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SUPERCONFORMAL MINIMAL MODELS AND THEIR ROLE IN SUPERSTRING COMPACTIFICATION

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To F.

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The geometry of Tlön comprises two somewhat different disciplines: the visual and the tactile. The latter corresponds to our own geometry and it is subordinated to the first. The basis of visual geometry is the 2-dimensional surface, not the point...

J.L.Borges, Labyrinths

Chapter 1

Introduction

1.1 General Overlook

Conformally invariant two-dimensional quantum field theories (QFT) describe universality classes of two-dimensional critical phenomena and arise in string theory in the description of perturbative string propagation. In fact the renormalization group flow in the presence of background fields seems to provide perturbatively a complete set of field equations for string propagation [1,2,3,4,5], whose solutions are conformally invariant QFT. A large class of these theories are specific types of non-linear σ models with vanishing β functional [1,4,6]. Besides these examples one can argue that every conformal field theory is a candidate 'vacuum' configuration for a string and their classification appears to be a formidable task.

The approach of Belavin, Polyakov and Zamolodchikov [7], which is obligatory literature in the subject, is to solve the theory via the conformal bootstrap: starting with a Hilbert space representation of the Virasoro algebra, one looks for an algebra of field operators consistent with crossing symmetry. In the region $c < 1$ (c is the central charge of the Virasoro algebra) unitarity restricts the allowed QFT to discrete series [7,8]. This subclass of conformally invariant models, called 'minimal models'(m.m.), has the nice property that the operator algebra closes involving only a finite number of fields, called primary. The anomalous dimension of the primary fields is quantized. In this case, it turns out that the correlation functions can be, at least in principle, exactly calculated leading to the complete solution of the theory. Common methods consist in solving differential equations coming from the 'null-vector' [7] or using integral representation generated by an auxiliary QFT, the Feigin-Fuchs theory [9,10].

On the contrary, in the case $c > 1$ an infinite number of primary fields is involved in the operator algebra and the models are no longer solvable with the previous methods. However the combination of an additional symmetry with the conformal one gives in general a larger infinite dimensional algebra: classifying the fields of the theory according to this extended algebra, some models having $c > 1$ may become 'minimal' with respect to the higher algebra and so still exactly solvable. This happens, for instance, combining conformal symmetry with N=1 supersymmetry, which we will discuss in detail in chapter 2: one gets in this way the well-known N=1 superconformal algebra, which has a discrete unitary minimal models for $c < \frac{3}{2}$ [11,12,13,14]. For models in these series having $c \geq 1$ an infinite number of Virasoro representations organize themselves in a finite number of superconformal irreducible representations.

The superconformal N=1 algebra has two sectors: Neveu-Schwarz (NS) and Ramond (R), according to the choice of periodic or antiperiodic boundary conditions for the fermionic fields. The construction of the NS sector requires slight modification of the Virasoro algebra methods: it consists only in using superfields instead of the ordinary conformal fields. The description of the Ramond sector is more subtle since we face with the problem of double valuedness of the fermionic fields in the presence of the Ramond fields (called 'spin-fields' in the ref.[11,15]). The solution of this problem, discussed in Refs. [17,18], consists in the construction of a Coulomb gas representation for the Ramond fields, involving fluctuation fields of the critical Ising model (order-disorder fields and free Majorana fermion) and a free massless bosonic field. This construction allows us to calculate the multipoint functions of the theory and the structure constants of the operator algebra, thus getting the full description of the N=1 superconformal minimal models.

In the same spirit of enlarging the Virasoro algebra, other symmetries are explored: N=2 superconformal models [31,32], parafermionic theory with Z_n internal group [33,34], Wess-Zumino models [35,36], orbifolds [37], etc.

Among this fastly increasing variety of unitary discrete series of minimal models, the two-dimensional models with extended N=2 local supersymmetry are singlet out by their properties to describe all the classical super-(heterotic) string vacua leading to N=1 space-time supersymmetry in four dimensions. The vacuum structure of the heterotic string [38] in ten dimensions is quite simple and completely understood: the space-time supersymmetric theories come in only two versions with gauge group $E_8 \times E_8$ and $SO(32)$ respectively. This uniqueness is lost if one begins to discuss compactification of superstring theories down to four dimensions. Attempts to relate string theory with elementary particle physics [39,40] would require the knowledge of the vacuum structure. A phenomenologically attractive string compactification is to take a vacuum of the form $M \times K$ where M is the usual four dimensional Minkowski space and K is an internal manifold which is a complex manifold of vanishing first Chern class, the so called Calabi-Yau manifold [3]. Such manifolds are solution of the ten dimensional supergravity equations with unbroken four dimensional supersymmetry. The particular choice of the manifold has a deep influence on the physical predictions of the theory, as the number of generations, their gauge quantum numbers, the long-wave Lagrangian approximation, etc. [40]. There is now a classification of the Calabi-Yau manifolds obtained by complete intersection of polynomials in CP^n [49,50]. However the extraction of quantitative low-energy predictions from the corresponding sigma models is difficult since the metric of the Calabi-Yau is not known. Useful tools in this direction is offered by topological considerations based on the algebraic geometry [3,40,54].

A new approach to this problem comes from the works of Gepner [55,56,57,58] who claims that all the N=2 superconformal field theories with $c = 9$ and space-time supersymmetry correspond to a string propagation on Calabi-Yau manifolds. In particular the string models obtained by gluing N=2 minimal models are solvable Calabi-Yau string theories [56]. Since all the properties of the low-energy four dimensional theories are dictated by the two-dimensional models defining the corresponding compactification, the problem of description of these four dimensional models reduces to that of solving the chosen N=2 superconformal models. In this way, knowing the fusion rules (FR's) and the structure constants of the two-dimensional N=2 operator algebra, one can determine the Yukawa couplings of the massles low-enrgy particles [59,60,58,61]; similarly the N-point functions of the 2-dimensional fields (representing the vertices of these particles) give the four-dimensional scattering amplitudes, etc.

All these issues make interesting the classification of the N=2 unitary superconformal models with $c \leq 9$ and the systematic study of their properties, -FR's, structure constants, 4-point functions,additional symmetries, partition functions, etc. Different elements needed in the explicit construction of the N=2 minimal models can be found in various recent papers [34,67,68,69]. The remaining open problems are mainly related to the appropriate description of the Ramond and the twisted (T) sector, to the computation of the 4-point functions of all the fields of NS, R and T sector, the corresponding structure constants and the FR's.

As in the case of the N=1 superconformal minimal models the null-vector method and its analytic modification [17,18] are useful in the explicit calculations only for the fields at lowest level of degeneracy. Certainly more effective and elegant approach is the Coulomb gas representation developed for the NS sector of the N=2 m.m. by Yu and Zheng [68]. In Chapter 3 we extend this formalism to the other sectors and we success in finding the FR's in all sectors of the theory.

The basic ingredient of the N=2 Coulomb gas representation is a system of two free dimensionless scalar fields $\phi, \overline{\phi}$ and two free fermionic fields $\psi, \overline{\psi}$ (with total central charge $c = 3$). In the NS sector they can combined into two free dimensionless chiral N=2 superfields $S^{\pm}$. Following the analogy with the superstring [15] and orbifolds [37] vertices, the Ramond and twisted primary fields can be represented by vertex operators using the spin $\sigma^{\pm}$ and twisted σ^{T} fields of the $c = 3$ system having dimensions $\Delta = \frac{1}{8}$ and $\Delta = \frac{1}{16}$ respectively. These fields correspond to the lowest energy states in the Ramond and twisted sector of the algebra generated by $\partial\phi, \partial\overline{\phi}$ and $\psi, \overline{\psi}$. The non trivial dynamics of this free field construction of the N=2 m.m. (reflected in the quantized values of the central charge $c \leq 3$) is carried by two background charges $\beta, \overline{\beta}$ placed at infinity and by so called, for this reason, screening operators. By these operators it is possible to express the correlation function in terms of integral representation on the complex plane and to find, by analysing the possible ways to screen the 3-point functions, the FR's.

Despite of the simple and elegant idea of the Coulomb gas representations for the minimal models, often it is difficult to compute explicitly the multi-integral expressions associated with the high-level degenerate fields. In this respect the other approach to the N=2 superconformal m.m. based on the Fateev-Zamolodchikov D_{2n} parafermionic construction [33,34] is more powerfull in comparison with the

Coulomb gas method in the calculation of the 4-point functions and the structure constants of the 2-dimensional operator algebra due the relation of the D_{2n} parafermionic models with the $SU(2)$ Wess-Zumino-Witten models [35,36]. In fact, since the 4-point functions of all the fields of these models have been computed [70,71] the problem of computation of the 4-point functions of the NS and R fields in N=2 m.m. reduces to express these functions in terms of the $SU(2)$ WZW functions and the 4-point functions of the free field vertices $exp(i\alpha\varphi(z))$. The same is true for the structure constants. The twisted fields in this language are realized in terms of the so called C-disorder fields [34] of the parafermionic models, which are not related to the fields of the $SU(2)$ WZW models. Therefore the problem of the computation of the structure constants of the twisted fields using the rather complicated representation for the twisted 4-point functions remain in general open, with exception of few examples which we present in Chapter 3.

Original part of our discussion of N=2 systems is related to the origin of the Z_{p+2} discrete symmetry of N=2 m.m. parametrized by $p = 1, 2, 3\cdot$ [60,72]. This symmetry was discovered by Gepner [56] analysing the characters of the irreducible representations. It turns out that in each c_p model of the N=2 unitary series there exists a set of NS superfields N^p_{-m} which together with the super-stress energy tensor $\mathcal{W}(z, \theta^+, \theta^-)$ close an operatorial algebra of the Z_{p+2} parafermionic type. The spectrum of these theories has the important feature to have Ramond as well as Neveu-Schwarz order parameters corresponding to the different choices of boundary conditions for the supercurrents $G^{\pm}(z)$ of the super-stress energy tensor $\mathcal{W}$. The subject is still in its infancy and in Chapter 3 we present our attempt in the understanding of these kind of discrete symmetry sistems, which play an important role in the compactification of heterotic string. Expecially we analyse the relation between the N=2 superparafermionic models and the corresponding N=2 m.m.

Our study of N=2 superconformal m.m. ends with a section devoted to the local 2-dimensional superconformal models with central charge $c = \frac{3}{2}$ and $c = 3$ in which we point out the additional higher symmetries arising in these theories. The importance of these models is related to the fact that they are limit points in N=1 and N=2 discrete series respectively and there is a suggestion that in this points the conformal algebra can be enlarged, as we have actually discovered.

To summarize, the organization of this work is as follows. The remaining part of this chapter deals with the basic aspects of the conformal field theories and it is useful to fix the notation. Chapter 2 is devoted to the N=1 superconformal formalism: we skip the NS sector (for a general discussion we refer to [16] and the papers listed under it) and we concentrate on the problem of the description of the R sector, which is original part. Using our construction of the R vertices and analysing the singularities of the correlators we carry out explicit computations of the correlation functions and we extract the structure constants of the theory.

In chapter 3 we present a complete study of the N=2 m.m. with central charge $c \leq 3$ we compute many correlators and the FR's. The most important original result is the discovery of the Z_{p+2} symmetry present in the N=2 m.m.

Finally, in the chapter 4 we face the problem of compactification of heteroric string. After a general discussion on the subject, by using the Gepner's approach we construct the internal compactified space by N=2 m.m.; in particular we analyse the four and three generation case and give the allowed Yukawa couplings.

Finally, Appendix A is devoted to the monodromy properties of the simplest solutions of the Ramond correlators in N=1 superconformal m.m., Appendix B discusses the free fermionic systems and Appendix C deals with a brief introduction to the Calabi-Yau spaces.

1.2 Conformal Field Theory

The importance of conformal invariance in critical phenomena was pointed out firstly by Polyakov [73]. When a physical system is near its critical point the most important parameter is the corralation lenght ξ which is much larger than all other microscopic lenghts. When ξ is infinite the statistical system can be described in terms of massless euclidean quantum field theory since it has lost its only relevant scale, becoming globally scale invariant. Under a dilatation of the unit of lenght

$$a \to \lambda a$$

the fields transform as

$$\phi_{\Delta_i} \to \lambda^{\Delta_i}\phi_{\Delta_i}$$

where Δ_i are called the anomalous dimensions and their computation is the most important problem of the theory since these quantities determine the critical exponents. The bootstrap program, as formulated by Polyakov, consists in solving the theory using two hypothesis: conformal invariance and operator algebra. The reader interested to this point can find a detailed discussion in the book of Patashinski and Pokrowski [74] and in the recent report of Cardy [75].

In this section we reveiw some basic results of conformal quantum field theory, whose range of application is both in string theory and critical phenomena. In string theory the conformal invariance assures the decoupling of negative norm states and fixes the critical dimension in which string lives.

The operator which implements the infinitesimal conformal mapping

$$z \to w(z) = z + \epsilon(z) \tag{1.1}$$

is

$$T_\epsilon = \oint \frac{dz}{2\pi i}\epsilon(z)T(z) \tag{1.2}$$

where $T(z)$ is the stress-energy tensor of the theory [7]. The transformation (1.1) changes the conformal primary fields according to

$$\phi_j(z\overline{z}) \to \left(\frac{dw}{dz}\right)^{\Delta_i}\left(\frac{d\overline{w}}{d\overline{z}}\right)^{\overline{\Delta}_i}\phi_j(w,\overline{w}) \tag{1.3}$$

Since

$$\delta_\epsilon\phi_j = [T_\epsilon,\phi_j] = \oint\frac{dz}{2\pi i}\epsilon(z) < T(z)\phi_j(w) > \tag{1.4}$$

the transformation (1.3) is equivalent to the following operator product expansion (OPE)

$$T(z)\phi_j(w) = \left(\frac{\Delta_i}{(z-w)^2} + \frac{1}{z-w}\partial\right)\phi_j(w) \tag{1.5}$$

To recover the algebra of the conformal group we must consider the conformal properties of $T(z)$. This is a field with dimension 2: from euclidean invariance, i.e. translation invariance and correct scaling limit, one finds the following OPE

$$T(z)T(w) = \frac{c}{2(z-w)^4} + \frac{2}{(z-w)^2}T(w) + \frac{1}{z-w}\partial T(w) \tag{1.6}$$

Inserting eq.(1.6) into eq.(1.4) we find

$$\delta T = \epsilon(z)T' + 2\epsilon' T + \frac{1}{12}c\epsilon''' \tag{1.7}$$

The last term is the Schwinger term that one has to insert for having a consistent quantum field theory. The finite transformation of $T(z)$ is

$$T(z) \to (\frac{dw}{dz})^2 T(w) + \frac{1}{12}c\{w,z\} \tag{1.8}$$

where $\{w,z\}$ is the Schwartz derivative

$$\{w,z\} = (\frac{d^3w}{dz^3}/\frac{dw}{dz}) - \frac{3}{2}(\frac{d^2w}{dz^2}/\frac{dw}{dz})^2 \tag{1.9}$$

Note that the Schwartz derivative is zero only in the case of Moebius transformation, generating the group $SL(2,C)$

$$z \to \frac{\alpha z+\beta}{\gamma z+\delta} \qquad \alpha\delta - \beta\gamma = 1 \tag{1.10}$$

The parameter c is a real positive number called central charge. It corresponds to an anomaly of the theory, in the following sense. By translation invariance, the expectation value of T on the complex plane is zero, $< T(z) >= 0$. Under the transformation (1.1) we have

$$T \to T + \delta T$$

where

$$\delta T = \oint \frac{dz}{2\pi i}\epsilon(z) < T(z)T(\xi) >= \frac{c}{12}\epsilon'''(\xi) \tag{1.11}$$

and then for transformations not belonging to the global conformal mapping, eq.(1.10), we have a term which breaks the condition $< T(z) >= 0$, i.e. there is an anomaly for the diffeomorphism. It is possible to give an interesting interpretation of the central charge in statistical mechanics considering a conformal map of the plane into a strip of width L with periodic boundary condition (a cylinder)[76]

$$z = e^{\frac{2\pi i w}{L}} \tag{1.12}$$

Using the transformation law (1.11) we get

$$< T(w) >_{cyl} = (\frac{2\pi}{L})^2 \frac{c}{24} \tag{1.13}$$

i.e. there is a density of energy related with the finite geometry of the system (Casimir effect).

