace of $W^{s,p}(\Gamma_h)$ of functions whose periodic extension to $\Gamma_h^{\#}$ belongs to $W^{s,p}_{\mathrm{loc}}(\Gamma_h^{\#})$, endowed with the norm of $W^{s,p}(\Gamma_h)$. The dual spaces of $W^{s,p}(\Gamma_h)$ and of $W^{s,p}_{\#}(\Gamma_h)$ are denoted by $W^{-s,\frac{p}{p-1}}(\Gamma_h)$ and $W^{-s,\frac{p}{p-1}}_{\#}(\Gamma_h)$, respectively.

REMARK 1.15. We remark that, if $-1 < t \le s < 1$ and $p > 1$, the space $W^{s,p}(\Gamma_h)$ is continuously imbedded in $W^{t,p}(\Gamma_h)$. This follows directly from the definition.

THEOREM 1.16. *If $-1 \le t \le s \le 1$, $q \ge p$ and $s - \frac{N-1}{p} \ge t - \frac{N-1}{q}$, then $W^{s,p}(\Gamma_h)$ is continuously imbedded in $W^{t,q}(\Gamma_h)$. The imbedding constant depends only on s, t, p, q and on the C^1-norm of h.*

In particular, it follows that if $N \le 3$ then $H^1(\Gamma_h)$ is continuously imbedded in $L^q(\Gamma_h)$ for every $q \ge 1$. The proof of the theorem follows from [52, Theorem 1.4.4.1] by a change of variables, and taking into account Remark 1.15. The following theorem, which follows from [52, Theorem 1.5.1.2], deals with the trace operator on Γ_h.

THEOREM 1.17. *There exists a continuous linear operator $T : W^{1,p}(\Omega_h) \to W^{1-\frac{1}{p},p}(\Gamma_h)$ such that $Tu = u|_{\Gamma_h}$ whenever u is continuous on $\overline{\Omega}_h$. The norm of T is bounded by a constant depending only on p, c_0, and on the C^1-norm of h.*

Denoting by $W^{1,p}_{\#}(\Omega_h)$ the space of functions $u \in W^{1,p}(\Omega_h)$ whose periodic extension to $\Omega_h^{\#}$ belongs to $W^{1,p}_{\mathrm{loc}}(\Omega_h^{\#})$, we have in particular that $Tu \in W^{1-1/p,p}_{\#}(\Gamma_h)$ whenever $u \in W^{1,p}_{\#}(\Omega_h)$. *Viceversa*, we have the following extension theorem.

THEOREM 1.18. *For every $\vartheta \in W^{1-\frac{1}{p},p}_{\#}(\Gamma_h)$ there exists $u \in W^{1,p}_{\#}(\Omega_h)$ such that $Tu = \vartheta$ and*
$$\|u\|_{W^{1,p}(\Omega_h)} \le C\|\vartheta\|_{W^{1-\frac{1}{p},p}(\Gamma_h)}, \tag{1.15}$$
where C depends only on p, c_0, and on the C^1-norm of h.

We now state the 3-dimensional version of [45, Theorem 8.6].

THEOREM 1.19. *Let $N = 3$. For every $u \in W^{1,p}_{\#}(\Omega_h)$ and for $i = 1, 2$*
$$\Big\|\frac{\partial u}{\partial z_i}\nu_h^3 - \frac{\partial u}{\partial z_3}\nu_h^i\Big\|_{W^{-\frac{1}{p},p}_{\#}(\Gamma_h)} \le C\|\nabla u\|_{L^p(\Omega_h;\mathbb{R}^3)},$$
where $\nu_h = (\nu_h^1, \nu_h^2, \nu_h^3)$ is the upper unit normal to Γ_h and C depends only on p, c_0, and on the C^1-norm of h.

PROOF. Assume $u \in C^2(\overline{\Omega}_h)$. Given $\varphi \in W^{\frac{1}{p},\frac{p}{p-1}}_{\#}(\Gamma_h)$ we consider an extension in $W^{1,\frac{p}{p-1}}_{\#}(\Omega_h)$ (still denoted by φ), according to Theorem 1.18. We may also assume, by increasing the constant in (1.15), that $\varphi(x,0) = 0$. Then
$$\begin{aligned}
\int_{\Gamma_h} \Big(\frac{\partial u}{\partial z_1}\nu_h^3 - \frac{\partial u}{\partial z_3}\nu_h^1\Big)\varphi \,\mathrm{d}\mathcal{H}^2 &= \int_{\Gamma_h} \varphi\Big(-\frac{\partial u}{\partial z_3}, 0, \frac{\partial u}{\partial z_1}\Big)\cdot \nu \,\mathrm{d}\mathcal{H}^2 \\
&= \int_{\Omega_h} \mathrm{div}\Big(-\varphi\frac{\partial u}{\partial z_3}, 0, \varphi\frac{\partial u}{\partial z_1}\Big)\,\mathrm{d}z = \int_{\Omega_h} \nabla u \cdot \Big(\frac{\partial \varphi}{\partial z_3}, 0, -\frac{\partial \varphi}{\partial z_1}\Big)\,\mathrm{d}z \\
&\le \|\nabla u\|_{L^p(\Omega_h;\mathbb{R}^3)}\|\nabla\varphi\|_{L^{\frac{p}{p-1}}(\Omega_h;\mathbb{R}^3)} \le C\,\|\nabla u\|_{L^p(\Omega_h;\mathbb{R}^3)}\|\varphi\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_h)}
\end{aligned}$$
and this shows the claim in the case $i = 1$. The case $i = 2$ is similar, and an approximation argument concludes the proof of the theorem. $\square$

We conclude this section with the following lemma, whose proof follows from the definition of the Gagliardo seminorm by using a duality argument (in particular, for the first property one can argue similarly to the proof of Lemma 1.14).

Lemma 1.20. *Let $p > 1$ and let u be a smooth function. Then:*

(i) if $a \in C^{0,\alpha}(\Gamma_h)$ with $\alpha > \frac{1}{p}$, then

$$\|ua\|_{W^{-\frac{1}{p},p}(\Gamma_h)} \leq C\|a\|_{C^{0,\alpha}(\Gamma_h)}\|u\|_{W^{-\frac{1}{p},p}(\Gamma_h)},$$

for some constant C depending only on p, α and on the C^1-norm of h;

(ii) if $\Phi : \Gamma_h \to \Phi(\Gamma_h)$ is a C^1-diffeomorphism, then

$$\|u \circ \Phi^{-1}\|_{W^{-\frac{1}{p},p}(\Phi(\Gamma_h))} \leq C\|u\|_{W^{-\frac{1}{p},p}(\Gamma_h)},$$

for some constant C depending only on p and on the C^1-norms of Φ and of Φ^{-1}.

CHAPTER 2

The Mumford-Shah functional

The first model that we study is the prototypical free-discontinuity problem, *i.e.* the Mumford-Shah functional. In this chapter we improve the minimality criterion established in [19], by showing that a regular critical point, with positive definite second variation, is an isolated local minimizer with respect to competitors which are sufficiently close in the L^1-topology.

Organization of the chapter. In Section 2.1 we collect the necessary definitions and state the main result of this chapter. Section 2.2 is devoted to the computation of the second variation, when also boundary variations of the discontinuity set are allowed; some properties of the associated quadratic form are studied in Section 2.3. The proof of the main theorem starts in Section 2.4 (where the local $W^{2,\infty}$-minimality is addressed) and lasts for Sections 2.5 and 2.6 (where the $C^{1,\alpha}$ and the desired local L^1-minimality, respectively, are established). In Section 2.7 we describe some examples and applications of our minimality criterion. In the concluding section (Section 2.8) we prove some auxiliary technical lemmas.

2.1. Setting and main result

Let $\Omega\subset\mathbb{R}^2$ be an open, bounded, connected set with boundary of class C^3. We introduce the following space of *admissible pairs*

$$\mathcal{A}(\Omega):=\big\{(K,v): K\subset\mathbb{R}^2 \text{ closed}, \ v\in H^1(\Omega\setminus K)\big\}$$

on which is defined the (homogeneous) *Mumford-Shah functional*

$$\mathcal{MS}(K,v):=\int_{\Omega\setminus K}|\nabla v|^2\,\mathrm{d}x+\mathcal{H}^1(K\cap\Omega) \qquad \text{for } (K,v)\in\mathcal{A}(\Omega).$$

It will be useful to consider also a localized version of the functional: for $A\subset\Omega$ open we set

$$\mathcal{MS}((K,v);A):=\int_{A\setminus K}|\nabla v|^2\,\mathrm{d}x+\mathcal{H}^1(K\cap A).$$

Given an admissible pair $(K,v)\in\mathcal{A}(\Omega)$ and assuming that K is a regular curve connecting two points of $\partial\Omega$, we denote by ν a smooth vector field coinciding with the unit normal to K when restricted to the points of K, by H the curvature of K with respect to ν (defined as in (1.5)), and by η the unit co-normal of $K\cap\partial\Omega$ (see Section 1.1). For any function $z\in H^1(\Omega\setminus K)$ we denote the traces of z on the two sides of K by z^+ and z^-: precisely, for $\mathcal{H}^1$-a.e. $x\in K$ we set

$$z^{\pm}(x):=\lim_{r\to0^+}\frac{1}{|B_r(x)\cap V_x^{\pm}|}\int_{B_r(x)\cap V_x^{\pm}}z(y)\,\mathrm{d}y,$$

where $V_x^{\pm}:=\{y\in\mathbb{R}^2:\pm(y-x)\cdot\nu(x)\geq0\}$. With an abuse of notation, we denote by z^+ and z^- also the restrictions of z to Ω^+ and Ω^- respectively, where Ω^+ and Ω^- are the two connected components of $\Omega\setminus K$, with the normal vector field ν pointing into Ω^+. Finally we

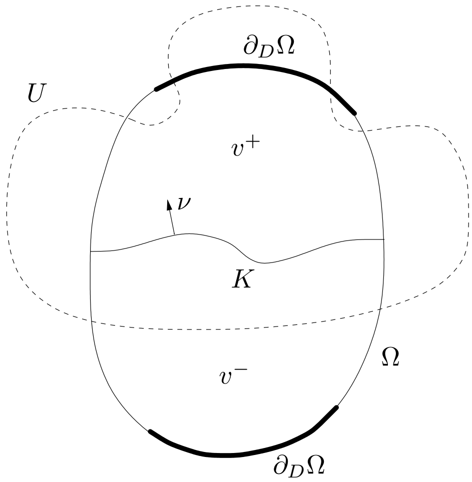

FIGURE 1. An admissible subdomain U for a regular pair (K, v) (see Definition 2.2). Notice that U excludes the relative boundary of $\partial_D\Omega$.

denote by $\nu_{\partial\Omega}$ the exterior unit normal vector to $\partial\Omega$ and by $H_{\partial\Omega}$ the curvature of $\partial\Omega$ with respect to $\nu_{\partial\Omega}$.

DEFINITION 2.1. We say that $(K, v) \in \mathcal{A}(\Omega)$ is a *regular pair* if K is a curve of class C^∞ connecting two points of $\partial\Omega$, and there exists $\partial_D\Omega \subset\subset \partial\Omega \setminus K$ relatively open in $\partial\Omega$ such that v is a solution to

$$\int_{\Omega\setminus K} \nabla v \cdot \nabla z \, \mathrm{d}x = 0 \qquad \text{for every } z \in H^1(\Omega \setminus K) \text{ with } z = 0 \text{ on } \partial_D\Omega, \tag{2.1}$$

that is, v is a weak solution to

$$\begin{cases} \Delta v = 0 & \text{in } \Omega \setminus K, \\ \partial_\nu v^\pm = 0 & \text{on } K \cap \Omega, \\ \partial_{\nu_{\partial\Omega}} v = 0 & \text{on } \partial_N\Omega := \partial\Omega \setminus \partial_D\Omega. \end{cases}$$

We denote by $\mathcal{A}_{reg}(\Omega)$ the space of all such pairs.

DEFINITION 2.2. Given a regular pair $(K, v) \in \mathcal{A}_{reg}(\Omega)$, we say that an open subset $U \subset \mathbb{R}^2$ with Lipschitz boundary is an *admissible subdomain* if $K \subset U$ and $\overline{U} \cap \mathcal{S} = \emptyset$, where $\mathcal{S}$ denotes the relative boundary of $\partial_D\Omega$ in $\partial\Omega$. In this case we define the space $H^1_U(\Omega \setminus K)$ consisting of all functions $v \in H^1(\Omega \setminus K)$ such that $v = 0$ in $(\Omega \setminus U) \cup \partial_D\Omega$ (the condition on $\partial_D\Omega$ has to be intended in the sense of traces). Notice that equation (2.1) holds for every $z \in H^1_U(\Omega \setminus K)$.

We now give the definition of regular critical pair, motivated by the formula for the first variation of the functional (see (2.12) and Remark 2.18).

DEFINITION 2.3. We say that a regular pair $(\Gamma, u) \in \mathcal{A}_{reg}(\Omega)$ is a *regular critical pair* for the Mumford-Shah functional $\mathcal{MS}$ if the following conditions are satisfied:

(i) Γ meets $\partial\Omega$ orthogonally,
(ii) *transmission condition*:

$$H = |\nabla_\Gamma u^+|^2 - |\nabla_\Gamma u^-|^2 \qquad \text{on } \Gamma \cap \Omega, \tag{2.2}$$

(iii) *non-vanishing jump condition*: $|u^+ - u^-| \geq c > 0$ on Γ.

REMARK 2.4. The assumption of C^∞-regularity of the curve Γ is not so restrictive as it may appear: indeed, as a consequence of the transmission condition (2.2) and of the fact that u satisfies (2.1), Γ is automatically analytical as soon as it is of class $C^{1,\alpha}$ (see [59]). Moreover, by (2.1) u is of class C^∞ up to $\Gamma \cap \Omega$ and the traces ∇u^+, ∇u^- of ∇u are well defined on both sides of Γ.

Besides the notion of critical pair, which amounts to the vanishing of the *first* variation of the functional, we also introduce the concept of stability, which is defined in terms of the positivity of the *second* variation. Its explicit expression at a regular critical pair (Γ, u) in an admissible subdomain U, which will be computed in Theorem 2.14, motivates the definition of the quadratic form $\partial^2\mathcal{MS}((\Gamma,u);U) : H^1(\Gamma\cap\Omega) \to \mathbb{R}$ given by

$$\begin{aligned}\partial^2\mathcal{MS}((\Gamma,u);U)[\varphi] := &-2\int_\Omega |\nabla v_\varphi|^2\,\mathrm{d}x + \int_{\Gamma\cap\Omega}|\nabla_\Gamma\varphi|^2\,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma\cap\Omega}H^2\varphi^2\,\mathrm{d}\mathcal{H}^1\\ &-\int_{\Gamma\cap\partial\Omega}H_{\partial\Omega}\,\varphi^2\,\mathrm{d}\mathcal{H}^0\end{aligned} \tag{2.3}$$

where $v_\varphi \in H^1_U(\Omega\setminus\Gamma)$ solves

$$\int_\Omega \nabla v_\varphi\cdot\nabla z\,\mathrm{d}x + \int_{\Gamma\cap\Omega}\big[z^+\mathrm{div}_\Gamma(\varphi\nabla_\Gamma u^+) - z^-\mathrm{div}_\Gamma(\varphi\nabla_\Gamma u^-)\big]\,\mathrm{d}\mathcal{H}^1 = 0 \tag{2.4}$$

for every $z \in H^1_U(\Omega\setminus\Gamma)$. Notice that the last integral in (2.3) in fact reduces to the sum $H_{\partial\Omega}(x_1)\varphi^2(x_1) + H_{\partial\Omega}(x_2)\varphi^2(x_2)$, where x_1 and x_2 are the intersections of Γ with $\partial\Omega$. The (nonlocal) dependence on U is realized through the function v_φ.

REMARK 2.5. The second integral in equation (2.4) has to be intended in the duality sense between $H^{-\frac12}(\Gamma\cap\Omega)$ and $H^{\frac12}(\Gamma\cap\Omega)$. Indeed, by Lemma 1.14 the product $\varphi\nabla_\Gamma u^\pm$ belongs to $H^{\frac12}(\Gamma\cap\Omega)$ as long as $\nabla_\Gamma u^\pm \in C^{0,\alpha}(\Gamma)$ for some $\alpha > \frac12$. In turn, the latter regularity property is guaranteed by Lemma 2.48, recalling that u solves (2.1).

DEFINITION 2.6. We say that a regular critical pair (Γ,u) (see Definition 2.3) is *strictly stable* in an admissible subdomain U if

$$\partial^2\mathcal{MS}((\Gamma,u);U)[\varphi] > 0 \qquad \text{for every } \varphi\in H^1(\Gamma\cap\Omega)\setminus\{0\}. \tag{2.5}$$

Our aim is to discuss the relation between the notion of strict stability of a regular critical pair and the one of local minimality. It is easily seen that the positive semidefiniteness of the quadratic form $\partial^2\mathcal{MS}((\Gamma,u);U)$ is a necessary condition for local minimality in U (see [19, Theorem 3.15]). In the main result of this chapter we prove that its *strict positivity* is in fact a *sufficient* condition for a regular critical pair to be a local minimizer in the L^1-sense:

THEOREM 2.7. *Let (Γ,u) be a strictly stable regular critical pair in an admissible subdomain U, according to Definition 2.6. Then (Γ,u) is an isolated local minimizer for $\mathcal{MS}$ in U, in the sense that there exists $\delta>0$ such that*

$$\mathcal{MS}(\Gamma,u) < \mathcal{MS}(K,v) \tag{2.6}$$

for every $(K,v)\in\mathcal{A}(\Omega)$ such that $v=u$ in $(\Omega\setminus U)\cup\partial_D\Omega$ and $0<\|u-v\|_{L^1(\Omega)}<\delta$.

REMARK 2.8. In order to simplify the proofs and the notations we decided to state and prove the previous result only in the simplified situation where Ω is connected and Γ is a regular curve joining two points of $\partial\Omega$. It is straightforward to check that Theorem 2.7 can be generalized to the case where Γ is a finite, disjoint union of curves of class C^∞, each one connecting two points of $\partial\Omega$ and meeting $\partial\Omega$ orthogonally.

REMARK 2.9. The non-vanishing jump condition (point (iii) of Definition 2.3) is not a technical assumption and cannot be dropped: indeed, it is possible to construct examples (see the Remark after Theorem 3.1 in [29]) satisfying all the assumptions of Theorem 2.7 except for this one, for which the conclusion of the theorem does not hold. In our strategy, this hypothesis is needed in order to deduce, in Proposition 2.36, by applying the calibration constructed in [67], that the unique solution of the penalization problem (2.65) is u itself , if β is sufficiently large.

We conclude with the following consequence of Theorem 2.7, which states that given any family of equicoercive functionals $\mathcal{F}_\varepsilon$ which Γ-converge to the relaxed version of $\mathcal{MS}$ with respect to the L^1-topology, we can approximate each strictly stable regular critical pair for $\mathcal{MS}$ by a sequence of local minimizers of the functionals $\mathcal{F}_\varepsilon$. This follows from the abstract result observed in [60, Theorem 4.1]. There is a vast literature concerning the approximation of the Mumford-Shah functional in the sense of Γ-convergence (see, for instance, [17]).

THEOREM 2.10 (link with Γ-convergence). *Let (Γ, u) be a strictly stable regular critical pair in an admissible subdomain U. Let $\mathcal{F}_\varepsilon : L^1(\Omega) \to \mathbb{R} \cup \{+\infty\}$ be a family of equicoercive and lower semi-continuous functionals which Γ-converge as $\varepsilon \to 0$ to the relaxed functional*

$$\overline{\mathcal{MS}}(v) := \begin{cases} \int_\Omega |\nabla v|^2 \,dx + \mathcal{H}^1(S_v) & \text{if } v \in SBV(\Omega),\ v = u \text{ on } (\Omega \setminus U) \cup \partial_D \Omega, \\ +\infty & \text{otherwise in } L^1(\Omega) \end{cases}$$

with respect to the L^1-topology. Then there exists $\varepsilon_0 > 0$ and a family $(u_\varepsilon)_{\varepsilon < \varepsilon_0}$ of local minimizers of $\mathcal{F}_\varepsilon$ such that $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$.

2.2. Computation of the second variation of the Mumford-Shah functional

This section is devoted to the computation of the second variation of the functional $\mathcal{MS}$. To start with, we fix some notation: for any one-parameter family of functions $(g_s)_{s\in\mathbb{R}}$ we denote the partial derivative with respect to the variable s of the map $(s, x) \mapsto g_s(x)$, evaluated at (t, x), by $\dot{g}_t(x)$. We usually omit the subscript when $t = 0$. In the following, we fix a regular pair $(K, v) \in \mathcal{A}_{reg}(\Omega)$ and an admissible subdomain U.

DEFINITION 2.11. A flow $(\Phi_t)_t$ is said to be *admissible* for (K, v) in U if it is generated by a vector field $X \in C^2(\mathbb{R}^2; \mathbb{R}^2)$ such that $\operatorname{supp} X \subset\subset U \setminus \partial_D \Omega$ and $X \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$, that is, Φ_t satisfies $\dot{\Phi}_t = X \circ \Phi_t$, $\Phi_0 = Id$.

REMARK 2.12. The condition $X \cdot \nu_{\partial\Omega} = 0$ guarantees that the trajectories of points in $\partial\Omega$ remain on $\partial\Omega$: thus $\Phi_t(\overline{\Omega}) = \overline{\Omega}$ for every t. Observe also that, since $\operatorname{supp} X \subset\subset U \setminus \partial_D \Omega$, we have that $K_{\Phi_t} \subset U \setminus \partial_D \Omega$ for every t, where we set $K_{\Phi_t} := \Phi_t(K)$.

Given an orientation preserving diffeomorphism $\Phi \in C^\infty(\overline{\Omega}; \overline{\Omega})$ such that $\operatorname{supp}(\Phi - Id) \subset\subset U \setminus \partial_D \Omega$, we define v_Φ as the unique solution in $H^1(\Omega \setminus K_\Phi)$ (up to additive constants in the connected components of $\Omega \setminus K_\Phi$ whose boundary does not contain $\partial_D \Omega$) to

$$\begin{cases} \displaystyle\int_{\Omega \setminus K_\Phi} \nabla v_\Phi \cdot \nabla z \,dx = 0 & \text{for every } z \in H^1_U(\Omega \setminus K_\Phi), \\ v_\Phi = v & \text{in } (\Omega \setminus U) \cup \partial_D \Omega. \end{cases} \tag{2.7}$$

DEFINITION 2.13. Let $(\Phi_t)_t$ be an admissible flow for (K, v) in U. We define the *first and second variations of $\mathcal{MS}$ at (K, v) in U along $(\Phi_t)_t$* to be

$$\frac{d}{dt}\mathcal{MS}((K_{\Phi_t}, v_{\Phi_t}); U)|_{t=0}, \qquad \frac{d^2}{dt^2}\mathcal{MS}((K_{\Phi_t}, v_{\Phi_t}); U)|_{t=0}$$

respectively, where v_{Φ_t} is defined as in (2.7) with Φ replaced by Φ_t.

Notice that this definition makes sense since the existence of the derivatives is guaranteed by the regularity result proved in [19, Proposition 8.1], which can be adapted to the present setting. In particular, this result implies that the map $(t,x)\mapsto v_{\Phi_t}(x)$ is differentiable with respect to the variable t and that $\dot v_{\Phi_t}\in H^1_U(\Omega\setminus K_{\Phi_t})$. We set $\dot v := \dot v_{\Phi_0}$

In the following theorem we compute explicitly the second variation of the functional $\mathcal{MS}$. We stress that, comparing with the analogous result obtained in [19, Theorem 3.6], we allow here the admissible variations to affect also the intersection of the discontinuity set K with the boundary of Ω, while in the quoted paper only variations compactly supported in Ω were considered. As a consequence, in the present situation boundary terms arise when integration by parts are performed: in particular this happens for the derivatives of the surface term, while the first and second variations of the volume term remain unchanged. We refer also to [78], where a similar computation for the second variation of the surface area was carried out taking into account boundary effects, in the case of a critical set (the novelty here is that we will be able to get an expression of the second variation at a generic regular pair, not necessarily critical).

THEOREM 2.14. *Let $(K,v)\in\mathcal{A}_{reg}(\Omega)$ be a regular pair, let U be an admissible subdomain, and let $(\Phi_t)_t$ be an admissible flow in U associated to a vector field X. Then the function $\dot v$ belongs to $H^1_U(\Omega\setminus K)$ and satisfies the equation*

$$\int_\Omega \nabla\dot v\cdot\nabla z\,\mathrm{d}x+\int_{K\cap\Omega}\Big[\mathrm{div}_K\big((X\cdot\nu)\nabla_K v^+\big)z^+-\mathrm{div}_K\big((X\cdot\nu)\nabla_K v^-\big)z^-\Big]\,\mathrm{d}\mathcal{H}^1=0 \tag{2.8}$$

for every $z\in H^1_U(\Omega\setminus K)$. Moreover, the second variation of $\mathcal{MS}$ at (K,v) in U along $(\Phi_t)_t$ is given by

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{MS}((K_{\Phi_t},v_{\Phi_t});U)|_{t=0}&=2\int_{K\cap\Omega}(\dot v^+\partial_\nu\dot v^+-\dot v^-\partial_\nu\dot v^-)\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\Omega}|\nabla_K(X\cdot\nu)|^2\,\mathrm{d}\mathcal{H}^1\\&+\int_{K\cap\Omega}H^2(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\Omega}f(Z\cdot\nu-2X^\parallel\cdot\nabla_K(X\cdot\nu)+D\nu[X^\parallel,X^\parallel]-H(X\cdot\nu)^2)\,\mathrm{d}\mathcal{H}^1\\&+\int_{K\cap\partial\Omega}(f-H)(X\cdot\nu)(X\cdot\eta)\,\mathrm{d}\mathcal{H}^0+\int_{K\cap\partial\Omega}Z\cdot\eta\,\mathrm{d}\mathcal{H}^0,\end{aligned}\tag{2.9}$$

where $f:=|\nabla_K v^-|^2-|\nabla_K v^+|^2+H$, $Z:=DX[X]$, and we split the field X in its tangential and normal components to K:

$$X=X^\parallel+(X\cdot\nu)\nu\quad \text{on } K. \tag{2.10}$$

REMARK 2.15. As in (2.4), the second integral in equation (2.8) has to be intended in the duality sense between $H^{-\frac12}(K\cap\Omega)$ and $H^{\frac12}(K\cap\Omega)$ (see Remark 2.5). Integrations by parts yields

$$-\int_\Omega|\nabla\dot v|^2\,\mathrm{d}x=\int_{K\cap\Omega}\big[\dot v^+\partial_\nu\dot v^+-\dot v^-\partial_\nu\dot v^-\big]\,\mathrm{d}\mathcal{H}^1.$$

Before proving Theorem 2.14, we collect in the following lemma some auxiliary identities which will be used in the computation of the second variation. We recall that, according to the notation introduced in Section 1.1, we denote by ν_{Φ_t} and η_{Φ_t} the unit normal to K_{Φ_t} and the unit co-normal of $K_{\Phi_t}\cap\partial\Omega$, respectively (see (1.7) and (1.10)).

LEMMA 2.16. *The following identities hold:*

(a) $\dot\nu=-(D_KX)^T[\nu]-D_K\nu[X]=-\nabla_K(X\cdot\nu)$ *on* K*;*

(b) $\frac{\partial}{\partial t}(\eta_{\Phi_t}\circ\Phi_t)|_{t=0}=(D_KX)^T[\nu,\eta]\nu$ *on* $K\cap\partial\Omega$*;*

(c) $(X\cdot\nu)\dot\nu\cdot\eta + X\cdot\frac{\partial}{\partial t}(\eta_{\Phi_t}\circ\Phi_t)|_{t=0} = -H(X\cdot\nu)(X\cdot\eta)$ *on* $K\cap\partial\Omega$;

(d) $DX[X,\nu_{\partial\Omega}] + D\nu_{\partial\Omega}[X,X] = 0$ *on* $K\cap\partial\Omega$.

PROOF. Equality (a) is proved in [19, Lemma 3.8, (f)]. To prove (b), we set $v_t := D\Phi_t[\eta]$ and recalling (1.10) we have

$$\begin{aligned}\frac{\partial}{\partial t}(\eta_{\Phi_t}\circ\Phi_t)|_{t=0} &= \frac{\partial}{\partial t}\left(\frac{v_t}{|v_t|}\right)\bigg|_{t=0} = \dot v - (\dot v\cdot\eta)\eta\\ &= DX[\eta] - DX[\eta,\eta]\eta = DX[\eta,\nu]\nu,\end{aligned}$$

which is (b). We obtain (c) by combining (a) and (b):

$$(X\cdot\nu)\dot\nu\cdot\eta + X\cdot\frac{\partial}{\partial t}(\eta_{\Phi_t}\circ\Phi_t)|_{t=0} = -(X\cdot\nu)D_K\nu[X,\eta] = -H(X\cdot\nu)(X\cdot\eta),$$

where the last equality follows by writing $X = (X\cdot\nu)\nu + (X\cdot\eta)\eta$ and observing that $D_K\nu[\nu] = 0$. Equation (d) follows by differentiating with respect to t at $t=0$ the identity

$$(X\circ\Phi_t)\cdot(\nu_{\partial\Omega}\circ\Phi_t) = 0,$$

which holds on $K\cap\partial\Omega$. □

PROOF OF THEOREM 2.14. We split the proof of the theorem into three steps.

Step 1. Derivation of the equation solved by $\dot v$. As already observed, the result contained in [19, Proposition 8.1] guarantees that $\dot v\in H^1_U(\Omega\setminus K)$. Given any test function $z\in H^1_U(\Omega\setminus K)$ with $\operatorname{supp} z\cap K=\emptyset$, for t small enough we have $\operatorname{supp} z\subset\Omega\setminus K_{\Phi_t}$, and in particular $z\in H^1_U(\Omega\setminus K_{\Phi_t})$. Hence by (2.7) we deduce

$$\int_\Omega \nabla v_{\Phi_t}\cdot\nabla z\,\mathrm{d}x = 0,$$

so that differentiating with respect to t at $t=0$ we obtain that $\dot v$ is harmonic in $(\Omega\cap U)\setminus K$ and $\nabla\dot v\cdot\nu_{\partial\Omega}=0$ on $(\partial\Omega\cap U)\setminus\partial_D\Omega$. In addition, it is shown in Step 1 of the proof of [19, Theorem 3.6] that

$$\partial_\nu\dot v^\pm = \operatorname{div}_K((X\cdot\nu)\nabla_K v^\pm) \qquad \text{on } K\cap\Omega.$$

By this expression we have that $\partial_\nu\dot v^\pm\in H^{-\frac12}(K\cap\Omega)$ (see Remark 2.5), and hence the previous conditions are equivalent to (2.8) by integration by parts.

Step 2. Computation of the first variation. The same computation carried out in Step 2 of the proof of [19, Theorem 3.6] leads to

$$\frac{d}{dt}\int_\Omega |\nabla v_{\Phi_t}|^2\,\mathrm{d}x = \int_\Omega \operatorname{div}\big(|\nabla v_{\Phi_t}|^2 X\big)\,\mathrm{d}y\,.$$

Hence, applying the divergence theorem we obtain

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla v_{\Phi_t}|^2\,\mathrm{d}x &= \int_{\partial\Omega}|\nabla v_{\Phi_t}|^2(X\cdot\nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^1 + \int_{K_{\Phi_t}\cap\Omega}\big(|\nabla v^-_{\Phi_t}|^2 - |\nabla v^+_{\Phi_t}|^2\big)(X\cdot\nu_{\Phi_t})\,\mathrm{d}\mathcal{H}^1\\ &= \int_{K_{\Phi_t}\cap\Omega}\big(|\nabla_{K_{\Phi_t}} v^-_{\Phi_t}|^2 - |\nabla_{K_{\Phi_t}} v^+_{\Phi_t}|^2\big)(X\cdot\nu_{\Phi_t})\,\mathrm{d}\mathcal{H}^1\end{aligned}$$

where to deduce the last equality we used $X\cdot\nu_{\partial\Omega}=0$ and the fact that $\partial_{\nu_{\Phi_t}}v^\pm_{\Phi_t}$ vanishes on K_{Φ_t}. Concerning the surface term, we start from the well known formula for the first variation of the area functional (see, for instance, [75, Chapter 2, Section 9]) and we use the divergence theorem on $K_{\Phi_t}\cap\Omega$ (see (1.9)) to obtain

$$\begin{aligned}\frac{d}{dt}\mathcal{H}^1(K_{\Phi_t}\cap\Omega) &= \int_{K_{\Phi_t}\cap\Omega}\mathrm{div}_{K_{\Phi_t}}X\,\mathrm{d}\mathcal{H}^1\\ &= \int_{K_{\Phi_t}\cap\Omega}H_{\Phi_t}(X\cdot\nu_{\Phi_t})\,\mathrm{d}\mathcal{H}^1+\int_{K_{\Phi_t}\cap\partial\Omega}X\cdot\eta_{\Phi_t}\,\mathrm{d}\mathcal{H}^0\,,\end{aligned}$$

where we recall that H_{Φ_t} stands for the curvature of K_{Φ_t}. Thus we can conclude that

$$\frac{d}{dt}\mathcal{MS}((K_{\Phi_t},v_{\Phi_t});U) = \int_{K_{\Phi_t}\cap\Omega}f_t(X\cdot\nu_{\Phi_t})\,\mathrm{d}\mathcal{H}^1+\int_{K_{\Phi_t}\cap\partial\Omega}X\cdot\eta_{\Phi_t}\,\mathrm{d}\mathcal{H}^0\,, \tag{2.11}$$

where $f_t := |\nabla_{K_{\Phi_t}}v^-_{\Phi_t}|^2-|\nabla_{K_{\Phi_t}}v^+_{\Phi_t}|^2+H_{\Phi_t}$. In particular, evaluating (2.11) at $t=0$ we obtain

$$\frac{d}{dt}\mathcal{MS}((K_{\Phi_t},v_{\Phi_t});U)|_{t=0} = \int_{K\cap\Omega}f(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\partial\Omega}X\cdot\eta\,\mathrm{d}\mathcal{H}^0\,. \tag{2.12}$$

Step 3. Computation of the second variation. We have to differentiate again (2.11) at $t=0$. By a change of variables we have

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{MS}((K_{\Phi_t},v_{\Phi_t});U)|_{t=0} &= \int_{K\cap\Omega}\frac{\partial}{\partial t}(f_t\circ\Phi_t)|_{t=0}(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1\\ &\quad+\int_{K\cap\Omega}f\frac{\partial}{\partial t}\big(\dot\Phi_t\cdot(\nu_{\Phi_t}\circ\Phi_t)J_{\Phi_t}\big)\big|_{t=0}\,\mathrm{d}\mathcal{H}^1+\frac{d}{dt}\Big(\int_{K_{\Phi_t}\cap\partial\Omega}X\cdot\eta_{\Phi_t}\,\mathrm{d}\mathcal{H}^0\Big)\Big|_{t=0}\\ &=: I_1+I_2+I_3\,.\end{aligned} \tag{2.13}$$

The first integral I_1 is equal to

$$I_1 = \int_{K\cap\Omega}\dot f(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\Omega}(\nabla f\cdot\nu)(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\Omega}(\nabla_K f\cdot X^{\|})(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1\,, \tag{2.14}$$

while using [19, Lemma 3.8, (g)] we have

$$I_2 = \int_{K\cap\Omega}f\mathrm{div}_K((X\cdot\nu)X)\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\Omega}f\big(Z\cdot\nu-2X^{\|}\cdot\nabla_K(X\cdot\nu)+D\nu[X^{\|},X^{\|}]\big)\,\mathrm{d}\mathcal{H}^1\,. \tag{2.15}$$

Applying the divergence formula (1.9) on $K\cap\Omega$ we obtain

$$\begin{aligned}\int_{K\cap\Omega}(\nabla_K f\cdot X^{\|})(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\Omega}f\mathrm{div}_K((X\cdot\nu)X)\,\mathrm{d}\mathcal{H}^1&\\ =\int_{K\cap\Omega}fH(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\partial\Omega}f(X\cdot\nu)(X\cdot\eta)\,\mathrm{d}\mathcal{H}^0\,,&\end{aligned} \tag{2.16}$$

while using [19, formula (3.17)] we get

$$\int_{K\cap\Omega}(\nabla f\cdot\nu)(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^1 = \int_{K\cap\Omega}(H^2-2fH)(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^1. \tag{2.17}$$

Differentiating f_t with respect to t we obtain

$$\int_{K\cap\Omega}\dot f(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1 = \int_{K\cap\Omega}(2\nabla_K v^-\cdot\nabla_K\dot v^- - 2\nabla_K v^+\cdot\nabla_K\dot v^+ + \dot H)(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1, \tag{2.18}$$

and an integration by parts yields

$$
\begin{aligned}
2\int_{K\cap\Omega}&(\nabla_K v^{\pm}\cdot\nabla_K \dot v^{\pm})(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1\\
&=-2\int_{K\cap\Omega}\dot v^{\pm}\mathrm{div}_K((X\cdot\nu)\nabla_K v^{\pm})\,\mathrm{d}\mathcal{H}^1+2\int_{K\cap\partial\Omega}\dot v^{\pm}(X\cdot\nu)(\nabla_K v^{\pm}\cdot\eta)\,\mathrm{d}\mathcal{H}^0\\
&=-2\int_{K\cap\Omega}\dot v^{\pm}\partial_\nu\dot v^{\pm}\,\mathrm{d}\mathcal{H}^1,
\end{aligned}
\tag{2.19}
$$

where the last equality follows by (2.8) and by observing that $\nabla v^{\pm}$ vanishes on $K\cap\partial\Omega$, as v satisfies homogeneous Neumann boundary conditions on K and on $\partial\Omega$ (∇v is regular up to $K\cap\partial\Omega$ by Lemma 2.48). Since $\partial_\nu\dot\nu\cdot\nu=-\dot\nu\cdot\partial_\nu\nu=0$, we have $\mathrm{div}\dot\nu=\mathrm{div}_K\dot\nu$ and in turn $\dot H=\mathrm{div}_K\dot\nu$. Hence, integrating by parts and using (a) of Lemma 2.16, we deduce

$$
\begin{aligned}
\int_{K\cap\Omega}\dot H(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1&=\int_{K\cap\Omega}\mathrm{div}_K\dot\nu(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1\\
&=-\int_{K\cap\Omega}\dot\nu\cdot\nabla_K(X\cdot\nu)\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\partial\Omega}(X\cdot\nu)(\dot\nu\cdot\eta)\,\mathrm{d}\mathcal{H}^0\\
&=\int_{K\cap\Omega}|\nabla_K(X\cdot\nu)|^2\,\mathrm{d}\mathcal{H}^1+\int_{K\cap\partial\Omega}(X\cdot\nu)(\dot\nu\cdot\eta)\,\mathrm{d}\mathcal{H}^0.
\end{aligned}
\tag{2.20}
$$

We finally compute I_3:

$$
\begin{aligned}
I_3&=\frac{d}{dt}\biggl(\int_{K_{\Phi_t}\cap\partial\Omega}X\cdot\eta_{\Phi_t}\,\mathrm{d}\mathcal{H}^0\biggr)\Big|_{t=0}=\sum_{x\in K\cap\partial\Omega}\frac{\partial}{\partial t}\Bigl(X(\Phi_t(x))\cdot\eta_{\Phi_t}(\Phi_t(x))\Bigr)\Big|_{t=0}\\
&=\sum_{x\in K\cap\partial\Omega}Z(x)\cdot\eta(x)+\sum_{x\in K\cap\partial\Omega}X(x)\cdot\frac{\partial}{\partial t}(\eta_{\Phi_t}\circ\Phi_t(x))|_{t=0}\,.
\end{aligned}
\tag{2.21}
$$

Collecting (2.13)–(2.21), and using equality (c) of Lemma 2.16, we finally obtain (2.9). $\square$

Remark 2.17. We observe that we can easily obtain an expression for the second variation of the functional $\mathcal{MS}$ at a generic t. Indeed, by exploiting the property $\Phi_{t+s}=\Phi_t\circ\Phi_s$ of the flow, we have

$$
\begin{aligned}
\frac{d^2}{dh^2}\mathcal{MS}((K_{\Phi_h},v_{\Phi_h});U)|_{h=t}&=\frac{d^2}{ds^2}\mathcal{MS}((\Phi_{t+s}(K),v_{\Phi_{t+s}});U)|_{s=0}\\
&=\frac{d^2}{ds^2}\mathcal{MS}(\Phi_s(K_{\Phi_t}),(v_{\Phi_t})_{\Phi_s})|_{s=0},
\end{aligned}
$$

and we can directly apply Theorem 2.14 to the regular pair (K_{Φ_t},v_{Φ_t}).

Remark 2.18. The formula (2.12) for the first variation of $\mathcal{MS}$ motivates the definition of critical pair (see Definition 2.3). Indeed, assuming that (2.12) vanishes for each vector field X which is tangent to $\partial\Omega$, we first obtain that $f=0$ on $K\cap\Omega$ by considering arbitrary vector fields with $\operatorname{supp}X\subset\subset\Omega$. Then, using this information and dropping the requirement on the support of X, we deduce the orthogonality of K and $\partial\Omega$.

Corollary 2.19. *Assume that (Γ,u) is a regular critical pair. Then*

$$
\begin{aligned}
\frac{d^2}{dt^2}\mathcal{MS}((\Gamma_{\Phi_t},u_{\Phi_t});U)|_{t=0}&=-2\int_\Omega|\nabla\dot u|^2\,\mathrm{d}x+\int_{\Gamma\cap\Omega}|\nabla_\Gamma(X\cdot\nu)|^2\,\mathrm{d}\mathcal{H}^1\\
&\quad+\int_{\Gamma\cap\Omega}H^2(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^1-\int_{\Gamma\cap\partial\Omega}H_{\partial\Omega}(X\cdot\nu)^2\,\mathrm{d}\mathcal{H}^0\,,
\end{aligned}
\tag{2.22}
$$

where $H_{\partial\Omega}:=\mathrm{div}\nu_{\partial\Omega}$ denotes the curvature of $\partial\Omega$.

PROOF. The first integral in (2.9) can be rewritten as $-2\int_\Omega |\nabla \dot{u}|^2 \,\mathrm{d}x$ thanks to (2.8) (see Remark 2.15). To obtain the expression in (2.22) it is now sufficient to observe that at a critical pair we have $f = 0$ on $K \cap \Omega$, $X \cdot \eta = X \cdot \nu_{\partial\Omega} = 0$ on $K \cap \partial\Omega$, and

$$Z \cdot \eta = DX[X, \nu_{\partial\Omega}] = -D\nu_{\partial\Omega}[X, X] = -(X \cdot \nu)^2 D\nu_{\partial\Omega}[\nu, \nu] = -H_{\partial\Omega}(X \cdot \nu)^2$$

on $K \cap \partial\Omega$ by (d) of Lemma 2.16. □

2.3. The stability condition

In the following we assume that (Γ, u) is a regular critical pair and U is an admissible subdomain. Notice that the expression of the second variation of $\mathcal{MS}$ at (Γ, u) proved in Corollary 2.19 motivates the definition of the quadratic form (2.3) and the notion of strict stability that we introduced in Definition 2.6.

Following the approach of [19], we start paving the way for the main result by proving two equivalent formulations of condition (2.5), one in terms of the first eigenvalue of a suitable compact linear operator defined on $H^1(\Gamma \cap \Omega)$ and the other in terms of a dual minimum problem. Let us start by introducing the following bilinear form on $H^1(\Gamma \cap \Omega)$:

$$(\varphi, \psi)_\sim := \int_{\Gamma\cap\Omega} \nabla_\Gamma \varphi \cdot \nabla_\Gamma \psi \,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma\cap\Omega} H^2\, \varphi\, \psi \,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma\cap\partial\Omega} H_{\partial\Omega}\, \varphi\, \psi \,\mathrm{d}\mathcal{H}^0 \tag{2.23}$$

for every $\varphi, \psi \in H^1(\Gamma \cap \Omega)$.

PROPOSITION 2.20. *Assume that*

$$(\varphi, \varphi)_\sim > 0 \qquad \textit{for every } \varphi \in H^1(\Gamma \cap \Omega) \setminus \{0\}. \tag{2.24}$$

Then $(\cdot,\cdot)_\sim$ is a scalar product which defines an equivalent norm on $H^1(\Gamma \cap \Omega)$, that will be denoted by $\|\cdot\|_\sim$.

PROOF. Clearly (2.24) implies that $(\cdot,\cdot)_\sim$ is a scalar product, and $\|\varphi\|_\sim \le C\|\varphi\|_{H^1(\Gamma\cap\Omega)}$ for every $\varphi \in H^1(\Gamma \cap \Omega)$, for some positive constant C. To complete the proof we have to show the opposite inequality.

Assume by contradiction the existence of a sequence $\varphi_n \in H^1(\Gamma\cap\Omega)$ such that $\|\varphi_n\|_\sim \le \frac{1}{n}$ and $\|\varphi_n\|_{H^1(\Gamma\cap\Omega)} = 1$ for every n. By compactness, φ_n converges weakly in $H^1(\Gamma \cap \Omega)$ to some φ, and uniformly on $\Gamma \cap \overline{\Omega}$, hence

$$\begin{aligned}
\int_{\Gamma\cap\Omega} H^2\, \varphi^2 \,\mathrm{d}\mathcal{H}^1 &= \lim_{n\to\infty} \int_{\Gamma\cap\Omega} H^2\, \varphi_n^2 \,\mathrm{d}\mathcal{H}^1, \\
\int_{\Gamma\cap\partial\Omega} H_{\partial\Omega}\, \varphi^2 \,\mathrm{d}\mathcal{H}^0 &= \lim_{n\to+\infty} \int_{\Gamma\cap\partial\Omega} H_{\partial\Omega}\, \varphi_n^2 \,\mathrm{d}\mathcal{H}^0, \\
\int_{\Gamma\cap\Omega} |\nabla_\Gamma \varphi|^2 \,\mathrm{d}\mathcal{H}^1 &\le \liminf_{n\to\infty} \int_{\Gamma\cap\Omega} |\nabla_\Gamma \varphi_n|^2 \,\mathrm{d}\mathcal{H}^1,
\end{aligned} \tag{2.25}$$

and recalling that $\|\varphi_n\|_\sim \to 0$ we get $\|\varphi\|_\sim = 0$, that is $\varphi = 0$ (thanks to (2.24)). Now from the first two equalities in (2.25) we deduce that $\int_{\Gamma\cap\Omega} H^2\, \varphi_n^2 \,\mathrm{d}\mathcal{H}^1 \to 0$, $\int_{\Gamma\cap\partial\Omega} H_{\partial\Omega}\, \varphi_n^2 \,\mathrm{d}\mathcal{H}^0 \to 0$, and since $\|\varphi_n\|_\sim \to 0$, we conclude that

$$\int_{\Gamma\cap\Omega} |\nabla_\Gamma \varphi_n|^2 \,\mathrm{d}\mathcal{H}^1 \to 0,$$

which is in contradiction with $\|\varphi_n\|_{H^1(\Gamma\cap\Omega)} = 1$. □

The announced equivalent formulations of the strict stability of a critical pair (condition (2.5)) are stated in the following proposition.

PROPOSITION 2.21. *The following statements are equivalent.*

(i) *Condition* (2.5) *is satisfied.*
(ii) *Condition* (2.24) *holds, and the monotone, compact, self-adjoint operator* $T : H^1(\Gamma \cap \Omega) \to H^1(\Gamma \cap \Omega)$ *defined by duality as*

$$(T\varphi, \psi)_\sim = -2 \int_{\Gamma\cap\Omega} \left[v_\varphi^+ \mathrm{div}_\Gamma(\psi \nabla_\Gamma u^+) - v_\varphi^- \mathrm{div}_\Gamma(\psi \nabla_\Gamma u^-) \right] d\mathcal{H}^1 \tag{2.26}$$

for every $\varphi, \psi \in H^1(\Gamma \cap \Omega)$ *(where* v_φ *is defined in* (2.4)*), satisfies*

$$\lambda_1(U) := \max_{\|\varphi\|_\sim = 1} (T\varphi, \varphi)_\sim < 1 \tag{2.27}$$

(the dependence on U *is realized through the function* v_φ*).*
(iii) *Condition* (2.24) *holds, and defining, for* $v \in H^1_U(\Omega \setminus \Gamma)$*,* Φ_v *as the unique solution in* $H^1(\Gamma \cap \Omega)$ *to*

$$(\Phi_v, \psi)_\sim = -2 \int_{\Gamma\cap\Omega} \left[v^+ \mathrm{div}_\Gamma(\psi \nabla_\Gamma u^+) - v^- \mathrm{div}_\Gamma(\psi \nabla_\Gamma u^-) \right] d\mathcal{H}^1$$

for every $\psi \in H^1(\Gamma \cap \Omega)$*, one has*

$$\mu(U) := \min\left\{ 2 \int_\Omega |\nabla v|^2 \, dx : v \in H^1_U(\Omega \setminus \Gamma), \ \|\Phi_v\|_\sim = 1 \right\} > 1. \tag{2.28}$$

We will omit the dependence on U *for* λ_1 *and* μ *where there is no risk of ambiguity.*

Remark 2.22. By (2.4) we immediately have

$$(T\varphi, \psi)_\sim = 2 \int_\Omega \nabla v_\varphi \cdot \nabla v_\psi \, dx. \tag{2.29}$$

Moreover comparing with (2.3) we see that

$$\partial^2 \mathcal{MS}((\Gamma, u); U)[\varphi] = -(T\varphi, \varphi)_\sim + \|\varphi\|_\sim^2.$$

Proof of Proposition 2.21. The linear map

$$\psi \in H^1(\Gamma \cap \Omega) \mapsto -2 \int_{\Gamma\cap\Omega} \left[v_\varphi^+ \mathrm{div}_\Gamma(\psi \nabla_\Gamma u^+) - v_\varphi^- \mathrm{div}_\Gamma(\psi \nabla_\Gamma u^-) \right] d\mathcal{H}^1$$

is continuous on $H^1(\Gamma \cap \Omega)$ (recall Remark 2.5). Hence, if condition (2.24) is satisfied, then by Proposition 2.20 and by the Riesz Theorem the operator T is well defined. The monotonicity and the self-adjointness of T follow immediately from (2.29). We prove that T is compact: let $\varphi_n \rightharpoonup \varphi$ weakly in $H^1(\Gamma \cap \Omega)$; then

$$\mathrm{div}_\Gamma(\varphi_n \nabla_\Gamma u^\pm) \to \mathrm{div}_\Gamma(\varphi \nabla_\Gamma u^\pm) \quad \text{in } H^{-\frac{1}{2}}(\Gamma \cap \Omega),$$

which implies that $v_{\varphi_n} \rightharpoonup v_\varphi$ weakly in $H^1(\Omega \setminus \Gamma)$, and, by compactness of the trace operator, $v_{\varphi_n}^\pm \to v_\varphi^\pm$ in $H^{\frac{1}{2}}(\Gamma \cap \Omega)$. It follows from (2.26) that $(T\varphi_n, \psi)_\sim \to (T\varphi, \psi)_\sim$ for every $\psi \in H^1(\Gamma \cap \Omega)$, that is, $T\varphi_n \rightharpoonup T\varphi$ weakly in $H^1(\Gamma \cap \Omega)$. Moreover, by taking $\varphi = \varphi_n$ and $\psi = T\varphi_n$ in (2.26), we also deduce that $\|T\varphi_n\|_\sim^2 \to \|T\varphi\|_\sim^2$. Hence $T\varphi_n \to T\varphi$ in $H^1(\Gamma \cap \Omega)$, which completes the proof of the compactness of the operator T. From what we have shown it follows that, under the assumption (2.24), λ_1 is well defined.

Assuming condition (2.5), we have immediately

$$(\varphi, \varphi)_\sim > 2 \int_\Omega |\nabla v_\varphi|^2 \, dx \geq 0 \qquad \text{for every } \varphi \in H^1(\Gamma \cap \Omega) \setminus \{0\}.$$

Hence the equivalence of (i) and (ii) amounts to show that, under condition (2.24), (2.5) and (2.27) are equivalent: in turn, this follows immediately from Remark 2.22.

To complete the proof, we show that, under condition (2.24), one has $\lambda_1 = \frac{1}{\mu}$. Notice first that, arguing as before, one can prove that the map $v \mapsto \Phi_v$ is compact, so that μ is well defined. Let $\varphi \in H^1(\Gamma \cap \Omega)$, with $\|\varphi\|_\sim = 1$, be such that $\lambda_1 = (T\varphi, \varphi)_\sim$. Then, observing that $\Phi_{v_\varphi} = T\varphi = \lambda_1 \varphi$, we have that $\frac{v_\varphi}{\lambda_1}$ is an admissible competitor in (2.28), and

$$\mu \leq \frac{2}{\lambda_1^2} \int_\Omega |\nabla v_\varphi|^2 \, dx = \frac{1}{\lambda_1^2} (T\varphi, \varphi)_\sim = \frac{1}{\lambda_1} \, .$$

Conversely, let $\bar{v} \in H^1_U(\Omega \setminus \Gamma)$ be a solution to the minimum problem (2.28). Then there exists a Lagrange multiplier μ_0 such that

$$2 \int_\Omega \nabla \bar{v} \cdot \nabla z \, dx = \mu_0 (\Phi_{\bar{v}}, \Phi_z)_\sim$$

for every $z \in H^1_U(\Omega \setminus \Gamma)$. By taking $z = \bar{v}$, we obtain $\mu_0 = \mu$. Moreover, it follows also that $v_{\Phi_{\bar{v}}} = \frac{\bar{v}}{\mu}$, and hence we conclude

$$\lambda_1 \geq (T\Phi_{\bar{v}}, \Phi_{\bar{v}})_\sim = 2 \int_\Omega |\nabla v_{\Phi_{\bar{v}}}|^2 \, dx = \frac{2}{\mu^2} \int_\Omega |\nabla \bar{v}|^2 \, dx = \frac{1}{\mu}.$$

This completes the proof of the proposition. □

Corollary 2.23. *Assume* (2.5). *Then there exists a constant* $C > 0$ *such that*

$$\partial^2 \mathcal{MS}((\Gamma, u); U)[\varphi] \geq C \|\varphi\|^2_{H^1(\Gamma\cap\Omega)} \qquad \textit{for every } \varphi \in H^1(\Gamma \cap \Omega).$$

Proof. By Remark 2.22

$$\partial^2 \mathcal{MS}((\Gamma, u); U)[\varphi] = \|\varphi\|^2_\sim - (T\varphi, \varphi)_\sim \geq (1 - \lambda_1) \|\varphi\|^2_\sim,$$

hence the conclusion follows by Proposition 2.21 and Proposition 2.20. □

From the definition in (2.28) it is clear that μ depends monotonically on the domain U. This is made explicit by the following corollary.

Corollary 2.24. *Let* U_1, U_2 *be admissible subdomains for* (Γ, u), *with* $U_1 \subset U_2$. *Then* $\mu(U_1) \geq \mu(U_2)$. *In particular, if condition* (2.5) *is satisfied in* U_2, *then it also holds in* U_1.

Corollary 2.25. *Assume that condition* (2.5) *holds in* U. *Let* U_n *be a decreasing sequence of admissible subdomains for* (Γ, u) *such that* U *is the interior part of* $\bigcap_n U_n$. *Then* (2.5) *holds in* U_n, *if* n *is sufficiently large.*

Proof. In view of (2.28) it is sufficient to show that $\lim_n \mu(U_n) \geq \mu(U)$. Let $v_n \in H^1_{U_n}(\Omega \setminus \Gamma)$ be a solution to (2.28) with U replaced by U_n. Then $v_n \in H^1_{U_1}(\Omega \setminus \Gamma)$ and $2\int_\Omega |\nabla v_n|^2 \, dx = \mu(U_n) \leq \mu(U)$, where the inequality follows from Corollary 2.24. Hence, up to subsequences, $v_n \rightharpoonup v \in H^1_{U_1}(\Omega \setminus \Gamma)$. Moreover, $v = 0$ a.e. in $U_1 \setminus U$, so that $v \in H^1_U(\Omega \setminus \Gamma)$ and v is admissible in problem (2.28) (by the compactness of the map $v \mapsto \Phi_v$): we conclude that

$$\lim_{n\to\infty} \mu(U_n) = \lim_{n\to\infty} 2 \int_\Omega |\nabla v_n|^2 \, dx \geq 2 \int_\Omega |\nabla v|^2 \, dx \geq \mu(U),$$

as claimed. □

2.4. Local $W^{2,\infty}$-minimality

In this section, as a first step toward the proof of Theorem 2.7, we show how the strategy developed in [19] can be adapted to the present setting in order to prove that strict stability is a sufficient condition for a regular critical pair to be a local minimizer with respect to variations of class $W^{2,\infty}$ of the discontinuity set. For the rest of the section (Γ, u) will be a fixed strictly stable regular critical pair in an admissible subdomain U. For $\eta > 0$, we denote by

$$\mathcal{N}_\eta(\Gamma) := \{x \in \mathbb{R}^2 : \operatorname{dist}(x, \Gamma) < \eta\}$$

the η-tubular neighborhood of Γ.

In order to give a proper notion of sets which are close to Γ in the $W^{2,\infty}$-sense, we now introduce a suitable flow in U whose trajectories intersect Γ orthogonally. To this aim, we start by fixing $\eta_0 > 0$ such that $\mathcal{N}_{\eta_0}(\Gamma) \subset\subset U \setminus \partial_D\Omega$, and a vector field $X \in C^2(\mathbb{R}^2; \mathbb{R}^2)$ such that $\operatorname{supp} X \subset\subset U \setminus \partial_D\Omega$, $X = \nu$ on Γ, $X \cdot \nu_{\partial\Omega} = 0$ on $\partial\Omega$, and $|X| = 1$ in $\mathcal{N}_{\eta_0}(\Gamma)$. We denote by $\Psi : \mathbb{R} \times \overline{\Omega} \to \overline{\Omega}$ the flow generated by X:

$$\frac{\partial}{\partial t}\Psi(t,x) = X(\Psi(t,x)), \qquad \Psi(0,x) = x.$$

Observe that (by taking a smaller η_0 if necessary) for every $y \in \mathcal{N}_{\eta_0}(\Gamma)$ are uniquely determined two points $\pi(y) \in \Gamma$ and $\tau(y) \in \mathbb{R}$ such that $y = \Psi(\tau(y), \pi(y))$. The existence of the maps π and τ, as well as the fact that they are of class C^2, is guaranteed by the Implicit Function Theorem.

We define, for $\delta > 0$, the following class of functions:

$$\mathcal{D}_\delta := \{\psi \in C^2(\Gamma) : \|\psi\|_{C^2(\Gamma)} < \delta\}.$$

We can extend each function $\psi \in \mathcal{D}_\delta$ to $\mathcal{N}_{\eta_0}(\Gamma)$ by setting $\psi(y) := \psi(\pi(y))$, in such a way that ψ is constant along the trajectories of the flow Ψ. We associate with ψ the diffeomorphism $\Phi^\psi(x) := \Psi(\psi(x), x)$, and we remark that

$$\|\Phi^\psi - Id\|_{C^2(\Gamma)} \le C\|\psi\|_{C^2(\Gamma)} \tag{2.30}$$

for some constant C independent of $\psi \in \mathcal{D}_\delta$. Finally, we define the set

$$\Gamma_\psi := \Phi^\psi(\Gamma) = \{\Psi(\psi(x), x) : x \in \Gamma\}, \tag{2.31}$$

and the function $u_\psi := u_{\Phi^\psi}$ as the unique solution in $H^1(\Omega \setminus \Gamma_\psi)$ to

$$\int_{\Omega\setminus\Gamma_\psi} \nabla u_\psi \cdot \nabla z \, dx = 0 \qquad \text{for every } z \in H^1_U(\Omega \setminus \Gamma_\psi)$$

with $u_\psi = u$ in $(\Omega \setminus U) \cup \partial_D\Omega$. We will also denote by $\nu_\psi := \nu_{\Phi^\psi}$ and $\eta_\psi := \eta_{\Phi^\psi}$ the unit normal to Γ_ψ and the unit co-normal of $\Gamma_\psi \cap \partial\Omega$ respectively, defined in (1.7) and (1.10), and by $H_\psi := \operatorname{div}_{\Gamma_\psi}\nu_\psi$ the curvature of Γ_ψ.

Remark 2.26. For $\psi \in \mathcal{D}_\delta$, the function u_ψ is a weak solution to the Neumann problem

$$\begin{cases} \Delta u_\psi = 0 & \text{in } (\Omega \cap U) \setminus \Gamma_\psi, \\ \partial_{\nu_\psi} u_\psi = 0 & \text{on } \Gamma_\psi \cap \Omega, \\ \partial_{\nu_{\partial\Omega}} u_\psi = 0 & \text{on } (\partial\Omega \cap \overline{U}) \setminus \partial_D\Omega, \end{cases}$$

and the sets Γ_ψ are uniformly bounded in C^2, by (2.30). Hence, by classical results and by using Lemma 2.48 to deal with the regularity in a neighborhood of the boundary $\Gamma_\psi \cap \partial\Omega$, we obtain that the functions $u^{\pm}_{\psi}$ are of class $C^{1,\gamma}$ up to $\Gamma_\psi\cap\overline{\Omega}$, for some $\gamma\in(\frac12,1)$, with $C^{1,\gamma}$-norm uniformly bounded with respect to $\psi\in\mathcal{D}_\delta$. More precisely,

$$\sup_{\psi\in\mathcal{D}_\delta}\|\nabla_\Gamma(u^{\pm}_{\psi}\circ\Phi^{\psi})\|_{C^{0,\gamma}(\Gamma\cap\overline{\Omega};\mathbb{R}^2)}<+\infty\,,$$

and, as an application of Ascoli–Arzelà Theorem, we also have

$$\sup_{\psi\in\mathcal{D}_\delta}\|\nabla_\Gamma(u^{\pm}_{\psi}\circ\Phi^{\psi})-\nabla_\Gamma u^{\pm}\|_{C^{0,\alpha}(\Gamma\cap\overline{\Omega};\mathbb{R}^2)}\to 0$$

for every $\alpha\in(0,\gamma)$, as $\delta\to0$.

The main result of this section is the following.

THEOREM 2.27. *Let (Γ,u) be a strictly stable regular critical pair in an admissible subdomain U, according to Definition 2.6. Then (Γ,u) is an isolated local $W^{2,\infty}$-minimizer in U, in the sense that there exist $\delta>0$ and $C>0$ such that*

$$\mathcal{MS}(\Gamma_\psi,v)\geq\mathcal{MS}(\Gamma,u)+C\,\|\psi\|^2_{H^1(\Gamma\cap\Omega)}$$

for every $\psi\in W^{2,\infty}(\Gamma\cap\Omega)$ such that $\|\psi\|_{W^{2,\infty}(\Gamma\cap\Omega)}<\delta$, and for every $v\in H^1(\Omega\setminus\Gamma_\psi)$ with $v=u$ in $(\Omega\setminus U)\cup\partial_D\Omega$ (where the set Γ_ψ is defined in (2.31)).

The remaining part of this section is entirely devoted to the proof of Theorem 2.27. We start by fixing $\delta_0>0$ such that $\Gamma_\psi\subset\mathcal{N}_{\eta_0}(\Gamma)$ for every $\psi\in\mathcal{D}_{\delta_0}$, where $\mathcal{N}_{\eta_0}(\Gamma)$ is the tubular neighborhood of Γ fixed at the beginning of this section. Our first task is to associate, with every $\psi\in\mathcal{D}_{\delta_0}$, an admissible flow $(\Phi_t)_t$ connecting Γ to Γ_ψ: this can be easily done by setting

$$\Phi_t(x):=\Psi(t\psi(x),x). \tag{2.32}$$

The flow Φ_t is admissible in U (according to Definition 2.11), as it is generated by the vector field

$$X_\psi:=\psi X, \tag{2.33}$$

where X is defined at the beginning of this section. Moreover it satisfies $\Phi_1(\Gamma)=\Gamma_\psi$, and

$$\|\Phi_t-Id\|_{C^2(\Gamma)}\leq C\,\|\psi\|_{C^2(\Gamma)} \tag{2.34}$$

for every $t\in[0,1]$, where C is a positive constant independent of $\psi\in\mathcal{D}_{\delta_0}$. We also introduce the vector field

$$Z_\psi:=DX_\psi[X_\psi]=\psi^2DX[X] \tag{2.35}$$

(the last equality follows by a direct computation, by observing that $\nabla\psi\cdot X=0$ since ψ is constant along the trajectories of the flow generated by X). Notice that by (2.33) and (2.35) we immediately have the estimates

$$|X_\psi|\leq|\psi|,\qquad |Z_\psi|\leq C\,|\psi|^2\qquad\text{in }\mathcal{N}_{\eta_0}(\Gamma), \tag{2.36}$$

where C is a positive constant independent of ψ. In the following lemma we collect some technical estimates concerning the above construction that will be used in the proof of the main result of this section.

LEMMA 2.28. *Given $\varepsilon>0$, there exists $\delta(\varepsilon)>0$ such that for every $\psi\in\mathcal{D}_{\delta(\varepsilon)}$ the following estimates hold:*

(a) $\frac12\|\psi\|^2_{H^1(\Gamma\cap\Omega)}\leq\|X_\psi\cdot\nu_\psi\|^2_{H^1(\Gamma_\psi\cap\Omega)}\leq2\|\psi\|^2_{H^1(\Gamma\cap\Omega)}$;

(b) $|X_\psi\cdot\eta_\psi|\leq\varepsilon\,|\psi|$ *on* $\Gamma_\psi\cap\partial\Omega$.

(c) $\frac12\,\|\psi\|^2_{H^1(\Gamma\cap\Omega)}\leq\|\psi\|^2_{H^1(\Gamma_\psi\cap\Omega)}\leq2\,\|\psi\|^2_{H^1(\Gamma\cap\Omega)}$.

PROOF. To prove (a), we first note that given $\sigma > 0$ we can find $\delta(\sigma) \in (0, \delta_0)$ such that for every $\psi \in \mathcal{D}_{\delta(\sigma)}$ we have on Γ_ψ

$$\nu_\psi = \nu \circ \Phi_1^{-1} + \tilde{\nu} \quad \text{with} \quad \|\tilde{\nu}\|_{C^1(\Gamma_\psi)} \leq \sigma \tag{2.37}$$

and

$$\|X - X \circ \Phi_1^{-1}\|_{C^1(\Gamma_\psi)} \leq \sigma \tag{2.38}$$

(where $\Phi_1 = \Phi^\psi$, by (2.32)). Hence on Γ_ψ

$$X_\psi \cdot \nu_\psi = \psi X \cdot \nu_\psi = \psi\big((X \cdot \nu) \circ \Phi_1^{-1} + (X - X \circ \Phi_1^{-1}) \cdot \nu \circ \Phi_1^{-1} + X \cdot \tilde{\nu}\big) =: \psi(1 + R_1)$$

(where we used the fact that $(X \cdot \nu) \circ \Phi_1^{-1} = 1$), and

$$\begin{aligned}
\nabla_{\Gamma_\psi}(X_\psi \cdot \nu_\psi) &= (\nabla_{\Gamma_\psi}\psi) X \cdot \nu_\psi + \psi \nabla_{\Gamma_\psi}(X \cdot \nu_\psi) \\
&= (\nabla_{\Gamma_\psi}\psi)(1 + R_1) + \psi \nabla_{\Gamma_\psi}(1 + (X - X \circ \Phi_1^{-1}) \cdot \nu \circ \Phi_1^{-1} + X \cdot \tilde{\nu}) \\
&=: (\nabla_{\Gamma_\psi}\psi)(1 + R_1) + \psi R_2.
\end{aligned}$$

Recalling (2.37) and (2.38), the L^∞-norm of R_1 and R_2 can be made as small as we want by taking σ small enough, and in turn from the previous identities we obtain (a).

To prove (b), we first observe that, by reducing $\delta(\sigma)$ if necessary, we can guarantee that for every $\psi \in \mathcal{D}_{\delta(\sigma)}$

$$\eta_\psi = \eta \circ \Phi_1^{-1} + \tilde{\eta} \quad \text{with} \quad |\tilde{\eta}| \leq \sigma \quad \text{on } \Gamma_\psi \cap \partial\Omega. \tag{2.39}$$

We deduce that on $\Gamma_\psi \cap \partial\Omega$

$$|X_\psi \cdot \eta_\psi| = |\psi X \cdot \eta_\psi| = \big|\psi\big((X \cdot \eta) \circ \Phi_1^{-1} + (X - X \circ \Phi_1^{-1}) \cdot \eta \circ \Phi_1^{-1} + X \cdot \tilde{\eta}\big)\big| \leq \varepsilon\,|\psi|$$

where the last inequality follows by observing that $(X \cdot \eta) \circ \Phi_1^{-1} = 0$, and by (2.38) and (2.39) (choosing σ small enough, depending on ε). This proves (b).

Finally, by a change of variables (using the area formula (1.8)) we have

$$\|\psi\|^2_{H^1(\Gamma_\psi \cap \Omega)} = \int_{\Gamma \cap \Omega} \Big(|\psi \circ \Phi^\psi|^2 + \frac{|\nabla_\Gamma(\psi \circ \Phi^\psi)|^2}{|D\Phi^\psi[\tau]|^2} \Big) J_{\Phi^\psi} \, d\mathcal{H}^1 ,$$

and (c) follows by (2.30) and recalling that $\psi \circ \Phi^\psi = \psi$ on Γ. □

Given $\psi \in \mathcal{D}_{\delta_0}$, we can define a bilinear form on $H^1(\Gamma_\psi \cap \Omega)$ as in (2.23), by setting

$$(\varphi, \vartheta)_{\sim,\psi} := \int_{\Omega \cap \Gamma_\psi} \nabla_{\Gamma_\psi}\varphi \cdot \nabla_{\Gamma_\psi}\vartheta \, d\mathcal{H}^1 + \int_{\Omega \cap \Gamma_\psi} H_\psi^2 \, \varphi \, \vartheta \, d\mathcal{H}^1 - \int_{\Gamma_\psi \cap \partial\Omega} D\nu_{\partial\Omega}[\nu_\psi, \nu_\psi] \, \varphi \, \vartheta \, d\mathcal{H}^0.$$

The positivity assumption (2.5) guarantees that, if δ is sufficiently small, it is possible to control the H^1-norm on Γ_ψ in terms of the norm $\|\cdot\|_{\sim,\psi}$ associated with $(\cdot,\cdot)_{\sim,\psi}$, uniformly with respect to $\psi \in \mathcal{D}_\delta$. This is the content of the following proposition, analogous to [19, Lemma 5.3].

PROPOSITION 2.29. *In the hypotheses of Theorem 2.27, there exist $C_1 > 0$ and $\delta_1 \in (0, \delta_0)$ such that for every $\psi \in \mathcal{D}_{\delta_1}$*

$$\|\varphi\|_{H^1(\Gamma_\psi \cap \Omega)} \leq C_1 \|\varphi\|_{\sim,\psi} \qquad \text{for every } \varphi \in H^1(\Gamma_\psi \cap \Omega).$$

PROOF. Condition (2.5) implies, by Proposition 2.21 and Proposition 2.20, that

$$\|\varphi\|^2_{H^1(\Gamma \cap \Omega)} \leq C \|\varphi\|^2_\sim \qquad \text{for every } \varphi \in H^1(\Gamma \cap \Omega) \tag{2.40}$$

for some positive constant C. Let us consider any $\psi \in \mathcal{D}_{\delta_0}$. Given $\varphi \in H^1(\Gamma_\psi \cap \Omega)$, by using the area formula (1.8) we obtain

$$
\begin{aligned}
\|\varphi\|^2_{H^1(\Gamma_\psi\cap\Omega)} &= \int_{\Gamma_\psi\cap\Omega} \Big(|\varphi\circ\Phi^\psi|^2 + \frac{|\nabla_\Gamma(\varphi\circ\Phi^\psi)|^2}{|D\Phi^\psi[\tau]|^2} \Big) J_{\Phi^\psi}\, d\mathcal{H}^1 \\
&\le M \int_{\Gamma\cap\Omega} \big(|\varphi\circ\Phi^\psi|^2 + |\nabla_\Gamma(\varphi\circ\Phi^\psi)|^2 \big)\, d\mathcal{H}^1 \le MC \|\varphi\circ\Phi^\psi\|^2_\sim
\end{aligned}
$$

where we set

$$
M := \sup_{\psi\in\mathcal{D}_{\delta_0}} \Big\| J_{\Phi^\psi} \Big(1 + \frac{1}{|D\Phi^\psi[\tau]|^2} \Big) \Big\|_{L^\infty(\Gamma\cap\Omega)} < +\infty
$$

and we used (2.40) in the last inequality. Fix $\varepsilon > 0$, and choose δ_1 so small that the following inequalities are satisfied:

$$
\sup_{\psi\in\mathcal{D}_{\delta_1}} \Big\| \frac{J_{\Phi^\psi}}{|D\Phi^\psi[\tau]|^2} - 1 \Big\|_{L^\infty(\Gamma\cap\Omega)} < \varepsilon, \qquad \sup_{\psi\in\mathcal{D}_{\delta_1}} \|(H^2_\psi\circ\Phi^\psi)J_{\Phi^\psi} - H^2\|_{L^\infty(\Gamma\cap\Omega)} < \varepsilon,
$$

$$
\sup_{\psi\in\mathcal{D}_{\delta_1}} \|(D\nu_{\partial\Omega}[\nu_\psi,\nu_\psi])\circ\Phi^\psi - H_{\partial\Omega}\|_{L^\infty(\Gamma\cap\partial\Omega)} < \varepsilon.
$$

Then

$$
\begin{aligned}
\|\varphi\circ\Phi^\psi\|^2_\sim &= \int_{\Gamma\cap\Omega} \big(H^2(\varphi\circ\Phi^\psi)^2 + |\nabla_\Gamma(\varphi\circ\Phi^\psi)|^2 \big)\, d\mathcal{H}^1 - \int_{\Gamma\cap\partial\Omega} H_{\partial\Omega}\,(\varphi\circ\Phi^\psi)^2\, d\mathcal{H}^0 \\
&= \|\varphi\|^2_{\sim,\psi} + \int_{\Gamma\cap\Omega} \big(H^2 - (H^2_\psi\circ\Phi^\psi)J_{\Phi^\psi} \big)(\varphi\circ\Phi^\psi)^2\, d\mathcal{H}^1 \\
&\quad + \int_{\Gamma\cap\Omega} \Big(1 - \frac{J_{\Phi^\psi}}{|D\Phi^\psi[\tau]|^2} \Big) |\nabla_\Gamma(\varphi\circ\Phi^\psi)|^2\, d\mathcal{H}^1 \\
&\quad - \int_{\Gamma\cap\partial\Omega} \big(H_{\partial\Omega} - D\nu_{\partial\Omega}[\nu_\psi,\nu_\psi]\circ\Phi^\psi \big)(\varphi\circ\Phi^\psi)^2\, d\mathcal{H}^0 \\
&\le \|\varphi\|^2_{\sim,\psi} + \varepsilon\, c\, \|\varphi\circ\Phi^\psi\|^2_{H^1(\Gamma\cap\Omega)} \le \|\varphi\|^2_{\sim,\psi} + \varepsilon\, c'\, \|\varphi\|^2_{H^1(\Gamma_\psi\cap\Omega)},
\end{aligned}
\tag{2.41}
$$

where we used also the fact that $\|\varphi\circ\Phi^\psi\|^2_{L^2(\Gamma\cap\partial\Omega)} \le c\|\varphi\circ\Phi^\psi\|^2_{H^1(\Gamma\cap\Omega)}$, and a change of variables in the last inequality. Now choosing ε such that $\varepsilon c' MC \le \frac{1}{2}$ and collecting the previous estimates, we obtain the desired inequality with $C_1 := \sqrt{2MC}$. □

The previous result allows us to introduce, for $\psi \in \mathcal{D}_{\delta_1}$, a compact, linear operator $T_\psi : H^1(\Gamma_\psi\cap\Omega) \to H^1(\Gamma_\psi\cap\Omega)$ defined by duality by

$$
(T_\psi\varphi, \vartheta)_{\sim,\psi} = -2 \int_{\Gamma_\psi\cap\Omega} \big[v^+_{\varphi,\psi} \mathrm{div}_{\Gamma_\psi}(\vartheta\nabla_{\Gamma_\psi} u^+_\psi) - v^-_{\varphi,\psi} \mathrm{div}_{\Gamma_\psi}(\vartheta\nabla_{\Gamma_\psi} u^-_\psi) \big]\, d\mathcal{H}^1 \tag{2.42}
$$

for every $\varphi, \vartheta \in H^1(\Gamma_\psi\cap\Omega)$, where $v_{\varphi,\psi} \in H^1_U(\Omega\setminus\Gamma_\psi)$ is the solution to

$$
\int_\Omega \nabla v_{\varphi,\psi}\cdot\nabla z\, dx + \int_{\Gamma_\psi\cap\Omega} \big[z^+ \mathrm{div}_{\Gamma_\psi}(\varphi\nabla_{\Gamma_\psi} u^+_\psi) - z^- \mathrm{div}_{\Gamma_\psi}(\varphi\nabla_{\Gamma_\psi} u^-_\psi) \big]\, d\mathcal{H}^1 = 0
$$

for every $z \in H^1_U(\Omega\setminus\Gamma_\psi)$ (the compactness of the operator T_ψ follows by the same argument contained in the first part of the proof of Proposition 2.21). We define also $\lambda_{1,\psi}$ similarly to (2.27). The following semicontinuity property of the eigenvalues $\lambda_{1,\psi}$ will be crucial in the proof of Theorem 2.27.

PROPOSITION 2.30. *In the hypotheses of Theorem 2.27,*

$$\limsup_{\|\psi\|_{C^2(\Gamma)}\to 0} \lambda_{1,\psi} \le \lambda_1.$$

PROOF. By contradiction, assume that there exists a sequence $\psi_n \to 0$ in $C^2(\Gamma)$ such that $\lambda_{1,\psi_n} \to \lambda_\infty > \lambda_1$. Let $\varphi_n \in C^\infty(\Gamma_{\psi_n})$, with $\|\varphi_n\|_{\sim,\psi_n} = 1$, satisfy

$$(T_{\psi_n}\varphi_n, \varphi_n)_{\sim,\psi_n} = 2\int_\Omega |\nabla w_n|^2\,\mathrm{d}x \to \lambda_\infty,$$

where $w_n := v_{\varphi_n,\psi_n}$, according to the previous notation. Setting $\Phi_n := \Phi^{\psi_n}$ and $\tilde{w}_n := w_n \circ \Phi_n$, we have that $\tilde{w}_n \in H^1_U(\Omega\setminus\Gamma)$ is a solution to

$$\int_\Omega A_n[\nabla\tilde{w}_n, \nabla z]\,\mathrm{d}x + \int_{\Gamma\cap\Omega} z^+\,(\mathrm{div}_{\Gamma_{\psi_n}}(\varphi_n\nabla_{\Gamma_{\psi_n}}u^+_{\psi_n}))\circ\Phi_n J_{\Phi_n}\,\mathrm{d}\mathcal{H}^1 \\ - \int_{\Gamma\cap\Omega} z^-\,(\mathrm{div}_{\Gamma_{\psi_n}}(\varphi_n\nabla_{\Gamma_{\psi_n}}u^-_{\psi_n}))\circ\Phi_n J_{\Phi_n}\,\mathrm{d}\mathcal{H}^1 = 0$$

for every $z \in H^1_U(\Omega\setminus\Gamma)$, where $A_n := \frac{D\Psi_n D\Psi_n^T}{\det D\Psi_n}\circ\Phi_n$ and $\Psi_n := \Phi_n^{-1}$. Moreover, as $A_n \to I$ uniformly, we have

$$\lim_{n\to\infty} 2\int_\Omega |\nabla\tilde{w}_n|^2\,\mathrm{d}x = \lim_{n\to\infty} 2\int_\Omega |\nabla w_n|^2\,\mathrm{d}x = \lambda_\infty. \tag{2.43}$$

We set also $\tilde{\varphi}_n := c_n\varphi_n\circ\Phi_n$, where

$$c_n := \|\varphi_n\circ\Phi_n\|_\sim^{-1} \to 1 \tag{2.44}$$

(this convergence follows arguing as in the proof of (2.41)). We claim that

$$\lim_{n\to\infty}\int_\Omega |\nabla(v_{\tilde{\varphi}_n} - \tilde{w}_n)|^2\,\mathrm{d}x = 0, \tag{2.45}$$

where $v_{\tilde{\varphi}_n}$ is defined as in (2.4) with φ replaced by $\tilde{\varphi}_n$. Notice that, if (2.45) holds, then by (2.43) we immediately obtain

$$\lambda_1 \ge \lim_{n\to\infty}(T\tilde{\varphi}_n, \tilde{\varphi}_n)_\sim = \lim_{n\to\infty} 2\int_\Omega |\nabla v_{\tilde{\varphi}_n}|^2\,\mathrm{d}x = \lim_{n\to\infty} 2\int_\Omega |\nabla\tilde{w}_n|^2\,\mathrm{d}x = \lambda_\infty > \lambda_1,$$

which is a contradiction. Hence we are left with the proof of (2.45).

The function $z_n := v_{\tilde{\varphi}_n} - \tilde{w}_n \in H^1_U(\Omega\setminus\Gamma)$ solves

$$\int_\Omega A_n[\nabla z_n, \nabla z]\,\mathrm{d}x - \int_\Omega (A_n - I)[\nabla v_{\tilde{\varphi}_n}, \nabla z]\,\mathrm{d}x + \int_{\Gamma\cap\Omega}(h_n^+z^+ - h_n^-z^-)\,\mathrm{d}\mathcal{H}^1 = 0$$

for all $z \in H^1_U(\Omega\setminus\Gamma)$, where $h_n^\pm := \mathrm{div}_\Gamma(\tilde{\varphi}_n\nabla_\Gamma u^\pm) - (\mathrm{div}_{\Gamma_{\psi_n}}(\varphi_n\nabla_{\Gamma_{\psi_n}}u^\pm_{\psi_n}))\circ\Phi_n J_{\Phi_n}$. Since $A_n - I \to 0$ uniformly and $v_{\tilde{\varphi}_n}$ is bounded in $H^1_U(\Omega\setminus\Gamma)$ (as $\|\tilde{\varphi}_n\|_\sim = 1$), we have that $(A_n - I)[\nabla v_{\tilde{\varphi}_n}]$ converges to 0 strongly in $L^2(\Omega;\mathbb{R}^2)$. Hence, to prove (2.45) it is sufficient to show that $h_n^\pm z^\pm \to 0$ in $L^1(\Gamma\cap\Omega)$.

We have that

$$\begin{aligned}(\mathrm{div}_{\Gamma_{\psi_n}}(\varphi_n\nabla_{\Gamma_{\psi_n}}u^\pm_{\psi_n}))\circ\Phi_n &= \big(\partial_{\tau_n}(\varphi_n\nabla_{\Gamma_{\psi_n}}u^\pm_{\psi_n})\cdot\tau_n\big)\circ\Phi_n \\ &= D\big((\varphi_n\nabla_{\Gamma_{\psi_n}}u^\pm_{\psi_n})\circ\Phi_n\big)(D\Phi_n)^{-1}[\tau_n\circ\Phi_n, \tau_n\circ\Phi_n] \\ &= |D\Phi_n[\tau]|^{-1}D\big(c_n^{-1}\tilde{\varphi}_n\nabla_{\Gamma_{\psi_n}}u^\pm_{\psi_n}\circ\Phi_n\big)[\tau, \tau_n\circ\Phi_n],\end{aligned}$$

where $\tau_n = (|D\Phi_n[\tau]|^{-1}D\Phi_n[\tau])\circ\Phi_n^{-1}$ is the tangent vector to Γ_{ψ_n}, hence

$$h_n^\pm = D\big(\tilde{\varphi}_n\nabla_\Gamma u^\pm\big)[\tau,\tau] - c_n^{-1}J_{\Phi_n}|D\Phi_n[\tau]|^{-1}D\big(\tilde{\varphi}_n\nabla_{\Gamma_{\psi_n}}u^\pm_{\psi_n}\circ\Phi_n\big)[\tau, \tau_n\circ\Phi_n]. \tag{2.46}$$

Recalling (2.44) and the convergence of Φ_n to Id in $C^2(\Gamma)$, it is now sufficient to show that

$$\|\partial_\tau(\tilde{\varphi}_n k_n^\pm)\|_{H^{-\frac{1}{2}}(\Gamma\cap\Omega)} \to 0, \tag{2.47}$$

where we set $k_n^\pm := \nabla_\Gamma u^\pm - \nabla_{\Gamma_{\psi_n}} u_{\psi_n}^\pm \circ \Phi_n$. Since the sequence $\tilde{\varphi}_n$ is bounded in $H^1(\Gamma\cap\Omega)$ and, by Remark 2.26, $k_n^\pm \to 0$ in $C^{0,\alpha}(\Gamma;\mathbb{R}^2)$ for some $\alpha \in \left(\frac{1}{2},1\right)$, we deduce from Lemma 1.14 that

$$\tilde{\varphi}_n k_n^\pm \to 0 \qquad \text{in } H^{\frac{1}{2}}(\Gamma\cap\Omega;\mathbb{R}^2),$$

which in turn yields (2.47). This completes the proof of the proposition. □

We are ready to prove the local $W^{2,\infty}$-minimality of a strictly stable regular critical pair.

Proof of Theorem 2.27. We divide the proof into two steps.

Step 1. We first show that there exist $\delta \in (0,\delta_1)$ and $c>0$ such that for every $\psi \in \mathcal{D}_\delta$

$$\mathcal{MS}(\Gamma_\psi, u_\psi) \geq \mathcal{MS}(\Gamma, u) + c\,\|\psi\|^2_{H^1(\Gamma\cap\Omega)}. \tag{2.48}$$

Given $\psi \in \mathcal{D}_\delta$, with $\delta \in (0,\delta_1)$ to be chosen, consider the admissible flow $(\Phi_t)_t$ associated with ψ, according to (2.32), and its tangent vector field X_ψ. Setting $g_\psi(t) := \mathcal{MS}(\Gamma_{\Phi_t}, u_{\Phi_t})$, we claim that there exist $c>0$ and $\delta>0$ such that

$$g''_\psi(t) \geq 2c\,\|\psi\|^2_{H^1(\Gamma\cap\Omega)} \qquad \text{for every } t\in[0,1] \text{ and } \psi \in \mathcal{D}_\delta. \tag{2.49}$$

Once this is proved, claim (2.48) will follow immediately: indeed, as $g'_\psi(0)=0$ since (Γ,u) is a critical pair, and recalling that $\Gamma_{\Phi_1} = \Gamma_\psi$, we deduce

$$\begin{aligned}\mathcal{MS}(\Gamma,u) = g_\psi(0) &= g_\psi(1) - \int_0^1 (1-t) g''_\psi(t)\,\mathrm{d}t \\ &\leq \mathcal{MS}(\Gamma_\psi,u_\psi) - c\,\|\psi\|^2_{H^1(\Gamma\cap\Omega)},\end{aligned}$$

which is (2.48).

We now come to the proof of (2.49). In order to simplify the notation, we set $\nu_t := \nu_{\Phi_t}$, $\eta_t := \eta_{\Phi_t}$, $\Gamma_t := \Gamma_{\Phi_t}$, and $H_t := H_{\Phi_t}$. By Remark 2.17, recalling the definition of $T_{t\psi}$ (see (2.42)), we deduce that

$$\begin{aligned} g''_\psi(t) = &-(T_{t\psi}(X_\psi\cdot\nu_t), X_\psi\cdot\nu_t)_{\sim,t\psi} + \int_{\Gamma_t\cap\Omega}\Big(H_t^2(X_\psi\cdot\nu_t)^2 + |\nabla_{\Gamma_t}(X_\psi\cdot\nu_t)|^2\Big)\,\mathrm{d}\mathcal{H}^1 \\ &+ \int_{\Gamma_t\cap\Omega} f_t\big(Z_\psi\cdot\nu_t - 2X_\psi^\parallel\cdot\nabla_{\Gamma_t}(X_\psi\cdot\nu_t) + D\nu_t[X_\psi^\parallel,X_\psi^\parallel] - H_t(X_\psi\cdot\nu_t)^2\big)\,\mathrm{d}\mathcal{H}^1 \\ &+ \int_{\Gamma_t\cap\partial\Omega}(f_t-H_t)(X_\psi\cdot\nu_t)(X_\psi\cdot\eta_t)\,\mathrm{d}\mathcal{H}^0 + \int_{\Gamma_t\cap\partial\Omega} Z_\psi\cdot\eta_t\,\mathrm{d}\mathcal{H}^0,\end{aligned}$$

where $f_t = |\nabla_{\Gamma_t} u^-_{\Phi_t}|^2 - |\nabla_{\Gamma_t} u^+_{\Phi_t}|^2 + H_t$. Since

$$\begin{aligned} 0 &= Z_\psi\cdot\nu_{\partial\Omega} + D\nu_{\partial\Omega}[X_\psi,X_\psi] \\ &= Z_\psi\cdot\nu_{\partial\Omega} + D\nu_{\partial\Omega}[\nu_t,\nu_t](X_\psi\cdot\nu_t)^2 + \big((X_\psi\cdot\eta_t)^2\eta_t + 2(X_\psi\cdot\nu_t)(X_\psi\cdot\eta_t)\nu_t\big)\cdot D\nu_{\partial\Omega}[\eta_t]\end{aligned}$$

on $\Gamma_t \cap \partial\Omega$ by Lemma 2.16 (d), we can rewrite $g''_\psi(t)$ as

$$\begin{aligned} g''_\psi(t) &= -(T_{t\psi}(X_\psi \cdot \nu_t), X_\psi \cdot \nu_t)_{\sim,t\psi} + \|X_\psi \cdot \nu_t\|^2_{\sim,t\psi} \\ &\quad + \int_{\Gamma_t\cap\Omega} f_t\big(Z_\psi \cdot \nu_t - 2X_\psi^{\|} \cdot \nabla_{\Gamma_t}(X_\psi \cdot \nu_t) + D\nu_t[X_\psi^{\|}, X_\psi^{\|}] - H_t(X_\psi \cdot \nu_t)^2\big)\, d\mathcal{H}^1 \\ &\quad + \int_{\Gamma_t\cap\partial\Omega} (f_t - H_t)(X_\psi \cdot \nu_t)(X_\psi \cdot \eta_t)\, d\mathcal{H}^0 + \int_{\Gamma_t\cap\partial\Omega} Z_\psi \cdot (\eta_t - \nu_{\partial\Omega})\, d\mathcal{H}^0 \\ &\quad - \int_{\Gamma_t\cap\partial\Omega} \big((X_\psi \cdot \eta_t)^2\eta_t + 2(X_\psi \cdot \nu_t)(X_\psi \cdot \eta_t)\nu_t\big) \cdot D\nu_{\partial\Omega}[\eta_t]\, d\mathcal{H}^0 \,. \end{aligned} \tag{2.50}$$

We now carefully estimate each term in the previous expression. In the following, C will denote a generic positive constant, independent of $\psi \in \mathcal{D}_{\delta_1}$, which may change from line to line.

As (Γ, u) satisfies condition (2.5), Proposition 2.21 implies that $\lambda_1 < 1$, so that by Proposition 2.30 we can find $\delta_2 \in (0, \delta_1)$ such that for every $\psi \in \mathcal{D}_{\delta_2}$

$$\lambda_{1,\psi} < \frac{1}{2}(\lambda_1 + 1) < 1. \tag{2.51}$$

Fix $\varepsilon > 0$ to be chosen later, and let $\delta(\varepsilon) > 0$ be given by Lemma 2.28 (assume without loss of generality that $\delta(\varepsilon) < \delta_2$). We remark that, if $\psi \in \mathcal{D}_{\delta(\varepsilon)}$, then $t\psi \in \mathcal{D}_{\delta(\varepsilon)}$ for every $t \in [0,1]$, and $X_{t\psi} = tX_\psi$: hence we can apply (a) and (b) of Lemma 2.28 to $t\psi$, and we easily obtain that

$$\frac{1}{2}\|\psi\|^2_{H^1(\Gamma\cap\Omega)} \le \|X_\psi \cdot \nu_t\|^2_{H^1(\Gamma_t\cap\Omega)} \le 2\|\psi\|^2_{H^1(\Gamma\cap\Omega)}, \tag{2.52}$$

$$|X_\psi \cdot \eta_t| \le \varepsilon\, |\psi| \qquad \text{on } \Gamma_t \cap \partial\Omega, \tag{2.53}$$

for every $\psi \in \mathcal{D}_{\delta(\varepsilon)}$ and for every $t \in [0,1]$.

Fix now any $\psi \in \mathcal{D}_{\delta(\varepsilon)}$. From the definition of $\lambda_{1,\psi}$ and (2.51) we have

$$\begin{aligned} &-(T_{t\psi}(X_\psi \cdot \nu_t), X_\psi \cdot \nu_t)_{\sim,t\psi} + \|X_\psi \cdot \nu_t\|^2_{\sim,t\psi} \ge (1 - \lambda_{1,t\psi})\|X_\psi \cdot \nu_t\|^2_{\sim,t\psi} \\ &\qquad \ge \frac{1-\lambda_1}{2}\|X_\psi \cdot \nu_t\|^2_{\sim,t\psi} \ge \frac{1-\lambda_1}{2C_1^2}\|X_\psi \cdot \nu_t\|^2_{H^1(\Gamma_t\cap\Omega)} \ge \frac{1-\lambda_1}{4C_1^2}\|\psi\|^2_{H^1(\Gamma\cap\Omega)}, \end{aligned} \tag{2.54}$$

where the last two inequalities follow from Proposition 2.29 and from (2.52).

By Remark 2.26 the map

$$\psi \in \mathcal{D}_{\delta(\varepsilon)} \mapsto \||\nabla_{\Gamma_\psi} u_\psi^-|^2 - |\nabla_{\Gamma_\psi} u_\psi^+|^2 + H_\psi\|_{L^\infty(\Gamma_\psi\cap\Omega)}$$

is continuous with respect to the C^2-topology; hence, as it vanishes for $\psi = 0$ by (2.2), possibly reducing $\delta(\varepsilon)$ we have that for every $\psi \in \mathcal{D}_{\delta(\varepsilon)}$

$$\||\nabla_{\Gamma_\psi} u_\psi^-|^2 - |\nabla_{\Gamma_\psi} u_\psi^+|^2 + H_\psi\|_{L^\infty(\Gamma_\psi\cap\Omega)} < \varepsilon.$$

We deduce that

$$\begin{aligned} &\int_{\Gamma_t\cap\Omega} f_t\big(Z_\psi \cdot \nu_t - 2X_\psi^{\|} \cdot \nabla_{\Gamma_t}(X_\psi \cdot \nu_t) + D\nu_t[X_\psi^{\|}, X_\psi^{\|}] - H_t(X_\psi \cdot \nu_t)^2\big)\, d\mathcal{H}^1 \\ &\qquad \ge -\varepsilon\, \|Z_\psi \cdot \nu_t - 2X_\psi^{\|} \cdot \nabla_{\Gamma_t}(X_\psi \cdot \nu_t) + D\nu_t[X_\psi^{\|}, X_\psi^{\|}] + H_t(X_\psi \cdot \nu_t)^2\|_{L^1(\Gamma_t\cap\Omega)} \\ &\qquad \ge -\varepsilon\, C\left(\|\psi\|^2_{L^2(\Gamma_t\cap\Omega)} + \|\nabla_{\Gamma_t}(X_\psi \cdot \nu_t)\|_{L^2(\Gamma_t\cap\Omega)}\, \|\psi\|_{L^2(\Gamma_t\cap\Omega)}\right) \\ &\qquad \ge -\varepsilon\, C\, \|\psi\|^2_{H^1(\Gamma\cap\Omega)}, \end{aligned} \tag{2.55}$$

where we used also (2.36), (2.52), and (c) of Lemma 2.28.

By (2.36), (2.53) and (c) of Lemma 2.28 we have

$$\int_{\Gamma_t\cap\partial\Omega}(f_t-H_t)(X_\psi\cdot\nu_t)(X_\psi\cdot\eta_t)\,\mathrm{d}\mathcal{H}^0\geq -\varepsilon\, C\int_{\Gamma_t\cap\partial\Omega}\psi^2\,\mathrm{d}\mathcal{H}^0\geq -\varepsilon\, C\,\|\psi\|^2_{H^1(\Gamma\cap\Omega)}\,. \tag{2.56}$$

By reducing $\delta(\varepsilon)$ if necessary we can assume

$$\max_{x\in\Gamma_\psi\cap\partial\Omega}|\eta_\psi(x)-\nu_{\partial\Omega}(x)|<\varepsilon\qquad\text{for every }\psi\in\mathcal{D}_{\delta(\varepsilon)},$$

so that using again (2.36) and (c) of Lemma 2.28 we obtain

$$\int_{\Gamma_t\cap\partial\Omega}Z_\psi\cdot(\eta_t-\nu_{\partial\Omega})\,\mathrm{d}\mathcal{H}^0\geq -\varepsilon\, C\int_{\Gamma_t\cap\partial\Omega}\psi^2\,\mathrm{d}\mathcal{H}^0\geq -\varepsilon\, C\,\|\psi\|^2_{H^1(\Gamma\cap\Omega)}\,. \tag{2.57}$$

Finally, we proceed in a similar way to estimate the last integral in (2.50): by (2.36) and (2.53)

$$-\int_{\Gamma_t\cap\partial\Omega}\big((X_\psi\cdot\eta_t)^2\eta_t+2(X_\psi\cdot\nu_t)(X_\psi\cdot\eta_t)\nu_t\big)\cdot D\nu_{\partial\Omega}[\eta_t]\,\mathrm{d}\mathcal{H}^0\geq -\varepsilon\, C\,\|\psi\|^2_{H^1(\Gamma\cap\Omega)}. \tag{2.58}$$

Collecting (2.54)–(2.58), by (2.50) we conclude that for every $\psi\in\mathcal{D}_{\delta(\varepsilon)}$ and for every $t\in[0,1]$

$$g''_\psi(t)\geq\left(\frac{1-\lambda_1}{4C_1^2}-\varepsilon\, C\right)\|\psi\|^2_{H^1(\Gamma\cap\Omega)}$$

for some positive constant C (independent of ψ), so that by choosing ε sufficiently small we obtain the claim (2.49) and, in turn, (2.48).

Step 2. The conclusion of the theorem follows now by approximation: given any $\psi\in W^{2,\infty}(\Gamma\cap\Omega)$ with $\|\psi\|_{W^{2,\infty}(\Gamma\cap\Omega)}<\delta$, we can find a sequence $\psi_n\in\mathcal{D}_\delta$ converging to ψ in $W^{1,\infty}(\Gamma\cap\Omega)$ for which the conclusion obtained in the previous step holds:

$$\mathcal{MS}(\Gamma_{\psi_n},u_{\psi_n})\geq\mathcal{MS}(\Gamma,u)+c\,\|\psi_n\|^2_{H^1(\Gamma\cap\Omega)}.$$

Noting that $\mathcal{MS}(\Gamma_{\psi_n},u_{\psi_n})\to\mathcal{MS}(\Gamma_\psi,u_\psi)$ as a consequence of the $W^{1,\infty}$-convergence of ψ_n to ψ, by passing to the limit we conclude that the same estimate holds for ψ. Hence the conclusion of the theorem follows by recalling that $\mathcal{MS}(\Gamma_\psi,v)\geq\mathcal{MS}(\Gamma_\psi,u_\psi)$ for every $v\in H^1(\Omega\setminus\Gamma_\psi)$ with $v=u$ in $(\Omega\setminus U)\cup\partial_D\Omega$, by definition of u_ψ. ☐

2.5. Local $C^{1,\alpha}$-minimality

In this section we show that the $W^{2,\infty}$-minimality property proved in the previous section implies that (Γ,u) is also a minimizer with respect to small $C^{1,\alpha}$-perturbations of the discontinuity set. We start with a preliminary construction that will be needed in the proof.

Remark 2.31. Let X be the vector field defined at the beginning of Section 2.4, which, we recall, coincides with ν on Γ and is tangent to $\partial\Omega$, and let Ψ be the flow generated by X. We want to define a one-parameter family of smooth curves $(\Gamma_\delta)_\delta$, for $\delta\in(-\delta_0,\delta_0)$, with $\Gamma_0=\Gamma$, such that X is normal to each curve of the family, and whose union is a tubular neighborhood of Γ. In order to do this, let $x_0\in\Gamma\cap\partial\Omega$ and let $x_\delta:=\Psi(\delta,x_0)$. We then define Γ_δ as the trajectory of the flow generated by $X^\perp$ starting from x_δ, where the vector field $X^\perp$ is obtained by a rotation of X by $\frac{\pi}{2}$. This construction provides a family of curves with the desired properties.

We can then define a family of tubular neighborhoods of Γ in Ω whose boundaries meet $\partial\Omega$ orthogonally, by setting for $\delta\in(-\delta_0,\delta_0)$

$$\mathcal{I}_\delta(\Gamma):=\bigcup_{|s|<\delta}\Gamma_s\,.$$

PROPOSITION 2.32. *Let (Γ, u) be a strictly stable regular critical pair in an admissible subdomain U, and let $\alpha \in (0,1)$. There exists $\delta > 0$ such that*

$$\mathcal{MS}(\Gamma, u) < \mathcal{MS}(\Phi(\Gamma), v)$$

for every diffeomorphism $\Phi \in C^{1,\alpha}(\overline{\Omega}; \overline{\Omega})$ with $0 < \|\Phi - Id\|_{C^{1,\alpha}(\overline{\Omega})} \leq \delta$ and $\operatorname{supp}(\Phi - Id) \subset\subset U \setminus \partial_D\Omega$, *and for every $v \in H^1(\Omega \setminus \Phi(\Gamma))$ with $v = u$ in $(\Omega \setminus U) \cup \partial_D\Omega$.*

PROOF. Assume by contradiction that there exist sequences $\sigma_n \to 0$ and $\Phi_n \in C^{1,\alpha}(\overline{\Omega}; \overline{\Omega})$, with

$$\operatorname{supp}(\Phi_n - Id) \subset\subset U \setminus \partial_D\Omega, \quad 0 < \|\Phi_n - Id\|_{C^{1,\alpha}(\overline{\Omega})} \leq \sigma_n,$$

such that $\mathcal{MS}(\Phi_n(\Gamma), u_n) \leq \mathcal{MS}(\Gamma, u)$, where $u_n := u_{\Phi_n}$ is defined as in (2.7). Notice that, arguing as in Remark 2.26, we have that $u_n^\pm$ are of class $C^{1,\alpha}$ up to $\Phi_n(\Gamma)$, and

$$\|\nabla(u_n^\pm \circ \Phi_n) - \nabla u^\pm\|_{L^\infty(\Omega^\pm)} \to 0.$$

We first extend u^+ and u^- to $C^{1,\alpha}$-functions in Ω^- and Ω^+, respectively, by using [52, Theorem 6.2.5]. We similarly extend $u_n^\pm \circ \Phi_n$ to $C^{1,\alpha}$-functions $\tilde{u}_n^\pm$ in $\Omega^\mp$, and we set $v_n^\pm := \tilde{u}_n^\pm \circ \Phi_n^{-1}$: since the extension operator constructed in [52, Theorem 6.2.5] is continuous, we have that

$$\|\nabla v_n^\pm - \nabla u^\pm\|_{L^\infty(\Omega)} \leq \delta_n$$

for some $\delta_n \to 0$. Finally, as $\|\Phi_n - Id\|_{C^{1,\alpha}} \to 0$, we can also assume that $\Phi_n(\Gamma) \subset \mathcal{I}_{\delta_n}(\Gamma)$.

Consider the following obstacle problems

$$\min\Big\{ J(E, v^+, v^-) : E \subset \Omega,\ \Omega^+ \setminus \mathcal{I}_{\delta_n}(\Gamma) \subset E \subset \Omega^+ \cup \mathcal{I}_{\delta_n}(\Gamma),\ v^\pm - u^\pm \in W^{1,\infty}(\Omega),$$
$$v^+\chi_E + v^-\chi_{E^c} = u \ \text{ in } \ (\Omega \setminus U) \cup \partial_D\Omega,\ \|\nabla v^\pm - \nabla u^\pm\|_{L^\infty(\Omega)} \leq 1 \Big\}, \tag{2.59}$$

where

$$J(E, v^+, v^-) := \int_E |\nabla v^+|^2 + \int_{\Omega\setminus E} |\nabla v^-|^2 + \mathcal{P}(E; \Omega),$$

and let (F_n, w_n^+, w_n^-) be a solution to (2.59), whose existence is guaranteed by the direct method of the Calculus of Variations. Since $(\Phi_n(\Omega^+), v_n^+, v_n^-)$ is an admissible competitor, we deduce that

$$J(F_n, w_n^+, w_n^-) \leq J(\Phi_n(\Omega^+), v_n^+, v_n^-) = \mathcal{MS}(\Phi_n(\Gamma), u_n) \leq \mathcal{MS}(\Gamma, u). \tag{2.60}$$

We now divide the proof into three steps.

Step 1. We claim that, if $\gamma > 0$ is sufficiently large (independently of n), then (F_n, w_n^+, w_n^-) is also a solution to the penalized problem (without obstacle)

$$\min\Big\{ \widetilde{J}(E, v^+, v^-) \ :\ E \subset \Omega,\ v^\pm - u^\pm \in W^{1,\infty}(\Omega),\ \|\nabla v^\pm - \nabla u^\pm\|_{L^\infty(\Omega)} \leq 1,$$
$$v^\pm = u \ \text{ in } \ (\Omega^\pm \setminus U) \cup (\partial_D\Omega \cap \overline{\Omega}^\pm) \Big\}, \tag{2.61}$$

where

$$\widetilde{J}(E, v^+, v^-) := \int_E |\nabla v^+|^2 + \int_{\Omega\setminus E} |\nabla v^-|^2 + \mathcal{P}(E; \Omega) + \gamma |E \triangle T_n(E)|$$

and $T_n(E) := E \cup (\Omega^+ \setminus \mathcal{I}_{\delta_n}(\Gamma)) \cap (\Omega^+ \cup \mathcal{I}_{\delta_n}(\Gamma))$.

In order to prove the claim, we fix any competitor (F, w^+, w^-) for problem (2.61). We recall that ν_E denotes the generalized outer unit normal to a finite perimeter set E. Since

$X = -\nu_{T_n(F)}$ almost everywhere on $\partial^* T_n(F) \cap \Gamma_{\delta_n}$, and $|X| \le 1$, we can estimate the difference of the perimeters of F and $T_n(F)$ in Ω^+ as follows:

$$\begin{aligned}\mathcal{P}(F;\Omega^+) - \mathcal{P}(T_n(F);\Omega^+) &= \int_{(\partial^* F \setminus \partial^* T_n(F))\cap\Omega^+} \mathrm{d}\mathcal{H}^1 - \int_{(\partial^* T_n(F)\setminus\partial^* F)\cap\Omega^+} \mathrm{d}\mathcal{H}^1 \\ &\ge -\int_{(\partial^* F \setminus \partial^* T_n(F))\cap\Omega^+} X\cdot\nu_F \,\mathrm{d}\mathcal{H}^1 + \int_{(\partial^* T_n(F)\setminus\partial^* F)\cap\Omega^+} X\cdot\nu_{T_n(F)}\,\mathrm{d}\mathcal{H}^1 \\ &= \int_{(F\triangle T_n(F))\cap\Omega^+} \mathrm{div}X \ge -\|\mathrm{div}X\|_\infty |(F\triangle T_n(F))\cap\Omega^+|,\end{aligned}$$

where we used the divergence theorem taking into account that $X\cdot\nu_{\partial\Omega} = 0$ on $\partial\Omega$. A similar estimate holds in Ω^-, and we conclude that

$$\mathcal{P}(F;\Omega) - \mathcal{P}(T_n(F);\Omega) \ge -\|\mathrm{div}X\|_\infty |F\triangle T_n(F)|.$$

Concerning the volume terms, since $\nabla w^\pm$ are uniformly bounded in L^∞ by a constant Λ depending only on $\|\nabla u\|_\infty$ we have

$$\left| \int_F |\nabla w^+|^2 - \int_{T_n(F)} |\nabla w^+|^2 \right| \le \Lambda^2 |F\triangle T_n(F)|,$$

and a similar estimate holds for w^- in the complements of the sets F and $T_n(F)$. Hence we deduce by minimality of (F_n, w_n^+, w_n^-)

$$\begin{aligned}\widetilde{J}(F,w^+,w^-) - \widetilde{J}(F_n,w_n^+,w_n^-) &\ge J(F,w^+,w^-) - J(T_n(F),w^+,w^-) + \gamma|F\triangle T_n(F)| \\ &\ge \big(\gamma - 2\Lambda^2 - \|\mathrm{div}X\|_\infty\big)|F\triangle T_n(F)| \ge 0\end{aligned}$$

if $\gamma > 2\Lambda^2 + \|\mathrm{div}X\|_\infty$. This shows that (F_n, w_n^+, w_n^-) is also a solution to (2.61).

Step 2. We now show that each set F_n is an ω-minimizer of the area functional in Ω (see Definition 1.1 and Remark 1.5), for some constant ω (independent of n). Indeed, given any finite perimeter set $F \subset \Omega$ and comparing the value of the functional $\widetilde{J}$ on (F_n, w_n^+, w_n^-) and (F, w_n^+, w_n^-), we have by minimality of (F_n, w_n^+, w_n^-)

$$\begin{aligned}\mathcal{P}(F_n;\Omega) &\le \mathcal{P}(F;\Omega) + \int_F |\nabla w_n^+|^2 + \int_{\Omega\setminus F} |\nabla w_n^+|^2 \\ &\quad - \int_{F_n} |\nabla w_n^+|^2 - \int_{\Omega\setminus F_n} |\nabla w_n^+|^2 + \gamma|F\triangle T_n(F)| \\ &\le \mathcal{P}(F;\Omega) + (2\Lambda^2+\gamma)|F\triangle F_n|,\end{aligned}$$

where we used in particular the L^∞-bound on $\nabla w_n^\pm$ to estimate the difference of the Dirichlet integrals by $|F\triangle F_n|$, and the fact that $F\triangle T_n(F) \subset F\triangle F_n$.

Combining the quasi-minimality of F_n with the Hausdorff convergence of F_n to Ω^+ (whose boundary inside Ω is regular), we deduce by Theorem 1.4 (see also Remark 1.5) that, for n sufficiently large, F_n has boundary of class $C^{1,\alpha}$ inside Ω which converges to Γ in the $C^{1,\alpha}$-sense for all $\alpha \in (0,\frac{1}{2})$.

Hence there exist diffeomorphisms $\Psi_n : \overline{\Omega} \to \overline{\Omega}$ of class $C^{1,\alpha}$ such that $F_n = \Psi_n(\Omega^+)$, $\overline{\partial F_n \cap \Omega} = \Psi_n(\Gamma)$ and $\|\Psi_n - Id\|_{C^{1,\alpha}(\Gamma)} \to 0$. In turn, by Lemma 2.47 we conclude that $\overline{\partial F_n \cap \Omega} = \Gamma_{\psi_n}$ for some functions $\psi_n \in C^{1,\alpha}(\Gamma)$ such that $\psi_n \to 0$ in $C^{1,\alpha}(\Gamma)$.

We also observe that $\nabla w_n^\pm$ are Hölder continuous up to Γ_{ψ_n}, and they converge uniformly to $\nabla u^\pm$. Indeed, by considering the Dirichlet minimizer u_{Ψ_n} in $\Omega \setminus \Psi_n(\Gamma)$ under the usual boundary conditions, we have by elliptic regularity (as in Remark 2.26) that $\nabla_\Gamma(u^\pm_{\Psi_n} \circ \Psi_n)$ is Hölder continuous and converges uniformly to $\nabla_\Gamma u^\pm$. Hence, for n large enough, and also taking into account the continuity of the extension operator, u_{Ψ_n} satisfies the constraint $\|\nabla u^\pm_{\Psi_n} - \nabla u^\pm\|_\infty \leq 1$; we can conclude that $w_n^\pm = u^\pm_{\Psi_n}$.

Step 3. By Lemma 1.8, the curvatures H_{ψ_n} of the sets Γ_{ψ_n} are uniformly bounded by the constant ω. In turn, this provides the $W^{2,\infty}$-regularity of Γ_{ψ_n}, by standard regularity of the C^1-solutions to the mean curvature equation (see [8, Theorem 7.57]).

If we now write the Euler-Lagrange equations for problem (2.59), we get

$$H_{\psi_n} = \begin{cases} |\nabla w_n^+|^2 - |\nabla w_n^-|^2 & \text{on } \Gamma_{\psi_n} \cap \mathcal{I}_{\delta_n}(\Gamma), \\ H_{\Gamma_{\pm\delta_n}} & \text{on } \Gamma_{\psi_n} \cap \Gamma_{\pm\delta_n}, \end{cases}$$

where $H_{\Gamma_{\pm\delta_n}}$ denotes the curvature of the curve $\Gamma_{\pm\delta_n}$. Moreover, as (Γ, u) is a critical pair, by (2.2) we have

$$H_\Gamma = |\nabla u^+|^2 - |\nabla u^-|^2 \quad \text{on } \Gamma.$$

Hence, by the uniform convergence of $\nabla w_n^\pm$ to $\nabla u^\pm$ and observing that the curvature $H_{\Gamma_{\pm\delta_n}}$ is uniformly close to H_Γ by the regularity of the flow generating the family of curves $(\Gamma_\delta)_\delta$, we deduce that

$$\|H_{\psi_n} \circ \Psi_n - H_\Gamma\|_{L^\infty(\Gamma)} \to 0 \qquad \text{as } n \to \infty,$$

which implies, by Lemma 2.47, that $\|\psi_n\|_{W^{2,\infty}(\Gamma)} \to 0$.

We can conclude that, setting $w_n := w_n^+ \chi_{F_n} + w_n^- \chi_{F_n^c}$, by (2.60)

$$\mathcal{MS}(\Gamma_{\psi_n}, w_n) = J(F_n, w_n^+, w_n^-) \leq \mathcal{MS}(\Gamma, u).$$

This inequality implies, by the isolated local $W^{2,\infty}$-minimality of (Γ, u) proved in Theorem 2.27, that for all n large enough $\psi_n = 0$ and $w_n = u$. As a consequence, $(\Phi_n(\Omega^+), v_n^+, v_n^-)$ is itself a solution to (2.59): by repeating all the previous argument for this sequence instead of (F_n, w_n^+, w_n^-), we conclude as before that $\Phi_n = Id$, which is the desired contradiction. $\square$

2.6. Local L^1-minimality

This section is entirely devoted to the proof of Theorem 2.7. It will be useful to introduce the relaxed functional

$$\overline{\mathcal{MS}}(u) := \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(S_u) \qquad \text{for } u \in SBV(\Omega)$$

and, for $B \subset \Omega$ Borel set, its local version

$$\overline{\mathcal{MS}}(u; B) := \int_{\Omega \cap B} |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(S_u \cap B).$$

In the sequel we will consider the following notion of convergence in the space SBV, motivated by the compactness theorem [8, Theorem 4.8].

Definition 2.33. We say that $u_n \to u$ in $SBV(\Omega)$ if $u_n \to u$ strongly in $L^1(\Omega)$, $\nabla u_n \rightharpoonup \nabla u$ weakly in $L^2(\Omega; \mathbb{R}^2)$, and $D^j u_n \rightharpoonup D^j u$ weakly* in the sense of measures in Ω.

Remark 2.34. If $(K, v) \in \mathcal{A}(\Omega)$ is an admissible pair with $\mathcal{H}^{N-1}(K) < +\infty$ and $v \in L^\infty(\Omega)$, then the function v is in $SBV(\Omega)$ and satisfies $\mathcal{H}^1(S_v \setminus K) = 0$ (see [8, Proposition 4.4]); in particular, we have $\overline{\mathcal{MS}}(v) \leq \mathcal{MS}(K, v)$. On the other hand, if (Γ, u) is a regular critical pair, then $u \in SBV(\Omega)$, $\overline{S}_u = \Gamma$ and $\overline{\mathcal{MS}}(u) = \mathcal{MS}(\Gamma, u)$.

Remark 2.35. We observe that, in proving Theorem 2.7, we can assume without loss of generality that U is an open set of class C^∞ and that ∂U and $\partial\Omega$ are orthogonal where they intersect. Indeed, assume to have proved the theorem under these additional assumptions. If U is any admissible subdomain for (Γ, u), we can find a decreasing sequence of admissible subdomains U_n of class C^∞, with boundaries meeting $\partial\Omega$ orthogonally, such that U is the interior part of $\bigcap_n U_n$. It follows from Corollary 2.25 that the second variation is strictly positive in U_n for n large enough, and hence (Γ, u) is an isolated local minimizer in U_n. This immediately yields the conclusion also in the initial domain U.

We can now start the proof of Theorem 2.7. By Remark 2.35 we are allowed to perform the proof under the additional assumption that U has boundary of class C^∞ intersecting $\partial\Omega$ orthogonally. Moreover, from Remark 2.34 it follows that in order obtain the result it is sufficient to show that there exists $\delta > 0$ such that $\overline{\mathcal{MS}}(u) < \overline{\mathcal{MS}}(v)$ for every $v \in SBV(\Omega)$ with $v = u$ in $(\Omega \setminus U) \cup \partial_D\Omega$ and $0 < \|v - u\|_{L^1(\Omega)} < \delta$.

Hence we assume by contradiction that there exists a sequence $v_n \in SBV(\Omega)$, with $v_n = u$ in $(\Omega \setminus U) \cup \partial_D\Omega$, such that $0 < \|v_n - u\|_{L^1(\Omega)} \to 0$ and

$$\overline{\mathcal{MS}}(v_n) \leq \overline{\mathcal{MS}}(u). \tag{2.62}$$

By truncation, we can assume that $\|v_n\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} =: M < +\infty$.

We introduce a bounded open set Ω' such that $\Omega \subset \Omega'$ and $\Omega' \cap \partial\Omega = \partial_D\Omega$, in order to enforce the boundary condition on $\partial_D\Omega$. We can extend u in $\Omega' \setminus \Omega$ to a function $u \in SBV(\Omega')$ such that $\mathcal{H}^1(S_u \cap \partial_D\Omega) = 0$ and $\|u\|_{L^\infty(\Omega')} \leq M$. Moreover, we can also assume that $v_n \in SBV(\Omega')$ and $v_n = u$ in $\Omega' \setminus (U \cap \Omega)$. In particular, $\mathcal{H}^1(S_{v_n} \cap \partial_D\Omega) = 0$ and hence $\overline{\mathcal{MS}}(v_n; \Omega') \leq \overline{\mathcal{MS}}(u; \Omega')$.

We set $\varepsilon_n := \|v_n - u\|^2_{L^2(\Omega)} \to 0$,

$$h_n(t) := \begin{cases} \sqrt{(t - \varepsilon_n)^2 + \varepsilon_n^2} - \varepsilon_n, & \text{if } t > \varepsilon_n, \\ 0, & \text{if } 0 \leq t \leq \varepsilon_n, \end{cases}$$

and we consider, for $\beta > 0$ to be chosen later, a solution w_n to the following penalized minimum problem:

$$\min\Big\{ \overline{\mathcal{MS}}(w; \Omega') + \beta h_n\big(\|w - u\|^2_{L^2(\Omega)}\big) \,:\, w \in SBV(\Omega'),\, w = u \text{ in } \Omega' \setminus (U \cap \Omega) \Big\}. \tag{2.63}$$

The existence of a solution to (2.63) is guaranteed by the lower semi-continuity and compactness theorems in SBV (see [8, Theorem 4.7 and Theorem 4.8]), and we can also assume $\|w_n\|_{L^\infty(\Omega')} \leq M$. Observe in addition that, by (2.62) and by minimality of w_n, we have

$$\overline{\mathcal{MS}}(w_n; \Omega') \leq \overline{\mathcal{MS}}(w_n; \Omega') + \beta h_n\big(\|w_n - u\|^2_{L^2(\Omega)}\big) \leq \overline{\mathcal{MS}}(v_n; \Omega') \leq \overline{\mathcal{MS}}(u; \Omega'). \tag{2.64}$$

In particular the energies $\overline{\mathcal{MS}}(w_n; \Omega')$ are equibounded, and in turn, again by the compactness and lower semi-continuity theorems in SBV, up to subsequences w_n converges in $SBV(\Omega')$ (see Definition 2.33) and in $L^p(\Omega')$ for every $p \in [1, +\infty)$ to a function $z \in SBV(\Omega')$ which solves the minimum problem

$$\min\Big\{ \overline{\mathcal{MS}}(w; \Omega') + \beta \int_\Omega |w - u|^2 \, dx \,:\, w \in SBV(\Omega'),\, w = u \text{ in } \Omega' \setminus (U \cap \Omega) \Big\}. \tag{2.65}$$

Indeed, if $w \in SBV(\Omega')$ is an admissible function for problem (2.65), then by semi-continuity and by minimality of w_n we immediately deduce that

$$\begin{aligned} \overline{\mathcal{MS}}(z; \Omega') + \beta \int_\Omega |z - u|^2 \, dx &\leq \liminf_{n\to\infty} \Big(\overline{\mathcal{MS}}(w_n; \Omega') + \beta h_n\big(\|w_n - u\|^2_{L^2(\Omega)}\big) \Big) \\ &\leq \liminf_{n\to\infty} \Big(\overline{\mathcal{MS}}(w; \Omega') + \beta h_n\big(\|w - u\|^2_{L^2(\Omega)}\big) \Big). \end{aligned}$$

By the result in [67], based on the construction of a suitable calibration, we can identify the solution to the limiting problem.

Proposition 2.36. *If β is sufficiently large, then the unique solution to (2.65) is u itself.*

Notice that in [67] only pure Neumann boundary conditions are considered (*i.e.*, $\partial_D\Omega = \emptyset$). Nevertheless, exactly the same construction applies to our setting without any change (see also [66, Remark 4.3.5]).

Hence, by choosing $\beta > 0$ sufficiently large, we have that $w_n \to u$ in $SBV(\Omega')$. In addition, by lower semi-continuity of $\overline{\mathcal{MS}}$ and by (2.64) we deduce that $\overline{\mathcal{MS}}(w_n;\Omega') \to \overline{\mathcal{MS}}(u;\Omega')$ as $n \to \infty$, which combined with the lower semi-continuity of the two terms in the functional (which holds separately, by [8, Theorem 4.7]) yields

$$\lim_{n\to+\infty}\int_{\Omega'}|\nabla w_n|^2\,\mathrm{d}x = \int_{\Omega'}|\nabla u|^2\,\mathrm{d}x, \qquad \lim_{n\to+\infty}\mathcal{H}^1(S_{w_n}) = \mathcal{H}^1(S_u). \tag{2.66}$$

In the following lemma we localize the previous convergence in open sets and we prove a continuity property that will be used subsequently.

Lemma 2.37. *For every open set $A \subset \mathbb{R}^2$ such that $|\partial A| = 0$ and $\mathcal{H}^1(S_u \cap \partial A) = 0$ we have*

$$\int_{\Omega'\cap A}|\nabla w_n|^2\,\mathrm{d}x \to \int_{\Omega'\cap A}|\nabla u|^2\,\mathrm{d}x, \quad \mathcal{H}^1(S_{w_n}\cap A) \to \mathcal{H}^1(S_u \cap A)$$

as $n \to +\infty$. Moreover, for every bounded continuous function $f \in C^0(\Omega')$ we have

$$\int_{S_{w_n}\cap A} f\,\mathrm{d}\mathcal{H}^1 \to \int_{S_u\cap A} f\,\mathrm{d}\mathcal{H}^1.$$

Proof. The first part of the statement follows easily from the lower semi-continuity of both terms in the functional, which holds in every open set, combined with (2.66). To prove the second part, fix any continuous and bounded function $f : \Omega' \to \mathbb{R}$. Assuming without loss of generality that $f \geq 0$ (for the general case, one can split f into positive and negative parts), we have to show that

$$\int_0^{\max f}\mathcal{H}^1(S_{w_n}\cap A\cap\{f>t\})\,\mathrm{d}t \to \int_0^{\max f}\mathcal{H}^1(S_u\cap A\cap\{f>t\})\,\mathrm{d}t.$$

The sets $A_t = \{f > t\}$ are open and they satisfy $|\partial A_t| = 0$, $\mathcal{H}^1(S_u \cap \partial A_t) = 0$ for all except at most for countable many t. Then, by the assumptions on A, the same is true for the sets $A \cap A_t$, and hence by the first part of the lemma we have

$$\mathcal{H}^1(S_{w_n}\cap A\cap\{f>t\}) \to \mathcal{H}^1(S_u\cap A\cap\{f>t\}) \qquad \text{for a.e. } t \in (0, \max f)\,,$$

and by the Dominated Convergence Theorem we obtain the conclusion. □

In the following proposition we show that w_n is a quasi-minimizer of the Mumford-Shah functional, according to Definition 1.10. This is an essential step in our strategy to prove Theorem 2.7: indeed, as a consequence of the regularity theory for quasi-minimizers we obtain firstly that a uniform lower bound on the 1-dimensional density of S_{w_n} holds, and moreover we will be able to deduce the $C^{1,\alpha}$-convergence of S_{w_n} to S_u (see Proposition 2.40).

Proposition 2.38. *There exists a positive constant ω (independent of n) such that if $x \in \overline{\Omega}'$ and $\rho > 0$ then*

$$\overline{\mathcal{MS}}(w_n; B_\rho(x)\cap\Omega') \leq \overline{\mathcal{MS}}(v; B_\rho(x)\cap\Omega') + \omega\rho^2 \tag{2.67}$$

for every $v \in SBV(\Omega')$ with $v = u$ in $\Omega' \setminus (U \cap \Omega)$ and $\{v \neq w_n\} \subset\subset B_\rho(x)$.

Proof. Let v be as in the statement, and set $v^M := (-M) \vee (v \wedge M)$, where $M = \|u\|_\infty$. Then, since $v^M \in SBV(\Omega')$ is an admissible competitor in problem (2.63), $\{v^M \neq w_n\} \subset \{v \neq w_n\}$ (as $\|w_n\|_\infty \leq M$) and $\overline{\mathcal{MS}}(v^M) \leq \overline{\mathcal{MS}}(v)$, we have by minimality of w_n

$$
\begin{aligned}
\overline{\mathcal{MS}}(w_n; B_\rho(x) \cap \Omega') &\leq \overline{\mathcal{MS}}(v^M; B_\rho(x) \cap \Omega') + \beta h_n\Big(\int_\Omega |v^M - u|^2 \,\mathrm{d}y \Big) - \beta h_n \Big(\int_\Omega |w_n - u|^2 \,\mathrm{d}y \Big) \\
&\leq \overline{\mathcal{MS}}(v; B_\rho(x) \cap \Omega') + \beta \Big| \int_{B_\rho(x)\cap\Omega} |v^M - u|^2 \,\mathrm{d}y - \int_{B_\rho(x)\cap\Omega} |w_n - u|^2 \,\mathrm{d}y \Big| \\
&\leq \overline{\mathcal{MS}}(v; B_\rho(x) \cap \Omega') + 8M^2 \beta\pi\rho^2 \,,
\end{aligned}
$$

where we used the fact that h_n is 1-Lipschitz in the second inequality. Hence (2.67) follows by choosing $\omega := 8M^2\beta\pi$. $\square$

Corollary 2.39. *Each set S_{w_n} is essentially closed: $\mathcal{H}^1(\overline{S}_{w_n} \setminus S_{w_n}) = 0$. Moreover, the sets $\overline{S}_{w_n}$ converge to $\overline{S}_u$ in $\overline{\Omega}'$ in the sense of Kuratowski:*

(i) for every $x_n \in \overline{S}_{w_n}$ such that $x_n \to x$, then $x \in \overline{S}_u$;
(ii) for every $x \in \overline{S}_u$ there exist $x_n \in \overline{S}_{w_n}$ such that $x_n \to x$.

Proof. Thanks to the quasi-minimality property proved in the previous proposition and to the fact that ∂U and $\partial\Omega$ meet orthogonally, we can apply Theorem 1.13 to infer the existence of constants $\vartheta_0 > 0$, $\rho_0 > 0$ (independent of n) such that for every $x \in \overline{S}_{w_n} \cap \overline{\Omega}'$ and for every $\rho \leq \rho_0$

$$
\mathcal{H}^1(S_{w_n} \cap B_\rho(x)) \geq \vartheta_0 \rho. \tag{2.68}
$$

The properties in the statement are now standard consequences of (2.68). The first part follows from the fact that the 1-dimensional density of S_{w_n} is zero $\mathcal{H}^1$-a.e. in the complement of S_{w_n}:

$$
\limsup_{\rho\to 0} \frac{\mathcal{H}^1(S_{w_n} \cap B_\rho(x))}{\rho} = 0 \qquad \text{for } \mathcal{H}^1\text{-a.e. } x \in \mathbb{R}^2 \setminus S_{w_n}
$$

(see [8, Theorem 2.56]). We set $\mu_n := \mathcal{H}^1 \llcorner \overline{S}_{w_n}$ and $\mu := \mathcal{H}^1 \llcorner \overline{S}_u$. To prove (i), consider any $\rho > 0$ such that $\mu(\partial B_\rho(x)) = 0$. Then for n sufficiently large $x_n \in B_{\rho/2}(x)$ and by (2.68)

$$
\mu_n(B_\rho(x)) \geq \mu_n(B_{\frac{\rho}{2}}(x_n)) \geq \frac{\vartheta_0\rho}{2}
$$

(if ρ is sufficiently small). By Lemma 2.37 we conclude that $\mu(B_\rho(x)) \geq \vartheta_0\rho/2$, hence $x \in \operatorname{supp}\mu = \overline{S}_u$. Finally, we can easily prove that each $x \in \overline{S}_u$ is the limit of a sequence of points $x_n \in \overline{S}_{w_n}$: indeed, if not we could find a ball $B_\rho(x)$ such that $\mu_n(B_\rho(x)) = 0$ for infinite indices n, and without loss of generality $\mu(\partial B_\rho(x)) = 0$; then, by Lemma 2.37 we would have $\mu(B_\rho(x)) = 0$, which is a contradiction. $\square$

Corollary 2.39 provides the Hausdorff convergence of $\overline{S}_{w_n}$ to $\overline{S}_u$ in $\overline{\Omega}'$, which allows us to assume, from now on, that $\overline{S}_{w_n}$ is contained in a tubular neighborhood of $\overline{S}_u$ contained in U. We now come to the main consequence of the regularity theory for quasi-minimizers.

We first observe that, using the good description of $\overline{S}_{w_n}$ near $\partial\Omega$ given by Theorem 1.12, we can find $\tau > 0$ such that $\overline{S}_{w_n} \cap \Omega(\tau)$ is a $C^{1,\alpha}$-curve for some $\alpha \in (0,1)$, with $C^{1,\alpha}$-norm uniformly bounded with respect to n, meeting $\partial\Omega$ orthogonally. Combining this information with the Hausdorff convergence to $\overline{S}_u$, we deduce that the sets $\overline{S}_{w_n}$ converge to $\overline{S}_u$ in $\Omega(\tau)$ in the $C^{1,\beta}$-sense, for every $\beta < \alpha$. In the following proposition we obtain the same convergence in the interior of Ω (the notation is the one introduced in Chapter 1).

PROPOSITION 2.40. *There exists a finite covering of* $\Gamma \cap (\Omega \setminus \Omega(\tau))$ *of the form* $\bigcup_{i=1}^{N_0}(x_i + C_{\nu_i,\rho_i})$ *where* $x_i \in \Gamma$, $\nu_i = \nu_\Gamma(x_i)$, *and functions* $f_i^{(n)} : (-\rho_i, \rho_i) \to (-\rho_i, \rho_i)$ *of class* $C^{1,\alpha}$ *(for some* $\alpha \in (0,1)$*) such that*

$$(\overline{S}_{w_n} - x_i) \cap C_{\nu_i,\rho_i} = \mathrm{gr}_{\nu_i}(f_i^{(n)})$$

for n *sufficiently large and* $i = 1, \ldots, N_0$. *Moreover, the sequence* $f_i^{(n)}$ *converges to* f_i *in* $C^{1,\beta}$ *as* $n \to +\infty$ *for every* $\beta < \alpha$, *where* $f_i : (-\rho_i, \rho_i) \to (-\rho_i, \rho_i)$ *is such that*

$$(\Gamma - x_i) \cap C_{\nu_i,\rho_i} = \mathrm{gr}_{\nu_i}(f_i).$$

PROOF. Fix any point $x_0 \in \Gamma \cap (\Omega \setminus \Omega(\tau))$. By the regularity of u and $\Gamma = \overline{S}_u$, we can find $r_0 > 0$ such that $B_{r_0}(x_0) \subset \Omega \cap U$, $\mathcal{H}^1(S_u \cap \partial B_{r_0}(x_0)) = 0$ and

$$E_u(x_0, r_0) < \varepsilon_0 \frac{r_0}{8}\,,$$

where ε_0 is given by Theorem 1.11. Lemma 2.37 immediately implies that $D_{w_n}(x_0, r_0) \to D_u(x_0, r_0)$ and that for every affine plane T

$$\int_{\overline{S}_{w_n} \cap B_{r_0}(x_0)} \mathrm{dist}^{\,2}(y, T)\, \mathrm{d}\mathcal{H}^1(y) \to \int_{\overline{S}_u \cap B_{r_0}(x_0)} \mathrm{dist}^{\,2}(y, T)\, \mathrm{d}\mathcal{H}^1(y)\,.$$

From the previous convergence it follows also that $\limsup_{n\to\infty} A_{w_n}(x_0, r_0) \le A_u(x_0, r_0)$, since if the minimum value defining $A_u(x_0, r_0)$ is attained at an affine plane T_0, then

$$\begin{aligned}\limsup_{n\to\infty} A_{w_n}(x_0, r_0) &\le \lim_{n\to\infty} \int_{\overline{S}_{w_n} \cap B_{r_0}(x_0)} \mathrm{dist}^{\,2}(y, T_0)\, \mathrm{d}\mathcal{H}^1(y) \\ &= \int_{\overline{S}_u \cap B_{r_0}(x_0)} \mathrm{dist}^{\,2}(y, T_0)\, \mathrm{d}\mathcal{H}^1(y) = A_u(x_0, r_0)\,.\end{aligned}$$

Hence $\limsup_{n\to\infty} E_{w_n}(x_0, r_0) \le E_u(x_0, r_0)$, so that for n sufficiently large we have

$$E_{w_n}(x_0, r_0) < \varepsilon_0 \frac{r_0}{8}\,.$$

By Corollary 2.39 we can find a sequence $x_n \in \overline{S}_{w_n}$ converging to x_0, so that $B_{r_0/2}(x_n) \subset B_{r_0}(x_0)$ for n large enough and thus

$$E_{w_n}(x_n, r_0/2) = D_{w_n}(x_n, r_0/2) + \frac{4}{r_0^2} A_{w_n}(x_n, r_0/2) \le 4E_{w_n}(x_0, r_0) < \varepsilon_0 \frac{r_0}{2}.$$

We are now in position to apply Theorem 1.11: we find a radius $r_1 \in (0, r_0)$ and functions $g_n : (-r_1, r_1) \to \mathbb{R}$ uniformly bounded in $C^{1,\frac{1}{4}}$, with $g_n(0) = g_n'(0) = 0$, such that $(\overline{S}_{w_n} - x_n) \cap C_{\nu_n, r_1} = \mathrm{gr}_{\nu_n}(g_n)$, where ν_n is the normal to $\overline{S}_{w_n}$ at x_n.

By compactness, $\nu_n \to \bar{\nu}$ (up to subsequences). For n large enough $C_{\bar{\nu}, r_1/2} \subset C_{\nu_n, r_1} + x_n - x_0$, and there exist functions f_n uniformly bounded in $C^{1,\frac{1}{4}}$ such that $\mathrm{gr}_{\bar{\nu}}(f_n) \cap C_{\bar{\nu}, r_1/2} = (\mathrm{gr}_{\nu_n}(g_n) + x_n - x_0) \cap C_{\bar{\nu}, r_1/2}$. Hence

$$(\overline{S}_{w_n} - x_0) \cap C_{\bar{\nu}, r_1/2} = \mathrm{gr}_{\bar{\nu}}(f_n),$$

and by Ascoli-Arzelà Theorem f_n converges to some function f in $C^{1,\beta}$ for every $\beta < \frac{1}{4}$, with $f(0) = f'(0) = 0$. Using the Kuratowski convergence of $\overline{S}_{w_n}$ to Γ, we deduce that $(\Gamma - x_0) \cap C_{\bar{\nu}, r_1/2} = \mathrm{gr}_{\bar{\nu}}(f)$, and since $f'(0) = 0$ it must be $\bar{\nu} = \nu_\Gamma(x_0)$. $\square$

From what we have proved it follows that for every $n \in \mathbb{N}$ there exists a diffeomorphism $\Phi_n : \overline{\Omega} \to \overline{\Omega}$, with $\operatorname{supp}(\Phi_n - Id) \subset\subset (U \setminus \partial_D \Omega)$, such that $\overline{S}_{w_n} = \Phi_n(\Gamma)$ and $\|\Phi_n - Id\|_{C^{1,\alpha}(\Gamma)} \to 0$.

With this information, we can finally conclude the proof of the isolated local minimality of u. Indeed, since $\mathcal{H}^1(\overline{S}_{w_n} \setminus S_{w_n}) = 0$ by Corollary 2.39, we have that $(\Phi_n(\Gamma), w_n) \in \mathcal{A}(\Omega)$ and $\mathcal{MS}(\Phi_n(\Gamma), w_n) = \overline{\mathcal{MS}}(w_n)$. Hence for n large enough, using (2.64),

$$\mathcal{MS}(\Phi_n(\Gamma), w_n) = \overline{\mathcal{MS}}(w_n) \le \overline{\mathcal{MS}}(u) = \mathcal{MS}(\Gamma, u)\,,$$

which implies that $\Phi_n = Id$ and $w_n = u$ for all (large) n by Proposition 2.32. Hence u itself is a solution to (2.63), and as a consequence of (2.62) also v_n solves the same minimum problem. We can then repeat all the previous argument for the sequence v_n instead of w_n, which leads, as before, to $v_n = u$ for n sufficiently large. This is the desired contradiction, since we are assuming $v_n \neq u$ for every n.

2.7. Applications and examples

We start this section by showing that any regular critical pair (Γ, u) satisfying (2.24) is strictly stable in a sufficiently small tubular neighborhood $\mathcal{N}_\varepsilon(\Gamma)$ of the discontinuity set. As a consequence of our main result, we deduce the local minimality of (Γ, u) in $\mathcal{N}_\varepsilon(\Gamma)$, and also that (Γ, u) is in fact a global minimizer in a smaller neighborhood. This is in analogy with the result proved in [65], where it is shown, by means of a calibration method, that a critical point is a Dirichlet minimizer in small domains.

Proposition 2.41 (local and global minimality in small neighborhoods). *Let (Γ, u) be a regular critical pair satisfying condition (2.24). Then there exists $\varepsilon_0 > 0$ such that the tubular neighborhood $\mathcal{N}_\varepsilon(\Gamma)$ of Γ is an admissible subdomain and (Γ, u) is strictly stable in $\mathcal{N}_\varepsilon(\Gamma)$ for every $\varepsilon < \varepsilon_0$. In particular, there exists $\delta > 0$ such that $\mathcal{MS}(\Gamma, u) < \mathcal{MS}(K, v)$ for every $(K, v) \in \mathcal{A}(\Omega)$ with $0 < \|u - v\|_{L^1(\Omega)} < \delta$ and $v = u$ in $\Omega \setminus \mathcal{N}_\varepsilon(\Gamma)$.*

Moreover, there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that (Γ, u) is a global minimizer in $\mathcal{N}_\varepsilon(\Gamma)$ for every $\varepsilon < \varepsilon_1$, in the sense that $\mathcal{MS}(\Gamma, u) \le \mathcal{MS}(K, v)$ for every $(K, v) \in \mathcal{A}(\Omega)$ with $v = u$ in $\Omega \setminus \mathcal{N}_\varepsilon(\Gamma)$.

Proof. Clearly $\mathcal{N}_\varepsilon(\Gamma)$ is an admissible subdomain for ε small enough, and in view of Proposition 2.21 we shall prove that

$$\lim_{\varepsilon \to 0} \mu(\mathcal{N}_\varepsilon(\Gamma)) = +\infty$$

in order to obtain the first part of the statement. Assume by contradiction that there exist $\varepsilon_n \to 0^+$, $C > 0$ and $v_n \in H^1_{U_n}(\Omega \setminus \Gamma)$ such that $\|\Phi_{v_n}\|_\sim = 1$ and

$$2 \int_\Omega |\nabla v_n|^2 \le C$$

for every n, where we set $U_n := \mathcal{N}_{\varepsilon_n}(\Gamma)$. Then v_n is a bounded sequence in $H^1_{U_1}(\Omega \setminus \Gamma)$, which converges weakly to 0 since the measure of U_n goes to 0. By compactness of the map $v \mapsto \Phi_v$, we have that Φ_{v_n} converge to 0 strongly in $H^1(\Gamma \cap \Omega)$, which is in contradiction to the fact that $\|\Phi_{v_n}\|_\sim = 1$ for every n.

To prove the second part of the statement, let u_ε be a solution to the minimum problem

$$\min\{\overline{\mathcal{MS}}(v) : v \in SBV(\Omega),\ v = u \text{ in } \Omega \setminus \mathcal{N}_\varepsilon(\Gamma)\}, \tag{2.69}$$

where $\overline{\mathcal{MS}}$ is the relaxed functional introduced at the beginning of Section 2.6. We remark that, by classical regularity results for minimizers of the Mumford-Shah functional, $\mathcal{H}^1(\overline{S}_{u_\varepsilon} \setminus S_{u_\varepsilon}) = 0$ and thus $\overline{\mathcal{MS}}(u_\varepsilon) = \mathcal{MS}(\overline{S}_{u_\varepsilon}, u_\varepsilon)$. Hence, since $u_\varepsilon \to u$ in $L^1(\Omega)$ as $\varepsilon \to 0$ because the measure of $\mathcal{N}_\varepsilon(\Gamma)$ goes to 0, we conclude that $u_\varepsilon = u$ for ε small enough, as a consequence of the isolated local minimality of (Γ, u). Then u is a solution to (2.69), and the conclusion follows by Remark 2.34. $\square$

REMARK 2.42. Let (Γ, u) be a regular critical pair, and assume that

$$-2\int_\Omega |\nabla v_\varphi|^2 \,\mathrm{d}x + \int_{\Gamma\cap\Omega} |\nabla_\Gamma \varphi|^2 \,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma\cap\Omega} H^2\varphi^2 \,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma\cap\partial\Omega} H_{\partial\Omega}\varphi^2 \,\mathrm{d}\mathcal{H}^0 > 0$$

for every $\varphi \in H^1(\Gamma\cap\Omega)\setminus\{0\}$, where $v_\varphi \in H^1(\Omega\setminus\Gamma)$, $v_\varphi = 0$ on $\partial_D\Omega$, solves

$$\int_\Omega \nabla v_\varphi \cdot \nabla z \,\mathrm{d}x + \int_{\Gamma\cap\Omega} \left[z^+ \mathrm{div}_\Gamma(\varphi\nabla_\Gamma u^+) - z^- \mathrm{div}_\Gamma(\varphi\nabla_\Gamma u^-) \right] \mathrm{d}\mathcal{H}^1 = 0$$

for every $z \in H^1(\Omega\setminus\Gamma)$ with $z = 0$ on $\partial_D\Omega$. Then (Γ, u) is strictly stable in every admissible subdomain U. Hence, under the previous assumptions we can conclude that for every neighborhood $\mathcal{N}_\eta(\mathcal{S})$, where $\mathcal{S}$ is the relative boundary of $\partial_D\Omega$ in $\partial\Omega$, there exists $\delta(\eta) > 0$ such that $\mathcal{MS}(\Gamma, u) < \mathcal{MS}(K, v)$ for every $(K, v) \in \mathcal{A}(\Omega)$ with $\|v-u\|_{L^1(\Omega)} < \delta$ and $v = u$ in $\mathcal{N}_\eta(\mathcal{S})$.

We now provide some explicit examples of critical point to which Theorem 2.7 can be applied. In particular, in Example 2.43 we discuss how the stability of constant critical pairs depends on the geometry of the domain Ω, while in Remark 2.44 we discuss how to construct families of (non-constant) critical pairs by a perturbing the Dirichlet data.

EXAMPLE 2.43. Let Γ be a straight line contained in Ω connecting two points $x_1, x_2 \in \partial\Omega$ of minimal distance, and let u be equal to two different constants in the two connected components of $\Omega\setminus\Gamma$. Assume that Ω is strictly concave at x_1 and x_2 (that is, the curvature $H_{\partial\Omega}$ with respect to the exterior normal is strictly negative at x_1 and x_2). Then (Γ, u) is a regular critical pair such that for every admissible subdomain U

$$\partial^2\mathcal{MS}((\Gamma,u);U)[\varphi] = \int_\Gamma |\nabla_\Gamma\varphi|^2 - H_{\partial\Omega}(x_1)\varphi^2(x_1) - H_{\partial\Omega}(x_2)\varphi^2(x_2) > 0$$

for every $\varphi \in H^1(\Gamma)\setminus\{0\}$. Hence it follows by Theorem 2.7 that (Γ, u) is an isolated local minimizer for $\mathcal{MS}$ in every admissible subdomain U.

If the domain Ω is strictly convex, then a straight line connecting two points on $\partial\Omega$ of minimal distance is never a local minimizer: indeed, if U is any admissible subdomain, by evaluating the quadratic form $\partial^2\mathcal{MS}((\Gamma,u);U)$ at the constant function $\varphi = 1$ we get

$$\partial^2\mathcal{MS}((\Gamma,u);U)[1] = -H_{\partial\Omega}(x_1) - H_{\partial\Omega}(x_2) < 0.$$

We remark that this is not in contradiction to the result of Proposition 2.41, since in the present situation condition (2.24) is not satisfied.

REMARK 2.44 (families of stable critical pairs by perturbation of the Dirichlet data). Let (Γ, u) be a strictly stable regular critical pair in an admissible subdomain U, and assume in addition that u^+ and u^- are of class C^2 in a neighborhood of Γ.

We fix a function $\psi_0 \in C^\infty_c(\partial_D\Omega)$ and we consider a perturbation of the Dirichlet datum of the form $u_\varepsilon := u + \varepsilon\psi_0$, for $\varepsilon > 0$. As an application of the Implicit Function Theorem, one can show that for every ε sufficiently small there exists a strictly stable regular critical pair $(\Gamma_\varepsilon, v_\varepsilon)$ with $v_\varepsilon = u_\varepsilon$ in $(\Omega\setminus U)\cup\partial_D\Omega$.

The idea of the proof is to associate, with every $\varepsilon > 0$ and $\psi \in C^{2,\alpha}(\Gamma)$, the curve Γ_ψ defined as in (2.31) and the function $u_{\varepsilon,\psi}$ which minimizes the Dirichlet integral in $H^1(\Omega\setminus\Gamma_\psi)$ and attains the boundary condition $u_{\varepsilon,\psi} = u_\varepsilon$ in $(\Omega\setminus U)\cup\partial_D\Omega$. Then one considers the map

$$G : \mathbb{R}\times C^{2,\alpha}(\Gamma) \to C^{0,\alpha}(\Gamma)\times\mathbb{R}\times\mathbb{R}$$

defined by

$$G(\varepsilon,\psi) := \big((H_\psi - |\nabla_{\Gamma_\psi} u^+_{\varepsilon,\psi}|^2 + |\nabla_{\Gamma_\psi} u^-_{\varepsilon,\psi}|^2) \circ \Phi^\psi,\ (\eta_\psi \cdot \tau_{\partial\Omega})(\Phi^\psi(x_1)),\ (\eta_\psi \cdot \tau_{\partial\Omega})(\Phi^\psi(x_2)) \big)\,.$$

Here Φ^ψ is a diffeomorphism mapping Γ onto Γ_ψ (see (2.31)), H_ψ denotes the curvature of Γ_ψ, η_ψ is the unit co-normal of $\Gamma_\psi \cap \partial\Omega$, $\{x_1, x_2\} = \Gamma \cap \partial\Omega$, and $\tau_{\partial\Omega}$ is the tangent vector to $\partial\Omega$, oriented in such a way that it coincides with ν in a neighborhood of $\Gamma \cap \partial\Omega$.

It can be shown that the map G is of class C^1 in a neighborhood of $(0,0)$, satisfies $G(0,0) = 0$ (as (Γ, u) is a critical pair), and the partial derivative

$$\partial_\psi G(0,0) : C^{2,\alpha}(\Gamma) \to C^{0,\alpha}(\Gamma) \times \mathbb{R} \times \mathbb{R}$$

is an invertible bounded linear operator. In order to show this last property, we need to prove that for every $(f, c_1, c_2) \in C^{0,\alpha}(\Gamma) \times \mathbb{R} \times \mathbb{R}$ there exists a unique $\psi \in C^{2,\alpha}(\Gamma)$ such that

$$\partial_\psi G(0,0)[\psi] = (f, c_1, c_2),$$

or equivalently, setting $\partial_\psi G(0,0)[\psi] = (h(\psi), a_1(\psi), a_2(\psi))$, that

$$\int_{\Gamma\cap\Omega} h(\psi)\phi \,\mathrm{d}\mathcal{H}^1 + a_1(\psi)\phi(x_1) + a_2(\psi)\phi(x_2) = \int_{\Gamma\cap\Omega} f\phi \,\mathrm{d}\mathcal{H}^1 + c_1\phi(x_1) + c_2\phi(x_2) \tag{2.70}$$

for every $\phi \in H^1(\Gamma \cap \Omega)$. By repeating the computations contained in the proof of Theorem 2.14, one can show that

$$\begin{aligned}&\int_{\Gamma\cap\Omega} h(\psi)\phi \,\mathrm{d}\mathcal{H}^1 + a_1(\psi)\phi(x_1) + a_2(\psi)\phi(x_2)\\ &\qquad = -2\int_\Omega \nabla v_\psi \cdot \nabla v_\phi \,\mathrm{d}x + \int_{\Gamma\cap\Omega} \nabla_\Gamma \psi \cdot \nabla_\Gamma \phi \,\mathrm{d}\mathcal{H}^1 + \int_{\Gamma\cap\Omega} H^2 \psi\phi \,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma\cap\partial\Omega} H_{\partial\Omega}\psi\phi \,\mathrm{d}\mathcal{H}^0\,.\end{aligned}$$

Hence, the existence of a function $\psi \in H^1(\Gamma)$ satisfying (2.70) for every $\phi \in H^1(\Gamma)$ is guaranteed by Lax-Milgram Lemma and by the assumption of strict positivity of the second variation at (Γ, u). Such a function ψ is a weak solution to the system

$$\begin{cases} -\Delta_\Gamma \psi + H^2\psi = 2\nabla u^+ \cdot \nabla_\Gamma v^+_\psi - 2\nabla u^- \cdot \nabla_\Gamma v^-_\psi + f & \text{on } \Gamma\cap\Omega,\\ \partial_{\nu_{\partial\Omega}}\psi - H_{\partial\Omega}\psi = -2(v^+_\psi \nabla u^+ - v^-_\psi \nabla u^-)\cdot \nu_{\partial\Omega} + c_i & \text{at } x_i,\ i=1,2,\\ \Delta v_\psi = 0 & \text{in } \Omega\setminus\Gamma,\\ v_\psi = 0 & \text{in } (\Omega\setminus U)\cup \partial_D\Omega,\\ \partial_\nu v_\psi = 0 & \text{on } \partial_N\Omega,\\ \partial_\nu v^\pm_\psi = \mathrm{div}_\Gamma(\psi\nabla u^\pm) & \text{on } \Gamma\cap\Omega. \end{cases}$$

By elliptic regularity we deduce that $\psi \in C^{2,\alpha}(\Gamma)$, completing the proof of the invertibility of $\partial_\psi G(0,0)$.

The previous conditions allow us to apply the Implicit Function Theorem and to obtain the desired family of critical pairs.

We conclude this section by observing, in the following remark, that our analysis can be extended to the periodic case: more precisely, we assume that the domain is a rectangle, Γ is a curve joining two opposite points on the boundary, and the Neumann boundary conditions are replaced by periodicity conditions on the sides connected by Γ. The remaining pair of sides represents the Dirichlet part of the boundary. We also discuss an explicit example in this different setting. In the remaining part of this section, with a slight abuse of notation we denote the generic point of $\mathbb{R}^2$ by (x,y).

REMARK 2.45. Let $R := [0,b) \times (-a,a)$, where $a,b>0$ are positive real numbers. We define the infinite strip $\widetilde{R} := \mathbb{R} \times (-a,a)$, the Dirichlet boundary $\partial_D R := [0,b] \times \{-a,a\}$, and the class of admissible pairs

$$\mathcal{A}(R) := \big\{(K,v) : K \subset \mathbb{R}^2 \text{ closed},\ K + (b,0) = K,\ v \in H^1_{\mathrm{loc}}(\widetilde{R} \setminus K) \cap H^1(R \setminus K),$$
$$v_x(x+b,y) = v_x(x,y) \text{ for every } (x,y) \in \widetilde{R} \setminus K\big\}\,.$$

We denote by $H^1_{\mathrm{per}}(R \setminus K)$ the class of functions $z \in H^1_{\mathrm{loc}}(\widetilde{R} \setminus K) \cap H^1(R \setminus K)$ such that the map $x \mapsto z(x,y)$ is b-periodic for every $y \in (-a,a)$. Finally we consider the functional

$$\mathcal{MS}(K,v) := \int_{R \setminus K} |\nabla v|^2 + \mathcal{H}^1(K \cap R) \qquad \text{for } (K,v) \in \mathcal{A}(R).$$

Similarly to what we did in Section 2.1, we say that $(\Gamma,u) \in \mathcal{A}(R)$ is a regular critical pair if $\Gamma \subset \widetilde{R}$ is a curve of class C^∞ such that $\Gamma \cap R$ connects two opposite points on ∂R, u satisfies

$$\int_{R \setminus \Gamma} \nabla u \cdot \nabla z = 0 \qquad \text{for every } z \in H^1_{\mathrm{per}}(R \setminus \Gamma) \text{ with } z = 0 \text{ on } \partial_D R,$$

and moreover the transmission condition and the non-vanishing jump condition (see Definition 2.3) hold on Γ. Setting $H^1_{\mathrm{per}}(\Gamma) := \{\varphi \in H^1_{\mathrm{loc}}(\Gamma) : \varphi(x+b,y) = \varphi(x,y) \text{ for every } (x,y) \in \Gamma\}$, we say that a regular critical pair (Γ,u) is strictly stable if

$$\partial^2 \mathcal{MS}(\Gamma,u)[\varphi] := -2\int_R |\nabla v_\varphi|^2 + \int_{\Gamma \cap R} |\nabla_\Gamma \varphi|^2 \, d\mathcal{H}^1 + \int_{\Gamma \cap R} H^2 \varphi^2 \, d\mathcal{H}^1 > 0$$

for every $\varphi \in H^1_{\mathrm{per}}(\Gamma) \setminus \{0\}$, where $v_\varphi \in H^1_{\mathrm{per}}(R \setminus \Gamma)$, $v_\varphi = 0$ on $\partial_D R$, is the solution to

$$\int_R \nabla v_\varphi \cdot \nabla z + \int_{\Gamma \cap R} \Big[z^+ \mathrm{div}_\Gamma\big(\varphi \nabla_\Gamma u^+\big) - z^- \mathrm{div}_\Gamma\big(\varphi \nabla_\Gamma u^-\big)\Big] \, d\mathcal{H}^1 = 0 \tag{2.71}$$

for every $z \in H^1_{\mathrm{per}}(R \setminus \Gamma)$, $z = 0$ on $\partial_D R$.

Then one can prove that every strictly stable regular critical pair (Γ,u) is a local minimizer, in the sense that there exists $\delta > 0$ such that $\mathcal{MS}(\Gamma,u) < \mathcal{MS}(K,v)$ for every $(K,v) \in \mathcal{A}(R)$ with $v = u$ on $\partial_D R$ and $0 < \|u - v\|_{L^1(R)} < \delta$. We omit the proof of this result, since it can be obtained by repeating all the arguments which lead to the proof of Theorem 2.7 with the natural modifications (notice that the proof in the present setting is in fact simpler, since by periodicity we can work in the whole strip $\widetilde{R}$ avoiding the technical difficulties related to the presence of Neumann boundary conditions).