A simple example of conformal theory is given by a free bosonic field with correlation function

$$< \phi(z)\phi(o) > = -lnz \tag{1.14}$$

and stress-energy tensor

$$T(z) = -\frac{1}{2} : (\partial\phi)^2 : \tag{1.15}$$

The central charge, computing the OPE of $T(z)$ with itself, is $c = 1$. Although the field $\phi(z)$ has not well defined scaling dimension, the operators $\partial\phi$ and $exp(ip\phi)$ are well objects with scaling dimensions 1 and $p^2/2$. The easiest way to compute these dimensions is to consider the expression of the two-point function, for example

$$< e^{ip\phi(z)} e^{-ip\phi(w)} > = \frac{1}{(z-w)^{p^2}}$$

Using eq.(1.6) we recover the algebra of conformal transformations

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ L_n = & \oint \frac{dz}{2\pi i} z^{n+1} T(z) \end{aligned} \tag{1.16}$$

and from eq.(1.5) the commutator with the conformal field

$$[L_n, \phi_j] = (z^{n+1}\partial + \Delta_j(n+1)z^n)\phi_j \tag{1.17}$$

The theory of the representations of the infinite conformal algebra can be carried on in a way that resembles the theory of the representations of the compact Lie algebra [77]. So, as easily seen from (1.17) the lowering operators for L_o are the L_n, $n > 0$. An eigenvector of L_0 annihilated by all the lowering operators is called *highest weight vector* (HWV). Explicitly, it satisfies

$$\begin{aligned} L_0 \mid \Delta > &= \Delta \mid \Delta > \\ L_n \mid \Delta > &= 0 \qquad\qquad n > 0 \end{aligned} \tag{1.18}$$

The vacuum $\mid 0 >$ is a HWV because it has the lowest eigenvalue for the 'Hamiltonian' L_0. There is a one-to-one correspondence between the conformal primary fields and the HWV. Namely, the state

$$\mid \Delta_n > = \phi_{\Delta_n}(0) \mid 0 > \tag{1.19}$$

is a HWV with eigenvalue Δ_n as it can be seen using eq.(1.17). The space of states is a sum of irreducible representations of the algebra of L_n each generated from one of the HWV's. A representation space of the Virasoro algebra is built from a HWV by applying the raising operators, i.e. L_{-n} $n \geq 1$. A state is said to be in the $k-th$ level if its L_0 eigenvalue is $\Delta + k$ and the $k-th$ level is spanned by the vectors

$$L_{-j_1} \cdots L_{-j_m} \qquad \phi_\Delta(0) \mid 0 > \tag{1.20}$$
$$j_1 \geq j_2 \geq \cdots j_m \qquad\qquad \sum_{i=1}^{m} j_i = k$$

There are P(n) such states, where P(n) is the number of ways of writing an integer n as a sum of positive integers. These higher level states correspond to operators of higher scaling dimension obtained by the OPE of stress-energy tensor with the primary field ϕ_{Δ_n}: they are called descendant fields [7]. These fields, together with the primary fields ϕ_{Δ_n} constitute a conformal family $[\phi_{\Delta_n}]$. A distinguished feature of a conformal family is that under conformal transformations every member of the family is mapped into a representative of the same conformal family, so each conformal family is a representation of the conformal algebra.

When $c < 1$ the structure provided by the conformal algebra and unitarity completely determine the theory [7,8], i.e. both the allowed representations and the correlation functions. In fact, there is an infinite series of models characterized by the quantized values of c

$$c = 1 - \frac{6}{m(m+1)} \qquad\qquad m = 3, 4, \cdots \tag{1.21}$$

and by the following anomalous dimensions

$$\Delta_{p,q} = \frac{[p(m+1) - qm]^2 - 1}{4m(m+1)} = \Delta_{m-p,m+1-q} \tag{1.22}$$
$$1 \leq p \leq m \quad ; \qquad 1 \leq q \leq m-1$$

These anomalous dimensions are the zeros of the Kac determinant [78] given by the inner products of the states at the various level. This implies that the conformal field $\phi_{p,q}$ with dimension $\Delta_{p,q}$ generates a null-state at level $p \times q$ of the form

$$\sum_{\sum_i \lambda_i = pq} a_{\lambda_1 \cdots \lambda_n} L_{-1}^{\lambda_1} L_{-2}^{\lambda_2} \cdots \mid \Delta_{p,q} >$$

where $a_{\lambda_i \cdots}$ are constants which are determined by the condition that the null-state is orthogonal to all the other states at that level. Inserting this state into correlation functions and using the commutation relation (1.17) one can show that the correlators of the field $\phi_{p,q}$ satisfy linear differential equation of order $p \times q$ [7]. At the lowest order the solutions are the hypergeometric functions.

The finite set of fields identified by their anomalous dimension (1.22) forms a closed operatorial algebra [7]. In fact by the characteristic equation associated to the differential equations, we get the following composition laws (the so called *Fusion Rules* (FR))

$$[\phi_{p_1,q_1}][\phi_{p_2,q_2}] \sim \sum_{k_1=|p_1-q_1|+1}^{p_1+q_1-1} \sum_{k_2=|p_2-q_2|+1}^{p_2+q_2-1} c_{(pqk)} [\phi_{k_1,k_2}] \tag{1.23}$$

where $c_{(pqk)}$ are the only nonzero operator product coefficients (structure constants) and the coordinate dependence is determined by dimensional arguments. The constant $c_{(pqk)}$ of the primary fields coincides with the one which appears in the 3-point functions

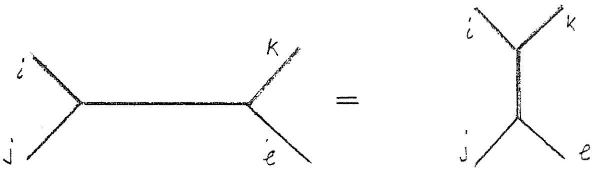
$$< \phi_p \phi_q \phi_k >$$

and those of the descendant fields are determined by recursive equations. Eq.(1.23) is essentially a Clebsh-Gordan series for the decomposition of two representations of the conformal algebra with HWV ϕ_p and ϕ_q.

It is important to mention that there exists an interesting Coulomb gas representation for these models [10] which gives an integral expression of the multipoint correlation functions and allows the computation of the structure constants of the OPE algebra. This method can be extended successfully to the superconformal minimal models with N=1 supersymmetry [13,17,18] as we will discuss in the next chapter, and to the minimal models coming from N=2 superconformal invariance [68,60,72] which will be the subject of the chapter 3.

It is rather peculiar that a conformal field theory does not need a Lagrangian for being specified but it is sufficient its spectrum of anomalous dimensions Δ_i and the structure constants of the OPE. In the case of minimal models only the first one ($c = \frac{1}{2}$), corresponding to Ising model, has a known Lagrangian in terms of free Majorana fermions. The others are defined by their operatorial algebra. The associativity of the operatorial algebra is an equivalent condition of the duality of the 4-point function, i.e.

$$< \phi_i \phi_j \phi_k \phi_l >=$$



which gives quadratic equations for $c_{(lpk)}$ involving the dimensions Δ_p. This is the bootstrap equation [7,73].

Actually the spectrum of 2-D conformal field theory is obtained combining the analytic part of the theory with the antianalytic one. One needs a principle to select the right combination. By a conformal transformation we can map all the complex plane into a finite width strip and the quantum field theory can be analized with the methods of transfer matrix along the strip and the finite size scaling method [75,79]. Moreover Cardy [80] has proposed to consider the system restricted to a strip with periodic boundary condition described by a torus on the complex plane with modulo parameter τ. A non trivial constraint on the spectrum and its degeneracy arises if one imposes the modular invariance of the partition function. This is given by

$$Z(\tau) = \sum_{pq\overline{pq}} M_{pq,\overline{pq}} \chi_{pq}(\tau) \chi^*_{\overline{pq}}(\tau) \tag{1.24}$$

where $\chi_{pq}(\tau)$ is the character of the (p,q) representation and $N_{pq,\overline{pq}}$ are integer numbers giving the molteplicity of the states. The two generators of the modular group are

$$S : \tau \to -\frac{1}{\tau} \qquad\qquad T : \tau \to \tau + 1 \tag{1.25}$$

under which the characters transform non trivially.

After, Gepner [81] and Cappelli et al. [82] solve the problem to construct modular invariant partition functions for the minimal models. With the same method it is also possible to classify the superconformal models [83,84] and the parafermionic system [67].

Chapter 2

N=1 Superconformal Symmetry

2.1 Introduction

In this chapter we analyse one of the most interesting extensions of the Virasoro algebra, namely the N=1 superconformal algebra. Supersymmetry, first invented in string theory [19,20], gives remarkable relations between the bosonic and fermionic sectors of quantum system and it is very attractive that there exist supersymmetric statistical models. In fact the first member of the N=1 supersymmetric discrete series, having a central charge equal to $c = \frac{7}{10}$, corresponds to the universality class of tricritical Ising model [14], which describes phenomena like superconductor mixture of He_3 and He_4 [21] or the adsorbed phase of the He on Kripton surface [22]. These are the first examples of experimentally observed supersymmetric critical systems.

Interesting features are also present in the second model of the superconformal series, with $c = 1$. There are some indications for suspecting that the $c = 1$ supersymmetric model is a gaussian one [23,24,25,26,27,28]. First, the gaussian model and its equivalent (Ashkin-Teller [29] and the 8-vertex model [28,30]) are the only known models with $c = 1$. Second, the presence of a field with anomalous dimension 1 resembles the U(1) current of the gaussian model.

This U(1) supercurrent can be combined with the super-stress energy tensor to produce a field theory with N=2 superconformal symmetry [31,32,68,90]. Then the field content of this model can be conveniently organized in N=2 supermultiplet and it will be analysed in the next chapter.

2.2 N=1 Algebra

The infinite N=1 superconformal algebra in 2-D splits into a direct sum of two algebras: a "left" one generated by the stress-energy tensor T(z) ($\Delta = 2$) and its fermionic superpartner G(z) ($\Delta = \frac{3}{2}$) [11,13]:

$$T(z_1)T(z_2) = \frac{c}{2z_{12}^4} + \frac{2T(z_2)}{z_{12}^2} + \frac{1}{z_{12}}\partial T(z_2) + \cdots \tag{2.1}$$

$$T(z_1)G(z_2) = \frac{3}{2z_{12}^2}G(z_2) + \frac{1}{z_{12}}\partial G(z_2) + \cdots \quad (2.2)$$

$$G(z_1)G(z_2) = \frac{2c}{3z_{12}^3} + \frac{2}{z_{12}}T(z_2) + \cdots \quad (2.3)$$

and a "right" one defined by the corresponding singular terms in the OPE of $\overline{T}(\overline{z})$ and $\overline{G}(\overline{z})$. We shall restrict our discussion to the properties of "1-dimensional models", leaving the 2-D construction to Sec.2.6.

The two possible boundary conditions for the supercurrent: $G(e^{2\pi i}z) = \pm G(z)$ imply two different Laurent mode expansions

$$G^{(p)}(z) = \sum_{n\in Z} \frac{G^{(p)}_{n+1/2}}{z^{n+2}} \qquad G^{(a)}(z) = \sum_{n\in Z} \frac{G^{(a)}_{n}}{z^{n+3/2}}$$

while the stress-energy tensor has the usual mode expansion

$$T(z) = \sum_{n\in Z} \frac{L_n}{z^{n+2}}$$

Then the OPE's (2.1-3) take the well-known form of the NS and R algebras

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \quad (2.4)$$

$$[L_n, G_\alpha] = (\frac{n}{2} - \alpha)G_{n+\alpha} \quad (2.5)$$

$$\{G_\alpha, G_\beta\} = 2L_{\alpha+\beta} + \frac{c}{3}(\alpha^2 - \frac{1}{4})\delta_{\alpha+\beta,0} \quad (2.6)$$

($\alpha, \beta \in Z + 1/2$ for the NS sector and $\alpha, \beta \in Z$ for the R sector). Following Friedan et al. analysis [11] of the highest weight representations of the superalgebra, (2.4-6) we define the corresponding primary states $\mid \Delta >$ requiring the conditions

$$L_n \mid \Delta > = G_\alpha \mid \Delta > = 0 \qquad (n, \alpha > 0) \quad (2.7)$$
$$L_0 \mid \Delta > = \Delta \mid \Delta >$$

(for each fixed value of the central charge c). Since in the Ramond case we have in addition the following G(z)-zero mode properties

$$[G_0, L_0] = 0 \qquad G_0^2 = L_0 - \frac{c}{24}$$

the lowest energy Ramond state is doubly degenerate (for $\Delta \neq \frac{c}{24}$), i.e. both states $\mid \Delta, \pm >$

$$G_0 \mid \Delta, + > = \mid \Delta, - >, \qquad G_0 \mid \Delta, - > = (\Delta - \frac{c}{24}) \mid \Delta, + > \quad (2.8)$$

correspond to the same eigenvalue Δ of L_0. Introducing the invariant vacuum state $\mid 0>$

$$L_n \mid 0 >= G_\alpha \mid 0 >= 0 \qquad n \geq -1, \alpha \geq -\frac{1}{2}$$

(which belongs to the NS sector [11]) we can represent each NS state $\mid \Delta >$ in terms of the NS primary fields $\phi_\Delta(z,\theta) = \varphi_\Delta(z) + \theta\psi_{\Delta+1/2}(z)$

$$\mid \Delta >= \varphi(0) \mid 0 > \qquad G_{-\frac{1}{2}} \mid \Delta >= \psi(0) \mid 0 >$$

These algebraic properties of the primary fields in the OPE language have the form (Ward identity)[11,13]

$$T(z_1)\varphi(z_2) = \frac{\Delta}{z_{12}^2}\varphi(z_2) + \frac{1}{z_{12}}\partial\varphi(z_2) + \cdots \tag{2.9}$$

$$G(z_1)\varphi(z_2) = \frac{1}{z_{12}}\psi(z_2) + \cdots \tag{2.10}$$

$$G(z_1)\psi(z_2) = \frac{2\Delta}{z_{12}^2}\varphi(z_2) + \frac{1}{z_{12}}\partial\varphi(z_2) + \cdots \tag{2.11}$$

Using the mode expansions of T(z) and G(z) we obtain the following "discrete" form of the Ward identities

$$[L_n, \phi(z,\theta)] = (z^{n+1}\partial + (n+1)z^n(\Delta + \frac{1}{2}\theta\frac{\partial}{\partial\theta}))\phi(z,\theta) \tag{2.12}$$

$$[\epsilon G_{n+\frac{1}{2}}, \phi(z,\theta)] = \epsilon z^n(z(\frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial z}) - 2\Delta(n+1)\theta)\phi(z,\theta) \tag{2.13}$$

(ϵ is a Grassmann parameter).

The R primary states can be created from the NS vacuum $\mid 0 >$ by the so called spin fields $R^{\pm}_\Delta(z)$[11]

$$\mid \Delta, \pm >= R^{\pm}_\Delta(0) \mid 0 >$$

These have the following OPE with G(z):

$$G(z_1)R^{\pm}_\Delta(z_2) = \frac{a^{\pm}(\Delta)}{z_{12}^{3/2}}R^{\mp}_\Delta(z_2) + ... \tag{2.14}$$

where

$$a^+(\Delta) = 1, \qquad a^-(\Delta) = (\Delta - \frac{c}{24})$$

and eq.(2.9) for the OPE $T(z_1)R^{\pm}(z_2)$. Therefore the main difference between R and NS fields is that they produce different analytic behaviour of G(z), originated by its Z_2 boundary condition

$$G(e^{2\pi i}z)S_k(0) = e^{ik\pi}G(z)S_k(0)$$

Thus the fields from the NS sector, Z_2 even, (k=0, $S_0 = \phi(z,\theta)$) and R sector, Z_2 odd, (k=1,$S_1 = R^{\pm}$) can be labelled by the elements of the simplest discrete group $Z_2 = \{\pm 1\}$.

An important element in the construction of local 2-D minimal models is the Grassmann parity of the fields. For example we have $\Gamma_{R^\pm} = \pm 1$ (since $\Gamma_{G_0} = -1$) and the usual fermionic parity for the components of the NS superfields. Neglecting these parity properties we shall also use the diagonal basis for G_0 (non-diagonal for the parity operator Γ). On this basis we have

$$G(z_1)\tilde{R}_\Delta(z_2) = \mp \frac{1}{z_{12}^{3/2}} \sqrt{\Delta - \frac{c}{24}} \tilde{R}_\Delta(z_2) + \cdots \tag{2.15}$$

with $\Delta \neq \frac{c}{24}$ and

$$\tilde{R}_\Delta = \sqrt{\Delta - \frac{c}{24}} R^+ \pm R^-$$

2.3 The modified null vector's method

As it is well known [11,13], the reducible unitary representations of the N=1 superconformal algebras given by

$$c = \frac{3}{2} - \frac{12}{p(p+2)} \qquad p = 3, 4, \ldots \tag{2.16}$$

$$\Delta_{n,m} = \frac{[(p+2)n - mp]^2 - 4}{8p(p+2)} + \frac{1}{32}[1 - (-1)^{n-m}] \tag{2.17}$$

($n - m \in 2Z$ for the NS sector and $n - m \in 2Z + 1$ for the R sector) determine an infinite series of exactly solvable minimal models. The basic property of the superconformal families $[\phi_{\Delta_{n,m}}]$ is that at level $\frac{1}{2}nm$ (and $\frac{1}{2}(p-n)(p+2-m)$) there exist descendent fields $\phi_{\Delta_{n,m} + \frac{1}{2}nm}$ which are again primary fields. Then the covariant condition

$$\phi_{\Delta_{n,m} + \frac{1}{2}nm} = 0 \tag{2.18}$$

separates the irreducible part of the families $[\phi_{\Delta_{n,m}}]$. In the NS sector these null vector's conditions, together with the Ward identities (2.13-14) lead to differential equations for the n-point functions of the fields $\phi_{\Delta_{n,m}}$. These equations allow one to find explicity the corresponding fusion rules [13,14,86] and the 4-point functions [14].