EXAMPLE 2.46. Here we adapt to the periodic setting described in Remark 2.45 the example discussed in [19, Section 7]. Setting $R = [0,b) \times (-a,a)$, we consider the regular critical pair $(\Gamma,u) \in \mathcal{A}(R)$ where $\Gamma = \mathbb{R} \times \{0\}$ and $u : \mathbb{R}^2 \to \mathbb{R}$ is the function

$$u(x,y) := \begin{cases} x+1 & \text{for } y \geq 0, \\ -x & \text{for } y < 0. \end{cases}$$

Notice that the energy of (Γ,u) is invariant along vertical translations of the discontinuity set. Nevertheless, we shall prove in fact that if

$$\tfrac{2b}{\pi} \tanh\big(\tfrac{2\pi a}{b}\big) < 1\,, \tag{2.72}$$

then (Γ,u) is an isolated local minimizer up to vertical translations: precisely, there exists $\delta > 0$ such that $\mathcal{MS}(\Gamma,u) < \mathcal{MS}(K,v)$ for every $(K,v) \in \mathcal{A}(R)$ with $v = u$ on $\partial_D R$ and $\|u - v\|_{L^1(R)} < \delta$, unless K coincides with a vertical translation of Γ. Moreover, (2.72) is sharp in the sense that if $\frac{2b}{\pi} \tanh\big(\frac{2\pi a}{b}\big) > 1$ then (Γ,u) is unstable.

To this aim, we will test the strict positivity of second variation at (Γ, u) on the subspace $H^1_0(0,b)$ of $H^1_{\rm per}(\Gamma)$ of the functions vanishing at the endpoints, showing that

$$\partial^2\mathcal{MS}(\Gamma,u)[\varphi] \geq C_0\|\varphi\|^2_{H^1(0,b)} \text{ for every } \varphi \in H^1_0(0,b)\setminus\{0\} \quad \text{iff} \quad \tfrac{2b}{\pi}\tanh\bigl(\tfrac{2\pi a}{b}\bigr) < 1\,. \tag{2.73}$$

In turn, setting $\Gamma_\varepsilon := \mathbb{R}\times\{\varepsilon\}$ and

$$u_\varepsilon(x,y) := \begin{cases} x+1 & \text{for } y \geq \varepsilon, \\ -x & \text{for } y < \varepsilon, \end{cases}$$

we have that $(\Gamma_\varepsilon, u_\varepsilon)$ is still a critical pair with the same energy of (Γ, u), and, assuming (2.72) and (2.73), there exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ we have

$$\partial^2\mathcal{MS}(\Gamma_\varepsilon,u_\varepsilon)[\varphi] \geq \frac{C_0}{2}\|\varphi\|^2_{H^1(0,b)} \qquad \text{for every } \varphi \in H^1_0(0,b)\setminus\{0\}. \tag{2.74}$$

This can be deduced by comparing the explicit expressions of the second variation at (Γ, u) and at $(\Gamma_\varepsilon, u_\varepsilon)$ and observing that

$$\sup_{\|\varphi\|_{H^1(0,b)}=1}\left|\int_R |\nabla v^\varepsilon_\varphi|^2 - \int_R |\nabla v_\varphi|^2\right| \to 0 \qquad \text{as } \varepsilon \to 0$$

(where v_φ and v^ε_φ are the solutions to (2.71) corresponding to (Γ, u) and $(\Gamma_\varepsilon, u_\varepsilon)$ respectively); this last estimate is obtained by subtracting the equations satisfied by v_φ and v^ε_φ. From (2.74) it follows that any configuration which is close in $W^{2,\infty}$ and coincides with Γ_ε at the endpoints has strictly larger energy than $(\Gamma_\varepsilon, u_\varepsilon)$: more precisely, there exists $\delta_0 > 0$ such that for every $|\varepsilon| < \varepsilon_0$, for every b-periodic function $h \in W^{2,\infty}_{\rm loc}(\mathbb{R})$ with $0 < \|h-\varepsilon\|_{W^{2,\infty}(0,b)} < \delta_0$, $h(0) = h(b) = \varepsilon$, and for every v such that $(\Gamma_h, v) \in \mathcal{A}(R)$ and $v = u$ on $\partial_D R$, we have $\mathcal{MS}(\Gamma_h, v) > \mathcal{MS}(\Gamma_\varepsilon, u_\varepsilon) = \mathcal{MS}(\Gamma, u)$, where we denoted by Γ_h the graph of h. This can be deduced by repeating the arguments for the proof of Theorem 2.27, paying attention to the fact that the local minimality neighborhood can be chosen uniform with respect to n.

In turn, from this property easily follows the isolated local $W^{2,\infty}$-minimality of (Γ, u), since it implies the existence of a positive δ such that for every $(\Gamma_h, v) \in \mathcal{A}(R)$ with $0 < \|h\|_{W^{2,\infty}(0,b)} < \delta$ and $v = u$ on $\partial_D R$ we have $\mathcal{MS}(\Gamma_h, v) > \mathcal{MS}(\Gamma, u)$, unless $\Gamma_h = \Gamma_\varepsilon$ for some $\varepsilon > 0$ and $v = u_\varepsilon$. Finally, this property implies also the local L^1-minimality (up to translations), by the same argument developed in Sections 2.5 and 2.6.

We are left with the proof of (2.73). Condition (2.24) is automatically satisfied on the subspace $H^1_0(0,b)$, and we can discuss the sign of $\partial^2\mathcal{MS}(\Gamma,u)$ in terms of the eigenvalue λ_1 introduced in (2.27). We will prove that

$$\lambda_1(R) = \frac{2b}{\pi}\tanh\frac{2\pi a}{b}\,. \tag{2.75}$$

We remark that λ_1 coincides with the greatest λ such that there exists a nontrivial solution $(v,\varphi) \in H^1_{\rm per}(R\setminus\Gamma)\times H^1_0(0,b)$, $v = 0$ in $\partial_D R$, to the equations

$$\lambda\int_R \nabla v\cdot\nabla z + \int_0^b (\varphi' z^+ + \varphi' z^-)\,dx = 0, \qquad \int_0^b (\varphi'\psi' + 2\psi' v^+ + 2\psi' v^-)\,dx = 0$$

for every $z \in H^1_{\rm per}(R\setminus\Gamma)$ with $z = 0$ on $\partial_D R$, and for every $\psi \in H^1_0(0,b)$. By symmetry, $v(x,y) = v(x,-y)$, so that by setting $R^+ := (0,b)\times(0,a)$, we look for a solution to

$$\begin{cases} \Delta v = 0 & \text{in } R^+, \\ v = 0 & \text{on } \partial_D R, \\ \lambda\partial_y v = \varphi' & \text{on } \Gamma, \\ \varphi'' = -4\partial_x v & \text{on } \Gamma. \end{cases}$$

The last two conditions say that

$$\lambda\partial_y v(x,0) = -4\big(v(x,0)-c\big), \qquad c := \frac{1}{b}\int_0^b v(x,0)\,\mathrm{d}x\,.$$

Hence we are left with the determination of the greatest λ such that there exists a nontrivial periodic solution v to the system

$$\begin{cases} \Delta v = 0 & \text{in } R^+, \\ v = 0 & \text{on } \partial_D R, \\ \lambda\partial_y v = -4\big(v-c\big) & \text{on } \Gamma. \end{cases}$$

We expand $v(\cdot,y)$ in series of cosines:

$$v(x,y) = \sum_{n=0}^{+\infty} c_n(y)\cos\big(\tfrac{n\pi}{b}x\big),$$

and by the first two condition of the system we have that $c_n(y) = c_n\sinh\big(\frac{n\pi}{b}(a-y)\big)$, with $c_n\in\mathbb{R}$. Hence

$$v(x,y) = \sum_{n=0}^{+\infty} c_n\cos\big(\tfrac{n\pi}{b}x\big)\sinh\big(\tfrac{n\pi}{b}(a-y)\big)$$

and by imposing the last condition of the system we have

$$\lambda\sum_{n=0}^{+\infty} c_n\tfrac{n\pi}{b}\cos\big(\tfrac{n\pi}{b}x\big)\cosh\big(\tfrac{n\pi}{b}a\big) = 4\sum_{n=0}^{+\infty} c_n\cos\big(\tfrac{n\pi}{b}x\big)\sinh\big(\tfrac{n\pi}{b}a\big) - 4c\,.$$

By expanding also c in series of cosines, we deduce from the previous inequality that $c=0$, and also

$$\lambda c_n\tfrac{n\pi}{b}\cosh\big(\tfrac{n\pi a}{b}\big) = 4c_n\sinh\big(\tfrac{n\pi a}{b}\big)$$

for all $n\geq 1$. Hence, since we are looking for a positive λ, it follows that $\lambda = \frac{4b}{n\pi}\tanh\big(\frac{n\pi a}{b}\big)$ whenever $c_n\neq 0$. Thus only one of the coefficients c_n can be different from 0, and by periodicity it must correspond to an even index (here we used also the fact that the function $t\mapsto\frac{4b}{t\pi}\tanh\big(\frac{t\pi a}{b}\big)$ is monotone decreasing). Hence there exists $\bar n\geq 2$ even such that $c_{\bar n}\neq 0$ and

$$\lambda = \tfrac{4b}{\bar n\pi}\tanh\big(\tfrac{\bar n\pi a}{b}\big)\,,$$

and clearly the largest value of λ corresponds to $\bar n = 2$. This completes the proof of (2.75) and, in turn, of (2.73).

2.8. Additional technical results

We collect here some technical results which have been used in this chapter. In the following lemma we assume to be in the same setting as described at the beginning of Section 2.4.

LEMMA 2.47. *Let $(\Gamma_n)_n$ be a sequence of curves of class $C^{1,\alpha}$, for some $\alpha\in(0,1)$, converging to Γ in $C^{1,\alpha}$, in the sense that there exist diffeomorphisms $\Phi_n:\overline{\Omega}\to\overline{\Omega}$ of class $C^{1,\alpha}$ such that $\Gamma_n=\Phi_n(\Gamma)$ and $\|\Phi_n - Id\|_{C^{1,\alpha}(\Gamma)}\to 0$.*

Then there exist $\psi_n\in C^{1,\alpha}(\Gamma)$, with $\psi_n\to 0$ in $C^{1,\alpha}(\Gamma)$, such that $\Gamma_n=\Gamma_{\psi_n}$, where Γ_{ψ_n} is the set defined according to (2.31).

Moreover, denoting by H_{Γ_n} and H the curvatures of Γ_n and of Γ respectively, if

$$\|H_{\Gamma_n}\circ\Phi_n - H\|_{L^\infty(\Gamma)}\to 0 \tag{2.76}$$

then ψ_n is of class $W^{2,\infty}$ and $\psi_n\to 0$ in $W^{2,\infty}(\Gamma)$.

PROOF. We first extend each curve Γ_n (and Γ itself) outside $\overline{\Omega}$ as a straight line so that the resulting curves are of class $C^{1,\alpha}$ and still converge to Γ in the $C^{1,\alpha}$ sense. We can then localize in a small square $R = (-\rho, \rho) \times (-\rho, \rho)$ (which we assume for simplicity centered at the origin) in which we can express Γ and Γ_n as graphs of $C^{1,\alpha}$ functions:

$$\Gamma_n \cap R = \{(x, f_n(x)) : x \in (-\rho, \rho)\}, \qquad \Gamma \cap R = \{(x, f(x)) : x \in (-\rho, \rho)\}$$

with $f_n \to f$ in $C^{1,\alpha}$. By a covering argument it is sufficient to prove the result in R (notice that, by our extension of the curves outside $\overline{\Omega}$, in this way we can cover also a neighborhood of the intersection of Γ with $\partial\Omega$).

We recall that in a sufficiently small tubular neighborhood $\mathcal{N}_{\eta_0}(\Gamma)$ of Γ are well defined two maps $\pi : \mathcal{N}_{\eta_0}(\Gamma) \to \Gamma$, $\tau : \mathcal{N}_{\eta_0}(\Gamma) \to \mathbb{R}$ of class C^2 (thank to the C^2 regularity of the vector field X generating the flow Ψ) such that $y = \Psi(\tau(y), \pi(y))$ for every y.

Taking $\rho' < \rho$, for n sufficiently large we can define a map $\tilde{\pi}_n : (-\rho', \rho') \to (-\rho, \rho)$ by setting $\tilde{\pi}_n(x) := \pi_1 \circ \pi(x, f_n(x))$, where $\pi_1(x, y) := x$. Notice that $\tilde{\pi}_n$ tends to the identity in $C^{1,\alpha}$, hence it is invertible and also its inverse converges to the identity in $C^{1,\alpha}$. Defining

$$\phi_n(x) := \tau\big(\tilde{\pi}_n^{-1}(x), f_n(\tilde{\pi}_n^{-1}(x))\big)$$

for $x \in (-\rho', \rho')$, since τ is regular and vanishes on Γ we deduce that $\phi_n \to 0$ in $C^{1,\alpha}(-\rho', \rho')$.

Hence the map $\psi_n(x, f(x)) := \phi_n(x)$, for $|x| < \rho'$, is of class $C^{1,\alpha}$ on $\Gamma \cap ((-\rho', \rho') \times (-\rho, \rho))$, converges to 0 in $C^{1,\alpha}$ and satisfies $\Gamma_{\psi_n} = \Gamma_n$. This proves the first part of the statement.

The second part follows similarly: indeed, since the sets Γ_n are locally one-dimensional graphs, the boundedness in L^∞ of the curvatures of Γ_n yields the $W^{2,\infty}$-regularity of the functions f_n, and the convergence (2.76) implies in addition that $f_n \to f$ in $W^{2,\infty}$. Hence the conclusion follows from the explicit expression of ψ_n obtained above. $\square$

We conclude with two regularity results for the Neumann problem and for the mixed Dirichlet-Neumann problem in planar domains with angles.

LEMMA 2.48. *Let A be an open subset of the unit ball B_1 such that $\partial A \cap B_1 = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two curves of class $C^{1,\beta}$ meeting at the origin with an internal angle $\alpha \in (0, \pi)$. Let $u \in H^1(A)$ be a weak solution to*

$$\begin{cases} \Delta u = 0 & \text{in } A, \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \cup \Gamma_2. \end{cases}$$

Then ∇u has a $C^{0,\gamma}$ extension up to $\Gamma_1 \cup \Gamma_2$, for $\gamma = \min\{\beta, \frac{\pi}{\alpha} - 1\}$, with $C^{0,\gamma}$-norm bounded by a constant depending only on the $C^{1,\beta}$-norm of Γ_1 and Γ_2.

PROOF. We consider A as a subset of the complex plane $\mathbb{C}$ (we can assume without loss of generality that the positive real axis coincides with the tangent to Γ_1 at the origin, and that the tangent to Γ_2 at the origin is the line $\{z = \rho e^{i\theta} : \rho > 0, \theta = \alpha\}$). Consider the map $\Phi : \overline{A} \to \Phi(\overline{A})$ given by $\Phi(z) := z^{\frac{\pi}{\alpha}} = \rho^{\frac{\pi}{\alpha}} e^{i\frac{\pi}{\alpha}\theta}$, where $z = \rho e^{i\theta}$. The map Φ is of class $C^{1,\frac{\pi}{\alpha}-1}(\overline{A})$, and since it is conformal out of the origin, the function $v := u \circ \Phi^{-1}$ is harmonic in $\Phi(A)$ and satisfies a homogenous Neumann condition on $\Phi(\Gamma_1 \cup \Gamma_2)$. Moreover $\Phi(\Gamma_1 \cup \Gamma_2)$ is a curve of class $C^{1,\gamma}$, hence by classical regularity results (see, *e.g.*, [8, Theorem 7.49]) ∇v has a $C^{0,\gamma}$ extension up to $\Phi(\Gamma_1 \cup \Gamma_2)$, with $C^{0,\gamma}$-norm bounded by a constant depending only on the $C^{1,\gamma}$-norm of $\Phi(\Gamma_1 \cup \Gamma_2)$. The conclusion follows since $u = v \circ \Phi$, using the regularity of Φ. $\square$

LEMMA 2.49. *Let A be an open subset of the unit ball B_1 such that $\partial A \cap B_1 = \Gamma_1 \cup \Gamma_2$, where Γ_1 and Γ_2 are two curves of class $C^{1,\beta}$ meeting at the origin with an internal angle equal to $\frac{\pi}{2}$. Let $u \in H^1(A)$ be a weak solution to*

$$\begin{cases} \Delta u = f & \text{in } A \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \\ u = u_0 & \text{on } \Gamma_2 \end{cases} \qquad \text{or to} \qquad \begin{cases} \Delta u = f & \text{in } A \\ \partial_\nu u = 0 & \text{on } \Gamma_1 \cup \Gamma_2 \end{cases}$$

where $f \in L^\infty(A)$, and $u_0 \in C^2(\overline{A})$ is such that $\partial_\nu u_0 = 0$ on Γ_1. Then ∇u has a $C^{0,\beta}$ extension up to $\Gamma_1 \cup \Gamma_2$, with $C^{0,\beta}$-norm bounded by a constant depending only on $\|f\|_\infty$, on the $C^{1,\beta}$-norm of Γ_1 and Γ_2, and on $\|u_0\|_{C^2(\overline{A})}$ in the first case.

PROOF. Let u solve the first problem, and let $\tilde{u} := u - u_0$. Then $\tilde{u}$ is a solution to

$$\begin{cases} \Delta \tilde{u} = \tilde{f} & \text{in } A, \\ \partial_\nu \tilde{u} = 0 & \text{on } \Gamma_1, \\ \tilde{u} = 0 & \text{on } \Gamma_2, \end{cases}$$

where $\tilde{f} := f - \Delta u_0$. We can find a radius $\rho > 0$ and a $C^{1,\beta}$ conformal mapping Φ in $\overline{A} \cap \overline{B}_\rho$ such that $\Phi(\Gamma_1)$ is a straight line meeting $\Phi(\Gamma_2)$ orthogonally. Then the function $v := \tilde{u} \circ \Phi^{-1}$ solves

$$\begin{cases} \Delta v = g & \text{in } \Phi(A), \\ \partial_\nu v = 0 & \text{on } \Phi(\Gamma_1), \\ v = 0 & \text{on } \Phi(\Gamma_2), \end{cases}$$

where $g := (\tilde{f} \circ \Phi^{-1}) |\det \nabla \Phi^{-1}|$. By even reflection across $\Phi(\Gamma_1)$ and by applying classical regularity results, we can conclude that ∇v has a $C^{0,\beta}$ extension up to $\Phi(\Gamma_1 \cup \Gamma_2)$, with $C^{0,\beta}$-norm bounded by a constant depending only on $\|g\|_\infty$ and on the $C^{1,\beta}$-norm of $\Phi(\Gamma_1 \cup \Gamma_2)$. Now the conclusion follows by using the regularity of the map Φ.

The regularity for the solution to the second problem can be obtained by a similar (and, in fact, simpler) argument. $\square$

CHAPTER 3

A variational model in epitaxial films theory

The object of investigation of this chapter is a variational model for the epitaxial growth of elastically strained films. The results obtained in [45] in a two-dimensional, linearized framework are extended here firstly to the case of anisotropic surface energies, and subsequently to the higher dimensional case, when also nonlinear elastic energies are taken into account. A particular attention is reserved to the case of crystalline surface energies.

Notation warning. Throughout this chapter, the generic vector $z \in \mathbb{R}^N$, $N \geq 2$, is often indicated as $z = (x, y)$, where $x \in \mathbb{R}^{N-1}$ is the orthogonal projection of z on the hyperplane spanned by $\{e_1, \dots, e_{N-1}\}$ and $y \in \mathbb{R}$. We also denote by $\mathbb{R}^N_+ := \{(x, y) \in \mathbb{R}^N : y > 0\}$ and $\mathbb{R}^N_- := \{(x, y) \in \mathbb{R}^N : y < 0\}$ the upper and lower half-space, respectively.

In this chapter we also deal with *fourth order tensors*, which are linear transformations of the space $\mathbb{M}^N$ into itself. We denote the action of such a tensor C on a matrix M by CM.

Organization of the chapter. The first part of the chapter (Section 3.1) is devoted to the study of the two-dimensional model, when anisotropic surface energies are taken into consideration: after having described the variational setting in details, we compute the second variation of the total energy and we prove the local minimality criterion, which will be applied to the study of the local minimality of the flat configuration.

In Section 3.2 we start the analysis of the higher dimensional case, in presence of a nonlinear elastic energy. In Section 3.3 it is shown how to find deformations which locally minimize the elastic energy in the perturbed reference configurations, thanks to the Implicit Function Theorem. The explicit computation of the second variation is carried out in Section 3.4, and the associated quadratic form is analyzed in Section 3.5. In Section 3.6 we start the proof of the main result, showing that the strict stability of a critical pair implies local minimality in the $W^{2,p}$-sense; in Section 3.7 we prove that, in any dimension, local $W^{2,p}$-minimizers are strong local minimizers, and we show how the results can be strengthen in the linear elastic case. Section 3.8 is devoted to the study of the stability of flat morphologies.

Finally, Section 3.9 deals with the case of crystalline anisotropic surface energies.

3.1. The case of anisotropic surface energies in two dimensions

3.1.1. Description of the model. We start with the extension of the two-dimensional setting considered in [16, 45] to the case of regular anisotropic surface energies. The reference configuration of the film is modeled as the subgraph of a lower semicontinuous function with finite pointwise total variation: given $b > 0$, we set

$$AP(0, b) := \{g : \mathbb{R} \to [0, +\infty) : g \text{ is lower semicontinuous and } b\text{-periodic}, \mathrm{Var}(g; 0, b) < +\infty\},$$

where

$$\mathrm{Var}(g; 0, b) := \sup \left\{ \sum_{i=1}^{k} |g(x_i) - g(x_{i-1})| : 0 < x_0 < x_1 < \ldots < x_k < b,\, k \in \mathbb{N} \right\}.$$

For an admissible profile $g \in AP(0,b)$, we introduce the sets

$$\begin{aligned}
\Omega_g &:= \{(x,y) \in \mathbb{R}^2 : x \in (0,b),\, 0 < y < g(x)\},\\
\Gamma_g &:= \{(x,y) \in \mathbb{R}^2 : x \in [0,b)\,,\, g^-(x) \le y \le g^+(x)\},\\
\Sigma_g &:= \{(x,y) \in \mathbb{R}^2 : x \in [0,b)\,,\, g(x) < g^-(x),\, g(x) \le y \le g^-(x)\},
\end{aligned}$$

which will be referred to as the *reference configuration* of the film, the *free profile* of the film, and the set of *vertical cuts*, respectively (here $g^+(x) = g(x+) \vee g(x-)$ and $g^-(x) = g(x+) \wedge g(x-)$, where $g(x+)$ and $g(x-)$ denote the right and the left limits of g at x, respectively, which exist at every point). We consider also the b-periodic extension of the reference configuration:

$$\Omega_g^\# := \{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R},\, 0 < y < g(x)\}$$

(the sets $\Gamma_g^\#$, $\Sigma_g^\#$ are defined similarly). If g is Lipschitz, we denote by ν the exterior unit normal vector to Ω_g on Γ_g, and by $\tau = \nu^\perp$ the unit tangent vector to Γ_g (obtained rotating ν clockwise by $\frac{\pi}{2}$). We recall that for a sufficiently regular g the curvature of Γ_g (with respect to the upper normal) has the expression

$$H = \mathrm{div}\nu = -\Big(\frac{g'}{\sqrt{1+(g')^2}}\Big)' \circ \pi_1 \qquad \text{on } \Gamma_g,$$

where $\pi_1 : \mathbb{R}^2 \to \mathbb{R}$ is the orthogonal projection on the x-axis.

In order to introduce the space of admissible elastic variations, we define for a given admissible profile $g \in AP(0,b)$

$$\begin{aligned}
LD_\#(\Omega_g;\mathbb{R}^2) := \big\{v \in L^2_{\mathrm{loc}}(\Omega_g^\#;\mathbb{R}^2) : v(x,y) = v(x+b,y) \text{ for } (x,y) \in \Omega_g^\#,\\
E(v)|_{\Omega_g} \in L^2(\Omega_g;\mathbb{M}^2)\big\},
\end{aligned}$$

where $E(v) := \frac{1}{2}(Dv + (Dv)^T)$ denotes the symmetrized gradient of v. We assign at the interface between the film and the substrate a boundary Dirichlet datum, which forces the film to be strained, of the form $u_0(x,0) = (e_0 x + q(x), 0)$, where $e_0 > 0$ and $q : \mathbb{R} \to \mathbb{R}$ is a b-periodic function of class C^∞ (the constant e_0 measures the lattice mismatch between film and substrate). Finally, let us introduce the following spaces of admissible pairs *film profile-deformation*:

$$\begin{aligned}
Y(u_0;0,b) &:= \{(g,v) \,:\, g \in AP(0,b),\, v : \Omega_g^\# \to \mathbb{R}^2,\, v - u_0 \in LD_\#(\Omega_g;\mathbb{R}^2)\},\\
X(u_0;0,b) &:= \{(g,v) \in Y(u_0;0,b) \,:\, v(x,0) = u_0(x,0) \text{ for all } x \in \mathbb{R}\},\\
X_L(u_0;0,b) &:= \{(g,v) \in X(u_0;0,b) \,:\, g \text{ is Lipschitz continuous}\}.
\end{aligned}$$

We consider the following notion of convergence in $Y(u_0;0,b)$: we say that a sequence (h_n,u_n) tends to (h,u) in Y iff

- $\sup_n \mathrm{Var}(h_n;0,b) < +\infty$,
- $d_H(\mathbb{R}^2_+ \setminus \Omega^\#_{h_n}, \mathbb{R}^2_+ \setminus \Omega^\#_h) \to 0$, where d_H is the Hausdorff distance defined as[1]

$$d_H(A,B) = \inf\{\varepsilon > 0 : A \subset \mathcal{N}_\varepsilon(B) \text{ and } B \subset \mathcal{N}_\varepsilon(A)\},$$

- $u_n \rightharpoonup u$ weakly in $H^1_{\mathrm{loc}}(\Omega^\#_h;\mathbb{R}^2)$

(note that this implies also that $h_n \to h$ in $L^1(0,b)$: see [42, Lemma 2.5]). We have the following compactness theorem (see [16], [42]):

[1]Here $\mathcal{N}_\varepsilon(C)$ denotes the ε-neighborhood of a set C.

THEOREM 3.1. *Assume that* $(h_n, u_n) \in X(u_0; 0, b)$ *satisfy*

$$\sup\Big\{\int_{\Omega_{h_n}} |E(u_n)|^2 \,\mathrm{d}z + \mathrm{Var}(h_n; 0, b) + |\Omega_{h_n}|\Big\} < +\infty\,.$$

Then there exists $(h, u) \in X(u_0; 0, b)$ *such that, up to subsequences,* $(h_n, u_n) \to (h, u)$ *in* Y.

We are now ready to introduce the functional on X, which is the sum of the bulk elastic energy and of the energy of the free surface of the film. In our investigation, anisotropy is incorporated only in the surface term and neglected in the volume energy. This reflects the observation that surface anisotropy is more considerable than anisotropy in the elastic field. Hence we consider an elastic energy density of the form $W(u) := \frac{1}{2}\mathbb{C}E(u) : E(u)$, where

$$\mathbb{C}\xi := \begin{pmatrix} (2\mu+\lambda)\xi_{11} + \lambda\xi_{22} & 2\mu\xi_{12} \\ 2\mu\xi_{12} & (2\mu+\lambda)\xi_{22} + \lambda\xi_{11} \end{pmatrix} \qquad \text{for } \xi \in \mathbb{M}^2_{sym}\,.$$

Here μ and λ denote the Lamé coefficients, which are assumed to satisfy the ellipticity conditions $\mu > 0$, $\lambda + \mu > 0$ (note that $W(u) \geq \min\{\mu, \lambda+\mu\}|E(u)|^2$ and thus W is coercive).

We add to the elastic energy a (regular) anisotropic surface term: we consider a convex and positively 1-homogeneous function $\psi : \mathbb{R}^2 \to [0, +\infty)$ satisfying the following conditions:

(1) ψ is of class C^3 away from the origin;
(2) for every $v \in \mathbb{S}^1$

$$D^2\psi(v)[w, w] > c_0\,|w|^2 \qquad \text{for all } w \perp v, \tag{3.1}$$

for some constant $c_0 > 0$;

(3) there exist positive constants m and M such that

$$m|z| \leq \psi(z) \leq M|z| \qquad \text{for every } z \in \mathbb{R}^2. \tag{3.2}$$

We notice for later use that by homogeneity

$$D^2\psi(v)[v] = 0 \qquad \text{for every } v \in \mathbb{R}^2\backslash\{0\}. \tag{3.3}$$

We finally introduce the functional

$$\widetilde{G}(h, u) = \int_{\Omega_h} W(u)\,\mathrm{d}z + \int_{\Gamma_h} \psi(\nu)\,\mathrm{d}\mathcal{H}^1 \qquad \text{for } (h, u) \in X_L(u_0; 0, b).$$

The functional $\widetilde{G}$, originally defined only for Lipschitz admissible profiles, can be extended to the whole space $X(u_0; 0, b)$, by relaxation: we set for $(h, u) \in X(u_0; 0, b)$

$$\begin{aligned} G(h, u) := \inf\Big\{\liminf_{n\to\infty} \widetilde{G}(h_n, u_n) : (h_n, u_n) \in X_L(u_0; 0, b),& \\ |\Omega_{h_n}| = |\Omega_h|,\ (h_n, u_n) \to (h, u) \text{ in } Y&\Big\}. \end{aligned}$$

The following theorem provides an explicit representation of the relaxed functional.

THEOREM 3.2. *Let* $\sigma = \psi(1, 0) + \psi(-1, 0)$. *The following representation formula for* G *holds:*

$$G(h, u) = \int_{\Omega_h} W(u)\,\mathrm{d}z + \int_{\Gamma_h} \psi(\nu_h)\,\mathrm{d}\mathcal{H}^1 + \sigma\,\mathcal{H}^1(\Sigma_h) \tag{3.4}$$

where ν_h *is the generalized outer normal to* $\Omega_h^\# \cup \mathbb{R}^2_-$ *at the points of its reduced boundary (which coincides, in the strip* $[0, b) \times \mathbb{R}$, *with* Γ_h *up to an* $\mathcal{H}^1$*-negligible set).*

The proof can be obtained arguing as in [42, Theorem 2.8] and [16, Lemma 2.1], using Reshetnyak's lower semicontinuity and continuity theorems (see [8, Theorem 2.38 and Theorem 2.39]) to treat the presence of anisotropy in the surface term (we refer also to the recent works [18, 23] for related relaxation results in higher dimension).

We now define the notions of local minimizer, critical pair and flat configuration.

DEFINITION 3.3. We say that $(h,u) \in X(u_0;0,b)$ is a *b-periodic local minimizer* for the functional G if there exists $\delta > 0$ such that

$$G(h,u) \le G(g,v)$$

for all $(g,v) \in X(u_0;0,b)$ with $|\Omega_g| = |\Omega_h|$ and $\|g-h\|_\infty < \delta$; if the inequality is strict when $g \neq h$, we say that (h,u) is an *isolated b-periodic local minimizer.*

DEFINITION 3.4. We say that an element $(h,u) \in X(u_0;0,b)$ with $h \in C^2(\mathbb{R})$ is a *critical pair* for the functional G if u minimizes the elastic energy in Ω_h, that is, u satisfies the equation

$$\int_{\Omega_h} \mathbb{C}E(u) : E(w)\,\mathrm{d}z = 0 \qquad \text{for every } w \in A(\Omega_h), \tag{3.5}$$

where

$$A(\Omega_h) := \{w \in LD_\#(\Omega_h;\mathbb{R}^2) : w(\cdot,0) \equiv 0\},$$

and the following transmission condition holds:

$$W(u) + H^\psi = \mathrm{const} \qquad \text{on } \Gamma_h \cap \{y > 0\}, \tag{3.6}$$

where H^ψ is the anisotropic mean curvature of Γ_h (see (1.6)).

REMARK 3.5. The definition of critical pair is motivated by the Euler-Lagrange equation satisfied by a sufficiently regular (local) minimizer of G (see the formula for the first variation of G deduced in Step 1 of the proof of Theorem 3.8). Notice that if $h > 0$ and Γ_h is of class $C^{1,\alpha}$ for all $\alpha \in (0,1/2)$, then equation (3.5) (which is a linear elliptic system satisfying the Legendre-Hadamard condition) implies that $u \in C^{1,\alpha}(\overline{\Omega}_h)$ for all $\alpha \in (0,1/2)$ (see [45, Proposition 8.9]). Moreover, if both ψ and u_0 are of class C^∞ (analytic, respectively), and equation (3.6) holds in the distributional sense, then (h,u) is of class C^∞ (analytic, respectively) by the results contained in [59, Subsection 4.2]. Observe that condition (3.1) is exactly the assumption needed in the regularity result of [59].

REMARK 3.6. We will make repeated use of the following explicit formula for H^ψ:

$$H^\psi(x,h(x)) = \big(\partial_1\psi(-h'(x),1)\big)'. \tag{3.7}$$

Indeed, from condition (3.3) it follows that $D^2\psi(-h',1)[(-h',1)] = 0$, which in turn implies $\partial^2_{12}\psi(-h',1) = \partial^2_{11}\psi(-h',1)h'$; hence

$$\begin{aligned} H^\psi &= \partial_\tau(\nabla\psi \circ \nu)\cdot\tau = -\frac{h''}{1+h'^2}\Big[\partial^2_{11}\psi(-h',1) + h'\partial^2_{12}\psi(-h',1)\Big] \circ \pi_1 \\ &= -\Big(h''\,\partial^2_{11}\psi(-h',1)\Big) \circ \pi_1, \end{aligned}$$

which is (3.7).

DEFINITION 3.7. The *flat configuration* corresponding to a given volume $d > 0$ and a boundary Dirichlet datum $u_0(x,0) = (e_0x,0)$, $e_0 > 0$, is the pair $(\frac{d}{b}, v_{e_0})$ with

$$v_{e_0}(x,y) := e_0\left(x, \frac{-\lambda}{2\mu+\lambda}\,y\right). \tag{3.8}$$

Notice that the flat configuration is a critical pair for G.

In Sections 3.1.2 and 3.1.3 we will prove a local minimality criterion for the functional G expressed in terms of the positivity of its second variation. The result will be established by implementing, in our anisotropic framework, the general strategy described in [45] to deal with the isotropic case. From this we will be able to deduce, in Section 3.1.4, a stability property for the flat configuration, showing that the qualitative results obtained in [45] hold also in the case of regular anisotropies. The case of crystalline anisotropies will be treated in Section 3.9, together with the three-dimensional and nonlinear case.

3.1.2. Second variation and local $W^{2,\infty}$-minimality. In this section, following [45], we introduce a suitable notion of second variation of the functional G along volume preserving deformations. Fix $(h,u) \in X(u_0;0,b)$ with $h \in C^\infty(\mathbb{R})$, $h>0$, and such that the displacement u minimizes the elastic energy in Ω_h. Given $\phi : \mathbb{R} \to \mathbb{R}$ of class C^∞, b-periodic and such that $\int_0^b \phi(x)\,dx = 0$, define $h_t := h + t\phi$ for $t \in \mathbb{R}$ and let u_{h_t} be the elastic equilibrium in Ω_{h_t}. We define the *second variation of G at (h,u) along the direction ϕ* to be the value of

$$\frac{d^2}{dt^2}\left[G(h_t,u_{h_t})\right]|_{t=0}.$$

In the following theorem we compute explicitly the second variation defined as above. Denote by ν_t the outer unit normal vector to Ω_{h_t} on Γ_{h_t}, and by $H_t^\psi := \operatorname{div}(\nabla\psi \circ \nu_t)$ the anisotropic curvature of Γ_{h_t}.

THEOREM 3.8. *Let (h,u), ϕ, and (h_t,u_{h_t}) be as above, and let $\varphi := \frac{\phi}{\sqrt{1+h'^2}} \circ \pi_1$. Then the function $\dot u$ belongs to $A(\Omega_h)$ and satisfies the equation*

$$\int_{\Omega_h} \mathbb{C}E(\dot u) : E(w)\,\mathrm{d}z = \int_{\Gamma_h} \operatorname{div}_{\Gamma_h}(\varphi\,\mathbb{C}E(u))\cdot w\,\mathrm{d}\mathcal{H}^1 \qquad \text{for all } w \in A(\Omega_h). \tag{3.9}$$

Moreover, the second variation of G at (h,u) along the direction ϕ is given by

$$\begin{aligned}\frac{d^2}{dt^2}G(h_t,u_{h_t})|_{t=0} = &-\int_{\Omega_h} \mathbb{C}E(\dot u) : E(\dot u)\,\mathrm{d}z + \int_{\Gamma_h} (D^2\psi\circ\nu)[\nabla_{\Gamma_h}\varphi,\nabla_{\Gamma_h}\varphi]\,\mathrm{d}\mathcal{H}^1 \\ &+ \int_{\Gamma_h} \left(\partial_\nu[W(u)] - HH^\psi\right)\varphi^2\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_h}\left(W(u)+H^\psi\right)\partial_\tau\left((h'\circ\pi_1)\,\varphi^2\right)\mathrm{d}\mathcal{H}^1. \end{aligned}\tag{3.10}$$

PROOF. The computation is carried out in [45, Theorem 3.2] in the case of an isotropic surface energy. The equation solved by $\dot u$ is deduced exactly in the same way, and also the same computation for the elastic energy yields

$$\begin{aligned}\frac{d^2}{dt^2}\left[\int_{\Omega_{h_t}} W(u_{h_t})\,\mathrm{d}z\right]\Bigg|_{t=0} = &-\int_{\Omega_h}\mathbb{C}E(\dot u):E(\dot u)\,\mathrm{d}z \\ &+\int_{\Gamma_h}\partial_\nu[W(u)]\,\varphi^2\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_h} W(u)\,\partial_\tau\left((h'\circ\pi_1)\,\varphi^2\right)\mathrm{d}\mathcal{H}^1.\end{aligned}\tag{3.11}$$

We are only left with the computation of the first and second derivatives of the surface energy.

Step 1. We compute the first variation of the surface term. Using the positive 1-homogeneity of ψ, we have

$$\begin{aligned}\frac{d}{dt}\int_{\Gamma_{h_t}}\psi(\nu_t)\,\mathrm{d}\mathcal{H}^1 &= \frac{d}{dt}\int_0^b \psi(-h_t'(x),1)\,\mathrm{d}x = -\int_0^b \partial_1\psi(-h_t'(x),1)\phi'(x)\,\mathrm{d}x \\ &= \int_0^b \phi(x)H_t^\psi(x,h_t(x))\,\mathrm{d}x,\end{aligned}$$

where the last equality follows by integration by parts and by (3.7). Hence the first variation of the complete functional G is

$$\frac{d}{dt}G(h_t, u_{h_t}) = \int_0^b \phi(x)\big[W(u_{h_t}) + H_t^\psi\big]|_{(x,h_t(x))}\,\mathrm{d}x\,.$$

Step 2. Before starting the computation of the second variation, we deduce some useful identities that will be used in the following. Observe first that, thanks to the fact that $D\nu[\nu] = 0$, we have

$$D\nu = D_{\Gamma_h}\nu = H\tau \otimes \tau \qquad \text{on } \Gamma_h. \tag{3.12}$$

Moreover, for the same reason we have also $D\left(\nabla\psi \circ \nu\right)[\nu] = 0$; differentiating we get

$$\partial_\nu\left(D\left(\nabla\psi\circ\nu\right)\right) = -D\left(\nabla\psi\circ\nu\right)D\nu\,,$$

thus

$$\begin{aligned}\partial_\nu H^\psi &= \partial_\nu\left[\operatorname{div}(\nabla\psi\circ\nu)\right] = \partial_\nu\left[\operatorname{trace}\left(D\left(\nabla\psi\circ\nu\right)\right)\right]\\ &= \operatorname{trace}\left[\partial_\nu\left(D\left(\nabla\psi\circ\nu\right)\right)\right] = -\operatorname{trace}\left[D\left(\nabla\psi\circ\nu\right)D\nu\right]\\ &= -HH^\psi,\end{aligned}$$

where the last equality follows using (3.12).

Differentiating with respect to t the identity

$$\nu_t(x, y + t\phi(x)) = \frac{(-h_t'(x), 1)}{\sqrt{1 + (h_t'(x))^2}} \qquad \text{for } (x,y) \in \Gamma_h,$$

and evaluating the result at $t = 0$, we get

$$\dot\nu + (\phi\circ\pi_1)\,\partial_2\nu = -\left(\frac{\phi'}{1+(h')^2}\circ\pi_1\right)\tau \qquad \text{on } \Gamma_h.$$

Now from this equality and from (3.12) we obtain

$$\dot\nu = -\left((\phi\circ\pi_1)\,H\tau_2 + \frac{\phi'}{1+(h')^2}\circ\pi_1\right)\tau = -\nabla_{\Gamma_h}\varphi. \tag{3.13}$$

As a consequence of (3.3) we have $(D^2\psi\circ\nu)[\nu,\dot\nu] = 0$, and differentiating this identity in the direction ν we get

$$\nu\cdot\partial_\nu\big((D^2\psi\circ\nu)[\dot\nu]\big) = -(D^2\psi\circ\nu)[\dot\nu, \partial_\nu\nu] = 0$$

(recall that $\partial_\nu\nu = 0$). Hence

$$\begin{aligned}\dot H^\psi &= \frac{\partial}{\partial t}H_t^\psi|_{t=0} = \frac{\partial}{\partial t}\big[\operatorname{div}(\nabla\psi\circ\nu_t)\big]|_{t=0} = \operatorname{div}\big((D^2\psi\circ\nu)[\dot\nu]\big)\\ &= \operatorname{div}_{\Gamma_h}\big((D^2\psi\circ\nu)[\dot\nu]\big) + \nu\cdot\partial_\nu\big((D^2\psi\circ\nu)[\dot\nu]\big)\\ &= \operatorname{div}_{\Gamma_h}\big((D^2\psi\circ\nu)[\dot\nu]\big) = -\operatorname{div}_{\Gamma_h}\big((D^2\psi\circ\nu)[\nabla_{\Gamma_h}\varphi]\big)\,,\end{aligned} \tag{3.14}$$

where in the last equality we used (3.13).

Step 3. We finally pass to the second variation. Differentiating the formula for the first variation of the surface term with respect to t and evaluating at $t = 0$ we get

$$\begin{aligned}\frac{d^2}{dt^2}\left[\int_{\Gamma_{h_t}}\psi(\nu_t)\,\mathrm{d}\mathcal{H}^1\right]\Bigg|_{t=0} &= \frac{d}{dt}\left[\int_0^b \phi(x)H_t^\psi(x, h_t(x))\,\mathrm{d}x\right]\Bigg|_{t=0}\\ &= \int_0^b \phi(x)\dot H^\psi(x, h(x))\,\mathrm{d}x + \int_0^b \phi(x)\,\nabla H^\psi(x,h(x))\cdot(0,\phi(x))\,\mathrm{d}x\\ &= I_1 + I_2.\end{aligned}$$

Changing variables in I_1 and using the equality (3.14) we obtain

$$I_1 = -\int_{\Gamma_h} \varphi \operatorname{div}_{\Gamma_h} \big((D^2\psi \circ \nu)[\nabla_{\Gamma_h}\varphi]\big)\, d\mathcal{H}^1 = \int_{\Gamma_h} (D^2\psi \circ \nu)[\nabla_{\Gamma_h}\varphi, \nabla_{\Gamma_h}\varphi]\, d\mathcal{H}^1, \tag{3.15}$$

where the last equality follows by integration by parts, using the periodicity of φ.

For the second integral, we can decompose $\nabla H^\psi = (\partial_\nu H^\psi)\nu + (\partial_\tau H^\psi)\tau$, so that after a change of variables

$$\begin{aligned} I_2 &= \int_{\Gamma_h} (\partial_\nu H^\psi)\varphi^2\, d\mathcal{H}^1 + \int_{\Gamma_h} (\partial_\tau H^\psi)(h' \circ \pi_1)\varphi^2\, d\mathcal{H}^1 \\ &= -\int_{\Gamma_h} H H^\psi \varphi^2\, d\mathcal{H}^1 - \int_{\Gamma_h} H^\psi \partial_\tau\big((h'\circ\pi_1)\varphi^2\big)\, d\mathcal{H}^1, \end{aligned} \tag{3.16}$$

where we used the identity $\partial_\nu H^\psi = -HH^\psi$ satisfied on Γ_h and we integrated by parts in the last integral (using again the periodicity of the functions involved).

Collecting (3.11), (3.15) and (3.16), the formula in the statement follows. □

Let us introduce the following subspace of $H^1(\Gamma_h)$:

$$\widetilde{H}^1_\#(\Gamma_h) := \Big\{\varphi \in H^1(\Gamma_h) : \varphi(0,h(0)) = \varphi(b,h(b)), \int_{\Gamma_h} \varphi\, d\mathcal{H}^1 = 0\Big\}$$

(note that the function φ defined in the statement of Theorem 3.8 belongs to this space). Having the formula for the second variation in hand, and observing that the last integral in (3.10) vanishes if (h,u) is a critical pair thanks to (3.6) and to the periodicity of the functions involved, we can define the quadratic form $\partial^2 G(h,u) : \widetilde{H}^1_\#(\Gamma_h) \to \mathbb{R}$ associated with the second variation at a critical pair (h,u) as

$$\begin{aligned} \partial^2 G(h,u)[\varphi] := -\int_{\Omega_h} \mathbb{C}E(v_\varphi) : E(v_\varphi)\, dz + \int_{\Gamma_h} (D^2\psi\circ\nu)[\nabla_{\Gamma_h}\varphi, \nabla_{\Gamma_h}\varphi]\, d\mathcal{H}^1 \\ + \int_{\Gamma_h} \big(\partial_\nu[W(u)] - HH^\psi\big)\,\varphi^2\, d\mathcal{H}^1 \end{aligned}$$

for $\varphi \in \widetilde{H}^1_\#(\Gamma_h)$, where v_φ is the unique solution in $A(\Omega_h)$ to

$$\int_{\Omega_h} \mathbb{C}E(v_\varphi) : E(w)\, dz = \int_{\Gamma_h} \operatorname{div}_{\Gamma_h}(\varphi\, \mathbb{C}E(u)) \cdot w\, d\mathcal{H}^1 \qquad \text{for every } w \in A(\Omega_h). \tag{3.17}$$

It is easily seen that the positive semi-definiteness of the quadratic form $\partial^2 G(h,u)$ is a necessary condition for local minimality (see [45, Corollary 3.4]). On the other hand, we have the following minimality criterion (see [45, Theorem 4.6]).

THEOREM 3.9. *Let $(h,u) \in X(u_0;0,b)$, with $h \in C^\infty(\mathbb{R})$, $h > 0$, be a critical pair for G such that*

$$\partial^2 G(h,u)[\varphi] > 0 \qquad \textit{for every } \varphi \in \widetilde{H}^1_\#(\Gamma_h)\setminus\{0\}. \tag{3.18}$$

Then there exists $\delta > 0$ such that for any $(g,v) \in X(u_0;0,b)$, with $\|g-h\|_{W^{2,\infty}(0,b)} < \delta$, $|\Omega_g| = |\Omega_h|$ and $g \neq h$ we have

$$G(h,u) < G(g,v)$$

(we say that the critical pair (h,u) is an isolated local $W^{2,\infty}$-minimizer *for G).*

We remark that, if ψ is of class C^∞, the regularity assumption on h is not restrictive (see Remark 3.5).

The strategy developed in [45] to prove the theorem (which, in turn, borrows some ideas from [19]) can be repeated here with some changes. We only recall what are the main steps, suggesting the modifications that are necessary to adapt the proof to our setting.

First of all, one can show that the positiveness condition (3.18) can be equivalently formulated in terms of the first eigenvalue of a suitable compact linear operator defined on $\widetilde{H}^1_{\#}(\Gamma_h)$. This is done by introducing the bilinear form on $\widetilde{H}^1_{\#}(\Gamma_h)$

$$(\varphi, \theta)_\sim := \int_{\Gamma_h} \left(\partial_\nu[W(u)] - HH^\psi\right) \varphi\, \theta \, \mathrm{d}\mathcal{H}^1 + \int_{\Gamma_h} (D^2\psi \circ \nu)[\nabla_{\Gamma_h}\varphi, \nabla_{\Gamma_h}\theta] \, \mathrm{d}\mathcal{H}^1 \tag{3.19}$$

which, if positive definite, defines an equivalent norm $\|\cdot\|_\sim$ on $\widetilde{H}^1_{\#}(\Gamma_h)$ (this can be shown using condition (3.1) and following the lines of the proof of Proposition 2.20). Then, one has the following equivalent formulation of condition (3.18) (see [45, Proposition 3.6]):

Proposition 3.10. *Condition (3.18) is satisfied if and only if the bilinear form $(\cdot,\cdot)_\sim$ is positive definite and the compact, monotone, self-adjoint operator $T : \widetilde{H}^1_{\#}(\Gamma_h) \to \widetilde{H}^1_{\#}(\Gamma_h)$, defined by duality as*

$$(T\varphi, \theta)_\sim := \int_{\Omega_h} \mathbb{C}E(v_\varphi) : E(v_\theta) \, \mathrm{d}z = \int_{\Omega_h} \mathbb{C}E(v_\theta) : E(v_\varphi) \, \mathrm{d}z$$

for every $\varphi, \theta \in \widetilde{H}^1_{\#}(\Gamma_h)$, satisfies $\lambda_1 := \max\{(T\varphi, \varphi)_\sim : \|\varphi\|_\sim = 1\} < 1$.

The proof of this proposition relies, essentially, on the following representation formula of $\partial^2 G(h,u)$ in terms of T:

$$\partial^2 G(h,u)[\varphi] = (\varphi,\varphi)_\sim - (T\varphi,\varphi)_\sim. \tag{3.20}$$

Moreover, using (3.20) it is easily seen that condition (3.18) implies the existence of a constant $C > 0$ such that

$$\partial^2 G(h,u)[\varphi] \geq C\|\varphi\|^2_{H^1(\Gamma_h)} \qquad \text{for all } \varphi \in \widetilde{H}^1_{\#}(\Gamma_h). \tag{3.21}$$

Having this equivalent formulation in hand, the proof of Theorem 3.9 is obtained arguing similarly to [45, Proposition 4.5], with some natural modifications. Notice that the elliptic estimates provided by the technical lemmas [45, Lemma 4.1, Lemma 4.4] are valid also in our setting, because they are concerned only with the volume term which we left unchanged.

Proof of Theorem 3.9 (sketch). The main steps in the proof are the following.
Step 1. For g in a C^2-neighborhood of h, let v_g be the elastic equilibrium in Ω_g and consider a diffeomorphism $\Phi_g : \overline{\Omega}_h \to \overline{\Omega}_g$ of class C^2 such that $\Phi_g - Id$ is b-periodic in x, $\Phi_g(x,0) = (x,0)$, $\Phi_g(x,y) = (x, y + g_n(x) - h(x))$ in a neighborhood of $\overline{\Gamma}_h$, and $\|\Phi_g - Id\|_{C^2(\overline{\Omega}_h;\mathbb{R}^2)} \leq 2\|g-h\|_{C^2([0,b])}$. The same elliptic estimates proved in [45, Lemma 4.1] yield the following convergence (compare with [45, (4.21)]):

$$\|\partial_{\nu_g}[W(v_g)] \circ \Phi_g J_{\Phi_g} - \partial_{\nu_h}[W(u)]\|_{H^{-\frac{1}{2}}_{\#}(\Gamma_h)} \to 0 \quad \text{as } \|g-h\|_{C^2([0,b])} \to 0, \tag{3.22}$$

where J_{Φ_g} denotes the 1-dimensional Jacobian of Φ_g on Γ_h.
Step 2. Let us introduce, for g in a C^2-neighborhood of h, a scalar product $(\cdot,\cdot)_{\sim,g}$ on $\widetilde{H}^1_{\#}(\Gamma_g)$ defined as in (3.19) with h replaced by g. We claim that the positivity condition (3.18) guarantees that it is possible to control the H^1-norm on Γ_g in terms of the norm associated with $(\cdot,\cdot)_{\sim,g}$, uniformly with respect to g in a C^2-neighborhood of h:

$$\|\varphi\|^2_{H^1(\Gamma_g)} \leq C\|\varphi\|^2_{\sim,g} \qquad \text{for every } \varphi \in \widetilde{H}^1_{\#}(\Gamma_g)$$

(here and in the following steps C denotes a generic positive constant, independent of g in a C^2-neighborhood of h, which may change from line to line). Indeed, given $\varphi \in \widetilde{H}^1_\#(\Gamma_g)$, set $\tilde{\varphi} := (\varphi \circ \Phi_g) J_{\Phi_g}$; then $\tilde{\varphi} \in \widetilde{H}^1_\#(\Gamma_h)$ and

$$
\begin{aligned}
\|\varphi\|^2_{H^1(\Gamma_g)} &= \int_{\Gamma_h} \left(|\varphi \circ \Phi_g|^2 + |(\partial_{\tau_g}\varphi) \circ \Phi_g|^2 \right) J_{\Phi_g} \, d\mathcal{H}^1 \\
&\le (1+\delta_g) \int_{\Gamma_h} \left(\tilde{\varphi}^2 + (\partial_{\tau_h}\tilde{\varphi})^2 \right) d\mathcal{H}^1 \\
&\le (1+\delta_g)\, C\, \|\tilde{\varphi}\|^2_{\sim},
\end{aligned}
$$

where in the last inequality we used (3.20) and (3.21) to deduce that

$$
\|\tilde{\varphi}\|^2_{\sim} \ge \partial^2 G(h,u)[\tilde{\varphi}] \ge C \|\tilde{\varphi}\|^2_{H^1(\Gamma_h)},
$$

and δ_g is a constant depending only on $\|g-h\|_{C^2([0,b])}$, tending to 0 as $\|g-h\|_{C^2([0,b])} \to 0$.

Now, setting $a_h := \partial_{\nu_h}[W(u)] - HH^\psi$, $a_g := \partial_{\nu_g}[W(v_g)] - HH^\psi_g$ (we denote by H^ψ_g the anisotropic mean curvature of g), we obtain from Step 1 that

$$
\|(a_g \circ \Phi_g) J_{\Phi_g} - a_h (J_{\Phi_g})^2\|_{H^{-\frac12}_\#(\Gamma_h)} \to 0 \qquad \text{as } \|g-h\|_{C^2([0,b])} \to 0.
$$

Hence

$$
\begin{aligned}
\|\tilde{\varphi}\|^2_{\sim} &= \int_{\Gamma_h} \left(a_h \tilde{\varphi}^2 + (D^2\psi \circ \nu_h)[\nabla_{\Gamma_h}\tilde{\varphi}, \nabla_{\Gamma_h}\tilde{\varphi}] \right) d\mathcal{H}^1 \\
&\le \int_{\Gamma_h} (a_g \circ \Phi_g)(\varphi \circ \Phi_g)^2 J_{\Phi_g} \, d\mathcal{H}^1 + \int_{\Gamma_g} (D^2\psi \circ \nu_g)[\nabla_{\Gamma_g}\varphi, \nabla_{\Gamma_g}\varphi] \, d\mathcal{H}^1 + \delta_g \|\varphi\|^2_{H^1(\Gamma_g)} \\
&\qquad + \|(a_g \circ \Phi_g) J_{\Phi_g} - a_h (J_{\Phi_g})^2\|_{H^{-\frac12}_\#(\Gamma_h)} \|(\varphi \circ \Phi_g)^2\|_{H^{\frac12}(\Gamma_h)} \\
&\le \|\varphi\|^2_{\sim,g} + \delta_g \|\varphi\|^2_{H^1(\Gamma_g)} + C(1+\delta_g) \|(a_g \circ \Phi_g) J_{\Phi_g} - a_h (J_{\Phi_g})^2\|_{H^{-\frac12}_\#(\Gamma_h)} \|\varphi\|^2_{H^1(\Gamma_g)},
\end{aligned}
$$

where, as before, δ_g tends to 0 as $\|g-h\|_{C^2} \to 0$, and in the last inequality we used the estimate

$$
\|(\varphi \circ \Phi_g)^2\|_{H^{\frac12}(\Gamma_h)} \le C \|(\varphi \circ \Phi_g)^2\|_{H^1(\Gamma_h)} \le C \|(\varphi \circ \Phi_g)\|^2_{H^1(\Gamma_h)} \le C(1+\delta_g)\|\varphi\|^2_{H^1(\Gamma_g)}.
$$

Combining the previous estimates the claim follows.

Step 3. The previous step allows us to introduce a compact linear operator T_g also on $\widetilde{H}^1_\#(\Gamma_g)$, as we did for T on $\widetilde{H}^1_\#(\Gamma_h)$; denoting by $\lambda_{1,g}$ its first eigenvalue, one can prove, arguing exactly as in Step 3 of the proof of [45, Proposition 4.5], that

$$
\limsup_{\|g-h\|_{C^2} \to 0} \lambda_{1,g} \le \lambda_1 < 1,
$$

where the last inequality follows by Proposition 3.10.

Step 4. We claim that the following estimate holds for g close to h in C^2:

$$
G(h,u) + C\|\varphi_g\|^2_{H^1(\Gamma_g)} \le G(g, v_g),
$$

where $\varphi_g := \frac{g-h}{\sqrt{1+g'^2}} \circ \pi_1$. In order to prove this estimate, we define $h_t := h + t(g-h)$ and u_t as the corresponding elastic equilibrium, and setting $f(t) := G(h_t, u_t)$, we can show that a careful estimate of the second variation combined with the previous steps yields

$$
f''(t) > C(1-\lambda_1)\|\varphi_g\|^2_{H^1(\Gamma_g)} \tag{3.23}
$$

for g sufficiently close to h in C^2. From this the claim will follow immediately, since (using $f'(0) = 0$, being (h, u) a critical pair)

$$G(h, u) = f(0) = f(1) - \int_0^1 (1 - t)f''(t)\,\mathrm{d}t < G(g, v_g) - \frac{C(1 - \lambda_1)}{2}\|\varphi_g\|^2_{H^1(\Gamma_g)}.$$

In order to prove (3.23), we have by Theorem 3.8

$$f''(t) = -(T_{h_t}\varphi_{g,t}, \varphi_{g,t})_{\sim,h_t} + \|\varphi_{g,t}\|^2_{\sim,h_t} - \int_{\Gamma_{h_t}} (W(u_t) + H_t^\psi)\partial_{\tau_{h_t}}\big((h_t' \circ \pi_1)\varphi^2_{g,t}\big)\,\mathrm{d}\mathcal{H}^1 \tag{3.24}$$

where we set $\varphi_{g,t} := \frac{g-h}{\sqrt{1+(h_t')^2}} \circ \pi_1$ and H_t^ψ denotes the anisotropic mean curvature of Γ_{h_t}. Using Step 2, Step 3 and the fact that

$$\frac{1}{2}\|\varphi_g\|^2_{H^1(\Gamma_g)} \le \|\varphi_{g,t}\|^2_{H^1(\Gamma_{h_t})} \le 2\|\varphi_g\|^2_{H^1(\Gamma_g)},$$

we deduce that

$$\begin{aligned} &-(T_{h_t}\varphi_{g,t}, \varphi_{g,t})_{\sim,h_t} + \|\varphi_{g,t}\|^2_{\sim,h_t} \ge (1 - \lambda_{1,h_t})\|\varphi_{g,t}\|^2_{\sim,h_t} \\ &\quad \ge \frac{1-\lambda_1}{2}\|\varphi_{g,t}\|^2_{\sim,h_t} \ge \frac{C(1-\lambda_1)}{2}\|\varphi_{g,t}\|^2_{H^1(\Gamma_{h_t})} \ge \frac{C(1-\lambda_1)}{4}\|\varphi_g\|^2_{H^1(\Gamma_g)} \end{aligned} \tag{3.25}$$

if $\|g - h\|_{C^2([0,b])}$ is sufficiently small. Moreover, since (h, u) is a critical pair, there exists a constant Λ such that $W(u) + H^\psi \equiv \Lambda$ on Γ_h, and it can be also shown that

$$\sup_{t\in(0,1]} \|W(u_t) + H_t^\psi - \Lambda\|_{L^\infty(\Gamma_{h_t})} \to 0 \quad \text{as } g \to h \text{ in } C^2. \tag{3.26}$$

We then have

$$\begin{aligned} -&\int_{\Gamma_{h_t}} (W(u_t) + H_t^\psi)\partial_{\tau_{h_t}}\big((h_t' \circ \pi_1)\varphi^2_{g,t}\big)\,\mathrm{d}\mathcal{H}^1 \\ &\qquad = -\int_{\Gamma_{h_t}} (W(u_t) + H_t^\psi - \Lambda)\partial_{\tau_{h_t}}\big((h_t' \circ \pi_1)\varphi^2_{g,t}\big)\,\mathrm{d}\mathcal{H}^1 \\ &\qquad \ge -C\|W(u_t) + H_t^\psi - \Lambda\|_{L^\infty(\Gamma_{h_t})}\|\varphi_{g,t}\|^2_{H^1(\Gamma_{h_t})} \\ &\qquad \ge -2C\|W(u_t) + H_t^\psi - \Lambda\|_{L^\infty(\Gamma_{h_t})}\|\varphi_g\|^2_{H^1(\Gamma_g)}. \end{aligned} \tag{3.27}$$

Hence (3.23) follows combining (3.24), (3.25) and (3.27), taking into account (3.26).

Step 5. Finally, using the estimate proved in Step 4, one obtains the local $W^{2,\infty}$-minimality by an approximation argument, as in [45, Theorem 4.6]. □

3.1.3. Improvement of the local minimality result. The improvement of the minimality Theorem 3.9 requires a careful review of the arguments developed in [45, Section 6], which lead to the following result.

THEOREM 3.11. *Let $(h, u) \in X(u_0; 0, b)$, with $h \in C^\infty(\mathbb{R})$, $h > 0$, be a critical pair for G such that condition (3.18) is satisfied. Then (h, u) is an isolated b-periodic local minimizer for G, in the sense of Definition 3.3.*

As in [45], the main idea of the proof is to consider a solution (g_n, v_n) to the penalized minimum problem

$$\min\Big\{G(k, w) + \Lambda\big||\Omega_k| - |\Omega_h|\big| : (k, w) \in X(u_0; 0, b),\ k \ge h - \frac{1}{n}\Big\}.$$

Assuming by contradiction that we can find a sequence of pairs $(\tilde{g}_n, \tilde{v}_n) \in X(u_0; 0, b)$ such that $|\Omega_{\tilde{g}_n}| = |\Omega_h|$, $G(\tilde{g}_n, \tilde{v}_n) < G(h, u)$ and $\|\tilde{g}_n - h\| \leq \frac{1}{n}$, we then have, since $(\tilde{g}_n, \tilde{v}_n)$ is an admissible competitor for the penalized problem,

$$G(g_n, v_n) \leq G(g_n, v_n) + \Lambda \big| |\Omega_{g_n}| - |\Omega_h| \big| \leq G(\tilde{g}_n, \tilde{v}_n) < G(h, u).$$

The conclusion will follow by showing, via regularity estimates, that the functions g_n converge to h in $W^{2,\infty}$, a contradiction with the local $W^{2,\infty}$-minimality of (h, u) given by Theorem 3.9.

We start to carry out the previous strategy with an approximation lemma which can be easily deduced from the second part of the proof of [16, Lemma 2.1] by means of Reshetnyak's Continuity Theorem.

Lemma 3.12. *Given any $h \in AP(0, b)$ with $h = h^-$, there exists a sequence of b-periodic and Lipschitz functions $h_n \uparrow h$ pointwise such that*

$$\lim_{n \to +\infty} \int_{\Gamma_{h_n}} \psi(\nu_{h_n}) \, d\mathcal{H}^1 = \int_{\Gamma_h} \psi(\nu_h) \, d\mathcal{H}^1.$$

Another preliminary result that we will need in the following is an easy consequence of condition (3.1).

Lemma 3.13. *For any $\xi \in \mathbb{R}$ we have*

$$\partial^2_{11} \psi(\xi, 1) \geq \frac{c_0}{(1 + \xi^2)^{\frac{3}{2}}} \,,$$

where c_0 is the constant appearing in (3.1).

Proof. We split the vector $(1, 0)$ into its components parallel and orthogonal to the direction $(\xi, 1)$:

$$(1, 0) = \tfrac{\xi}{\sqrt{1+\xi^2}} \Big(\tfrac{\xi}{\sqrt{1+\xi^2}}, \tfrac{1}{\sqrt{1+\xi^2}} \Big) + \Big(1 - \tfrac{\xi^2}{1+\xi^2}, -\tfrac{\xi}{1+\xi^2} \Big).$$

From this decomposition, using (3.3), we get

$$\begin{aligned}
\partial^2_{11} \psi(\xi, 1) &= D^2 \psi(\xi, 1) \big[(1, 0), (1, 0) \big] \\
&= \tfrac{1}{\sqrt{1+\xi^2}} \, D^2 \psi \Big(\tfrac{\xi}{\sqrt{1+\xi^2}}, \tfrac{1}{\sqrt{1+\xi^2}} \Big) \Big[\Big(1 - \tfrac{\xi^2}{1+\xi^2}, -\tfrac{\xi}{1+\xi^2} \Big), \Big(1 - \tfrac{\xi^2}{1+\xi^2}, -\tfrac{\xi}{1+\xi^2} \Big) \Big] \\
&\geq \tfrac{c_0}{\sqrt{1+\xi^2}} \, \Big| \Big(1 - \tfrac{\xi^2}{1+\xi^2}, -\tfrac{\xi}{1+\xi^2} \Big) \Big|^2 = \tfrac{c_0}{(1+\xi^2)^{\frac{3}{2}}} \,,
\end{aligned}$$

which is the inequality in the statement. ☐

Remark 3.14. Using the previous lemma and formula (3.7), a straightforward computation shows that the anisotropic mean curvature of a circumference of radius ρ is bounded from below by the constant $\frac{c_0}{\rho}$.

We now prove an "anisotropic version" of [45, Lemma 6.5].

Lemma 3.15. *Let $h \in C^\infty(\mathbb{R})$ be a b-periodic function, and let $\Lambda_0 = \|H^\psi\|_{L^\infty(\Gamma_h)}$, where H^ψ denotes the anisotropic mean curvature of Γ_h. Then for any admissible profile $k \in AP(0, b)$*

$$\int_{\Gamma_k} \psi(\nu_k) \, d\mathcal{H}^1 + \Lambda_0 \int_0^b |k - h| \, dx \geq \int_{\Gamma_h} \psi(\nu_h) \, d\mathcal{H}^1 \,.$$

PROOF. If k is Lipschitz, then using the 1-homogeneity and convexity of ψ we get

$$\begin{aligned}\int_{\Gamma_k} \psi(\nu_k)\, d\mathcal{H}^1 - \int_{\Gamma_h} \psi(\nu_h)\, d\mathcal{H}^1 &= \int_0^b \big[\psi(-k',1) - \psi(-h',1)\big]\, dx \\ &\geq \int_0^b (h'-k')\,\partial_1\psi(-h',1)\, dx = \int_0^b |k-h|\,\mathrm{sign}(k-h)\, H^{\psi}(x,h(x))\, dx \\ &\geq -\Lambda_0 \int_0^b |k-h|\, dx\,,\end{aligned}$$

where we integrated by parts using the periodicity of h, k and formula (3.7). If $k \in AP(0,b)$ and $\Sigma_k = \emptyset$, then the conclusion follows by approximation using Lemma 3.12. Finally, if $\Sigma_k \neq \emptyset$, one can simply replace k with k^- (for which $\Sigma_{k^-} = \emptyset$ and $\Gamma_{k^-} = \Gamma_k$), and apply again Lemma 3.12. $\square$

One essential point in the regularization procedure which leads to the $W^{2,\infty}$ convergence is that the solutions to the penalized problems that we will consider satisfy an inner ball condition. This is the content of the following lemma, which borrows some ideas from [22].

LEMMA 3.16 (Uniform inner ball condition). *Let $h \in AP(0,b) \cap C^2(\mathbb{R})$, $\Lambda > 0$, $d > 0$; let $(g,v) \in X(u_0;0,b)$ be a solution to*

$$\min\Big\{G(k,w) + \Lambda\big||\Omega_k| - d\big| : (k,w) \in X(u_0;0,b),\ k \geq h\Big\}.$$

Then there exists $\rho_0 = \rho_0(\Lambda,h)$ such that for every $\rho < \rho_0$ and for every $z \in \Gamma_g \cup \Sigma_g$ there exists a ball $B_\rho(z_0) \subset \Omega_g^{\#} \cup (\mathbb{R} \times (-\infty;0])$ such that $\partial B_\rho(z_0) \cap (\Gamma_g \cup \Sigma_g) = \{z\}$.

PROOF. As in [45, Lemma 6.7], the proof is based on a suitable isoperimetric inequality which in our anisotropic framework reads as follows (see [45, Lemma 6.6]):

let $k \in AP(0,b)$, $B_\rho(z_0) \subset \Omega_k^{\#} \cup \mathbb{R}^2_-$, and let $z_1 = (x_1,y_1)$, $z_2 = (x_2,y_2)$ be points in $\partial B_\rho(z_0) \cap (\Gamma_k^{\#} \cup \Sigma_k^{\#})$ (with $x_1 < x_2$). Let $S = (x_1,x_2) \times \mathbb{R}$, let γ be the shortest arc on $\partial B_\rho(z_0)$ connecting z_1 and z_2 (if z_1 and z_2 are antipodal, the arc which stays above), let γ' be the arc on $\Gamma_k^{\#} \cup \Sigma_k^{\#}$ connecting z_1 and z_2, and let D be the region enclosed by $\gamma \cup \gamma'$. Then

$$\begin{aligned}\int_{\Gamma_k^{\#}\cap S} \psi(\nu_k)\, d\mathcal{H}^1 &+ \psi(-1,0)\big(k(x_1+) - y_1\big) \\ &+ \psi(1,0)\big(k(x_2-) - y_2\big) - \int_\gamma \psi(\nu)\, d\mathcal{H}^1 \geq \frac{c_0}{\rho}\,|D|\,, \qquad (3.28)\end{aligned}$$

where c_0 is the constant appearing in (3.1).

Let us prove (3.28). Assume first that k is Lipschitz in $[x_1,x_2]$: let h be the function in (x_1,x_2) whose graph coincides with γ, then arguing as in the proof of Lemma 3.15 we obtain (observe that $k(x_1) = h(x_1)$, $k(x_2) = h(x_2)$, and $k \geq h$)

$$\begin{aligned}\int_{\Gamma_k^{\#}\cap S} \psi(\nu_k)\, d\mathcal{H}^1 - \int_{\Gamma_h\cap S} \psi(\nu_h)\, d\mathcal{H}^1 &= \int_{x_1}^{x_2} \big[\psi(-k',1) - \psi(-h',1)\big]\, dx \\ &\geq \int_{x_1}^{x_2} (h'-k')\,\partial_1\psi(-h',1)\, dx = \int_{x_1}^{x_2} (k-h)\,\big(\partial_1\psi(-h',1)\big)'\, dx \\ &\geq \frac{c_0}{\rho}\int_{x_1}^{x_2} (k-h)\, dx\,,\end{aligned}$$

which is (3.28) (in the last inequality we used Remark 3.14). For a general k, we can proceed by approximation using the following property: given $g : [x_1, x_2] \to \mathbb{R}$ lower semicontinuous with finite total variation, there exists a sequence of Lipschitz functions $g_n : [x_1, x_2] \to \mathbb{R}$ such that $g_n(x_1) = g(x_1)$, $g_n(x_2) = g(x_2)$, $g_n \to g$ in $L^1((x_1, x_2))$, and

$$\int_{\Gamma_{g_n}\cap S} \psi(\nu_{g_n})\, \mathrm{d}\mathcal{H}^1 \to \int_{\Gamma_g \cap S} \psi(\nu_g)\, \mathrm{d}\mathcal{H}^1 + \psi(-1,0)\big(g(x_1+) - g(x_1)\big) + \psi(1,0)\big(g(x_2-) - g(x_2)\big).$$

This can be obtained from [45, Lemma 6.2] using Reshetnyak's Continuity Theorem. Thus (3.28) follows.

Now the proof of the lemma can be obtained arguing exactly as in [45, Lemma 6.7], taking $\rho_0 < \min\{c_0/\Lambda, 1/\|h''\|_\infty\}$. In particular, one can use (3.28) to show that, if $B_{\rho_0}(z) \subset \Omega_g^\# \cup (\mathbb{R} \times (-\infty; 0])$, then $\partial B_{\rho_0}(z) \cap (\Gamma_g^\# \cup \Sigma_g^\#)$ is empty or consists of a single point. Then, the conclusion follows by showing that

$$\bigcup \Big\{ B_{\rho_0}(z) : B_{\rho_0}(z) \subset \Omega_g^\# \cup (\mathbb{R} \times (-\infty; 0]) \Big\} = \Omega_g^\# \cup (\mathbb{R} \times (-\infty; 0])$$

as in [22, Lemma 2] or [42, Proposition 3.3, Step 2]. □

The following proposition contains the main regularization result which allows us to get $W^{2,\infty}$-convergence of the sequence of penalized minima.

Proposition 3.17. *Let $(h, u) \in X(u_0; 0, b)$, $h > 0$, be a critical pair for G. Let $\Lambda > \Lambda_0 := \|H^\psi\|_{L^\infty(\Gamma_h)}$, where H^ψ is the anisotropic mean curvature of Γ_h. Let $(g_n, v_n) \in X(u_0; 0, b)$ be a solution to the penalization problem*

$$\min\Big\{ G(g, v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| : (g, v) \in X(u_0; 0, b),\, g \geq h - a_n \Big\} \tag{3.29}$$

where $(a_n)_n$ is a sequence of positive numbers converging to zero. Assume also that $g_n \to h$ in $L^1(0, b)$, $Dv_n \rightharpoonup Du$ in $L^2_{\mathrm{loc}}(\Omega_h; \mathbb{M}^2)$,

$$\lim_{n\to+\infty} \int_{\Gamma_{g_n}} \psi(\nu_{g_n})\, \mathrm{d}\mathcal{H}^1 = \int_{\Gamma_h} \psi(\nu_h)\, \mathrm{d}\mathcal{H}^1, \qquad \lim_{n\to+\infty} \mathcal{H}^1(\Sigma_{g_n}) = 0, \tag{3.30}$$

$$\text{and} \quad \lim_{n\to+\infty} \int_{\Omega_{g_n}} W(v_n)\, dz = \int_{\Omega_h} W(u). \tag{3.31}$$

Then $g_n \in W^{2,\infty}(0, b)$ for n large enough, and $g_n \to h$ in $W^{2,\infty}(0, b)$.

Proof. We review the proof of [45, Theorem 6.9], underlining the main changes needed to treat the present situation.

Step 1. We show that $\sup_{[0,b]} |g_n - h| \to 0$ as $n \to +\infty$. We may assume that $\overline{\Gamma_{g_n} \cup \Sigma_{g_n}}$ converge in the Hausdorff metric (up to subsequences) to some compact connected set K containing Γ_h. We claim that $\mathcal{H}^1(K \setminus \Gamma_h) = 0$. In fact, the approximate normal vector ν_K is defined at $\mathcal{H}^1$-a.e. point of K, coinciding with ν_h on Γ_h, and applying [47, Theorem 3.1] we get

$$\int_{\Gamma_h} \psi(\nu_h)\, \mathrm{d}\mathcal{H}^1 \leq \int_K \psi(\nu_K)\, \mathrm{d}\mathcal{H}^1 \leq \liminf_{n\to+\infty} \int_{\Gamma_{g_n}} \psi(\nu_{g_n})\, \mathrm{d}\mathcal{H}^1 + M\mathcal{H}^1(\Sigma_{g_n}) = \int_{\Gamma_h} \psi(\nu_h)\, \mathrm{d}\mathcal{H}^1,$$

from which the claim immediately follows. Now, since K is the Hausdorff limit of graphs, for every $x \in [0, b)$ the section $K \cap (\{x\} \times \mathbb{R})$ is connected; hence $\mathcal{H}^1(K \setminus \Gamma_h) = 0$ implies that $K = \overline{\Gamma}_h$. The uniform convergence of g_n to h follows using this equality, the definition of Hausdorff convergence and the continuity of h.

Step 2. We have $g_n \in C^0([0,b])$ and $\Sigma_{g_n,c} = \emptyset$ for n large enough, where

$$\Sigma_{g_n,c} := \{(x, g_n(x)) : x \in [0,b)\,,\, g_n(x) = g_n^-(x),\, (g_n)'_+(x) = -(g_n)'_-(x) = +\infty\}$$

is the set of *cusps*. The argument relies only on the inner ball condition, proved above (Lemma 3.16), and can be obtained repeating word for word the second step in the proof of [45, Theorem 6.9].

Step 3. We claim that $g_n \in C^1([0,b])$ for n large enough. In fact, using again the inner ball condition we first obtain that g_n is Lipschitz and admits left and right derivatives at every point, which are left and right continuous respectively: this is proved in [22, Lemma 3] (notice that the second situation described in the quoted result can be excluded thanks to the fact that $\Sigma_{g_n} \cup \Sigma_{g_n,c} = \emptyset$, as proved in the previous step).

From this we can also obtain the following decay estimate for v_n: for all $z_0 \in \Gamma_{g_n}$ there exists $c_n > 0$, a radius $r_n > 0$ and an exponent $\alpha_n \in (1/2, 1)$ such that

$$\int_{B_r(z_0)\cap\Omega_{g_n}} |Dv_n|^2 \,\mathrm{d}z \le c_n r^{2\alpha_n} \tag{3.32}$$

for all $r < r_n$ (see [42, Theorem 3.12]).