The difficulties in the application of this method to the Ramond fields come from the branch-cut singularity in the OPE (2.14). In this section we describe a modification of the null-vector's method based on the specific analytic properties of the Ramond fields.

In order to find the fusion rules for the first level degenerated R fields (i.e. nm=2) we take the first level null vector

$$| \chi^{\pm}_{\tilde{\Delta}} >= \left(L_{-1} - \frac{4\tilde{\Delta}}{3(\tilde{\Delta} - \frac{c}{24})} G_{-1} G_0 \right) R^{\pm}_{\tilde{\Delta}}(0) \mid 0 > \tag{2.19}$$

and we have to analyse the solutions of the following equation

$$< 0 \mid N_{\Delta_x}(z_1) R^{\pm}_{\tilde{\Delta}}(z_2) \left(L_{-1} - \frac{4\tilde{\Delta}}{3(\tilde{\Delta} - \frac{c}{24})} G_{-1} G_0 \right) R^{\pm}_{\tilde{\Delta}}(0) \mid 0 >= 0 \tag{2.20}$$

The Γ parity properties of the fields imply that only the bosonic components of the NS superfields contribute to the OPE of two R fields with equal parities.

The first term in eq. (2.20) can be written in the form

$$< 0 \mid N_{\Delta_x}(z_1)R^{\pm}_{\tilde{\Delta}}(z_2)L_{-1} \mid \tilde{\Delta}^{\pm} >= -(\partial_1+\partial_2) < 0 \mid N_{\Delta_x}(z_1)R^{\pm}_{\tilde{\Delta}}(z_2) \mid \tilde{\Delta}^{\pm} > \quad (2.21)$$

To obtain the explicit expression for the second term in eq. (2.20) we consider the following auxiliary function

$$F^{(I)}(v \mid z_1, z_2) = \sqrt{v(z_2-v)}(z_1-v) < 0 \mid N_{\Delta_x}(z_1)R^{\pm}_{\tilde{\Delta}}(z_2)G(v) \mid \tilde{\Delta}^{\pm} > \quad (2.22)$$

The Ward identities (2.14) and the asymptotic behaviour of the fields allow us to express this function only in terms of the 3-point functions

$$f^{\pm} =< 0 \mid N_{\Delta_x}(z_1)R^{\pm}_{\tilde{\Delta}}(z_2) \mid \tilde{\Delta}^{\pm} >$$

i.e.

$$F^{(I)}(v \mid z_1, z_2) = \frac{A(z_1,z_2)}{z_2-v} + \frac{B(z_1,z_2)}{v} \quad (2.23)$$

where

$$\begin{aligned} A &= \oint_{z_2} F dv = \sqrt{z_2}(z_1-z_2)f^{\mp}a^{\pm}(\Delta) \\ B &= \oint_0 F dv = \sqrt{z_2}z_1 f^{\pm}a^{\mp}(\Delta) \end{aligned}$$

Using the OPE (2.14) in the form

$$G(v) \mid \tilde{\Delta}^{\pm} >= \left(\frac{G_0}{v^{\frac{3}{2}}} + \frac{G_{-1}}{v^{\frac{1}{2}}} + \cdots\right) \mid \tilde{\Delta}^{\pm} > \quad (2.24)$$

and eq.(2.23), we can calculate the residuum of the function $F^{(I)}(v \mid z_1, z_2)$ in two different ways:

$$\oint_0 F\frac{dv}{v} = \frac{A(z_1,z_2)}{z_2} \quad (2.25)$$

and

$$\oint_0 F\frac{dv}{v} = -\frac{z_1+2z_2}{2\sqrt{z_2}a^{\mp}}(\tilde{\Delta})f^{\pm}(z_1,z_2) + \sqrt{z_2}z_1 < 0 \mid N_{\Delta_x}(z_1)R^{\pm}(z_2)G_{-1} \mid \tilde{\Delta} > \quad (2.26)$$

Finally, eq.(2.20) takes the form

$$(\partial_1+\partial_2)f^{\pm}(z_1,z_2) = \frac{4\tilde{\Delta}}{3(\tilde{\Delta}-\frac{c}{24})}a^{\pm}(\tilde{\Delta})[\frac{z_1-z_2}{z_1z_2}a^{\pm}(\Delta)f^{\mp} + \frac{z_1+2z_2}{2z_1z_2}a^{\mp}(\tilde{\Delta})f^{\pm}] \quad (2.27)$$

Taking into account the explicit conformal invariant form of the 3-point function $f^{\pm}$

$$f^{\pm}(z_1,z_2) = h^{\pm}(z_1-z_2)^{\tilde{\Delta}-\Delta_1-\Delta_x}(z_1)^{\Delta-\Delta_x-\tilde{\Delta}}(z_2)^{\Delta_x-\Delta-\tilde{\Delta}}$$

and substituting it in eq. (2.27), we obtain the following system of algebraic equations

$$\begin{aligned} (\Delta - \Delta_x + \tfrac{1}{3}\tilde{\Delta})h^+ - \tfrac{4\tilde{\Delta}}{3(\tilde{\Delta}-\frac{c}{24})}h^- &= 0 \\ \tfrac{4}{3}\tilde{\Delta}(\Delta - \tfrac{c}{24})h^+ - (\Delta - \Delta_x + \tfrac{1}{3}\tilde{\Delta})h^- &= 0 \end{aligned}$$

The consistence condition of this system gives us the unknown dimensions $\Delta_x^\pm$ that we looked for

$$\Delta_x^\pm = \Delta + \frac{\tilde{\Delta}}{3} \mp \frac{4}{3}\tilde{\Delta}\sqrt{\frac{\Delta - \frac{c}{24}}{\tilde{\Delta} - \frac{c}{24}}} \tag{2.28}$$

In the first model of the series, which describes Tricritical Ising Model (TIM $(c = \frac{7}{10})$ [8,11,14,16] the first level degenerated R fields have dimensions $\Delta_{1,2} = \frac{3}{80}$, $\Delta_{2,1} = \frac{7}{16}$. Then according to (2.28) we get the following fusion rules

$$\begin{aligned} (\tfrac{3}{80})_R^\pm(\tfrac{3}{80})_R^\pm &= [0]_{NS}^1 + [\frac{1}{10}]_{NS}^1 \\ (\tfrac{3}{80})_R^\pm(\tfrac{7}{16})_R^\pm &= [\frac{1}{10}]_{NS}^1 + [\frac{4}{5}]_{NS}^1 \\ (\tfrac{7}{16})_R^\pm(\tfrac{3}{80})_R^\pm &= [\frac{1}{10}]_{NS}^1 + [\frac{4}{15}]_{NS}^1 \\ (\tfrac{7}{16})_R^\pm(\tfrac{7}{16})_R^\pm &= [0]_{NS}^1 + [\frac{7}{6}]_{NS}^1 \end{aligned} \tag{2.29}$$

where $[\Delta]^1$ denotes the family of the first component of the NS superfield. In order to obtain the true fusion rules we have to take the intersection of the two OPE relations $(\frac{3}{80})(\frac{7}{16})$ and $(\frac{7}{16})(\frac{3}{80})$ that is

$$(\frac{3}{80})_R^\pm(\frac{7}{16})_R^\pm = (\frac{7}{16})_R^\pm(\frac{3}{80})_R^\pm = [\frac{1}{10}]_{NS}^1$$

The restricted fusion rule

$$(\frac{7}{16})_R^\pm(\frac{7}{16})_R^\pm = [0]_{NS}^1$$

is a consequence of the intersection of the corresponding fusion rules given above and those coming from the analysis of the equation generated by the second level null vector

$$\begin{aligned} \mid \chi_{\tilde{\Delta}_{1,4}}^\pm >= \{\frac{1}{15}(4\tilde{\Delta} + \frac{11}{4})(\frac{2}{5}\tilde{\Delta} + \frac{c}{4} + \frac{7}{5})L_{-2} - \frac{1}{3}(\frac{2}{5}\tilde{\Delta} + \frac{c}{4} + \frac{7}{5})L_{-1}^2 \\ + \frac{4}{5}(\tilde{\Delta} + 1)G_{-2}G_0 + L_{-1}G_{-1}G_0\} \mid \tilde{\Delta}^\pm > \end{aligned}$$

since $\Delta_{2,1} = \Delta_{1,4} = \frac{7}{16}$ has this degeneracy, too.

In the case $c = 1$, $\Delta_{1,2} = \frac{3}{8}, \Delta_{2,1} = \frac{1}{16}$ we have

$$\begin{aligned} (\tfrac{1}{16})_R^\pm(\tfrac{3}{8})_R^\pm &= [\frac{1}{16}]_{NS}^1 \\ (\tfrac{1}{16})_R^\pm(\tfrac{1}{16})_R^\pm &= [0]_{NS}^1 + [\frac{1}{6}]_{NS}^1 \\ (\tfrac{3}{8})_R^\pm(\tfrac{3}{8})_R^\pm &= [0]_{NS}^1 + [1]_{NS}^1 \end{aligned} \tag{2.30}$$

We shall use the same procedure to calculate the FR's for the Ramond fields of different Γ parity. The parity conservation implies that only the fermionic component of the NS superfields contribute to these FR's. In this case we introduce a new auxiliary function

$$F^{(II)}(v \mid x, z) = \sqrt{v(z-v)}(x-v)^2 < 0 \mid N_{II}(x) R^{\pm}_{\tilde{\Delta}}(z) G(v) \mid \tilde{\Delta}^{\pm} >$$

As before we insert the square-root in order to avoid the branch-cuts of the correlation function when $v \to 0$ and $v \to z$ and the factor $(x-v)^2$ for cancelling the second order pole at $v \to x$:

$$G(v) N_{II}(x) = \frac{2\Delta}{(v-x)^2} N_I(x) + \cdots \tag{2.31}$$

Then $F^{(II)}$ has only a simple pole at $v = z$ and at the origin and it is a constant asymptotically at infinity, i.e.

$$F^{(II)}(v \mid x, z) = \frac{A(x,z)}{z-v} + \frac{B(x,z)}{v} + C(x,z) \tag{2.32}$$

The "coefficients" A,B,C computed by the Cauchy theorem and the Ward identity (2.14),(2.31) have the form

$$\begin{aligned} A(x,z) \; &= a^{\pm}(\Delta)\sqrt{z}(x-z)^2 \tilde{f}^{\pm} \\ B(x,z) \; &= a^{\pm}(\tilde{\Delta})\sqrt{z} x^2 \tilde{f}^{\pm} \\ C(x,z) &= \sqrt{z} x^2 < 0 \mid N_{II}(x) R^{\pm}(z) G_{-1} \mid \tilde{\Delta}^{\pm} > + \\ & -\frac{a^{\pm}(\tilde{\Delta})}{2\sqrt{z}}(x^2 + 4zx)\tilde{f}^{\pm} - \frac{a^{\pm}(\Delta)}{\sqrt{z}}(x-z)^2 \tilde{f}^{\mp} \end{aligned}$$

where

$$\tilde{f}^{\pm} =< 0 \mid N_{II}(x) R^{\pm}_{\tilde{\Delta}}(z) \mid \tilde{\Delta}^{\mp} >= \tilde{h}^{\pm}(x-z)^{\tilde{\Delta}-\Delta-\Delta_x} x^{\Delta-\Delta_x-\tilde{\Delta}}$$

Taking the integral

$$\oint_0 \frac{F^{(II)}}{v^2} dv$$

in two different ways and using the identity

$$G_{-2} \mid \tilde{\Delta}^{\pm} >= \left(\frac{3}{2\tilde{\Delta}} a^{\pm}(\tilde{\Delta}) L_{-1}^2 - \frac{8}{3} \frac{\tilde{\Delta}}{a^{\mp}(\tilde{\Delta})} L_{-2} \right) \mid \tilde{\Delta}^{\mp} > \tag{2.33}$$

(valid only for $\Delta = \Delta_{1,2}$ or $\Delta = \Delta_{2,1}$) we reduce eq.(2.20) to the system of algebraic equations with the following consistency condition

$$\begin{aligned} & \frac{1}{8} + \frac{3}{8\tilde{\Delta}}(\Delta_x - \Delta - \tilde{\Delta}) + \frac{3}{2\tilde{\Delta}}(\Delta_x - \Delta - \tilde{\Delta})(\Delta_x - \Delta - \tilde{\Delta} - 1) + \\ & -\frac{8\tilde{\Delta}}{3\tilde{\Delta} - \frac{c}{24}}(2\Delta - \Delta_x + \tilde{\Delta}) = \pm\sqrt{\frac{\Delta - \frac{c}{24}}{\tilde{\Delta} - \frac{c}{24}}} \end{aligned}$$

The solutions of these equations give the dimensions of the second components of the NS fields in the FR's (2.29),(2.30) together with a few "wrong" dimensions, which can be eliminated considering the null vectors at the higher levels of degeneracy. We omit the discussion of the higher level fusion rules since the Coulomb gas representation developed in the next section allows us to avoid the hard computations in the modified null-vector method described above.

Our experience in the application of this method to the computation of the Ramond 4-point function showed that there exist technical obstructions which make it a non-effective one. The only case in which these null-vector techniques work well is in the calculation of the Ising model's 4-point functions. We present here the explicit computation of these functions since they are important elements in our Ramond Coulomb gas construction (see next sections).

Let us consider the semi-direct sum of the Virasoro algebra and the algebra of the Laurent coefficients ψ_n of the antiperiodic Majorana field

$$\psi(z) = \sum_n \frac{\psi_n}{z^{n+\frac{1}{2}}} \qquad \psi(e^{2\pi i}z) = -\psi(z)$$

$$\begin{aligned} [L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m,0} \\ [L_n, \psi_m] &= -(\frac{n}{2}+m)\psi_{n+m} \\ \{\psi_n, \psi_m\} &= \delta_{n+m,0} \end{aligned} \tag{2.34}$$

which is known [87] as the 'Ramond counterpart' of the full algebra of symmetries of the critical Ising model. Since there are zero-modes, i.e.

$$[L_0, \psi_0] = 0 \qquad \psi_0^2 = \frac{1}{2}$$

the lowest energy state of this algebra, $\mid \sigma^\pm >$ is doubly degenerated and has dimension $\Delta^\pm = \frac{1}{16}$, The 'spin fields' corresponding to these 'Ramond' states

$$\mid \sigma^\pm >= \sigma^\pm(0) \mid 0 > \tag{2.35}$$

produce a branch-cut singularity of the antiperiodic fermionic field

$$\psi(z)\sigma^\pm(w) = \frac{1}{\sqrt{2(z-w)}}\sigma^\mp(w) + \cdots \tag{2.36}$$

We shall use also the diagonal basis

$$\tilde{\sigma} = \frac{\sigma^+ \mp \sigma^-}{\sqrt{2}}$$

with the following OPE

$$\psi(z)\tilde{\sigma}(w) = \mp\frac{1}{\sqrt{2(z-w)}}\tilde{\sigma}(w) + \cdots \tag{2.37}$$

The SL(2,R) invariance allows us to write the 4-point function of the field $\sigma(z)$ in the form

$$\mathcal{F}(z) = \lim_{w\to\infty} w^{\frac{1}{8}} < \sigma(w)\sigma(z)\sigma(1)\sigma(0) >= z^{\frac{1}{8}}(z-1)^{-\frac{1}{8}} f(z)$$

The singular part in the OPE (2.36) and the first level null vector of the Ising algebra

$$(L_{-1} - \frac{1}{2}\psi_{-1}\psi_0) \mid \sigma >= 0$$

(for $c = \frac{1}{2}, \Delta = \frac{1}{16}$) allow us to find a first order differential equation for the unknown function $f(z)$. In order to compute explicity the second term in the equation

$$< \sigma(\infty)\sigma(z)\sigma(1)(L_{-1} - \frac{1}{2}\psi_{-1}\psi_0)\sigma(0) >= 0 \tag{2.38}$$

we introduce the auxiliary 5-point function

$$F(v \mid z) = \sqrt{\frac{2(z-v)}{v(1-v)z}} < \sigma(\infty)\sigma(z)\sigma(1)\psi(v)\sigma(0) > \tag{2.39}$$

The analytic properties and the behaviour at infinity fix its form to be

$$F(v \mid z) = \frac{A(z)}{v} + \frac{B(z)}{1-v}$$

where

$$A(z) = \mathcal{F}(z) \qquad B(z) = \sqrt{1-\frac{1}{z}}\mathcal{F}(z)$$

Using the explicit expression (2.39) of the function $F(v \mid z)$ and the OPE

$$\psi(v) \mid \sigma >= (\frac{\psi_0}{\sqrt{v}} + \sqrt{v}\psi_{-1} + \cdots) \mid \sigma >$$

and taking the integral $\oint_{C_0} dv \frac{F}{v}$ in two different ways we get

$$< \sigma(\infty) \mid \sigma(z)\sigma(1)\psi_{-1} \mid \sigma >= \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{z}}\left(1 - \frac{1}{\sqrt{2}}\sqrt{1-\frac{1}{z}}\right)\mathcal{F}(z)$$

Then eq. (2.38) reduces to the following simple first order differential equation

$$4z\frac{df}{dz} - \frac{1}{\sqrt{1-\frac{1}{z}}}(1 - \sqrt{1-\frac{1}{z}})f(z) = 0$$

and finally we have

$$\mathcal{F}(z) = \frac{1}{\sqrt{2}} z^{\frac{1}{8}}(z-1)^{-\frac{1}{8}}\sqrt{1+\sqrt{1-\frac{1}{z}}} \tag{2.40}$$

The normalization is fixed by the cluster decomposition property [87], the OPE (2.36) and the explicit form of the 2-point function

$$< \sigma(z_1)\sigma(z_2) >= (z_{12})^{-\frac{1}{8}}$$

Repeating the same procedure for the 4-point function

$$\tilde{\mathcal{F}}(z) =< \tilde{\sigma}(\infty)\sigma(z)\tilde{\sigma}(1)\sigma(0) >$$

we obtain a similar result

$$\tilde{\mathcal{F}}(z) = \frac{1}{\sqrt{2}} z^{\frac{1}{8}}(z-1)^{-\frac{1}{8}}\sqrt{1-\sqrt{1-\frac{1}{z}}} \qquad (2.41)$$

This method, together with the OPE

$$\psi(z_1)\psi(z_2) = \frac{1}{z_{12}} + 2z_{12}T(z_2) + \cdots$$

leads to the recursive equation for the following correlation function

$$G^N(z, v_i) =< \sigma(\infty)\sigma(z)\sigma(1)\prod_{i=1}^{N}\psi(v_i)\sigma(0) >=$$

$$\sqrt{\frac{(1-v_1)v_1}{z-v_1}}[\frac{1}{\sqrt{z}}\left(\frac{\sqrt{z-1}}{1-v_1} + (-1)^{N+1}\frac{\sqrt{z}}{v_1}\right) < \sigma(\infty)\sigma(z)\sigma(1)\prod_{i=2}^{N}\psi(v_i)\sigma(0) >$$

$$(2.42)$$

$$+\sum_{k=2}^{N}\frac{(-1)^k}{v_{1k}}\sqrt{\frac{z-v_k}{(1-v_k)v_k}} < \sigma(\infty)\sigma(z)\sigma(1)\prod_{j=2,j\neq k}^{N}\psi(v_j)\sigma(0) >]$$

The solution of this equation can be written in the form of the "Wick-like theorem"

$$G^N(z, v_i) = (\Pi_{i=1}^N f_i(v_i, N-i+1) + \Sigma_{all<ij>}\Pi_{l\neq i,j}^{N-2} f_l g_{ij} +$$

$$+\Sigma_{all<ij>,<kl>}\Pi_{p\neq ijkl}^{N-4} f_p g_{ij} g_{kl} + \cdots)\mathcal{F}(z) \qquad (2.43)$$

where

$$f(v_i, n) = \frac{1}{\sqrt{2}}(\frac{\sqrt{z-1}}{1-v_i} + (-1)^{n+1}\frac{\sqrt{z}}{v_i})\sqrt{\frac{(1-v_i)v_i}{z-v_i}}$$

$$g_{ij} = \frac{(-1)^{i+j-1}}{v_{ij}}\sqrt{\frac{v_i(1-v_i)(z-v_i)}{v_j(z-v_j)(1-v_j)}}$$

As we shall see in the next sections, these functions are an important ingredient in the calculation of the 4-point functions of the R fields.

2.4 Supersymmetric Coulomb gas

The Coulomb gas method [10] seems to be the natural language in the description of the two dimensional minimal models. The generalization of the Coulomb gas to the case of N=1 superconformal models [13] is based on the properties of the free dimensionless superfield

$$S(z,\theta) = \phi(z) + \theta\psi(z)$$

(and $\overline{S}(\overline{z},\overline{\theta})$) with action

$$A(S,\overline{S}) = \frac{2}{\pi}\int dz d\overline{z}(\frac{1}{2}\partial\phi\overline{\partial}\overline{\phi} - \psi\overline{\partial}\overline{\psi}) \tag{2.44}$$

As follows from this expression, the propagator of the field is

$$\begin{aligned} &< 0 \mid S(z_1,\theta_1)S(z_2,\theta_2) \mid 0 >= -\ln\frac{\hat{z}_{12}}{R} \\ &\hat{z}_{12} = z_1 - z_2 - \theta_1\theta_2 \end{aligned} \tag{2.45}$$

(R is an infrared cut-off). We note that eq. (2.45) takes place only when ϕ and ψ are both periodic fields

$$\phi(e^{2\pi i}z) = \phi(z) \qquad \psi(e^{2\pi i}z) = \psi(z)$$

In order to construct the superfiels of the conformal grid we consider the so-called NS vertices [13]

$$\mathcal{V}_{\alpha_j}(z_j,\theta_j) = e^{i\alpha_j S(z_j,\theta_j)} \qquad \alpha_j \in R$$

Since their N-point functions have both infrared and ultraviolet singularities

$$\begin{aligned} \lessdot \prod_{i=1}^{N} \mathcal{V}_{\alpha_j}(z_j,\theta_j) >&= \int \mathcal{D}S\mathcal{D}\overline{S} \prod_{i=1}^{N} e^{i\alpha_i S(z_i,\theta_i)} e^{-A(S,\overline{S})} \\ &= \left(\frac{a}{R}\right)^{(\sum_k \alpha_k)^2} \prod_{i<j=2} \left(\frac{\hat{z}_{ij}}{a}\right)^{\alpha_i\alpha_j} \end{aligned} \tag{2.46}$$

(a is an ultraviolet cut-off), we have to introduce the renormalized vertex

$$V_\alpha(z,\theta) = \lim_{a\to 0} a^{-\frac{\alpha^2}{2}} e^{i\alpha S(z,\theta)} \equiv: e^{i\alpha S(z,\theta)} : \tag{2.47}$$

and to imply the neutrality condition

$$\sum_k \alpha_k = 0$$

In this way we eliminate the cut-off dependence of the N-point functions. Then the vertex propagator takes the form

$$< V_\alpha(z_1,\theta_1)V_{-\alpha}(z_2,\theta_2) >= (\hat{z}_{12})^{-\alpha^2}$$

and consequently the fields $V_{\pm\alpha}$ have dimension $\Delta(\pm\alpha) = \frac{\alpha^2}{2}$ Since from the action we get

$$T = -\frac{1}{2}(:(\partial\phi)^2: - :\psi\partial\psi:) \qquad G = i:\partial\phi\psi: \tag{2.48}$$

all the properties (2.9-2.11) of the NS vertex (2.47) can be verified explicity using the Wick theorem and the free propagator (2.45).Therefore this vertex represents a primary superfield with dimension $\alpha^2/2$. A defect of this construction is that the central charges of the model is $c = \frac{3}{2}$.

The construction which leads to the anomalous central charges

$$c = \frac{3}{2} - \frac{12}{p(p+2)}$$

is generated by the following modified action

$$\mathcal{A}(S, \overline{S}, \hat{R}) = \frac{2}{\pi}\int dz d\overline{z} d\theta d\overline{\theta}(\frac{1}{2}DS\overline{DS} - 2\alpha_0 i\hat{R}(S + \overline{S})) \tag{2.49}$$

where $\hat{R} = -\tau + \theta\chi + \overline{\theta}\overline{\chi} + \theta\overline{\theta}R$ plays a role of supercurvature ($\Delta_{\hat{R}} = 1$) and

$$\alpha_0^2 = \frac{1}{p(p+2)}$$

To see this let take the last component of $\hat{R}$ to be a curvature corresponding to the singular metric

$$g_{zz} = g_{\overline{z}\overline{z}} = 0 \qquad g_{z\overline{z}} = \frac{4}{(1 + z\overline{z})^2}$$

that is

$$\begin{aligned} gR &= -\partial\overline{\partial}\ln g = 4\pi\delta(z - z_\infty)\delta(\overline{z} - \overline{z}_\infty) \\ g^2 &= det g_{ab} = g_{z\overline{z}}^2 \end{aligned}$$

Then, the action (2.49) can be rewritten in the form

$$\mathcal{A}(S, \overline{S}, \hat{R}) = 2\alpha_0 i(S(z_\infty, \theta) + \overline{S}(\overline{z}_\infty, \overline{\theta})) + A(S, \overline{S}) \tag{2.50}$$

Therefore the "super-curvature" term in the eq. (2.49) introduces a vertex $e^{(-2\alpha_0 iS)}$ (with charge $-2\alpha_0$) at infinity in the new partition function

$$Z(-2\alpha_0) = \int \mathcal{D}S\mathcal{D}\overline{S}e^{-\mathcal{A}} \equiv \int \mathcal{D}S \lim_{|z|\to\infty} e^{-2\alpha_0 i(S(z,\theta)+\overline{S}(\overline{z},\overline{\theta}))}e^{-A}\ |_{\theta=\overline{\theta}=0}$$

Thus the correlation functions (2.46), calculated with the new action (2.49),(2.50), remain the same but should satisfy a new neutrality condition

$$\sum_{i=1}^{N}\alpha_i = 2\alpha_0 \tag{2.51}$$

For instance, we obtain that the only non-zero 2-point function is

$$\ll V_\alpha(z_1)V_{2\alpha_0-\alpha}(z_2)\gg=(\hat{z}_{12})^{-\alpha(\alpha-2\alpha_0)} \tag{2.52}$$

and therefore the vertices V_α and $V_{2\alpha_0-\alpha}$ have equal dimensions

$$\Delta(\alpha)=\Delta(2\alpha_0-\alpha)=\frac{\alpha(\alpha-2\alpha_0)}{2} \tag{2.53}$$

The action (2.49) gives a new expression for the super stress-energy tensor

$$T(z) \;=-\frac{1}{2}:((\partial\phi)^2:-:\psi\partial\psi:)+i\alpha_0\partial^2\phi \tag{2.54}$$
$$G(z) \;=i:\partial\phi\psi:+2\alpha_0:\partial\psi: \tag{2.55}$$

and the central charge is a function of the charge at infinity α_0:

$$c=\frac{3}{2}-12\alpha_0^2 \tag{2.56}$$

Thus the different superconformal minimal models are parametrized by their charges at infinity

$$\alpha_0^2=\frac{1}{p(p+2)}$$

The 4-point functions of the NS fields can be constructed in terms of the vertices (2.47) by the following procedure [13]

$$<\prod_{k=1}^{4}\phi_\Delta(z_k,\theta_k)>=\oint_{C_i}\prod_{i=1}^{n-1}d\zeta_i dv_i\oint_{C_j}\prod_{j=1}^{m-1}d\eta_j dw_j \ll V_\alpha(z_1,\theta_1)$$
$$V_\alpha(z_2,\theta_2)V_{2\alpha_0-\alpha}(z_3,\theta_3)V_\alpha(z_4,\theta_4)\prod_{i=1}^{n-1}V_{\alpha_+}(v_i,\zeta_i)\prod_{j=1}^{m-1}V_{\alpha_-}(w_j,\eta_j)\gg \tag{2.57}$$

The superinvariant dimensionless screening operators

$$J_\pm=\oint_{C_\pm}d\theta dz V_{\alpha_\pm}(z,\theta)\sim\oint_{C_\pm}dz\psi(z)e^{i\alpha\phi(z)} \tag{2.58}$$

with charges and dimensions

$$\alpha_\pm=\alpha_0\pm\sqrt{\alpha_0^2+1} \qquad \Delta(\alpha_\pm)=\frac{\alpha_\pm(\alpha_\pm-2\alpha_0)}{2}=\frac{1}{2} \tag{2.59}$$

are introduced in the eq. (2.57) in order to screen the extra charge 2α. They generate non-trivial solutions of the neutrality condition (2.51)

$$\alpha_{n,m}=\frac{[(1-n)\alpha_++(1-m)\alpha_-]}{2} \tag{2.60}$$

This quantization of the charges of the superfields leads to the well-known quantization of the dimension of the minimal models

$$\Delta_{n,m} = \frac{(n\alpha_+ + m\alpha_-)^2 - (\alpha_- + \alpha_+)^2}{8} \tag{2.61}$$

which exactly coincides with the Kac's formula (2.17) if $(n-m) \in 2Z$.

The Ramond fields of the minimal models should have the same stress-energy tensor T and supercurrent G as the NS fields. The only difference is that in this case G(z) has to be an antiperiodic field and therefore we have to impose antiperiodic boundary conditions on the free Majorana field $\psi(z)$. Then the fields $\phi(z)$ and $\psi(z)$ cannot be combined in a superfield multiplet.

The "spin fields" σ and $\tilde{\sigma}$, (corresponding to the lowest energy state of $\psi(z)$) play an important role in the construction of the Ramond vertices. Following the analogy with the Ramond sector of the superstring models [15,88], we define these vertices as follows [17]

$$\tilde{R}_\alpha(z) = \tilde{\sigma}(z) : e^{i\alpha\phi(z)} :\equiv \tilde{\sigma}(z) \lim_{a\to 0} a^{-\frac{\alpha^2}{2}} e^{i\alpha\phi(z)} \tag{2.62}$$

The direct inspection, based on the expression of the super stress energy tensor (2.54),(2.55) eq.(2.37) and on the Wick theorem for free fields, shows that the Ramond vertices satisfy the eqs.(2.9),(2.14) and their dimensions are

$$\Delta_R(\alpha) = \frac{1}{16} + \frac{\alpha(\alpha - 2\alpha_0)}{2} \tag{2.63}$$

In fact, since we have

$$G(z_1)R_\alpha(z_2) = \frac{\alpha - \alpha_0}{\sqrt{2}(z_1 - z_2)^{\frac{3}{2}}} R_\alpha(z_2) + \cdots$$

a simple algebra gives

$$\frac{\alpha - \alpha_0}{\sqrt{2}} = \pm\sqrt{\Delta_R(\alpha) - \frac{c}{24}}$$

Therefore the vertices (2.62) form a correct representation of the Ramond algebra.

Accepting that the screening operators $J_\pm$ are the same as for the NS sector

$$J_\pm = \oint_{C_\pm} dz \psi(z) e^{i\alpha_\pm \phi(z)} \equiv \oint_{C_\pm} R_\pm(z) dz$$

(with antiperiodic $\psi(z)$) we can construct the correlation functions of four Ramond fields $R_{\Delta_{n,m}}$, modifing the NS average procedure (2.57):

$$< \prod_{k=1}^{4} R_\Delta(z_k) >= \oint_{C_i} \prod_{i=1}^{n-1} dv_i \oint_{C_j} \prod_{j=1}^{m-1} dw_j \ll R_\alpha(z_1)$$

$$R_\alpha(z_2,)R_{2\alpha_0-\alpha}(z_3)R_\alpha(z_4) \prod_{i=1}^{n-1} R_{\alpha_-}(v_i) \prod_{j=1}^{m-1} R_{\alpha_+}(w_j) \gg\equiv$$

$$\equiv \{\alpha, \alpha, 2\alpha_0 - \alpha, \alpha; (n-1)\alpha_-; (m-1)\alpha_+\} \tag{2.64}$$

Since the neutrality condition implies that one has again the same charges quantization (2.60) the dimensions (2.63) are quantized in accordance with the Kac formula (2.17)

$$\Delta^R_{n,m} = \frac{1}{16} + \frac{[(n\alpha_- + m\alpha_+)^2 - (\alpha_- + \alpha_+)^2]}{8} \tag{2.65}$$

$(n - m \in 2Z + 1)$ This screening procedure works well also in the case of mixed R-NS correlation functions and in the general case of multipoint functions.