Finally, the argument which leads to the C^1-regularity of g_n goes as follows. It consists in showing that the left and right tangent lines at any point z_0 coincide, comparing the energy of (g_n, v_n) with the energy of a suitable competitor obtained by replacing the graph of g_n in a neighborhood of z_0 with an affine function. Assume by contradiction that the left and right tangent lines at a point $z_0 = (x_0, g_n(x_0)) \in \Gamma_{g_n}$ are distinct, and form an angle $\theta \in (0, \pi)$. Extend v_n out of Ω_{g_n} to a function $\tilde{v}_n$ which still satisfies the estimate

$$\int_{B_r(z_0)} |D\tilde{v}_n|^2 \,\mathrm{d}z \le c_n r^{2\alpha_n}. \tag{3.33}$$

For $r < r_n$, consider the points $z'_r = (x'_r, g_n(x'_r))$, $z''_r = (x''_r, g_n(x''_r))$ on $\Gamma_{g_n} \cap \partial B_r(z_0)$ such that the arcs γ'_r, γ''_r on Γ_{g_n} connecting z'_r to z_0, and z''_r to z_0 respectively, are contained in $\Gamma_{g_n} \cap B_r(z_0)$. Let s be the affine function whose graph connects z'_r and z''_r, denote by ν_r, ν'_r and ν''_r the upper-pointing normals to the segments $[z'_r, z''_r]$, $[z'_r, z_0]$ and $[z''_r, z_0]$ respectively and define

$$\tilde{g}_n(x) = \begin{cases} g_n(x) & \text{if } x \in [0,b)\setminus(x'_r, x''_r), \\ \max\{s(x), h(x) - a_n\} & \text{if } x \in (x'_r, x''_r). \end{cases}$$

Then $(\tilde{g}_n, \tilde{v}_n)$ is an admissible competitor in problem (3.29), and by the minimality of (g_n, v_n) we get

$$\begin{aligned} 0 &\ge G(g_n, v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| - G(\tilde{g}_n, \tilde{v}_n) - \Lambda\big||\Omega_{\tilde{g}_n}| - |\Omega_h|\big| \\ &\ge \int_{\gamma'_r\cup\gamma''_r} \psi(\nu_{g_n})\,\mathrm{d}\mathcal{H}^1 - \int_{\Gamma_{\tilde{g}_n}\cap((x'_r,x''_r)\times\mathbb{R})} \psi(\nu_{\tilde{g}_n})\,\mathrm{d}\mathcal{H}^1 - \int_{B_r(z_0)} W(\tilde{v}_n)\,dz - \Lambda|\Omega_{g_n}\triangle\Omega_{\tilde{g}_n}| \\ &\ge |z'_r - z_0|\,\psi(\nu'_r) + |z''_r - z_0|\,\psi(\nu''_r) - |z'_r - z''_r|\,\psi(\nu_r) \\ &\qquad - \int_{(x'_r,x''_r)\cap\{h>s+a_n\}} \big(\psi(-h'(x),1) - \psi(-s',1)\big)\,\mathrm{d}x - c_n r^{2\alpha_n} - \Lambda\pi r^2\,, \end{aligned} \tag{3.34}$$

where we used (3.33) and the inequality

$$\int_{\gamma'_r\cup\gamma''_r} \psi(\nu_{g_n})\,\mathrm{d}\mathcal{H}^1 \ge |z'_r - z_0|\,\psi(\nu'_r) + |z''_r - z_0|\,\psi(\nu''_r)\,,$$

which can be deduced arguing as in the proof of Lemma 3.15.

Now, observe that $|z_r' - z_0|\nu_r' + |z_r'' - z_0|\nu_r'' = |z_r' - z_r''|\nu_r$; therefore, applying [41, Proposition 8.1] (notice that the assumption (3.1) guarantees that the sublevel set $\{\psi \le 1\}$ is strictly convex) we get

$$|z_r' - z_0|\,\psi(\nu_r') + |z_r'' - z_0|\,\psi(\nu_r'') - |z_r' - z_r''|\,\psi(\nu_r) \ge r\,\omega(1 - \nu_r' \cdot \nu_r'')\,,$$

where $\omega : [0,2] \to [0,+\infty)$ is a modulus of continuity. From (3.34) we deduce

$$r\,\omega(1 - \nu_r' \cdot \nu_r'') \le \int_{(x_r', x_r'')\cap\{h > s + a_n\}} \big(\psi(-h'(x),1) - \psi(-s',1)\big)\,\mathrm{d}x + c_n' r^{2\alpha_n}\,,$$

and, in turn,

$$\omega(1 - \nu_r' \cdot \nu_r'') \le 2\,\mathrm{Lip}(\psi)\,\mathrm{osc}_{(x_r', x_r'')} h' + c_n' r^{2\alpha_n - 1},$$

and since $\alpha_n > \frac{1}{2}$ and h' is continuous, letting $r \to 0$ we obtain $\omega(1 - \nu_r' \cdot \nu_r'') \to 0$, which is a contradiction since $\nu_r' \cdot \nu_r'' \to \cos\theta < 1$. This completes the proof of the C^1-regularity of g_n.

Step 4. We have $g_n \to h$ in $C^1([0,b])$. The purely geometric argument that leads to this claim relies only on the inner ball condition, and is contained in the fourth step of the proof of [45, Theorem 6.9].

Step 5. We now prove that for all $\alpha \in (0,1/2)$, $g_n \to h$ in $C^{1,\alpha}([0,b])$, $v_n \in C^{1,\alpha}(\overline{\Omega}_{g_n})$ for n large enough, and $\sup_n \|v_n\|_{C^{1,\alpha}(\overline{\Omega}_{g_n})} < +\infty$.

The first claim follows by a comparison argument. Fix any point $z_0 = (x_0, g_n(x_0)) \in \Gamma_{g_n}$, $r > 0$, denote by γ_r the open arc contained in Γ_{g_n} of endpoints z_0 and $(x_0 + r, g_n(x_0 + r))$, and define $\tilde{g}_n$ as

$$\tilde{g}_n(x) = \begin{cases} g_n(x) & \text{if } x \in [0,b)\setminus(x_0, x_0 + r), \\ \max\{s(x), h(x) - a_n\} & \text{if } x \in (x_0, x_0 + r), \end{cases}$$

where s is the affine function whose graph connects z_0 and $(x_0 + r, g_n(x_0 + r))$. Then, comparing the energies of g_n and $\tilde{g}_n$ (as we did in Step 3), one can see that inequality (6.8) in [45] becomes in our case

$$\int_{x_0}^{x_0+r} \psi(-g_n', 1)\,\mathrm{d}x - \int_{x_0}^{x_0+r} \psi(-s', 1)\,\mathrm{d}x \le c' r^{2\sigma}\,. \tag{3.35}$$

Now, observe that for every a, b there exists a point ξ in the interval $[a \wedge b, a \vee b]$ such that

$$\begin{aligned} \psi(b,1) - \psi(a,1) &= \partial_1\psi(a,1)\,(b-a) + \frac{1}{2}\,\partial_{11}^2\psi(\xi,1)\,(b-a)^2 \\ &\ge \partial_1\psi(a,1)\,(b-a) + \frac{c_0 (b-a)^2}{2(1+\xi^2)^{3/2}} \\ &\ge \partial_1\psi(a,1)\,(b-a) + \frac{c_0 (b-a)^2}{2(1+\max\{a^2,b^2\})^{3/2}} \end{aligned} \tag{3.36}$$

(in the first inequality we used Lemma 3.13). Applying (3.36) with $a = -\frac{1}{r}\int_{x_0}^{x_0+r} g_n'\,\mathrm{d}x$ and $b = -g_n'(x)$, integrating in (x_0, x_0+r) and using (3.35), we get

$$\begin{aligned} &\frac{c_0}{2(1+M_1^2)^{3/2}}\,\frac{1}{r}\int_{x_0}^{x_0+r}\Big(g_n'(x) - \frac{1}{r}\int_{x_0}^{x_0+r} g_n'\,\mathrm{d}s\Big)^2\,\mathrm{d}x \\ &\qquad \le \frac{1}{r}\int_{x_0}^{x_0+r}\psi(-g_n',1)\,\mathrm{d}x - \frac{1}{r}\int_{x_0}^{x_0+r}\psi(-s',1)\,\mathrm{d}x \le c' r^{2\sigma-1}\,. \end{aligned}$$

From this inequality, arguing as in Step 5 of the proof of [45, Theorem 6.9], it follows that the sequence $(g_n)_n$ is equibounded in $C^{1,\sigma-\frac{1}{2}}([0,b])$ for all $\sigma \in (1/2,1)$, thus proving the first claim. The other claims are obtained using standard elliptic estimates (see [45, Proposition 8.9]).

Step 6. The conclusion ($g_n \to h$ in $W^{2,\infty}(0,b)$) follows by using the Euler-Lagrange equation (3.6) satisfied by a critical pair.

Indeed, setting $K_n = \{x \in [0,b] : g_n(x) = h(x) - a_n\}$ and assuming without loss of generality that $A_n = (0,b) \setminus K_n$ is not empty, it is easily seen that $g_n'(x) = h'(x)$ for every $x \in K_n$, while for $x \in A_n$ the following Euler-Lagrange equations are satisfied by g_n and h respectively:

$$\begin{aligned} \big(\partial_1 \psi(-g_n'(x), 1)\big)' &= -W(v_n)(x, g_n(x)) + \lambda_n, \\ \big(\partial_1 \psi(-h'(x), 1)\big)' &= -W(u)(x, h(x)) + \lambda, \end{aligned}$$

for some Lagrange multipliers λ_n, λ (the first equation follows by the minimality of (g_n, v_n), the second one by the fact that (h,u) is a critical pair: see (3.6)). Observe that, thanks to the results contained in [8, Section 7.7], g_n' is a Lipschitz function for all n. Now using the fact that the anisotropic mean curvature is expressed as a derivative (see (3.7)), we first deduce from the previous equations that, splitting A_n into the union of its connected components $(\alpha_{i,n}, \beta_{i,n})$,

$$\begin{aligned} \lambda_n|A_n| - \int_{A_n} W(v_n)(x, g_n(x))\,\mathrm{d}x &= \sum_i \int_{\alpha_{i,n}}^{\beta_{i,n}} \big(\partial_1\psi(-g_n'(x),1)\big)'\,\mathrm{d}x \\ &= \sum_i \Big(\partial_1\psi(-g_n'(\beta_{i,n}),1) - \partial_1\psi(-g_n'(\alpha_{i,n}),1)\Big) \\ &= \sum_i \Big(\partial_1\psi(-h'(\beta_{i,n}),1) - \partial_1\psi(-h'(\alpha_{i,n}),1)\Big) \\ &= \int_{A_n} \big(\partial_1\psi(-h'(x),1)\big)'\,\mathrm{d}x = \lambda|A_n| - \int_{A_n} W(u)(x,h(x))\,\mathrm{d}x\,, \end{aligned}$$

which, in turn, gives

$$\lambda_n - \lambda = \frac{1}{|A_n|}\int_{A_n} \Big[W(v_n)(x, g_n(x)) - W(u)(x, h(x))\Big]\,\mathrm{d}x.$$

From assumption (3.31) and Step 5 one can deduce that $W(v_n)(\cdot, g_n(\cdot)) \to W(u)(\cdot, h(\cdot))$ uniformly in $[0,b]$, hence we conclude that $\lambda_n \to \lambda$. Now the Euler-Lagrange equations, the convergence $\lambda_n \to \lambda$ and the uniform convergence of $W(v_n)(\cdot, g_n(\cdot))$ to $W(u)(\cdot, h(\cdot))$ imply

$$\big(\partial_1\psi(-g_n'(x),1)\big)' \to \big(\partial_1\psi(-h'(x),1)\big)' \quad \text{uniformly in } [0,b].$$

Finally, from this we deduce that $g_n'' \to h''$ in $L^\infty(0,b)$, since

$$\|g_n'' - h''\|_{L^\infty(0,b)} = \left\| \frac{\big(\partial_1\psi(-g_n',1)\big)'}{\partial^2_{11}\psi(-g_n',1)} - \frac{\big(\partial_1\psi(-h',1)\big)'}{\partial^2_{11}\psi(-h',1)} \right\|_{L^\infty(0,b)} \to 0$$

(using the fact that the denominators are uniformly bounded away from 0 by Lemma 3.13). This concludes the proof of the proposition. ☐

Proof of Theorem 3.11. By contradiction, let $(\tilde g_n, \tilde v_n) \in X(u_0; 0, b)$ be such that $|\Omega_{\tilde g_n}| = |\Omega_h|$, $0 < \|\tilde g_n - h\|_{L^\infty(0,b)} \le \frac{1}{n}$ and

$$G(\tilde g_n, \tilde v_n) \le G(h,u). \tag{3.37}$$

Fix $\Lambda > \max\{\Lambda_0, W_0\}$, where Λ_0 is defined in Proposition 3.17 and

$$W_0 = \frac{1}{b}\int_0^b W(U_0(x,y))\,\mathrm{d}x\,, \qquad U_0(x,y) = u_0(x,0) + e_0\Big(0, \frac{-\lambda}{2\mu+\lambda}\,y\Big)$$

(notice that W_0 is finite since u_0 is Lipschitz), and let (g_n, v_n) be a solution to the minimum problem

$$\min\Big\{G(g,v) + \Lambda\big||\Omega_g| - |\Omega_h|\big| : (g,v) \in X(u_0;0,b),\, g \geq h - \frac{1}{n}\Big\}\,; \tag{3.38}$$

then

$$G(g_n,v_n) \leq G(g_n,v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| \leq G(\tilde g_n,\tilde v_n) \leq G(h,u). \tag{3.39}$$

We claim that $(g_n,v_n) \to (h,u)$ in Y, up to subsequences. Indeed, by (3.39) we have a uniform bound

$$\int_{\Omega_{g_n}} |E(v_n)|^2\,\mathrm{d}z + \mathrm{Var}(g_n;0,b) + |\Omega_{g_n}| \leq C$$

(the bound on the variation of g_n follows using condition (3.2), which gives a uniform bound on $\mathcal{H}^1(\Gamma_{g_n})$), so that by Theorem 3.1 we have $(g_n,v_n) \xrightarrow{Y} (k,v) \in X(u_0;0,b)$ up to subsequences. Taken any $(g,w) \in X(u_0;0,b)$ with $g \geq h$ (it is an admissible competitor for all the penalized problems) we have, by the l.s.c. of G with respect to the convergence in Y and the minimality of (g_n,v_n),

$$G(k,v) + \Lambda\big||\Omega_k| - |\Omega_h|\big| \leq \liminf_{n\to+\infty}\big(G(g_n,v_n) + \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big|\big) \leq G(g,w) + \Lambda\big||\Omega_g| - |\Omega_h|\big|\,. \tag{3.40}$$

From the previous inequality with $(g,w) = (h,v)$ we get, since $k \geq h$,

$$\int_{\Gamma_k} \psi(\nu_k)\,\mathrm{d}\mathcal{H}^1 + \Lambda\int_0^b |k-h| \leq \int_{\Gamma_h} \psi(\nu_h)\,\mathrm{d}\mathcal{H}^1,$$

from which it follows $k = h$ by Lemma 3.15 (using $\Lambda > \Lambda_0$), and in turn $v = u$. Thus the claim is proved.

Moreover, using again (3.40) with $(g,w) = (h,u)$, combined with the l.s.c. of the volume energy and of the map $g \to \int_{\Gamma_g} \psi(\nu_g)\,\mathrm{d}\mathcal{H}^1$ with respect to the convergence in Y (the second one follows from Reshetnyak's Lower Semicontinuity Theorem), we deduce that conditions (3.30) and (3.31) hold. By Proposition 3.17 we can conclude that $g_n \to h$ in $W^{2,\infty}(0,b)$.

We now deal with the volume constraint. Suppose first by contradiction that $|\Omega_{g_n}| < |\Omega_h|$. In this case, consider the competitor $(\bar g_n, \bar v_n)$, where $\bar g_n = g_n + (|\Omega_h| - |\Omega_{g_n}|)/b$ and

$$\bar v_n(x,y) = \begin{cases} U_0(x,y) & \text{if } 0 \leq y < (|\Omega_h| - |\Omega_{g_n}|)/b\,, \\ v_n\Big(x, y - \frac{|\Omega_h| - |\Omega_{g_n}|}{b}\Big) + e_0\Big(0, \frac{-\lambda(|\Omega_h| - |\Omega_{g_n}|)}{b(2\mu+\lambda)}\Big) & \text{if } y \geq (|\Omega_h| - |\Omega_{g_n}|)/b\,, \end{cases}$$

for $(x,y) \in \Omega_{\bar g_n}$: then

$$G(\bar g_n,\bar v_n) + \Lambda\big||\Omega_{\bar g_n}| - |\Omega_h|\big| - G(g_n,v_n) - \Lambda\big||\Omega_{g_n}| - |\Omega_h|\big| = \big(|\Omega_h| - |\Omega_{g_n}|\big)(W_0 - \Lambda) < 0$$

(since $\Lambda > W_0$), which contradicts the minimality of (g_n,v_n).

Thus, $|\Omega_{g_n}| \geq |\Omega_h|$ for every n. We define $\hat g_n := g_n - (|\Omega_{g_n}| - |\Omega_h|)/b$, so that

$$|\Omega_{\hat g_n}| = |\Omega_h|, \quad \hat g_n \to g \text{ in } W^{2,\infty}(0,b), \quad \text{and} \quad G(\hat g_n, v_n) \leq G(h,u)\,.$$

From the isolated $W^{2,\infty}$-minimality of (h,u) (given by Theorem 3.9) we get $(\hat g_n, v_n) = (h,u)$ for n large. By (3.39) this implies that $G(g_n,v_n) = G(\tilde g_n,\tilde v_n) = G(h,u)$ for n large, thus the pair $(\tilde g_n,\tilde v_n)$ is a solution to the minimum problem (3.38). Hence, the previous compactness argument applied now to the sequence $(\tilde g_n,\tilde v_n)$ instead of (g_n,v_n) leads to $\tilde g_n \to h$ in $W^{2,\infty}$, which contradicts (3.37) since (h,u) is an isolated local $W^{2,\infty}$-minimizer. □

3.1.4. Stability of the flat configuration. Now we come to the study of the stability of the flat configuration $(\frac{d}{b}, v_{e_0})$ (see Definition 3.7): we show the existence a volume threshold of minimality, which can be analitically determined in terms of the *Grinfeld function* K, defined for $y \geq 0$ by

$$K(y) := \max_{n \in \mathbb{N}} \frac{1}{n} J(ny), \qquad J(y) := \frac{y + (3 - 4\nu_p) \sinh y \cosh y}{4(1-\nu_p)^2 + y^2 + (3 - 4\nu_p) \sinh^2 y},$$

where $\nu_p = \frac{\lambda}{2(\lambda+\mu)}$ (the function K is strictly increasing and continuous, $K(y) \leq Cy$ for some positive constant C, and $\lim_{y \to +\infty} K(y) = 1$: see [45, Corollary 5.3]).

THEOREM 3.18. *For any $b > 0$ and $e_0 > 0$, let $d(b, e_0) \in (0, +\infty]$ be defined as $d(b, e_0) = +\infty$ if $0 < b \leq \frac{\pi}{4} \frac{(2\mu+\lambda)\, \partial^2_{11}\psi(0,1)}{e_0^2 \mu(\mu+\lambda)}$, and as the solution to*

$$K\left(\frac{2\pi d(b, e_0)}{b^2}\right) = \frac{\pi}{4} \frac{(2\mu + \lambda)\, \partial^2_{11}\psi(0, 1)}{e_0^2\, \mu\, (\mu + \lambda)} \frac{1}{b}$$

otherwise. Then the flat configuration $(\frac{d}{b}, v_{e_0})$ is an isolated b-periodic local minimizer for G, in the sense of Definition 3.3, if $0 < d < d(b, e_0)$,

The threshold $d(b, e_0)$ is critical: indeed, for $d > d(b, e_0)$ there exists a sequence $(g_n, v_n) \in X(u_0; 0, b)$ such that $|\Omega_{g_n}| = d$, $\|g_n - \frac{d}{b}\|_\infty \leq \frac{1}{n}$ and $G(g_n, v_n) < G(\frac{d}{b}, v_{e_0})$.

In order to prove the theorem, we start by noticing that we can consider without loss of generality variations in the subspace

$$\widetilde{H}^1_0(\Gamma_{d/b}) := \{\varphi \in \widetilde{H}^1_\#(\Gamma_{d/b}) : \varphi(0, d/b) = \varphi(b, d/b) = 0\},$$

(see [45, Remark 4.8]); in turn, this space can be identified with

$$\widetilde{H}^1_0(0, b) := \left\{\varphi \in H^1(0, b) : \varphi(0) = \varphi(b) = 0,\ \int_0^b \varphi = 0\right\}.$$

Observe moreover that the quadratic form associated with the second variation of the functional G at the flat configuration is given by

$$\partial^2 G\,(d/b, v_{e_0})\,[\varphi] = -\int_{(0,b)\times(0,\frac{d}{b})} \mathbb{C}E(v_\varphi) : E(v_\varphi)\,\mathrm{d}z + \partial^2_{11}\psi(0, 1) \int_0^b {\varphi'}^2(x)\,\mathrm{d}x$$

for all $\varphi \in \widetilde{H}^1_0(0, b)$, where $v_\varphi \in A(\Omega_{d/b})$ is the solution to

$$\int_{(0,b)\times(0,\frac{d}{b})} \mathbb{C}E(v_\varphi) : E(w)\,\mathrm{d}z = \tau \int_0^b \varphi'(x)\, w_1(x, d/b)\,\mathrm{d}x \quad \text{for every } w = (w_1, w_2) \in A(\Omega_{d/b})$$

with $\tau = \frac{4\mu(\mu+\lambda)e_0}{2\mu+\lambda}$. Observe that, by Lemma 3.13, the coefficient $\partial^2_{11}\psi(0, 1)$ is strictly positive.

PROOF OF THEOREM 3.18. Arguing as in the proof of [45, Theorem 5.1], we get an explicit expression of the second variation in terms of the Fourier coefficients of φ, namely

$$\partial^2 G(d/b, v_{e_0})[\varphi] = \sum_{n \in \mathbb{Z}} n^2 \varphi_n \varphi_{-n} \left[\partial^2_{11}\psi(0, 1) - \frac{\tau^2(1-\nu_p) b J(2\pi n d/b^2)}{2\pi\mu n}\right], \tag{3.41}$$

where the φ_n's are the Fourier coefficients of φ in $(0, b)$. Now by definition of K

$$\sup_{n \in \mathbb{Z}} \frac{\tau^2(1-\nu_p) b J(2\pi n d/b^2)}{2\pi\mu n} \gtrless \partial^2_{11}\psi(0, 1) \iff K\left(\frac{2\pi d}{b^2}\right) \gtrless \frac{\pi}{4} \frac{(2\mu + \lambda)\partial^2_{11}\psi(0, 1)}{e_0^2 \mu(\mu + \lambda)} \frac{1}{b},$$

which implies by (3.41)

$$\partial^2 G(d/b, v_{e_0})[\varphi] > 0 \quad \forall \varphi \in \widetilde{H}^1_0(0,b) \iff K\left(\frac{2\pi d}{b^2}\right) < \frac{\pi}{4}\,\frac{(2\mu+\lambda)\partial^2_{11}\psi(0,1)}{e_0^2\mu(\mu+\lambda)}\,\frac{1}{b}\,,$$

$$K\left(\frac{2\pi d}{b^2}\right) > \frac{\pi}{4}\,\frac{(2\mu+\lambda)\partial^2_{11}\psi(0,1)}{e_0^2\mu(\mu+\lambda)}\,\frac{1}{b} \implies \partial^2 G(d/b, v_{e_0})[\varphi] < 0 \quad \text{for some } \varphi \in \widetilde{H}^1_0(0,b)\,.$$

Then the conclusion follows by Theorem 3.11. □

Remark 3.19. It can be interesting to study what can be said, in this anisotropic contest, about the issue of the *global minimality* of the flat configuration, that is, whether $(\frac{d}{b}, v_{e_0})$ minimizes G among *all* b-periodic competitors satisfying the same volume constraint. One can check that the corresponding result proved in the first part of [45, Theorem 2.11] can be extended to the anisotropic functional considered in this section, with no particular changes in the proof: precisely, we have that for every $b > 0$ and $e_0 > 0$ there exists $d_{\rm glob}(b, e_0) \in (0, d(b, e_0)]$ such that the flat configuration $(\frac{d}{b}, v_{e_0})$ is a b-periodic global minimizer if and only if $d \le d_{\rm glob}(b, e_0)$, and it is the *unique* global minimizer if $d < d_{\rm glob}$.

3.2. The case of nonlinear elastic energies in two and three dimensions: setting

In the remaining part of this chapter, except for Section 3.9, we will extend the results of the previous section to the three-dimensional case, taking into account also nonlinear elastic energies. We start by describing the setting that will be the subject of investigation of the rest of this chapter.

Let $Q = (0,1)^{N-1}$ be the unit square in $\mathbb{R}^{N-1}$. For $p \in [1, +\infty]$ and $k \ge 0$, we denote by $W^{k,p}_{\#}(Q)$ the set of functions $h : \mathbb{R}^{N-1} \to (0, +\infty)$ of class $W^{k,p}_{\rm loc}(\mathbb{R}^{N-1})$ which are one-periodic with respect to all the coordinate directions, endowed with the norm $\|\cdot\|_{W^{k,p}(Q)}$. Similarly, $C^k_{\#}(Q)$ and $C^{k,\alpha}_{\#}(Q)$, for $\alpha \in (0,1)$, denote the sets of one-periodic functions $h : \mathbb{R}^{N-1} \to (0, +\infty)$ of class C^k and $C^{k,\alpha}$, respectively.

We first introduce the class of *admissible profiles*, given by Lipschitz, strictly positive and periodic functions:

$$AP(Q) := \Big\{ h : \mathbb{R}^{N-1} \to (0, +\infty) \,:\, h \text{ is Lipschitz continuous,}$$
$$h(x + e_i) = h(x) \text{ for every } x \in \mathbb{R}^{N-1} \text{ and } i = 1, \dots, N-1 \Big\}.$$

Given $h \in AP(Q)$, we define the associated reference configuration Ω_h and its periodic extension $\Omega_h^\#$ to be the sets

$$\Omega_h := \big\{(x,y) \in \mathbb{R}^N : x \in Q,\ 0 < y < h(x)\big\}, \quad \Omega_h^\# := \big\{(x,y) \in \mathbb{R}^N : 0 < y < h(x)\big\}$$

respectively, and the graph Γ_h of h and its periodic extension $\Gamma_h^\#$, representing the *free profile*,

$$\Gamma_h := \big\{(x, h(x)) \in \mathbb{R}^N : x \in Q\big\}, \quad \Gamma_h^\# := \big\{(x, h(x)) \in \mathbb{R}^N : x \in \mathbb{R}^{N-1}\big\}.$$

We also introduce the following space of admissible elastic variations:

$$\mathcal{V}(\Omega_h) := \Big\{ w \in W^{1,\infty}(\Omega_h^\#; \mathbb{R}^N) : \ w(x,0) = 0,\ w(x + e_i, y) = w(x,y)$$
$$\text{for all } (x,y) \in \Omega_h^\# \text{ and } i = 1, \dots, N-1 \Big\},$$

and we denote by $\widetilde{\mathcal{V}}(\Omega_h)$ the completion of $\mathcal{V}(\Omega_h)$ with respect to the norm of $H^1(\Omega_h;\mathbb{R}^N)$. Since we assume to be in presence of a mismatch strain at the interface $\{y=0\}$, we prescribe a boundary Dirichlet datum in the form

$$u_0(x,y) := (A[x]+q(x),0),$$

where $A \in \mathbb{M}^{N-1}_+$ and $q : \mathbb{R}^{N-1} \to \mathbb{R}^{N-1}$ is a smooth function, one-periodic with respect to the coordinate directions. We can finally define the space of *admissible pairs*

$$X = \Big\{(h,u) \in AP(Q) \times W^{1,\infty}(\Omega_h^\#;\mathbb{R}^N) \,:\, u-u_0 \in \mathcal{V}(\Omega_h),\ \det Du(z) > 0 \text{ for a.e. } z \in \Omega_h\Big\}.$$

In order to introduce the functional on X which represents the total energy of the system, we define the *elastic energy density* and the *anisotropic surface energy density* to be, respectively:

- $W : \mathbb{M}^N_+ \to [0,+\infty)$ of class C^3,
- $\psi : \mathbb{R}^N \to [0,+\infty)$, of class C^3 away from the origin, positively 1-homogeneous, such that

$$m|z| \le \psi(z) \le M|z| \qquad \text{for all } z \in \mathbb{R}^N \tag{3.42}$$

  for some positive constants m, M, and satisfying the following condition of uniform convexity: for every $v \in \mathbb{S}^{N-1}$

$$D^2\psi(v)[w,w] > \bar{c}\,|w|^2 \quad \text{for all } w \perp v, \tag{3.43}$$

  for some constant $\bar{c} > 0$.

Finally, we define the functional on X

$$F(h,u) := \int_{\Omega_h} W(Du)\,\mathrm{d}z + \int_{\Gamma_h} \psi(\nu_h)\,\mathrm{d}\mathcal{H}^{N-1}, \tag{3.44}$$

where ν_h denotes the exterior unit normal vector to Ω_h on Γ_h (we shall omit the subscript h when there is no risk of ambiguity).

Remark 3.20. Although, for the sake of simplicity, we assume that W is defined on the space $\mathbb{M}^N_+$ of the matrices with positive determinant, the results that we are going to prove are valid also for a general nonlinear density W of class C^3, defined only on an open subset $\mathcal{O}$ of $\mathbb{M}^N$; in this case the space X should be replaced by the following space of admissible pairs:

$$\Big\{(h,u) \in AP(Q) \times W^{1,\infty}(\Omega_h^\#;\mathbb{R}^N) \,:\, u-u_0 \in \mathcal{V}(\Omega_h),\ \nabla u(z) \in \mathcal{O} \text{ for a.e. } z \in \Omega_h\Big\}.$$

The physically relevant condition that $W(\xi) \to +\infty$ as $\det\xi \to 0^+$, which is customary in Finite Elasticity, is compatible with our assumption. When W is a quasi-convex function defined on the whole space $\mathbb{M}^N$ and satisfying standard p-growth conditions, the definition of the functional F can be extended to a larger class of admissible pairs by a relaxation procedure (see [23]).

We will denote the derivatives of W by

$$W_\xi(\xi) := DW(\xi) = \left(\frac{\partial W}{\partial \xi_{ij}}(\xi)\right)_{ij}, \qquad W_{\xi\xi}(\xi) := D^2W(\xi) = \left(\frac{\partial^2 W}{\partial \xi_{ij}\partial \xi_{hk}}(\xi)\right)_{ijhk}.$$

We now give the definitions of critical point for the elastic energy in a given reference configuration Ω_h, and of critical pair for the functional F.

Definition 3.21. Let $(h,u)\in X$ with $u\in C^1(\overline{\Omega}_h^\#;\mathbb{R}^N)$. The function u is said to be a *critical point for the elastic energy in* Ω_h if

$$\int_{\Omega_h} W_\xi(Du) : Dw\,\mathrm{d}z = 0 \qquad \text{for every } w\in\mathcal{V}(\Omega_h). \tag{3.45}$$

Notice that, by periodicity, (3.45) is equivalent to

$$\begin{cases} \operatorname{div}[W_\xi(Du)] = 0 & \text{in } \Omega_h^\#, \\ W_\xi(Du)[\nu] = 0 & \text{on } \Gamma_h^\#. \end{cases}$$

Definition 3.22. We say that a pair $(h,u)\in X$ is a *(regular) critical pair* for F if $h\in C^2_\#(Q)$, $u\in C^2(\overline{\Omega}_h^\#;\mathbb{R}^N)$ is a critical point for the elastic energy in Ω_h, and the following condition holds:

$$W(Du) + H^\psi = \text{const} \qquad \text{on } \Gamma_h. \tag{3.46}$$

The regularity assumptions on a critical pair (h,u) allow us to extend u to a slightly larger domain, preserving the property that the deformation gradient Du has positive determinant. More precisely, given a critical pair (h,u) we can find an open set Ω' of the form $\Omega_{h+\eta}$, for some $\eta>0$, with the following property: denoting by $\Omega'_\#$ the periodic extension of Ω', we can extend u to a periodic function of class C^1 in $\overline{\Omega}'_\#$ in such a way that $\det Du(z)>0$ for every $z\in\overline{\Omega}'$. This induces us to consider the following class of competitors:

$$X' := \big\{(g,v)\in X \,:\, \Omega_g\subset\Omega',\ v\in W^{1,\infty}(\Omega'_\#;\mathbb{R}^N),\ \det Dv(z)>0 \text{ for a.e. } z\in\Omega'\big\}. \tag{3.47}$$

We then consider the following notion of local minimality.

Definition 3.23. Let $(h,u)\in X$ be a critical pair for F. We say that (h,u) is a *local minimizer* for F if there exists $\delta>0$ such that

$$F(h,u)\le F(g,v) \tag{3.48}$$

for all $(g,v)\in X'$ with $\|g-h\|_\infty<\delta$, $|\Omega_g|=|\Omega_h|$, and $\|Dv-Du\|_{L^\infty(\Omega';\mathbb{M}^N)}<\delta$. We say that (h,u) is an *isolated local minimizer* if (3.48) holds with strict inequality when $g\neq h$.

Remark 3.24. The following construction will be used several times throughout this chapter. Given any admissible profile $h\in AP(Q)$, we associate with every $g\in AP(Q)$ in a sufficiently small L^∞-neighborhood of h a map $\Phi_g:\overline{\Omega}_h^\#\to\overline{\Omega}_g^\#$ with the properties:

- $\Phi_g(x,0)=(x,0)$ for every $x\in\mathbb{R}^{N-1}$;
- $\Phi_g(x,y)=(x,y+g(x)-h(x))$ in a neighborhood of $\Gamma_h^\#$;
- $\Phi_g(x+e_i,y)=\Phi_g(x,y)+(e_i,0)$ for $(x,y)\in\overline{\Omega}_h^\#$ and $i=1,\dots,N-1$;
- Φ_g satisfies the following estimate:

$$\|\Phi_g - Id\|_{L^\infty(\Omega_h;\mathbb{R}^N)} \le \|g-h\|_{L^\infty(Q)}. \tag{3.49}$$

We can explicitly construct the diffeomorphism Φ_g as follows. Setting $m_0:=\min h>0$, we fix a nonnegative cut-off function $\rho\in C^\infty_c(-\frac{m_0}{2},\frac{m_0}{2})$ with $\rho\equiv1$ in $(\frac{m_0}{4},\frac{m_0}{4})$. Then it is easily seen that, if $\|g-h\|_\infty<\frac{m_0}{4}$, the map

$$\Phi_g(x,y) := \big(x, y+\rho(y-h(x))(g(x)-h(x))\big)$$

satisfies all the previous conditions.

REMARK 3.25. We note here for later use that, as a consequence of the positive 1-homogeneity of the anisotropy ψ,

$$D^2\psi(v)[v] = 0 \quad \text{for every } v \in \mathbb{R}^N \setminus \{0\}. \tag{3.50}$$

Moreover, given a sufficiently regular admissible profile h, we can prove the following explicit formula for the anisotropic mean curvature of Γ_h (see (1.6)), analogous to the expression obtained in Remark 3.6 for the two-dimensional case:

$$H^\psi(x, h(x)) = \sum_{i=1}^{N-1} \frac{\partial}{\partial x_i}\Big(\frac{\partial \psi}{\partial z_i}(-\nabla h(x), 1)\Big). \tag{3.51}$$

Indeed, observe that by (3.50) we have $D^2\psi(-\nabla h, 1)[(-\nabla h, 1)] = 0$, that is,

$$\sum_{j=1}^{N-1} \frac{\partial^2 \psi}{\partial z_i \partial z_j}(-\nabla h, 1)\frac{\partial h}{\partial x_j} = \frac{\partial^2 \psi}{\partial z_i \partial z_N}(-\nabla h, 1)$$

for $i = 1, \dots, N$. Hence, as $\nabla\psi$ is 0-homogeneous, a straightforward computation yields

$$\begin{aligned} H^\psi(x, h(x)) = \operatorname{div}_{\Gamma_h}(\nabla\psi \circ \nu)|_{(x,h(x))} &= -\sum_{j,k=1}^{N-1} \frac{\partial^2 \psi}{\partial z_j \partial z_k}(-\nabla h, 1)\frac{\partial^2 h}{\partial x_k \partial x_j} \\ &+ \frac{1}{1+|\nabla h|^2}\sum_{i,k=1}^{N-1}\Big(\sum_{j=1}^{N-1}\frac{\partial^2 \psi}{\partial z_k \partial z_j}(-\nabla h, 1)\frac{\partial h}{\partial x_j} - \frac{\partial^2 \psi}{\partial z_k \partial z_N}(-\nabla h, 1)\Big)\frac{\partial^2 h}{\partial x_k \partial x_i}\frac{\partial h}{\partial x_i}, \end{aligned}$$

from which (3.51) follows by using the previous equality.

3.3. Critical points for the elastic energy

The purpose of this section is to associate with every g close to h (in some norm, to be specified) a deformation u_g such that, if g is fixed, the map $v \mapsto F(g, v)$ has a local minimum at u_g. If this is the case, then in order to prove the local minimality of an admissible pair (h, u) it will be sufficient to compare $F(h, u)$ only with the values of F at pairs of the form (g, u_g), avoiding in some sense the dependence on the second variable. The Implicit Function Theorem guarantees that this is in fact possible, under suitable assumptions on the starting pair (h, u).

DEFINITION 3.26. Let $(h, u) \in X$, and assume that u is a critical point for the elastic energy in Ω_h, according to Definition 3.21. We say that u is a *strict δ-local minimizer for the elastic energy in Ω_h*, for $\delta > 0$, if

$$\int_{\Omega_h} W(Du)\,\mathrm{d}z < \int_{\Omega_h} W(Du + Dw)\,\mathrm{d}z$$

whenever $w \in \mathcal{V}(\Omega_h)$ and $0 < \|Dw\|_{L^\infty(\Omega_h;\mathbb{M}^N)} \leq \delta$.

We now provide suitable assumptions on a pair (h, u), with u critical point for the elastic energy in Ω_h, which guarantee that if g is a small $W^{2,p}$-perturbation of the profile h then we can find a critical point u_g for the elastic energy in Ω_g which in addition locally minimizes the elastic energy. In order to do this, we introduce a fourth order symmetric tensor field, associated with a deformation u in a domain Ω_h, setting

$$C_u(z) := W_{\xi\xi}(Du(z)) \qquad \text{for every } z \in \overline{\Omega}_h^\# .$$

DEFINITION 3.27. Let $(h,u) \in X$. We say that the elastic second variation is uniformly positive at u in Ω_h if there exists a positive constant c_0 such that

$$\int_{\Omega_h} C_u Dw : Dw \, dz \geq c_0 \|w\|^2_{H^1(\Omega_h;\mathbb{R}^N)} \qquad \text{for every } w \in \widetilde{\mathcal{V}}(\Omega_h)\,, \tag{3.52}$$

where we recall that $\widetilde{\mathcal{V}}(\Omega_h)$ denotes the completion of $\mathcal{V}(\Omega_h)$ with respect to the norm of $H^1(\Omega_h;\mathbb{R}^N)$.

Arguing as in [76, Theorem 1], it is possible to prove[2] the following equivalent formulation of condition (3.52).

THEOREM 3.28. *Let $(h,u) \in X$ be such that $h \in C^2_\#(Q)$ and $u \in C^2(\overline{\Omega}^\#_h;\mathbb{R}^N)$ is a critical point for the elastic energy in Ω_h. Then (3.52) holds (with some positive constant c_0 depending only on the pair (h,u)) if and only if the following three conditions are satisfied:*

(H1) for all $z \in \Omega_h$ the fourth order tensor $C_u(z)$ satisfies the strong ellipticity condition, *that is*

$$C_u(z)M : M > 0$$

whenever $M = a \otimes b$ with $a \neq 0$, $b \neq 0$;

(H2) for all $z_0 \in \Gamma_h$ the boundary value problem

$$\begin{cases} \operatorname{div}\left[C_u(z_0)Dv\right] = 0 & \text{in } H_{\nu(z_0)}, \\ \left(C_u(z_0)Dv\right)\left[\nu(z_0)\right] = 0 & \text{on } \partial H_{\nu(z_0)}, \end{cases}$$

where

$$H_{\nu(z_0)} := \{z \in \mathbb{R}^N : z \cdot \nu(z_0) > 0\},$$

satisfies the complementing condition, i.e., *the only bounded exponential solution to the previous equation is $v \equiv 0$. By* bounded exponential *we mean a solution of the form*

$$v(z) = \mathrm{Re}\big[f(z \cdot \nu(z_0))\, e^{i(z\cdot b)}\big]$$

for some $b \in \partial H_{\nu(z_0)} \setminus \{0\}$ and $f \in C^\infty([0,+\infty),\mathbb{C}^N)$ satisfying $\sup_s |f(s)| < \infty$;

(H3) the elastic second variation is strictly positive, that is, for every $w \in \widetilde{\mathcal{V}}(\Omega_h)\setminus\{0\}$

$$\int_{\Omega_h} C_u Dw : Dw \, dz > 0\,.$$

We are now ready to explain the construction announced at the beginning of this section.

PROPOSITION 3.29. *Let $(h,u) \in X$ be such that $h \in C^2_\#(Q)$, $u \in C^2(\overline{\Omega}^\#_h;\mathbb{R}^N)$ is a critical point for the elastic energy in Ω_h, and condition (3.52) holds. Let $p \in (N,+\infty)$. There exists a neighborhood $\mathcal{U}$ of h in $W^{2,p}_\#(Q)$ and a map $g \in \mathcal{U} \mapsto u_g \in W^{2,p}(\Omega_g;\mathbb{R}^N)$ such that:*

(i) u_g is a critical point for the elastic energy in Ω_g, according to Definition 3.21;
(ii) $u_h = u$;
(iii) the map $g \mapsto u_g \circ \Phi_g$ is of class C^1 from $W^{2,p}_\#(Q)$ to $W^{2,p}(\Omega_h;\mathbb{R}^N)$.

[2]In view of the Remark following [76, Proposition 9.4], our regularity assumptions on W and (h,u) are sufficient to guarantee the validity of the stated result.

Proof. We start by observing that if $g \in W^{2,p}_{\#}(Q)$ is close to h in the $W^{2,p}$-topology, the maps Φ_g introduced in Remark 3.24 are orientation preserving diffeomorphisms of class $W^{2,p}$ satisfying an estimate

$$\|\Phi_g - Id\|_{W^{2,p}(\Omega_h;\mathbb{R}^N)} \le c\,\|g-h\|_{W^{2,p}(Q)} \tag{3.53}$$

for some constant $c>0$ depending only on h. Moreover, by construction the map $g \mapsto \Phi_g$ is affine, and hence of class C^∞ from a neighborhood of h in $W^{2,p}_{\#}(Q)$ to $W^{2,p}(\Omega_h;\mathbb{R}^N)$.

Our aim is to associate, with every g in a sufficiently small $W^{2,p}$-neighborhood of h, a solution u_g to (3.45) (where h is replaced by g) with $u_g - u_0 \in \mathcal{V}(\Omega_g)$. A change of variables shows that a function v is a solution to (3.45) with $v - u_0 \in \mathcal{V}(\Omega_g)$ if and only if the function $\tilde v = v\circ\Phi_g - u_0$ belongs to $\mathcal{V}(\Omega_h)$ and solves

$$\int_{\Omega_h} W_\xi\big((D\tilde v + Du_0)(D\Phi_g)^{-1}\big)(D\Phi_g)^{-T} : D\tilde w \,\det D\Phi_g \,dz = 0 \qquad \text{for every } \tilde w \in \mathcal{V}(\Omega_h). \tag{3.54}$$

Notice that an equivalent formulation of (3.54) is

$$\begin{cases} \operatorname{div}\big[Q_{\Phi_g}(z, D\tilde v(z))\big] = 0 & \text{in } \Omega_h^{\#},\\ Q_{\Phi_g}(z, D\tilde v(z))[\nu] = 0 & \text{on } \Gamma_h^{\#}, \end{cases}$$

where we set, for $z \in \overline{\Omega_h^{\#}}$ and $M \in \mathbb{M}^N$,

$$Q_{\Phi_g}(z, M) := \det D\Phi_g(z)\, W_\xi\big((M + Du_0(z))(D\Phi_g(z))^{-1}\big)\,(D\Phi_g(z))^{-T}. \tag{3.55}$$

Our strategy will be to get a solution to this boundary value problem by means of the Implicit Function Theorem. To this aim, let us define the open subsets

$$\begin{aligned} A &:= \big\{\Phi \in W^{2,p}(\Omega_h;\mathbb{R}^N) : \det D\Phi > 0 \text{ in } \Omega_h^{\#},\ D\Phi(x+e_i, y) = D\Phi(x,y) \\ &\qquad\qquad \text{for } (x,y) \in \Omega_h^{\#} \text{ and } i = 1,\dots,N-1\big\},\\ B &:= \{v \in \mathcal{V}(\Omega_h) \cap W^{2,p}(\Omega_h;\mathbb{R}^N) : \det(Dv + Du_0) > 0 \text{ in } \Omega_h\}, \end{aligned}$$

both equipped with the norm $\|\cdot\|_{W^{2,p}(\Omega_h;\mathbb{R}^N)}$ (notice that the pointwise conditions on the determinants in the definition of the spaces A and B make sense thank to the embedding of $W^{2,p}$ in $C^{1,\alpha}$). Observing that, for $(\Phi, v) \in A\times B$, the map $z \mapsto Q_\Phi(z, Dv(z))$ is of class $W^{1,p}$ in Ω_h (here Q_Φ is defined as in (3.55) with Φ_g replaced by Φ), we introduce the spaces

$$\begin{aligned} Y_1 &:= \big\{f \in L^p(\Omega_h;\mathbb{R}^N) : f(x+e_i, y) = f(x,y) \text{ for a.e. } (x,y) \in \Omega_h^{\#} \text{ and } i = 1,\dots,N-1\big\},\\ Y_2 &:= \big\{\eta \in W^{1-\frac1p,p}(\Gamma_h;\mathbb{R}^N) : \eta(x+e_i, h(x+e_i)) = \eta(x, h(x)) \text{ for a.e. } x \in \mathbb{R}^{N-1}\big\}, \end{aligned}$$

and the map $G : A\times B \to Y_1\times Y_2$ defined as

$$G(\Phi, v) := \Big(\operatorname{div}\big[Q_\Phi(\cdot, Dv(\cdot))\big], Q_\Phi(\cdot, Dv(\cdot))[\nu]\Big).$$

It can be checked that G is a map of class C^1, and $G(Id, u-u_0) = (0,0)$ (as u solves (3.45)). In order to apply the Implicit Function Theorem, we need to verify that the partial derivative $\partial_v G(Id, u-u_0)$ is an invertible bounded linear operator. Since for every $v \in \mathcal{V}(\Omega_h)\cap W^{2,p}(\Omega_h;\mathbb{R}^N)$

$$\begin{aligned} \partial_v G(Id, u-u_0)[v] &= \Big(\operatorname{div}\left[W_{\xi\xi}(Du)Dv\right], \big(W_{\xi\xi}(Du)Dv\big)[\nu]\Big)\\ &= \big(\operatorname{div}\left[C_u Dv\right], (C_u Dv)[\nu]\big), \end{aligned}$$

the invertibility of the operator $\partial_v G(Id, u-u_0)$ corresponds to prove existence and uniqueness in $\mathcal{V}(\Omega_h)\cap W^{2,p}(\Omega_h;\mathbb{R}^N)$ of solutions to the problem

$$\begin{cases} \operatorname{div}[C_u Dv] = f & \text{in } \Omega_h^{\#}, \\ (C_u Dv)[\nu] = \eta & \text{on } \Gamma_h^{\#}, \end{cases}$$

for any given $(f,\eta)\in Y_1\times Y_2$. The proof of this fact relies on the regularity theory for elliptic systems with mixed boundary conditions, and in particular on the regularity estimates of Agmon, Douglis and Nirenberg (see [2, Theorem 10.5]), which can be applied thank to the assumption (3.52), which is equivalent to the three conditions (H1)–(H3) by Theorem 3.28, and to the regularity of h and u (we refer also to [81] for a clear presentation of the theory in the context of linear elasticity).

We are now in position to apply the Implicit Function Theorem: there exist a neighborhood $\mathcal{V}$ of Id in A, a neighborhood $\mathcal{W}$ of $u-u_0$ in B and a map

$$\Phi\in\mathcal{V}\longmapsto u_\Phi\in\mathcal{W}$$

of class C^1 such that $u_{Id}=u-u_0$ and $G(\Phi,u_\Phi)=(0,0)$ for all $\Phi\in\mathcal{V}$. Finally, thank to (3.53), we can determine a neighborhood $\mathcal{U}$ of h in $W^{2,p}_{\#}(Q)$ such that if $g\in\mathcal{U}$ then $\Phi_g\in\mathcal{V}$. Setting $u_g:=(u_{\Phi_g}+u_0)\circ\Phi_g^{-1}$ for any $g\in\mathcal{U}$, we obtain the conclusion of the proposition. $\square$

REMARK 3.30. From the proof of the previous proposition it follows in particular that there exists a compact set $K\subset\mathbb{M}^N_+$ such that

$$Du_g(z)\in K \qquad \text{for every } g\in\mathcal{U} \text{ and } z\in\overline{\Omega}_g.$$

We conclude this section by showing that the critical points u_g constructed in Proposition 3.29 are also local minimizers of the elastic energy, in the sense of Definition 3.26.

PROPOSITION 3.31. *Let $\mathcal{U}$ be as in Proposition 3.29. There exist $\delta>0$ and $\varepsilon>0$ such that, if $g\in\mathcal{U}$ and $\|g-h\|_{W^{2,p}(Q)}<\varepsilon$, then u_g is a strict δ-local minimizer for the elastic energy in Ω_g, according to Definition 3.26.*

PROOF. We start by observing that, if $g\in\mathcal{U}$ and $\|g-h\|_{W^{2,p}(Q)}<\varepsilon$, then from (3.52) and from the smoothness of the map $g\mapsto u_g\circ\Phi_g$ one can easily deduce that

$$\int_{\Omega_g} C_{u_g}Dw : Dw \,dz > \frac{c_0}{4}\|w\|^2_{H^1(\Omega_g;\mathbb{R}^N)} \tag{3.56}$$

for every $w\in\mathcal{V}(\Omega_g)$, provided $\varepsilon>0$ is small enough.

Let now $w\in\mathcal{V}(\Omega_g)$ satisfy $0<\|Dw\|_{L^\infty(\Omega_g;\mathbb{M}^N)}\le\delta$, with $\delta>0$ to be chosen. We set

$$f(t):=\int_{\Omega_g} W(Du_g+tDw)\,dz, \quad t\in[0,1].$$

Notice that, since u_g is a critical point, $f'(0)=0$. Hence, there exists $\tau\in(0,1)$ such that

$$\begin{aligned}\int_{\Omega_g} W(Du_g+Dw)\,dz &= f(1) = f(0)+\frac{f''(\tau)}{2} \\ &= \int_{\Omega_g} W(Du_g)\,dz + \frac12\int_{\Omega_g} C_{u_g}[Dw,Dw]\,dz \\ &\quad + \frac12\int_{\Omega_g}\left(W_{\xi\xi}(Du_g+\tau Dw)-W_{\xi\xi}(Du_g)\right)[Dw,Dw]\,dz \\ &\ge \int_{\Omega_g} W(Du_g)\,dz + \Big(\frac{c_0}{8}-\omega(\delta)\Big)\|w\|^2_{H^1(\Omega_g;\mathbb{R}^N)}, \end{aligned} \tag{3.57}$$

where we used (3.56) and we set

$$\omega(\delta) := \max\Big\{\|W_{\xi\xi}(A+\tau B) - W_{\xi\xi}(A)\|_\infty : A \in K,\, B \in \mathbb{M}^N,\, |B| \le \delta,\, 0 \le \tau \le 1\Big\},$$

with K as in Remark 3.30. Note that $\omega(\delta) \to 0$ as $\delta \to 0^+$. Therefore, choosing δ so small that $\omega(\delta) < \frac{c_0}{8}$ it follows from (3.57) that u_g is a strict δ-local minimizer. □

3.4. The second variation

The main result of this section is the explicit computation of the second variation of the functional F along volume-preserving deformations. Here and in the following we assume that $(h,u) \in X$ satisfies the assumptions of Proposition 3.29: $h \in C^2_\#(Q)$, $u \in C^2(\overline{\Omega}^\#_h;\mathbb{R}^N)$ is a critical point for the elastic energy in Ω_h, and condition (3.52) holds.

Given $\phi \in C^2_\#(Q)$ with $\int_Q \phi\,\mathrm{d}x = 0$, for $t \in \mathbb{R}$ we set $h_t := h + t\phi$. According to Proposition 3.29, for t so small that $h_t \in \mathcal{U}$ we may consider a critical point u_{h_t} for the elastic energy in Ω_{h_t}. To simplify the notation, we set $u_t := u_{h_t}$. We define the *second variation of F at (h,u) along the direction ϕ* to be the value of

$$\frac{d^2}{dt^2}\left[F(h_t,u_t)\right]|_{t=0}.$$

We remark that the existence of the derivative is guaranteed by the regularity result contained in Proposition 3.29 (see the first step of the proof of Theorem 3.32).

As usual, for any one-parameter family of functions $\{g_t\}_{t\in\mathbb{R}}$ we denote by $\dot g_t(z)$ the partial derivative with respect to t of the function $(t,z) \mapsto g_t(z)$. We omit the subscript when $t=0$. In particular we let

$$\dot u_t := \frac{\partial u_t}{\partial t}, \qquad \dot u := \frac{\partial u_t}{\partial t}\Big|_{t=0}.$$

We introduce also the following subspace of $H^1(\Gamma_h)$:

$$\widetilde{H}^1_\#(\Gamma_h) := \Big\{\vartheta \in H^1_{\mathrm{loc}}(\Gamma^\#_h) : \vartheta(x+e_i, h(x+e_i)) = \vartheta(x,h(x)) \text{ for a.e. } x \in \mathbb{R}^{N-1}$$
$$\text{and for every } i = 1,\dots,N-1,\ \int_{\Gamma_h} \vartheta\,\mathrm{d}\mathcal{H}^{N-1} = 0\Big\},$$

and we define $\varphi \in \widetilde{H}^1_\#(\Gamma_h)$ to be

$$\varphi := \frac{\phi}{\sqrt{1+|\nabla h|^2}} \circ \pi,$$

where $\pi : \mathbb{R}^N \to \mathbb{R}^{N-1}$ is the projection on the hyperplane spanned by $e_1,\dots,e_{N-1}$. Denote also by ν_t the outer unit normal vector to Ω_{h_t} on Γ_{h_t}, and by $H^\psi_t := \operatorname{div}(\nabla\psi \circ \nu_t)$ the anisotropic curvature of Γ_{h_t}. It will be convenient to consider, as we did before, a family of diffeomorphisms $\Phi_t : \overline{\Omega}_h \to \overline{\Omega}_{h_t}$ of class C^2 such that $\Phi_0 = Id$ and $\Phi_t(x,y) = (x, y + t\phi(x))$ in a neighborhood of Γ_h (see Remark 3.24).

In the following theorem we deduce an explicit expression of the second variation.

THEOREM 3.32. *Let (h,u), ϕ, φ and (h_t,u_t) be as above. Then the function $\dot u$ belongs to $\mathcal{V}(\Omega_h)$ and satisfies the equation*

$$\int_{\Omega_h} \mathbb{C}_u D\dot u : Dw\,\mathrm{d}z = \int_{\Gamma_h} \operatorname{div}_{\Gamma_h}(\varphi\, W_\xi(Du))\cdot w\,\mathrm{d}\mathcal{H}^{N-1} \qquad \textit{for all } w \in \widetilde{\mathcal{V}}(\Omega_h). \tag{3.58}$$

Moreover, the second variation of F at (h,u) along the direction ϕ is given by

$$\begin{aligned}\frac{d^2}{dt^2}F(h_t,u_t)|_{t=0} = &-\int_{\Omega_h} C_u D\dot{u} : D\dot{u}\,\mathrm{d}z + \int_{\Gamma_h}(D^2\psi\circ\nu)[\nabla_{\Gamma_h}\varphi,\nabla_{\Gamma_h}\varphi]\,\mathrm{d}\mathcal{H}^{N-1}\\ &+\int_{\Gamma_h}\big(\partial_\nu(W\circ Du) - \operatorname{trace}(\mathbf{B}^\psi\mathbf{B})\big)\,\varphi^2\,\mathrm{d}\mathcal{H}^{N-1}\\ &-\int_{\Gamma_h}\big(W\circ Du + H^\psi\big)\operatorname{div}_{\Gamma_h}\left[\left(\frac{(\nabla h,|\nabla h|^2)}{\sqrt{1+|\nabla h|^2}}\circ\pi\right)\varphi^2\right]\mathrm{d}\mathcal{H}^{N-1},\end{aligned} \tag{3.59}$$

where H^ψ, $\mathbf{B}$ and $\mathbf{B}^\psi$ are the anisotropic mean curvature, the second fundamental form and the anisotropic second fundamental form of Γ_h, respectively (see (1.4) and (1.6)).

Before proving the theorem, we collect in the following lemma some identities that will be used in the computation of the second variation.

Lemma 3.33. *The following identities are satisfied on Γ_h:*

(a) $\partial_\nu H^\psi = -\operatorname{trace}\big(\mathbf{B}^\psi\mathbf{B}\big) = -\operatorname{trace}\big(\mathbf{B}^2(D^2\psi\circ\nu)\big)$;

(b) $\dot{\nu} = -\nabla_{\Gamma_h}\varphi$;

(c) $\dot{H}^\psi = \operatorname{div}_{\Gamma_h}\big((D^2\psi\circ\nu)[\dot{\nu}]\big) = -\operatorname{div}_{\Gamma_h}\big((D^2\psi\circ\nu)[\nabla_{\Gamma_h}\varphi]\big)$.

Proof. Recalling that $D\nu[\nu]=0$, we easily deduce that $D(\nabla\psi\circ\nu)[\nu]=0$. By differentiating,

$$\partial_\nu\left(D\left(\nabla\psi\circ\nu\right)\right) = -D\left(\nabla\psi\circ\nu\right)D\nu = -\mathbf{B}^\psi\mathbf{B},$$

and from this we obtain (a), since

$$\begin{aligned}\partial_\nu H^\psi &= \partial_\nu\left[\operatorname{div}(\nabla\psi\circ\nu)\right] = \partial_\nu\left[\operatorname{trace}\left(D\left(\nabla\psi\circ\nu\right)\right)\right]\\ &= \operatorname{trace}\left[\partial_\nu\left(D\left(\nabla\psi\circ\nu\right)\right)\right] = -\operatorname{trace}\left[\mathbf{B}^\psi\mathbf{B}\right].\end{aligned}$$

Let us prove (b). Differentiating with respect to t the identity

$$\nu_t\circ\Phi_t = \frac{(-\nabla h_t,1)}{\sqrt{1+|\nabla h_t|^2}}\circ\pi \qquad \text{on } \Gamma_h,$$

and evaluating the result at $t=0$, we get that on Γ_h holds

$$\begin{aligned}\dot{\nu}+(\phi\circ\pi)\partial_y\nu &= \left(\frac{-\nabla\phi}{\sqrt{1+|\nabla h|^2}}+\frac{(\nabla h\cdot\nabla\phi)\nabla h}{(1+|\nabla h|^2)^{\frac32}},\frac{-\nabla h\cdot\nabla\phi}{(1+|\nabla h|^2)^{\frac32}}\right)\circ\pi\\ &= \frac{(-\nabla\phi,0)}{\sqrt{1+|\nabla h|^2}}\circ\pi - \left(\frac{\nabla h\cdot\nabla\phi}{1+|\nabla h|^2}\circ\pi\right)\nu\\ &= \left(-\frac{1}{\sqrt{1+|\nabla h|^2}}\circ\pi\right)\Big[\nabla(\phi\circ\pi)-\big(\nabla(\phi\circ\pi)\cdot\nu\big)\nu\Big]\\ &= \left(-\frac{1}{\sqrt{1+|\nabla h|^2}}\circ\pi\right)\nabla_{\Gamma_h}(\phi\circ\pi).\end{aligned}$$

Hence, using the identity

$$\partial_y\nu = \nabla_{\Gamma_h}\left(\frac{1}{\sqrt{1+|\nabla h|^2}}\circ\pi\right) \qquad \text{on } \Gamma_h,$$

we finally get

$$
\begin{aligned}
\dot{\nu} &= -\left(\frac{1}{\sqrt{1+|\nabla h|^2}}\circ\pi\right)\nabla_{\Gamma_h}(\phi\circ\pi) - \nabla_{\Gamma_h}\left(\frac{1}{\sqrt{1+|\nabla h|^2}}\circ\pi\right)(\phi\circ\pi) \\
&= -\nabla_{\Gamma_h}\left(\frac{\phi}{\sqrt{1+|\nabla h|^2}}\circ\pi\right) = -\nabla_{\Gamma_h}\varphi,
\end{aligned}
$$

that is (b).

Let us prove (c). Differentiating in the direction ν the identity $(D^2\psi\circ\nu)[\nu,\dot{\nu}]=0$ (which follows by (3.50)), we obtain

$$
\nu\cdot\partial_\nu\left((D^2\psi\circ\nu)[\dot{\nu}]\right) = -(D^2\psi\circ\nu)[\dot{\nu},\partial_\nu\nu]=0,
$$

where we recall that $\partial_\nu\nu=0$. Hence

$$
\begin{aligned}
\dot{H}^\psi = \frac{\partial}{\partial t}H_t^\psi|_{t=0} &= \frac{\partial}{\partial t}\left[\operatorname{div}(\nabla\psi\circ\nu_t)\right]|_{t=0} = \operatorname{div}\left((D^2\psi\circ\nu)[\dot{\nu}]\right) \\
&= \operatorname{div}_{\Gamma_h}\left((D^2\psi\circ\nu)[\dot{\nu}]\right) + \nu\cdot\partial_\nu\left((D^2\psi\circ\nu)[\dot{\nu}]\right) \\
&= \operatorname{div}_{\Gamma_h}\left((D^2\psi\circ\nu)[\dot{\nu}]\right) = -\operatorname{div}_{\Gamma_h}\left((D^2\psi\circ\nu)[\nabla_{\Gamma_h}\varphi]\right),
\end{aligned}
$$

where in the last equality we used (b). $\square$

We are now ready to perform the computation of the second variation of the functional.

Proof of Theorem 3.32. We divide the proof into several steps.

Step 1. We claim that the regularity property stated in Proposition 3.29-(iii) guarantees that the map $(t,z)\mapsto w_t(z):=u_t\circ\Phi_t(z)$ is of class C^1 in $(-\varepsilon_0,\varepsilon_0)\times\overline{\Omega}_h$ for some ε_0 small enough.

Indeed, denoting by w'_{t_0} the derivative of the map $t\mapsto w_t$ with respect to the $W^{2,p}$-norm, evaluated at some t_0 (small), we have that

$$
\tfrac{1}{s}\left(w_{t_0+s}-w_{t_0}\right)\to w'_{t_0} \qquad \text{in } W^{2,p}(\Omega_h), \text{ as } s\to 0. \tag{3.60}
$$

In particular, $w_{t_0+s}\to w_{t_0}$ in $C^1(\overline{\Omega}_h)$ as $s\to 0$, showing that the map $(t,z)\mapsto Dw_t(z)$ is continuous in $(-\varepsilon_0,\varepsilon_0)\times\overline{\Omega}_h$. Moreover, (3.60) implies that $w'_{t_0}=\dot{w}_{t_0}$, and the continuity of $t\mapsto w'_t$ yields $\dot{w}_{t_0+s}\to\dot{w}_{t_0}$ in $C^0(\overline{\Omega}_h)$ as $s\to 0$, showing that the map $(t,z)\mapsto\dot{w}_t(z)$ is continuous in $(-\varepsilon_0,\varepsilon_0)\times\overline{\Omega}_h$. The claim follows.

This provides a justification to all the differentiations that will be performed throughout the proof. Moreover, it is also easily seen that $\dot{u}_t\in\mathcal{V}(\Omega_{h_t})$ for $t\in(-\varepsilon_0,\varepsilon_0)$.

Step 2. We prove (3.58). Let us recall that u_t satisfies equation (3.45):

$$
\int_{\Omega_{h_t}} W_\xi(Du_t):Dw\,\mathrm{d}z = 0 \qquad \text{for every } w\in\mathcal{V}(\Omega_{h_t}). \tag{3.61}
$$

Fix $w\in\mathcal{V}(\Omega_h)$. Then w may be extended outside Ω_h in such a way that $w\in\mathcal{V}(\Omega_{h_t})$ for t small. We can differentiate (3.61) with respect to t and evaluate the result at $t=0$ to obtain

$$
\begin{aligned}
0 &= \int_{\Omega_h} C_u D\dot{u}:Dw\,\mathrm{d}z + \int_Q \phi(x)\left[W_\xi(Du):Dw\right]|_{(x,h(x))}\,\mathrm{d}x \\
&= \int_{\Omega_h} C_u D\dot{u}:Dw\,\mathrm{d}z + \int_{\Gamma_h}\varphi\, W_\xi(Du):Dw\,\mathrm{d}\mathcal{H}^{N-1}.
\end{aligned} \tag{3.62}
$$

Recalling that $W_\xi(Du)[\nu] = 0$ along Γ_h, the second integral in the above formula can be rewritten as

$$\begin{aligned}\int_{\Gamma_h} \varphi\, W_\xi(Du) : Dw \,\mathrm{d}\mathcal{H}^{N-1} &= \int_{\Gamma_h} \varphi\, W_\xi(Du) : D_{\Gamma_h} w \,\mathrm{d}\mathcal{H}^{N-1} \\ &= -\int_{\Gamma_h} \mathrm{div}_{\Gamma_h}(\varphi\, W_\xi(Du)) \cdot w \,\mathrm{d}\mathcal{H}^{N-1}.\end{aligned}$$

This concludes the proof of (3.58).

Step 3. We compute the first variation. By the positive one-homogeneity of ψ we have on Γ_{h_t}

$$\psi(\nu_t) = \psi\left(\frac{(-\nabla h_t, 1)}{\sqrt{1+|\nabla h_t|^2}} \circ \pi\right) = \frac{\psi\left((-\nabla h_t, 1)\right)}{\sqrt{1+|\nabla h_t|^2}} \circ \pi.$$

Hence,

$$\begin{aligned}\frac{d}{dt} F(h_t, u_t) &= \frac{d}{dt}\left[\int_Q \int_0^{h_t} W(Du_t)\,\mathrm{d}y\,\mathrm{d}x + \int_Q \psi((-\nabla h_t, 1))\,\mathrm{d}x\right] \\ &= \int_Q \phi(x)\,[W(Du_t)]\,|_{(x,h_t(x))}\,\mathrm{d}x + \int_Q \int_0^{h_t} W_\xi(Du_t) : D\dot{u}_t\,\mathrm{d}y\,\mathrm{d}x \\ &\quad - \int_Q \nabla\psi((-\nabla h_t, 1)) \cdot (\nabla\phi, 0)\,\mathrm{d}x.\end{aligned}$$

Since $\dot{u}_t \in \mathcal{V}(\Omega_{h_t})$ the second integral vanishes by (3.61). Then, integrating by parts in the last integral and recalling the expression for the anisotropic mean curvature provided by (3.51), we obtain

$$\frac{d}{dt} F(h_t, u_t) = \int_Q \phi(x)\big[W(Du_t) + H_t^\psi\big]|_{(x,h_t(x))}\,\mathrm{d}x. \tag{3.63}$$

Step 4. We finally pass to the second variation. Differentiating (3.63) with respect to t and evaluating the result at $t = 0$ we get

$$\begin{aligned}\frac{d^2}{dt^2} F(h_t, u_t)|_{t=0} &= \int_Q \phi(x)\,[W_\xi(Du) : D\dot{u}]\,|_{(x,h(x))}\,\mathrm{d}x + \int_Q \phi(x) \dot{H}^\psi|_{(x,h(x))}\,\mathrm{d}x \\ &\quad + \int_Q \phi(x)\left[\nabla(W \circ Du + H^\psi)\right]|_{(x,h(x))} \cdot (0, \phi(x))\,\mathrm{d}x \\ &=: I_1 + I_2 + I_3.\end{aligned}$$

Since $\dot{u} \in \mathcal{V}(\Omega_h)$, thanks to (3.62) the first integral is

$$I_1 = -\int_{\Omega_h} C_u D\dot{u} : D\dot{u}\,\mathrm{d}z.$$

For the second integral, changing variables, using identity (c) of Lemma 3.33 and integrating by parts we get

$$I_2 = -\int_{\Gamma_h} \varphi\, \mathrm{div}_{\Gamma_h}\left((D^2\psi \circ \nu)[\nabla_{\Gamma_h}\varphi]\right)\,\mathrm{d}\mathcal{H}^{N-1} = \int_{\Gamma_h} (D^2\psi \circ \nu)[\nabla_{\Gamma_h}\varphi, \nabla_{\Gamma_h}\varphi]\,\mathrm{d}\mathcal{H}^{N-1}.$$

To conclude, we observe that along Γ_h the vector $(0, \varphi)$ can be decomposed as

$$(0, \varphi) = (0, \varphi)_{\Gamma_h} + (0, \varphi)_\nu,$$

with $(0,\varphi)_{\Gamma_h}$ tangent to Γ_h and $(0,\varphi)_\nu$ parallel to ν, *i.e.*,

$$(0,\varphi)_{\Gamma_h} = \varphi\left[\frac{(\nabla h, |\nabla h|^2)}{1+|\nabla h|^2}\circ\pi\right], \qquad (0,\varphi)_\nu = \varphi\left[\frac{(-\nabla h, 1)}{1+|\nabla h|^2}\circ\pi\right].$$

Hence, recalling the definition of φ, changing variables in I_3 and integrating by parts:

$$\begin{aligned}
I_3 &= \int_{\Gamma_h} \varphi\,\nabla(W\circ Du + H^\psi)\cdot(0,\varphi)\left(\sqrt{1+|\nabla h|^2}\circ\pi\right)\mathrm{d}\mathcal{H}^{N-1}\\
&= \int_{\Gamma_h} \varphi^2\,\nabla_{\Gamma_h}(W\circ Du + H^\psi)\cdot\left(\frac{(\nabla h, |\nabla h|^2)}{\sqrt{1+|\nabla h|^2}}\circ\pi\right)\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad + \int_{\Gamma_h} \varphi^2\,\partial_\nu(W\circ Du + H^\psi)\,\mathrm{d}\mathcal{H}^{N-1}\\
&= -\int_{\Gamma_h} (W\circ Du + H^\psi)\,\mathrm{div}_{\Gamma_h}\left[\left(\frac{(\nabla h, |\nabla h|^2)}{\sqrt{1+|\nabla h|^2}}\circ\pi\right)\varphi^2\right]\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad + \int_{\Gamma_h} \varphi^2\left[\partial_\nu(W\circ Du) - \mathrm{trace}\,(\mathbf{B}^\psi\mathbf{B})\right]\mathrm{d}\mathcal{H}^{N-1},
\end{aligned}$$

where in the last equality we used identity (a) of Lemma 3.33. □

REMARK 3.34. For a fixed $s\in\mathbb{R}$ sufficiently small, we deduce from Theorem 3.32 that

$$\begin{aligned}
\frac{d^2}{dt^2}F(h_t,u_t)|_{t=s} &= \frac{d^2}{dt^2}F(h_{s+t},u_{s+t})|_{t=0}\\
&= -\int_{\Omega_{h_s}} C_{u_s}D\dot u_s : D\dot u_s\,\mathrm{d}z + \int_{\Gamma_{h_s}}(D^2\psi\circ\nu_s)[\nabla_{\Gamma_{h_s}}\varphi_s, \nabla_{\Gamma_{h_s}}\varphi_s]\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad + \int_{\Gamma_{h_s}}\left(\partial_{\nu_s}(W\circ Du_s) - \mathrm{trace}\,(\mathbf{B}_s^\psi\mathbf{B}_s)\right)\varphi_s^2\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad - \int_{\Gamma_{h_s}}(W\circ Du_s + H_s^\psi)\,\mathrm{div}_{\Gamma_{h_s}}\left[\left(\frac{(\nabla h_s, |\nabla h_s|^2)}{\sqrt{1+|\nabla h_s|^2}}\circ\pi\right)\varphi_s^2\right]\mathrm{d}\mathcal{H}^{N-1},
\end{aligned}$$

where $\varphi_s := \frac{\phi}{\sqrt{1+|\nabla h_s|^2}}\circ\pi\in\widetilde H^1_\#(\Gamma_{h_s})$, $\mathbf{B}_s := D\nu_s$ and $\mathbf{B}_s^\psi := D(\nabla\psi\circ\nu_s)$. Moreover, the function $\dot u_s$ belongs to $\mathcal{V}(\Omega_{h_s})$ and satisfies the equation

$$\int_{\Omega_{h_s}} C_{u_s}D\dot u_s : Dw\,\mathrm{d}z = \int_{\Gamma_{h_s}}\mathrm{div}_{\Gamma_{h_s}}(\varphi_s\,W_\xi(Du_s))\cdot w\,d\mathcal{H}^{N-1} \qquad \text{for all } w\in\widetilde{\mathcal{V}}(\Omega_{h_s}).$$

3.5. The stability condition

The expression of the second variation at a critical pair (see Definition 3.22) simplifies, as the last integral in (3.59) vanishes by the divergence formula. This observation suggests to associate with every critical pair $(h,u)\in X$ a quadratic form $\partial^2F(h,u) : \widetilde H^1_\#(\Gamma_h)\to\mathbb{R}$ defined as

$$\begin{aligned}
\partial^2F(h,u)[\varphi] := &-\int_{\Omega_h} C_u Dv_\varphi : Dv_\varphi\,\mathrm{d}z + \int_{\Gamma_h}(D^2\psi\circ\nu)[\nabla_{\Gamma_h}\varphi, \nabla_{\Gamma_h}\varphi]\,\mathrm{d}\mathcal{H}^{N-1}\\
&+ \int_{\Gamma_h}\left(\partial_\nu(W\circ Du) - \mathrm{trace}\,(\mathbf{B}^\psi\mathbf{B})\right)\varphi^2\,\mathrm{d}\mathcal{H}^{N-1},
\end{aligned} \tag{3.64}$$

where $v_\varphi \in \widetilde{\mathcal{V}}(\Omega_h)$ is the unique solution to

$$\int_{\Omega_h} C_u Dv_\varphi : Dw \,\mathrm{d}z = \int_{\Gamma_h} \mathrm{div}_{\Gamma_h}(\varphi W_\xi(Du)) \cdot w \,\mathrm{d}\mathcal{H}^{N-1} \qquad \text{for every } w \in \widetilde{\mathcal{V}}(\Omega_h). \tag{3.65}$$

It is easily seen that the positivity of the quadratic form (3.64) is a necessary condition for local minimality: this is made precise by the following theorem.

THEOREM 3.35. *Let $(h,u) \in X$, with $h \in C^2_\#(Q)$ and $u \in C^2(\overline{\Omega}^\#_h;\mathbb{R}^N)$, be a local minimizer for F, according to Definition 3.23, and assume in addition that u satisfies (3.52). Then the quadratic form (3.64) is positive semidefinite,* i.e.,

$$\partial^2 F(h,u)[\varphi] \geq 0 \qquad \textit{for every } \varphi \in \widetilde{H}^1_\#(\Gamma_h).$$

PROOF. Given any $\varphi \in \widetilde{H}^1_\#(\Gamma_h) \cap C^\infty(\Gamma^\#_h)$, we can consider the deformation $h_t = h + t\phi$, where $\phi(x) = (1 + |\nabla h(x)|^2)^{\frac{1}{2}}\, \varphi(x, h(x))$, and, for t small, the corresponding critical points for the elastic energy u_{h_t}. It follows from equation (3.59) and from the local minimality of (h,u) (which is in particular a critical pair) that

$$\partial^2 F(h,u)[\varphi] = \frac{d^2}{dt^2} F(h_t, u_{h_t})|_{t=0} \geq 0.$$

For a general φ the result follows by approximation with functions in $\widetilde{H}^1_\#(\Gamma_h) \cap C^\infty(\Gamma^\#_h)$ (observe that $\partial^2 F(h,u)$ is continuous with respect to strong convergence in H^1). ☐

DEFINITION 3.36. Let $(h,u) \in X$ be a critical pair for the functional F, according to Definition 3.22. We say that (h,u) is *strictly stable* if the elastic second variation is uniformly positive at u in Ω_h (see Definition 3.27) and in addition

$$\partial^2 F(h,u)[\varphi] > 0 \qquad \text{for every } \varphi \in \widetilde{H}^1_\#(\Gamma_h) \setminus \{0\}. \tag{3.66}$$

Our main result (Theorem 3.45) states that a strictly stable critical pair is a local minimizer for F, according to Definition 3.23. This will be proved in Sections 3.6 and 3.7, while we now focus on condition (3.66) providing two equivalent formulations.

Given a critical pair $(h,u) \in X$ satisfying (3.52), we define the bilinear form on $\widetilde{H}^1_\#(\Gamma_h)$

$$(\varphi, \vartheta)_\sim := \int_{\Gamma_h} (D^2\psi \circ \nu)[\nabla_{\Gamma_h}\varphi, \nabla_{\Gamma_h}\vartheta] \,\mathrm{d}\mathcal{H}^{N-1} + \int_{\Gamma_h} a\, \varphi\, \vartheta \,\mathrm{d}\mathcal{H}^{N-1} \tag{3.67}$$

for $\varphi, \vartheta \in \widetilde{H}^1_\#(\Gamma_h)$, where $a := \partial_\nu (W \circ Du) - \mathrm{trace}\,(\mathbf{B}^\psi \mathbf{B})$ on Γ_h. Arguing as in Proposition 2.20, one can show that if

$$(\varphi, \varphi)_\sim > 0 \qquad \text{for every } \varphi \in \widetilde{H}^1_\#(\Gamma_h) \setminus \{0\}, \tag{3.68}$$

then $(\cdot,\cdot)_\sim$ is a scalar product which defines an equivalent norm on $\widetilde{H}^1_\#(\Gamma_h)$, denoted by $\|\cdot\|_\sim$. We omit the proof also of the following result, since it can be deduced by repeating the proof of Proposition 2.21 (see also [45, Proposition 3.6]).

THEOREM 3.37. *The following statement are equivalent.*

(i) Condition (3.66) holds.