2.5 Fusion rules and 4-point functions

The screening procedure and the neutrality condition (2.51) applied to the 3-point functions generate the fusion rules for the fields of a given minimal model. In fact the primary field $\phi_{x,y}$ which enters the OPE of two given fields ϕ_{n_1,m_1} and ϕ_{n_2,m_2} should have a non-zero 3- point function

$$< \phi_{n_1,m_1}(z_1)\phi_{n_2,m_2}(z_2)\phi_{x,y}(z_3) >$$

The Z_2 charge conservation implies the following qualitative description of the N=1 supersymmetric OPE algebra of the fields [11]

$$[R][R] \sim [NS], \qquad [R][NS] \sim [R], \qquad [NS][NS] \sim [NS]$$

We begin with the fusion rules in the pure NS sector considering the correlation function of three superfields. It is known that there exist two different structure in it, an even part and an odd one

$$\begin{gathered} < N(z_1,\theta_1)N(z_2,\theta_2)N(z_3,\theta_3) >= \\ (\hat{z}_{12})^{\Delta_3-\Delta_2-\Delta_1}(\hat{z}_{13})^{\Delta_2-\Delta_3-\Delta_1}(\hat{z}_{23})^{\Delta_1-\Delta_2-\Delta_3}(a_1 + a_2\eta) \\ \eta = (\hat{z}_{12}\hat{z}_{13}\hat{z}_{23})^{-\frac{1}{2}}(\theta_1\hat{z}_{23} + \theta_2\hat{z}_{13} + \theta_3\hat{z}_{12} + \theta_1\theta_2\theta_3) \end{gathered} \tag{2.66}$$

(a_1 and a_2 are arbitrary constants). These structure give rise to different fusion rules -odd and even- generated by the corresponding odd and even part of the function (2.66) [86].

In the Coulomb gas picture with the screening operators, there exist at least three different ways to construct the 3-point function of the fields $N_{n_1,m_1}, N_{n_2,m_2}, N_{n_3,m_3}$ depending which vertex has conjugate charge. In each case there exists a number of screening operators which assure the neutrality condition (2.51). Denoting the 3-point function

$$\begin{aligned} &< N_{n_1,m_1}(z_1,\theta_1)N_{n_2,m_2}(z_2,\theta_2)N_{n_3,m_3}(z_3,\theta_3) >= \\ &= \prod_{i=1}^{k-1}\prod_{j=1}^{l-1}\oint_{C_i} d\epsilon_i dv_i \oint_{C_j} d\eta_j dw_j \ll V_{\alpha_{n_1,m_1}}(z_1,\theta_1) \\ &V_{\alpha_{n_2,m_2}}(z_2,\theta_2)V_{2\alpha_0-\alpha_{x,y}}(z_3,\theta_3)V_{\alpha_-}(v_i,\epsilon_i)V_{\alpha_+}(w_j,\eta_j) \gg\equiv \\ &\{\alpha_{n_1,m_1},\alpha_{n_2,m_2},2\alpha_0-\alpha_{x,y};(k-1)\alpha_-;(l-1)\alpha_+\} \end{aligned} \tag{2.67}$$

we have a chain of equalities

$$\begin{aligned}
&\{\alpha_{n_1,m_1}, \alpha_{n_2,m_2}, 2\alpha_0 - \alpha_{x,y}; (k-1)\alpha_-; (l-1)\alpha_+\} = \\
&= \{2\alpha_0 - \alpha_{n_1,m_1}, \alpha_{n_2,m_2}, \alpha_{x,y}; (a-1)\alpha_-; (b-1)\alpha_+\} = \\
&= \{\alpha_{n_1,m_1}, 2\alpha_0 - \alpha_{n_2,m_2}, \alpha_{x,y}; (p-1)\alpha_-; (q-1)\alpha_+\}
\end{aligned}$$

The neutrality condition (2.51) selects the following expressions for the unknown charge $\alpha_{x,y}$

$$\begin{aligned}
\alpha^1_{x,y} &= \frac{1}{2}\{[1-(n_1+n_2-2k+1)]\alpha_- + [1-(m_1+m_2-2l+1)]\alpha_+\} \\
\alpha^2_{x,y} &= \frac{1}{2}\{[1-(n_2-n_1+2p-1)]\alpha_- + [1-(m_2-m_1+2q-1)]\alpha_+\} \\
\alpha^3_{x,y} &= \frac{1}{2}\{[1-(n_1-n_2+2a-1)]\alpha_- + [1-(m_1-m_2+2b-1)]\alpha_+\}
\end{aligned}$$

with the conditions

$$\begin{aligned}
&n_1+n_2-2k+1>0 \\
&n_2-n_1+2p-1>0 \\
&n_1-n_2+2a-1>0
\end{aligned}$$

and analogous ones for the α_+ part. We have to take the common solutions of these equations, that is

$$\begin{aligned}
x = &\mid n_1-n_2 \mid +1, \mid n_1-n_2 \mid +3, \cdots n_1+n_2-1 \\
y = &\mid m_1-m_2 \mid +1, \mid m_1-m_2 \mid +3, \cdots m_1+m_2-1
\end{aligned}$$

This gives the fusion rules for the fields N_{n_1,m_1}, N_{n_2,m_2}

$$[N_{n_1,m_1}][N_{n_2,m_2}] = \sum_{x=|n_1-n_2|+1}^{n_1+n_2-1} \sum_{y=|m_1-m_2|+1}^{m_1+m_2-1} [N_{x,y}] \tag{2.68}$$

The even fusion rules are recovered when there is an even number of screening operators (i.e. $x+y \in 2Z_+$) while the odd ones correspond to an odd number of these operators (i.e. $x+y \in 2Z_+ + 1$) , as can be seen from the following examples.

We first consider the top term of the sum (2.68) when there is no screening operator at all:

$$\begin{aligned}
&\ll V_{\alpha_1}(z_1,\theta_1)V_{\alpha_2}(z_2,\theta_2)V_{2\alpha_0-\alpha_3}(z_3,\theta_3) \gg = \\
&= (\hat{z}_{12})^{\alpha_1\alpha_2}(\hat{z}_{13})^{\alpha_1(2\alpha_0-\alpha_3)}(\hat{z}_{23})^{\alpha_2(2\alpha_0-\alpha_3)}
\end{aligned}$$

From the neutrality condition we have

$$\alpha_1 + \alpha_2 = \alpha_3$$

and hence this 3-point function can be rewritten in terms of the anomalous dimensions

$$\ll V_{\alpha_1}(z_1,\theta_1)V_{\alpha_2}(z_2,\theta_2)V_{2\alpha_0-\alpha_3}(z_3,\theta_3) \gg =$$
$$= (\hat{z}_{12})^{\Delta_3-\Delta_1-\Delta_2}(\hat{z}_{13})^{\Delta_2-\Delta_1-\Delta_3}(\hat{z}_{23})^{\Delta_1-\Delta_2-\Delta_3}$$

Now we consider the case in which there is only one screening operator, say J_+. For simplicity we put $z_1 \to \infty, z_2 = z, z_3 = 0$ and $\theta_3 = 0$. This choice is always possible due to the Osp(2,1) invariance. The 3-point function is given by

$$\oint d\theta dw \lim_{z_1\to\infty} z_1^{2\Delta_1} \ll V_{\alpha_1}(z_1,\theta_1)V_{\alpha_2}(z_2,\theta_2)V_{2\alpha_0-\alpha_3}(z_3,0)V_{\alpha_+}(w,\theta) \gg =$$
$$= (z_{23})^{\alpha_2(2\alpha_0-\alpha_3)} \oint d\theta dw(z_2 - w - \theta_2\theta)^{\alpha_2\alpha_+}w^{\alpha_+(2\alpha_0-\alpha_3)}$$

Taking the θ-integral, we get

$$< N_1(\infty)N_2(z,\theta_2)N_3(z_3,0) >= \theta_2(z_{23})^{\alpha_2(2\alpha_0-\alpha_3)} \oint dw w^{\alpha_+(2\alpha_0-\alpha_3)}(z-w)^{\alpha_+\alpha_2-1}$$

and the remaining integral is a particular case of integral representation of the hypergeometric function

$$I(a,b,c,z) = \int_0^z dv v^a(1-v)^b(z-v)^c = const\ z^{1+a+c}$$

with b=0, so

$$< N_1(\infty)N_2(z,\theta_2)N_3(z_3,0) >= \theta_2(z_{23})^{\alpha_2(2\alpha_0-\alpha_3)+(2\alpha_0-\alpha_3)\alpha_+ +\alpha_2\alpha_+} =$$
$$= \theta_2(z_{23})^{\Delta_1-\Delta_2-\Delta_3-\frac{1}{2}}$$

This function coincides with the corresponding limit of the odd part of eq. (2.66)

We shall give two examples of the FR's (2.68) which present the NS sector of the OPE algebras of TIM and the N=2 supersymmetric point of the Ashkin-Teller model (AT)

$$p = 3 \qquad c = \frac{7}{10} \qquad (TIM)$$

$$[\frac{1}{10}]_{NS}[\frac{1}{10}]_{NS} = [0]_{NS}^{even} + [\frac{1}{10}]_{NS}^{odd}$$

$$p = 4 \qquad c = 1 \qquad (AT)$$

$$[1]_{NS}[1]_{NS} = [0]_{NS}^{even} \quad , [\frac{1}{6}]_{NS}[\frac{1}{6}]_{NS} = [0]_{NS}^{even} + [\frac{1}{6}]_{NS}^{odd} + [1]_{NS}^{even}$$
$$[1]_{NS}[\frac{1}{6}]_{NS} = [\frac{1}{6}]_{NS}^{even} \quad , [\frac{1}{6}]_{NS}[\frac{1}{16}]_{NS} = [\frac{1}{16}]_{NS}^{even} + [\frac{1}{16}]_{NS}^{odd}$$
$$[1]_{NS}[\frac{1}{16}]_{NS} = [\frac{1}{16}]_{NS}^{odd} \quad , [\frac{1}{16}]_{NS}[\frac{1}{16}]_{NS} = [\frac{1}{6}]_{NS}^{even} + [\frac{1}{6}]_{NS}^{odd} + [0]_{NS}^{even} + [1]_{NS}^{odd}$$

To find the fusion rules of two given Ramond fields R_{n_1,m_1} and R_{n_2,m_2} we have to look at the non-zero 3-point functions with the NS superfield $N_{x,y}$,

$$< R_{n_1,m_1}(z_1) R_{n_2,m_2}(z_2) N_{x,y}(z_3,\theta_3) >$$

The same procedure as before leads to the following FR's of two R-fields [17]

$$[R_{n_1,m_1}][R_{n_2,m_2}] = \sum_{x=|n_1-n_2|+1}^{n_1+n_2-1} \sum_{y=|m_1-m_2|+1}^{m_1+m_2-1} [N_{x,y}] \tag{2.69}$$

The corresponding mixed FR's $[R][NS] \sim [R]$ are direct consequences of eqs. (2.68),(2.69). The set of these fusion rules completes the structure of the associative OPE algebra of the fields of the corresponding supersymmetric minimal models. In the case p=3 our formula (2.69) reproduces the well-known fusion rules of TIM [7], while in the case p=4 (c=1) the structure of the Ramond sector is determined by the following fusion rules

$$\begin{array}{llll}
(\frac{1}{16})_R(\frac{1}{16})_R & = [0]_{NS} + [\frac{1}{6}]_{NS} & (\frac{1}{16})_R(\frac{3}{8})_R = [\frac{1}{16}]_{NS} \\
(\frac{3}{8})_R(\frac{3}{8})_R & = [0]_{NS} + [1]_{NS} & (\frac{9}{16})_R(\frac{1}{16})_R = [\frac{1}{6}]_{NS} + [1]_{NS} \\
(\frac{9}{16})_R(\frac{9}{16})_R & = [0]_{NS} + [\frac{1}{6}]_{NS} & (\frac{1}{24})_R(\frac{1}{16})_R = [\frac{1}{16}]_{NS} \\
(\frac{1}{24})_R(\frac{1}{24})_R & = [0]_{NS} + [1]_{NS} + [\frac{1}{6}]_{NS} & (\frac{9}{16})_R(\frac{1}{24})_R = [\frac{1}{16}]_{NS}
\end{array}$$

The NS FR's were discussed in the ref.[86]. The corresponding mixed FR's have the form

$$\begin{array}{llll}
(\frac{1}{16})_R(\frac{1}{16})_{NS} & = [\frac{3}{8}]_R + [\frac{1}{24}]_R & (\frac{9}{16})_R(\frac{1}{6})_{NS} = [\frac{1}{16}]_R + [\frac{9}{16}]_R \\
(\frac{1}{16})_R(\frac{1}{6})_{NS} & = [\frac{9}{16}]_R + [\frac{1}{16}]_R & (\frac{9}{16})_R(\frac{1}{16})_{NS} = [\frac{1}{24}]_R
\end{array}$$

We note that the same screening procedure works well also in the case of the FR's for the Virasoro minimal models.

Using the vertex representation and the screening procedure, we can compute the 4-point function of the R fields [17]

$$\begin{aligned}
&< R_{n_1,m_1}(z_1) R_{n_2,m_2}(z_2) R_{n_3,m_3}(z_3) R_{n_4,m_4}(z_4) >= \\
&= \oint_{C_i} dv_i \oint_{C_j} dw_j < \sigma(z_1)\sigma(z_2)\sigma(z_3) \prod_{i=1}^{n-1} \psi(v_i) \prod_{j=1}^{m-1} \psi(w_j)\sigma(z_4) > \\
&\ll e^{i\alpha_{n_1,m_1}\phi(z_1)} e^{i\alpha_{n_2,m_2}\phi(z_2)} e^{i\alpha_{n_3,m_3}\phi(z_3)} e^{i(2\alpha_0-\alpha_{n_4,m_4})\phi(z_4)} \\
&\prod_{i=1}^{n-1} e^{i\alpha_-\phi(v_i)} \prod_{j=1}^{m-1} e^{i\alpha_+\phi(w_j)} \gg
\end{aligned} \tag{2.70}$$

The second factor in the integrand is the well-known multipoint function of the modified Coulomb system

$$\ll \prod_{k=1}^{N} e^{i\alpha_k\phi(z_k)} \gg = \prod_{l<n=2}^{N} (z_{ln})^{\alpha_n\alpha_l} \qquad \sum_{i=1}^{N} \alpha_i = 2\alpha_0$$

while the first factor is calculated by the recursive equations (2.42) for the Ising model. Then the 4-point function of the field $R_{n,m}$ takes the form

$$< R_{n_1,m_1}(\infty)R_{n_2,m_2}(z)R_{n_3,m_3}(1)R_{n_4,m_4}(0) >= z^{\alpha^2}(z-1)^{\alpha(2\alpha_0-\alpha)}$$

$$\oint_{C_i} dw_i \oint_{C_j} dv_j < \sigma(\infty)\sigma(z)\sigma(1)\prod_{i=1}^{n-1}\psi(w_i)\prod_{j=1}^{m-1}\psi(v_j)\sigma(0) >=$$

$$= \prod_{l<k=2}^{n-1} w_{lk}^{\alpha_-^2}\prod_{s<t=2}^{m-1} v_{st}^{\alpha_+^2}\prod_{i=1}^{n-1}\prod_{j=1}^{m-1}[((w_i-z)w_i)^{\alpha-\alpha}(w-1)^{\alpha-(2\alpha_0-\alpha)}$$

$$((v_j-z)v_j)^{\alpha+\alpha}(v_j-1)^{\alpha+(2\alpha_0-\alpha)}(w_i-v_j)^{\alpha-\alpha+}] \quad (2.71)$$

The integration contours C_i are fixed by the branch cut singularities of the integrand. Thus for the general expression of the 4-point function of the Ramond fields we should take a linear combination of all 4-point functions corresponding to the possible independent choices of the contours C_i.

In the simplest example of one screening operator , say J_+, we get from (2.71) (taking the sum of the functions $< RRRR >$ and $< \tilde{R}R\tilde{R}R >$) the 4-point function of the R-field $R_{1,2}$

$$G_{1,2}^p(z) = z^{\alpha_{12}(2\alpha_0-\alpha_{12})+\frac{1}{8}}(z-1)^{\alpha_{12}^2-\frac{1}{8}}$$

$$\left[\sqrt{1+\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2}A^a(\tilde{I}_a+I_a)+\sqrt{1-\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2}(\tilde{I}_a-I_a)\right] \quad (2.72)$$

where

$$\tilde{I}_a = \sqrt{z}\oint_{C_a} dv v^{\frac{3}{p}}(1-v)^{-\frac{1}{p}}(z-v)^{-(1+\frac{1}{p})}$$

$$I_a = \sqrt{z-1}\oint_{C_a} dv v^{1+\frac{3}{p}}[(1-v)(z-v)]^{-(1+\frac{1}{p})}$$

and A^a and B^a are arbitrary constants. Then only in the case of TIM (p=3), $\Delta_{1,2} = \frac{7}{16}$, we have a unique branch-cut from 1 to z and correspondingly only one independent integration contour. This case is a good test for our construction since the expression for the 4-point function of the sub-leading magnetization field $\Delta = \frac{7}{16}$ can be calculated from the null-vector's method of the Virasoro algebra ($\frac{7}{16}$ is a representation's weight which appears in the Virasoro algebra too). The solution of the corresponding differential equation is

$$< R_{\frac{7}{16}}(\infty)R_{\frac{7}{16}}(z)R_{\frac{7}{16}}(1)R_{\frac{7}{16}}(0) >= z^{-\frac{5}{8}}(z-1)^{-\frac{7}{8}}$$

$$\{C_1\sqrt{1+\sqrt{1-\frac{1}{z}}[2(z^2-z+1)-(2z-1)\sqrt{z(z-1)}]}+$$

$$C_2\sqrt{1-\sqrt{1-\frac{1}{z}}[2(z^2-z+1)+(2z-1)\sqrt{z(z-1)}]}\} \quad (2.73)$$

The comparison of this expression with the corresponding one from eq.(2.72) shows their full coincidence.

For the other superconformal models, with $p > 3$, we have two independent contours, one from 0 to z ant the other from 1 to ∞ and we have four independent solutions for the corresponding 4-point function

$$G_{12}^{p\neq 3}(z) = (z-1)^{\frac{p+4}{8p}} z^{-\frac{p+12}{8p}} \sum_{i=1}^{4} A^i W_i(z) \tag{2.74}$$

where, putting $h = \frac{1}{p}$

$$\begin{aligned}
W_1(z) &= \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(1-h,h)\sqrt{z}F(1+h,-h,1-2h,z)+ \\
&\quad -B(-h,-h)\sqrt{z-1}F(1+h,-h,-2h,z)] \\
W_2(z) &= \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(1+3h,-h)\sqrt{z}z^{2h}F(h,1+h,1+2h,z)+ \\
&\quad +B(2+3h,-h)\sqrt{z-1}z^{-2-2h}F(1+h,2+3h,2+2h,z)] \\
W_3(z) &= \sqrt{1-\sqrt{1-\frac{1}{z}}}[B(1-h,-h)\sqrt{z}F(1+h,-h,1-2h,z)+ \\
&\quad +B(-h,-h)\sqrt{z-1}F(1+h,-h,-2h,z)] \\
W_4(z) &= \sqrt{1-\sqrt{1-\frac{1}{z}}}[B(1+3h,-h)\sqrt{z}z^{2h}F(h,1+3h,1+2h,z)+ \\
&\quad -B(2+3h,-h)\sqrt{z-1}z^{-2-2h}F(1+h,2+3h,2+2h,z)]
\end{aligned} \tag{2.75}$$

In the same way we can calculate the 4-point function of the Ramond field $\Delta_{2,1}$, since it needs only the insertion of one J_- screening operator

$$\begin{aligned}
G_{21}^{p}(z) &= z^{\alpha_{21}(2\alpha_0-\alpha_{21})+\frac{1}{8}}(z-1)^{\alpha_{21}^2-\frac{1}{8}} \\
&[\sqrt{1+\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2} A^a(\tilde{K}_a+K_a)+\sqrt{1-\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2} B^a(\tilde{K}_a-K_a)]
\end{aligned} \tag{2.76}$$

where

$$\begin{aligned}
\tilde{K}_a &= \sqrt{z}\oint_{C_a} dv v^{-\frac{3}{p+2}}(1-v)^{\frac{1}{p+2}}(z-v)^{\frac{1}{p+2}-1} \\
K_a &= \sqrt{z-1}\oint_{C_a} dv v^{1-\frac{3}{p+2}}(1-v)^{\frac{1}{p+2}-1}(z-v)^{\frac{1}{p+2}-1}
\end{aligned}$$

From the branch-cut analysis of the integrand one can see that there are two independent integration contours for each p . Putting $a = \frac{1}{p+2}$, we have four independent solutions

$$G_{21}^{p}(z) = z^{\frac{10-p}{8(p+2)}}(z-1)^{\frac{p-2}{8(p+2)}} \sum_{i=1}^{4} A^i Y_i(z) \tag{2.77}$$

where

$$Y_1(z) = \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(a,1+a)\sqrt{z}F(1-a,a,1+2a,z)+$$
$$-B(a,a)\sqrt{z-1}F(1-a,a,2a,z)]$$
$$Y_2(z) = \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(1-3a,a)\sqrt{z}z^{-2a}F(-a,1-3a,1-2a,z)+$$
$$+B(2-3a,a)\sqrt{z-1}z^{1-2a}F(1-a,2-3a,2-2a,z)]$$
$$Y_3(z) = \sqrt{1-\sqrt{1-\frac{1}{z}}}[B(a,1+a)\sqrt{z}F(1-a,a,1+2a,z)+$$
$$+B(a,a)\sqrt{z-1}F(1-a,a,2a,z)]$$
$$Y_4(z) = \sqrt{1-\sqrt{1-\frac{1}{z}}}[B(1-3a,a)\sqrt{z}z^{-2a}F(-a,1-3a,1-2a,z)+$$
$$-B(2-3a,a)\sqrt{z-1}z^{1-2a}F(1-a,2-3a,2-2a,z)] \qquad (2.78)$$

According to this formula, the 4-point function of the magnetization field $\Delta_{2,1} = \frac{3}{80}$ of TIM is

$$< R_{\frac{3}{80}}(\infty)R_{\frac{3}{80}}(z)R_{\frac{3}{80}}(1)R_{\frac{3}{80}}(0) >= z^{\frac{7}{40}}(z-1)^{\frac{1}{40}}$$
$$[\sqrt{1+\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2}C^a L_a(z) + \sqrt{1-\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2}D^a \tilde{L}_a(z)] \qquad (2.79)$$

where

$$\tilde{L}_1 = [B(\frac{1}{5},\frac{6}{5})\sqrt{z}F(\frac{4}{5},\frac{1}{5},\frac{7}{5},z) \pm B(\frac{1}{5},\frac{1}{5})\sqrt{z-1}F(\frac{4}{5},\frac{1}{5},\frac{2}{5},z)]$$
$$\tilde{L}_2 = [B(\frac{2}{5},\frac{1}{5})\sqrt{z}z^{-\frac{2}{5}}F(-\frac{1}{5},\frac{2}{5},\frac{3}{5},z) \mp B(\frac{7}{5},\frac{1}{5})\sqrt{z-1}z^{\frac{3}{5}}F(\frac{4}{5},\frac{7}{5},\frac{8}{5},z)]$$

Similarly the 4-point function of the order parameter field $R_{\frac{1}{16}}$ in the AT model (c=1) is

$$< R_{\frac{1}{16}}(\infty)R_{\frac{1}{16}}(z)R_{\frac{1}{16}}(1)R_{\frac{1}{16}}(0) >= z^{\frac{1}{8}}(z-1)^{\frac{1}{24}}$$
$$[\sqrt{1+\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2}C^a M_a(z) + \sqrt{1-\sqrt{1-\frac{1}{z}}}\sum_{a=1}^{2}D^a \tilde{M}_a(z)] \qquad (2.80)$$

where

$$\tilde{M}_1 = [B(\frac{1}{6},\frac{7}{6})\sqrt{z}F(\frac{5}{6},\frac{1}{6},\frac{4}{3},z) \pm B(\frac{1}{6},\frac{1}{6})\sqrt{z-1}F(\frac{5}{6},\frac{1}{6},\frac{1}{3},z)]$$
$$\tilde{M}_2 = [B(\frac{1}{2},\frac{1}{6})\sqrt{z}z^{-\frac{1}{3}}F(-\frac{1}{6},\frac{1}{2},\frac{4}{3},z) \mp B(\frac{3}{2},\frac{1}{6})\sqrt{z-1}z^{\frac{2}{3}}F(\frac{5}{6},\frac{3}{2},\frac{5}{3},z)]$$

We note that the field $R_{2,1}$, $\Delta_{2,1} = \frac{1}{16}$, considered as the N=2 extended supersymmetric field is a twisted field, while in the N=1, c=1 model it is a Ramond field. Thus, the eq. (2.80) gives an example of 4-point function of the N=2 twisted field (see [18] for more details).

2.6 Physical correlators and structure constants of the OPE's algebras

One of the specific features of the 2-D conformal theories is their factorization in 1-D ones. We have to remind that the conformal fields $\phi^c_{d,s}(z,\bar{z})$ having anomalous dimension d, spin s and belonging to a theory with central charge c can be represented as a product of two 1-D fields $\phi^c_\Delta(z), \phi^c_{\overline{\Delta}}(\bar{z})$ where $d = \Delta + \overline{\Delta}$ and $s = \Delta - \overline{\Delta}$. In order to construct 2-D minimal models we have to take two copies ("left" and "right") of a given 1-D minimal model and to find a subset of fields $\phi^c_{d,s}(z,\bar{z})$ which close an associative algebra. One way to achieve this is to impose the condition of modular invariance of the partition function on the torus [80,89]. This condition gives us the operator content of the 2-D minimal model of fields with integer spins.

Another method is based on the crossing symmetry of the 4-point functions. For scalar fields ($\Delta = \overline{\Delta}$)this condition has the following form

$$G^{lk}_{nm}(z,\bar{z}) = G^{mk}_{nl}(1-z, 1-\bar{z}) = \left(\frac{1}{z}\right)^{2\Delta_n} \left(\frac{1}{\bar{z}}\right)^{2\overline{\Delta}_n} G^{lm}_{nk}\left(\frac{1}{z}, \frac{1}{\bar{z}}\right)$$

where

$$G^{lk}_{nm}(z,\bar{z}) =< \phi_k(\infty)\phi_l(1,1)\phi_n(z,\bar{z})\phi_m(0,0) >$$

As it is known [10] the crossing symmetry is equivalent to the monodromy invariance of the 4-point functions . Denoting by $\{W_i(z)\}$ the set of functions corresponding to the possible independent contours, one can write 2-D 4-point function in the form

$$G(z,\bar{z}) = \sum_{i,j} I_{ij} W_i(z) \overline{W_j(z)}$$

where I_{ij} are unknown coefficients. Since the functions $W_i(z)$ have branch cut singularities in the point $0,1$, and ∞ they transform nontrivially along a closed curves enclosing the singular points

$$W_i(z) \to (g_l)_{ik} W_k(z) \qquad l = 0,1,\infty \tag{2.81}$$

The matrices g_l generate the monodromy group of the functions $W_i(z)$. The correlation functions of scalar fields should be uniquely defined in the 2-D space, i.e. they have to be monodromy invariant [10]

$$\begin{aligned} G(z,\bar{z}) = & \sum_{i,j} I_{ij} W_i \overline{W}_j = \sum_{ij}\sum_{k,l} I_{ij}(g_l)_{ik} W_k (\bar{g}_l)_{jp} \overline{W}_p = \\ & = \sum_{k,l}\left(\sum_{i,j}(g^t_l)_{ki} I_{ij} (\bar{g}_l)_{jl}\right) W_k \overline{W}_p \end{aligned} \tag{2.82}$$

Thus we obtain the following equations for the unknown coefficients I_{ij}

$$I_{kp} = \sum_{ij} (g_l^t)_{ki} I_{ij} (\overline{g}_l)_{jp} \tag{2.83}$$

These equations determine $\{I_{ij}\}$ up to an overall factor related to the normalization of the 2-point function.

Let us start with the simplest example: the 4-point function of the Ramond field with $\Delta = \frac{7}{16}$ in the TIM

$$G_{\frac{7}{16},\frac{7}{16}} =| z |^{-\frac{5}{4}} | z-1 |^{-\frac{7}{4}} \sum_{i,j=1}^{2} I_{ij} W_i(z) \overline{W}_j(\overline{z})$$

where (see eq.(2.73)

$$W_1 = \sqrt{1 + \sqrt{1 - \frac{1}{z}[2(z^2 - z + 1) - (2z-1)\sqrt{z(z-1)}]}}$$

$$W_3 = \sqrt{1 - \sqrt{1 - \frac{1}{z}[2(z^2 - z + 1) + (2z-1)\sqrt{z(z-1)}]}}$$

In this case it is sufficient to calculate only two monodromy matrices, say g_1 and g_∞

$$g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad g_\infty = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The monodromy invariant correlator is

$$G_{\frac{7}{16},\frac{7}{16}}(z,\overline{z}) = \lambda \mid z \mid^{-\frac{5}{4}} \mid z-1 \mid^{-\frac{7}{4}} (W_1\overline{W}_1 + W_3\overline{W}_3) \tag{2.84}$$

and the overall factor λ is fixed to be

$$\lambda = \frac{1}{8}$$

if we normalize

$$< R_{\frac{7}{16}}(\infty) R_{\frac{7}{16}}(0) >= 1$$

One can works out, using the formulas given in the Appendix A, the general case of the 4-point function of the Ramond fields $R^p_{1,2}$, for each value of the central charge c_p

$$G^p_{1,2}(z,\overline{z}) = \lambda_{1,2}(p) \mid z \mid^{-\frac{p+12}{4p}} \mid z-1 \mid^{\frac{p+4}{4p}}$$
$$[W_1\overline{W}_1 + W_3\overline{W}_3 + (4\cos^2(\frac{\pi}{p}) - 1)(W_2\overline{W}_2 + W_4\overline{W}_4)] \tag{2.85}$$

where $W_i(z)$ are those of eq.(2.75).This formula includes also the precedent case of $\Delta = \frac{7}{16}$ (p=3), since

$$4\cos^2(\frac{\pi}{3}) - 1 = 0$$

and the hypergeometric functions W_1 and W_3 are elementary ones. In the case of the Ramond fields $R^p_{2,1}$ similar calculations lead to the following 4-point function

$$G^p_{2,1}(z,\bar{z}) = \lambda_{2,1}(p) \mid z \mid^{\frac{10-p}{4(p+2)}} \mid z-1 \mid^{\frac{p-2}{4(p+2)}}$$
$$[Y_1\overline{Y}_1 + Y_3\overline{Y}_3 + (a\cos^2(\frac{\pi}{p+2}) - 1)(Y_2\overline{Y}_2 + Y_4\overline{Y}_4)] \qquad (2.86)$$

where $Y_i(z)$ are the functions defined in the equations (2.78).

Analogously the monodromy invariant expression for the correlator involving the Ramond fields $R^p_{1,2}$ and $R^p_{2,1}$

$$G^p_{12,21}(z,\bar{z}) =< R^p_{12}(\infty)R^p_{12}(z,\bar{z})R^p_{21}(1,1)R^p_{21}(0,0) >$$

is

$$\begin{aligned} G^p_{12,21}(z,\bar{z}) &= \frac{1}{8} \mid z \mid^{-\frac{3}{4}} \mid z-1 \mid^{-\frac{3}{4}} (N_1\overline{N}_1 + N_2\overline{N}_2) \qquad (2.87) \\ N_1(z) &= \sqrt{1+\sqrt{1-\frac{1}{z}(\frac{1-2z+2\sqrt{z(z-1)}}{\sqrt{z}})}} \\ N_2(z) &= \sqrt{1-\sqrt{1-\frac{1}{z}(\frac{1-2z-2\sqrt{z(z-1)}}{\sqrt{z}})}} \end{aligned}$$

with the usual normalization to unity of the 2-point functions.

Looking at the singularities of the 4-point functions, in the limit $\mid z \mid \rightarrow 1$, we can recover the operator product expansion

$$[R^p_{21}][R^p_{21}] = 1 + c_1(p)\Phi^p_{31} \qquad (2.88)$$
$$[R^p_{12}][R^p_{12}] = 1 + c_2(p)\Phi^p_{13} \qquad (2.89)$$
$$[R^p_{12}][R^p_{21}] = c_3(p)\Phi^p_{22} \qquad (2.90)$$

and the structure constants c_1, c_2, c_3. In fact, using the OPE

$$\phi_{\Delta_1}(z,\bar{z})\phi_{\Delta_2}(1,1) \simeq \sum_k \frac{c_k}{\mid z-1 \mid^{2(\Delta_1+\Delta_2-\Delta_k)}}\phi_k(1,1)$$

with normalization fixed by the identity family, i.e.

$$c_0 = 1$$

we have

$$G(z,\bar{z}) =< R_\Delta(\infty)R_\Delta(z,\bar{z})R_\Delta(1,1)R_\Delta(0,0) >\simeq$$
$$\simeq \sum_k \frac{c_k}{\mid z-1 \mid^{2(2\Delta-\Delta_k)}} < R_\Delta(\infty)\Phi_k(1,1)R_\Delta(0,0) >= \sum_k \frac{c_k^2}{\mid z-1 \mid^{2(2\Delta-\Delta_k)}}$$

Hence the contributions of the different conformal families is identified by the power singularities.