(ii) Condition (3.68) is satisfied and $T : \widetilde{H}^1_\#(\Gamma_h) \to \widetilde{H}^1_\#(\Gamma_h)$, defined by duality as

$$(T\varphi, \vartheta)_\sim := \int_{\Gamma_h} \mathrm{div}_{\Gamma_h}(\vartheta\, W_\xi(Du)) \cdot v_\varphi \,\mathrm{d}\mathcal{H}^{N-1} = \int_{\Omega_h} C_u Dv_\varphi : Dv_\vartheta \,\mathrm{d}z \tag{3.69}$$

for every $\varphi, \vartheta \in \widetilde{H}^1_\#(\Gamma_h)$*, is a compact, monotone, self-adjoint linear operator such that*

$$\lambda_1 < 1, \qquad \text{where } \lambda_1 := \max_{\|\varphi\|_\sim = 1} (T\varphi, \varphi)_\sim. \tag{3.70}$$

(iii) Condition (3.68) is satisfied and defining, for $v \in \widetilde{\mathcal{V}}(\Omega_h)$*,* Φ_v *to be the unique solution in* $\widetilde{H}^1_\#(\Gamma_h)$ *to the equation*

$$(\Phi_v, \vartheta)_\sim = \int_{\Gamma_h} \mathrm{div}_{\Gamma_h}(\vartheta\, W_\xi(Du)) \cdot v \, d\mathcal{H}^{N-1} \qquad \text{for every } \vartheta \in \widetilde{H}^1_\#(\Gamma_h),$$

we have

$$\mu_1 := \min\left\{ \int_{\Omega_h} C_u Dv : Dv \, dz \ : \ v \in \widetilde{\mathcal{V}}(\Omega_h), \ \|\Phi_v\|_\sim = 1 \right\} > 1. \tag{3.71}$$

Remark 3.38. We remark that, by definition of T, we have

$$\partial^2 F(h,u)[\varphi] = \|\varphi\|^2_\sim - (T\varphi, \varphi)_\sim \qquad \text{for every } \varphi \in \widetilde{H}^1_\#(\Gamma_h). \tag{3.72}$$

Observe also that λ_1 coincides with the greatest λ such that the following system

$$\begin{cases} \lambda \int_{\Omega_h} C_u Dv : Dw = \int_{\Gamma_h} \mathrm{div}_{\Gamma_h}(\varphi\, W_\xi(Du)) \cdot w \, d\mathcal{H}^{N-1} & \text{for every } w \in \widetilde{\mathcal{V}}(\Omega_h), \\ (\varphi, \psi)_\sim = \int_{\Gamma_h} \mathrm{div}_{\Gamma_h}(\psi\, W_\xi(Du)) \cdot v \, d\mathcal{H}^{N-1} & \text{for every } \psi \in \widetilde{H}^1_\#(\Gamma_h) \end{cases} \tag{3.73}$$

admits a nontrivial solution $(v, \varphi) \in \widetilde{\mathcal{V}}(\Omega_h) \times \widetilde{H}^1_\#(\Gamma_h)$: in fact, λ is an eigenvalue of T with eigenfunction φ if and only if the pair $(\frac{v_\varphi}{\lambda}, \varphi)$ is a nontrivial solution to (3.73).

Corollary 3.39. *If (3.66) holds, then* $\partial^2 F(h,u)$ *is uniformly positive: that is, there exists a constant* $C > 0$ *such that*

$$\partial^2 F(h,u)[\varphi] \geq C \|\varphi\|^2_{H^1(\Gamma_h)} \qquad \text{for every } \varphi \in \widetilde{H}^1_\#(\Gamma_h).$$

Proof. By (3.72), recalling that $\|\cdot\|_\sim$ is an equivalent norm on $\widetilde{H}^1_\#(\Gamma_h)$ and that $\lambda_1 < 1$ we have

$$\partial^2 F(h,u)[\varphi] = \|\varphi\|^2_\sim - (T\varphi, \varphi)_\sim \geq (1 - \lambda_1)\|\varphi\|^2_\sim \geq C\|\varphi\|^2_{H^1(\Gamma_h)},$$

which is the conclusion. □

3.6. Local $W^{2,p}$-minimality

In this section we prove the first part of the main result, namely that the strict stability of a critical pair (h,u) is a sufficient condition for local minimality, in the following weaker sense:

Definition 3.40. Let $p \in [1, \infty)$. We say that a critical pair $(h,u) \in X$ is a *local* $W^{2,p}$*-minimizer* for F if there exists $\delta > 0$ such that

$$F(h,u) \leq F(g,v) \tag{3.74}$$

for all $(g,v) \in X$ with $0 < \|g - h\|_{W^{2,p}(Q)} < \delta$, $|\Omega_g| = |\Omega_h|$, and $\|Dv - Du\|_{L^\infty(\Omega_g; \mathbb{M}^N)} < \delta$. We say that (h,u) is an *isolated local* $W^{2,p}$*-minimizer* if the inequality in (3.74) is strict when $g \neq h$.

Theorem 3.41. *Let* $N = 2, 3$*, and let* $p > 2N$*. If* $(h,u) \in X$ *is a strictly stable critical pair for* F*, according to Definition 3.36, then* (h,u) *is an isolated local* $W^{2,p}$*-minimizer.*

As has been observed in [45], the main difficulty in proving Theorem 3.41 comes from the presence, in the expression of the quadratic form associated with the second variation, of the trace of the gradient of $W(Du)$ on Γ_h: the crucial estimate is provided by Lemma 3.42, where it is shown how to control this term in a proper Sobolev space of fractional order, uniformly with respect to small $W^{2,p}$-variations of the profile h (we refer to Section 1.4 for the definition and properties of fractional Sobolev spaces).

Let $\mathcal{U}_\delta := \{g \in C^\infty_\#(Q) : \|g-h\|_{W^{2,p}(Q)} < \delta,\ |\Omega_g| = |\Omega_h|\}$, where $\delta > 0$ is so small that $\mathcal{U}_\delta$ is contained in the neighborhood $\mathcal{U}$ of h determined by Proposition 3.29: this allows us to consider, for $g \in \mathcal{U}_\delta$, a critical point u_g for the elastic energy in Ω_g. We denote by c_0 a positive constant such that $g \geq 2c_0$ in Q for every $g \in \mathcal{U}_\delta$.

Lemma 3.42. *Under the assumptions of Theorem 3.41, we have that*

$$\sup_{g\in\mathcal{U}_\delta} \|\partial_{\nu_g}(W(Du_g))\circ\Phi_g - \partial_\nu(W(Du))\|_{W^{-\frac{1}{p},p}_\#(\Gamma_h)} \to 0 \quad \textit{as } \delta\to 0.$$

Proof. We set, for $g \in \mathcal{U}_\delta$, $v_g := u_g - u\circ\Psi_g$ (where $\Psi_g := \Phi_g^{-1}$), and we denote by v_g^i the i-th component of v_g. We remark that, by Proposition 3.29,

$$\sup_{g\in\mathcal{U}_\delta} \|v_g\|_{W^{2,p}(\Omega_g;\mathbb{R}^N)} \to 0 \qquad \text{as } \delta\to 0, \tag{3.75}$$

and moreover, since $p > 2N$, $u_g\circ\Phi_g \to u$ in $C^{1,\alpha}(\overline{\Omega}_h;\mathbb{R}^N)$ as $\delta\to 0$, for $\alpha = 1-\frac{N}{p}$, uniformly with respect to $g\in\mathcal{U}_\delta$.

Step 1. We start by observing that, using the equations satisfied by u and u_g and performing a change of variable, we get

$$\int_{\Omega_g} \big[W_\xi(Du_g) - W_\xi(D(u\circ\Psi_g))\big] : Dw\,\mathrm{d}z = \int_{\Omega_g} d_g : Dw\,\mathrm{d}z \qquad \text{for all } w\in\mathcal{V}(\Omega_g),$$

where $d_g := W_\xi(D(u\circ\Psi_g)(D\Psi_g)^{-1})(D\Psi_g)^{-T}\det D\Psi_g - W_\xi(D(u\circ\Psi_g))$. Observe in particular that, by using the explicit construction of the diffeomorphism Ψ_g (see Remark 3.24) and the regularity of u,

$$\sup_{g\in\mathcal{U}_\delta} \|d_g\|_{W^{1,p}(\Omega_g;\mathbb{M}^N)} \to 0, \qquad \sup_{g\in\mathcal{U}_\delta} \Big\|\frac{\partial d_g}{\partial z_k}\Big\|_{L^p(\Gamma_g;\mathbb{M}^N)} \to 0 \qquad \text{as } \delta\to 0. \tag{3.76}$$

Fix now any $\varphi \in W^{\frac{1}{p},\frac{p}{p-1}}_\#(\Gamma_g;\mathbb{R}^N)$, and consider an extension of φ (which we still denote by φ) such that $\varphi\in W^{1,\frac{p}{p-1}}_\#(\Omega_g;\mathbb{R}^N)$, φ vanishes in Ω_{g-c_0} and

$$\|\varphi\|_{W^{1,\frac{p}{p-1}}(\Omega_g;\mathbb{R}^N)} \leq C\|\varphi\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_g;\mathbb{R}^N)} \tag{3.77}$$

for some constant $C>0$ which can be chosen independently of $g\in\mathcal{U}_\delta$ (see Theorem 1.18).

Differentiating the equation $\mathrm{div}[W_\xi(Du_g) - W_\xi(D(u\circ\Psi_g))] = \mathrm{div} d_g$ with respect to z_k, multiplying by φ and integrating by parts on Ω_g we get

$$\begin{aligned}\int_{\Gamma_g} C_{u_g} D\Big(\frac{\partial v_g}{\partial z_k}\Big)[\nu_g]\cdot\varphi\,\mathrm{d}\mathcal{H}^{N-1} &= \int_{\Gamma_g}(C_{u\circ\Psi_g} - C_{u_g})D\Big(\frac{\partial(u\circ\Psi_g)}{\partial z_k}\Big)[\nu_g]\cdot\varphi\,\mathrm{d}\mathcal{H}^{N-1}\\ &\quad+\int_{\Omega_g}\Big[C_{u_g}D\Big(\frac{\partial u_g}{\partial z_k}\Big) - C_{u\circ\Psi_g}D\Big(\frac{\partial(u\circ\Psi_g)}{\partial z_k}\Big) - \frac{\partial d_g}{\partial z_k}\Big] : D\varphi\,\mathrm{d}z + \int_{\Gamma_g}\frac{\partial d_g}{\partial z_k}[\nu_g]\cdot\varphi\,\mathrm{d}\mathcal{H}^{N-1}\\ &\le C\bigg(\|C_{u\circ\Psi_g} - C_{u_g}\|_\infty\|D^2(u\circ\Psi_g)\|_{L^p(\Gamma_g)} + \|d_g\|_{W^{1,p}(\Omega_g;\mathbb{M}^N)} + \Big\|\frac{\partial d_g}{\partial z_k}\Big\|_{L^p(\Gamma_g;\mathbb{M}^N)}\\ &\qquad + \Big\|C_{u_g}D\Big(\frac{\partial u_g}{\partial z_k}\Big) - C_{u\circ\Psi_g}D\Big(\frac{\partial(u\circ\Psi_g)}{\partial z_k}\Big)\Big\|_{L^p(\Omega_g;\mathbb{M}^N)}\bigg)\|\varphi\|_{W^{\frac1p,\frac{p}{p-1}}(\Gamma_g;\mathbb{R}^N)},\end{aligned}$$

where we repeatedly used (3.77) (here the constant C is independent of $g \in \mathcal{U}_\delta$). Hence, recalling (3.75) and (3.76), we deduce that for $k = 1,\dots,N$

$$\sup_{g\in\mathcal{U}_\delta}\Big\|C_{u_g}D\Big(\frac{\partial v_g}{\partial z_k}\Big)[\nu_g]\Big\|_{W_\#^{-\frac1p,p}(\Gamma_g;\mathbb{R}^N)} \to 0 \qquad \text{as } \delta\to 0. \tag{3.78}$$

Step 2. We now claim that for $k = 1,\dots,N$

$$\sup_{g\in\mathcal{U}_\delta}\Big\|D\Big(\frac{\partial u}{\partial z_k}\Big) - D\Big(\frac{\partial u_g}{\partial z_k}\Big)\circ\Phi_g\Big\|_{W_\#^{-\frac1p,p}(\Gamma_h;\mathbb{M}^N)} \to 0 \qquad \text{as } \delta\to 0. \tag{3.79}$$

We first note that, thanks to the uniform convergence of $C_{u_g}\circ\Phi_g$ to C_u and to the strong ellipticity of C_u, also the tensors C_{u_g} are strongly elliptic for every $g\in\mathcal{U}_\delta$, if δ is sufficiently small; in particular, there exists a positive constant m_0 such that

$$C_{u_g}(z)\,a\otimes b : a\otimes b \ge m_0\,|a|^2\,|b|^2 \qquad \text{for every } a, b\in\mathbb{R}^N,$$

for every $z\in\overline{\Omega}_g$ and for every $g\in\mathcal{U}_\delta$. Hence the $N\times N$ matrix $Q_g(z)$, whose entries are defined by

$$q_{ih}(z) := \sum_{j,k=1}^N C_{ijhk}(z)\nu_g^j(z)\nu_g^k(z), \qquad i,h = 1,\dots,N \tag{3.80}$$

(C_{ijhk} denoting the components of the tensor C_{u_g}), is positive definite, and $\det Q_g(z)$ is uniformly positive with respect to $z\in\overline{\Omega}_g$ and $g\in\mathcal{U}_\delta$.

Setting, for $i,j,k = 1,\dots,N$,

$$\sigma_{ijk} := \frac{\partial^2 v_g^k}{\partial z_i\partial z_j},$$

by Lemma 1.20 our claim reduces to show that

$$\sup_{g\in\mathcal{U}_\delta}\|\sigma_{ijk}\|_{W_\#^{-\frac1p,p}(\Gamma_g)} \to 0 \qquad \text{as } \delta\to 0.$$

We start from the case $N = 2$. Consider the following system of equations at the points of Γ_g:

$$\begin{pmatrix}\eta_1\\ \eta_2\\ \vartheta_{11}\\ \vartheta_{12}\\ \vartheta_{21}\\ \vartheta_{22}\end{pmatrix} := \begin{pmatrix}0 & 0 & a & b & c & d\\ 0 & 0 & a' & b' & c' & d'\\ \nu_g^2 & 0 & -\nu_g^1 & 0 & 0 & 0\\ 0 & \nu_g^2 & 0 & -\nu_g^1 & 0 & 0\\ 0 & 0 & \nu_g^2 & 0 & -\nu_g^1 & 0\\ 0 & 0 & 0 & \nu_g^2 & 0 & -\nu_g^1\end{pmatrix}\cdot\begin{pmatrix}\sigma_{111}\\ \sigma_{112}\\ \sigma_{121}\\ \sigma_{122}\\ \sigma_{221}\\ \sigma_{222}\end{pmatrix}, \tag{3.81}$$

where the coefficients in the first two rows of the matrix are defined by

$$\begin{aligned} a &= C_{1111}\nu_g^1 + C_{1211}\nu_g^2, & b &= C_{1121}\nu_g^1 + C_{1221}\nu_g^2,\\ c &= C_{1112}\nu_g^1 + C_{1212}\nu_g^2, & d &= C_{1122}\nu_g^1 + C_{1222}\nu_g^2,\\ a' &= C_{2111}\nu_g^1 + C_{2211}\nu_g^2, & b' &= C_{2121}\nu_g^1 + C_{2221}\nu_g^2,\\ c' &= C_{2112}\nu_g^1 + C_{2212}\nu_g^2, & d' &= C_{2122}\nu_g^1 + C_{2222}\nu_g^2 \end{aligned}$$

in such a way that

$$\eta_1 = \Big(C_{u_g} D\Big(\frac{\partial v_g}{\partial z_2}\Big)[\nu_g]\Big)\cdot e_1, \qquad \eta_2 = \Big(C_{u_g} D\Big(\frac{\partial v_g}{\partial z_2}\Big)[\nu_g]\Big)\cdot e_2.$$

Hence by (3.78) we have

$$\|\eta_i\|_{W_\#^{-\frac{1}{p},p}(\Gamma_g)} \to 0 \qquad \text{as } \delta\to 0 \tag{3.82}$$

(uniformly with respect to $g\in\mathcal{U}_\delta$). Moreover, observe that we can write each ϑ_{ij} as a tangential derivative on Γ_g:

$$\vartheta_{ij} = \partial_{\tau_g}\Big(\frac{\partial v_g^j}{\partial z_i}\Big) = \nabla\Big(\frac{\partial v_g^j}{\partial z_i}\Big)\cdot(\nu_g^2, -\nu_g^1),$$

so that by [45, Theorem 8.6] we also have

$$\|\vartheta_{ij}\|_{W_\#^{-\frac{1}{p},p}(\Gamma_g)} \le C\Big\|\nabla\Big(\frac{\partial v_g^j}{\partial z_i}\Big)\Big\|_{L^p(\Omega_g;\mathbb{R}^2)} \to 0 \qquad \text{as } \delta\to 0 \tag{3.83}$$

(uniformly with respect to $g\in\mathcal{U}_\delta$). To conclude, observe that the 6×6 matrix in (3.81) has coefficients uniformly bounded in $C^{0,\alpha}$ with respect to $g\in\mathcal{U}_\delta$, for $\alpha = 1-\frac{2}{p} > \frac{1}{p}$ (as $p>4$); if we are able to show that its determinant is uniformly positive, then we can invert the relations in (3.81) and express σ_{ijk} as linear combinations of the quantities estimated in (3.82) and (3.83), and in turn (3.79) follows by Lemma 1.20. Hence we are left with the computation of the determinant of the 6×6 matrix M appearing in (3.81), which turns out to be equal to

$$\det M = (\nu_g^2(z))^2 \det Q_g(z),$$

which is uniformly positive as observed before. This concludes the proof of this step in the case $N=2$.

In the three-dimensional case we follow the same strategy. We observe that, setting

$$\eta_{ik} := \Big(C_{u_g} D\Big(\frac{\partial v_g}{\partial z_k}\Big)[\nu_g]\Big)\cdot e_i \qquad \text{for } i,k=1,2,3, \tag{3.84}$$

by (3.78) we have

$$\|\eta_{ik}\|_{W_\#^{-\frac{1}{p},p}(\Gamma_g)} \to 0 \qquad \text{as } \delta\to 0$$

(uniformly with respect to $g\in\mathcal{U}_\delta$). Moreover by Theorem 1.19 we have also a similar estimate for the quantities $\vartheta_{ijk} := \sigma_{ikj}\nu_g^3 - \sigma_{i3j}\nu_g^k$ for $i,j=1,2,3$ and $k=1,2$, namely

$$\|\vartheta_{ijk}\|_{W_\#^{-\frac{1}{p},p}(\Gamma_g)} \le C\Big\|\nabla\Big(\frac{\partial v_g^j}{\partial z_i}\Big)\Big\|_{L^p(\Omega_g;\mathbb{R}^3)} \to 0$$

as $\delta\to 0$, uniformly with respect to $g\in\mathcal{U}_\delta$. Hence we can write a linear system similar to (3.81) by choosing 18 among the 27 quantities ϑ_{ijk}, η_{ik} to be expressed as combinations of the 18 (different) terms σ_{ijk}: precisely, we consider η_{ik} for $k=3$ and $i=1,2,3$, and all the ϑ_{ijk} except for ϑ_{211}, ϑ_{221}, ϑ_{231}. As before, the (computer assisted) computation of the determinant of the 18×18 matrix of the system obtained in this way shows that this coincides (up to a sign) with $(\nu_g^3(z))^{12} \det Q_g(z)$, which is uniformly positive (see Appendix B for more details). Inverting these relations we can then write each term σ_{ijk} as a linear combination of the quantities ϑ_{ijk}, η_{ik}, and from the previous estimates the claim follows, again using Lemma 1.20.

Step 3. We claim that there exists a constant C, independent of $g \in \mathcal{U}_\delta$, such that for every $\varphi \in W_\#^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_h)$

$$\left\|W_\xi(Du_g \circ \Phi_g)\varphi\right\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_h;\mathbb{M}^N)} \leq C\,\|\varphi\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_h)}. \tag{3.85}$$

In fact, we use Theorem 1.18 to extend φ to a function $\tilde{\varphi} \in W_\#^{1,\frac{p}{p-1}}(\Omega_h)$. Note that, by the Sobolev Imbedding Theorem, setting $q := \frac{Np}{Np-N-p}$ we have

$$\|\tilde{\varphi}\|_{L^q(\Omega_h)} \leq C\|\tilde{\varphi}\|_{W^{1,\frac{p}{p-1}}(\Omega_h)} \leq C\|\varphi\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_h)}$$

for some constant C independent of g (the second inequality still follows from Theorem 1.18). Hence, using Hölder inequality, we deduce that

$$\begin{aligned}
&\|W_\xi(Du_g \circ \Phi_g)\varphi\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_h;\mathbb{M}^N)} \leq C\,\|W_\xi(Du_g \circ \Phi_g)\tilde{\varphi}\|_{W^{1,\frac{p}{p-1}}(\Omega_h;\mathbb{M}^N)} \\
&\quad\leq C\,\|W_\xi(Du_g \circ \Phi_g)\|_{L^\infty(\Omega_h;\mathbb{M}^N)}\|\tilde{\varphi}\|_{L^{\frac{p}{p-1}}(\Omega_h)} + C\left\|D\big(W_\xi(Du_g \circ \Phi_g)\tilde{\varphi}\big)\right\|_{L^{\frac{p}{p-1}}(\Omega_h)} \\
&\quad\leq C\,\|W_\xi(Du_g \circ \Phi_g)\|_{L^\infty(\Omega_h;\mathbb{M}^N)}\|\tilde{\varphi}\|_{W^{1,\frac{p}{p-1}}(\Omega_h)} + C\,\|\tilde{\varphi}\|_{L^q(\Omega_h)}\left\|D\big(W_\xi(Du_g \circ \Phi_g)\big)\right\|_{L^N(\Omega_h)} \\
&\quad\leq C\Big[\|W_\xi(Du_g \circ \Phi_g)\|_{L^\infty(\Omega_h;\mathbb{M}^N)} + \|C_{u_g} \circ \Phi_g\|_{L^\infty(\Omega_h)}\|D^2 u_g \circ \Phi_g\|_{L^p(\Omega_h)}\Big]\|\varphi\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_h)}.
\end{aligned}$$

From this estimate, recalling the equiboundedness of $u_g \circ \Phi_g$ in $W^{2,p}(\Omega_h)$, we obtain that (3.85) holds with a constant C depending also on the C^2-norm of W on K, where K is the compact subset of $\mathbb{M}_+^N$ given by Remark 3.30.

Step 4. We now conclude the proof of the lemma. For every $\varphi \in W_\#^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_h)$, and for $k = 1,\dots,N$ we have

$$\begin{aligned}
&\int_{\Gamma_h}\left[\frac{\partial}{\partial z_k}W(Du) - \Big(\frac{\partial}{\partial z_k}W(Du_g)\Big)\circ \Phi_g\right]\varphi\,\mathrm{d}\mathcal{H}^{N-1} \\
&\qquad= \int_{\Gamma_h}\Big(W_\xi(Du) - W_\xi(Du_g)\circ\Phi_g\Big) : D\Big(\frac{\partial u}{\partial z_k}\Big)\varphi\,\mathrm{d}\mathcal{H}^{N-1} \\
&\qquad\quad+ \int_{\Gamma_h} W_\xi(Du_g)\circ\Phi_g : \Big[D\Big(\frac{\partial u}{\partial z_k}\Big) - D\Big(\frac{\partial u_g}{\partial z_k}\Big)\circ\Phi_g\Big]\varphi\,\mathrm{d}\mathcal{H}^{N-1} \\
&\qquad\leq C\big\|W_\xi(Du) - W_\xi(Du_g)\circ\Phi_g\big\|_{L^\infty(\Gamma_h;\mathbb{M}^N)}\|\varphi\|_{L^{\frac{p}{p-1}}(\Gamma_h)} \\
&\qquad\quad+ \big\|W_\xi(Du_g\circ\Phi_g)\varphi\big\|_{W^{\frac{1}{p},\frac{p}{p-1}}(\Gamma_h;\mathbb{M}^N)}\Big\|D\Big(\frac{\partial u}{\partial z_k}\Big) - D\Big(\frac{\partial u_g}{\partial z_k}\Big)\circ\Phi_g\Big\|_{W_\#^{-\frac{1}{p},p}(\Gamma_h;\mathbb{M}^N)},
\end{aligned}$$

where C is a positive constant depending only on the C^2-norm of u and on $\mathcal{H}^{N-1}(\Gamma_h)$. Hence, since $\sup_{g\in\mathcal{U}_\delta}\big\|W_\xi(Du) - W_\xi(Du_g)\circ\Phi_g\big\|_{L^\infty(\Gamma_h;\mathbb{M}^N)} \to 0$ as $\delta \to 0$, recalling (3.79) and (3.85) we obtain that

$$\sup_{g\in\mathcal{U}_\delta}\big\|\nabla(W(Du)) - \nabla(W(Du_g))\circ\Phi_g\big\|_{W_\#^{-\frac{1}{p},p}(\Gamma_h;\mathbb{R}^N)} \to 0 \qquad \text{as } \delta \to 0,$$

and the conclusion of the lemma follows from Lemma 1.20. $\square$

We can now prove Theorem 3.41 by reproducing the strategy of [45] with easy modifications. For the sake of completeness and for the reader's convenience we will work out all the details of the proof.

Proof of Theorem 3.41. Let $\delta > 0$ to be chosen and consider any $g \in \mathcal{U}_\delta$. We will denote by $\mathbf{B}_g$ and H_g the second fundamental form and the mean curvature of Γ_g respectively, and by $\mathbf{B}_g^\psi$, H_g^ψ the "anisotropic versions" of the same quantities. We define the bilinear form on $\widetilde{H}^1_\#(\Gamma_g)$

$$(\varphi,\vartheta)_{\sim,g} := \int_{\Gamma_g} (D^2\psi\circ\nu_g)[\nabla_{\Gamma_g}\varphi,\nabla_{\Gamma_g}\vartheta]\,\mathrm{d}\mathcal{H}^{N-1} + \int_{\Gamma_g} a_g\,\varphi\,\vartheta\,\mathrm{d}\mathcal{H}^{N-1}$$

where $a_g := \partial_{\nu_g}(W\circ Du_g) - \mathrm{trace}\,(\mathbf{B}_g^\psi\mathbf{B}_g)$ on Γ_g, and we set $\|\varphi\|^2_{\sim,g} := (\varphi,\varphi)_{\sim,g}$. We omit the subscript in all the analogous quantities defined on Γ_h, according to the notation introduced in Section 3.4. We now split the proof into four steps.

Step 1. We start by observing that for every $\varphi \in \widetilde{H}^1_\#(\Gamma_g)$

$$\int_{\Gamma_h} \big(a(J_{\Phi_g})^2 - (a_g\circ\Phi_g)J_{\Phi_g}\big)(\varphi\circ\Phi_g)^2\,\mathrm{d}\mathcal{H}^{N-1} \le c(\delta)\|\varphi\|^2_{H^1(\Gamma_g)}\,, \tag{3.86}$$

where $c(\delta)\to 0$ as $\delta\to 0$ (independently of $g\in\mathcal{U}_\delta$). Indeed, by using Lemma 3.42 and recalling that $\|J_{\Phi_g}-1\|_{L^\infty(\Gamma_h)}\to 0$ as $\delta\to 0$, we have

$$\begin{aligned}
\int_{\Gamma_h} \big(\partial_\nu(W(Du))(J_{\Phi_g})^2 &- (\partial_{\nu_g}(W(Du_g))\circ\Phi_g)J_{\Phi_g}\big)(\varphi\circ\Phi_g)^2\,\mathrm{d}\mathcal{H}^{N-1}\\
&\le c'(\delta)\|(\varphi\circ\Phi_g)^2\|_{W^{\frac1p,\frac{p}{p-1}}_\#(\Gamma_h)} \le c'(\delta)\|(\varphi\circ\Phi_g)^2\|_{W^{1,\frac{p}{p-1}}(\Gamma_h)}\\
&\le c''(\delta)\|\varphi\circ\Phi_g\|^2_{H^1(\Gamma_h)} \le c'''(\delta)\|\varphi\|^2_{H^1(\Gamma_g)}\,,
\end{aligned}$$

where the third inequality can be deduced by recalling the imbedding of $H^1(\Gamma_h)$ in $L^q(\Gamma_h)$ for every q, which holds in dimension $N\le 3$. Here $c'(\delta), c''(\delta), c'''(\delta)\to 0$ as $\delta\to 0$, independently of $g\in\mathcal{U}_\delta$. Moreover, it is not hard to see that

$$\sup_{g\in\mathcal{U}_\delta}\big\|\mathrm{trace}\,(\mathbf{B}_g^\psi\mathbf{B}_g)\circ\Phi_g - \mathrm{trace}\,(\mathbf{B}^\psi\mathbf{B})\big\|_{L^{p/2}(\Gamma_h)} \to 0 \quad \text{as } \delta\to 0\,,$$

from which follows by Hölder inequality (again using $\|J_{\Phi_g}-1\|_{L^\infty(\Gamma_h)}\to 0$)

$$\begin{aligned}
\int_{\Gamma_h} \big(\mathrm{trace}\,(\mathbf{B}_g^\psi\mathbf{B}_g)\circ\Phi_g &- \mathrm{trace}\,(\mathbf{B}^\psi\mathbf{B})J_{\Phi_g}\big)J_{\Phi_g}(\varphi\circ\Phi_g)^2\,\mathrm{d}\mathcal{H}^{N-1}\\
&\le c'(\delta)\|(\varphi\circ\Phi_g)^2\|_{L^{\frac{p}{p-2}}(\Gamma_h)} = c'(\delta)\|\varphi\circ\Phi_g\|^2_{L^{\frac{2p}{p-2}}(\Gamma_h)}\\
&\le c''(\delta)\|\varphi\circ\Phi_g\|^2_{H^1(\Gamma_h)} \le c'''(\delta)\|\varphi\|^2_{H^1(\Gamma_g)}\,,
\end{aligned}$$

where the second inequality is justified, as before, by the Sobolev Embedding Theorem. By combining the previous estimates, (3.86) follows.

Step 2. We claim that if δ is sufficiently small then for every $g\in\mathcal{U}_\delta$

$$\|\varphi\|^2_{\sim,g} \ge C_1\|\varphi\|^2_{H^1(\Gamma_g)} \qquad \text{for every } \varphi\in\widetilde{H}^1_\#(\Gamma_g) \tag{3.87}$$

for some positive constant C_1. To prove (3.87), we first note that for every $\vartheta\in\widetilde{H}^1_\#(\Gamma_h)$ one has, thanks to (3.72) and to Corollary 3.39,

$$\|\vartheta\|^2_\sim \ge \partial^2F(h,u)[\vartheta] \ge C\|\vartheta\|^2_{H^1(\Gamma_h)}.$$

For $\varphi \in \widetilde{H}^1_\#(\Gamma_g)$ we define $\tilde{\varphi} := (\varphi \circ \Phi_g) J_{\Phi_g} \in \widetilde{H}^1_\#(\Gamma_h)$; then, using the area formula we have

$$
\begin{aligned}
\|\varphi\|^2_{H^1(\Gamma_g)} &= \int_{\Gamma_g} (\varphi^2 + |\nabla_{\Gamma_g}\varphi|^2)\, d\mathcal{H}^{N-1} = \int_{\Gamma_h} \big((\varphi \circ \Phi_g)^2 + |(\nabla_{\Gamma_g}\varphi)\circ\Phi_g|^2\big) J_{\Phi_g}\, d\mathcal{H}^{N-1} \\
&\leq C'\|\tilde{\varphi}\|^2_{H^1(\Gamma_h)} \leq \frac{C'}{C}\,\|\tilde{\varphi}\|^2_\sim
\end{aligned}
$$

for some positive constant C' independent of $g \in \mathcal{U}_\delta$. Now

$$
\begin{aligned}
\|\tilde{\varphi}\|^2_\sim &= \int_{\Gamma_h} \big(a\,\tilde{\varphi}^2 + (D^2\psi\circ\nu)[\nabla_{\Gamma_h}\tilde{\varphi}, \nabla_{\Gamma_h}\tilde{\varphi}]\,\big)\, d\mathcal{H}^{N-1} \\
&= \|\varphi\|^2_{\sim,g} + \int_{\Gamma_h} \big(a(J_{\Phi_g})^2 - (a_g\circ\Phi_g)J_{\Phi_g}\big)(\varphi\circ\Phi_g)^2\, d\mathcal{H}^{N-1} \\
&\qquad + \int_{\Gamma_h} (D^2\psi\circ\nu)[\nabla_{\Gamma_h}\tilde{\varphi}, \nabla_{\Gamma_h}\tilde{\varphi}]\, d\mathcal{H}^{N-1} \\
&\qquad - \int_{\Gamma_h} (D^2\psi\circ\nu_g\circ\Phi_g)[(\nabla_{\Gamma_g}\varphi)\circ\Phi_g, (\nabla_{\Gamma_g}\varphi)\circ\Phi_g] J_{\Phi_g}\, d\mathcal{H}^{N-1} \\
&\leq \|\varphi\|^2_{\sim,g} + c(\delta)\|\varphi\|^2_{H^1(\Gamma_g)},
\end{aligned}
\tag{3.88}
$$

where $c(\delta)$ tends to 0 as $\delta \to 0$. To deduce the last inequality in the previous estimate we used in particular (3.86) and the fact that $\|\Phi_g - Id\|_{W^{2,p}(\Gamma_h;\mathbb{R}^N)} \to 0$. Choosing δ sufficiently small and combining the previous estimates the claim follows.

Step 3. By Step 2 we can define a compact linear operator $T_g : \widetilde{H}^1_\#(\Gamma_g) \to \widetilde{H}^1_\#(\Gamma_g)$ by duality:

$$
(T_g\varphi, \vartheta)_{\sim,g} = \int_{\Gamma_g} \mathrm{div}_{\Gamma_g}(\vartheta\, W_\xi(Du_g))\cdot v_\varphi\, d\mathcal{H}^{N-1} = \int_{\Omega_g} C_{u_g} Dv_\varphi : Dv_\vartheta\, dz \tag{3.89}
$$

for every $\varphi, \vartheta \in \widetilde{H}^1_\#(\Gamma_g)$, where for $\zeta \in \widetilde{H}^1_\#(\Gamma_g)$ we denote by v_ζ the unique solution in $\widetilde{\mathcal{V}}(\Omega_g)$ to the equation

$$
\int_{\Omega_g} C_{u_g} Dv_\zeta : Dw\, dz = \int_{\Gamma_g} \mathrm{div}_{\Gamma_g}(\zeta\, W_\xi(Du_g))\cdot w\, d\mathcal{H}^{N-1} \qquad \text{for every } w \in \widetilde{\mathcal{V}}(\Omega_g). \tag{3.90}
$$

Setting, similarly to (3.70),

$$
\lambda_{1,g} := \max_{\|\varphi\|_{\sim,g}=1} (T_g\varphi, \varphi)_{\sim,g},
$$

we claim that

$$
\lambda_\infty := \limsup_{\|g-h\|_{W^{2,p}(Q)}\to 0} \lambda_{1,g} \leq \lambda_1. \tag{3.91}
$$

Indeed, let $(g_n)_n$ be a sequence in $C^\infty_\#(Q)$ converging to h in $W^{2,p}(Q)$, $|\Omega_{g_n}| = |\Omega_h|$, such that

$$
\lambda_\infty = \lim_{n\to+\infty} \lambda_{1,g_n},
$$

and let u_n be the corresponding critical points for the elastic energy in Ω_{g_n}. Let $\varphi_n \in \widetilde{H}^1_\#(\Gamma_{g_n})$, with $\|\varphi_n\|_{\sim,g_n} = 1$, be such that

$$
\lambda_{1,g_n} = (T_{g_n}\varphi_n, \varphi_n)_{\sim,g_n} = \int_{\Omega_{g_n}} C_{u_n} Dv_{\varphi_n} : Dv_{\varphi_n}\, dz,
$$

where v_{φ_n} is defined as in (3.90). We set $\tilde{\varphi}_n := c_n(\varphi_n \circ \Phi_{g_n})J_{\Phi_{g_n}}$, where $c_n := \|(\varphi_n \circ \Phi_{g_n})J_{\Phi_{g_n}}\|^{-1}_\sim$, so that $\tilde{\varphi}_n \in \widetilde{H}^1_\#(\Gamma_h)$ and $\|\tilde{\varphi}_n\|_\sim = 1$. Setting also $w_n := v_{\varphi_n} \circ \Phi_{g_n}$, by a change of variables it follows that for every $w \in \widetilde{\mathcal{V}}(\Omega_{g_n})$

$$\int_{\Omega_{g_n}} C_{u_n} Dv_{\varphi_n} : Dw \,\mathrm{d}z = \int_{\Omega_h} A_n Dw_n : D(w \circ \Phi_{g_n})\,\mathrm{d}z,$$

where A_n is the fourth order tensor defined by

$$A_n M = \Big(W_{\xi\xi}\big(D(u_n \circ \Phi_{g_n})(D\Phi_{g_n})^{-1}\big)\big(M(D\Phi_{g_n})^{-1}\big)\Big)(D\Phi_{g_n})^{-T} \det D\Phi_{g_n} \qquad \text{for } M \in \mathbb{M}^N.$$

Hence by (3.90) we see that $w_n \in \widetilde{\mathcal{V}}(\Omega_h)$ solves the equation

$$\int_{\Omega_h} A_n Dw_n : Dw \,\mathrm{d}z = \int_{\Gamma_h} \big(\mathrm{div}_{\Gamma_{g_n}}(\varphi_n W_\xi(Du_n)) \circ \Phi_{g_n}\big) \cdot w \, J_{\Phi_{g_n}} \,\mathrm{d}\mathcal{H}^{N-1} \tag{3.92}$$

for every $w \in \widetilde{\mathcal{V}}(\Omega_h)$. Let us observe also that $A_n \to C_u$ uniformly in $\overline{\Omega}_h$. We now claim that

$$\lim_{n\to\infty} \int_{\Omega_h} C_u Dv_{\tilde\varphi_n} : Dv_{\tilde\varphi_n} \,\mathrm{d}z = \lim_{n\to\infty} \int_{\Omega_h} A_n Dw_n : Dw_n \,\mathrm{d}z. \tag{3.93}$$

Notice that this implies (3.91), since

$$\begin{aligned}\lambda_1 &\geq \lim_{n\to\infty} (T\tilde\varphi_n, \tilde\varphi_n)_\sim = \lim_{n\to\infty} \int_{\Omega_h} C_u Dv_{\tilde\varphi_n} : Dv_{\tilde\varphi_n} \,\mathrm{d}z \\ &= \lim_{n\to\infty} \int_{\Omega_h} A_n Dw_n : Dw_n \,\mathrm{d}z = \lim_{n\to\infty} \int_{\Omega_{g_n}} C_{u_n} Dv_{\varphi_n} : Dv_{\varphi_n} \,\mathrm{d}z = \lambda_\infty.\end{aligned}$$

In order to prove (3.93), we need to deduce some preliminary estimates. Using the equation satisfied by v_{φ_n} and recalling (3.56) we have

$$\begin{aligned}\frac{c_0}{4}\|v_{\varphi_n}\|^2_{H^1(\Omega_{g_n};\mathbb{R}^N)} &\leq \int_{\Omega_{g_n}} C_{u_n} Dv_{\varphi_n} : Dv_{\varphi_n} \,\mathrm{d}z = \int_{\Gamma_{g_n}} \mathrm{div}_{\Gamma_{g_n}}\big(\varphi_n W_\xi(Du_n)\big) \cdot v_{\varphi_n} \,\mathrm{d}\mathcal{H}^{N-1} \\ &\leq \|\mathrm{div}_{\Gamma_{g_n}}\big(\varphi_n W_\xi(Du_n)\big)\|_{H^{-\frac12}_{\#}(\Gamma_{g_n};\mathbb{R}^N)} \|v_{\varphi_n}\|_{H^{\frac12}(\Gamma_{g_n};\mathbb{R}^N)},\end{aligned}$$

and since the $H^{-\frac12}$-norm in the previous expression is uniformly bounded by Lemma 1.14 (recall that φ_n are uniformly bounded in $H^1(\Gamma_{g_n})$, and that $W_\xi(Du_n)$ are uniformly bounded in $C^{0,\alpha}(\overline{\Omega}_{g_n};\mathbb{M}^N)$ with $\alpha = 1 - \frac{N}{p} > \frac12$), we deduce that

$$\sup_n \|v_{\varphi_n}\|_{H^1(\Omega_{g_n};\mathbb{R}^N)} < \infty. \tag{3.94}$$

Moreover we have also

$$\sup_n \|w_n\|_{H^1(\Omega_h;\mathbb{R}^N)} < \infty, \qquad \sup_n \|v_{\tilde\varphi_n}\|_{H^1(\Omega_h;\mathbb{R}^N)} < \infty, \tag{3.95}$$

where the first estimate follows from (3.94), using the definition of w_n. Finally, arguing as in the proof of the estimate (3.88) with $\tilde\varphi$ replaced by $\frac{\tilde\varphi_n}{c_n}$ and φ replaced by φ_n, we obtain $c_n \to 1$.

Now we are ready to prove (3.93), from which the conclusion follows. Observe that, thanks to the uniform bound (3.95) and to the uniform convergence of A_n to C_u, we have

$$\lim_{n\to\infty} \int_{\Omega_h} C_u Dw_n : Dw_n \,\mathrm{d}z = \lim_{n\to\infty} \int_{\Omega_h} A_n Dw_n : Dw_n \,\mathrm{d}z,$$

thus claim (3.93) will follow from

$$\lim_{n\to\infty} \int_{\Omega_h} C_u D(v_{\tilde\varphi_n} - w_n) : D(v_{\tilde\varphi_n} - w_n)\,\mathrm{d}z = 0, \tag{3.96}$$

since this implies that $v_{\tilde\varphi_n} - w_n$ tends to 0 strongly in $H^1(\Omega_h;\mathbb{R}^N)$. Hence we are left with the proof of (3.96).

Observe that, as $v_{\tilde\varphi_n} - w_n$ is an admissible test function for both the equations satisfied by $v_{\tilde\varphi_n}$ and w_n, we have

$$
\begin{aligned}
&\int_{\Omega_h} C_u D(v_{\tilde\varphi_n} - w_n) : D(v_{\tilde\varphi_n} - w_n)\,\mathrm{d}z \\
&\qquad = \int_{\Omega_h} C_u Dv_{\tilde\varphi_n} : D(v_{\tilde\varphi_n} - w_n)\,\mathrm{d}z - \int_{\Omega_h} (C_u - A_n)Dw_n : D(v_{\tilde\varphi_n} - w_n)\,\mathrm{d}z \\
&\qquad\qquad - \int_{\Omega_h} A_n Dw_n : D(v_{\tilde\varphi_n} - w_n)\,\mathrm{d}z \\
&\qquad = \int_{\Gamma_h} \mathrm{div}_{\Gamma_h}(\tilde\varphi_n W_\xi(Du))\cdot(v_{\tilde\varphi_n} - w_n)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Omega_h} (C_u - A_n)Dw_n : D(v_{\tilde\varphi_n} - w_n)\,\mathrm{d}z \\
&\qquad\qquad - \int_{\Gamma_h} \big(\mathrm{div}_{\Gamma_{g_n}}(\varphi_n W_\xi(Du_n))\circ\Phi_{g_n}\big)\cdot(v_{\tilde\varphi_n} - w_n)J_{\Phi_{g_n}}\,\mathrm{d}\mathcal{H}^{N-1} \\
&\qquad =: I_1 - I_2 - I_3.
\end{aligned}
$$

It is clear, from the bounds in (3.95) and from the uniform convergence of A_n to C_u, that the second integral I_2 tends to 0. Since, thanks to (3.95), $v_{\tilde\varphi_n} - w_n$ is bounded in $H^{\frac12}(\Gamma_h;\mathbb{R}^N)$, to prove that also the difference $I_1 - I_3$ tends to 0 it will be sufficient to show that

$$
\big\|\mathrm{div}_{\Gamma_h}(\tilde\varphi_n W_\xi(Du)) - \mathrm{div}_{\Gamma_{g_n}}(\varphi_n W_\xi(Du_n))\circ\Phi_{g_n}\big\|_{H^{-\frac12}_{\#}(\Gamma_h;\mathbb{R}^N)} \to 0\,.
$$

In turn, by Lemma 1.20 the previous convergence will follow from

$$
\|\tilde\varphi_n h_n\|_{H^{\frac12}_{\#}(\Gamma_h;\mathbb{M}^N)} \to 0, \tag{3.97}
$$

where

$$
h_n := W_\xi(Du) - c_n^{-1}(J_{\Phi_{g_n}})^{-1} W_\xi(Du_n)\circ\Phi_{g_n}.
$$

Recalling that $c_n \to 1$, we have that $h_n \to 0$ in $C^{0,\alpha}(\Gamma_h;\mathbb{M}^N)$ for $\alpha = 1 - \frac{N}{p}$; hence by Lemma 1.14 we obtain (3.97), which concludes the proof of Step 3.

Step 4. We define $h_t := h + t(g-h)$ for $t\in[0,1]$. Setting $f(t) := F(h_t, u_{h_t})$, we claim that if δ is sufficiently small then

$$
f''(t) > 2C_2\|\varphi_g\|^2_{H^1(\Gamma_g)} \qquad \text{for every } t\in[0,1] \tag{3.98}
$$

for some positive constant C_2, where $\varphi_g := \big((g-h)/\sqrt{1+|\nabla g|^2}\big)\circ\pi$. In fact, the quantity $f''(t)$ is nothing but the second variation of F at (h_t,u_{h_t}) along the direction $g-h$, hence by Remark 3.34

$$
\begin{aligned}
f''(t) &= -(T_{h_t}\varphi_t,\varphi_t)_{\sim,h_t} + \|\varphi_t\|^2_{\sim,h_t} \\
&\quad - \int_{\Gamma_{h_t}} \big(W(Du_{h_t}) + H^\psi_{h_t}\big)\,\mathrm{div}_{\Gamma_{h_t}}\left[\left(\frac{(\nabla h_t, |\nabla h_t|^2)}{\sqrt{1+|\nabla h_t|^2}}\circ\pi\right)\varphi_t^2\right]\mathrm{d}\mathcal{H}^{N-1},
\end{aligned} \tag{3.99}
$$

where $\varphi_t := \big((g-h)/\sqrt{1+|\nabla h_t|^2}\big)\circ\pi \in \widetilde H^1_{\#}(\Gamma_{h_t})$. Observe that, as $\lambda_1 < 1$ by Theorem 3.37, combining Step 2 and Step 3 we have that for δ sufficiently small

$$
\begin{aligned}
-(T_{h_t}\varphi_t,\varphi_t)_{\sim,h_t} + \|\varphi_t\|^2_{\sim,h_t} &\ge (1-\lambda_{1,h_t})\|\varphi_t\|^2_{\sim,h_t} > \frac{1-\lambda_1}{2}\|\varphi_t\|^2_{\sim,h_t} \\
&\ge \frac{C_1(1-\lambda_1)}{2}\|\varphi_t\|^2_{H^1(\Gamma_{h_t})} \ge \frac{C_1(1-\lambda_1)}{4}\|\varphi_g\|^2_{H^1(\Gamma_g)},
\end{aligned} \tag{3.100}
$$

where in the last inequality we used the fact that, for δ small enough,

$$\frac{1}{2}\|\varphi_g\|^2_{H^1(\Gamma_g)} \leq \|\varphi_t\|^2_{H^1(\Gamma_{h_t})} \leq 2\|\varphi_g\|^2_{H^1(\Gamma_g)}. \tag{3.101}$$

In addition, as (h,u) is a critical pair, there exists a constant Λ such that $W(Du)+H^\psi \equiv \Lambda$ on Γ_h, and moreover

$$\sup_{g\in\mathcal{U}_\delta}\sup_{t\in[0,1]}\big\|W(Du_{h_t})+H^\psi_{h_t}-\Lambda\big\|_{L^p(\Gamma_{h_t})}\to 0 \qquad \text{as } \delta\to 0. \tag{3.102}$$

From this it follows that if δ is sufficiently small, by Hölder inequality

$$\begin{aligned}
&-\int_{\Gamma_{h_t}}\big(W(Du_{h_t})+H^\psi_{h_t}\big)\operatorname{div}_{\Gamma_{h_t}}\Big[\Big(\tfrac{(\nabla h_t,|\nabla h_t|^2)}{\sqrt{1+|\nabla h_t|^2}}\circ\pi\Big)\varphi_t^2\Big]\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad=-\int_{\Gamma_{h_t}}\big(W(Du_{h_t})+H^\psi_{h_t}-\Lambda\big)\operatorname{div}_{\Gamma_{h_t}}\Big[\Big(\tfrac{(\nabla h_t,|\nabla h_t|^2)}{\sqrt{1+|\nabla h_t|^2}}\circ\pi\Big)\varphi_t^2\Big]\,\mathrm{d}\mathcal{H}^{N-1}\\
&\quad\geq-\big\|W(Du_{h_t})+H^\psi_{h_t}-\Lambda\big\|_{L^p(\Gamma_{h_t})}\bigg\{\Big\|\operatorname{div}_{\Gamma_{h_t}}\Big(\tfrac{(\nabla h_t,|\nabla h_t|^2)}{\sqrt{1+|\nabla h_t|^2}}\circ\pi\Big)\Big\|_{L^p(\Gamma_{h_t})}\|\varphi_t\|^2_{L^{\frac{2p}{p-2}}(\Gamma_{h_t})}\\
&\qquad\qquad+2\big\|\nabla_{\Gamma_{h_t}}\varphi_t\big\|_{L^2(\Gamma_{h_t};\mathbb{R}^N)}\Big\|\varphi_t\tfrac{(\nabla h_t,|\nabla h_t|^2)}{\sqrt{1+|\nabla h_t|^2}}\circ\pi\Big\|_{L^{\frac{2p}{p-2}}(\Gamma_{h_t};\mathbb{R}^N)}\bigg\}\\
&\quad\geq-\frac{C_1(1-\lambda_1)}{8}\|\varphi_g\|^2_{H^1(\Gamma_g)},
\end{aligned} \tag{3.103}$$

where in the last inequality we used also the boundedness of h_t in $W^{2,p}(Q)$, the Sobolev imbedding theorem, (3.102) and (3.101). Collecting (3.99), (3.100) and (3.103) we conclude that the claim (3.98) holds with $C_2=\frac{C_1(1-\lambda_1)}{16}$.

Finally, thank to the fact that $f'(0)=0$ (as (h,u) is a critical pair), we have

$$F(h,u)=f(0)=f(1)-\int_0^1(1-t)f''(t)\,\mathrm{d}t<F(g,u_g)-C_2\|\varphi_g\|^2_{H^1(\Gamma_g)}. \tag{3.104}$$

This inequality is valid for every $g\in\mathcal{U}_\delta$, for a sufficiently small δ. Now, by an approximation argument, if $g\in AP(Q)$ is such that $\|g-h\|_{W^{2,p}(Q)}<\delta$ and $|\Omega_g|=|\Omega_h|$, we set $\tilde g:=h+\rho_\varepsilon*(g-h)$, where ρ_ε is a standard mollifier with support in $B_\varepsilon(0)$. Then $\tilde g\in\mathcal{U}_\delta$, and ε can be chosen so small that

$$F(\tilde g,u_{\tilde g})\leq F(g,u_g)+\frac{C_2}{2}\|\varphi_{\tilde g}\|^2_{H^1(\Gamma_{\tilde g})},$$

hence by (3.104)

$$F(h,u)<F(g,u_g)-\frac{C_2}{2}\|\varphi_{\tilde g}\|^2_{H^1(\Gamma_{\tilde g})}.$$

Now the minimality with respect to a generic pair (g,v) follows from Proposition 3.31. □

3.7. Strong local minimality

In the main result of this section (Theorem 3.44) we prove that the local $W^{2,p}$-minimality (see Definition 3.40) implies local minimality in the stronger sense of Definition 3.23. In particular, by Theorem 3.41 we deduce that the strict stability of a critical pair (h,u) is a sufficient condition for local minimality (Theorem 3.45). We will also observe, in Theorem 3.46, that our methods provide the isolated local minimality in the case of the linear elasticity.

The following lemma, which can be proved by standard elliptic estimates, contains a preliminary result that we will need in this section.

LEMMA 3.43. *Let* $h \in C^2_\#(Q)$*, and let* $h_n \in C^{1,\alpha}_\#(Q)$ *be such that* $h_n \to h$ *in* $C^{1,\alpha}$*, for some* $\alpha \in (0,1)$*. Assume also that the anisotropic mean curvature* $H^\psi_{h_n}$ *of* h_n *is bounded. Then*

(i) *if* $H^\psi_{h_n}(\cdot, h_n(\cdot)) \to H^\psi(\cdot, h(\cdot))$ *in* $L^p(Q)$*, then* $h_n \to h$ *in* $W^{2,p}(Q)$*;*
(ii) *if* $\sup_n \|H^\psi_{h_n}\|_{L^p(Q)} < \infty$*, then* $\sup_n \|h_n\|_{W^{2,p}(Q)} < \infty$*.*

PROOF. The function h_n is a weak solution to the equation

$$-\int_Q \nabla\psi(-\nabla h_n, 1)\cdot(\nabla\eta, 0)\,\mathrm{d}x = \int_Q H^\psi_{h_n}(x, h_n(x))\eta(x)\,\mathrm{d}x \qquad \text{for all } \eta \in C^\infty_\#(Q)$$

with $H^\psi_{h_n}(\cdot, h_n(\cdot)) \in L^\infty(Q)$, which implies, by elliptic regularity (see, *e.g.*, [8, Proposition 7.56]), that $h_n \in W^{2,2}_\#(Q)$. Hence it makes sense to perform the differentiation and rewrite the equation in non-divergence form:

$$\sum_{i,j=1}^{N-1} \frac{\partial^2\psi}{\partial z_i \partial z_j}(-\nabla h_n(x), 1)\frac{\partial^2 h_n}{\partial x_i \partial x_j}(x) = -H^\psi_{h_n}(x, h_n(x)) \qquad \text{a.e. in } Q.$$

By elliptic regularity results for equations in non-divergence form with continuous coefficients, we deduce that $h_n \in W^{2,p}_\#(Q)$ for every $p \in [1,\infty)$ (see [8, Theorem 7.48]), and in turn the conclusion follows from [48, Theorem 9.11] recalling that $h_n \to h$ in $C^{1,\alpha}$. □

We recall that we associated, with a critical pair (h,u), an open set Ω' containing Ω_h in terms of which we defined in (3.47) the class of competitors X'. Our strategy requires now the extension of the functional F to a larger class of admissible pairs: in particular, we shall consider not just subgraphs of Lipschitz functions, but generic periodic sets with locally finite perimeter. More precisely, let $\widetilde{X}$ be the set of all pairs (Ω, v) such that:

- $\Omega \subset \Omega'$ is a set of finite perimeter; we will denote by $\Omega^\#$ the periodic extension of $\Omega \cup (Q \times (-\infty; 0])$ in the first $N-1$ directions;
- $v \in W^{1,\infty}(\Omega'_\#; \mathbb{R}^N)$ is such that $v - u_0 \in \mathcal{V}(\Omega')$ and $\det Dv > 0$ a.e. in Ω'.

For $(\Omega, v) \in \widetilde{X}$ we define

$$\widetilde{F}(\Omega, v) := \int_\Omega W(Dv)\,\mathrm{d}z + \int_{\Gamma_\Omega} \psi(\nu_\Omega)\,\mathrm{d}\mathcal{H}^{N-1}$$

where $\Gamma_\Omega := \partial^*\Omega^\# \cap \big([0,1)^{N-1} \times \mathbb{R}\big)$ and ν_Ω is the generalized outer unit normal to the reduced boundary of $\Omega^\#$. We remark that, if $(g,v) \in X'$, then $(\Omega_g, v) \in \widetilde{X}$ and $\widetilde{F}(\Omega_g, v) = F(g, v)$.

We are now ready to state and prove the main result of this section.

THEOREM 3.44. *Let* $p \in (1,\infty)$*, and assume that a critical pair* $(h,u) \in X$ *is a local* $W^{2,p}$*-minimizer, in the sense of Definition 3.40. Then* (h,u) *is a local minimizer for* F*, according to Definition 3.23.*

PROOF. We argue by contradiction, assuming the existence of a decreasing sequence $\sigma_n \to 0$ and of a sequence $(g_n, u_n) \in X'$ such that

$$0 < \|g_n - h\|_\infty \le \sigma_n, \quad \|Du_n - Du\|_{L^\infty(\Omega';\mathbb{M}^N)} \le \sigma_n, \quad |\Omega_{g_n}| = |\Omega_h|,$$

and

$$F(g_n, u_n) < F(h,u). \tag{3.105}$$

We now split the proof into several steps.

Step 1. We claim that we can find new sequences $\delta_n \to 0$ and $v_n \in C^\infty(\overline{\Omega}';\mathbb{R}^N)$ such that $(g_n, v_n) \in X'$, $\|g_n - h\|_\infty \le \delta_n$, $\|Dv_n - Du\|_{L^\infty(\Omega';\mathbb{M}^N)} \le \delta_n$, and for which we still have

$$F(g_n, v_n) < F(h, u). \tag{3.106}$$

Indeed, for every n we can construct an approximating sequence v_n^k, $k \in \mathbb{N}$, in the following way: we let $\rho_{1/k}$ be the standard mollifier in $\mathbb{R}^N$ with support compactly contained in $B_{1/k}$, and we set

$$v_n^k := w_n^k * \rho_{1/k} + u_0, \qquad \text{where} \quad w_n^k(x,y) := \begin{cases} (u_n - u_0)(x, y - 1/k) & \text{if } y \ge 0, \\ 0 & \text{if } y < 0 \end{cases}$$

(where we extended $u_n - u_0$ to 0 in $\mathbb{R}^N_-$). Then by the properties of the convolution product we have $v_n^k \in C^\infty(\overline{\Omega}';\mathbb{R}^N)$, $v_n^k - u_0 \in \mathcal{V}(\Omega')$, and

$$\|Dv_n^k - Du\|_{L^\infty(\Omega';\mathbb{M}^N)} \le 2\sigma_n$$

for every k sufficiently large. Moreover, $F(g_n, v_n^k) \to F(g_n, u_n)$ as $k \to \infty$ by the Lebesgue Dominated Convergence Theorem. Hence, for every n we can find k_n such that the function $v_n := v_n^{k_n}$ satisfies the desired properties with $\delta_n = 2\sigma_n$. We set $M_n := \|D^2 v_n\|_\infty$.

Step 2. Let $(\Omega_n, w_n) \in \widetilde{X}$ be a solution to the penalized problem

$$\min\Big\{ J_\beta(\Omega, v) : (\Omega, v) \in \widetilde{X},\ \Omega_{h-\delta_n} \subset \Omega \subset \Omega_{h+\delta_n},\ v \in W^{2,\infty}(\Omega';\mathbb{R}^N), \\ \|D^2 v\|_\infty \le M_n,\ \|Dv - Du\|_{L^\infty(\Omega';\mathbb{M}^N)} \le \delta_n \Big\}, \tag{3.107}$$

where

$$J_\beta(\Omega, v) := \widetilde{F}(\Omega, v) + \beta\big||\Omega| - |\Omega_h|\big|$$

and β is a positive constant, to be chosen later. Observe that problem (3.107) admits a solution by the direct method of the Calculus of Variations: indeed, if (Ω^k, w^k) is a minimizing sequence, then up to subsequences we have that $\Omega^k \to \Omega^0$ in L^1 and $w^k \to w^0$ weakly* in $W^{2,\infty}(\Omega';\mathbb{R}^N)$; the pair (Ω^0, w^0) satisfies all the constraints and is a minimizer of (3.107) by the lower semicontinuity of the functional (which follows in particular from Reshetnyak's Lower Semicontinuity Theorem, as stated in [8, Theorem 2.38], for the surface term).

Since (Ω_{g_n}, v_n) is an admissible competitor for (3.107), the minimality of (Ω_n, w_n) and (3.106) yield

$$\widetilde{F}(\Omega_n, w_n) \le J_\beta(\Omega_n, w_n) \le J_\beta(\Omega_{g_n}, v_n) = F(g_n, v_n) < F(h, u). \tag{3.108}$$

Step 3. We claim that, for β large enough (independently of n), (Ω_n, w_n) is also a solution to the minimum problem

$$\min\Big\{ \widetilde{J}_\beta(\Omega, v) : (\Omega, v) \in \widetilde{X},\ v \in W^{2,\infty}(\Omega';\mathbb{R}^N),\ \|D^2 v\|_\infty \le M_n,\ \|Dv - Du\|_{L^\infty(\Omega';\mathbb{M}^N)} \le \delta_n \Big\}, \tag{3.109}$$

where

$$\widetilde{J}_\beta(\Omega, v) := J_\beta(\Omega, v) + 2\beta\,|\Omega \triangle T_n(\Omega)|$$

and $T_n(\Omega) := \big(\Omega \cup \Omega_{h-\delta_n}\big) \cap \Omega_{h+\delta_n}$.

To prove the claim, consider any competitor (Ω, v) for problem (3.109). Then we have, since $T_n(\Omega_n) = \Omega_n$,

$$\begin{aligned}
\widetilde{J}_\beta(\Omega, v) - \widetilde{J}_\beta(\Omega_n, w_n) &= J_\beta(T_n(\Omega), v) - J_\beta(\Omega_n, w_n) + 2\beta\,|\Omega \triangle T_n(\Omega)| \\
&+ \int_\Omega W(Dv)\,\mathrm{d}z - \int_{T_n(\Omega)} W(Dv)\,\mathrm{d}z + \int_{\Gamma_\Omega} \psi(\nu_\Omega)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Gamma_{T_n(\Omega)}} \psi(\nu_{T_n(\Omega)})\,\mathrm{d}\mathcal{H}^{N-1} \\
&+ \beta\Big(\big||\Omega| - |\Omega_h|\big| - \big||T_n(\Omega)| - |\Omega_h|\big|\Big) \\
&\geq (2\beta - W_0 - \beta)\,|\Omega \triangle T_n(\Omega)| + \int_{\Gamma_\Omega} \psi(\nu_\Omega)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Gamma_{T_n(\Omega)}} \psi(\nu_{T_n(\Omega)})\,\mathrm{d}\mathcal{H}^{N-1},
\end{aligned}$$

where in the last inequality we used the fact that $J_\beta(T_n(\Omega), v) - J_\beta(\Omega_n, w_n) \geq 0$ by the minimality of (Ω_n, w_n), and W_0 is a positive constant depending only on W and u.

Now recalling the 1-homogeneity of ψ, the Euler's theorem $\psi(\nu) = \nabla\psi(\nu) \cdot \nu$ and the convexity of ψ yield

$$\psi(\nu_\Omega) \geq \psi(\nu_h) + \nabla\psi(\nu_h) \cdot (\nu_\Omega - \nu_h) = \nabla\psi(\nu_h) \cdot \nu_\Omega \quad \text{on } \Gamma_\Omega,$$

where, for every $z \in \mathbb{R}^N$, we denote by $\nu_h(z)$ the upper unit normal to the graph of h at the point $(\pi(z), h(\pi(z)))$. Hence, using again Euler's theorem and observing that $\mathcal{H}^{N-1}$-almost everywhere on $\Gamma_{T_n(\Omega)} \setminus \Gamma_\Omega$ the normal to $\Gamma_{T_n(\Omega)}$ coincides with ν_h , we obtain

$$\begin{aligned}
&\int_{\Gamma_\Omega} \psi(\nu_\Omega)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Gamma_{T_n(\Omega)}} \psi(\nu_{T_n(\Omega)})\,\mathrm{d}\mathcal{H}^{N-1} \\
&\qquad \geq \int_{\Gamma_\Omega \setminus \Gamma_{T_n(\Omega)}} \nabla\psi(\nu_h) \cdot \nu_\Omega\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Gamma_{T_n(\Omega)} \setminus \Gamma_\Omega} \nabla\psi(\nu_h) \cdot \nu_h\,\mathrm{d}\mathcal{H}^{N-1} \\
&\qquad \geq -\int_{\Omega \triangle T_n(\Omega)} \big|\mathrm{div}(\nabla\psi \circ \nu_h)\big|\,\mathrm{d}z \geq -\Lambda_0 |\Omega \triangle T_n(\Omega)|\,. \qquad (3.110)
\end{aligned}$$

Here $\Lambda_0 := \|H^\psi\|_{L^\infty(\Gamma_h)}$, where H^ψ denotes the anisotropic mean curvature of Γ_h. Hence we can conclude

$$\widetilde{J}_\beta(\Omega, v) - \widetilde{J}_\beta(\Omega_n, w_n) \geq (\beta - W_0 - \Lambda_0)\,|\Omega \triangle T_n(\Omega)|,$$

so that by choosing $\beta > W_0 + \Lambda_0$ (notice that this constant depends only on W, ψ, h and u) we deduce that (Ω_n, w_n) is a solution to (3.109).

Step 4. We claim that each Ω_n satisfies the volume constraint

$$|\Omega_n| = |\Omega_h|. \qquad (3.111)$$

Suppose by contradiction that $|\Omega_h| - |\Omega_n| =: d > 0$ for some n. We can find $\delta \in (-\delta_n, \delta_n)$ such that $|\Omega_n \cup \Omega_{h+\delta}| = |\Omega_h|$. Define $U := \Omega_n \cup \Omega_{h+\delta}$. Then, as $|U| = |\Omega_h|$, we have

$$\begin{aligned}
J_\beta(U, w_n) - J_\beta(\Omega_n, w_n) &= \int_U W(Dw_n)\,\mathrm{d}z - \int_{\Omega_n} W(Dw_n)\,\mathrm{d}z \\
&+ \int_{\Gamma_U} \psi(\nu_U)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Gamma_{\Omega_n}} \psi(\nu_{\Omega_n})\,\mathrm{d}\mathcal{H}^{N-1} - \beta d \\
&\leq (W_0 - \beta)\,d + \int_{\Gamma_U} \psi(\nu_U)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Gamma_{\Omega_n}} \psi(\nu_{\Omega_n})\,\mathrm{d}\mathcal{H}^{N-1} \qquad (3.112)
\end{aligned}$$

where W_0 is the same constant as in Step 3. Now, arguing as in (3.110), we have

$$\int_{\Gamma_U} \psi(\nu_U)\,\mathrm{d}\mathcal{H}^{N-1} - \int_{\Gamma_{\Omega_n}} \psi(\nu_{\Omega_n})\,\mathrm{d}\mathcal{H}^{N-1} \leq \Lambda_0 d.$$

Hence (3.112) implies that

$$J_\beta(U,w_n)-J_\beta(\Omega_n,w_n)\le (W_0+\Lambda_0-\beta)\,d<0$$

(recall that $\beta>W_0+\Lambda_0$), which is a contradiction to the minimality of (Ω_n,w_n).

In the case $|\Omega_n|>|\Omega_h|$, we can find $\delta\in(-\delta_n,\delta_n)$ such that $|\Omega_n\cap\Omega_{h+\delta}|=|\Omega_h|$. Then, setting $U:=\Omega_n\cap\Omega_{h+\delta}$ and arguing as before, we still contradict the minimality of (Ω_n,w_n).

Step 5. We claim that $\Omega_n^\#$ is an (ω,r_0)-minimizer for the anisotropic perimeter (see Remark 1.6), with ω and r_0 independent of n. Indeed, consider any ball $B_r(x)$ and any set F such that $\Omega_n^\#\triangle F\subset\subset B_r(x)$. By a translation argument we can assume $B_r(x)\subset Q\times\mathbb{R}$; moreover, by taking a sufficiently small r_0 we can also assume without loss of generality that $B_r(x)\subset\Omega'$. Hence, setting $F':=F\cap\Omega'$, we have that $(F',w_n)\in\widetilde X$ is an admissible competitor in problem (3.109). By the minimality of (Ω_n,w_n), we have $\widetilde J_\beta(F',w_n)-\widetilde J_\beta(\Omega_n,w_n)\ge 0$, which yields

$$\begin{aligned}\int_{\partial^*\Omega_n\cap B_r(x)}\psi(\nu_{\Omega_n})\,\mathrm{d}\mathcal{H}^{N-1}&\le\int_{\partial^*F\cap B_r(x)}\psi(\nu_F)\,\mathrm{d}\mathcal{H}^{N-1}+\int_{F'}W(Dw_n)\,\mathrm{d}z-\int_{\Omega_n}W(Dw_n)\,\mathrm{d}z\\&\qquad+\beta\big||F'|-|\Omega_n|\big|+2\beta|F'\triangle T_n(F')|\\&\le\int_{\partial^*F\cap B_r(x)}\psi(\nu_F)\,d\mathcal{H}^{N-1}+(W_0+3\beta)|F'\triangle\Omega_n|,\end{aligned}$$

where we used the fact that $F'\triangle T_n(F')\subset F'\triangle\Omega_n$. Since $|F'\triangle\Omega_n|=|F\triangle\Omega_n^\#|$, the previous inequality proves the claim with $\omega=W_0+3\beta$.

Hence, by Theorem 1.4 (see also Remark 1.6 and Remark 1.7), we deduce that for n large enough Ω_n is a set of class $C^{1,\alpha}$ and it converges to Ω_h in $C^{1,\alpha}$, for all $\alpha\in(0,\frac12)$. In turn, this implies that for n large the set Ω_n is in fact the subgraph of a function $k_n\in C^{1,\alpha}_\#(Q)$ (that is, $\Omega_n=\Omega_{k_n}$), and $k_n\to h$ in $C^{1,\alpha}$ for all $\alpha\in(0,\frac12)$.

Step 6. We claim that $k_n\to h$ in $W^{2,p}$ for every $p\in(1,\infty)$.

Fix $\eta\in C^\infty_\#(Q)$ and set $k_n^\varepsilon:=k_n+\varepsilon\eta$, for $\varepsilon>0$. By the quasi-minimality property of Γ_{k_n} proved in the previous step we have

$$\int_{\Gamma_{k_n}}\psi(\nu_{k_n})\,\mathrm{d}\mathcal{H}^{N-1}\le\int_{\Gamma_{k_n^\varepsilon}}\psi(\nu_{k_n^\varepsilon})\,\mathrm{d}\mathcal{H}^{N-1}+(W_0+3\beta)\,\varepsilon\int_Q|\eta(x)|\,dx.$$

Dividing by ε and letting $\varepsilon\to0$, we deduce

$$\int_Q\nabla\psi(-\nabla k_n,1)\cdot(\nabla\eta,0)\,\mathrm{d}x\le(W_0+3\beta)\|\eta\|_{L^1(Q)}\,.$$

Hence, the left-hand side in the previous inequality defines a continuous linear functional on $L^1_\#(Q)$, that is, denoting by $H^\psi_{k_n}$ the anisotropic mean curvature of Γ_{k_n} and recalling (3.51),

$$-H^\psi_{k_n}(\cdot,k_n(\cdot))=H_n\qquad\text{on }Q$$

in the sense of distributions, for some bounded function H_n whose L^∞-norm is bounded by $W_0+3\beta$. This uniform bound, combined with the convergence of the functions k_n to h in $C^{1,\alpha}$, implies by standard elliptic estimates (see Lemma 3.43) that the functions k_n are equibounded in $W^{2,p}$ for every $p>1$.

We can now write the Euler-Lagrange equations for problem (3.107): since k_n is of class $W^{2,p}$ we have

$$H^\psi_{k_n}(x,k_n(x))=\begin{cases}-W\big(Dw_n(x,k_n(x))\big)+\lambda_n & \text{in } A_n:=\big\{|k_n-h|<\delta_n\big\},\\-W\big(Du(x,h(x)\big)+\lambda & \text{in }\big\{|k_n-h|=\delta_n\big\},\end{cases}$$

where λ_n, λ are the Lagrange multipliers due to the volume constraint. To deduce the equation in A_n we considered variations only of the profile k_n, compactly supported in A_n, while the equation in the complement of A_n easily follows from the fact that (h,u) satisfies (3.46). Notice that the sequence λ_n is bounded, by the uniform bounds on $H^\psi_{k_n}$ and on Dw_n.

Now, if $\mathcal{H}^{N-1}(A_n)\to 0$, we immediately have

$$H^\psi_{k_n}(\cdot,k_n(\cdot))\to H^\psi(\cdot,h(\cdot)) \qquad \text{in } L^p(Q) \text{ for all } p>1. \tag{3.113}$$

Otherwise, assuming that $\mathcal{H}^{N-1}(A_n)\geq c>0$ for all n, integrating the Euler-Lagrange equation in Q we deduce by periodicity that

$$\begin{aligned}
-\int_{A_n} W(Dw_n(x,k_n(x)))\,\mathrm{d}x + \lambda_n\mathcal{H}^{N-1}(A_n) - \int_{Q\setminus A_n} W(Du(x,h(x)))\,\mathrm{d}x + \lambda\mathcal{H}^{N-1}(Q\setminus A_n)\\
= \int_Q H^\psi_{k_n}(x,k_n(x))\,\mathrm{d}x = 0 = \int_Q H^\psi(x,h(x))\,\mathrm{d}x\\
= -\int_Q W(Du(x,h(x)))\,\mathrm{d}x + \lambda\mathcal{H}^{N-1}(Q).
\end{aligned}$$

Now the uniform convergence of Dw_n to Du on Γ_{k_n} and the convergence of k_n to h in $C^{1,\alpha}$ yield $(\lambda_n-\lambda)\mathcal{H}^{N-1}(A_n)\to 0$, and in turn $\lambda_n\to\lambda$ since $\mathcal{H}^{N-1}(A_n)\geq c>0$. Hence, using again the Euler-Lagrange equations, we can conclude that (3.113) holds. In turn, by elliptic regularity (Lemma 3.43) this implies that $k_n\to h$ in $W^{2,p}$ for every $p>1$, as claimed.

Step 7. We are now in position to conclude the proof of the theorem. Since

$$\|k_n-h\|_{W^{2,p}(Q)}\to 0, \quad \|Dw_n-Du\|_{L^\infty(\Omega_{k_n};\mathbb{M}^N)}\to 0,$$

and, by Step 4, $|\Omega_{k_n}|=|\Omega_h|$, inequality (3.108) contradicts the local $W^{2,p}$-minimality of (h,u). ☐

Combining the previous result with Theorem 3.41, we immediately obtain the announced local minimality condition.

THEOREM 3.45. *Assume $N=2,3$. If $(h,u)\in X$ is a strictly stable critical pair, according to Definition 3.36, then (h,u) is a local minimizer for the functional F, in the sense of Definition 3.23.*

We conclude this section by observing that Theorem 3.45 can be extended to the linear elastic case, where we have the following stronger result. Given a set A and a constant $M>0$, we denote by $\mathrm{Lip}_M(A;\mathbb{R}^N)$ the class of Lipschitz functions $v:A\to\mathbb{R}^N$ whose Lipschitz constant is bounded by M.

THEOREM 3.46. *Assume that the elastic energy density has the form*

$$W(\xi):=\frac{1}{2}C\left(\frac{\xi+\xi^T}{2}\right):\left(\frac{\xi+\xi^T}{2}\right), \qquad \xi\in\mathbb{M}^N,$$

for some constant fourth-order tensor C such that

$$C\xi:\xi\geq c_0|\xi|^2 \quad \text{for every } \xi\in\mathbb{M}^N_{\mathrm{sym}}, \quad c_0>0, \tag{3.114}$$

where $\mathbb{M}^N_{\mathrm{sym}}$ denotes the subset of $\mathbb{M}^N$ of the symmetric matrices. If $N=2,3$ and (h,u) is a strictly stable critical pair, then (h,u) is an isolated local minimizer for F in the following sense: for every $M>\|Du\|_\infty$ there exists $\delta=\delta(M)>0$ such that

$$F(h,u)<F(g,v) \tag{3.115}$$

for every $(g,v)\in X$ with $0<\|g-h\|_\infty<\delta$, $|\Omega_g|=|\Omega_h|$, and $v\in\mathrm{Lip}_M(\Omega_g;\mathbb{R}^N)$.

REMARK 3.47. Notice that, by Korn's inequality, the positive definiteness of the tensor C on the space of symmetric matrices implies that condition (3.52) is automatically satisfied. We suspect that, as in the two-dimensional case, in the linearized framework the following stronger result should hold: there exists $\delta > 0$ such that (3.115) is satisfied for every $(g, v) \in X$ with $0 < \|g - h\|_\infty < \delta$, $|\Omega_g| = |\Omega_h|$, and $v \in \mathrm{Lip}(\Omega_g; \mathbb{R}^N)$. In order to prove such a result, we would need a regularity theory for minimizing configurations, which is not yet available in the three-dimensional case.

PROOF OF THEOREM 3.46. We first observe that the conclusion of Theorem 3.41 holds also in this case. Indeed, the construction provided by Proposition 3.29 is now unnecessary, since for every admissible profile g we can consider the unique minimizer u_g of the elastic energy in the corresponding reference configuration Ω_g. By standard elliptic regularity, the map $g \mapsto u_g$ satisfies the conclusions of Proposition 3.29, so that we can repeat the proof of Theorem 3.41 without changes. Notice also that the estimate provided by Lemma 3.42 remains valid in this case, since the fourth order tensor $W_{\xi\xi}$ satisfies the strong ellipticity condition, as a consequence of (3.114).

At this point we can follow the strategy of the proof of Theorem 3.44, where the contradiction hypothesis consists now in assuming the existence of a sequence $(g_n, v_n) \in X$ such that $\delta_n := \|g_n - v_n\|_\infty \to 0$, $|\Omega_{g_n}| = |\Omega_h|$, $v_n \in \mathrm{Lip}_M(\Omega_{g_n})$, and $F(g_n, v_n) \leq F(h, u)$

The approximation argument contained in Step 1 of the previous proof is in this case unnecessary, so that we do not need the strict inequality in (3.106). Indeed, each function v_n can be extended to Ω' without increasing the Lipschitz constant, and we can now consider the penalized minimum problems

$$\min\Big\{ J_\beta(\Omega, v) : (\Omega, v) \in \widetilde{X},\ \Omega_{h-\delta_n} \subset \Omega \subset \Omega_{h+\delta_n},\ v \in \mathrm{Lip}_M(\Omega'; \mathbb{R}^N) \Big\} \tag{3.116}$$

which admits a solution without assuming any *a priori* $W^{2,\infty}$-bound, as we did before. Replacing (3.107) by (3.116), the proof goes exactly as in the previous case, yielding the $C^{1,\alpha}$-convergence of k_n to h at the end of the fifth step; moreover, $k_n \in W^{2,p}(Q)$, as proved in the first part of Step 6.

Observe now that, denoting by $\tilde{w}_n$ the unique minimizer of the (linear) elastic energy in Ω_{k_n}, by the standard regularity of the elliptic system associated with the first variation of the elastic energy we have that $D\tilde{w}_n \circ \Phi_{k_n}$ converge uniformly to Du in $\overline{\Omega}_h$, so that for n sufficiently large the constraint $\tilde{w}_n \in \mathrm{Lip}_M(\Omega')$ is satisfied. Hence we necessarily have $w_n = \tilde{w}_n$: thus w_n is in fact of class C^1 up to Γ_{k_n}, and we can conclude as before, by writing the Euler-Lagrange equations for the penalized problems, that $k_n \to h$ in $W^{2,p}(Q)$.

Finally, in the last step of the proof we deduce, by the isolated local minimality of (h, u) proved in Theorem 3.41, that $k_n = h$ and $w_n = u$ for all sufficiently large n. It follows that (h, u) and, in turn, (g_n, v_n) are solutions to the penalized minimum problem: repeating the same argument for the sequence (g_n, v_n), we conclude that for n sufficiently large $g_n = h$ and $v_n = u$, which is the final contradiction. □

3.8. Stability of the flat configuration

In this section, as an application of our local minimality criterion, we deal with the issue of the stability of the flat configuration. Given a volume $d > 0$, we will assume the existence of an affine critical point for the elastic energy in the domain $\Omega_d = Q \times (0, d)$, namely (recall

Definition 3.21) an affine function $v_0(z) = M[z]$ for some $M \in \mathbb{M}^N_+$ solution to the problem

$$\begin{cases} \operatorname{div}(W_\xi(Dv_0)) = 0 & \text{in } \Omega_d, \\ W_\xi(Dv_0)[e_N] = 0 & \text{on } \Gamma_d, \\ v_0 - u_0 \in \mathcal{V}(\Omega_d), \end{cases} \tag{3.117}$$

where $u_0(x, y) = (A[x], 0)$ is the boundary Dirichlet datum. Notice that an affine function automatically satisfies the first condition (as Dv_0 is constant), but this is not always the case for the second one, that can be rewritten as

$$\frac{\partial W}{\partial \xi_{iN}}(Dv_0) = 0 \qquad \text{for every } i = 1, \dots, N. \tag{3.118}$$

Definition 3.48. A pair $(d, v_0) \in X$, with $v_0(z) = M[z]$, satisfying (3.117) and condition (3.52) will be referred to as *flat configuration* with volume d.

We remark that, whenever it exists, (d, v_0) is obviously a critical pair.

Example 3.49. We now show the existence of an affine critical point for the elastic energy in a flat domain, for boundary data close to the identity, under the assumption that the identical deformation is a strict local minimum of the elastic energy. More precisely, we assume that $W(I) = 0$ and that

$$\int_{\Omega_d} W_{\xi\xi}(I)Dw : Dw \, \mathrm{d}z \geq k\|w\|^2_{H^1(\Omega_d;\mathbb{R}^N)} \qquad \text{for every } w \in \widetilde{\mathcal{V}}(\Omega_d), \tag{3.119}$$

for some $k > 0$. Notice that, as $W \geq 0$ and $W(I) = 0$, necessarily $W_\xi(I) = 0$. We claim that, if $|A - I| < \varepsilon_0$ for some $\varepsilon_0 > 0$ sufficiently small, then there exists an affine solution to (3.117) corresponding to the boundary datum $u_0(x, y) = (A[x], 0)$.

Indeed, given $A \in \mathbb{M}^{N-1}$, we look for a vector $\mathbf{b} = (b_1, \dots, b_N)$ such that the affine function

$$v_{A,\mathbf{b}}(x, y) = (A[x], 0) + y\mathbf{b}$$

satisfies (3.118). We define a map $G : (A, \mathbf{b}) \mapsto W_\xi(Dv_{A,\mathbf{b}})[e_N]$. As $W_\xi(I) = 0$, we have that $G(I, e_N) = 0$. Moreover the matrix $\partial_{\mathbf{b}} G(I, e_N)$ is positive definite (hence invertible), since for every vector $w \in \mathbb{R}^N \setminus \{0\}$

$$\partial_{\mathbf{b}} G(I, e_N)[w, w] = \sum_{i,j=1}^N \frac{\partial^2 W}{\partial \xi_{iN} \partial \xi_{jN}}(I) w_i w_j = W_{\xi\xi}(I)(w \otimes e_N) : (w \otimes e_N) > 0\,,$$

where the last inequality follows from the fact that the tensor $W_{\xi\xi}(I)$ satisfies the strong ellipticity condition (by Theorem 3.28 and (3.119)). Hence the claim follows by applying the Implicit Function Theorem (notice also that the affine critical point constructed in this way satisfies condition (3.52), up to taking a smaller ε_0 if necessary, by continuity and by (3.119)).

When dealing with the flat configuration (d, v_0), it is convenient to identify the space $\widetilde{H}^1_\#(\Gamma_d)$ with the space

$$\widetilde{H}^1_\#(Q) := \Big\{ \varphi \in H^1_{\mathrm{loc}}(\mathbb{R}^{N-1}) : \varphi(x + e_i) = \varphi(x) \text{ for a.e. } x \in \mathbb{R}^{N-1}, \\ \text{for every } i = 1, \dots, N-1, \int_Q \varphi(x) \, \mathrm{d}x = 0 \Big\}.$$

Notice that condition (3.68) is always fulfilled (the coefficient a in (3.67) vanishes), so that

$$\|\varphi\|^2_\sim = \int_Q D^2\psi(e_N)[(\nabla\varphi, 0), (\nabla\varphi, 0)] \, \mathrm{d}x \qquad \text{for every } \varphi \in \widetilde{H}^1_\#(Q)$$

is an equivalent norm on $\widetilde{H}^1_\#(Q)$; in particular, this allows us to discuss the positivity of the second variation at the flat configuration in terms of the quantity $\lambda_1(d)$ defined by (3.70) (here we make explicit the dependence on the height d of the reference configuration).

We now prove a couple of propositions concerning the stability of the flat configuration. Precisely, we show that the flat configuration, whenever it exists, is strictly stable if the volume is sufficiently small, while condition (3.66) is not satisfied if the domain is large enough. In the following, we will always assume to deal with elastic energy densities W which admit a flat configuration.

PROPOSITION 3.50. *There exists $d_0 > 0$ such that for every $d < d_0$*

$$\partial^2 F(d, v_0)[\varphi] > 0 \qquad \text{for every } \varphi \in \widetilde{H}^1_\#(Q)\setminus\{0\}.$$

PROOF. Denote by $\mu_1(d)$ the value of the minimum in (3.71) corresponding to the critical pair (d, v_0); by Theorem 3.37 it is sufficient to show that

$$\lim_{d\to 0^+} \mu_1(d) = +\infty.$$

Assume by contradiction that there exist $C > 0$, a sequence $d_n \to 0^+$ and a sequence $v_n \in \widetilde{\mathcal{V}}(\Omega_{d_n})$ such that $\|\Phi_{v_n}\|_\sim = 1$ and

$$\int_{\Omega_{d_n}} W_{\xi\xi}(Dv_0)Dv_n : Dv_n \,\mathrm{d}z \le C.$$

Then the functions

$$\tilde{v}_n(x,y) := \begin{cases} 0 & \text{if } 0 \le y \le 1 - d_n \\ v_n(x, y-1+d_n) & \text{if } 1-d_n < y \le 1 \end{cases}$$

belong to $\widetilde{\mathcal{V}}(\Omega_1)$, $\|\Phi_{\tilde{v}_n}\|_\sim = \|\Phi_{v_n}\|_\sim = 1$ and satisfy

$$\int_{\Omega_1} W_{\xi\xi}(Dv_0)D\tilde{v}_n : D\tilde{v}_n \,\mathrm{d}z \le C.$$

It follows that, up to subsequences, $\tilde{v}_n$ converges weakly to 0 in $\widetilde{\mathcal{V}}(\Omega_1)$. From the compactness of the map $v \mapsto \Phi_v$ we conclude that $\Phi_{\tilde{v}_n} \to 0$ strongly in $\widetilde{H}^1_\#(Q)$, a contradiction with the fact that $\|\Phi_{\tilde{v}_n}\|_\sim = 1$. □

In order to show a situation where the flat configuration is no longer a local minimizer, we slightly modify the setting of the problem defining, for $d > 0$, $Q_d = (0,d)^{N-1}$ and $\Omega_d = (0,d)^N$; all the notions considered up to now are extended to this situation in the natural way.

PROPOSITION 3.51. *There exists $d_1 > 0$ such that the quadratic form $\partial^2 F(d, v_0)$ is not positive semidefinite for all $d > d_1$. In particular, for all $d > d_1$ the flat configuration (d, v_0) is not a local minimizer for F.*

PROOF. Consider a nontrivial solution $(v, \varphi) \in \widetilde{\mathcal{V}}(\Omega_1) \times \widetilde{H}^1_\#(Q)$ of (3.73) in Ω_1 with $\lambda = \lambda_1(1)$. Setting $v_d(z) = v(\frac{z}{d})$, $\varphi_d(x) = d\,\varphi(\frac{x}{d})$, a direct computation shows that (v_d, φ_d) is a nontrivial solution of (3.73) in Ω_d corresponding to $\lambda = d\,\lambda_1(1)$. Hence $\lambda_1(d) \ge d\,\lambda_1(1)$, and taking $d_1 = \frac{1}{\lambda_1(1)}$ we get that $\lambda_1(d) > 1$ for every $d > d_1$. From this it is easily seen, using (3.72), that the quadratic form $\partial^2 F(d, v_0)$ is not positive semidefinite for all $d > d_1$. The last part of the statement follows from Theorem 3.35. □

3.9. The crystalline case

We conclude our analysis by discussing the local minimality of the flat configuration in the case of *crystalline* anisotropies, that is when we assume less regularity in the anisotropic surface density. Precisely, we assume here that $\psi_c : \mathbb{R}^N \to [0, +\infty)$ is a positively 1-homogeneous and convex function, such that the associated Wulff shape W_{ψ_c} contains a neighborhood of the origin and its boundary has a flat horizontal facet intersecting the y-axis. Under these assumptions, the model exhibits a different qualitative behavior: we can show that the flat configuration is always a local minimizer, whatever the volume $d > 0$.

We will first prove the result in details in the setting considered in Section 3.1, that is, in the two-dimensional case and in the framework of linearized elasticity (Theorem 3.53). Then we show that the same conclusion holds also in the general setting introduced in Section 3.2 (Theorem 3.56).

Let $\psi_c : \mathbb{R}^N \to [0, +\infty)$ be a surface energy density satisfying the following assumptions:

(C1) ψ_c is a positively 1-homogeneous and convex function;
(C2) the associated Wulff shape W_{ψ_c} contains a neighborhood of the origin;
(C3) the boundary of W_{ψ_c} contains a horizontal facet: precisely, we assume that

$$\big\{(x, y) \in \mathbb{R}^N : |x| \le a_1,\, y = a_2\big\} \subset \partial W_{\psi_c} \qquad \text{for some } a_1, a_2 > 0.$$

Remark 3.52. We recall (see [40], [44], [80]) that the *Wulff shape* associated with a function $\psi : \mathbb{S}^{N-1} \to (0, +\infty)$ is the convex set

$$W_\psi = \{z \in \mathbb{R}^N \,:\, z \cdot v \le \psi(v) \text{ for every } v \in \mathbb{S}^{N-1}\}\,, \tag{3.120}$$

which coincides with the unique minimizer (up to translations) of the "anisotropic isoperimetric problem"

$$\min\left\{ \int_{\partial^* E} \psi(\nu_E)\, \mathrm{d}\mathcal{H}^{N-1} : E \subset \mathbb{R}^N \text{ has finite perimeter, } |E| = |W_\psi| \right\}.$$

Viceversa, every compact convex set K containing a neighborhood of the origin is the Wulff set associated with the convex function

$$\psi_K(v) = \sup\{z \cdot v : z \in K\}. \tag{3.121}$$

We start by considering the linearized two-dimensional framework introduced in Section 3.1. We first remark that the relaxation result stated Theorem 3.2 still holds under the current assumptions on ψ_c: hence, setting $\sigma_c = \psi_c(1, 0) + \psi_c(-1, 0)$, we consider the functional

$$G_c(h, u) = \int_{\Omega_h} W(u)\, \mathrm{d}z + \int_{\Gamma_h} \psi_c(\nu_h)\, \mathrm{d}\mathcal{H}^1 + \sigma_c\, \mathcal{H}^1(\Sigma_h), \qquad (h, u) \in X(u_0; 0, b).$$

Theorem 3.53. *Assume that ψ_c satisfies* (C1), (C2), (C3). *For every $b > 0$, $d > 0$ and $e_0 > 0$, the flat configuration $(\frac{d}{b}, v_{e_0})$ corresponding to the volume d and the boundary Dirichlet datum $u_0(x, 0) = (e_0 x, 0)$ is an isolated b-periodic local minimizer for G_c, in the sense of Definition 3.3.*

Proof. The strategy of the proof will be the following: first of all, we show that we do not lose in generality if we prove the theorem for crystalline anisotropies of a particular form (namely, whose Wulff shape is a rectangle with sides parallel to the coordinate axes). Then, we conclude using an approximation argument combined with the results obtained in Section 3.1 for the regular case. We divide the proof into three steps.

Step 1. From the assumptions on ψ_c it follows that we can find $0 < b_1 \leq a_1$, $b_2 > 0$ such that the rectangle $R = \{(x, y) : |x| \leq b_1, -b_2 \leq y \leq a_2\}$ is contained in the Wulff shape W_{ψ_c}. Denote by ψ_R the function whose Wulff shape is R, given by

$$\psi_R(\nu_1, \nu_2) = \begin{cases} b_1|\nu_1| + a_2|\nu_2| & \text{if } \nu_2 \geq 0, \\ b_1|\nu_1| + b_2|\nu_2| & \text{if } \nu_2 < 0, \end{cases}$$

(see (3.121)), and by G_R the functional corresponding to this anisotropic surface density. Notice that, since $R \subset W_{\psi_c}$, by (3.121) it follows immediately that $\psi_R \leq \psi_c$; moreover

$$\psi_R(0, 1) = a_2 = \psi_c(0, 1) \tag{3.122}$$

(concerning the second equality see, for instance, [40, Proposition 3.5 (iv)]).

Step 2. We introduce a family of "approximating" functionals, defined as follows. We consider, for $\varepsilon > 0$, the family of anisotropic surface densities

$$\psi_\varepsilon(x, y) = b_1\sqrt{\varepsilon^2 y^2 + x^2} + (a_2 - b_1\varepsilon)|y|,$$

and the associated functionals

$$G_\varepsilon(h, u) = \int_{\Omega_h} W(u)\,\mathrm{d}z + \int_{\Gamma_h} \psi_\varepsilon(\nu_h)\,\mathrm{d}\mathcal{H}^1 + 2b_1\,\mathcal{H}^1(\Sigma_h)\,.$$

The functions ψ_ε converge monotonically as $\varepsilon \to 0^+$ to ψ_R in $\mathbb{R} \times [0, +\infty)$: indeed, it is sufficient to observe that for $(x, y) \in \mathbb{R} \times [0, +\infty)$

$$\begin{aligned} \psi_\varepsilon(x, y) &= b_1\sqrt{\varepsilon^2 y^2 + x^2} + (a_2 - b_1\varepsilon)y \\ &= \frac{b_1^2 x^2}{b_1\sqrt{\varepsilon^2 y^2 + x^2} + b_1\varepsilon y} + a_2 y \ \nearrow\ b_1|x| + a_2 y = \psi_R(x, y)\,. \end{aligned} \tag{3.123}$$

From a geometrical point of view, this means that the Wulff shapes associated with the functions ψ_ε are converging monotonically from the interior to the corresponding one associated with ψ_R in the upper half-plane (see Figure 1).

Consider now the functionals $\widehat{G}_\varepsilon$ corresponding to the regular surface densities

$$\hat\psi_\varepsilon(x, y) = b_1\sqrt{\varepsilon^2 y^2 + x^2};$$

the functions $\hat\psi_\varepsilon$ satisfy all the assumptions considered in the regular case: in particular, condition (3.1) follows after some computations from the formula

$$D^2\hat\psi_\varepsilon(v)[w, w] = \frac{b_1}{\sqrt{v_1^2 + \varepsilon^2 v_2^2}}\left[(w_1^2 + \varepsilon^2 w_2^2) - \frac{(v_1 w_1 + \varepsilon^2 v_2 w_2)^2}{v_1^2 + \varepsilon^2 v_2^2}\right],$$

where $v = (v_1, v_2)$ and $w = (w_1, w_2)$. The general analysis developed in Section 3.1 applies to the functional $\widehat{G}_\varepsilon$: in particular, since $\partial^2_{11}\hat\psi_\varepsilon(0, 1) = \frac{b_1}{\varepsilon}$, from Theorem 3.18 it follows that, given any $b > 0$ and $e_0 > 0$, there exists $\varepsilon_0 = \varepsilon_0(b, e_0) > 0$ such that if $0 < \varepsilon \leq \varepsilon_0$ the flat configuration $(\frac{d}{b}, v_{e_0})$ is an isolated b-periodic local minimizer for $\widehat{G}_\varepsilon$ for every volume $d > 0$. The same is true also for G_ε, since the energies G_ε and $\widehat{G}_\varepsilon$ differ only by a constant value: $G_\varepsilon = \widehat{G}_\varepsilon + (a_2 - b_1\varepsilon)b$.

Step 3. Given $b > 0$, $d > 0$, $e_0 > 0$, let $\varepsilon_0 = \varepsilon_0(b, e_0)$ be as above, and let $\delta > 0$ be such that the flat configuration minimizes the energy G_{ε_0} among all competitors satisfying the volume constraint and whose L^∞-distance from the flat configuration is less than δ.

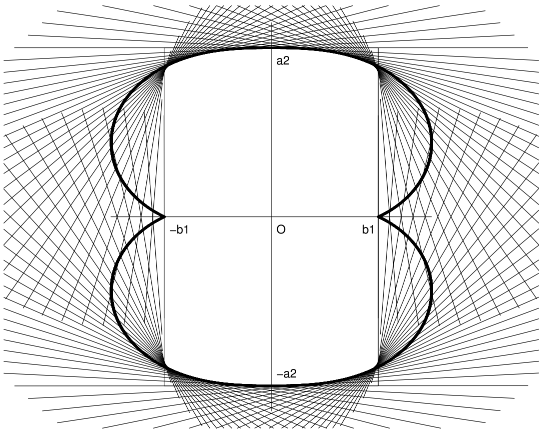

FIGURE 1. The Wulff shape corresponding to the anisotropy ψ_ε is an approximation from the interior of the symmetric rectangle $R_0 = \{|x| \le b_1, |y| \le a_2\}$. To construct the Wulff shape associated with a function ψ, consider at every point $\psi(\nu)\nu$, $\nu \in \mathbb{S}^1$, of the polar plot of ψ (the bold curve in the figure), the line orthogonal to the radius vector and passing through that point: the Wulff shape is the intersection of all the halfplanes containing the origin and whose boundary is one of these lines (see (3.120)).

Then, for all $(g, v) \in X(u_0; 0, b)$ such that $|\Omega_g| = d$ and $0 < \|g - \frac{d}{b}\|_\infty < \delta$ we have, using condition (3.122),

$$G_c\big(\tfrac{d}{b}, v_{e_0}\big) = \int_{\Omega_{d/b}} W(v_{e_0})\, \mathrm{d}z + b\, \psi_c(0,1) = \int_{\Omega_{d/b}} W(v_{e_0})\, \mathrm{d}z + b\, \psi_R(0,1)$$
$$= G_R\big(\tfrac{d}{b}, v_{e_0}\big) = G_{\varepsilon_0}\left(\tfrac{d}{b}, v_{e_0}\right) < G_{\varepsilon_0}(g,v) \le G_R(g,v) \le G_c(g,v)\,,$$

where the first inequality follows from the local minimality of the flat configuration for G_{ε_0}, the second one is a straight consequence of (3.123) and the last one follows using $\psi_R \le \psi_c$. From the previous chain of inequalities the conclusion follows. □

REMARK 3.54. Concerning the global minimality of the flat configuration in the crystalline case, an argument similar to the one used in the previous proof combined with the result stated in Remark 3.19 shows that, for every $b > 0$ and $e_0 > 0$, the flat configuration $(\frac{d}{b}, v_{e_0})$ is a global minimizer if the volume d is sufficiently small.

REMARK 3.55. A natural question arising from the previous analysis is whether in the crystalline case the flat configuration is always a *global* minimizer. This is in fact not true, at least if the interval of periodicity is sufficiently large. Indeed, we first recall that in [45, Proposition 2.12] was proved that, for b sufficiently large, the threshold of global minimality is strictly smaller than the threshold of local minimality. The same comparison argument used to prove that result shows that, if ψ_R is an anisotropy whose associated Wulff shape is a rectangle (as in Step 1 of the proof of Theorem 3.53), then for every $s > 0$ there exists $b > 0$ such that one can construct a b-periodic competitor (g, v) whose energy is strictly below the energy of the flat configuration (s, v_{e_0}): indeed, it is sufficient to observe that the surface energy corresponding to ψ_R coincides, up to constant factors, with the isotropic surface energy when evaluated on the flat configuration and on the competitor constructed in the proof of [45, Proposition 2.12]. Finally, the same is true for a general anisotropy ψ_c

satisfying assumptions (C1)–(C3): in fact, one can always find a rectangle R containing the associated Wulff shape whose upper side contains the horizontal facet, in such a way that

$$\psi_R(0,1)=\psi_c(0,1), \qquad \psi_c\leq\psi_R,$$

hence $G_c(g,v)\leq G_R(g,v)<G_R(s,v_{e_0})=G_c(s,v_{e_0})$.

We conclude this section by proving the result analogous to Theorem 3.53 in the more general context introduced in Section 3.2.

THEOREM 3.56. *Let $N=2,3$ and let F_c be the functional defined in (3.44) associated to a surface energy density ψ_c satisfying (C1), (C2), (C3). Then for every $d>0$ the flat configuration (d,v_0) is a local minimizer for F_c, in the sense of Definition 3.23.*

PROOF (SKETCH). The proof is the same as for Theorem 3.53.

Since we always evaluate the function ψ_c at vectors whose last component is nonnegative, without loss of generality we can assume that the Wulff shape W_{ψ_c} is symmetric with respect to the hyperplane $\{y=0\}$. From the assumptions on ψ_c it follows that the cylinder $C=\{(x,y):|x|\leq a_1,|y|\leq a_2\}$ is contained in W_{ψ_c}. Let $\psi_C(x,y)=a_1|x|+a_2|y|$ be an anisotropy whose Wulff shape is exactly the cylinder C, and observe that

$$\psi_C\leq\psi_c, \qquad \psi_c(0,1)=\psi_C(0,1)=a_2\,. \tag{3.124}$$

As we did before, we now introduce a family of "approximating" functionals: let F_ε and $\widehat{F}_\varepsilon$, for $\varepsilon>0$, be the functionals associated to the anisotropies

$$\psi_\varepsilon(x,y)=a_1\sqrt{\varepsilon^2y^2+|x|^2}+(a_2-a_1\varepsilon)|y|, \qquad \hat{\psi}_\varepsilon(x,y)=a_1\sqrt{\varepsilon^2y^2+|x|^2},$$

respectively (notice that ψ_ε converges monotonically from below to ψ_C as $\varepsilon\to0^+$, and, geometrically, the Wulff shapes associated with the functions ψ_ε converge monotonically from the interior to the cylinder C).

The functions $\hat{\psi}_\varepsilon$ satisfy all the assumptions of Section 3.2 (in particular, the uniform convexity condition (3.43) follows from the explicit computation of the hessian of $\hat{\psi}_\varepsilon$), and the quadratic form associated to the second variation of $\widehat{F}_\varepsilon$ at the flat configuration turns out to be

$$\partial^2\widehat{F}_\varepsilon(d,v_0)[\varphi]=-\int_{Q\times(0,d)}W_{\xi\xi}(Dv_0)Dv_\varphi:Dv_\varphi\,\mathrm{d}z+\frac{a_1}{\varepsilon}\int_Q|\nabla\varphi|^2\,\mathrm{d}\mathcal{H}^{N-1}.$$

Since

$$\int_{Q\times(0,d)}W_{\xi\xi}(Dv_0)Dv_\varphi:Dv_\varphi\,\mathrm{d}z\leq C\|v_\varphi\|^2_{H^1(\Omega_d;\mathbb{R}^2)}\leq C'\|\varphi\|^2_{H^1(Q)}$$

(where C,C' are positive constants depending only on the boundary Dirichlet datum), it follows that there exists $\varepsilon_0>0$ such that the quadratic form $\partial^2\widehat{F}_{\varepsilon_0}(d,v_0)$ is positive definite. Hence, by Theorem 3.45, the flat configuration (d,v_0) is a local minimizer for $\widehat{F}_{\varepsilon_0}$ for every volume $d>0$. The same is true also for F_{ε_0}, since the energies F_{ε_0} and $\widehat{F}_{\varepsilon_0}$ differ only by a constant value: $F_{\varepsilon_0}=\widehat{F}_{\varepsilon_0}+(a_2-a_1\varepsilon_0)$.

To conclude, let $\delta>0$ be such that the flat configuration minimizes the energy F_{ε_0} among all competitors $(g,v)\in X'$ such that $|\Omega_g|=d$, $0<\|g-d\|_\infty<\delta$, and $\|Dv-Dv_0\|_{L^\infty(\Omega';\mathbb{M}^N)}<\delta$. Then for every such (g,v) we have

$$\begin{aligned}F_c(d,v_0)&=\int_{Q\times(0,d)}W(Dv_0)\,\mathrm{d}z+\psi_c(0,1)=\int_{Q\times(0,d)}W(Dv_0)\,\mathrm{d}z+\psi_C(0,1)\\&=F_C(d,v_0)=F_{\varepsilon_0}(d,v_0)\leq F_{\varepsilon_0}(g,v)\leq F_C(g,v)\leq F_c(g,v),\end{aligned}$$

where the first inequality follows from the local minimality of the flat configuration for F_{ε_0}, the second one from $\psi_\varepsilon \leq \psi_C$ and the last one using $\psi_C \leq \psi_c$. From the previous chain of inequalities the conclusion follows. □

REMARK 3.57. If W is as in Theorem 3.46 and under the assumptions of Theorem 3.56, we conclude that for every $d > 0$ the flat configuration satisfies the isolated local minimality property stated in Theorem 3.46.

CHAPTER 4

A nonlocal isoperimetric problem

In this chapter we provide a description of the energy landscape of the family of functionals

$$\mathcal{F}(E) := \mathcal{P}(E) + \gamma \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \,\mathrm{d}x\mathrm{d}y\,, \qquad \alpha \in (0, N-1),\, \gamma > 0 \tag{4.1}$$

defined over finite perimeter sets $E \subset \mathbb{R}^N$, $N \geq 2$. We will usually denote the nonlocal term in the total energy (4.1) by

$$\mathcal{NL}_\alpha(E) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\chi_E(x)\chi_E(y)}{|x-y|^\alpha} \,\mathrm{d}x\mathrm{d}y\,, \tag{4.2}$$

and we will omit the subscript α when there is no risk of ambiguity. When needed, we will also underline the dependence of the functional on the parameters α and γ by writing $\mathcal{F}_{\alpha,\gamma}$ instead of $\mathcal{F}$.

Organization of the chapter. In Section 4.1 we set up the problem and we list the main results of this chapter. The notion of second variation of the functional $\mathcal{F}$ is introduced in Section 4.2 (the explicit computation is carried out in Section 4.6); here we present also the first part of the proof of the main result, which is completed in Section 4.3. In Section 4.4 we compute the second variation at the ball, and we discuss its local minimality by applying our sufficiency criterion. Finally, Section 4.5 is devoted to the proof of the results concerning the global minimality issues.

4.1. Statements of the results

We start our analysis with some preliminary observations about the features of the energy functional (4.1), before listing the main results of this chapter.

Given a measurable set $E \subset \mathbb{R}^N$, we introduce an auxiliary function v_E by setting

$$v_E(x) := \int_E \frac{1}{|x-y|^\alpha} \,\mathrm{d}y \qquad \text{for } x \in \mathbb{R}^N. \tag{4.3}$$

The function v_E can be characterized as the solution to the equation

$$(-\Delta)^s v_E = c_{N,s}\, \chi_E\,, \qquad s = \frac{N-\alpha}{2} \tag{4.4}$$

where $(-\Delta)^s$ denotes the fractional laplacian and $c_{N,s}$ is a constant depending on the dimension and on s (see [37] for an introductory account on this operator and the references contained therein). Notice that we are interested in those values of s which range in the interval $(\frac{1}{2}, \frac{N}{2})$. We collect in the following proposition some regularity properties of the function v_E.

PROPOSITION 4.1. *Let $E \subset \mathbb{R}^N$ be a measurable set with $|E| \leq m$. Then there exists a constant C, depending only on N, α and m, such that*

$$\|v_E\|_{W^{1,\infty}(\mathbb{R}^N)} \leq C\,.$$

Moreover, $v_E \in C^{1,\beta}(\mathbb{R}^N)$ for every $\beta < N-\alpha-1$ and

$$\|v_E\|_{C^{1,\beta}(\mathbb{R}^N)} \le C'$$

for some positive constant C' depending only on N, α, m and β.

PROOF. The first part of the result is proved in [57, Lemma 4.4], but we repeat here the easy proof for the reader's convenience. By (4.3),

$$v_E(x) = \int_{B_1(x)\cap E} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y + \int_{E\setminus B_1(x)} \frac{1}{|x-y|^\alpha}\,\mathrm{d}y \le \int_{B_1} \frac{1}{|y|^\alpha}\,\mathrm{d}y + m \le C.$$

By differentiating (4.3) in x and arguing similarly, we obtain

$$|\nabla v_E(x)| \le \alpha \int_E \frac{1}{|x-y|^{\alpha+1}}\,\mathrm{d}y \le \alpha \int_{B_1} \frac{1}{|y|^{\alpha+1}}\,\mathrm{d}y + \alpha\, m \le C.$$

Finally, by adding and subtracting the term $\frac{(x-y)|z-y|^\beta}{|x-y|^{\alpha+\beta+2}} - \frac{(z-y)|x-y|^\beta}{|z-y|^{\alpha+\beta+2}}$, we can write

$$\begin{aligned}
|\nabla v_E(x) - \nabla v_E(z)| &\le \alpha \int_E \left| \frac{x-y}{|x-y|^{\alpha+2}} - \frac{z-y}{|z-y|^{\alpha+2}} \right| dy \\
&\le \alpha \int_E \left(\frac{1}{|x-y|^{\alpha+\beta+1}} + \frac{1}{|z-y|^{\alpha+\beta+1}} \right) \big| |x-y|^\beta - |z-y|^\beta \big|\,\mathrm{d}y \qquad (4.5)\\
&\quad + \alpha \int_E \left| \frac{(x-y)|z-y|^\beta}{|x-y|^{\alpha+\beta+2}} - \frac{(z-y)|x-y|^\beta}{|z-y|^{\alpha+\beta+2}} \right| \mathrm{d}y
\end{aligned}$$

Observe now that for every $v, w \in \mathbb{R}^N \setminus \{0\}$

$$\begin{aligned}
\left| \frac{v}{|v|}|w|^{\alpha+2\beta+1} - \frac{w}{|w|}|v|^{\alpha+2\beta+1} \right| &= \big| v\,|v|^{\alpha+2\beta} - w\,|w|^{\alpha+2\beta} \big| \le C \max\{|v|,|w|\}^{\alpha+2\beta}|v-w| \\
&\le C \max\{|v|,|w|\}^{\alpha+\beta+1}|v-w|^\beta
\end{aligned}$$

where C depends on N, α and β. Using this inequality to estimate the second term in (4.5) we deduce

$$\begin{aligned}
&|\nabla v_E(x) - \nabla v_E(z)| \\
&\quad \le \alpha |x-z|^\beta \int_E \left(\frac{1}{|x-y|^{\alpha+\beta+1}} + \frac{1}{|z-y|^{\alpha+\beta+1}} + \frac{C}{\min\{|x-y|,|z-y|\}^{\alpha+\beta+1}} \right) \mathrm{d}y
\end{aligned}$$

which completes the proof of the proposition, since the last integral is bounded by a constant depending only on N, α, m and β. □

REMARK 4.2. In the case $\alpha = N-2$, the function v_E solves the equation $-\Delta v_E = c_N\,\chi_E$, and the nonlocal term is exactly

$$\mathcal{NL}_{N-2}(E) = \int_{\mathbb{R}^N} |\nabla v_E(x)|^2\,\mathrm{d}x\,.$$

By standard elliptic regularity, $v_E \in W^{2,p}_{\mathrm{loc}}(\mathbb{R}^N)$ for every $p \in [1,+\infty)$.

PROPOSITION 4.3 (Lipschitzianity of the nonlocal term). *Given $\bar\alpha \in (0, N-1)$ and $m \in (0,+\infty)$, there exists a constant c_0, depending only on N, $\bar\alpha$ and m such that for every measurable sets $E, F \subset \mathbb{R}^N$ with $|E|, |F| \le m$ and for every $\alpha \le \bar\alpha$*

$$|\mathcal{NL}_\alpha(E) - \mathcal{NL}_\alpha(F)| \le c_0 |E \triangle F|\,.$$

PROOF. We have that

$$\begin{aligned}\mathcal{NL}_\alpha(E)-\mathcal{NL}_\alpha(F)&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{\chi_E(x)(\chi_E(y)-\chi_F(y))}{|x-y|^\alpha}+\frac{\chi_F(y)(\chi_E(x)-\chi_F(x))}{|x-y|^\alpha}\right)\mathrm{d}x\mathrm{d}y\\&=\int_{E\setminus F}\big(v_E(x)+v_F(x)\big)\,\mathrm{d}x-\int_{F\setminus E}\big(v_E(x)+v_F(x)\big)\,\mathrm{d}x\\&\le\int_{E\triangle F}\big(v_E(x)+v_F(x)\big)\,\mathrm{d}x\le 2C\,|E\triangle F|,\end{aligned}$$

where the constant C is provided by Proposition 4.1, whose proof shows also that it can be chosen independently of $\alpha\le\bar\alpha$. □

The issue of existence and characterization of *global minimizers* of the problem

$$\min\left\{\mathcal{F}(E)\,:\,E\subset\mathbb{R}^N,\,|E|=m\right\}\,, \tag{4.6}$$

for $m>0$, is not at all an easy task. A principal source of difficulty in applying the direct method of the Calculus of Variations comes from the lack of compactness of the space with respect to L^1-convergence of sets (with respect to which the functional is lower semi-continuous). It is in fact well known that the minimum problem (4.6) does not admit a solution for certain ranges of masses.

Besides the notion of global minimality, we will address also the study of sets which minimize locally the functional with respect to small L^1-perturbations. By translation invariance, we measure the L^1-distance of two sets modulo translations by the quantity

$$\alpha(E,F):=\min_{x\in\mathbb{R}^N}|E\triangle(x+F)|\,. \tag{4.7}$$

DEFINITION 4.4. We say that $E\subset\mathbb{R}^N$ is a *local minimizer* for the functional (4.1) if there exists $\delta>0$ such that

$$\mathcal{F}(E)\le\mathcal{F}(F)$$

for every $F\subset\mathbb{R}^N$ such that $|F|=|E|$ and $\alpha(E,F)\le\delta$. We say that E is an *isolated local minimizer* if the previous inequality is strict whenever $\alpha(E,F)>0$.

The first order condition for minimality, coming from the first variation of the functional (see (4.12), and also [26, Theorem 2.3]), requires a C^2-minimizer E (local or global) to satisfy the Euler-Lagrange equation

$$H_{\partial E}(x)+2\gamma v_E(x)=\lambda\qquad\text{for every } x\in\partial E \tag{4.8}$$

for some constant λ which plays the role of a Lagrange multiplier associated with the volume constraint. Here $H_{\partial E}:=\mathrm{div}_{\partial E}\nu_E$ denotes the curvature of E with respect to the outer normal ν_E, according to (1.5). Following [1], we define *critical sets* as those satisfying (4.8) in a weak sense, for which further regularity can be gained *a posteriori* (see Remark 4.6).

DEFINITION 4.5. We say that $E\subset\mathbb{R}^N$ is a *regular critical set* for the functional (4.1) if E is a bounded set of class C^1 and (4.8) holds weakly on ∂E, *i.e.*,

$$\int_{\partial E}\mathrm{div}_{\partial E}\zeta\,\mathrm{d}\mathcal{H}^{N-1}=-2\gamma\int_{\partial E}v_E\,\langle\zeta,\nu_E\rangle\,\mathrm{d}\mathcal{H}^{N-1}$$

for every $\zeta\in C^1(\mathbb{R}^N;\mathbb{R}^N)$ such that $\int_{\partial E}\langle\zeta,\nu_E\rangle\,\mathrm{d}\mathcal{H}^{N-1}=0$.

REMARK 4.6. By Proposition 4.1 and by standard regularity (see, *e.g.*, [8, Proposition 7.56 and Theorem 7.57]) a critical set E is of class $W^{2,2}$ and $C^{1,\beta}$ for all $\beta\in(0,1)$. In turn, recalling Proposition 4.1, by Schauder estimates (see [48, Theorem 9.19]) we have that E is of class $C^{3,\beta}$ for all $\beta\in(0,N-\alpha-1)$.

We collect in the following theorem some regularity properties of local and global minimizers, which are mostly known (see, for instance, [57, 62, 77] for global minimizers, and [1] for local minimizers in a periodic setting). The basic idea is to show that a minimizer solves a suitable penalized minimum problem, where the volume constraint is replaced by a penalization term in the functional, and to deduce that a quasi-minimality property is satisfied (see Definition 1.1).

THEOREM 4.7. *Let $E\subset\mathbb{R}^N$ be a global or local minimizer for the functional (4.1) with volume $|E|=m$. Then the reduced boundary ∂^*E is a $C^{3,\beta}$-manifold for all $\beta<N-\alpha-1$, and the Hausdorff dimension of the singular set satisfies $\dim_{\mathcal{H}}(\partial E\setminus\partial^*E)\leq N-8$. Moreover, E is (essentially) bounded. Finally, every global minimizer is connected, and every local minimizer has at most a finite number of connected components* [1].

PROOF. We divide the proof into three steps, following the ideas contained in [1, Proposition 2.7 and Theorem 2.8] in the first part.

Step 1. We claim that there exists $\Lambda>0$ such that E is a solution to the penalized minimum problem

$$\min\left\{\mathcal{F}(F)+\Lambda\big||F|-|E|\big| \,:\, F\subset\mathbb{R}^N,\, \alpha(F,E)\leq\frac{\delta}{2}\right\},$$

where δ is as in Definition 4.4 (the obstacle $\alpha(F,E)\leq\frac{\delta}{2}$ is not present in the case of a global minimizer). To obtain this, it is in fact sufficient to show that there exists $\Lambda>0$ such that if $F\subset\mathbb{R}^N$ satisfies $\alpha(F,E)\leq\frac{\delta}{2}$ and $\mathcal{F}(F)+\Lambda\big||F|-|E|\big|\leq\mathcal{F}(E)$, then $|F|=|E|$.

Assume by contradiction that there exist sequences $\Lambda_h\to+\infty$ and $E_h\subset\mathbb{R}^N$ such that $\alpha(E_h,E)\leq\frac{\delta}{2}$, $\mathcal{F}(E_h)+\Lambda_h\big||E_h|-|E|\big|\leq\mathcal{F}(E)$, and $|E_h|\neq|E|$. Notice that, since $\Lambda_h\to+\infty$, we have $|E_h|\to|E|$.

We define new sets $F_h:=\lambda_hE_h$, where $\lambda_h=\left(\frac{|E|}{|E_h|}\right)^{\frac{1}{N}}\to1$, so that $|F_h|=|E|$. Then we have, for h sufficiently large, that $\alpha(F_h,E)\leq\delta$ and

$$\begin{aligned}\mathcal{F}(F_h)&=\mathcal{F}(E_h)+(\lambda_h^{N-1}-1)\mathcal{P}(E_h)+\gamma(\lambda_h^{2N-\alpha}-1)\mathcal{NL}_\alpha(E_h)\\&\leq\mathcal{F}(E)+(\lambda_h^{N-1}-1)\mathcal{P}(E_h)+\gamma(\lambda_h^{2N-\alpha}-1)\mathcal{NL}_\alpha(E_h)-\Lambda_h\big||E_h|-|E|\big|\\&=\mathcal{F}(E)+|\lambda_h^N-1|\,|E_h|\left(\frac{\lambda_h^{N-1}-1}{|\lambda_h^N-1|}\,\frac{\mathcal{P}(E_h)}{|E_h|}+\gamma\,\frac{\lambda_h^{2N-\alpha}-1}{|\lambda_h^N-1|}\,\frac{\mathcal{NL}_\alpha(E_h)}{|E_h|}-\Lambda_h\right)<\mathcal{F}(E),\end{aligned}$$

which contradicts the local minimality of E (notice that the same proof works also in the case of global minimizers).

Step 2. From the previous step, it follows that E is an (ω,r_0)-minimizer of the area functional for suitable $\omega>0$ and $r_0>0$ (see Definition 1.1). Indeed, choose r_0 such that $\omega_Nr_0^N\leq\frac{\delta}{2}$: then if F is such that $F\triangle E\subset\subset B_r(x)$ with $r<r_0$, we clearly have that $\alpha(F,E)\leq\frac{\delta}{2}$ and by minimality of E we deduce that

$$\begin{aligned}\mathcal{P}(E)&\leq\mathcal{P}(F)+\gamma\big(\mathcal{NL}_\alpha(F)-\mathcal{NL}_\alpha(E)\big)+\Lambda\big||F|-|E|\big|\\&\leq\mathcal{P}(F)+\big(\gamma c_0+\Lambda\big)|E\triangle F|\end{aligned}$$

(using Proposition 4.3), and the claim follows with $\omega:=\gamma c_0+\Lambda$.

[1] Here and in the rest of this chapter *connectedness* is intended in a measure-theoretic sense: E is said to be *connected* (or *indecomposable*) if $E=E_1\cup E_2$, $|E|=|E_1|+|E_2|$ and $\mathcal{P}(E)=\mathcal{P}(E_1)+\mathcal{P}(E_2)$ imply $|E_1||E_2|=0$. A *connected component* of E is any connected subset $E_0\subset E$ such that $|E_0|>0$ and $\mathcal{P}(E)=\mathcal{P}(E_0)+\mathcal{P}(E\setminus E_0)$.

Step 3. The $C^{1,\frac{1}{2}}$-regularity of $\partial^* E$, as well as the condition on the Hausdorff dimension of the singular set, follows from classical regularity results for (ω, r_0)-minimizers of the area functional (see the discussion in Section 1.2 or, *e.g.*, [79, Theorem 1]). In turn, the $C^{3,\beta}$-regularity follows from the Euler-Lagrange equation, as in Remark 4.6.

To show the essential boundedness, we use the density estimates for (ω, r_0)-minimizers, which guarantee the existence of a positive constant $\vartheta_0 > 0$ (depending only on N) such that for every point $y \in \partial^* E$ and $r < \min\{r_0, 1/(2N\omega)\}$

$$\mathcal{P}(E; B_r(y)) \geq \vartheta_0 r^{N-1} \tag{4.9}$$

(see, *e.g.*, [64, Theorem 21.11]). Assume by contradiction that there exists a sequence of points $x_n \in \mathbb{R}^N \setminus E^{(0)}$, where

$$E^{(0)} := \left\{ x \in \mathbb{R}^N : \limsup_{r \to 0^+} \frac{|E \cap B_r(x)|}{r^N} = 0 \right\},$$

such that $|x_n| \to +\infty$. Fix $r < \min\{r_0, 1/(2N\omega)\}$ and assume without loss of generality that $|x_n - x_m| > 4r$. It is easily seen that for infinitely many n we can find $y_n \in \partial^* E \cap B_r(x_n)$; then

$$\mathcal{P}(E) \geq \sum_n \mathcal{P}(E, B_r(y_n)) \geq \sum_n \vartheta_0 r^{N-1} = +\infty,$$

which is a contradiction.

Connectedness of global minimizers follows easily from their boundedness, since if a global minimizer had at least two connected components one could move one of them far apart from the others without changing the perimeter but decreasing the nonlocal term in the energy (see [62, Lemma 3] for a formal argument).

Finally, given a local minimizer $|E|$, assume that $E_0 \subset E$ satisfies $|E_0| > 0$, $|E \setminus E_0| > 0$, and $\mathcal{P}(E) = \mathcal{P}(E_0) + \mathcal{P}(E \setminus E_0)$: then, denoting by B_r a ball with volume $|B_r| = |E_0|$, using the isoperimetric inequality and the fact that E is a (ω, r_0)-minimizer of the area functional we obtain

$$\begin{aligned} \mathcal{P}(E \setminus E_0) + N\omega_N r^{N-1} &\leq \mathcal{P}(E \setminus E_0) + \mathcal{P}(E_0) = \mathcal{P}(E) \\ &\leq \mathcal{P}(E \setminus E_0) + \omega|E_0| = \mathcal{P}(E \setminus E_0) + \omega\omega_N r^N, \end{aligned}$$

which is a contradiction if r is small enough. This shows an uniform lower bound on the volume of each connected component of E, from which we deduce that E can have at most a finite number of connected components. $\square$

We are now ready to state the main results of this chapter. The central theorem, whose proof lasts for Sections 4.2 and 4.3, provides a sufficiency local minimality criterion based on the second variation of the functional. Following [1] (see also [26]), we introduce a quadratic form associated with the second variation of the functional at a regular critical set (see Definition 4.18); then we show that its strict positivity (on the orthogonal complement to a suitable finite dimensional subspace of directions where the second variation degenerates, due to translation invariance) is a sufficient condition for isolated local minimality, according to Definition 4.4, by proving a *quantitative* stability inequality. The result reads as follows.

THEOREM 4.8. *Assume that E is a regular critical set for $\mathcal{F}$ with positive second variation, in the sense of Definition 4.22. Then there exist $\delta > 0$ and $C > 0$ such that*

$$\mathcal{F}(F) \geq \mathcal{F}(E) + C\big(\alpha(E, F)\big)^2 \tag{4.10}$$

for every $F \subset \mathbb{R}^N$ such that $|F| = |E|$ and $\alpha(E, F) < \delta$.

The local minimality criterion in Theorem 4.8 can be applied to obtain information about local and global minimizers of the functional (4.1). We start with the following theorem, which shows the existence of a critical mass $m_{\rm loc} > 0$ such that the ball B_R is an isolated local minimizer if $|B_R| < m_{\rm loc}$, but is no longer a local minimizer for larger masses. We also determine explicitly the volume threshold in the three-dimensional case. The result, which to the best of our knowledge provides the first characterization of the local minimality of the ball, will be proved in Section 4.4.

THEOREM 4.9 (Local minimality of the ball). *Given $N \geq 2$, $\alpha \in (0, N-1)$ and $\gamma > 0$, there exists a critical threshold $m_{\rm loc} = m_{\rm loc}(N, \alpha, \gamma) > 0$ such that the ball B_R is an isolated local minimizer for $\mathcal{F}_{\alpha,\gamma}$, in the sense of Definition 4.4, if $0 < |B_R| < m_{\rm loc}$.*

If $|B_R| > m_{\rm loc}$, there exist sets $E \subset \mathbb{R}^N$ with $|E| = |B_R|$ and $\alpha(E, B_R)$ arbitrarily small such that $\mathcal{F}_{\alpha,\gamma}(E) < \mathcal{F}_{\alpha,\gamma}(B_R)$.

Finally $m_{\rm loc}(N, \alpha, \gamma) \to \infty$ as $\alpha \to 0^+$, and in dimension $N = 3$ we have

$$m_{\rm loc}(3, \alpha, \gamma) = \frac{4}{3}\pi \left(\frac{(6-\alpha)(4-\alpha)}{2^{3-\alpha}\gamma\alpha\pi} \right)^{\frac{3}{4-\alpha}}.$$

Our local minimality criterion allows us to deduce further properties about global minimizers, which will be proved in Section 4.5. The first result states that the ball is the unique global minimizer of the functional for small masses. We provide an alternative proof of this fact (which was already known in the literature in some particular cases, as explained in the Introduction) which holds in full generality and removes the restrictions on the parameters N and α which were present in the previous partial results.

THEOREM 4.10 (Global minimality of the ball). *Let $m_{\rm glob}(N, \alpha, \gamma)$ be the supremum of the masses $m > 0$ such that the ball of volume m is a global minimizer of $\mathcal{F}_{\alpha,\gamma}$ in $\mathbb{R}^N$. Then $m_{\rm glob}(N, \alpha, \gamma)$ is positive and finite, and the ball of volume m is a global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $m \leq m_{\rm glob}(N, \alpha, \gamma)$. Moreover, it is the unique (up to translations) global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $m < m_{\rm glob}(N, \alpha, \gamma)$.*

In the following theorems we analyze the global minimality issue for α close to 0, showing in particular that in this case the unique minimizer, as long as a minimizer exists, is the ball, and characterizing the infimum of the energy when the problem does not have a solution.

THEOREM 4.11 (Characterization of global minimizers for α small). *Given $N \geq 2$ and $\gamma > 0$, there exists $\bar{\alpha} = \bar{\alpha}(N, \gamma) > 0$ such that for every $\alpha < \bar{\alpha}$ the ball with volume m is the unique (up to translations) global minimizer of $\mathcal{F}_{\alpha,\gamma}$ if $m \leq m_{\rm glob}(N, \alpha, \gamma)$, while for $m > m_{\rm glob}(N, \alpha, \gamma)$ the minimum problem for $\mathcal{F}_{\alpha,\gamma}$ does not have a solution.*

THEOREM 4.12 (Characterization of minimizing sequences for α small). *Let $\bar{\alpha}$ be given by Theorem 4.11, let $\alpha < \bar{\alpha}$ and let*

$$f_k(m) := \min_{\substack{\mu_1,\ldots,\mu_k \geq 0 \\ \mu_1+\ldots+\mu_k = m}} \left\{ \sum_{j=1}^{k} \mathcal{F}(B^i) : B^i \text{ ball}, \ |B^i| = \mu_i \right\}.$$

There exists an increasing sequence $(m_k)_k$, with $m_0 = 0$, $m_1 = m_{\rm glob}$, such that $\lim_k m_k = \infty$ and

$$\inf_{|E|=m} \mathcal{F}(E) = f_k(m) \qquad \text{for every } m \in [m_{k-1}, m_k], \text{ for all } k \in \mathbb{N}, \tag{4.11}$$

that is, for every $m \in [m_{k-1}, m_k]$ a minimizing sequence for the total energy is obtained by a configuration of at most k disjoint balls with diverging mutual distance. Moreover, the number of non-degenerate balls tends to $+\infty$ as $m \to +\infty$.

REMARK 4.13. Since $m_{\rm loc}(N,\alpha,\gamma)\to+\infty$ as $\alpha\to 0^+$ and the non-existence threshold is uniformly bounded for $\alpha\in(0,1)$ (see Proposition 4.34), we immediately deduce that, for α small, $m_{\rm glob}(N,\alpha,\gamma)<m_{\rm loc}(N,\alpha,\gamma)$. Moreover, by comparing the energy of a ball of volume m with the energy of two disjoint balls of volume $\frac{m}{2}$, and sending to infinity the distance between the balls, we deduce after a straightforward computation (and estimating $\mathcal{NL}_\alpha(B_1)\geq\omega_N^2 2^{-\alpha}$) that the following upper bound for the global minimality threshold of the ball holds:

$$m_{\rm glob}(N,\alpha,\gamma)<\omega_N\left(\frac{2^\alpha N(2^{\frac{1}{N}}-1)}{\omega_N\gamma\big(1-(\frac{1}{2})^{\frac{N-\alpha}{N}}\big)}\right)^{\frac{N}{N+1-\alpha}}.$$

Hence, by comparing this value with the explicit expression of $m_{\rm loc}$ in the physical interesting case $N=3$, $\alpha=1$ (see Theorem 4.9), we deduce that $m_{\rm glob}(3,1,\gamma)<m_{\rm loc}(3,1,\gamma)$.

REMARK 4.14. In the planar case, one can also consider a Newtonian potential in the nonlocal term, *i.e.*

$$\int_E\int_E\log\frac{1}{|x-y|}\,dxdy\,.$$

It is clear that the infimum of the corresponding functional on $\mathbb{R}^2$ is $-\infty$ (consider, for instance, a minimizing sequence obtained by sending to infinity the distance between the centers of two disjoint balls). Moreover, also the notion of local minimality considered in Definition 4.4 becomes meaningless in this situation, since, given any finite perimeter set E, it is always possible to find sets with total energy arbitrarily close to $-\infty$ in every L^1-neighborhood of E. Nevertheless, by reproducing the arguments of this paper one can show that, given a bounded regular critical set E with positive second variation, and a radius $R>0$ such that $E\subset B_R$, there exists $\delta>0$ such that E minimizes the energy with respect to competitors $F\subset B_R$ with $\alpha(F,E)<\delta$.

4.2. Second variation and local $W^{2,p}$-minimality

We start this section by introducing the notions of first and second variation of the functional $\mathcal{F}$ along families of deformations as in the following definition.

DEFINITION 4.15. Let $X:\mathbb{R}^N\to\mathbb{R}^N$ be a C^2-vector field. The *admissible flow* associated with X is the function $\Phi:\mathbb{R}^N\times(-1,1)\to\mathbb{R}^N$ defined by the equations

$$\frac{\partial\Phi}{\partial t}=X\circ\Phi\,,\qquad \Phi(x,0)=x\,.$$

DEFINITION 4.16. Let $E\subset\mathbb{R}^N$ be a set of class C^2, and let Φ be an admissible flow. We define the *first and second variation of $\mathcal{F}$ at E with respect to the flow Φ* to be

$$\frac{d}{dt}\mathcal{F}(E_t)_{|t=0}\qquad\text{and}\qquad\frac{d^2}{dt^2}\mathcal{F}(E_t)_{|t=0}$$

respectively, where we set $E_t:=\Phi_t(E)$.

Given a regular set E, we denote by $X_\tau:=X-\langle X,\nu_E\rangle\nu_E$ the tangential part to ∂E of a vector field X. We will also denote by $\mathbf{B}_{\partial E}:=D_{\partial E}\nu_E$ and $H_{\partial E}:=\operatorname{div}_{\partial E}\nu_E$ the second fundamental form and the mean curvature of ∂E respectively, according to (1.4) and (1.5).

The following theorem contains the explicit formula for the first and second variations of $\mathcal{F}$. The computation, which is postponed to Section 4.6, is performed by a regularization approach which is slightly different from the technique used, in the case $\alpha=N-2$, in [26] (for a critical set, see also [71]) and in [1] (for a general regular set): here we introduce a family of regularized potentials (depending on a small parameter $\delta\in\mathbb{R}$) to avoid the problems in the differentiation of the singularity in the nonlocal part, recovering the result by letting the parameter tend to 0.

THEOREM 4.17. *Let $E\subset\mathbb{R}^N$ be a bounded set of class C^2, and let Φ be the admissible flow associated with a C^2-vector field X. Then the first variation of $\mathcal{F}$ at E with respect to the flow Φ is*

$$\frac{\mathrm{d}\mathcal{F}(E_t)}{\mathrm{d}t}_{|t=0}=\int_{\partial E}(H_{\partial E}+2\gamma v_E)\langle X,\nu_E\rangle\,\mathrm{d}\mathcal{H}^{N-1}\,, \tag{4.12}$$

and the second variation of $\mathcal{F}$ at E with respect to the flow Φ is

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{F}(E_t)}{\mathrm{d}t^2}_{|t=0}&=\int_{\partial E}\big(|\nabla_{\partial E}\langle X,\nu_E\rangle|^2-|\mathbf{B}_{\partial E}|^2\langle X,\nu_E\rangle^2\big)\,\mathrm{d}\mathcal{H}^{N-1}\\&\quad+2\gamma\int_{\partial E}\int_{\partial E}G(x,y)\langle X(x),\nu_E(x)\rangle\langle X(y),\nu_E(y)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\\&\quad+2\gamma\int_{\partial E}\partial_{\nu_E}v_E\,\langle X,\nu_E\rangle^2\,\mathrm{d}\mathcal{H}^{N-1}-\int_{\partial E}(2\gamma v_E+H_{\partial E})\,\mathrm{div}_{\partial E}\big(X_\tau\langle X,\nu_E\rangle\big)\,\mathrm{d}\mathcal{H}^{N-1}\\&\quad+\int_{\partial E}(2\gamma v_E+H_{\partial E})(\mathrm{div}X)\langle X,\nu_E\rangle\,\mathrm{d}\mathcal{H}^{N-1}\,,\end{aligned}$$

where $G(x,y):=\frac{1}{|x-y|^\alpha}$ is the potential in the nonlocal part of the energy.

If E is a regular critical set (as in Definition 4.5) it holds

$$\int_{\partial E}(2\gamma v_E+H_{\partial E})\mathrm{div}_{\partial E}\big(X_\tau\langle X,\nu_E\rangle\big)\,\mathrm{d}\mathcal{H}^{N-1}=0\,.$$

Moreover if the admissible flow Φ preserves the volume of E, *i.e.* if $|\Phi_t(E)|=|E|$ for all $t\in(-1,1)$, then (see [26, equation (2.30)])

$$0=\frac{\mathrm{d}^2}{\mathrm{d}t^2}|E_t|_{|t=0}=\int_{\partial E}(\mathrm{div}X)\langle X,\nu_E\rangle\,\mathrm{d}\mathcal{H}^{N-1}\,.$$

Hence we obtain the following expression for the second variation at a regular critical set with respect to a volume-preserving admissible flow:

$$\begin{aligned}\frac{\mathrm{d}^2\mathcal{F}(E_t)}{\mathrm{d}t^2}_{|t=0}&=\int_{\partial E}\big(|\nabla_{\partial E}\langle X,\nu_E\rangle|^2-|\mathbf{B}_{\partial E}|^2\langle X,\nu_E\rangle^2\big)\,\mathrm{d}\mathcal{H}^{N-1}+2\gamma\int_{\partial E}\partial_{\nu_E}v_E\langle X,\nu_E\rangle^2\,\mathrm{d}\mathcal{H}^{N-1}\\&\quad+2\gamma\int_{\partial E}\int_{\partial E}G(x,y)\langle X(x),\nu_E(x)\rangle\langle X(y),\nu_E(y)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\,.\end{aligned}$$

Following [1], we introduce the space

$$\widetilde{H}^1(\partial E):=\left\{\varphi\in H^1(\partial E)\;:\;\int_{\partial E}\varphi\,\mathrm{d}\mathcal{H}^{N-1}=0\right\}$$

endowed with the norm $\|\varphi\|_{\widetilde{H}^1(\partial E)}:=\|\nabla\varphi\|_{L^2(\partial E)}$, and we define on it the following quadratic form associated with the second variation.

DEFINITION 4.18. Let $E\subset\mathbb{R}^N$ be a regular critical set. We define the quadratic form $\partial^2\mathcal{F}(E):\widetilde{H}^1(\partial E)\to\mathbb{R}$ by

$$\begin{aligned}\partial^2\mathcal{F}(E)[\varphi]&=\int_{\partial E}\big(|\nabla_{\partial E}\varphi|^2-|\mathbf{B}_{\partial E}|^2\varphi^2\big)\,\mathrm{d}\mathcal{H}^{N-1}+2\gamma\int_{\partial E}(\partial_{\nu_E}v_E)\varphi^2\,\mathrm{d}\mathcal{H}^{N-1}\\&\quad+2\gamma\int_{\partial E}\int_{\partial E}G(x,y)\varphi(x)\varphi(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\,.\end{aligned}\tag{4.13}$$

If E is a regular critical set and Φ preserves the volume of E, then

$$\partial^2\mathcal{F}(E)[\langle X,\nu_E\rangle] = \frac{\mathrm{d}^2\mathcal{F}(E_t)}{\mathrm{d}t^2}_{|t=0}. \tag{4.14}$$

We remark that the last integral in the expression of $\partial^2\mathcal{F}(E)$ is well defined for $\varphi\in\widetilde{H}^1(\partial E)$, thanks to the following result.

LEMMA 4.19. *Let E be a bounded set of class C^1. There exists a constant $C>0$, depending only on E, N and α, such that for every $\varphi,\psi\in\widetilde{H}^1(\partial E)$*

$$\int_{\partial E}\int_{\partial E} G(x,y)\varphi(x)\psi(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\le C\|\varphi\|_{L^2}\|\psi\|_{L^2}\le C\|\varphi\|_{\widetilde{H}^1}\|\psi\|_{\widetilde{H}^1}. \tag{4.15}$$

PROOF. The proof lies on [48, Lemma 7.12], which states that if $\Omega\subset\mathbb{R}^n$ is a bounded domain and $\mu\in(0,1]$, the operator $f\mapsto V_\mu f$ defined by

$$(V_\mu f)(x) := \int_\Omega |x-y|^{n(\mu-1)}f(y)\,\mathrm{d}y$$

maps $L^p(\Omega)$ continuously into $L^q(\Omega)$ provided that $0\le\delta:=p^{-1}-q^{-1}<\mu$, and

$$\|V_\mu f\|_{L^q(\Omega)}\le\left(\frac{1-\delta}{\mu-\delta}\right)^{1-\delta}\omega_n^{1-\mu}|\Omega|^{\mu-\delta}\|f\|_{L^p(\Omega)}.$$

In our case, from the fact that our set has compact boundary, we can simply reduce to the above case using local charts and partition of unity. In particular we have that $\mu=\frac{N-1-\alpha}{N-1}$, and applying this result with $p=q=2$ we easily obtain the estimate in the statement. $\square$

REMARK 4.20. Using the estimate contained in the previous lemma it is easily seen that $\partial^2\mathcal{F}(E)$ is continuous with respect to the strong convergence in $\widetilde{H}^1(\partial E)$ and lower semicontinuous with respect to the weak convergence in $\widetilde{H}^1(\partial E)$. Moreover, it is also clear from the proof that, given $\bar{\alpha}<N-1$, the constant C in (4.15) can be chosen independently of $\alpha\in(0,\bar{\alpha})$.

Equality (4.14) suggests that at a regular local minimizer the quadratic form (4.13) must be nonnegative on the space $\widetilde{H}^1(\partial E)$. This is the content of the following corollary, whose proof is analogous to [1, Corollary 3.4].

COROLLARY 4.21. *Let E be a local minimizer of $\mathcal{F}$ of class C^2. Then*

$$\partial^2\mathcal{F}(E)[\varphi]\ge 0 \qquad \textit{for all } \varphi\in\widetilde{H}^1(\partial E).$$

Now we want to look for a sufficient condition for local minimality. First of all we notice that, since our functional is translation invariant, if we compute the second variation of $\mathcal{F}$ at a regular set E with respect to a flow of the form $\Phi(x,t):=x+t\eta e_i$, where $\eta\in\mathbb{R}$ and e_i is an element of the canonical basis of $\mathbb{R}^N$, setting $\nu_i:=\langle\nu_E,e_i\rangle$ we obtain that

$$\partial^2\mathcal{F}(E)[\eta\nu_i] = \frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(\Phi(E,t))_{|t=0} = 0.$$

Following [1], since we aim to prove that the *strict* positivity of the second variation is a sufficient condition for local minimality, we shall exclude the finite dimensional subspace of $\widetilde{H}^1(\partial E)$ generated by the functions ν_i, which we denote by $T(\partial E)$. Hence we split

$$\widetilde{H}^1(\partial E) = T^\perp(\partial E)\oplus T(\partial E),$$

where $T^{\perp}(\partial E)$ is the orthogonal complement to $T(\partial E)$ in the L^2-sense, *i.e.*,

$$T^{\perp}(\partial E) := \left\{\varphi \in \widetilde{H}^1(\partial E)\,:\, \int_{\partial E} \varphi \nu_i \,\mathrm{d}\mathcal{H}^{N-1} = 0 \ \text{ for each } i = 1, \dots, N\right\}.$$

It can be shown (see [1, Equation (3.7)]) that there exists an orthonormal frame $(\varepsilon_1, \dots, \varepsilon_N)$ such that

$$\int_{\partial E} \langle \nu, \varepsilon_i \rangle \langle \nu, \varepsilon_j \rangle \,\mathrm{d}\mathcal{H}^{N-1} = 0 \qquad \text{for all } i \neq j\,,$$

so that the projection on $T^{\perp}(\partial E)$ of a function $\varphi \in \widetilde{H}^1(\partial E)$ is

$$\pi_{T^{\perp}(\partial E)}(\varphi) = \varphi - \sum_{i=1}^{N} \left(\int_{\partial E} \varphi \langle \nu, \varepsilon_i \rangle \,\mathrm{d}\mathcal{H}^{N-1}\right) \frac{\langle \nu, \varepsilon_i \rangle}{\|\langle \nu, \varepsilon_i \rangle\|^2_{L^2(\partial E)}}$$

(notice that $\langle \nu, \varepsilon_i \rangle \not\equiv 0$ for every i, since on the contrary the set E would be translation invariant in the direction ε_i).

Definition 4.22. We say that $\mathcal{F}$ has *positive second variation* at the regular critical set E if

$$\partial^2 \mathcal{F}(E)[\varphi] > 0 \qquad \text{for all } \varphi \in T^{\perp}(\partial E) \setminus \{0\}.$$

One could expect that the positiveness of the second variation implies also a sort of coercivity; this is shown in the following lemma.

Lemma 4.23. *Assume that $\mathcal{F}$ has positive second variation at a regular critical set E. Then*

$$m_0 := \inf\bigl\{\partial^2 \mathcal{F}(E)[\varphi]\,:\, \varphi \in T^{\perp}(\partial E),\, \|\varphi\|_{\widetilde{H}^1(\partial E)} = 1\bigr\} > 0\,,$$

and

$$\partial^2 \mathcal{F}(E)[\varphi] \geq m_0 \|\varphi\|^2_{\widetilde{H}^1(\partial E)} \qquad \textit{for all } \varphi \in T^{\perp}(\partial E)\,.$$

Proof. Let $(\varphi_h)_h$ be a minimizing sequence for m_0. Up to a subsequence we can suppose that $\varphi_h \rightharpoonup \varphi_0$ weakly in $\widetilde{H}^1(\partial E)$, with $\varphi_0 \in T^{\perp}(\partial E)$. By the lower semicontinuity of $\partial^2 \mathcal{F}(E)$ with respect to the weak convergence in $\widetilde{H}^1(\partial E)$ (see Remark 4.20), we have that if $\varphi_0 \neq 0$

$$m_0 = \lim_{h \to \infty} \partial^2 \mathcal{F}(E)[\varphi_h] \geq \partial^2 \mathcal{F}(E)[\varphi_0] > 0\,,$$

while if $\varphi_0 = 0$

$$m_0 = \lim_{h \to \infty} \partial^2 \mathcal{F}(E)[\varphi_h] = \lim_{h \to \infty} \int_{\partial E} |\nabla_{\partial E} \varphi_h|^2 \,\mathrm{d}\mathcal{H}^{N-1} = 1\,.$$

The second part of the statement follows from the fact that $\partial^2 \mathcal{F}(E)$ is a quadratic form. □

We now start the proof of the local minimality criterion stated in Theorem 4.8. In the remaining part of this section we prove that a regular critical pair with positive second variation satisfies a weaker minimality property, that is minimality with respect to sets whose boundaries are graphs over the boundary of E with sufficiently small $W^{2,p}$-norm (Theorem 4.25). In order to do this, we start by recalling the technical result contained in [1, Theorem 3.7], which provides the construction of an admissible flow connecting a regular set $E \subset \mathbb{R}^N$ with an arbitrary set sufficiently close in the $W^{2,p}$-sense.

Theorem 4.24. *Let $E \subset \mathbb{R}^N$ be a bounded set of class C^3 and let $p > N - 1$. For all $\varepsilon > 0$ there exist a tubular neighborhood $\mathcal{U}$ of ∂E and two positive constants δ, C with the* *following properties: if* $\psi \in C^2(\partial E)$ *and* $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$ *then there exists a field* $X \in C^2$ *with* $\operatorname{div} X = 0$ *in* $\mathcal{U}$ *such that*

$$\|X - \psi\nu_E\|_{L^2(\partial E)} \leq \varepsilon \|\psi\|_{L^2(\partial E)} .$$

Moreover the associated flow

$$\Phi(x,0) = 0, \quad \frac{\partial \Phi}{\partial t} = X \circ \Phi$$

satisfies $\Phi(\partial E, 1) = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$*, and for every* $t \in [0,1]$

$$\|\Phi(\cdot, t) - Id\|_{W^{2,p}(\partial E)} \leq C\|\psi\|_{W^{2,p}(\partial E)} .$$

If in addition E_1 *has the same volume as* E*, then for every* t *we have* $|E_t| = |E|$ *and*

$$\int_{\partial E_t} \langle X, \nu_{E_t}\rangle \, \mathrm{d}\mathcal{H}^{N-1} = 0 .$$

We are now in position to prove the following local $W^{2,p}$-minimality theorem, analogous to [1, Theorem 3.9]. The proof contained in [1] can be repeated here with minor changes, and we will only give a sketch of it for the reader's convenience.

THEOREM 4.25. *Let* $p > \max\{2, N-1\}$ *and let* E *be a regular critical set for* $\mathcal{F}$ *with positive second variation, according to Definition 4.22. Then there exist* δ*,* $C_0 > 0$ *such that*

$$\mathcal{F}(F) \geq \mathcal{F}(E) + C_0(\alpha(E,F))^2 ,$$

for each $F \subset \mathbb{R}^N$ *such that* $|F| = |E|$ *and* $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$ *for some* ψ *with* $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta$*.*

PROOF (SKETCH). We just describe the strategy of the proof, which is divided into two steps.
Step 1. There exists $\delta_1 > 0$ such that if $\partial F = \{x + \psi(x)\nu_E(x) \; : \; x \in \partial E\}$ with $|F| = |E|$ and $\|\psi\|_{W^{2,p}(\partial E)} \leq \delta_1$, then

$$\inf\left\{\partial^2\mathcal{F}(F)[\varphi] \; : \; \varphi \in \widetilde{H}^1(\partial F),\, \|\varphi\|_{\widetilde{H}^1(\partial F)} = 1,\, \left|\int_{\partial F} \varphi\nu_F \, \mathrm{d}\mathcal{H}^{N-1}\right| \leq \delta_1\right\} \geq \frac{m_0}{2},$$

where m_0 is defined in Lemma 4.23. To prove this we suppose by contradiction that there exist a sequence $(F_n)_n$ of subsets of $\mathbb{R}^N$ such that $\partial F_n = \{x + \psi_n(x)\nu_E(x) : x \in \partial E\}$, $|F_n| = |E|$, $\|\psi_n\|_{W^{2,p}(\partial E)} \to 0$, and a sequence of functions $\varphi_n \in \widetilde{H}^1(\partial F_n)$ with $\|\varphi_n\|_{\widetilde{H}^1(\partial F_n)} = 1$, $|\int_{\partial F_n} \varphi_n \nu_{F_n} \, \mathrm{d}\mathcal{H}^{N-1}| \to 0$, such that

$$\partial^2\mathcal{F}(F_n)[\varphi_n] < \frac{m_0}{2}.$$

We consider a sequence of diffeomorphisms $\Phi_n : E \to F_n$, with $\Phi_n \to Id$ in $W^{2,p}$, and we set

$$\tilde{\varphi}_n := \varphi_n \circ \Phi_n - a_n, \qquad a_n := \fint_{\partial E} \varphi_n \circ \Phi_n \, \mathrm{d}\mathcal{H}^{N-1}.$$

Hence $\tilde{\varphi}_n \in \widetilde{H}^1(\partial E)$, $a_n \to 0$, and since $\nu_{F_n} \circ \Phi_n - \nu_E \to 0$ in $C^{0,\beta}$ for some $\beta \in (0,1)$ and a similar convergence holds for the tangential vectors, we have that

$$\int_{\partial E} \tilde{\varphi}_n \langle \nu_E, \varepsilon_i\rangle \, \mathrm{d}\mathcal{H}^{N-1} \to 0$$

for every $i = 1, \ldots, N$, so that $\|\pi_{T^\perp(\partial E)}(\tilde{\varphi}_n)\|_{\widetilde{H}^1(\partial E)} \to 1$. Moreover it can be proved that

$$\left|\partial^2\mathcal{F}(F_n)[\varphi_n] - \partial^2\mathcal{F}(E)[\tilde{\varphi}_n]\right| \to 0.$$

Indeed, the convergence of the first integral in the expression of the quadratic form follows easily from the fact that $\mathbf{B}_{\partial F_n} \circ \Phi_n - \mathbf{B}_{\partial E} \to 0$ in $L^p(\partial E)$, and from the Sobolev Embedding Theorem (recall that $p > \max\{2, N-1\}$). For the second integral, it is sufficient to observe that, as a consequence of Proposition 4.1, the functions v_{F_h} are uniformly bounded in $C^{1,\beta}(\mathbb{R}^N)$ for some $\beta \in (0,1)$ and hence they converge to v_E in $C^{1,\gamma}(B_R)$ for all $\gamma < \beta$ and $R > 0$. Finally, the difference of the last integrals can be written as

$$
\begin{aligned}
&\int_{\partial F_n}\int_{\partial F_n} G(x,y)\varphi_n(x)\varphi_n(y)\,\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}\mathcal{H}^{N-1} - \int_{\partial E}\int_{\partial E} G(x,y)\tilde{\varphi}_n(x)\tilde{\varphi}_n(y)\,\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad = \int_{\partial E}\int_{\partial E} g_n(x,y)G(x,y)\tilde{\varphi}_n(x)\tilde{\varphi}_n(y)\,\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}\mathcal{H}^{N-1}\\
&\qquad + a_n \int_{\partial E}\int_{\partial E} G(\Phi_n(x),\Phi_n(y))J_{\Phi_n}(x)J_{\Phi_n}(y)\big(\tilde{\varphi}_n(x)+\tilde{\varphi}_n(y)+a_n\big)\,\mathrm{d}\mathcal{H}^{N-1}\mathrm{d}\mathcal{H}^{N-1}
\end{aligned}
$$

where J_{Φ_n} is the $(N-1)$-dimensional jacobian of Φ_n on ∂E, and

$$
g_n(x,y) := \frac{|x-y|^\alpha}{|\Phi_n(x)-\Phi_n(y)|^\alpha}\, J_{\Phi_n}(x)J_{\Phi_n}(y) - 1\,.
$$

Thus the desired convergence follows from the fact that $g_n \to 0$ uniformly, $a_n \to 0$, and from the estimate provided by Lemma 4.19.

Hence

$$
\begin{aligned}
\frac{m_0}{2} \geq \lim_{n\to\infty} \partial^2\mathcal{F}(F_n)[\varphi_n] &= \lim_{n\to\infty} \partial^2\mathcal{F}(E)[\tilde{\varphi}_n] = \lim_{n\to\infty} \partial^2\mathcal{F}(E)[\pi_{T^\perp(\partial E)}(\tilde{\varphi}_n)]\\
&\geq m_0 \lim_{n\to\infty} \|\pi_{T^\perp(\partial E)}(\tilde{\varphi}_n)\|_{\widetilde{H}^1(\partial E)} = m_0,
\end{aligned}
$$

which is a contradiction.

Step 2. If F is as in the statement of the theorem, we can use the vector field X provided by Theorem 4.24 to generate a flow connecting E to F by a family of sets E_t, $t \in [0,1]$. Recalling that E is critical and that X is divergence free, we can write

$$
\begin{aligned}
\mathcal{F}(F) - \mathcal{F}(E) &= \mathcal{F}(E_1) - \mathcal{F}(E_0) = \int_0^1 (1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{F}(E_t)\,\mathrm{d}t\\
&= \int_0^1 (1-t)\Big(\partial^2\mathcal{F}(E_t)[\langle X,\nu_{E_t}\rangle] - \int_{\partial E_t} (2\gamma v_{E_t} + H_{\partial E_t})\mathrm{div}_{\partial E_t}(X_{\tau_t}\langle X,\nu_{E_t}\rangle)\,\mathrm{d}\mathcal{H}^{N-1}\Big)\,\mathrm{d}t.
\end{aligned}
$$

It is now possible to bound from below the previous integral in a quantitative fashion: to do this we use, in particular, the result proved in Step 1 for the first term, and we proceed as in Step 2 of [1, Theorem 3.9] for the second one. In this way we obtain the desired estimate. $\square$

4.3. Local L^1-minimality

In this section we complete the proof of Theorem 4.8 by using a contradiction argument which relies on the regularity properties of sequences of quasi-minimizers of the area functional. We premise to the proof the following lemma, similar to Lemma 3.43, which is a consequence of the classical elliptic regularity theory (see [1, Lemma 7.2]).

LEMMA 4.26. *Let E be a bounded set of class C^2 and let E_n be a sequence of sets of class $C^{1,\beta}$ for some $\beta \in (0,1)$ such that $\partial E_n = \{x + \psi_n(x)\nu_E(x) : x \in \partial E\}$, with $\psi_n \to 0$ in $C^{1,\beta}(\partial E)$. Assume also that $H_{\partial E_n} \in L^p(\partial E_n)$ for some $p \geq 1$. We have:*

- *if $H_{\partial E_n}(\cdot + \psi_n(\cdot)\nu_E(\cdot)) \to H_{\partial E}$ in $L^p(\partial E)$, then $\psi_n \to 0$ in $W^{2,p}(\partial E)$;*
- *if $\sup_n \|H_{\partial E_n}\|_{L^p(\partial E_n)} < \infty$, then $\sup_n \|\psi_n\|_{W^{2,p}(\partial E)} < \infty$.*

An intermediate step in the proof of Theorem 4.8 consists in showing that the local $W^{2,p}$-minimality proved in Theorem 4.25 implies local minimality with respect to competing sets which are sufficiently close in the Hausdorff distance. This is the content of the following theorem, whose proof can be easily adapted from [1, Theorem 4.3] (notice, indeed, that the difficulties coming from the fact of working in the whole space $\mathbb{R}^N$ are not present, due to the constraint $F \subset \mathcal{I}_{\delta_0}(E)$).