In the case $[R^p_{21}][R^p_{21}]$, taking the limit $\mid z \mid\to 1$ in eq.(2.86), we have $(a = \frac{1}{p+2})$

$$Y_1\overline{Y}_1 + Y_3\overline{Y}_3 \to 2\{\mid B(a,1+a)\frac{\Gamma(1+2a)\Gamma(2a)}{\Gamma(3a)\Gamma(1+a)}\mid^2 + $$
$$+\mid z-1\mid^{4a-1}\mid B(a,a)\frac{\Gamma(2a)\Gamma(1-2a)}{\Gamma(1-a)\Gamma(a)}\mid^2\}$$
$$Y_2\overline{Y}_2 + Y_4\overline{Y}_4 \to 2\{\mid B(1-3a,a)\frac{\Gamma(1-2a)\Gamma(2a)}{\Gamma(a)\Gamma(1-a)}\mid^2 + $$
$$+\mid z-1\mid^{4a-1}\mid B(2-3a,a)\frac{\Gamma(2-2a)\Gamma(1-2a)}{\Gamma(1-a)\Gamma(2-3a)}\mid^2\}$$

The second block in both of the expressions comes from the unit operator and from this we can normalize the function

$$\lambda_{2,1}(a) = \frac{1}{32\cos^2(\pi a)}\mid\frac{\Gamma(-a)}{\Gamma(a)\Gamma(-2a)}\mid^2$$

while the first block in both of the expressions is identified as the contribution of the $\Phi^p_{3,1}$ operator. So we obtain

$$c_1(p) = \frac{1}{2}\frac{\Gamma(\frac{2}{p+2})\Gamma(\frac{p-1}{p+2})}{\Gamma(-\frac{2}{p+2})\Gamma(\frac{p+3}{p+2})}\sqrt{4\cos^2(\frac{\pi}{p+2})-1} \tag{2.91}$$

Similarly in the case $[R_{12}][R_{12}]$ we get

$$\lambda_{12}(p) = \frac{1}{32\cos^2(\frac{\pi}{p})}\mid\frac{\Gamma(\frac{1}{p})}{\Gamma(-\frac{1}{p})\Gamma(\frac{2}{p})}\mid^2 \tag{2.92}$$

$$c_2(p) = \frac{3}{2}\frac{\Gamma(\frac{3}{p})\Gamma(-\frac{2}{p})}{\Gamma(\frac{2}{p})\Gamma(-\frac{1}{p})}\sqrt{4\cos^2(\frac{\pi}{p})-1} \tag{2.93}$$

and $c_2(3)$ is correctly zero, since the only contribution in the OPE of $R_{\frac{7}{16}}$ is due to the the identity family. In this case the operator Φ_{13} decouples from the conformal grid. The structure constant of the OPE (2.90) is $c_3(p) = \frac{1}{2}$.

Using the mixed R-NS 4-point functions we can also extract the structure constants of the NS fields Φ^p_{31} with itself. To show this, we start with the computation of the following correlation function

$$< \Phi^p_{31}(\infty)\Phi^p_{31}(1)R^p_{21}(z)R^p_{21}(0) >=$$
$$= \sum_i \oint_{C_i} dv < V_{2\alpha_0-\alpha_{31}}(\infty)V_{\alpha_{31}}(1)V_{\alpha_{21}}(z)V_{\alpha_{21}}(0)V_{\alpha_-}(v) >=$$
$$= z^{\alpha^2_{21}}(z-1)^{\alpha_{31}\alpha_{21}}\sum_i \oint_{C_i} dv v^{\alpha-\alpha_{21}}(1-v)^{\alpha-\alpha_{31}}(z-v)^{\alpha-\alpha_{21}} < \sigma(z)\psi(v)\sigma(0) >=$$
$$= z^{\alpha^2_{21}+\frac{3}{8}}(z-1)^{\alpha_{31}\alpha_{21}}\sum_i \oint_{C_i} dv v^{\alpha-\alpha_{21}-\frac{1}{2}}(1-v)^{\alpha-\alpha_{31}}(z-v)^{\alpha-\alpha_{21}-\frac{1}{2}} =$$

$$= z^{\frac{5p+6}{8(p+2)}}(z-1)^{\frac{p}{2(p+2)}}(b_1 B(\frac{2p}{p+2},\frac{2}{p+2})F(\frac{p+1}{p+2},\frac{2p}{p+2},\frac{2p+2}{p+2},z)+$$
$$b_2 B(\frac{1}{p+2},\frac{1}{p+2})z^{-\frac{p}{p+2}}F(\frac{p}{p+2},\frac{1}{p+2},\frac{2}{p+2},z))$$

The monodromy invariant solution with the usual normalization is

$$G(z,\bar{z}) = | z |^{\frac{5p+6}{4(p+2)}} | z-1 |^{\frac{p}{p+2}} \{ | z^{-\frac{2p}{p+2}} | F(\frac{p}{p+2},\frac{1}{p+2},\frac{2}{p+2},z) |^2 +$$
$$+\frac{s(\frac{2}{p+2})s(\frac{4}{p+2})}{s^2(\frac{1}{p+2})}\left(\frac{\Gamma^2(\frac{2}{p+2})\Gamma(\frac{2p}{p+2})}{\Gamma^2(\frac{1}{p+2})\Gamma(\frac{2p+2}{p+2})}\right)^2 | F(\frac{p+1}{p+2},\frac{2p}{p+2},\frac{2p+2}{p+2},z) |^2\} \tag{2.94}$$

($s(x) \equiv \sin(\pi x)$) In the limit $z \to 0$ the first term in (2.94) gives the contribution of the identity family while the second one the contribution of the Φ_{31} operator. In this way we get

$$\begin{aligned} G(z,\bar{z}) &= < \Phi^p_{31}(\infty)\Phi^p_{31}(1,1)R^p_{21}(z,\bar{z})R^p_{21}(0,0) > \simeq \\ &\simeq c_1(p) \mid z \mid^{-2(2\Delta_{21}-\Delta_{31}} < \Phi^p_{31}(\infty)\Phi^p_{31}(1)\Phi^p_{31}(0) > = \\ &= c_1(p)[a_1 + a_2 \mid \tilde{\eta} \mid^2] \mid z \mid^{-2(2\Delta_{21}-\Delta_{31})} \end{aligned} \tag{2.95}$$

Since the value of $c_1(p)$ is known,eq.(2.91), we can extract the structure constant a_1 and a_2 of the even and odd part of the field Φ^p_{31}:

$$a_1 = 0$$
$$a_2(a) = \frac{\Gamma^3(2a)\Gamma^2(2-4a)}{\Gamma(4a)\Gamma(1-4a)\Gamma(2a-1)\Gamma^2(2-2a)}\sqrt{\frac{\Gamma(1-a)\Gamma(3a)}{\Gamma^3(a)\Gamma(1-3a)}}$$

($a \equiv \frac{1}{p+2}$). In the case p=3 (TIM) these results coincides with the values obtained in ref.[14].

$$a_2(p=3) = \frac{1}{15}\sqrt{\frac{\Gamma(\frac{4}{5})\Gamma^3(\frac{2}{5})}{\Gamma(\frac{1}{5})\Gamma^3(\frac{3}{5})}}$$

The complete description of the R sector of this model is obtained from the remaining operator product expansions

$$\begin{aligned} (\tfrac{7}{16})_R(\tfrac{7}{16})_R &= [0]_{NS} \\ (\tfrac{3}{80})_R(\tfrac{7}{16})_R &= \frac{1}{2}[\frac{1}{10}]_{NS} \\ (\tfrac{3}{80})_R(\tfrac{3}{80})_R &= [0]_{NS} + \frac{5}{2}\frac{\Gamma^2(\frac{2}{5})}{\Gamma(-\frac{2}{5})\Gamma(\frac{1}{5})}\sqrt{4\cos^2(\frac{\pi}{5})-1}\,[\frac{1}{10}]_{NS} \end{aligned}$$

Considering the appropriate correlation functions and the analytic properties of the independent solutions W_i, one can extract all the structure constants of the superconformal OPE's, as in the Virasoro case. We list here some of them for the second model of the superconformal series, identified as the supersymmetric point of the Ashkin-Teller model on the critical line [90]

$$\begin{aligned}
[\tfrac{1}{6}]_{NS}[\tfrac{1}{6}]_{NS} &= [0]_{NS} + \frac{1}{\sqrt{2}}[\frac{1}{6}]_{NS} + \frac{1}{3}[1]_{NS} \\
[\tfrac{1}{16}]_{R}[\tfrac{1}{16}]_{R} &= [0]_{NS} + \frac{1}{2^{1/6}}[\frac{1}{6}]_{NS} \\
[\tfrac{3}{8}]_{R}[\tfrac{3}{8}]_{R} &= [0]_{NS} + \frac{3}{4}[1]_{NS} \\
[\tfrac{3}{8}]_{R}[\tfrac{1}{16}]_{R} &= \frac{1}{2}[\frac{1}{16}]_{NS} \\
[\tfrac{1}{16}]_{R}[\tfrac{9}{16}]_{R} &= \frac{2^{5/6}}{3}[\frac{1}{6}]_{NS} + \frac{1}{2}[1]_{NS} \\
[\tfrac{1}{24}]_{R}[\tfrac{1}{24}]_{R} &= [0]_{NS} + \frac{1}{\sqrt{2}}[\frac{1}{6}]_{NS} + \frac{1}{12}[1]_{NS}
\end{aligned} \tag{2.96}$$

An important peculiarity of this model is its N=2 superconformal symmetry [13,32,86] generated by the N=1 superstress energy tensor and by the N=1 superfield

$$V = J(z) + \theta\overline{G}(z)$$

where $\Delta_J = 1$. One can use this symmetry to compute some 4-point functions of the Ashlin-Teller model.For example, the 4-point function of the Ramond fields with dimension $\Delta = \frac{1}{24}$ and U(1) charge equals to $q = \pm\frac{1}{12}$ is

$$\begin{aligned}
&< R_{+}(z_1)R_{-}(z_2)R_{+}(z_3)R_{-}(z_4) >= (z_{13}z_{24})^{-\frac{1}{12}} \\
&\quad [\eta(1-\eta)]^{-\frac{1}{12}}[c_1 + c_2\eta^{\frac{1}{6}} + c_3(1-\eta)^{\frac{1}{6}}] \\
&\quad \eta = \frac{z_{12}z_{34}}{z_{13}z_{24}}
\end{aligned}$$

The physical correlator of these fields, satisfying the crossing symmetry condition, is

$$G(z,\overline{z})_{\frac{1}{24}} = \frac{1}{2} \mid z_{13}z_{24} \mid^{-\frac{1}{6}} \left[\mid \eta(1-\eta) \mid^{-\frac{1}{6}} + \mid \frac{\eta}{1-\eta} \mid^{\frac{1}{6}} + \mid \frac{1-\eta}{\eta} \mid^{\frac{1}{6}} \right]$$

from where one obtains the last expression in the OPE of the model.

Chapter 3

Minimal models of N=2 superconformal series

3.1 Introduction

A natural extension of N=1 superconformal invariance is to include an extended supersymmetry, giving rise to N=2 superconformal invariance. In addition there is also a U(1) current which mixes the two supercharges. There are various reasons to consider such kind of extended algebra.

First, the N=2 superconformal algebra is the gauge algebra of the U(1) string, [91], which despite its phenomenological failure, is a consistent string theory in its critical dimension which is d=2.

Moreover, as we will discuss in the next chapter, N=2 superconformal models play an important role in discussing string compactification, since one needs N=2 superconformal invariance in the associated non linear σ model describing the string propagation on a six dimensional compact manifold to ensure N=1 space-time supersymmetry in four dimensions [3]. There is a serious hope that N=2 superconformal structure will provide useful hints in disentangling various questions concerning to the nature of different possible compactification schemes.

Other reasons come from the statistical mechanics, since N=2 invariance might be relevant in some critical phenomena, as we discovered at the end of last chapter.

In this chapter we discuss in detail the properties of this class of theories, which have deep relations with the parafermionic systems [33,34] and the Wess-Zumino-Witten models [35,36] and in particular we point out the existence of a discrete symmetry Z_{p+2} in the p-th model of the discrete unitary series. A clear understanding of the realization of this discrete abelian symmetry in terms of the operator algebra requires the introduction of N=2 superparafermionic systems, which have the interesting feature to have Ramond as well as Neveu-Schwarz order parameters. In addition to a self-evident interest, this system enters in an important way to construct solvable Calabi-Yau compactification as it will be evident in the discussion of our last chapter.

3.2 N=2 Algebra

The set of generators of the N=2 extended superconformal algebra contains in addition to the stress-energy tensor $T(z)$ and the supercurrent $G^1(z)$ (generating N=1 SUSY) a U(1)-current $J(z)$ of conformal dimension 1 and its superpartner $G^2(z)$ of dimension 3/2 (and also the corresponding "right moving modes" $\overline{T}(z), \overline{G}^{1,2}(z), \overline{J}(z)$). For the time being our considerations will concern the "left" part only keeping in mind that similar results hold also for the "right" part. We leave the discussion of the 2-D properties of the theory to the Sec.(3.4.5). This algebra is completely determined by the singular terms in the OPE's of the generators:

$$\begin{aligned}
T(z_1)T(z_2) &= \frac{c}{2z_{12}^4} + \frac{2}{z_{12}^2}T(z_2) + \frac{1}{z_{12}}\partial T(z_2) \\
T(z_1)G^{1,2}(z_2) &= \frac{3}{2z_{12}^2}G^{1,2}(z_2) + \frac{1}{z_{12}}\partial G^{1,2}(z_2) \\
T(z_1)J(z_2) &= \frac{1}{z_{12}^2}J(z_2) + \frac{1}{z_{12}}\partial J(z_2) \\
J(z_1)J(z_2) &= \frac{c}{12}\frac{1}{z_{12}^2} \\
J(z_1)G^{1,2}(z_2) &= \pm\frac{i}{2z_{12}}G^{2,1} \\
G^{1,2}(z_1)G^{1,2}(z_2) &= \frac{2c}{3z_{12}^3} + \frac{2}{z_{12}}T(z_2) \\
G^1(z_1)G^2(z_2) &= \frac{4}{z_{12}^2}J(z_2) + \frac{2}{z_{12}}\partial J(z_2)
\end{aligned} \tag{3.1}$$

It is convenient to use the U(1)-diagonal basis of the algebra (3.1) for the description of the NS and R sectors of the N=2 algebra:

$$G^{\pm}(z) = \frac{1}{\sqrt{2}}(G^1 \pm iG^2)(z) \tag{3.2}$$

so that:

$$\begin{aligned}
J(z_1)G^{\pm}(z_2) &= \pm\frac{1}{2z_{12}}G^{2,1} \\
G^+(z_1)G^-(z_2) &= \frac{2c}{3z_{12}^3} + \frac{4}{z_{12}^2}J(z_2) + \frac{2}{z_{12}}(T(z_2) + \partial J(z_2)) \\
G^+(z_1)G^-(z_2) &= G^-(z_1)G^+(z_2) = O(z_{12})
\end{aligned} \tag{3.3}$$

Similarly to the case of N=1 SUSY one may choose periodic ($G^{\pm}(e^{2\pi i}z) = G^{\pm}(z)$) or antiperiodic ($G^{\pm}(e^{2\pi i}z) = -G(z)$) boundary conditions for the supercurrents with corresponding Laurent expansions:

$$G_P^{\pm}(z) = \sum_{n\in Z} z^{-n-2}G_{n+1/2}^{\pm} \qquad\qquad G_A^{\pm}(z) = \sum_{n\in Z} z^{-n-3/2}G_n^{\pm}$$

The coefficients $G^{\pm}_{\alpha}$ of these expansions close NS and R parts of the discrete N=2 superconformal algebra respectively [31,32,91]

$$\begin{aligned}
[L_n, L_m] &= (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{n+m} \\
[L_n, J_m] &= -mJ_{n+m}, \qquad\qquad m[L_n, G^{\pm}_{\alpha}] = (\frac{n}{2}-\alpha)G^{\pm}_{n+\alpha} \\
[J_n, J_m] &= \frac{c}{12}\delta_{n+m}, \qquad\qquad [J_n, G^{\pm}_{\alpha}] = \pm\frac{1}{2}G^{\pm}_{n+\alpha} \\
\{G^{+}_{\alpha}, G^{-}_{\beta}\} &= 2L_{n+m} + 2(\alpha-\beta)J_{\alpha+\beta} + \frac{c}{3}(\alpha^2-\frac{1}{4})\delta_{\alpha+\beta} \\
\{G^{+}_{\alpha}, G^{+}_{\beta}\} &= \{G^{+}_{\alpha}, G^{+}_{\beta}\} = 0
\end{aligned} \tag{3.4}$$