THEOREM 4.27. *Let $E \subset \mathbb{R}^N$ be a bounded regular set, and assume that there exists $\delta > 0$ such that*

$$\mathcal{F}(E) \le \mathcal{F}(F) \tag{4.16}$$

for every set $F \subset \mathbb{R}^N$ with $|F| = |E|$ and $\partial F = \{x + \psi(x)\nu_E(x) : x \in \partial E\}$, for some function ψ with $\|\psi\|_{W^{2,p}(\partial E)} \le \delta$.

Then there exists $\delta_0 > 0$ such that (4.16) holds for every finite perimeter set F with $|F| = |E|$ and such that $\mathcal{I}_{-\delta_0}(E) \subset F \subset \mathcal{I}_{\delta_0}(E)$, where for $\delta \in \mathbb{R}$ we set (d denoting the signed distance to E)

$$\mathcal{I}_\delta(E) := \{x : d(x) < \delta\}\,.$$

PROOF. Assume by contradiction that there exist sequences $\delta_h \to 0$ and $E_h \subset \mathbb{R}^N$ such that $|E_h| = |E|$, $\mathcal{I}_{-\delta_h}(E) \subset E_h \subset \mathcal{I}_{\delta_h}(E)$, and $\mathcal{F}(E_h) < \mathcal{F}(E)$. We consider a solution F_h to the obstacle problem

$$\min\left\{\mathcal{F}(F) + \Lambda\big||F| - |E|\big| \,:\, \mathcal{I}_{-\delta_h}(E) \subset F \subset \mathcal{I}_{\delta_h}(E)\right\}, \tag{4.17}$$

where we choose

$$\Lambda > \max\bigl\{\mathcal{F}(E),\, \|\mathrm{div}\,\nu\|_\infty + c_0\gamma\bigr\}\,. \tag{4.18}$$

Here ν denotes a regular extension of the normal vector field ν_E (given, for instance, by ∇d), while c_0 is the constant provided by Proposition 4.3 corresponding to the fixed values of N and α and to $m := |E|+1$. Notice that Λ depends only on the set E, and that by minimality of F_h

$$\mathcal{F}(F_h) \le \mathcal{F}(E_h) < \mathcal{F}(E). \tag{4.19}$$

Step 1. We claim that $|F_h| = |E|$ for every h. Indeed, assuming that $|F_h| < |E|$ for some h (the argument in the opposite case is similar), we can find $\tau_h \in (-\delta_h, \delta_h)$ such that setting $\widetilde{F}_h := F_h \cup \mathcal{I}_{\tau_h}(E)$ one has $|\widetilde{F}_h| = |E|$. By Proposition 4.3, observing that without loss of generality we have $|F_h| \le |E| + 1$,

$$|\mathcal{NL}(\widetilde{F}_h) - \mathcal{NL}(F_h)| \le c_0|\widetilde{F}_h \triangle F_h| = c_0\bigl(|\widetilde{F}_h| - |F_h|\bigr)\,.$$

Moreover, since $\partial^*\widetilde{F}_h \setminus \partial^* F_h$ is contained in $\partial\mathcal{I}_{\tau_h}(E)$ and $\nu_{\mathcal{I}_{\tau_h}(E)} = \nu$,

$$\begin{aligned}
\mathcal{P}(\widetilde{F}_h) - \mathcal{P}(F_h) &= \int_{\partial^*\widetilde{F}_h \setminus \partial^* F_h} \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial^* F_h \setminus \partial^*\widetilde{F}_h} \mathrm{d}\mathcal{H}^{N-1} \\
&\le \int_{\partial^*\widetilde{F}_h \setminus \partial^* F_h} \nu \cdot \nu_{\widetilde{F}_h}\, \mathrm{d}\mathcal{H}^{N-1} - \int_{\partial^* F_h \setminus \partial^*\widetilde{F}_h} \nu \cdot \nu_{F_h}\, \mathrm{d}\mathcal{H}^{N-1} \\
&\le \int_{\widetilde{F}_h \triangle F_h} |\mathrm{div}\nu|\, \mathrm{d}x \le \|\mathrm{div}\nu\|_\infty\bigl(|\widetilde{F}_h| - |F_h|\bigr).
\end{aligned}$$

Hence, combining the estimates above and recalling the choice of Λ, we have

$$\mathcal{F}(\widetilde{F}_h) - \mathcal{F}(F_h) - \Lambda\big||F_h| - |E|\big| \le \bigl(\|\mathrm{div}\nu\|_\infty + c_0\gamma - \Lambda\bigr)\bigl(|\widetilde{F}_h| - |F_h|\bigr) < 0,$$

which contradicts the minimality of F_h.

Step 2. We now show that each set F_h is a solution to the penalized minimum problem

$$\min\left\{\mathcal{J}_h(F) := \mathcal{F}(F) + \Lambda\big||F|-|E|\big| + 2\Lambda|F\triangle T_h(F)| \,:\, F\subset\mathbb{R}^N\right\},$$

where for a set F we define $T_h(F) := \big(F\cap\mathcal{I}_{\delta_h}(E)\big)\cup\mathcal{I}_{-\delta_h}(E)$. Indeed, we have as in the previous step that for every $F\subset\mathbb{R}^N$

$$\mathcal{P}(F)-\mathcal{P}(T_h(F)) \geq -\|\mathrm{div}\nu\|_\infty|F\triangle T_h(F)|\,.$$

Moreover we can assume without loss of generality that $|F|\leq|E|+1$, since on the contrary

$$\mathcal{J}_h(F)\geq\Lambda\big||F|-|E|\big|>\Lambda\geq\mathcal{F}(E)>\mathcal{F}(F_h)=\mathcal{J}_h(F_h)$$

by the choice of Λ in (4.18) and by (4.19). Hence by Proposition 4.3

$$|\mathcal{NL}(F)-\mathcal{NL}(T_h(F))|\leq c_0|F\triangle T_h(F)|$$

and we conclude that

$$\begin{aligned}\mathcal{J}_h(F)-\mathcal{J}_h(F_h) &\geq \mathcal{F}(F)-\mathcal{F}(T_h(F))+\Lambda\Big(\big||F|-|E|\big|-\big||T_h(F)|-|E|\big|\Big)+2\Lambda|F\triangle T_h(F)|\\ &\geq \big(\Lambda-\|\mathrm{div}\nu\|_\infty-c_0\gamma\big)|F\triangle T_h(F)|\geq 0,\end{aligned}$$

where we used the fact that F_h solves (4.17) in the first inequality.

Step 3. We have that each set F_h is a $(4\Lambda,r_0)$-minimizer of the area functional (see Definition 1.1), where $r_0>0$ is such that $\omega_N {r_0}^N\leq 1$. Indeed, with this choice we have that for every ball $B_r(x)$ with $r\leq r_0$ and for every finite perimeter set F such that $F\triangle F_h\subset\subset B_r(x)$

$$|\mathcal{NL}(F)-\mathcal{NL}(F_h)|\leq c_0|F\triangle F_h|$$

by Proposition 4.3, where c_0 is the same constant as before since we can bound the volume of F by $|F|\leq|F_h|+\omega_N {r_0}^N\leq|E|+1$. Hence, as F_h is a minimizer of $\mathcal{J}_h$, we deduce

$$\begin{aligned}\mathcal{P}(F_h) &\leq \mathcal{P}(F)+\gamma\big(\mathcal{NL}(F)-\mathcal{NL}(F_h)\big)+\Lambda\big||F|-|E|\big|+2\Lambda|F\triangle T_h(F)|\\ &\leq \mathcal{P}(F)+c_0\gamma|F\triangle F_h|+3\Lambda|F\triangle F_h|\leq\mathcal{P}(F)+4\Lambda|F\triangle F_h|.\end{aligned}$$

Hence, since $\chi_{F_h}\to\chi_E$ in $L^1(\mathbb{R}^N)$, by Theorem 1.4 we deduce that for h sufficiently large F_h is a set of class $C^{1,\beta}$ and

$$\partial F_h=\{x+\psi_h(x)\nu_E(x)\,:\,x\in\partial E\}$$

for some ψ_h such that $\psi_h\to 0$ in $C^{1,\beta}(\partial E)$, for every $\beta\in(0,\frac{1}{2})$.

Step 4. Another consequence of the quasi-minimality of F_h is that, by Lemma 1.8,

$$\|H_{\partial F_h}\|_{L^\infty(\partial F_h)}\leq 4\Lambda,$$

and in turn, by Lemma 4.26, we obtain that

$$\sup_h\|\psi_h\|_{W^{2,p}(\partial E)}<\infty$$

for every $p\geq 1$. Hence we can write the Euler-Lagrange equation for problem (4.17):

$$H_{\partial F_h}=\left\{\begin{array}{ll}\lambda_h-2\gamma v_{F_h} & \text{in } A_h:=\partial F_h\cap\big(\mathcal{I}_{\delta_h}(E)\setminus\overline{\mathcal{I}_{-\delta_h}(E)}\big),\\ \lambda-2\gamma v_E+\rho_h & \text{otherwise,}\end{array}\right. \tag{4.20}$$

where λ_h, λ are the Lagrange multipliers corresponding to F_h and E, respectively, and ρ_h is a reminder term tending uniformly to 0. On the other hand we have on ∂E

$$H_{\partial E}=\lambda-2\gamma v_E. \tag{4.21}$$

Observe now that, since the functions v_{F_h} are equibounded in $C^{1,\beta}(\mathbb{R}^N)$ for some $\beta \in (0,1)$ (see Proposition 4.1) and they converge pointwise to v_E since $\chi_{F_h} \to \chi_E$ in $L^1(\mathbb{R}^N)$, we deduce that

$$v_{F_h} \to v_E \qquad \text{in } C^1(\overline{B}_R) \text{ for all } R > 0, \tag{4.22}$$

and moreover the sequence λ_h is bounded. We now show that

$$H_{\partial F_h}(\cdot + \psi_h(\cdot)\nu_E(\cdot)) \to H_{\partial E} \qquad \text{in } L^p(\partial E) \tag{4.23}$$

for every $p \geq 1$. This is trivial if $\mathcal{H}^{N-1}(A_h) \to 0$. On the contrary, assume without loss of generality that $\mathcal{H}^{N-1}(A_h) \geq c > 0$ for every h. Then we can find a cylinder $C = B' \times]-L, L[$, where $B' \subset \mathbb{R}^{N-1}$ is a ball centered at the origin, such that $\mathcal{H}^{N-1}(A_h \cap C) \geq c' > 0$ and in a suitable coordinate system we have

$$\begin{aligned} F_h \cap C &= \{(x', x_N) \in C : x' \in B',\, x_N < g_h(x')\}, \\ E \cap C &= \{(x', x_N) \in C : x' \in B',\, x_N < g(x')\} \end{aligned}$$

for some functions $g_h, g \in W^{2,p}(B')$ such that $g_h \to g$ in $C^{1,\beta}(\overline{B'})$ for every $\beta \in (0, \frac{1}{2})$. Let now A'_h be the projection of $A_h \cap C$ on B'. Then, by integrating (4.20) we obtain

$$\begin{aligned} &\lambda_h \mathcal{H}^{N-1}(A'_h) - 2\gamma \int_{A'_h} v_{F_h}(x', g_h(x'))\, \mathrm{d}\mathcal{H}^{N-1}(x') \\ &\quad + \lambda \mathcal{H}^{N-1}(B' \setminus A'_h) - 2\gamma \int_{B' \setminus A'_h} v_E(x', g_h(x'))\, \mathrm{d}\mathcal{H}^{N-1}(x') + \int_{B' \setminus A'_h} \rho_h \, \mathrm{d}\mathcal{H}^{N-1} \\ &\quad = - \int_{B'} \mathrm{div}\Big(\frac{\nabla g_h}{\sqrt{1 + |\nabla g_h|^2}} \Big)\, \mathrm{d}\mathcal{H}^{N-1}(x') = - \int_{\partial B'} \frac{\nabla g_h}{\sqrt{1 + |\nabla g_h|^2}} \cdot \frac{x'}{|x'|}\, \mathrm{d}\mathcal{H}^{N-2}, \end{aligned}$$

and the last integral in the previous expression converges as $h \to \infty$ to

$$\begin{aligned} - \int_{\partial B'} \frac{\nabla g}{\sqrt{1 + |\nabla g|^2}} \cdot \frac{x'}{|x'|}\, \mathrm{d}\mathcal{H}^{N-2} &= - \int_{B'} \mathrm{div}\Big(\frac{\nabla g}{\sqrt{1 + |\nabla g|^2}} \Big)\, \mathrm{d}\mathcal{H}^{N-1}(x') \\ &= \lambda \mathcal{H}^{N-1}(B') - 2\gamma \int_{B'} v_E(x', g(x'))\, \mathrm{d}\mathcal{H}^{N-1}(x'), \end{aligned}$$

where the last equality follows by (4.21). Then, recalling (4.22) and that ρ_h tends uniformly to 0, we conclude that $(\lambda_h - \lambda)\mathcal{H}^{N-1}(A'_h) \to 0$, which in turn gives $\lambda_h \to \lambda$.

Hence (4.23) is proved and, in turn, we conclude that $\psi_h \to 0$ in $W^{2,p}(\partial E)$ by Lemma 4.26. The thesis now follows since (4.19) contradicts the assumption that E is a local $W^{2,p}$-minimizer. $\square$

We are finally ready to complete the proof of Theorem 4.8. The strategy follows closely [1, Theorem 1.1], with the necessary technical modifications due to the fact that here we have to deal with a more general exponent α and with the lack of compactness of the ambient space.

Proof of Theorem 4.8. We assume by contradiction that there exists a sequence of sets $E_h \subset \mathbb{R}^N$, with $|E_h| = |E|$ and $\alpha(E_h, E) > 0$, such that $\varepsilon_h := \alpha(E_h, E) \to 0$ and

$$\mathcal{F}(E_h) < \mathcal{F}(E) + \frac{C_0}{4} \big(\alpha(E_h, E)\big)^2, \tag{4.24}$$

where C_0 is the constant provided by Theorem 4.25. By approximation we can assume without loss of generality that each set of the sequence is bounded, that is, there exist $R_h > 0$ (which we can also take satisfying $R_h \to +\infty$) such that $E_h \subset B_{R_h}$.

We now define $F_h \subset \mathbb{R}^N$ as a solution to the penalization problem

$$\min\left\{\mathcal{J}_h(F) := \mathcal{F}(F) + \Lambda_1\sqrt{\big(\alpha(F,E)-\varepsilon_h\big)^2+\varepsilon_h} + \Lambda_2\big||F|-|E|\big| \,:\, F\subset B_{R_h}\right\}, \tag{4.25}$$

where Λ_1 and Λ_2 are positive constant, to be chosen (notice that the constraint $F \subset B_{R_h}$ guarantees the existence of a solution). We first fix

$$\Lambda_1 > C_E + c_0\gamma\,. \tag{4.26}$$

Here C_E is given by Lemma 1.9, while c_0 is the constant provided by Proposition 4.3 corresponding to the fixed values of N and α and to $m := |E|+1$. We remark that with this choice Λ_1 depends only on the set E. We will consider also the sets $\widetilde{F}_h$ obtained by translating F_h in such a way that $\alpha(F_h, E) = |\widetilde{F}_h \triangle E|$ (clearly $\mathcal{J}_h(\widetilde{F}_h) = \mathcal{J}_h(F_h)$).

Step 1. We claim that, if Λ_2 is sufficiently large (depending on Λ_1, but not on h), then $|F_h| = |E|$ for every h large enough. This can be deduced by adapting an argument from [38, Section 2] (see also [1, Proposition 2.7]). Indeed, assume by contradiction that there exist $\Lambda_h \to \infty$ and F_h solution to the minimum problem (4.25) with Λ_2 replaced by Λ_h such that $|F_h| < |E|$ (a similar argument can be performed in the case $|F_h| > |E|$). Up to subsequences, we have that $F_h \to F_0$ in L^1_{loc} and $|F_h| \to |E|$.

As each set F_h minimizes the functional

$$\mathcal{F}(F) + \Lambda_1\sqrt{\big(\alpha(F,E)-\varepsilon_h\big)^2+\varepsilon_h}$$

in B_{R_h} under the constraint $|F| = |F_h|$, it is easily seen that F_h is a quasi-minimizer of the perimeter with volume constraint, so that by the regularity result contained in [73, Theorem 1.4.4] we have that the $(N-1)$-dimensional density of $\partial^* F_h$ is uniformly bounded from below by a constant independent of h. This observation implies that we can assume without loss of generality that the limit set F_0 is not empty and that there exists a point $x_0 \in \partial^* F_0$, so that, by repeating an argument contained in [38], we obtain that given $\varepsilon > 0$ we can find $r > 0$ and $\bar{x} \in \mathbb{R}^N$ such that

$$|F_h \cap B_{r/2}(\bar{x})| < \varepsilon r^N, \quad |F_h \cap B_r(\bar{x})| > \frac{\omega_N r^N}{2^{N+2}}$$

for every h sufficiently large (and we assume $\bar{x} = 0$ for simplicity).

Now we modify F_h in B_r by setting $G_h := \Phi_h(F_h)$, where Φ_h is the bilipschitz map

$$\Phi_h(x) := \begin{cases} \big(1-\sigma_h(2^N-1)\big)x & \text{if } |x| \le \frac{r}{2},\\ x + \sigma_h\big(1-\frac{r^N}{|x|^N}\big)x & \text{if } \frac{r}{2} < x < r,\\ x & \text{if } |x| \ge r, \end{cases}$$

and $\sigma_h \in (0, \frac{1}{2^N})$. It can be shown (see [38, Section 2], [1, Proposition 2.7] for details) that ε and σ_h can be chosen in such a way that $|G_h| = |E|$, and moreover there exists a dimensional constant $C > 0$ such that

$$J_{\Lambda_h}(F_h) - J_{\Lambda_h}(G_h) \ge \sigma_h\Big(C\Lambda_h r^N - (2^N N + C\gamma + C\Lambda_1)\mathcal{P}(F_h; B_r)\Big)$$

(where J_{Λ_h} denotes the functional in (4.25) with Λ_2 replaced by Λ_h). This contradicts the minimality of F_h for h sufficiently large.

Step 2. We now show that

$$\lim_{h\to+\infty} \alpha(F_h, E) = 0. \tag{4.27}$$

Indeed, by Lemma 1.9 we have that

$$\mathcal{P}(E) \le \mathcal{P}(\widetilde{F}_h) + C_E|\widetilde{F}_h \triangle E|,$$

while by Proposition 4.3

$$|\mathcal{NL}(E) - \mathcal{NL}(\widetilde{F}_h)| \le c_0 |\widetilde{F}_h \triangle E|.$$

Combining the two estimates above, using the minimality of F_h and recalling that $|F_h| = |E|$ we deduce

$$\begin{aligned}\mathcal{P}(\widetilde{F}_h) + \gamma\mathcal{NL}(\widetilde{F}_h) + \Lambda_1\sqrt{\big(|\widetilde{F}_h\triangle E| - \varepsilon_h\big)^2 + \varepsilon_h} &= \mathcal{J}_h(F_h) \le \mathcal{J}_h(E)\\ &= \mathcal{P}(E) + \gamma\mathcal{NL}(E) + \Lambda_1\sqrt{\varepsilon_h^2 + \varepsilon_h}\\ &\le \mathcal{P}(\widetilde{F}_h) + \gamma\mathcal{NL}(\widetilde{F}_h) + (C_E + c_0\gamma)|\widetilde{F}_h\triangle E| + \Lambda_1\sqrt{\varepsilon_h^2 + \varepsilon_h},\end{aligned}$$

which yields

$$\Lambda_1\sqrt{\big(|\widetilde{F}_h\triangle E| - \varepsilon_h\big)^2 + \varepsilon_h} \le (C_E + c_0\gamma)|\widetilde{F}_h\triangle E| + \Lambda_1\sqrt{\varepsilon_h^2 + \varepsilon_h}.$$

Passing to the limit as $h \to +\infty$, we conclude that

$$\Lambda_1 \limsup_{h\to+\infty} |\widetilde{F}_h\triangle E| \le (C_E + c_0\gamma) \limsup_{h\to+\infty} |\widetilde{F}_h\triangle E|,$$

which implies $|\widetilde{F}_h\triangle E| \to 0$ by the choice of Λ_1 in (4.26). Hence (4.27) is proved, and this shows in particular that $\chi_{\widetilde{F}_h} \to \chi_E$ in $L^1(\mathbb{R}^N)$.

Step 3. We claim that each set F_h is an (ω, r_0)-minimizer of the area functional (see Definition 1.1), for suitable $\omega > 0$ and $r_0 > 0$ independent of h. Indeed, choose r_0 such that $\omega_N {r_0}^N \le 1$, and consider any ball $B_r(x)$ with $r \le r_0$ and any finite perimeter set F such that $F\triangle F_h \subset\subset B_r(x)$. We have

$$|\mathcal{NL}(F) - \mathcal{NL}(F_h)| \le c_0|F\triangle F_h|$$

by Proposition 4.3, where c_0 is the same constant as before since we can bound the volume of F by $|F| \le |F_h| + \omega_N {r_0}^N \le |E| + 1$. Moreover

$$\begin{aligned}\mathcal{P}(F) - \mathcal{P}(F\cap B_{R_h}) &= \int_{\partial^* F\setminus B_{R_h}} \mathrm{d}\mathcal{H}^{N-1}(x) - \int_{\partial^*(F\cap B_{R_h})\cap\partial B_{R_h}} \mathrm{d}\mathcal{H}^{N-1}(x)\\ &\ge \int_{\partial^* F\setminus B_{R_h}} \frac{x}{|x|}\cdot\nu_F\,\mathrm{d}\mathcal{H}^{N-1}(x) - \int_{\partial^*(F\cap B_{R_h})\cap\partial B_{R_h}} \frac{x}{|x|}\cdot\nu_{F\cap B_{R_h}}\,\mathrm{d}\mathcal{H}^{N-1}(x)\\ &= \int_{\partial^*(F\setminus B_{R_h})} \frac{x}{|x|}\cdot\nu_{F\setminus B_{R_h}}\,\mathrm{d}\mathcal{H}^{N-1}(x) = \int_{F\setminus B_{R_h}} \operatorname{div}\frac{x}{|x|}\,\mathrm{d}x \ge 0.\end{aligned}$$

Hence, as F_h is a minimizer of $\mathcal{J}_h$ among sets contained in B_{R_h}, we deduce

$$\begin{aligned}\mathcal{P}(F_h) &\le \mathcal{P}(F\cap B_{R_h}) + \gamma\big(\mathcal{NL}(F\cap B_{R_h}) - \mathcal{NL}(F_h)\big) + \Lambda_2\big||F\cap B_{R_h}| - |E|\big|\\ &\quad + \Lambda_1\sqrt{\big(\alpha(F\cap B_{R_h}, E) - \varepsilon_h\big)^2 + \varepsilon_h} - \Lambda_1\sqrt{\big(\alpha(F_h, E) - \varepsilon_h\big)^2 + \varepsilon_h}\\ &\le \mathcal{P}(F) + \big(c_0\gamma + \Lambda_1 + \Lambda_2\big)|(F\cap B_{R_h})\triangle F_h|\\ &\le \mathcal{P}(F) + \big(c_0\gamma + \Lambda_1 + \Lambda_2\big)|F\triangle F_h|\end{aligned}$$

for h large enough. This shows the claim with $\omega = c_0\gamma + \Lambda_1 + \Lambda_2$ (the same property holds obviously also for $\widetilde{F}_h$).

Recalling that $\chi_{\widetilde{F}_h} \to \chi_E$ in L^1, we can apply Theorem 1.4 and we obtain that for h sufficiently large $\widetilde{F}_h$ is a set of class $C^{1,\beta}$ and

$$\partial\widetilde{F}_h = \{x + \psi_h(x)\nu_E(x) \,:\, x \in \partial E\}$$

for some ψ_h such that $\psi_h \to 0$ in $C^{1,\beta}(\partial E)$, for every $\beta \in (0, \frac{1}{2})$. We remark also that the sets $\widetilde{F}_h$ are uniformly bounded, and for h large enough $\widetilde{F}_h \subset\subset B_{R_h}$: in particular, $\widetilde{F}_h$ solves the minimum problem (4.25).

Step 4. We now claim that

$$\lim_{h \to +\infty} \frac{\alpha(F_h, E)}{\varepsilon_h} = 1. \tag{4.28}$$

Indeed, assuming by contradiction that $|\alpha(F_h, E) - \varepsilon_h| \geq \sigma\varepsilon_h$ for some $\sigma > 0$ and for infinitely many h, we would obtain

$$\begin{aligned} \mathcal{F}(F_h) + \Lambda_1\sqrt{\sigma^2\varepsilon_h^2 + \varepsilon_h} &\leq \mathcal{F}(F_h) + \Lambda_1\sqrt{\big(\alpha(F_h, E) - \varepsilon_h\big)^2 + \varepsilon_h} \\ &\leq \mathcal{F}(E_h) + \Lambda_1\sqrt{\varepsilon_h} < \mathcal{F}(E) + \frac{C_0}{4}\,\varepsilon_h^2 + \Lambda_1\sqrt{\varepsilon_h} \\ &\leq \mathcal{F}(\widetilde{F}_h) + \frac{C_0}{4}\,\varepsilon_h^2 + \Lambda_1\sqrt{\varepsilon_h} \end{aligned}$$

where the second inequality follows from the minimality of F_h, the third one from (4.24) and the last one from Theorem 4.27. This shows that

$$\Lambda_1\sqrt{\sigma^2\varepsilon_h^2 + \varepsilon_h} \leq \frac{C_0}{4}\,\varepsilon_h^2 + \Lambda_1\sqrt{\varepsilon_h}\,,$$

which is a contradiction for h large enough.

Step 5. We now show the existence of constants $\lambda_h \in \mathbb{R}$ such that

$$\|H_{\partial\widetilde{F}_h} + 2\gamma v_{\widetilde{F}_h} - \lambda_h\|_{L^\infty(\partial\widetilde{F}_h)} \leq 4\Lambda_1\sqrt{\varepsilon_h} \to 0. \tag{4.29}$$

We first observe that the function $f_h(t) := \sqrt{(t - \varepsilon_h)^2 + \varepsilon_h}$ satisfies

$$|f_h(t_1) - f_h(t_2)| \leq 2\sqrt{\varepsilon_h}\,|t_1 - t_2| \qquad \text{if} \quad |t_i - \varepsilon_h| \leq \varepsilon_h. \tag{4.30}$$

Hence for every set $F \subset \mathbb{R}^N$ with $|F| = |E|$, $F \subset B_{R_h}$ and $|\alpha(F, E) - \varepsilon_h| \leq \varepsilon_h$ we have

$$\begin{aligned} \mathcal{F}(\widetilde{F}_h) &\leq \mathcal{F}(F) + \Lambda_1\Big(\sqrt{\big(\alpha(F, E) - \varepsilon_h\big)^2 + \varepsilon_h} - \sqrt{\big(\alpha(\widetilde{F}_h, E) - \varepsilon_h\big)^2 + \varepsilon_h}\Big) \\ &\leq \mathcal{F}(F) + 2\Lambda_1\sqrt{\varepsilon_h}\,|\alpha(F, E) - \alpha(\widetilde{F}_h, E)| \\ &\leq \mathcal{F}(F) + 2\Lambda_1\sqrt{\varepsilon_h}\,|F \triangle \widetilde{F}_h| \end{aligned} \tag{4.31}$$

where we used the minimality of $\widetilde{F}_h$ in the first inequality, and (4.30) combined with the fact that $|\alpha(\widetilde{F}_h, E) - \varepsilon_h| \leq \varepsilon_h$ for h large (which, in turn, follows by (4.28)) in the second one.

Consider now any variation Φ_t, as in Definition 4.15, preserving the volume of the set $\widetilde{F}_h$, associated with a vector field X. For $|t|$ sufficiently small we can plug the set $\Phi_t(\widetilde{F}_h)$ in the inequality (4.31):

$$\mathcal{F}(\widetilde{F}_h) \leq \mathcal{F}(\Phi_t(\widetilde{F}_h)) + 2\Lambda_1\sqrt{\varepsilon_h}\,|\Phi_t(\widetilde{F}_h) \triangle \widetilde{F}_h|,$$

which gives

$$\mathcal{F}(\Phi_t(\widetilde{F}_h)) - \mathcal{F}(\widetilde{F}_h) + 2\Lambda_1\sqrt{\varepsilon_h}\,|t| \int_{\partial\widetilde{F}_h} |X \cdot \nu_{\widetilde{F}_h}|\,\mathrm{d}\mathcal{H}^{N-1} + o(t) \geq 0$$

for $|t|$ sufficiently small. Hence, dividing by t and letting $t \to 0^+$ and $t \to 0^-$, we get

$$\left| \int_{\partial\widetilde{F}_h} (H_{\partial\widetilde{F}_h} + 2\gamma v_{\widetilde{F}_h}) X \cdot \nu_{\widetilde{F}_h}\,\mathrm{d}\mathcal{H}^{N-1} \right| \leq 2\Lambda_1\sqrt{\varepsilon_h} \int_{\partial\widetilde{F}_h} |X \cdot \nu_{\widetilde{F}_h}|\,\mathrm{d}\mathcal{H}^{N-1},$$

and by density

$$\left| \int_{\partial \widetilde{F}_h} (H_{\partial \widetilde{F}_h} + 2\gamma v_{\widetilde{F}_h}) \varphi \, \mathrm{d}\mathcal{H}^{N-1} \right| \leq 2\Lambda_1 \sqrt{\varepsilon_h} \int_{\partial \widetilde{F}_h} |\varphi| \, \mathrm{d}\mathcal{H}^{N-1}$$

for every $\varphi \in C^\infty(\partial \widetilde{F}_h)$ with $\int_{\partial \widetilde{F}_h} \varphi \, \mathrm{d}\mathcal{H}^{N-1} = 0$. In turn, this implies (4.29) by a simple functional analysis argument.

Step 6. We are now close to the end of the proof. Recall that on ∂E

$$H_{\partial E} = \lambda - 2\gamma v_E \tag{4.32}$$

for some constant λ, while by (4.29)

$$H_{\partial \widetilde{F}_h} = \lambda_h - 2\gamma v_{\widetilde{F}_h} + \rho_h, \qquad \text{with } \rho_h \to 0 \text{ uniformly.} \tag{4.33}$$

Observe now that, since the functions $v_{\widetilde{F}_h}$ are equibounded in $C^{1,\beta}(\mathbb{R}^N)$ for some $\beta \in (0,1)$ (see Proposition 4.1) and they converge pointwise to v_E since $\chi_{\widetilde{F}_h} \to \chi_E$ in L^1, we have that

$$v_{\widetilde{F}_h} \to v_E \qquad \text{in } C^1(\overline{B}_R) \text{ for every } R > 0. \tag{4.34}$$

We consider a cylinder $C = B' \times]-L, L[$, where $B' \subset \mathbb{R}^{N-1}$ is a ball centered at the origin, such that in a suitable coordinate system we have

$$\widetilde{F}_h \cap C = \{(x', x_N) \in C : x' \in B',\, x_N < g_h(x')\},$$
$$E \cap C = \{(x', x_N) \in C : x' \in B',\, x_N < g(x')\}$$

for some functions $g_h \to g$ in $C^{1,\beta}(\overline{B'})$ for every $\beta \in (0, \frac{1}{2})$. By integrating (4.33) on B' we obtain

$$\begin{aligned}
&\lambda_h \mathcal{L}^{N-1}(B') - 2\gamma \int_{B'} v_{\widetilde{F}_h}(x', g_h(x')) \, \mathrm{d}\mathcal{L}^{N-1}(x') + \int_{B'} \rho_h(x', g_h(x')) \, \mathrm{d}\mathcal{L}^{N-1}(x') \\
&\quad = -\int_{B'} \operatorname{div}\left(\frac{\nabla g_h}{\sqrt{1 + |\nabla g_h|^2}} \right) \mathrm{d}\mathcal{L}^{N-1}(x') = -\int_{\partial B'} \frac{\nabla g_h}{\sqrt{1 + |\nabla g_h|^2}} \cdot \frac{x'}{|x'|} \, \mathrm{d}\mathcal{H}^{N-2},
\end{aligned}$$

and the last integral in the previous expression converges as $h \to 0$ to

$$\begin{aligned}
-\int_{\partial B'} \frac{\nabla g}{\sqrt{1 + |\nabla g|^2}} \cdot \frac{x'}{|x'|} \, \mathrm{d}\mathcal{H}^{N-2} &= -\int_{B'} \operatorname{div}\left(\frac{\nabla g}{\sqrt{1 + |\nabla g|^2}} \right) \mathrm{d}\mathcal{L}^{N-1}(x') \\
&= \lambda \mathcal{L}^{N-1}(B') - 2\gamma \int_{B'} v_E(x', g(x')) \, \mathrm{d}\mathcal{L}^{N-1}(x'),
\end{aligned}$$

where the last equality follows by (4.32). This shows, recalling (4.34) and that ρ_h tends to 0 uniformly, that $\lambda_h \to \lambda$, which in turn implies, by (4.32), (4.33) and (4.34),

$$H_{\partial \widetilde{F}_h}(\cdot + \psi_h(\cdot)\nu_E(\cdot)) \to H_{\partial E} \qquad \text{in } L^\infty(\partial E).$$

By Lemma 4.26 we conclude that $\psi_h \in W^{2,p}(\partial E)$ for every $p \geq 1$ and $\psi_h \to 0$ in $W^{2,p}(\partial E)$.

Finally, by minimality of $\widetilde{F}_h$ we have

$$\begin{aligned}
\mathcal{F}(\widetilde{F}_h) &\leq \mathcal{F}(\widetilde{F}_h) + \Lambda_1 \sqrt{\big(\alpha(\widetilde{F}_h, E) - \varepsilon_h\big)^2 + \varepsilon_h} - \Lambda_1 \sqrt{\varepsilon_h} \\
&\leq \mathcal{F}(E_h) < \mathcal{F}(E) + \frac{C_0}{4} \varepsilon_h^2 \leq \mathcal{F}(E) + \frac{C_0}{2} \big(\alpha(\widetilde{F}_h, E)\big)^2
\end{aligned}$$

where we used (4.24) in the third inequality and (4.28) in the last one. This is the desired contradiction with the conclusion of Theorem 4.25. $\square$

REMARK 4.28. It is important to remark that in the arguments of this section we have not made use of the assumption of strict positivity of the second variation: the quantitative local L^1-minimality follows in fact just from the local $W^{2,p}$-minimality.

4.4. Local minimality of the ball

This section is devoted to the proof of Theorem 4.9, which can be obtained as a consequence of Theorem 4.8 by computing the second variation of the functional $\mathcal{F}$ at the ball and studying the sign of the associated quadratic form.

4.4.1. Recalls on spherical harmonics. We first recall some basic facts about spherical harmonics, referring to [53] for an account on this topic.

DEFINITION 4.29. A *spherical harmonic of dimension* N is the restriction to $\mathbb{S}^{N-1}$ of a *harmonic polynomial* in N variables, *i.e.* a homogeneous polynomial p with $\Delta p = 0$.

We will denote by $\mathcal{H}_d^N$ the set of all spherical harmonics of dimension N that are obtained as restrictions to $\mathbb{S}^{N-1}$ of homogeneous polynomials of degree d. In particular $\mathcal{H}_0^N$ is the space of constant functions, and $\mathcal{H}_1^N$ is generated by the coordinate functions. The basic properties of spherical harmonics that we need are listed in the following theorem.

THEOREM 4.30. *The following properties hold.*

(1) *For each* $d \in \mathbb{N}$, $\mathcal{H}_d^N$ *is a finite dimensional vector space.*
(2) *If* $F \in \mathcal{H}_d^N$, $G \in \mathcal{H}_e^N$ *and* $d \neq e$, *then* F *and* G *are orthogonal (in the* L^2*-sense).*
(3) *If* $F \in \mathcal{H}_d^N$ *and* $d \neq 0$, *then*
$$\int_{\mathbb{S}^{N-1}} F \, \mathrm{d}\mathcal{H}^{N-1} = 0.$$
(4) *If* $(H_d^1, \ldots, H_d^{\dim(\mathcal{H}_d^N)})$ *is an orthonormal basis of* $\mathcal{H}_d^N$ *for every* $d \geq 0$, *then this sequence is complete,* i.e. *every* $F \in L^2(\mathbb{S}^{N-1})$ *can be written in the form*
$$F = \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} c_d^i H_d^i, \tag{4.35}$$
where $c_d^i := \langle F, H_d^i \rangle_{L^2}$.
(5) *If* H_d^i *are as in (4) and* $F, G \in L^2(\mathbb{S}^{N-1})$ *are such that*
$$F = \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} c_d^i H_d^i, \qquad G = \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} e_d^i H_d^i,$$
then
$$\langle F, G \rangle_{L^2} = \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} c_d^i e_d^i.$$
(6) *Spherical harmonics are eigenfunctions of the* Laplace-Beltrami operator $\Delta_{\mathbb{S}^{N-1}}$. *More precisely, if* $H \in \mathcal{H}_d^N$ *then*
$$-\Delta_{\mathbb{S}^{N-1}} H = d(d+N-2)H.$$
(7) *If* F *is a* C^2*-function on* $\mathbb{S}^{N-1}$ *represented as in* (4.35), *then*
$$\int_{\mathbb{S}^{N-1}} |\nabla_{\mathbb{S}^{N-1}} F|^2 \, \mathrm{d}\mathcal{H}^{N-1}(x) = \sum_{d=0}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} d(d+N-2)(c_d^i)^2.$$

We recall also the following important result in the theory of spherical harmonics.

THEOREM 4.31 (Funk-Hecke Formula). *Let $f : (-1,1) \to \mathbb{R}$ such that*

$$\int_{-1}^{1} |f(t)|(1-t^2)^{\frac{N-3}{2}}\,\mathrm{d}t < \infty\,.$$

Then if $H \in \mathcal{H}_d^N$ and $x_0 \in \mathbb{S}^{N-1}$ it holds

$$\int_{\mathbb{S}^{N-1}} f(\langle x_0, x\rangle)H(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) \;=\; \mu_d H(x_0)\,,$$

where the coefficient μ_d is given by

$$\mu_d = (N-1)\omega_{N-1}\int_{-1}^{1} P_{N,d}(t)f(t)(1-t^2)^{\frac{N-3}{2}}\,\mathrm{d}t.$$

Here $P_{N,d}$ is the Legendre polynomial *of dimension N and degree d given by*

$$P_{N,d}(t) = (-1)^d\frac{\Gamma(\frac{N-1}{2})}{2^d\Gamma(d+\frac{N-1}{2})}(1-t^2)^{-\frac{N-3}{2}}\Big(\frac{\mathrm{d}}{\mathrm{d}t}\Big)^d(1-t^2)^{d+\frac{N-3}{2}}\,,$$

where $\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\mathrm{d}t$ is the Gamma function.

4.4.2. Second variation of the ball. The quadratic form (4.13) associated with the second variation of $\mathcal{F}$ at the ball B_R, computed at a function $\tilde\varphi \in \widetilde{H}^1(\partial B_R)$ is

$$\begin{aligned}
\partial^2\mathcal{F}(B_R)[\tilde\varphi] = &\int_{\partial B_R}\Big(|\nabla_{\partial B_R}\tilde\varphi(x)|^2 - \frac{N-1}{R^2}\,\tilde\varphi^2(x)\Big)\,\mathrm{d}\mathcal{H}^{N-1}(x)\\
&+\;2\gamma\int_{\partial B_R}\int_{\partial B_R}\frac{1}{|x-y|^\alpha}\tilde\varphi(x)\tilde\varphi(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y)\\
&+\;2\gamma\int_{\partial B_R}\Big(\int_{B_R} -\alpha\frac{\langle x-y, \frac{x}{|x|}\rangle}{|x-y|^{\alpha+2}}\,\mathrm{d}y\Big)\tilde\varphi^2(x)\,\mathrm{d}\mathcal{H}^{N-1}(x).
\end{aligned}$$

Since we want to obtain a sign condition of $\partial^2\mathcal{F}(B_R)[\tilde\varphi]$ in terms of the radius R, we first make a change of variable:

$$\begin{aligned}
\partial^2\mathcal{F}(B_R)[\tilde\varphi] = R^{N-3}&\int_{\partial B_1}(|\nabla_{\partial B_1}\varphi(x)|^2 - (N-1)\varphi^2(x))\,\mathrm{d}\mathcal{H}^{N-1}(x)\\
&+\;2\gamma R^{2N-2-\alpha}\int_{\partial B_1}\int_{\partial B_1}\frac{1}{|x-y|^\alpha}\varphi(x)\varphi(x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y)\\
&+\;2\gamma R^{2N-2-\alpha}\int_{\partial B_1}\Big(\int_{B_1} -\alpha\frac{\langle x-y, x\rangle}{|x-y|^{\alpha+2}}\,\mathrm{d}y\Big)\varphi^2(x)\,\mathrm{d}\mathcal{H}^{N-1}(x),
\end{aligned}\tag{4.36}$$

where the function $\varphi \in \widetilde{H}^1(\mathbb{S}^{N-1})$ is defined as $\varphi(x) := \tilde\varphi(Rx)$. Since we are only interested in the sign of the second variation, which is continuous with respect to the strong convergence in $\widetilde{H}^1(\mathbb{S}^{N-1})$, we can assume $\varphi \in C^2(\mathbb{S}^{N-1}) \cap T^\perp(\mathbb{S}^{N-1})$.

The idea is now to expand φ with respect to an orthonormal basis of spherical harmonics, as in (4.35). First of all we notice that if $\varphi \in T^\perp(\mathbb{S}^{N-1})$, then its harmonic expansion does not contain spherical harmonics of order 0 and 1. Indeed, harmonics of order 0 are constant functions, that are not allowed by the null average condition. Moreover $\mathcal{H}_1^N = T(\mathbb{S}^{N-1})$, because $\nu_{\mathbb{S}^{N-1}}(x) = x$, and the functions x_i form an orthonormal basis of $\mathcal{H}_1^N$. Hence we can write the harmonic expansion of $\varphi \in C^2(\mathbb{S}^{N-1}) \cap T^\perp(\mathbb{S}^{N-1})$ as follows:

$$\varphi = \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} c_d^i H_d^i\,,$$

where $(H_d^1, \ldots, H_d^{\dim(\mathcal{H}_d^N)})$ is an orthonormal basis of $\mathcal{H}_d^N$ for each $d \in \mathbb{N}$. We can now compute each term appearing in (4.36) as follows: the first term, by property (7) of Theorem 4.30, is

$$\int_{\mathbb{S}^{N-1}} (|\nabla_{\mathbb{S}^{N-1}}\varphi|^2 - (N-1)\varphi^2)\,\mathrm{d}\mathcal{H}^{N-1} = \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} \big(d(d+N-2) - (N-1)\big)(c_d^i)^2.$$

For the second term we want to use the Funk-Hecke Formula to compute the inner integral; so we define the function

$$f(t) := \Big(2(1-t)\Big)^{-\frac{\alpha}{2}}$$

and we notice that

$$|x-y|^{-\alpha} = f(\langle x, y\rangle) \qquad \text{for } x, y \in \mathbb{S}^{N-1}\,,$$

and that, for $\alpha \in (0, N-1)$, f satisfies the integrability assumptions of Theorem 4.31. Hence for each $y \in \mathbb{S}^{N-1}$

$$\int_{\partial B_1} \frac{1}{|x-y|^\alpha}\varphi(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) = \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} \mu_d^{N,\alpha} c_d^i H_d^i(y)\,,$$

where the coefficient

$$\mu_d^{N,\alpha} := 2^{N-1-\alpha}\frac{(N-1)\omega_{N-1}}{2}\left(\prod_{i=0}^{d-1}\left(\frac{\alpha}{2}+i\right)\right)\frac{\Gamma(\frac{N-1-\alpha}{2})\Gamma(\frac{N-1}{2})}{\Gamma(N-1-\frac{\alpha}{2}+d)} \tag{4.37}$$

is obtained by direct computation just integrating by parts. Therefore

$$\int_{\partial B_1}\int_{\partial B_1} \frac{1}{|x-y|^\alpha}\varphi(x)\varphi(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,\mathrm{d}\mathcal{H}^{N-1}(y) = \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} \mu_d^{N,\alpha}(c_d^i)^2.$$

For the last term of (4.36), noticing that the integral

$$\mathcal{I}^{N,\alpha} := \int_{B_1} \frac{\langle x-y, x\rangle}{|x-y|^{\alpha+2}}\,\mathrm{d}y$$

is independent of $x \in \mathbb{S}^{N-1}$, we get

$$\int_{\partial B_1}\Big(\int_{B_1} -\alpha\frac{\langle x-y, x\rangle}{|x-y|^{\alpha+2}}\,\mathrm{d}y\Big)\varphi^2(x)\,\mathrm{d}\mathcal{H}^{N-1}(x) = -\alpha\mathcal{I}^{N,\alpha}\sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} (c_d^i)^2.$$

Combining all the previous equalities with (4.36) we obtain

$$\partial^2\mathcal{F}(B_R)[\tilde\varphi] = \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} R^{N-3}(c_d^i)^2\Big[d(d+N-2) - (N-1) + 2\gamma R^{N+1-\alpha}\Big(\mu_d^{N,\alpha} - \alpha\mathcal{I}^{N,\alpha}\Big)\Big].$$

4.4.3. Local minimality of the ball. From the above expression we deduce that the quadratic form $\partial^2\mathcal{F}(B_R)$ is strictly positive on $T^\perp(\partial B_R)$, that is, the second variation of $\mathcal{F}$ at B_R is positive according to Definition 4.22, if and only if

$$d(d+N-2)-(N-1)+2\gamma R^{N+1-\alpha}\Big(\mu_d^{N,\alpha}-\alpha\mathcal{I}^{N,\alpha}\Big)>0 \tag{4.38}$$

for all $d\geq 2$, where the "only if" part is due to the fact that $\mathcal{H}_d^N\subset T^\perp(S^{N-1})$ for each $d\geq 2$. On the contrary, $\partial^2\mathcal{F}(B_R)[\tilde\varphi]<0$ for some $\tilde\varphi\in T^\perp(\partial B_R)$ if and only if there exists $d\geq 2$ such that the left-hand side of (4.38) is negative.

We want to write (4.38) as a condition on R. Since $d(d+N-2)-(N-1)>0$ for $d\geq 2$, we have that (4.38) is certainly satisfied if $\mu_d^{N,\alpha}-\alpha\mathcal{I}^{N,\alpha}>0$. But this is not always the case, as the following lemma shows.

Lemma 4.32. *The sequence $\mu_d^{N,\alpha}$ strictly decreases to 0 as $d\to\infty$.*

Proof. First of all we note that

$$\mu_{d+1}^{N,\alpha}=\frac{\frac{\alpha}{2}+d}{N-1-\frac{\alpha}{2}+d}\,\mu_d^{N,\alpha}\,, \tag{4.39}$$

hence the sequence $(\mu_d^{N,\alpha})_{d\in\mathbb{N}}$ is decreasing since $\alpha<N-1$. Now

$$\begin{aligned}\mu_{d+1}^{N,\alpha}&=\Big(\prod_{k=1}^{d}\frac{\frac{\alpha}{2}+k}{N-1-\frac{\alpha}{2}+k}\Big)\mu_1^{N,\alpha}=\frac{\Gamma(N-\frac{\alpha}{2})\Gamma(1+\frac{\alpha}{2}+d)}{\Gamma(1+\frac{\alpha}{2})\Gamma(N-\frac{\alpha}{2}+d)}\mu_1^{N,\alpha}\\ &\sim_{d\to\infty}\frac{\Gamma(N-\frac{\alpha}{2})}{\Gamma(1+\frac{\alpha}{2})}\mu_1^{N,\alpha}\sqrt{\frac{\frac{\alpha}{2}+d}{N-1-\frac{\alpha}{2}+d}}\,\frac{e^{(\frac{\alpha}{2}+d)[\log(\frac{\alpha}{2}+d)-1]}}{e^{(N-1-\frac{\alpha}{2}+d)[\log(N-1-\frac{\alpha}{2}+d)-1]}}\,,\end{aligned}$$

where in the second equality we used the well known property $\Gamma(x+1)=x\Gamma(x)$, and in the last step we used the Stirling's formula. Since the previous quantity is infinitesimal as $d\to\infty$, we conclude the proof of the lemma. □

As a consequence of this lemma and of the fact that $\mathcal{I}^{N,\alpha}>0$, we have that the number

$$d_A^{N,\alpha}:=\min\{d\geq 2\,:\,\mu_d^{N,\alpha}<\alpha\mathcal{I}^{N,\alpha}\}$$

is well defined. This tells us that (4.38) is satisfied for every $R>0$ if $d<d_A^{N,\alpha}$, and for

$$R<\left(\frac{d(d+N-2)-(N-1)}{2\gamma\big(\alpha\mathcal{I}^{N,\alpha}-\mu_d^{N,\alpha}\big)}\right)^{\frac{1}{N+1-\alpha}}=:g^{N,\alpha}(d).$$

if $d\geq d_A^{N,\alpha}$. Moreover, by the previous lemma we get that $g^{N,\alpha}(d)\to\infty$ as $d\to\infty$. The following lemma tells us something more about the behaviour of the function $g^{N,\alpha}$.

Lemma 4.33. *There exists a natural number $d_I^{N,\alpha}$ such that for $d<d_I^{N,\alpha}$ the function $g^{N,\alpha}$ is decreasing, while for $d>d_I^{N,\alpha}$ is increasing.*

Proof. The condition $g^{N,\alpha}(d+1)>g^{N,\alpha}(d)$ is equivalent to

$$\frac{(d+1)(d+1+N-2)-(N-1)}{2\gamma\big(\alpha\mathcal{I}^{N,\alpha}-\mu_{d+1}^{N,\alpha}\big)}>\frac{d(d+N-2)-(N-1)}{2\gamma\big(\alpha\mathcal{I}^{N,\alpha}-\mu_d^{N,\alpha}\big)}.$$

Recalling (4.39), the above inequality can be rewritten, after some algebraic steps, as follows:

$$\alpha\mathcal{I}^{N,\alpha}>\frac{d^2(N-\alpha+1)+d(N^2-\alpha N+\alpha-1)+\frac{\alpha}{2}(N-1)}{(N-1-\frac{\alpha}{2}+d)(2d+N-1)}\,\mu_d^{N,\alpha}. \tag{4.40}$$

Using (4.39), it is easily seen that the right-hand side of the above inequality is decreasing and converges to 0 as $d \to \infty$. Hence the number

$$d_I^{N,\alpha} := \min\{d \in \mathbb{N} : (4.40) \text{ is satisfied}\}$$

is well defined and satisfies the requirement of the lemma. □

We are now in position to prove Theorem 4.9.

PROOF OF THEOREM 4.9. Define

$$\overline{R}(N,\alpha,\gamma) := \min_{d \geq d_A^{N,\alpha}} g^{N,\alpha}(d)\,,$$

which can be characterized, by the previous lemmas, as

$$\overline{R}(N,\alpha,\gamma) := \begin{cases} g^{N,\alpha}(d_A^{N,\alpha}) & \text{if } d_A^{N,\alpha} > d_I^{N,\alpha}, \\ g^{N,\alpha}(d_I^{N,\alpha}) & \text{if } d_A^{N,\alpha} \leq d_I^{N,\alpha}. \end{cases}$$

Now, from (4.38), we have that

$$\partial^2 \mathcal{F}(B_R)[\tilde\varphi] > 0 \text{ for every } \tilde\varphi \in T^\perp(\partial B_R) \quad \Longleftrightarrow \quad R < \overline{R}(N,\alpha,\gamma),$$

while

$$\partial^2 \mathcal{F}(B_R)[\tilde\varphi] < 0 \text{ for some } \tilde\varphi \in T^\perp(\partial B_R) \quad \Longleftrightarrow \quad R > \overline{R}(N,\alpha,\gamma).$$

By virtue of Theorem 4.8 and Corollary 4.21, we obtain the first part of the theorem, where $m_{\mathrm{loc}}(N,\alpha,\gamma)$ is the volume of the ball of radius $\overline{R}(N,\alpha,\gamma)$.

In order to show that the critical radius tends to ∞ as $\alpha \to 0$, we notice that

$$\partial^2 \mathcal{F}(B_R)[\tilde\varphi] \geq \sum_{d=2}^{\infty} \sum_{i=1}^{\dim(\mathcal{H}_d^N)} (c_d^i)^2 R^{N-3}\big(N+1-2\gamma\alpha\mathcal{I}^{N,\alpha}R^{N+1-\alpha}\big).$$

Since

$$\mathcal{I}^{N,\alpha} \overset{\alpha\to 0^+}{\longrightarrow} \int_{B_1} \frac{\langle x-y, x\rangle}{|x-y|^2}\,\mathrm{d}y < \infty\,,$$

we have that for each $R>0$ there exists $\overline{\alpha}(N,\gamma,R)>0$ such that for each $\alpha < \overline{\alpha}(N,\gamma,R)$

$$\alpha\mathcal{I}^{N,\alpha} < \frac{N+1}{2\gamma R^{N+1-\alpha}},$$

which immediately implies the claim. To conclude the proof we examine in more details the special case $N=3$, determining explicitly the critical mass m_{loc}. From (4.37) we have that

$$\mu_d^{3,\alpha} = 2^{2-\alpha}\pi\Big(\prod_{j=0}^{d-1}\Big(\frac{\alpha}{2}+j\Big)\Big)\frac{\Gamma(1-\frac{\alpha}{2})}{\Gamma(2+d-\frac{\alpha}{2})} = 2^{2-\alpha}\pi\alpha\frac{\Big(\prod_{j=1}^{d-1}\big(\frac{\alpha}{2}+j\big)\Big)}{\prod_{j=1}^{d-1}\big(1-\frac{\alpha}{2}+j\big)}\frac{1}{d+1-\frac{\alpha}{2}}\frac{1}{2-\alpha},$$

where we used the property $\Gamma(x+1) = x\Gamma(x)$. Moreover, by the explicit computation of the integral $\mathcal{I}^{3,\alpha}$, which is done below (see (4.42)), we have

$$\mathcal{I}^{3,\alpha} = 2\pi\frac{2^{2-\alpha}}{(4-\alpha)(2-\alpha)}. \tag{4.41}$$

It is now easily seen that $d_I^{3,\alpha} = d_A^{3,\alpha} = 2$ for every $\alpha \in (0,2)$. Hence

$$\overline{R}(3,\alpha,\gamma) = \left(\frac{(6-\alpha)(4-\alpha)}{2^{3-\alpha}\gamma\alpha\pi}\right)^{\frac{1}{4-\alpha}},$$

which completes the proof of the theorem. □

4.4.4. Computation of $\mathcal{I}^{N,\alpha}$. Here we want to get an explicit expression of the integral

$$\mathcal{I}^{N,\alpha} := \int_{B_1} \frac{\langle x-y, x\rangle}{|x-y|^{\alpha+2}}\,\mathrm{d}y$$

in the case $N=3$. First of all, since $\mathcal{I}^{N,\alpha}$ is independent of $x \in \mathbb{S}^{N-1}$, we fix $x = e_1$. By Fubini's Theorem we get

$$\mathcal{I}^{N,\alpha} = \int_{B_1} \frac{1-y_1}{|e_1-y|^{\alpha+2}}\,\mathrm{d}y = \int_{-1}^{1}\bigg(\int_{B_t} \frac{1-t}{\big((1-t)^2+|z|^2\big)^{\frac{\alpha+2}{2}}}\,\mathrm{d}\mathcal{L}^{N-1}(z)\bigg)\mathrm{d}t\,,$$

where $B_t := B^{N-1}(0,\sqrt{1-t^2})$ denotes a $(N-1)$-dimensional ball of radius $\sqrt{1-t^2}$ centered at the origin. To treat the inner integral, we apply the co-area formula (see [8, equation (2.74)]), by integrating on the level sets of the function $f_t(z) := \sqrt{(1-t)^2+|z|^2}$, $z \in \mathbb{R}^{N-1}$: setting $\delta(r) = \sqrt{r^2-(1-t)^2}$, we get

$$\begin{aligned}\int_{B_t} \frac{\mathrm{d}\mathcal{L}^{N-1}(z)}{\big((1-t)^2+|z|^2\big)^{\frac{\alpha+2}{2}}} &= \int_{1-t}^{\sqrt{2(1-t)}}\bigg(\int_{\partial B^{N-1}(0,\delta(r))} \frac{\mathrm{d}\mathcal{H}^{N-2}}{r^{\alpha+1}\sqrt{r^2-(1-t)^2}}\bigg)\,\mathrm{d}r\\ &= (N-1)\omega_{N-1}\int_{1-t}^{\sqrt{2(1-t)}} \frac{\big(r^2-(1-t)^2\big)^{\frac{N-3}{2}}}{r^{\alpha+1}}\,\mathrm{d}r.\end{aligned}$$

Therefore

$$\mathcal{I}^{N,\alpha} = (N-1)\omega_{N-1}\int_{-1}^{1}(1-t)\bigg(\int_{1-t}^{\sqrt{2(1-t)}} \frac{\big(r^2-(1-t)^2\big)^{\frac{N-3}{2}}}{r^{\alpha+1}}\,\mathrm{d}r\bigg)\,\mathrm{d}t. \tag{4.42}$$

By computing the previous expression for $N=3$, we end up with (4.41).

4.5. Global minimality

This section is devoted to the proof of the results concerning global minimality issues. We start by showing how the information gained in Theorem 4.9 can be used to prove the global minimality of the ball for small volumes.

Proof of Theorem 4.10. By scaling, we can equivalently prove that given $N \geq 2$ and $\alpha \in (0, N-1)$ and setting

$$\bar{\gamma} := \sup\big\{\gamma > 0 : B_1 \text{ is a global minimizer of } \mathcal{F}_{\alpha,\gamma} \text{ in } \mathbb{R}^N \text{ under volume constraint}\big\},$$

we have that $\bar{\gamma} \in (0,\infty)$ and B_1 is the unique global minimizer of $\mathcal{F}_{\alpha,\gamma}$ for every $\gamma < \bar{\gamma}$.

We start assuming by contradiction that there exist a sequence $\gamma_n \to 0$ and a sequence of sets E_n, with $|E_n| = |B_1|$ and $\alpha(E_n, B_1) > 0$, such that

$$\mathcal{F}_{\alpha,\gamma_n}(E_n) \leq \mathcal{F}_{\alpha,\gamma_n}(B_1). \tag{4.43}$$

By translating E_n so that $\alpha(E_n,B_1) = |E_n \triangle B_1|$, from (4.43) one immediately gets

$$C(N)\,|E_n \triangle B_1|^2 \leq \mathcal{P}(E_n) - \mathcal{P}(B_1) \leq \gamma_n\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E_n)\big) \leq \gamma_n c_0 |E_n \triangle B_1|$$

where the first inequality follows from the quantitative isoperimetric inequality and the last one from Proposition 4.3. Hence, as $\gamma_n \to 0$, we deduce that $\alpha(E_n, B_1) \to 0$.

From the results of Section 4.4 it follows that if $\gamma_0 > 0$ is sufficiently small then the functional $\mathcal{F}_{\alpha,\gamma_0}$ has positive second variation at B_1: by Theorem 4.8, this implies the existence of a positive δ such that

$$\mathcal{F}_{\alpha,\gamma_0}(B_1) < \mathcal{F}_{\alpha,\gamma_0}(E) \qquad \text{for every } E \text{ with } |E| = |B_1| \text{ and } 0 < \alpha(E,B_1) < \delta. \tag{4.44}$$

We now want to show that (4.44) holds for every $\gamma < \gamma_0$, with the same δ. Indeed, assuming by contradiction the existence of $\gamma < \gamma_0$ and $E \subset \mathbb{R}^N$ such that $|E| = |B_1|$, $0 < \alpha(E, B_1) < \delta$ and

$$\mathcal{F}_{\alpha,\gamma}(E) \leq \mathcal{F}_{\alpha,\gamma}(B_1), \tag{4.45}$$

since $\mathcal{P}(B_1) < \mathcal{P}(E)$ we necessarily have $\mathcal{NL}_\alpha(E) < \mathcal{NL}_\alpha(B_1)$. Hence by (4.45)

$$\mathcal{P}(E) - \mathcal{P}(B_1) \leq \gamma\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E)\big) < \gamma_0\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E)\big)\,, \tag{4.46}$$

that is, $\mathcal{F}_{\alpha,\gamma_0}(E) < \mathcal{F}_{\alpha,\gamma_0}(B_1)$, which contradicts (4.44).

Now, since for n large enough we have that $\gamma_n < \gamma_0$ and $0 < \alpha(E_n, B_1) < \delta$, the previous property is in contradiction with (4.43). This shows in particular that $\bar\gamma > 0$.

The fact that $\bar\gamma$ is finite follows from Theorem 4.9, which shows that for large masses the ball is not a local minimizer (and obviously not even a global minimizer).

Finally, assume by contradiction that for some $\gamma < \bar\gamma$ the ball is not the unique global minimizer, that is there exists a set E, with $|E| = |B_1|$ and $\alpha(E, B_1) > 0$, such that $\mathcal{F}_{\alpha,\gamma}(E) \leq \mathcal{F}_{\alpha,\gamma}(B_1)$. By definition of $\bar\gamma$, we can find $\gamma' \in (\gamma, \bar\gamma)$ such that B_1 is a global minimizer of $\mathcal{F}_{\alpha,\gamma'}$. Arguing as before, we have that by the isoperimetric inequality $\mathcal{P}(B_1) < \mathcal{P}(E)$, which by our contradiction assumption implies that $\mathcal{NL}_\alpha(E) < \mathcal{NL}_\alpha(B_1)$; this yields

$$\mathcal{P}(E) - \mathcal{P}(B_1) \leq \gamma\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E)\big) < \gamma'\big(\mathcal{NL}_\alpha(B_1) - \mathcal{NL}_\alpha(E)\big)\,,$$

which contradicts the fact that B_1 is a global minimizer for $\mathcal{F}_{\alpha,\gamma'}$. □

We now want to analyze what happens for small exponents α. Since for $\alpha = 0$ the functional is just the perimeter, which is uniquely minimized by the ball, the intuition suggests that the unique minimizer of $\mathcal{F}_{\alpha,\gamma}$, for α close to 0, is the ball itself, as long as a minimizer exists. In order to prove the theorem, we need an auxiliary result: the non-existence volume threshold is uniformly bounded for $\alpha \in (0,1)$. The proof is a simple adaptation of the argument contained in [62, Section 2], where just the three-dimensional case with $\alpha = 1$ is considered.

Proposition 4.34. *There exists $\bar m = \bar m(N,\gamma) < +\infty$ such that for every $m > \bar m$ the minimum problem*

$$I_m^\alpha := \inf\big\{\mathcal{F}_{\alpha,\gamma}(E)\,:\, E \subset \mathbb{R}^N,\ |E| = m\big\} \tag{4.47}$$

does not have a solution for every $\alpha \in (0,1)$.

Proof. During the proof we will denote by C a generic constant, depending only on N and γ, which may change from line to line.

Step 1. We claim that there exists a constant C_0, depending only on N and γ, such that

$$I_m^\alpha \leq C_0\, m \qquad \text{for every } 0 < \alpha < N-1 \text{ and } m \geq 1. \tag{4.48}$$

Indeed, if B is a ball of volume m, then

$$\mathcal{F}_{\alpha,\gamma}(B) = N{\omega_N}^{1/N} m^{(N-1)/N} + \gamma\, c_\alpha \left(\frac{m}{\omega_N}\right)^{\frac{2N-\alpha}{N}}, \qquad c_\alpha := \int_{B_1}\int_{B_1} \frac{1}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y\,.$$

It follows that for every $1 \leq m < 2$ we have $I_m^\alpha \leq C_0$, for some constant C_0 depending only on N and γ. It is now easily seen that $I_m^\alpha \leq I_{m_1}^\alpha + I_{m_2}^\alpha$ if $m = m_1 + m_2$ (see the proof of [62, Lemma 3]): hence by induction on k we obtain $I_m^\alpha \leq C_0 k$ for every $m \in [k, k+1)$.

Step 2. We claim that there exists a constant C_1, depending only on N and γ, such that for every $0 < \alpha < N-1$ and $m \geq 1$, if E is a solution to (4.47) then

$$|E \cap B_R(x)| \geq C_1 R^N \tag{4.49}$$

for every $R \leq 1$ and for every $x \in E$ such that $|E \cap B_r(x)| > 0$ for all $r > 0$.

To prove the claim, assume without loss of generality that $x = 0$. It is clearly sufficient to show (4.49) for $\mathcal{L}^1$-a.e. $R < \varepsilon_0$, where ε_0 will be fixed later in the proof. In particular, from now on we can assume without loss of generality that R is such that $\mathcal{H}^{N-1}(\partial E \cap \partial B_R) = 0$. We compare the energies of E and $E' := \lambda(E \setminus B_R)$, where $\lambda > 1$ is such that $|E'| = m$: by minimality of E we have $\mathcal{F}_{\alpha,\gamma}(E) \leq \mathcal{F}_{\alpha,\gamma}(E')$, which gives after a direct computation

$$\mathcal{H}^{N-1}(\partial E \cap B_R) \leq (\lambda^{2N-\alpha} - 1)\mathcal{F}_{\alpha,\gamma}(E) + \lambda^{N-1}\mathcal{H}^{N-1}(\partial B_R \cap E).$$

In turn this implies, by using $\mathcal{H}^{N-1}(\partial(E\cap B_R)) = \mathcal{H}^{N-1}(\partial E\cap B_R)+\mathcal{H}^{N-1}(\partial B_R\cap E)$ (recall that $\mathcal{H}^{N-1}(\partial E \cap \partial B_R) = 0$),

$$\mathcal{H}^{N-1}(\partial(E \cap B_R)) \leq (\lambda^{2N-\alpha} - 1)\mathcal{F}_{\alpha,\gamma}(E) + (\lambda^{N-1} + 1)\mathcal{H}^{N-1}(\partial B_R \cap E).$$

Now, choosing $\varepsilon_0 > 0$ so small that $|E \backslash B_R| \geq \frac{1}{2}m$, we obtain the following estimates:

$$\lambda^{2N-\alpha} - 1 = \left(\frac{m}{|E\backslash B_R|}\right)^{\frac{2N-\alpha}{N}} - 1 \leq C\left(\frac{m}{|E\backslash B_R|} - 1\right) \leq C\frac{|E\cap B_R|}{m}, \qquad \lambda^{N-1} \leq C\,.$$

Hence from the isoperimetric inequality, (4.48), and from the above estimates we deduce that

$$|E \cap B_R|^{\frac{N-1}{N}} \leq C|E \cap B_R| + C\mathcal{H}^{N-1}(\partial B_R \cap E).$$

Finally, observe that if ε_0 is sufficiently small we also have $|E\cap B_R| \leq \frac{1}{2C}|E\cap B_R|^{\frac{N-1}{N}}$, hence we obtain

$$|E \cap B_R|^{\frac{N-1}{N}} \leq C\mathcal{H}^{N-1}(\partial B_R \cap E) = C\frac{\mathrm{d}}{\mathrm{d}R}|E \cap B_R|,$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}R}|E \cap B_R|^{\frac{1}{N}} \geq C \qquad \text{for } \mathcal{L}^1\text{-a.e. } R < \varepsilon_0.$$

By integrating the previous inequality we conclude the proof of the claim.

Step 3. We claim that there exists a constant C_2, depending only on N and γ, such that for every $0 < \alpha < 1$ and $m \geq 1$, if E is a solution to (4.47) then

$$\mathcal{NL}_\alpha(E) \geq C_2\, m \log m - C_2\, m \tag{4.50}$$

(notice that the conclusion of the proposition follows immediately from (4.48) and (4.50)).

In order to prove the claim, we first observe that

$$|E \cap B_R(x)| \geq CR \qquad \text{for every } x \in E \text{ and } 1 < R < \tfrac{1}{2}\mathrm{diam}(E). \tag{4.51}$$

Indeed, as E is not contained in $B_R(x)$ and E is connected (see Theorem 4.7), we can find points $x_i \in E \cap \partial B_{R-i}(x)$ for $i = 1, \dots, \lfloor R \rfloor$ such that $|E \cap B_r(x_i)| > 0$ for every $r > 0$. Then by (4.49)

$$|E \cap B_R(x)| \geq \sum_{i=1}^{\lfloor R \rfloor} |E \cap B_{\frac{1}{2}}(x_i)| \geq C_1 \left(\frac{1}{2}\right)^N \lfloor R \rfloor\,.$$

Observe now that, if we set $E_R := \{(x, y) \in E \times E : |x - y| < R\}$, we have by (4.51) that for every $1 < R < \frac{1}{2}\mathrm{diam}(E)$

$$\mathcal{L}^{2N}(E_R) = \int_E |E \cap B_R(x)|\, \mathrm{d}x \geq C|E|R. \tag{4.52}$$

Hence

$$
\begin{aligned}
\mathcal{NL}_\alpha(E) &= \int_E\int_E \frac{1}{|x-y|^\alpha}\,\mathrm{d}x\mathrm{d}y = \frac{1}{\sqrt{2}}\int_0^{+\infty}\frac{1}{R^\alpha}\,\mathcal{H}^{2N-1}(\partial E_R)\,\mathrm{d}R \\
&= \frac{1}{2}\int_0^{+\infty}\frac{1}{R^\alpha}\frac{\mathrm{d}}{\mathrm{d}R}\mathcal{L}^{2N}(E_R)\,\mathrm{d}R \geq \frac{1}{2}\int_1^{+\infty}\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}R}\mathcal{L}^{2N}(E_R)\,\mathrm{d}R \\
&= -\frac{1}{2}\,\mathcal{L}^{2N}(E_1) + \frac{1}{2}\int_1^{+\infty}\frac{1}{R^2}\,\mathcal{L}^{2N}(E_R)\,\mathrm{d}R \\
&\geq -Cm + Cm\int_1^{\frac{1}{2}\mathrm{diam}(E)}\frac{1}{R}\,\mathrm{d}R\,,
\end{aligned}
$$

where in the first inequality we used the fact that $\alpha<1$, while the second one follows from (4.52). This completes the proof of the proposition. $\square$

An essential remark for the proof of Theorem 4.11 is contained in the following lemma, where it is shown that the local minimality neighborhood of the ball is in fact *uniform* with respect to γ and α.

LEMMA 4.35. *Given $\bar\gamma>0$, there exist $\bar\alpha>0$ and $\delta>0$ such that*

$$\mathcal{F}_{\alpha,\gamma}(B_1)<\mathcal{F}_{\alpha,\gamma}(E)$$

for every $\alpha\leq\bar\alpha$, for every $\gamma\leq\bar\gamma$ and for every set E with $|E|=|B_1|$ and $0<\alpha(E,B_1)<\delta$.

PROOF (SKETCH). Notice that, by the same argument used in the proof of Theorem 4.10, it is sufficient to show that, given $\bar\gamma>0$, there exist $\bar\alpha>0$ and $\delta>0$ such that

$$\mathcal{F}_{\alpha,\bar\gamma}(B_1)<\mathcal{F}_{\alpha,\bar\gamma}(E)$$

for every $\alpha\leq\bar\alpha$ and for every set E with $|E|=|B_1|$ and $0<\alpha(E,B_1)<\delta$.

In order to prove this property, we start by observing that there exists $\alpha_1>0$ such that

$$m_0 := \inf_{\alpha\leq\alpha_1}\inf\Big\{\partial^2\mathcal{F}_{\alpha,\bar\gamma}(B_1)[\varphi] \,:\, \varphi\in T^\perp(\partial B_1),\ \|\varphi\|_{\widetilde{H}^1(\partial B_1)}=1\Big\}>0\,. \tag{4.53}$$

In fact, assuming by contradiction the existence of $\alpha_n\to 0$ and $\varphi_n\in T^\perp(\partial B_1)$, with $\|\varphi_n\|_{\widetilde{H}^1}=1$, such that $\partial^2\mathcal{F}_{\alpha_n,\bar\gamma}(B_1)[\varphi_n]\to 0$, we have by compactness that, up to subsequences, $\varphi_n\rightharpoonup\varphi_0$ weakly in H^1 for some $\varphi_0\in T^\perp(\partial B_1)$. It is now not hard to show that the last two integrals in the quadratic form $\partial^2\mathcal{F}_{\alpha_n,\bar\gamma}(B_1)[\varphi_n]$ converge to 0 as $n\to\infty$: indeed, the second integral in (4.13) converges to 0, since it is equal to

$$-\alpha_n\int_{\partial B_1}\bigg(\int_{B_1}\frac{x-y}{|x-y|^{\alpha_n+2}}\,\mathrm{d}y\bigg)\cdot x\,\varphi_n^2(x)\,\mathrm{d}\mathcal{H}^{N-1}(x)\leq C\alpha_n\int_{\partial B_1}\varphi_n^2\,\mathrm{d}\mathcal{H}^{N-1}\to 0\quad\text{as } n\to\infty\,.$$

For the last integral in (4.13), denoting by $G_{\alpha_n}(x,y):=|x-y|^{-\alpha_n}$, we write

$$
\begin{aligned}
&\int_{\partial B_1}\int_{\partial B_1}G_{\alpha_n}(x,y)\varphi_n(x)\varphi_n(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y) \\
&\quad= \int_{\partial B_1}\int_{\partial B_1}G_{\alpha_n}(x,y)\varphi_n(x)(\varphi_n(y)-\varphi_0(y))\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y) \\
&\quad+ \int_{\partial B_1}\int_{\partial B_1}G_{\alpha_n}(x,y)(\varphi_n(x)-\varphi_0(x))\varphi_0(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y) \\
&\quad+ \int_{\partial B_1}\int_{\partial B_1}G_{\alpha_n}(x,y)\varphi_0(x)\varphi_0(y)\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\,;
\end{aligned}
$$

the potential estimates provided by Lemma 4.19, where the constant can be chosen independently of α_n by Remark 4.20, guarantee that the first two integrals in the above expression converge to zero, while also the third one vanishes in the limit by the Lebesgue's Dominate Convergence Theorem, recalling that $\int_{\partial B_1}\varphi_0=0$ and $\alpha_n\to 0$. Moreover, for the first integral in the quadratic form $\partial^2\mathcal{F}_{\alpha_n,\bar\gamma}(B_1)[\varphi_n]$, we have that

$$\int_{\partial B_1}|\nabla_{\partial B_1}\varphi_0|^2\leq\liminf_{n\to\infty}\int_{\partial B_1}|\nabla_{\partial B_1}\varphi_n|^2\,,\qquad\int_{\partial B_1}|\mathbf{B}_{\partial B_1}|^2\varphi_n^2\to\int_{\partial B_1}|\mathbf{B}_{\partial B_1}|^2\varphi_0^2\,.$$

Hence, if $\varphi_0=0$ we conclude that $\int_{\partial B_1}|\nabla_{\partial B_1}\varphi_n|^2\to 0$, which contradicts the fact that $\|\varphi_n\|_{\widetilde{H}^1}=1$ for every n. On the other hand, if $\varphi_0\neq 0$, we obtain

$$\int_{\partial B_1}|\nabla_{\partial B_1}\varphi_0|^2\,d\mathcal{H}^{N-1}-\int_{\partial B_1}|\mathbf{B}_{\partial B_1}|^2\varphi_0^2\,d\mathcal{H}^{N-1}\leq 0\,,$$

that is, the second variation of the area functional computed at the ball B_1 is not strictly positive, which is again a contradiction.

With condition (4.53), it is straightforward to check that the proof of Theorem 4.25 provides the existence of $\delta_1>0$ and $C_1>0$ such that

$$\mathcal{F}_{\alpha,\bar\gamma}(E)\geq\mathcal{F}_{\alpha,\bar\gamma}(B_1)+C_1\big(\alpha(E,B_1)\big)^2$$

for every $\alpha\leq\alpha_1$ and for every $E\subset\mathbb{R}^N$ with $|E|=|B_1|$ and $\partial E=\{x+\psi(x)x:x\in\partial B_1\}$ for some ψ with $\|\psi\|_{W^{2,p}(\partial B_1)}<\delta_1$.

In turn, having proved this property one can repeat the proofs of Theorem 4.27 and Theorem 4.8 to deduce that there exist $\alpha_2>0$, $\delta_2>0$ and $C_2>0$ such that

$$\mathcal{F}_{\alpha,\bar\gamma}(E)\geq\mathcal{F}_{\alpha,\bar\gamma}(B_1)+C_2\big(\alpha(E,B_1)\big)^2$$

for every $\alpha\leq\alpha_2$ and for every $E\subset\mathbb{R}^N$ with $|E|=|B_1|$ and $\alpha(E,B_1)<\delta_2$. The only small modifications consist in assuming, in the contradiction arguments, also the existence of sequences $\alpha_n\to 0$, instead of working with a fixed α. Then the essential remark is that the constant c_0 provided by Proposition 4.3 is independent of α_n. In addition, some small changes are required in the last part of the proof, since the functions v_{F_n} associated, according to (4.3), with the sets F_n constructed in the proof are defined with respect to different exponents α_n, but observe that the bounds provided by Proposition 4.1 are still uniform. The easy details are left to the reader.

These observations complete the proof of the lemma. □

We are now in position to complete the proof of Theorem 4.11.

Proof of Theorem 4.11. We assume by contradiction that there exist $\alpha_n\to 0$, $m_n>0$ and sets $E_n\subset\mathbb{R}^N$, with $|E_n|=m_n$, $\alpha(E_n,B_n)>0$ (where we denote by B_n a ball with volume m_n), such that E_n is a global minimizer of $\mathcal{F}_{\alpha_n,\gamma}$ under volume constraint. Note that, as the non-existence threshold is uniformly bounded for $\alpha\in(0,1)$ (Proposition 4.34), we can assume without loss of generality that $m_n\leq\bar m<+\infty$.

By scaling, we can rephrase our contradiction assumption as follows: there exist $\alpha_n\to 0$, $\gamma_n>0$ with $\bar\gamma:=\sup_n\gamma_n<+\infty$, and $F_n\subset\mathbb{R}^N$ with $|F_n|=|B_1|$, $\alpha(F_n,B_1)>0$ such that

$$\mathcal{F}_{\alpha_n,\gamma_n}(F_n)=\min\{\mathcal{F}_{\alpha_n,\gamma_n}(F)\,:\,|F|=|B_1|\}\,,$$

and in particular

$$\mathcal{F}_{\alpha_n,\gamma_n}(F_n)\leq\mathcal{F}_{\alpha_n,\gamma_n}(B_1).\tag{4.54}$$

We now claim that, since $\alpha_n \to 0$,

$$\lim_{n\to+\infty} |\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(F_n)| = 0. \tag{4.55}$$

Indeed, we observe that by adapting the first step of the proof of Theorem 4.7, we have that there exists $\Lambda > 0$ (independent of n) such that F_n is also a solution to the penalized minimum problem

$$\min\left\{\mathcal{F}_{\alpha_n,\gamma_n}(F) + \Lambda\big||F| - |B_1|\big| \,:\, F \subset \mathbb{R}^N\right\}$$

(for n large enough). In turn, this implies that each set F_n is an (ω, r_0)-minimizer for the area functional for some positive ω and r_0 (independent of n): in fact for every finite perimeter set F with $F \triangle F_n \subset\subset B_{r_0}(x)$ we have by minimality of F_n

$$\begin{aligned}\mathcal{P}(F_n) &\le \mathcal{P}(F) + \gamma_n\big(\mathcal{NL}_{\alpha_n}(F) - \mathcal{NL}_{\alpha_n}(F_n)\big) + \Lambda\big||F| - |B_1|\big| \\ &\le \mathcal{P}(F) + \big(\bar\gamma c_0 + \Lambda\big)|F \triangle F_n|,\end{aligned}$$

where we used Proposition 4.3 and the fact that the constant c_0 can be chosen independently of α_n. We can now use the uniform density estimates for (ω, r_0)-minimizers (see [64, Theorem 21.11]), combined with the connectedness of the sets F_n (see Theorem 4.7), to deduce that (up to translations) they are equibounded: there exists $\bar R > 0$ such that $F_n \subset B_{\bar R}$ for every n. Using this information, it is now easily seen that, since $\alpha_n \to 0$,

$$\mathcal{NL}_{\alpha_n}(F_n) = \int_{F_n}\int_{F_n} \frac{1}{|x-y|^{\alpha_n}}\, dx dy \to |B_1|^2\,,$$

from which (4.55) follows.

By (4.54), (4.55) and using the quantitative isoperimetric inequality we finally deduce

$$\begin{aligned}C_N\big(\alpha(F_n, B_1)\big)^2 \le \mathcal{P}(F_n) - \mathcal{P}(B_1) &\le \gamma_n\big(\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(F_n)\big) \\ &\le \bar\gamma\,\big|\mathcal{NL}_{\alpha_n}(B_1) - \mathcal{NL}_{\alpha_n}(F_n)\big| \to 0,\end{aligned}$$

that is, F_n converges to B_1 in L^1. Hence (4.54) is in contradiction with Lemma 4.35 for n large enough. ☐

We conclude this section with the proof of Theorem 4.12.

Proof of Theorem 4.12. First of all we notice that, since for masses smaller than m_{glob} the ball is the unique global minimizer, for each $m > 0$ there exists $k_m \in \mathbb{N}$ such that $f_{k_m}(m) = \min_i f_i(m)$. Setting $m_0 = 0$, $m_1 = m_{\mathrm{glob}}$, we have by Theorem 4.11 that (4.11) holds for $k = 1$. In the following, we denote by E_R^m a solution to the constrained minimum problem

$$\min\{\mathcal{F}(E) : E \subset B_R,\ |E| = m\}.$$

We remark that

$$\mathcal{F}(E_R^m) \to \inf\left\{\mathcal{F}(E) : E \subset \mathbb{R}^N,\ |E| = m\right\} \quad \text{as } R \to \infty\,, \tag{4.56}$$

and that, given $\bar m > 0$, for every $m < \bar m$ and for every $R > 0$ the volume of each connected component of E_R^m is bounded from below by a positive constant $M_{\bar m} > 0$ depending on $\bar m$ (this conclusion can be obtained by arguing as in the proof of Theorem 4.7, showing in particular that each set E_R^m is an (ω, r_0)-minimizer for some constant ω independent of $m \le \bar m$).

We now define

$$m_2 := \sup\big\{m \ge m_1 : f_2(m') = \inf_{|E|=m'}\mathcal{F}(E) \text{ for each } m' \in [m_1, m)\big\}$$

and we show that $m_2 > m_1$. Indeed, fix $\varepsilon > 0$ and $m \in (m_1, m_1 + \varepsilon)$. Observe that the sets $(E_R^m)_R$ cannot be equibounded, or otherwise they would converge (as $R \to \infty$) to a global minimizer of $\mathcal{F}$ with volume m, whose existence is excluded by Theorem 4.11. The fact that the diameter of E_R^m tends to infinity, combined with the uniform density lower bound satisfied by E_R^m (which, in turn, follows from the quasiminimality property), guarantees that for all R large enough the set E_R^m is not connected; moreover, if ε is small enough, each of its connected component has mass smaller than m_{glob}, as a consequence of the lower bound on the volume of the connected components. Then we can write $E_R^m = F_1 \cup F_2$, with $|F_1|, |F_2| < m_{\mathrm{glob}}$ and $F_1 \cap F_2 = \emptyset$, so that we can decrease the energy of E_R^m by replacing each F_i by a ball of the same volume, sufficiently far apart from each other, obtaining that $f_2(m) \leq \mathcal{F}(E_R^m)$. By (4.56) we easily conclude that $f_2(m) = \inf_{|E|=m} \mathcal{F}(E)$ for every $m \in (m_1, m_1+\varepsilon)$, from which follows that $m_2 > m_1$. Moreover, by definition of m_2, we have that (4.11) holds for $k = 2$.

We now proceed by induction, defining

$$m_{k+1} := \sup\{m \geq m_k \,:\, f_{k+1}(m') = \inf_{|E|=m'} \mathcal{F}(E) \text{ for each } m' \in [m_k, m)\}$$

and showing that $m_k < m_{k+1}$. Arguing as before, we consider $m \in (m_k, m_k + \varepsilon)$, for some $\varepsilon > 0$ small enough, and we observe that for R sufficiently large the set E_R^m is not connected, and each of its connected components has volume belonging to an interval $(m_{i-1}, m_i]$ for some $i \leq k$. By the inductive hypothesis we can obtain a new set F_R^m, union of a finite number of disjoint balls, such that $\mathcal{F}(F_R^m) \leq \mathcal{F}(E_R^m)$, simply by replacing each connected component of E_R^m by a disjoint union of balls. We can also assume that at least one of these balls, say B, has volume larger than ε (if we choose for instance $\varepsilon < \frac{m_1}{2}$); in this way $|F_R^m \setminus B| < m_k$ and we can decrease the energy of F_R^m by replacing $F_R^m \setminus B$ by a finite union of at most k balls. With this procedure we find a disjoint union of at most $k+1$ balls whose energy is smaller than $\mathcal{F}(F_R^m)$, so that, recalling (4.56) and that $\mathcal{F}(F_R^m) \leq \mathcal{F}(E_R^m)$, we conclude that $f_{k+1}(m) = \inf_{|E|=m} \mathcal{F}(E)$ for every $m \in (m_k, m_k + \varepsilon)$. This completes the proof of the inequality $m_k < m_{k+1}$, and shows also, by definition of m_k, that (4.11) holds.

Now, assume by contradiction that $m_k \to \bar{m} < \infty$ as $k \to \infty$. Since each interval (m_k, m_{k+1}) is not degenerate, the definition of m_k as a supremum ensures that we can find an increasing sequence of masses $\bar{m}_k \to \bar{m}$ such that an optimal configuration for $\min_i f_i(\bar{m}_k)$ is given by exactly $k+1$ balls. Recalling that the constant $M_{\bar{m}}$ provides a lower bound on the volume of each ball of an optimal configuration, the previous assertion is impossible for k large and shows that $\lim_{k\to\infty} m_k = \infty$. Finally, it is clear that the number of non-degenerate balls tends to ∞ as $m \to \infty$, since the volume of each ball in an optimal configuration for $\min_i f_i(m)$ must be not larger than m_1. □

4.6. Computation of the first and second variations of the functional

We conclude this chapter by proving Theorem 4.17, which consists in the computation of the first and second variations of the functional $\mathcal{F}$.

Proof of Theorem 4.17. The first and the second variations of the perimeter of a regular set E are standard calculations (see, *e.g.*, [75]) and lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{P}(E_t)_{|t=0} = \int_{\partial E} H_{\partial E}\langle X, \nu_E\rangle \, \mathcal{H}^{N-1} \tag{4.57}$$

and

$$\begin{aligned}\frac{\mathrm{d}^2}{\mathrm{d}t^2}\mathcal{P}(E_t)_{|t=0} &= \int_{\partial E} \big(|\nabla_{\partial E}\langle X, \nu_E\rangle|^2 - |\mathbf{B}_{\partial E}|^2 \langle X, \nu_E\rangle^2 \big) \, \mathrm{d}\mathcal{H}^{N-1} \\ &\quad + \int_{\partial E} H_{\partial E} \big(\langle X, \nu_E\rangle \operatorname{div} X - \operatorname{div}_{\partial E}\big(X_\tau \langle X, \nu_E\rangle\big)\big) \, \mathrm{d}\mathcal{H}^{N-1} . \end{aligned} \tag{4.58}$$

This particular form of the second variation is in fact obtained in [19, Proposition 3.9], and we rewrote the last term according to [1, equation (7.5)].

We now focus on the calculation of the first and the second variations of the nonlocal part. In order to compute these quantities we introduce the smoothed potential

$$G_\delta(a,b) := \frac{1}{(|a-b|^2+\delta^2)^{\frac{\alpha}{2}}}$$

for $\delta > 0$, and the associated nonlocal energy

$$\mathcal{NL}_\delta(F) := \int_F \int_F G_\delta(a,b)\,\mathrm{d}a\mathrm{d}b.$$

We remark that the following identities hold:

$$\nabla_x\big(G_\delta(\Phi_t(x),\Phi_t(y))\big) = \nabla_a G_\delta(\Phi_t(x),\Phi_t(y))\cdot D\Phi_t(x), \tag{4.59}$$

$$\nabla_b G_\delta(a,b) = \nabla_a G_\delta(b,a). \tag{4.60}$$

Step 1: first variation of the nonlocal term. The idea to compute the first variation of the nonlocal part is to prove the following two steps:

(1) $\mathcal{NL}_\delta(E_t) \overset{\delta\to 0}{\longrightarrow} \mathcal{NL}(E_t)$ uniformly for $t \in (-t_0, t_0)$,
(2) $\frac{\partial}{\partial t}\mathcal{NL}_\delta(E_t)$ converges uniformly for $t\in(-t_0,t_0)$ to some function $H(t)$ as $\delta \to 0$,

where $t_0 < 1$ is a fixed number. From (1) and (2) it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{NL}(E_t)_{|t=0} = H(0) = \lim_{\delta\to 0}\frac{\partial}{\partial t}\mathcal{NL}_\delta(E_t)_{|t=0}. \tag{4.61}$$

We prove (1). We have that

$$\begin{aligned}|\mathcal{NL}_\delta(E_t) - \mathcal{NL}(E_t)| &= \left|\int_{E_t}\int_{E_t}\big(G_\delta(x,y) - G(x,y)\big)\,\mathrm{d}x\mathrm{d}y\right| \\ &\le \int_{B_R}\int_{B_R} |G_\delta(x,y)-G(x,y)|\,\mathrm{d}x\mathrm{d}y\,,\end{aligned}$$

where we have used the fact that E is bounded and hence $E_t \subset B_R$ for some ball B_R. It is now easily seen that the last integral in the previous expression tends to 0 as $\delta\to 0$, thanks to the Lebesgue's Dominated Convergence Theorem, hence

$$\sup_{t\in(-t_0,t_0)} |\mathcal{NL}_\delta(E_t) - \mathcal{NL}(E_t)| \to 0 \qquad \text{as } \delta \to 0.$$

We now prove (2). By a change of variables and using (4.59) and (4.60) we have

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{NL}_\delta(E_t) &= 2\int_E\int_E \frac{\partial J\Phi_t}{\partial t}(x) J\Phi_t(y) G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y \\ &\quad + 2\int_E\int_E J\Phi_t(x)J\Phi_t(y)\langle\nabla_x\big(G_\delta(\Phi_t(x),\Phi_t(y))\big)\cdot(D\Phi_t(x))^{-1}, X(\Phi_t(x))\rangle\,\mathrm{d}x\mathrm{d}y \\ &= \int_E\int_E f(t,x,y) G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y \\ &\quad + \int_{\partial E}\left(\int_E g(t,x,y)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}y\right)\,\mathrm{d}\mathcal{H}^{N-1}(x)\,,\end{aligned}$$

where $J\Phi_t := \det(D\Phi_t)$ is the jacobian of the map Φ_t,

$$f(t,x,y) := 2\frac{\partial J\Phi_t}{\partial t}(x)J\Phi_t(y) - 2\mathrm{div}_x\big(J\Phi_t(x)J\Phi_t(y)X(\Phi_t(x))\cdot(D\Phi_t(x))^{-T}\big)\,,$$

$$g(t,x,y) := J\Phi_t(x)J\Phi_t(y)\langle X(\Phi_t(x))\cdot(D\Phi_t(x))^{-T},\nu(x)\rangle$$

and in the last step we used integration by parts and Fubini's Theorem. Now since f and g are uniformly bounded on $(-t_0, t_0) \times E \times E$ and $(-t_0, t_0) \times \partial E \times E$ respectively, it is easily seen that

$$\frac{\partial}{\partial t} \mathcal{NL}_\delta(E_t) \overset{\delta \to 0}{\longrightarrow} H(t) \ \text{ uniformly for } t \in (-t_0, t_0)\,,$$

where

$$\begin{aligned} H(t) := & \int_E \int_E f(t,x,y) G(\Phi_t(x), \Phi_t(y))\, \mathrm{d}x\mathrm{d}y \\ & + \int_{\partial E} \left(\int_E g(t,x,y) G(\Phi_t(x), \Phi_t(y))\, \mathrm{d}y \right) \mathrm{d}\mathcal{H}^{N-1}(x). \end{aligned}$$

We finally compute (4.61). Recalling that

$$\frac{\partial J\Phi_t}{\partial t}_{|t=0} = \mathrm{div} X\,, \tag{4.62}$$

we have

$$\begin{aligned} \frac{\partial}{\partial t} \mathcal{NL}_\delta(E_t)_{|t=0} &= 2 \int_E \int_E \left(\frac{\mathrm{div} X(x)}{(|x-y|^2 + \delta^2)^{\frac{\alpha}{2}}} - \alpha \frac{\langle X(x), x-y \rangle}{(|x-y|^2+\delta^2)^{\frac{\alpha+2}{2}}} \right) \mathrm{d}x\mathrm{d}y \\ &= 2 \int_E \int_E \mathrm{div}_x \left(\frac{X(x)}{(|x-y|^2+\delta^2)^{\frac{\alpha}{2}}} \right) \mathrm{d}x\mathrm{d}y \\ &= 2 \int_{\partial E} \left(\int_E \frac{\langle X(x), \nu(x) \rangle}{(|x-y|^2+\delta^2)^{\frac{\alpha}{2}}}\, \mathrm{d}y \right) \mathrm{d}\mathcal{H}^{N-1}(x) \end{aligned}$$

(where we used the divergence Theorem and Fubini's Theorem in the last equality), and hence by letting $\delta \to 0$ we conclude that

$$H(0) = 2 \int_{\partial E} \left(\int_E \frac{\langle X(x), \nu(x) \rangle}{|x-y|^\alpha}\, \mathrm{d}y \right) \mathrm{d}\mathcal{H}^{N-1}(x) = 2 \int_{\partial E} v_E \langle X, \nu \rangle\, \mathrm{d}\mathcal{H}^{N-1}\,.$$

This, combined with (4.57), concludes the proof of the formula for the first variation of $\mathcal{F}$.

Step 2: second variation of the nonlocal term. We will compute the second variation of the nonlocal term by showing that

$$\frac{\partial^2}{\partial t^2} \mathcal{NL}_\delta(E_t) \overset{\delta \to 0}{\longrightarrow} K(t) \ \text{ uniformly in } t \in (-t_0, t_0)$$

for some function K, hence getting

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \mathcal{NL}(E_t)_{|t=0} = K(0) = \lim_{\delta \to 0} \frac{\partial^2}{\partial t^2} \mathcal{NL}_\delta(E_t)_{|t=0}\,. \tag{4.63}$$

First of all we have that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t) = \frac{\partial}{\partial t}\Big[2\int_E\int_E \frac{\partial J\Phi_t}{\partial t}(x)J\Phi_t(y)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y \\ + 2\int_E\int_E J\Phi_t(x)J\Phi_t(y)\langle\nabla_a G_\delta(\Phi_t(x),\Phi_t(y)),X(\Phi_t(x))\rangle\,\mathrm{d}x\mathrm{d}y\Big] \\ = 2\int_E\int_E \frac{\partial}{\partial t}\Big(\frac{\partial J\Phi_t}{\partial t}(x)J\Phi_t(y)\Big)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y \\ + 2\int_E\int_E J\Phi_t(x)\frac{\partial}{\partial t}J\Phi_t(y)\Big(\langle\nabla_a G_\delta(\Phi_t(x),\Phi_t(y)),X(\Phi_t(x))\rangle \\ + \langle\nabla_b G_\delta(\Phi_t(x),\Phi_t(y)),X(\Phi_t(y))\rangle\Big)\,\mathrm{d}x\mathrm{d}y \\ + 2\int_E\int_E\langle\frac{\partial}{\partial t}\Big(J\Phi_t(x)J\Phi_t(y)X(\Phi_t(x))\Big),\nabla_a G_\delta(\Phi_t(x),\Phi_t(y))\rangle\,\mathrm{d}x\mathrm{d}y \\ + 2\int_E\int_E J\Phi_t(x)J\Phi_t(y)\Big(\sum_{i,j=1}^N\frac{\partial^2 G_\delta}{\partial a_i\partial a_j}(\Phi_t(x),\Phi_t(y))X_i(\Phi_t(x))X_j(\Phi_t(x)) \\ + \sum_{i,j=1}^N\frac{\partial^2 G_\delta}{\partial a_i\partial b_j}(\Phi_t(x),\Phi_t(y))X_i(\Phi_t(x))X_j(\Phi_t(y))\Big)\,\mathrm{d}x\mathrm{d}y\,.\end{aligned} \tag{4.64}$$

Using identity (4.59) and integrating by parts, we can rewrite this expression as

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t) = \int_E\int_E f(t,x,y)G_\delta(\Phi_t(x),\Phi_t(y))\,\mathrm{d}x\mathrm{d}y \\ + \int_E\int_E\Big(\langle\nabla_a G_\delta(\Phi_t(x),\Phi_t(y)),g_1(t,x,y)\rangle + \langle\nabla_b G_\delta(\Phi_t(x),\Phi_t(y)),g_2(t,x,y)\rangle\Big)\,\mathrm{d}x\mathrm{d}y \\ + \int_E\int_{\partial E}\Big(\langle\nabla_a G_\delta(\Phi_t(x),\Phi_t(y)),h_1(t,x,y)\rangle + \langle\nabla_b G_\delta(\Phi_t(x),\Phi_t(y)),h_2(t,x,y)\rangle\Big)\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}y\end{aligned}$$

for some functions f, g_1, g_2, h_1, h_2 uniformly bounded in $(-t_0,t_0)\times\overline{E}\times\overline{E}$. It is then easily seen that

$$\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t) \xrightarrow{\delta\to 0} K(t) \text{ uniformly in } t\in(-t_0,t_0)\,,$$

where $K(t)$ is simply obtained by replacing G_δ by G in the previous expression.

We finally compute (4.63). Setting $Z := \frac{\partial^2\Phi}{\partial t^2}_{|t=0}$ we have that

$$\frac{\partial^2 J\Phi_t}{\partial t^2}_{|t=0} = \mathrm{div}Z + (\mathrm{div}X)^2 - \sum_{i,j=1}^N\frac{\partial X_i}{\partial x_j}\frac{\partial X_j}{\partial x_i} = \mathrm{div}\big((\mathrm{div}X)X\big)\,.$$

Therefore, computing (4.64) at $t=0$, from this identity and recalling (4.62) we obtain

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t)_{|t=0} = 2\int_E\int_E\Big[\mathrm{div}\big((\mathrm{div}X)X\big)(x)G_\delta(x,y) + \mathrm{div}X(x)\mathrm{div}X(y)G_\delta(x,y)\Big]\,\mathrm{d}x\mathrm{d}y \\ + 4\int_E\int_E \mathrm{div}X(y)\sum_{i=1}^N\Big(\frac{\partial G_\delta}{\partial x_i}(x,y)X_i(x) + \frac{\partial G_\delta}{\partial y_i}(x,y)X_i(y)\Big)\,\mathrm{d}x\mathrm{d}y \\ + 2\int_E\int_E\sum_{i,j=1}^N\Big(\frac{\partial G_\delta}{\partial x_i}(x,y)\frac{\partial X_i}{\partial x_j}(x)X_j(x) + \frac{\partial^2 G_\delta}{\partial x_i\partial x_j}(x,y)X_i(x)X_j(x)\end{aligned}$$

$$+\frac{\partial^2 G_\delta}{\partial x_i \partial y_j}(x,y)X_i(x)X_j(y)\Big)\,\mathrm{d}x\mathrm{d}y =: I_1 + I_2 + I_3\,.$$

By integrating by parts in I_1, the sum of the first two integrals is equal to

$$\begin{aligned}I_1 + I_2 &= 2\int_E\int_E \langle \nabla_x G_\delta(x,y), X(x)\rangle\big(\mathrm{div}X(x)+\mathrm{div}X(y)\big)\,\mathrm{d}x\mathrm{d}y\\ &\quad + 2\int_E\int_{\partial E} G_\delta(x,y)\big(\mathrm{div}X(x)+\mathrm{div}X(y)\big)\langle X(x),\nu(x)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}y\,.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t)_{|_{t=0}} &= 2\int_E\int_{\partial E} G_\delta(x,y)\big(\mathrm{div}X(x)+\mathrm{div}X(y)\big)\langle X(x),\nu(x)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}y\\ &\quad + 2\int_E\int_E\Big(\mathrm{div}_x\big(\langle\nabla_x G_\delta(x,y),X(x)\rangle X(x)\big)+\mathrm{div}_y\big(\langle\nabla_x G_\delta(x,y),X(x)\rangle X(y)\big)\Big)\,\mathrm{d}x\mathrm{d}y\\ &= 2\int_E\Big(\int_{\partial E}\mathrm{div}_x\big(G_\delta(x,y)X(x)\big)\langle X(x),\nu(x)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x)\Big)\,\mathrm{d}y\\ &\quad + 2\int_E\Big(\int_{\partial E}\mathrm{div}_x\big(G_\delta(x,y)X(x)\big)\langle X(y),\nu(y)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(y)\Big)\,\mathrm{d}x\\ &= 2\int_{\partial E}\Big(\int_E \mathrm{div}_x\big(G_\delta(x,y)X(x)\big)\,\mathrm{d}y\Big)\,\langle X(x),\nu(x)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x)\\ &\quad + 2\int_{\partial E}\int_{\partial E} G_\delta(x,y)\langle X(x),\nu(x)\rangle\langle X(y),\nu(y)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\,,\end{aligned}$$

where the second equality follows after having applied the divergence theorem, and the last one by Fubini's Theorem and the divergence theorem. Thus, using the Lebesgue's Dominated Convergence Theorem to compute the limit of the previous quantity as $\delta \to 0$, and recalling that $\alpha \in (0, N-1)$, we obtain

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathcal{NL}_\delta(E_t)_{|_{t=0}} &= 2\int_{\partial E}\Big(\int_E \mathrm{div}_x\big(G(x,y)X(x)\big)\,\mathrm{d}y\Big)\,\langle X(x),\nu(x)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x)\\ &\quad + 2\int_{\partial E}\int_{\partial E} G(x,y)\langle X(x),\nu(x)\rangle\langle X(y),\nu(y)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x)\mathrm{d}\mathcal{H}^{N-1}(y)\,.\end{aligned}\tag{4.65}$$

We can rewrite the first integral in the previous expression as

$$\begin{aligned}2\int_{\partial E}\Big(\int_E \mathrm{div}_x\big(G(x,y)X(x)\big)\,\mathrm{d}y\Big)\langle X(x),\nu(x)\rangle\,\mathrm{d}\mathcal{H}^{N-1}(x) &= 2\int_{\partial E}\mathrm{div}\big(v_E X\big)\langle X,\nu\rangle\,\mathrm{d}\mathcal{H}^{N-1}\\ &= 2\int_{\partial E}\Big(v_E(\mathrm{div}X)\langle X,\nu\rangle + \langle\nabla v_E, X_\tau\rangle\langle X,\nu\rangle + \partial_\nu v_E\,\langle X,\nu\rangle^2\Big)\,\mathrm{d}\mathcal{H}^{N-1}\\ &= 2\int_{\partial E}\Big(v_E(\mathrm{div}X)\langle X,\nu\rangle - v_E\,\mathrm{div}_{\partial E}\big(X_\tau\langle X,\nu\rangle\big) + \partial_\nu v_E\,\langle X,\nu\rangle^2\Big)\,\mathrm{d}\mathcal{H}^{N-1}\,.\end{aligned}$$

Finally, combining this expression with (4.65) and (4.58), we obtain the formula in the statement. $\square$

APPENDIX A

Proof of the density lower bound

This section is entirely devoted to the proof of the density lower bound for quasi-minimizers of the Mumford-Shah functional (Theorem 1.13), in the case where we fix a Dirichlet condition on a part $\partial_D\Omega$ of the boundary of the domain and a Neumann condition on the remaining part $\partial_N\Omega$, under the assumption that $\partial_D\Omega$ and $\partial_N\Omega$ meet orthogonally. We recall that the relaxed version of the Mumford-Shah functional is defined on functions $u \in SBV(\Omega)$ by

$$\overline{\mathcal{MS}}(u) := \int_\Omega |\nabla u|^2 \,\mathrm{d}x + \mathcal{H}^1(S_u).$$

We start by observing that, if w satisfies the hypotheses of Theorem 1.13, the following energy upper bound holds in every ball $B_\rho(x)$ with $\rho \le R_0$ (it can be easily deduced by comparing the energies of w and of $w\chi_{\Omega'\setminus(B_\rho(x)\cap\Omega)}$):

$$\overline{\mathcal{MS}}(w; B_\rho(x) \cap \Omega') \le c_0\rho, \tag{A.1}$$

where c_0 depends only on R_0, ω, u and Ω. In the following, C will always denote a positive constant depending only on the previous quantities. We now show that we can replace the Dirichlet condition in $\Omega' \setminus \Omega$ by a homogeneous boundary condition.

Lemma A.1. *Set $\tilde{w} := w - u$. Then $\tilde{w} \in SBV(\Omega')$, $\tilde{w} = 0$ in $\Omega' \setminus \Omega$, and there exist $\eta > 0$, $\tilde{\omega} > 0$ (depending only on Ω, ω and u) such that for every $x \in \overline{\Omega} \cap \mathcal{N}_\eta(\partial_D\Omega)$ and for every $\rho < \eta$*

$$\overline{\mathcal{MS}}(\tilde{w}; B_\rho(x) \cap \Omega') \le \overline{\mathcal{MS}}(v; B_\rho(x) \cap \Omega') + \tilde{\omega}\rho^{\frac{3}{2}}$$

whenever $v \in SBV(\Omega')$ is such that $v = 0$ in $\Omega' \setminus \Omega$ and $\{v \ne \tilde{w}\} \subset\subset B_\rho(x)$.

Proof. By choosing η sufficiently small, we can guarantee that $\overline{S}_u \cap B_\rho(x) = \emptyset$ for each ball $B_\rho(x)$ as in the statement, hence $S_{\tilde{w}} \cap B_\rho(x) = S_w \cap B_\rho(x)$. By comparing the energies of w and $v + u$ we obtain

$$\overline{\mathcal{MS}}(\tilde{w}; B_\rho(x) \cap \Omega') \le \overline{\mathcal{MS}}(v; B_\rho(x) \cap \Omega') + 2\int_{B_\rho(x)\cap\Omega'} \nabla u \cdot (\nabla v - \nabla w)$$
$$+ 2\int_{B_\rho(x)\cap\Omega'} |\nabla u|^2 + \omega\rho^2.$$

Now, using the fact that $\nabla u \in L^\infty$ and the upper bound (A.1), we have

$$2\int_{B_\rho(x)\cap\Omega'} |\nabla u|^2 \le C\rho^2, \qquad -2\int_{B_\rho(x)\cap\Omega'} \nabla w \cdot \nabla u \le C\rho^{\frac{3}{2}},$$

while for every $\varepsilon > 0$ we have

$$2\int_{B_\rho(x)\cap\Omega'} \nabla v \cdot \nabla u \le \varepsilon^2 \|\nabla v\|^2_{L^2} + \frac{1}{\varepsilon^2}\|\nabla u\|^2_{L^2} \le \varepsilon^2 \overline{\mathcal{MS}}(v; B_\rho(x) \cap \Omega') + \frac{C}{\varepsilon^2}\rho^2.$$

It follows that

$$\overline{\mathcal{MS}}(\tilde{w}; B_\rho(x) \cap \Omega') \le (1 + \varepsilon^2)\overline{\mathcal{MS}}(v; B_\rho(x) \cap \Omega') + C\Big(1 + \frac{1}{\varepsilon^2}\Big)\rho^2 + C\rho^{\frac{3}{2}}.$$

Defining the deviation from minimality of $\tilde{w}$ in a Borel set B as

$$\begin{aligned}\mathrm{Dev}(\tilde{w};B) &:= \overline{\mathcal{MS}}(\tilde{w};B\cap\Omega') \\ &\quad - \inf\bigl\{\overline{\mathcal{MS}}(v;B\cap\Omega') : v\in SBV(\Omega'),\ v=0 \text{ in } \Omega'\setminus\Omega,\ \{v\neq\tilde{w}\}\subset\subset B\bigr\}, \quad \text{(A.2)}\end{aligned}$$

from the previous inequality we obtain, by taking the infimum over all v,

$$\begin{aligned}\mathrm{Dev}(\tilde{w};B_\rho(x)) &\le \varepsilon^2\overline{\mathcal{MS}}(\tilde{w};B_\rho(x)\cap\Omega') + C\Bigl(1+\frac{1}{\varepsilon^2}\Bigr)\rho^2 + C\rho^{\frac32} \\ &\le c_0\varepsilon^2\rho + C\Bigl(1+\frac{1}{\varepsilon^2}\Bigr)\rho^2 + C\rho^{\frac32} \le \tilde{\omega}\rho^{\frac32},\end{aligned}$$

where we used (A.1) in the second inequality and we choose $\varepsilon=\rho^{\frac14}$ in the last inequality. $\square$

In the proof of the main decay property in Lemma A.5 we will perform a blow-up in a sequence of balls whose centers converge to a point in $\overline{\partial_D\Omega}\cap\overline{\partial_N\Omega}$. This situation is examined in the following lemma.

LEMMA A.2. *Let $x_n\in\overline{\Omega}$, $x_n\to x_0\in\overline{\partial_D\Omega}\cap\overline{\partial_N\Omega}$, and $r_n\to 0^+$. Setting*

$$\Omega_n := \frac{\Omega'-x_n}{r_n}\cap B_1, \qquad D_n := \frac{(\Omega'\setminus\Omega)-x_n}{r_n}\cap B_1, \tag{A.3}$$

there exist $\delta_1,\delta_2\in[0,1]$ and a coordinate system such that (up to subsequences)

$$\Omega_n\to\Omega_0 := \{(\xi,\zeta)\in B_1 : \xi<\delta_1\}, \quad D_n\to D_0 := \{(\xi,\zeta)\in B_1 : \xi<\delta_1,\ \zeta>\delta_2\}$$

in L^1. Moreover, the constant of the relative isoperimetric inequality in Ω_n is the same for all the sets Ω_n (and we denote it by γ). Finally, assuming $\delta_2<1$, given $v\in W^{1,2}(\Omega_0)$ with $v=0$ in D_0 there exists a sequence $v_n\in W^{1,2}(B_1)$ such that $v_n\to v$ in $W^{1,2}(\Omega_0)$ and $v_n=0$ in D_n.

PROOF. In a suitable coordinate system and for r sufficiently small we have

$$\Omega'\cap B_r(x_0) = \{(\xi,\zeta)\in B_r(x_0) : \xi<f(\zeta)\}$$

for some function f of class C^1, with $f(\zeta_0)=\xi_0$, $f'(\zeta_0)=0$, $x_0=(\xi_0,\zeta_0)$. We then have, for n sufficiently large,

$$\Omega_n = \{(\xi,\zeta)\in B_1 : \xi<f_n(\zeta)\}, \qquad f_n(\zeta) := \frac{f(\zeta_n+r_n\zeta)-\xi_n}{r_n},$$

where $x_n=(\xi_n,\zeta_n)$. If $\partial\Omega'\cap B_{r_n}(x_n)=\emptyset$ for infinitely many n, then $\Omega_0=B_1$; otherwise, taken any point $z_n=(\xi_n',\zeta_n')\in\partial\Omega'\cap B_{r_n}(x_n)$, we have

$$f_n(\zeta) = \frac{f(\zeta_n+r_n\zeta)-f(\zeta_n)+f(\zeta_n)-f(\zeta_n')+\xi_n'-\xi_n}{r_n},$$

and since $|f(\zeta_n)-f(\zeta_n')|\le Cr_n$, $|\xi_n'-\xi_n|\le r_n$ and $(f(\zeta_n+r_n\zeta)-f(\zeta_n))/r_n$ converges to 0 uniformly, we deduce that $f_n\to\delta_1$ uniformly, for some $\delta_1\in\mathbb{R}$. Note that $\delta_1\ge 0$ since $f_n(0)\ge 0$ for every n, and $\delta_1\le 1$.

We can prove similarly the convergence of the sets D_n, by writing (using the orthogonality of $\overline{\partial_D\Omega}$ and $\overline{\partial_N\Omega}$)

$$(\Omega'\setminus\Omega)\cap B_r(x_0) = \{(\xi,\zeta)\in B_r(x_0) : \xi<f(\zeta),\ \zeta>g(\xi)\}$$

with g of class C^1, $g(\xi_0)=\zeta_0$, $g'(\xi_0)=0$, and

$$D_n = \{(\xi,\zeta)\in B_1 : \xi<f_n(\zeta),\ \zeta>g_n(\xi)\}, \qquad g_n(\xi) := \frac{g(\xi_n+r_n\xi)-\zeta_n}{r_n},$$

and arguing as before we prove that $g_n\to\delta_2$ uniformly for some $\delta_2\in[0,1]$.

The fact that the constant in the relative isoperimetric inequality is the same for all the sets Ω_n follows from the fact that, as $f_n' \to 0$ uniformly, the boundaries of the sets Ω_n are close to the boundary of Ω_0 in the C^1-sense.

Finally, we prove the last part of the statement, under the assumption $\delta_2 < 1$. We extend v to the set $\widetilde{\Omega} = \Omega_0 \cup \{\zeta > \delta_2\}$ by setting $v = 0$ outside Ω_0, and since $\widetilde{\Omega}$ satisfies the exterior cone condition we can find $\tilde{v} \in W^{1,2}(\mathbb{R}^2)$ such that $\tilde{v}_{|\widetilde{\Omega}} = v$. Setting, for $(\xi,\zeta) \in B_1$,

$$v_n(\xi,\zeta) := \tilde{v}(\xi,\zeta + a_n), \qquad a_n := \sup_{\xi} |g_n(\xi) - \delta_2| \to 0,$$

we obtain a sequence with the desired properties. □

In the following compactness property, which is a consequence of the Poincaré inequality, we adapt [8, Proposition 7.5] to our context.

LEMMA A.3. *Let x_n and r_n be as in Lemma A.2, and assume that $|D_n| \geq d_0 > 0$ for every n. Let $u_n \in SBV(\Omega_n)$, with $u_n = 0$ a.e. in D_n, be such that*

$$\sup_n \int_{\Omega_n} |\nabla u_n|^2 \,\mathrm{d}x < \infty, \qquad \lim_{n\to\infty} \mathcal{H}^1(S_{u_n}) = 0.$$

Setting $\bar{u}_n := (u_n \wedge \tau_n^+) \vee \tau_n^-$, where

$$\begin{aligned} \tau_n^+ &:= \inf\{t \in [-\infty,+\infty] : |\{u_n < t\}| \geq |\Omega_n| - (2\gamma\mathcal{H}^1(S_{u_n}))^2\}, \\ \tau_n^- &:= \inf\{t \in [-\infty,+\infty] : |\{u_n < t\}| \geq (2\gamma\mathcal{H}^1(S_{u_n}))^2\}, \end{aligned}$$

(here γ is the constant in the relative isoperimetric inequality in Ω_n), one has that $\bar{u}_n = 0$ in D_n for n large, and (up to subsequences) $\bar{u}_n \to v \in W^{1,2}(\Omega_0)$ in $L^2_{\mathrm{loc}}(\Omega_0)$, $u_n \to v$ a.e. in Ω_0, and for every $\rho \leq 1$

$$\int_{\Omega_0\cap B_\rho} |\nabla v|^2 \,\mathrm{d}x \leq \liminf_{n\to\infty} \int_{\Omega_n\cap B_\rho} |\nabla \bar{u}_n|^2 \,\mathrm{d}x. \tag{A.4}$$

PROOF. To show that $\bar{u}_n = 0$ in D_n, we fix $\varepsilon > 0$ and, since $\mathcal{H}^1(S_{u_n}) \to 0$, for all n sufficiently large (and independently of ε) we have

$$\begin{aligned} &|\{u_n < \varepsilon\}| \geq |D_n| \geq d_0 \geq (2\gamma\mathcal{H}^1(S_{u_n}))^2 \quad \Longrightarrow \quad \tau_n^- \leq \varepsilon, \\ &|\{u_n < -\varepsilon\}| \leq |\Omega_n| - |D_n| \leq |\Omega_n| - d_0 \leq |\Omega_n| - (2\gamma\mathcal{H}^1(S_{u_n}))^2 \quad \Longrightarrow \quad \tau_n^+ \geq -\varepsilon. \end{aligned}$$

Hence $\tau_n^- \leq 0$, $\tau_n^+ \geq 0$ for n large enough, and this implies that $\bar{u}_n = 0$ in D_n.

We now repeat the argument of the proof of the Poincaré inequality in SBV, following [8, Theorem 4.14], in order to deduce the compactness of the sequence $\bar{u}_n$. Let

$$m_n := \inf\{t \in [-\infty,+\infty] : |\{u_n < t\}| \geq |\Omega_n|/2\}$$

be a median of u_n in Ω_n, and observe that $\tau_n^- \leq m_n \leq \tau_n^+$ for n sufficiently large, since

$$(2\gamma\mathcal{H}^1(S_{u_n}))^2 < \frac{|\Omega_n|}{2}.$$

We have

$$|D\bar{u}_n|(\Omega_n) = \int_{\Omega_n} |\nabla \bar{u}_n| + \int_{S_{\bar{u}_n}} |\bar{u}_n^+ - \bar{u}_n^-| \,\mathrm{d}\mathcal{H}^1 \leq \int_{\Omega_n} |\nabla u_n| + (\tau_n^+ - \tau_n^-)\mathcal{H}^1(S_{u_n}),$$

while using the Coarea formula and the relative isoperimetric inequality in Ω_n

$$
\begin{aligned}
|D\bar{u}_n|(\Omega_n) &= \int_{\tau_n^-}^{\tau_n^+} P(\{\bar{u}_n > t\}, \Omega_n)\,\mathrm{d}t \\
&\geq \frac{1}{\gamma}\left[\int_{\tau_n^-}^{m_n} |\{u_n \leq t\}|^{\frac{1}{2}}\,\mathrm{d}t + \int_{m_n}^{\tau_n^+} |\{u_n > t\}|^{\frac{1}{2}}\,\mathrm{d}t\right] \geq 2(\tau_n^+ - \tau_n^-)\mathcal{H}^1(S_{u_n}).
\end{aligned}
$$

Combining the previous inequalities we deduce that

$$
|D\bar{u}_n|(\Omega_n) \leq 2\int_{\Omega_n} |\nabla u_n|. \tag{A.5}
$$

By the Poincaré inequality (notice that, since we are in dimension 2, we have $1^* = 2$)

$$
\|\bar{u}_n - m_n\|_{L^2(\Omega_n)} = \left(\int_{\Omega_n} (\bar{u}_n - m_n)^{1^*}\right)^{1/1^*} \leq \gamma\,|D\bar{u}_n|(\Omega_n) \leq 2\gamma\,|\Omega_n|^{\frac{1}{2}}\|\nabla u_n\|_{L^2(\Omega_n)}. \tag{A.6}
$$

Now from (A.5) and (A.6), since by assumption $\sup_n \|\nabla u_n\|_{L^2(\Omega_n)} < \infty$, we deduce that up to subsequences $\bar{u}_n - m_n \to v \in BV(\Omega_0)$ in $L^2_{\mathrm{loc}}(\Omega_0)$. Moreover, by setting $v^M := (v \wedge M) \vee (-M)$, by the compactness and lower semi-continuity theorems in SBV we deduce that $v^M \in SBV(\Omega_0)$ and

$$
\int_{\Omega_0} |\nabla v^M|^2 \leq \liminf_{n\to\infty} \int_{\Omega_n} |\nabla(\bar{u}_n - m_n)^M|^2 \leq \liminf_{n\to\infty} \int_{\Omega_n} |\nabla \bar{u}_n|^2,
$$

$$
\mathcal{H}^1(S_{v^M} \cap \Omega_0) \leq \liminf_{n\to\infty} \mathcal{H}^1(S_{(\bar{u}_n - m_n)^M} \cap \Omega_n) \leq \liminf_{n\to\infty} \mathcal{H}^1(S_{u_n}) = 0.
$$

Hence we obtain that, for every M, $v^M \in W^{1,1}(\Omega_0)$ and ∇v^M are equibounded in $L^2(\Omega_0)$, hence passing to the limit as $M \to +\infty$ we obtain that $v \in W^{1,2}(\Omega_0)$ and (A.4) holds. We remark also that, since $\{u_n \neq \bar{u}_n\} = \{u_n > \tau_n^+\} \cup \{u_n < \tau_n^-\}$,

$$
|\{u_n \neq \bar{u}_n\}| \leq 2(2\gamma\mathcal{H}^1(S_{u_n}))^2, \tag{A.7}
$$

and hence $u_n - m_n \to v$ a.e. in Ω_0.

To conclude the proof, it remains to show that the sequence m_n is bounded (indeed, in this case $m_n \to m \in \mathbb{R}$, and hence the sequence $\bar{u}_n$ converges to $v + m$). In turn, this follows from the fact that

$$
\limsup_{n\to+\infty} m_n^2|D_n| = \limsup_{n\to+\infty} \int_{D_n} |\bar{u}_n - m_n|^2 \leq \int_{\Omega_0} |v|^2 < +\infty,
$$

and $|D_n| \geq d_0 > 0$. □

The following lemma is a variant of [8, Theorem 7.7]. For $B \subset \mathbb{R}^2$ Borel set and $c > 0$ we set

$$
\overline{\mathcal{MS}}(v, c; B) := \int_B |\nabla v|^2\,\mathrm{d}x + c\mathcal{H}^1(S_v \cap B).
$$

LEMMA A.4. *Let x_n and r_n be as in Lemma A.2, and assume that $|D_n| \geq d_0 > 0$ for every n. Let $c_n > 0$, $u_n \in SBV(\Omega_n)$, with $u_n = 0$ in D_n, be such that*

$$
\sup_n \overline{\mathcal{MS}}(u_n, c_n; \Omega_n) < +\infty, \qquad \lim_{n\to+\infty} \mathrm{Dev}_{D_n}(u_n, c_n; B_1) = 0,
$$

$$
\lim_{n\to+\infty} \mathcal{H}^1(S_{u_n}) = 0, \qquad u_n \to v \in W^{1,2}(\Omega_0) \text{ a.e. in } \Omega_0,
$$

where

$$\begin{aligned}\mathrm{Dev}_{D_n}(v,c;B) &:= \overline{\mathcal{MS}}(v,c;B\cap\Omega_n)\\ &\quad - \inf\{\overline{\mathcal{MS}}(w,c;B\cap\Omega_n) : w\in SBV(\Omega_n),\ w=0 \text{ in } D_n,\ \{w\neq v\}\subset\subset B\}.\end{aligned}$$

Then

$$\int_{\Omega_0} |\nabla v|^2\,\mathrm{d}x \le \int_{\Omega_0} |\nabla w|^2\,\mathrm{d}x$$

for every $w\in W^{1,2}(\Omega_0)$ *such that* $w=0$ *in* D *and* $\{v\neq w\}\subset\subset B_1$, *and*

$$\lim_{n\to+\infty} \overline{\mathcal{MS}}(u_n,c_n;\Omega_n\cap B_\rho) = \int_{\Omega_0\cap B_\rho} |\nabla v|^2\,\mathrm{d}x \qquad \text{for every } \rho\in(0,1).$$

PROOF. The map $\rho\mapsto\overline{\mathcal{MS}}(u_n,c_n;B_\rho\cap\Omega_n)$ is increasing in $[0,1]$, hence up to subsequences

$$\lim_{n\to+\infty} \overline{\mathcal{MS}}(u_n,c_n;B_\rho\cap\Omega_n) = \alpha(\rho) \qquad \text{for every } \rho\in[0,1],$$

where $\alpha:[0,1]\to[0,+\infty)$ is increasing. By Lemma A.3 we have also $\bar u_n\to v$ in $L^2_{\mathrm{loc}}(\Omega_0)$ and

$$\int_{\Omega_0\cap B_\rho} |\nabla v|^2 \le \liminf_{n\to+\infty} \int_{\Omega_n\cap B_\rho} |\nabla \bar u_n|^2 \qquad \text{for every } \rho\le 1. \tag{A.8}$$

We now show that

$$\lim_{n\to+\infty} \overline{\mathcal{MS}}(\bar u_n,c_n;B_\rho\cap\Omega_n) = \alpha(\rho), \qquad \lim_{n\to+\infty} \mathrm{Dev}_{D_n}(\bar u_n,c_n;B_\rho) = 0 \tag{A.9}$$

for $\mathcal{L}^1$-a.e. $\rho\in(0,1)$. Denoting by $\tilde u_n$, $\tilde{\bar u}_n$ the Lebesgue representative of u_n and $\bar u_n$, respectively, we have

$$c_n\int_0^1 \mathcal{H}^1(\{\tilde u_n\neq\tilde{\bar u}_n\}\cap\partial B_\rho\cap\Omega_n)\,\mathrm{d}\rho = c_n|\{u_n\neq\bar u_n\}| \le 2c_n(2\gamma\mathcal{H}^1(S_{u_n}))^2 \to 0$$

(see (A.7)), from which it follows that (up to further subsequences)

$$c_n\mathcal{H}^1(\{\tilde u_n\neq\tilde{\bar u}_n\}\cap\partial B_\rho\cap\Omega_n)\to 0 \quad \text{for } \mathcal{L}^1\text{-a.e. } \rho\in(0,1).$$

Moreover, $\mathcal{H}^1(S_{\bar u_n}\cap\partial B_\rho)=0$ for every n and for $\mathcal{L}^1$-a.e. $\rho\in(0,1)$. Choosing $0<\rho<\rho'<1$ such that the previous properties hold, by comparing the energies of u_n and $\bar u_n\chi_{B_\rho}+u_n\chi_{B_{\rho'}\setminus B_\rho}$ we deduce

$$\begin{aligned}\overline{\mathcal{MS}}(\bar u_n,c_n;B_\rho\cap\Omega_n) &\le \overline{\mathcal{MS}}(u_n,c_n;B_\rho\cap\Omega_n)\\ &\le \overline{\mathcal{MS}}(\bar u_n,c_n;B_\rho\cap\Omega_n) + c_n\mathcal{H}^1(\{\tilde u_n\neq\tilde{\bar u}_n\}\cap\partial B_\rho\cap\Omega_n) + \mathrm{Dev}_{D_n}(u_n,c_n;B_{\rho'}),\end{aligned}$$

from which we obtain the first part of (A.9). Similarly, if $w\in SBV(\Omega_n)$, $w=0$ in D_n, $\{w\neq\bar u_n\}\subset\subset B_\rho$, by comparing the energies of u_n and $w\chi_{B_\rho}+u_n\chi_{B_{\rho'}\setminus B_\rho}$ we obtain

$$\begin{aligned}\overline{\mathcal{MS}}(\bar u_n,c_n;B_\rho\cap\Omega_n) &\le \overline{\mathcal{MS}}(u_n,c_n;B_\rho\cap\Omega_n)\\ &\le \overline{\mathcal{MS}}(w,c_n;B_\rho\cap\Omega_n) + c_n\mathcal{H}^1(\{\tilde w\neq\tilde u_n\}\cap\partial B_\rho\cap\Omega_n) + \mathrm{Dev}_{D_n}(u_n,c_n;B_{\rho'})\\ &\le \overline{\mathcal{MS}}(w,c_n;B_\rho\cap\Omega_n) + c_n\mathcal{H}^1(\{\tilde{\bar u}_n\neq\tilde u_n\}\cap\partial B_\rho\cap\Omega_n) + \mathrm{Dev}_{D_n}(u_n,c_n;B_{\rho'}),\end{aligned}$$

from which the second part of (A.9) follows.

We can now prove the minimality of v. Let $w\in W^{1,2}(\Omega_0)$, $w=0$ in D, $\{v\neq w\}\subset\subset B_1$. By Lemma A.2 we can find a sequence $w_n\in W^{1,2}(B_1)$, $w_n=0$ in D_n, such that $w_n\to w$ in $W^{1,2}(\Omega_0)$. Let $0<\rho<\rho'<1$ be such that $\{v\neq w\}\subset\subset B_\rho$, α is continuous in ρ, ρ' and

(A.9) holds. We fix a cut-off function ϕ between B_ρ and $B_{\rho'}$ and we compare the energies of $\bar{u}_n$ and $\phi w_n + (1-\phi)\bar{u}_n$:

$$\overline{\mathcal{MS}}(\bar{u}_n, c_n; B_\rho \cap \Omega_n) \le \int_{B_\rho\cap\Omega_n} |\nabla w_n|^2 + \mathrm{Dev}_{D_n}(\bar{u}_n, c_n; B_{\rho'}) + C\overline{\mathcal{MS}}(\bar{u}_n, c_n; (B_{\rho'}\setminus B_\rho)\cap\Omega_n)$$
$$+ C\Big[\int_{(B_{\rho'}\setminus B_\rho)\cap\Omega_n} |\nabla w_n|^2 + \frac{1}{(\rho'-\rho)^2}\int_{(B_{\rho'}\setminus B_\rho)\cap\Omega_n} |w_n - \bar{u}_n|^2\Big].$$

Letting $n \to +\infty$, we obtain

$$\alpha(\rho) \le \int_{B_\rho\cap\Omega_0} |\nabla w|^2 + C\big(\alpha(\rho') - \alpha(\rho)\big) + C\int_{(B_{\rho'}\setminus B_\rho)\cap\Omega_0} |\nabla w|^2,$$

and letting $\rho \nearrow \rho'$ we conclude that $\alpha(\rho') \le \int_{B_{\rho'}\cap\Omega_0} |\nabla w|^2$. By choosing, in particular, $w = v$, and recalling (A.8), we deduce that

$$\alpha(\rho') = \int_{B_{\rho'}\cap\Omega_0} |\nabla v|^2$$

and that v is a local minimum of the Dirichlet integral. Finally, since the monotone increasing functions $\rho \mapsto \alpha(\rho)$ and $\rho \mapsto \int_{B_\rho\cap\Omega_0} |\nabla v|^2$ coincide for $\mathcal{L}^1$-a.e. $\rho \in (0,1)$, and the second one is continuous, we conclude that they coincide everywhere. $\square$

The following lemma contains the main decay property used to prove Theorem 1.13.

LEMMA A.5. *There exists a positive constant C such that for every $\tau \in (0,1)$ there exist $\varepsilon(\tau) > 0$, $\theta(\tau) > 0$ and $r(\tau) > 0$ with the property that for every $x \in \overline{\Omega}$ and $\rho \le r(\tau)$, whenever $v \in SBV(\Omega' \cap B_\rho(x))$ is such that $v = 0$ in $(\Omega'\setminus\Omega)\cap B_\rho(x)$,*

$$\mathcal{H}^1(S_v \cap B_\rho(x) \cap \Omega') < \varepsilon(\tau)\rho, \quad \mathrm{Dev}(v; B_\rho(x)) < \theta(\tau)\overline{\mathcal{MS}}(v; B_\rho(x)\cap\Omega')$$

(the deviation from minimality is defined as in (A.2)) then

$$\overline{\mathcal{MS}}(v; B_{\tau\rho}(x)\cap\Omega') \le C\tau^2\overline{\mathcal{MS}}(v; B_\rho(x)\cap\Omega').$$

PROOF. By choosing C large enough, we can assume without loss of generality that $\tau < \frac{1}{4}$. The proof is by a contradiction argument: let $\varepsilon_n \to 0$, $\theta_n \to 0$, $r_n \to 0$, $x_n \in \overline{\Omega}$, $v_n \in SBV(B_{r_n}(x_n)\cap\Omega')$, $v_n = 0$ in $(\Omega'\setminus\Omega)\cap B_{r_n}(x_n)$, be such that

$$\mathcal{H}^1(S_{v_n} \cap B_{r_n}(x_n)\cap\Omega') = \varepsilon_n r_n, \quad \mathrm{Dev}(v_n; B_{r_n}(x_n)) = \theta_n\overline{\mathcal{MS}}(v_n; B_{r_n}(x_n)\cap\Omega'),$$

and

$$\overline{\mathcal{MS}}(v_n; B_{\tau r_n}(x_n)\cap\Omega') > C\tau^2\overline{\mathcal{MS}}(v_n; B_{r_n}(x_n)\cap\Omega'),$$

where C will be chosen later. By a change of variables, we set

$$w_n(y) := r_n^{-\frac{1}{2}}\, c_n^{\frac{1}{2}}\, v_n(x_n + r_n y), \qquad c_n := \frac{r_n}{\overline{\mathcal{MS}}(v_n; B_{r_n}(x_n)\cap\Omega')}.$$

We obtain a sequence $w_n \in SBV(\Omega_n)$ such that $\overline{\mathcal{MS}}(w_n, c_n; \Omega_n) = 1$, $\mathrm{Dev}_{D_n}(w_n, c_n; B_1) = \theta_n$, $\mathcal{H}^1(S_{w_n}\cap\Omega_n) = \varepsilon_n$, and

$$\overline{\mathcal{MS}}(w_n, c_n; B_\tau\cap\Omega_n) > C\tau^2$$

(here Ω_n and D_n are defined as in (A.3)). Up to subsequences, $x_n \to x_0$, and we are in one of the following cases:

- $x_0 \in \Omega$: in this case the balls $B_{r_n}(x_n)$ are contained in Ω for n large, hence the boundary does not play any role and the contradiction follows from [8, Lemma 7.14];

- $x_0 \in \partial_D\Omega$: the balls $B_{r_n}(x_n)$ intersect only the Dirichlet part of the boundary for n large, and the contradiction follows from [10, Lemma 6.6];
- $x_0 \in \partial_N\Omega$: we have that $\Omega_n \to \Omega_0 = \{(\xi,\zeta) \in B_1 : \xi < \delta_1\}$ for some $\delta_1 \in [0,1]$ (in a suitable coordinate system) and $D_n = \emptyset$ for n large enough. Adapting Lemma A.3 and Lemma A.4 to this situation (in which the Dirichlet condition does not play any role) we have that, up to further subsequences, $w_n - m_n \to w$ almost everywhere in Ω_0, where m_n are medians of w_n in Ω_n and $w \in W^{1,2}(\Omega_0)$, with
$$\int_{\Omega_0} |\nabla w|^2 \leq \liminf_n \int_{\Omega_n} |\nabla w_n|^2 \leq 1.$$
In addition, w is harmonic in Ω_0 and satisfies a homogeneous Neumann condition on $\{(\xi,\zeta) : \xi = \delta_1\}$, and hence (by the decay properties of harmonic functions)
$$C\tau^2 \leq \lim_{n\to+\infty} \overline{\mathcal{MS}}(w_n, c_n; B_\tau \cap \Omega_n) = \int_{B_\tau\cap\Omega_0} |\nabla w|^2 \leq 8\tau^2 \int_{B_{\frac{1}{2}}\cap\Omega_0} |\nabla w|^2 \leq 8\tau^2$$
which is a contradiction if we take $C > 8$.
- $x_0 \in \overline{\partial_D\Omega} \cap \overline{\partial_N\Omega}$: in this case we are under the assumptions of Lemma A.2. If $\delta_2 \in (\frac{1}{2}, 1]$, then $B_{1/2} \cap D_n = \emptyset$ for n large enough, and we can argue exactly as in the previous case, in the ball $B_{1/2}$. It remains only to deal with the case $\delta_2 \in [0, \frac{1}{2}]$.

To get a contradiction also in the case $\delta_2 \in [0, \frac{1}{2}]$, observe first that $|D_n| \geq d_0 > 0$. We can apply Lemma A.3 and Lemma A.4 to deduce that, up to subsequences, $w_n \to w_\infty \in W^{1,2}(\Omega_0)$ a.e. in Ω_0, with $w_\infty = 0$ in D,
$$\int_{\Omega_0} |\nabla w_\infty|^2 \leq \liminf_{n\to\infty} \int_{\Omega_n} |\nabla w_n|^2 \leq 1.$$
Moreover for every $w \in W^{1,2}(\Omega_0)$ such that $w = 0$ in D and $\{w \neq w_\infty\} \subset\subset B_1$
$$\int_{\Omega_0} |\nabla w_\infty|^2 \leq \int_{\Omega_0} |\nabla w|^2,$$
and
$$\overline{\mathcal{MS}}(w_n, c_n; B_r \cap \Omega_n) \to \int_{B_r\cap\Omega_0} |\nabla w_\infty|^2 \qquad \text{for every } r \in (0,1).$$
If $\tilde{w}_\infty$ is the harmonic function in B_1 obtained by applying firstly an even reflection of w_∞ across $\{(\xi,\zeta) : \xi = \delta_1\}$, and then an odd reflection across $\{(\xi,\zeta) : \zeta = \delta_2\}$, we conclude, by using the decay properties of harmonic functions, that
$$\begin{aligned} C\tau^2 &\leq \lim_{n\to\infty} \overline{\mathcal{MS}}(w_n, c_n; B_\tau \cap \Omega_n) = \int_{B_\tau\cap\Omega_0} |\nabla w_\infty|^2 \leq \int_{B_\tau} |\nabla \tilde{w}_\infty|^2 \\ &\leq (2\tau)^2 \int_{B_{\frac{1}{2}}} |\nabla \tilde{w}_\infty|^2 \leq 4(2\tau)^2 \int_{B_{\frac{1}{2}}\cap\Omega_0} |\nabla w_\infty|^2 \leq 16\tau^2, \end{aligned}$$
and this is a contradiction if we choose $C > 16$. □

We have now all the ingredients to conclude the proof of Theorem 1.13.

Proof of Theorem 1.13. Let η be given by Lemma A.1. We first observe that the density lower bound holds in any ball $B_\rho(x)$ with $x \in \overline{\Omega} \setminus \mathcal{N}_\eta(\partial_D\Omega)$ and $\rho \leq \rho_0$ (for some $\rho_0 < \eta$ depending only on ω, u and Ω): indeed, in this case the Dirichlet boundary condition does not play any role, and the result is classical. It is then sufficient to prove the lower bound for the function $\tilde{w}$ defined in Lemma A.1 in balls $B_\rho(x)$ centered at points $x \in \overline{S}_{\tilde{w}} \cap \mathcal{N}_\eta(\partial_D\Omega)$, since in such balls $S_w \cap B_\rho(x) = S_{\tilde{w}} \cap B_\rho(x)$ if $\rho < \eta$.

In order to do this, we first note that by a simple comparison argument the following energy upper bound holds for $\tilde{w}$:

$$\overline{\mathcal{MS}}(\tilde{w}; B_\rho(x)) \le 2\pi\rho + \tilde{\omega}\rho^{\frac{3}{2}}. \tag{A.10}$$

Let $\tau \in (0,1)$ be such that $C\tau^2 \le \tau^{\frac{3}{2}}$, where C is the constant provided by Lemma A.5, and let $\sigma \in (0,1)$ be such that $C\sigma(2\pi+1) \le \varepsilon(\tau)$. We define $\theta_0 := \varepsilon(\sigma)$,

$$\rho_0 := \min\Big\{\rho(\sigma), \frac{1}{\tilde{\omega}}, \Big(\frac{\varepsilon(\tau)\sigma\theta_0}{\tilde{\omega}}\Big)^2, \rho(\tau), \Big(\frac{\varepsilon(\tau)\theta(\tau)\tau^2}{\tilde{\omega}}\Big)^2\Big\}$$

and we prove the density lower bound for this choice of θ_0 and ρ_0 (here we are using the notation of Lemma A.5).

We first show by induction that, assuming $\mathcal{H}^1(S_{\tilde{w}} \cap B_\rho(x)) < \theta_0\rho$ for some $x \in \overline{\Omega} \cap \mathcal{N}_\eta(\partial_D\Omega)$ and $\rho \le \rho_0$, then

$$\overline{\mathcal{MS}}(\tilde{w}; B_{\sigma\tau^h\rho}(x)) \le \varepsilon(\tau)\tau^{\frac{h}{2}}(\sigma\tau^h\rho) \quad \text{for every } h \in \mathbb{N}. \tag{A.11}$$

In the case $h=0$, if $\mathrm{Dev}(\tilde{w}; B_\rho(x)) < \theta(\sigma)\overline{\mathcal{MS}}(\tilde{w}; B_\rho(x))$, then Lemma A.5 and (A.10) imply

$$\overline{\mathcal{MS}}(\tilde{w}; B_{\sigma\rho}(x)) \le C\sigma^2\overline{\mathcal{MS}}(\tilde{w}; B_\rho(x)) \le C\sigma^2(2\pi\rho + \tilde{\omega}\rho^{\frac{3}{2}}) \le C\sigma^2(2\pi+1)\rho \le \varepsilon(\tau)\sigma\rho,$$

while if $\mathrm{Dev}(\tilde{w}; B_\rho(x)) \ge \theta(\sigma)\overline{\mathcal{MS}}(\tilde{w}; B_\rho(x))$ then

$$\overline{\mathcal{MS}}(\tilde{w}; B_{\sigma\rho}(x)) \le \overline{\mathcal{MS}}(\tilde{w}; B_\rho(x)) \le \frac{1}{\theta(\sigma)}\mathrm{Dev}(\tilde{w}; B_\rho(x)) \le \frac{\tilde{\omega}\rho^{\frac{3}{2}}}{\theta(\sigma)} \le \varepsilon(\tau)\sigma\rho.$$

Hence (A.11) is proved if $h=0$. Assuming now that it holds for a given $h \ge 0$, we prove it for $h+1$. As before, if $\mathrm{Dev}(\tilde{w}; B_{\sigma\tau^h\rho}(x)) < \theta(\tau)\overline{\mathcal{MS}}(\tilde{w}; B_{\sigma\tau^h\rho}(x))$ then by Lemma A.5 we obtain

$$\begin{aligned}\overline{\mathcal{MS}}(\tilde{w}; B_{\sigma\tau^{h+1}\rho}(x)) &\le C\tau^2\overline{\mathcal{MS}}(\tilde{w}; B_{\sigma\tau^h\rho}(x)) \le C\tau^2\varepsilon(\tau)\tau^{\frac{h}{2}}(\sigma\tau^h\rho)\\ &\le \tau^{\frac{3}{2}}\varepsilon(\tau)\tau^{\frac{h}{2}}(\sigma\tau^h\rho) = \varepsilon(\tau)\tau^{\frac{h+1}{2}}(\sigma\tau^{h+1}\rho),\end{aligned}$$

while if $\mathrm{Dev}(\tilde{w}; B_{\sigma\tau^h\rho}(x)) \ge \theta(\tau)\overline{\mathcal{MS}}(\tilde{w}; B_{\sigma\tau^h\rho}(x))$ then

$$\begin{aligned}\overline{\mathcal{MS}}(\tilde{w}; B_{\sigma\tau^{h+1}\rho}(x)) &\le \overline{\mathcal{MS}}(\tilde{w}; B_{\sigma\tau^h\rho}(x)) \le \frac{1}{\theta(\tau)}\mathrm{Dev}(\tilde{w}; B_{\sigma\tau^h\rho}(x))\\ &\le \frac{\tilde{\omega}(\sigma\tau^h\rho)^{\frac{3}{2}}}{\theta(\tau)} \le \varepsilon(\tau)\tau^{\frac{h+1}{2}}(\sigma\tau^{h+1}\rho).\end{aligned}$$

Hence (A.11) is proved. By an iteration argument, we obtain that for $x \in \overline{\Omega} \cap \mathcal{N}_\eta(\partial_D\Omega)$ and $\rho \le \rho_0$

$$\mathcal{H}^1(S_{\tilde{w}} \cap B_\rho(x)) < \theta_0\rho \quad \Longrightarrow \quad r^{-1}\overline{\mathcal{MS}}(\tilde{w}; B_r(x)) \to 0 \quad \text{as } r \to 0^+.$$

Now, setting

$$I := \Big\{x \in \Omega \cap \mathcal{N}_\eta(\partial_D\Omega) \,:\, \limsup_{\rho\to 0}\frac{1}{|B_\rho(x)|}\int_{B_\rho(x)}|\tilde{w}(y)|^{1^*}\,\mathrm{d}y = \infty\Big\},$$

by [8, Theorem 7.8] we have that the lower bound holds in every point of $S_{\tilde{w}} \setminus I$, and by density also in every point of $\overline{S_{\tilde{w}} \setminus I}$. It is the sufficient to prove that $\overline{S_{\tilde{w}} \setminus I} = \overline{S}_{\tilde{w}}$. Let $x \notin \overline{S_{\tilde{w}} \setminus I}$, and let us prove that $x \notin \overline{S}_{\tilde{w}}$. Since $\mathcal{H}^1(I) = 0$ by [8, Lemma 3.75], we can find a neighborhood V of x such that $\mathcal{H}^1(S_{\tilde{w}} \cap V) = 0$. Hence $\tilde{w} \in W^{1,2}(V)$, and in each ball $B_r(y) \subset V$ we have (by using the energy upper bound (A.10) and the Poincaré inequality)

$$\int_{B_r(y)}|\tilde{w}(z) - \tilde{w}_{y,r}|^2\,dz \le cr^2\int_{B_r(y)}|\nabla\tilde{w}|^2 \le c'r^3,$$

where $\tilde{w}_{y,r}$ is the average of $\tilde{w}$ in $B_r(y)$. By [8, Theorem 7.51] we conclude that $\tilde{w} \in C^{0,\frac{1}{2}}_{\mathrm{loc}}(V)$, and hence $x \notin \overline{S}_{\tilde{w}}$. $\square$

On the invertibility of the linear system appearing in Lemma 3.42

The final part of the second step in the proof of Lemma 3.42 requires to invert the relations determined by an 18×18 linear system which we can write explicitly as

$$\xi = M\sigma,$$

where, according to the notation introduced in the proof of Lemma 3.42, ξ and σ are the column vectors

$$\begin{aligned}\xi &:= \big(\vartheta_{111}, \vartheta_{311}, \vartheta_{112}, \vartheta_{212}, \vartheta_{312}, \vartheta_{121}, \vartheta_{321}, \vartheta_{122}, \vartheta_{222}, \vartheta_{322}, \vartheta_{131}, \vartheta_{331}, \\ &\qquad\qquad\qquad \vartheta_{132}, \vartheta_{232}, \vartheta_{332}, \eta_{13}, \eta_{23}, \eta_{33}\big)^T, \\ \sigma &:= \big(\sigma_{111}, \sigma_{121}, \sigma_{131}, \sigma_{221}, \sigma_{231}, \sigma_{331}, \sigma_{112}, \sigma_{122}, \sigma_{132}, \sigma_{222}, \sigma_{232}, \sigma_{332}, \\ &\qquad\qquad\qquad \sigma_{113}, \sigma_{123}, \sigma_{133}, \sigma_{223}, \sigma_{233}, \sigma_{333}\big)^T,\end{aligned}$$

and M is the matrix

$$\begin{pmatrix}
\nu_g^3 & 0 & -\nu_g^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \nu_g^3 & 0 & 0 & -\nu_g^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \nu_g^3 & -\nu_g^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \nu_g^3 & -\nu_g^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \nu_g^3 & -\nu_g^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & 0 & -\nu_g^1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & 0 & 0 & -\nu_g^1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & -\nu_g^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & -\nu_g^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & -\nu_g^2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & 0 & -\nu_g^1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & 0 & 0 & -\nu_g^1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & -\nu_g^2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & -\nu_g^2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nu_g^3 & -\nu_g^2 \\
0 & 0 & a_1 & 0 & b_1 & c_1 & 0 & 0 & d_1 & 0 & e_1 & f_1 & 0 & 0 & g_1 & 0 & h_1 & i_1 \\
0 & 0 & a_2 & 0 & b_2 & c_2 & 0 & 0 & d_2 & 0 & e_2 & f_2 & 0 & 0 & g_2 & 0 & h_2 & i_2 \\
0 & 0 & a_3 & 0 & b_3 & c_3 & 0 & 0 & d_3 & 0 & e_3 & f_3 & 0 & 0 & g_3 & 0 & h_3 & i_3
\end{pmatrix}$$

The coefficients in the last three rows of M are defined by

$$\begin{aligned}
a_j &:= \textstyle\sum_{k=1}^3 C_{jk11}\nu_g^k, \quad & b_j &:= \textstyle\sum_{k=1}^3 C_{jk12}\nu_g^k, \quad & c_j &:= \textstyle\sum_{k=1}^3 C_{jk13}\nu_g^k, \\
d_j &:= \textstyle\sum_{k=1}^3 C_{jk21}\nu_g^k, \quad & e_j &:= \textstyle\sum_{k=1}^3 C_{jk22}\nu_g^k, \quad & f_j &:= \textstyle\sum_{k=1}^3 C_{jk23}\nu_g^k, \\
g_j &:= \textstyle\sum_{k=1}^3 C_{jk31}\nu_g^k, \quad & h_j &:= \textstyle\sum_{k=1}^3 C_{jk32}\nu_g^k, \quad & i_j &:= \textstyle\sum_{k=1}^3 C_{jk33}\nu_g^k,
\end{aligned}$$

for $j = 1, 2, 3$, so that the corresponding equations are exactly the equalities (3.84). In order to invert the relations determined by the previous system, we claimed that the determinant of M equals $(\nu_g^3)^{12} \det Q_g$, where Q_g is the 3×3 matrix defined by (3.80).

We present here the *Mathematica* code which allows us to check this equality. We first define the 18×18 matrix M: here the variables **n1**, **n2** and **n3** stand for the components $\nu_g^1, \nu_g^2, \nu_g^3$ of the normal vector, and the variables **Cijhk** for the coefficients C_{ijhk} of the tensor. We then define the matrix Q_g introduced in (3.80), whose entries are indicated by **qij**, and we compute its determinant (multiplied by $(\nu_g^3)^{12}$). Finally we evaluate the difference between the determinant of M and $(\nu_g^3)^{12} \det Q_g$, which turns out to be zero.

The *Mathematica* code is the following.

$$M = \begin{pmatrix}
\mathrm{n3} & 0 & -\mathrm{n1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \mathrm{n3} & 0 & 0 & -\mathrm{n1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \mathrm{n3} & -\mathrm{n2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \mathrm{n3} & -\mathrm{n2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{n3} & -\mathrm{n2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & 0 & -\mathrm{n1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & 0 & 0 & -\mathrm{n1} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & -\mathrm{n2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & -\mathrm{n2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & -\mathrm{n2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & 0 & -\mathrm{n1} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & 0 & 0 & -\mathrm{n1} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & -\mathrm{n2} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & -\mathrm{n2} & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathrm{n3} & -\mathrm{n2} \\
0 & 0 & \mathrm{a1} & 0 & \mathrm{b1} & \mathrm{c1} & 0 & 0 & \mathrm{d1} & 0 & \mathrm{e1} & \mathrm{f1} & 0 & 0 & \mathrm{g1} & 0 & \mathrm{h1} & \mathrm{i1} \\
0 & 0 & \mathrm{a2} & 0 & \mathrm{b2} & \mathrm{c2} & 0 & 0 & \mathrm{d2} & 0 & \mathrm{e2} & \mathrm{f2} & 0 & 0 & \mathrm{g2} & 0 & \mathrm{h2} & \mathrm{i2} \\
0 & 0 & \mathrm{a3} & 0 & \mathrm{b3} & \mathrm{c3} & 0 & 0 & \mathrm{d3} & 0 & \mathrm{e3} & \mathrm{f3} & 0 & 0 & \mathrm{g3} & 0 & \mathrm{h3} & \mathrm{i3}
\end{pmatrix};$$

DM = Det[M];

a1 = C1111n1 + C1211n2 + C1311n3;

b1 = C1112n1 + C1212n2 + C1312n3;

c1 = C1113n1 + C1213n2 + C1313n3;

d1 = C1121n1 + C1221n2 + C1321n3;

e1 = C1122n1 + C1222n2 + C1322n3;

f1 = C1123n1 + C1223n2 + C1323n3;

g1 = C1131n1 + C1231n2 + C1331n3;

h1 = C1132n1 + C1232n2 + C1332n3;

$$\mathrm{i1 = C1133n1 + C1233n2 + C1333n3;}$$

$$\mathrm{a2 = C2111n1 + C2211n2 + C2311n3;}$$

$$\mathrm{b2 = C2112n1 + C2212n2 + C2312n3;}$$

$$\mathrm{c2 = C2113n1 + C2213n2 + C2313n3;}$$

$$\mathrm{d2 = C2121n1 + C2221n2 + C2321n3;}$$

$$\mathrm{e2 = C2122n1 + C2222n2 + C2322n3;}$$

$$\mathrm{f2 = C2123n1 + C2223n2 + C2323n3;}$$

$$\mathrm{g2 = C2131n1 + C2231n2 + C2331n3;}$$

$$\mathrm{h2 = C2132n1 + C2232n2 + C2332n3;}$$

$$\mathrm{i2 = C2133n1 + C2233n2 + C2333n3;}$$

$$\mathrm{a3 = C3111n1 + C3211n2 + C3311n3;}$$

$$\mathrm{b3 = C3112n1 + C3212n2 + C3312n3;}$$

$$\mathrm{c3 = C3113n1 + C3213n2 + C3313n3;}$$

$$\mathrm{d3 = C3121n1 + C3221n2 + C3321n3;}$$

$$\mathrm{e3 = C3122n1 + C3222n2 + C3322n3;}$$

$$\mathrm{f3 = C3123n1 + C3223n2 + C3323n3;}$$

$$\mathrm{g3 = C3131n1 + C3231n2 + C3331n3;}$$

$$\mathrm{h3 = C3132n1 + C3232n2 + C3332n3;}$$

$$\mathrm{i3 = C3133n1 + C3233n2 + C3333n3;}$$

$$\mathrm{q11 = C1111n1n1 + C1212n2n2 + C1313n3n3 + (C1112 + C1211)n1n2+}$$
$$\mathrm{(C1113 + C1311)n1n3 + (C1213 + C1312)n2n3;}$$

$$\mathrm{q12 = C1121n1n1 + C1222n2n2 + C1323n3n3 + (C1122 + C1221)n1n2+}$$
$$\mathrm{(C1123 + C1321)n1n3 + (C1223 + C1322)n2n3;}$$

$$\mathrm{q13 = C1131n1n1 + C1232n2n2 + C1333n3n3 + (C1132 + C1231)n1n2+}$$
$$\mathrm{(C1133 + C1331)n1n3 + (C1233 + C1332)n2n3;}$$

$$\mathrm{q21 = C2111n1n1 + C2212n2n2 + C2313n3n3 + (C2112 + C2211)n1n2+}$$
$$\mathrm{(C2113 + C2311)n1n3 + (C2213 + C2312)n2n3;}$$

$$\mathrm{q22 = C2121n1n1 + C2222n2n2 + C2323n3n3 + (C2122 + C2221)n1n2+}$$
$$\mathrm{(C2123 + C2321)n1n3 + (C2223 + C2322)n2n3;}$$

$$\mathrm{q23 = C2131n1n1 + C2232n2n2 + C2333n3n3 + (C2132 + C2231)n1n2+}$$

$(\mathrm{C2133} + \mathrm{C2331})\mathrm{n1n3} + (\mathrm{C2233} + \mathrm{C2332})\mathrm{n2n3};$

$\mathrm{q31} = \mathrm{C3111n1n1} + \mathrm{C3212n2n2} + \mathrm{C3313n3n3} + (\mathrm{C3112} + \mathrm{C3211})\mathrm{n1n2} +$

$(\mathrm{C3113} + \mathrm{C3311})\mathrm{n1n3} + (\mathrm{C3213} + \mathrm{C3312})\mathrm{n2n3};$

$\mathrm{q32} = \mathrm{C3121n1n1} + \mathrm{C3222n2n2} + \mathrm{C3323n3n3} + (\mathrm{C3122} + \mathrm{C3221})\mathrm{n1n2} +$

$(\mathrm{C3123} + \mathrm{C3321})\mathrm{n1n3} + (\mathrm{C3223} + \mathrm{C3322})\mathrm{n2n3};$

$\mathrm{q33} = \mathrm{C3131n1n1} + \mathrm{C3232n2n2} + \mathrm{C3333n3n3} + (\mathrm{C3132} + \mathrm{C3231})\mathrm{n1n2} +$

$(\mathrm{C3133} + \mathrm{C3331})\mathrm{n1n3} + (\mathrm{C3233} + \mathrm{C3332})\mathrm{n2n3};$

$$Q = \begin{pmatrix} \mathrm{q11} & \mathrm{q12} & \mathrm{q13} \\ \mathrm{q21} & \mathrm{q22} & \mathrm{q23} \\ \mathrm{q31} & \mathrm{q32} & \mathrm{q33} \end{pmatrix};$$

$\mathrm{DQ} = \mathrm{n3}^{12}\mathrm{Det}[Q];$

$\mathrm{ExpandAll}[\mathrm{DQ}] - \mathrm{ExpandAll}[\mathrm{DM}]$

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