$\alpha, \beta \in Z + 1/2$ for the NS sector, $\alpha, \beta \in Z$ for the R sector and

$$T(z) = \sum_{-\infty}^{\infty} z^{-n-2}L_n, \qquad\qquad J(z) = \sum_{-\infty}^{\infty} z^{-n-1}J_n$$

Due to the presence of the U(1) current J(z) we have here one more possibility, that is we can choose Z_2 twisted boundary conditions for this current, i.e. $J(e^{2\pi i}z) = -J(z)$ and

$$J(z) = \sum_{-\infty}^{\infty} z^{-n-3/2}J_{n+1/2}$$

Then as a consequences of eq.(3.1) one of the currents $G^{1,2}(z)$ must have periodic and the other one antiperiodic boundary conditions. The two choices are equivalent and taking

$$G^1(z) = \sum_{-\infty}^{\infty} z^{-n-2}G^1_{n+1/2}, \qquad\qquad G^2(z) = \sum_{-\infty}^{\infty} z^{-n-3/2}G^2_{n+1/2}$$

we get the so-called N=2 twisted superconformal algebra:

$$\begin{aligned}
[J_\alpha, J_\beta] &= \frac{c}{12}\delta_{\alpha+\beta}, \qquad [J_\alpha, G^1_\beta] = -\frac{i}{2}G^2_{\alpha+\beta} \\
[J_\alpha, G^2_\beta] &= \frac{i}{2}G^2_{\alpha+\beta}, \qquad \{G^1_\alpha, G^2_m\} = 2(\alpha-m)J_{\alpha+m} \\
\{G^1_\alpha, G^1_\beta\} &= 2L_{\alpha+\beta} + \frac{c}{3}(\alpha^2-\frac{1}{4})\delta_{\alpha+\beta} \\
\{G^2_n, G^2_m\} &= 2L_{n+m} + \frac{c}{3}(n^2-\frac{1}{4})\delta_{n+m}
\end{aligned} \tag{3.5}$$

$\alpha, \beta \in Z + 1/2, n, m \in Z$ (the missed CR's coincide with those in (3.4))

The Cartan subalgebra of the NS and R algebras is two-dimensional and consist of the generators L_0 and J_0. Therefore the corresponding lowest weight representations for each fixed value of c are labeled by two parameters: the conformal dimension Δ and the U(1) charge q. The primary states $\mid \Delta, q >$, which generate these LWR's must satisfy the conditions:

$$\begin{aligned} L_0 \mid \Delta, q > &= \Delta \mid \Delta, q >, \qquad J_0 \mid \Delta, q >= q \mid \Delta, q > \\ L_n \mid \Delta, q > &= G^{\pm}_{\alpha} \mid \Delta, q >= J_m \mid \Delta, q >= 0, \qquad \alpha, n, m > 0 \end{aligned} \tag{3.6}$$

3.2.1 NS-sector

Introducing the $OSp(2 \mid 2)$ invariant vacuum of the theory which belongs to the NS sector

$$L_n \mid 0 >= J_m \mid 0 >= G_{\alpha} \mid 0 >= 0 \qquad n \geq -1, m \geq 0, \alpha \geq -1/2 \tag{3.7}$$

we can realize the NS $\mid \Delta, q >$ primary states in terms of NS primary superfields

$$N^q_{\Delta}(z, \theta^+, \theta^-) = \varphi^q_{\Delta}(z) + \theta^+ \psi^{q-1/2}_{\Delta+1/2}(z) + \theta^- \overline{\psi}^{q+1/2}_{\Delta+1/2}(z) + \theta^- \theta^+ \overline{\varphi}^q_{\Delta+1} \tag{3.8}$$

where

$$\begin{aligned} &\mid \Delta, q >= \varphi^q_{\Delta}(0) \mid 0 > && G^+_{-1/2} \mid \Delta, q >= \overline{\psi}^{q+1/2}_{\Delta+1/2}(0) \mid 0 > \\ &G^-_{-1/2} \mid \Delta, q >= \psi^{q-1/2}_{\Delta+1/2}(0) \mid 0 > && G^+_{-1/2} G^-_{-1/2} \mid \Delta, q >= \overline{\varphi}^q_{\Delta+1}(0) \mid 0 > \end{aligned}$$

These relations determine the following OPE's of the components fields with the generators:

$$\begin{aligned} G^+(z_1)\varphi^q_{\Delta}(z_2) &= \frac{1}{z_{12}} \overline{\psi}^{q+1/2}_{\Delta+1/2}(z_2) + \cdots \qquad G^-(z_1)\varphi^q_{\Delta}(z_2) = \frac{1}{z_{12}} \psi^{q-1/2}_{\Delta+1/2}(z_2) + \cdots \\ G^+(z_1)\psi^{q-1/2}_{\Delta+1/2}(z_2) &= \frac{2(\Delta+q)}{z^2_{12}} \varphi^q_{\Delta}(z_2) + \frac{1}{z_{12}} \overline{\varphi}^q_{\Delta+1}(z_2) + \cdots \\ G^-(z_1)\overline{\psi}^{q+1/2}_{\Delta+1/2}(z_2) &= \frac{2(\Delta-q)}{z^2_{12}} \varphi^q_{\Delta}(z_2) + \frac{1}{z_{12}} (\partial \varphi^q_{\Delta}(z_2) - \overline{\varphi}^q_{\Delta+1}(z_2)) + \cdots \\ G^+(z_1)\overline{\varphi}^q_{\Delta+1}(z_2) &= -\frac{2(\Delta+q)}{z^2_{12}} \overline{\psi}^{q+1/2}_{\Delta+1/2}(z_2) + \cdots, \\ G^-(z_1)\overline{\varphi}^q_{\Delta+1}(z_2) &= \frac{2(\Delta-q)}{z^2_{12}} \psi^{q-1/2}_{\Delta+1/2}(z_2) + \cdots \end{aligned} \tag{3.9}$$

In the NS sector one can combine the currents $T(z), G^{\pm}(z)$ and $J(z)$ in a N=2 supermultiplet i.e. the superstress energy tensor $\mathcal{W}(z, \theta^+, \theta^-)$ having conformal dimension 1 and U(1) charge 0:

$$\mathcal{W}(z, \theta^+, \theta^-) = J(z) + \frac{1}{2} \theta^- G^+(z) - \frac{1}{2} \theta^+ G^-(z) + \theta^+ \theta^- T(z)$$

Then the whole N=2 OPE algebra (3.3) can be easily coded in the singular terms of the following supersymmetric OPE:

$$\begin{aligned} \mathcal{W}(z_1, \theta^+_1, \theta^-_1) \mathcal{W}(z_2, \theta^+_2, \theta^-_2) = \frac{c}{12 \hat{z}^2_{12}} + \Big(\frac{\theta^-_{12} \theta^+_{12}}{\hat{z}^2_{12}} + \frac{1}{2} \frac{\theta^-_{12}}{\hat{z}_{12}} D^+ \\ - \frac{1}{2} \frac{\theta^+_{12}}{\hat{z}_{12}} D^- + \frac{\theta^-_{12} \theta^+_{12}}{\hat{z}_{12}} \partial \Big) \mathcal{W}(z_2, \theta^+_2, \theta^-_2) \end{aligned} \tag{3.10}$$

where

$$\hat{z}_{12} = z_1 - z_2 - \theta_1^+\theta_2^- - \theta_1^-\theta_2^+$$

and

$$D^{\pm} = \frac{\partial}{\partial\theta^{\mp}} + \theta^{\pm}\partial, \qquad \{D^+, D^-\} = 2\partial, \qquad \{D^{\pm}, D^{\pm}\} = 0$$

are the corresponding covariant derivatives. In this language all the OPE's (3.9) unify to give the N=2 supersymmetric Ward Identity (WI)

$$\mathcal{W}(z_1,\theta_1^+,\theta_1^-)N_\Delta^q(z_2,\theta_2^+,\theta_2^-) = \Big(\Delta\tfrac{\theta_{12}^-\theta_{12}^+}{\hat{z}_{12}^2} + \tfrac{1}{2}\tfrac{\theta_{12}^-}{\hat{z}_{12}}D^+ + \\ -\tfrac{1}{2}\tfrac{\theta_{12}^+}{\hat{z}_{12}}D^- + \tfrac{\theta_{12}^-\theta_{12}^+}{\hat{z}_{12}}\partial - q\tfrac{1}{\hat{z}_{12}}\Big) N_\Delta^q(z_2,\theta_2^+,\theta_2^-) \qquad (3.11)$$

The LWR in the NS-sector of the N=2 superconformal algebra corresponding to the LW state $| \Delta, q >$ (fixed c) consists of all the linear combinations of the vectors of the form

$$L_{-n_1}\cdots L_{-n_s}J_{-p_1}\cdots J_{-p_r}G^+_{-\alpha_1}\cdots G^+_{\alpha_t}G^-_{\beta_1}\cdots G^-_{\beta_\tau} \mid \Delta, q > \qquad (3.12)$$

$$n_i, p_i, \alpha_i, \beta_i \geq 0$$

The value $k = \sum_1^s n_i + \sum_1^r p_i + \sum_1^t \alpha_i + \sum_1^\tau \beta_i$ and $m = \frac{1}{2}(t-\tau)$ are called level and relative charge of the state (3.12) respectively. Because of the specific properties of the operators $G^{\pm}_\alpha$, namely $\{G^{\pm}_\alpha, G^{\pm}_\beta\} = 0$, it turns out that only the relative charges 0 (at integer level k) and $\pm 1/2$ (at half-integer level) are allowed. To each vector of the representation $\mathcal{V}(\Delta, q)$ at level k and with relative charge $l/2 (l = 0, \pm 1)$ one can put in correspondence NS superfields with dimension $\Delta + k$ and charge $q + l/2$ belonging to the superconformal family of the primary superfield N_Δ^q. This superconformal family realizes a LWR of the NS sector of N=2 superconformal algebra in terms of fields.

The two- and three-point functions of the primary superfields $N_{\Delta_i}^{q_i}(z,\theta^+,\theta^-)$ are determined (up to an arbitrary constant) by the conditions of infinitesimal $OSp(2 \mid 2)$-invariance, i.e. the finite subalgebra of (3.4) spanned by $J_0, L_0, L_{\pm 1}, G^{\pm}_{\pm 1/2}$. Using the superconformal WI (3.11) and the invariance of the vacuum $\mid 0, 0 >$ these conditions can be written as a system of differential equations:

$$\sum_{i=1}^{s}[z_i^{n+1}(\frac{\partial}{\partial\theta^-} - \theta^-\partial_{z_i}) - 2(n+1)z_i^n\theta_i^+(\Delta_i + q_i + \frac{1}{2}\theta^-\frac{\partial}{\partial\theta_i^-})] < \prod_{k=1}^{s} N_{\Delta_k}^{q_k} >= 0$$

$$\sum_{i=1}^{s}[z_i^{n+1}(\frac{\partial}{\partial\theta^+} - \theta^+\partial_{z_i}) - 2(n+1)z_i^n\theta_i^-(\Delta_i - q_i + \frac{1}{2}\theta^+\frac{\partial}{\partial\theta_i^+})] < \prod_{k=1}^{s} N_{\Delta_k}^{q_k} >= 0$$

where $n = 0$ or -1 since it is enough to consider the $G^{\pm}_{\pm 1/2}$ invariance only. Then the corresponding two-point function (s=2) has the form:

$$< N_{\Delta_1}^{q_1}(z_1,\theta_1^+,\theta_1^-)N_{\Delta_2}^{q_2}(z_2,\theta_2^+,\theta_2^-) >= C^{II}\hat{z}_{12}^{-2\Delta_1}(1 + 2q\frac{\theta_{12}^-\theta_{12}^+}{\hat{z}_{12}})\delta_{\Delta_1-\Delta_2}\delta_{q_1+q_2} \qquad (3.13)$$

In the case of 3-point functions (s=3) one can see that there are three independent solutions corresponding to the following three possibilities for the total U(1) charge $q_1 + q_2 + q_3 = 0, \pm 1/2$ which are dictated by the J_0-invariance

$$\sum_{i=1}^{s}(q_i - \frac{1}{2}\theta_i^+ \partial_{\theta_i^+} + \frac{1}{2}\theta_i^- \partial_{\theta_i^-}) < \prod_{k=1}^{3} N_{\Delta_k}^{q_k}(z_k, \theta_k^+, \theta_k^-) >= 0$$

Thus we have one even 3-point function

$$< N_{\Delta_1}^{q_1}(z_1) N_{\Delta_2}^{q_2}(z_2) N_{\Delta_3}^{-q_1-q_2}(z_3) >= C^{III} \Big[(q_1+q_2)\left(2q_1 \tfrac{\theta_{12}^-\theta_{12}^+}{\hat{z}_{12}} + 1\right) \hat{z}_{12}^{-\delta_3} \hat{z}_{23}^{-\delta_1} \hat{z}_{13}^{-\delta_2}$$
$$\left(2(q_1+q_2)\tfrac{\theta_{23}^-\theta_{23}^+}{\hat{z}_{23}} + 1\right) - \Delta_3\delta_2 \left(\tfrac{\theta_{12}^-}{\hat{z}_{12}} - \tfrac{\theta_{23}^-}{\hat{z}_{23}}\right)\left(\tfrac{\theta_{12}^+}{\hat{z}_{12}} - \tfrac{\theta_{23}^+}{\hat{z}_{23}}\right) \hat{z}_{12}^{-\delta_3+1} \hat{z}_{23}^{-\delta_1+1} \hat{z}_{13}^{-\delta_2-1} \tag{3.14}$$
$$\Big[-2\Delta_3\delta_2 \theta_{12}^-\theta_{12}^+\theta_{23}^-\theta_{23}^+ \left(\tfrac{q_1}{\hat{z}_{23}} + \tfrac{q_1+q_2}{\hat{z}_{12}}\right) \hat{z}_{12}^{-\delta_3} \hat{z}_{23}^{-\delta_1} \hat{z}_{13}^{-\delta_2-1} \Big]$$

$$\delta_1 = \Delta_2 + \Delta_3 - \Delta_1, \qquad \delta_2 = \Delta_1 + \Delta_3 - \Delta_2, \qquad \delta_3 = \Delta_1 + \Delta_2 - \Delta_3$$

and two odd ones

$$< N_{\Delta_1}^{q_1}(z_1) N_{\Delta_2}^{q_2}(z_2) N_{\Delta_3}^{-q_1-q_2+1/2}(z_3) >= C^{III,-} \hat{z}_{12}^{-\delta_3} \hat{z}_{23}^{-\delta_1} \hat{z}_{13}^{-\delta_2}$$
$$\left\{ \left(\tfrac{\theta_{12}^+}{\hat{z}_{12}} - \tfrac{\theta_{23}^+}{\hat{z}_{23}}\right) \left(\tfrac{\hat{z}_{12}\hat{z}_{23}}{\hat{z}_{13}}\right)^{1/2} - (\hat{z}_{12}\hat{z}_{23}\hat{z}_{13})^{1/2} \right. \tag{3.15}$$
$$\left. \{[2q_1\theta_{23}^+\theta_{12}^-\theta_{12}^+ + (1-2(q_1+q_2))\theta_{12}^+\theta_{23}^-\theta_{23}^+]\right\}$$

$$< N_{\Delta_1}^{q_1}(z_1) N_{\Delta_2}^{q_2}(z_2) N_{\Delta_3}^{-q_1-q_2-1/2}(z_3) >= C^{III,+} \hat{z}_{12}^{-\delta_3} \hat{z}_{23}^{-\delta_1} \hat{z}_{13}^{-\delta_2}$$
$$\left\{ \left(\tfrac{\theta_{12}^-}{\hat{z}_{12}} - \tfrac{\theta_{23}^-}{\hat{z}_{23}}\right) \left(\tfrac{\hat{z}_{12}\hat{z}_{23}}{\hat{z}_{13}}\right)^{1/2} - (\hat{z}_{12}\hat{z}_{23}\hat{z}_{13})^{1/2} \right. \tag{3.16}$$
$$\left. \{[2q_1\theta_{23}^-\theta_{12}^-\theta_{12}^+ - (1+2(q_1+q_2))\theta_{12}^-\theta_{23}^-\theta_{23}^+\right\}$$

where C^{III} and $C^{III,\pm}$ are arbitrary constants. The fact that we can have three different 3-point functions has an important consequences in the analysis of the fusion rules for the NS-fields. As we shall show in Sec.(3.3) it gives rise to three independent NS fusion rules - one even and two odd ones.

In what follows we will be interested in the so-called degenerate unitary representations only. Their main peculiarity is the existence of null-vectors at given levels with given relative charges, i.e. states which are again primary. This properties gives rise to a certain relation between the parameters characterizing these representations: the central charge c, dimension Δ and U(1) charge q ("Kac formula"), which completely classify them. For the NS sector of the N=2 superconformal algebra this formula reads [31,32,34]

$$c = 3 - \frac{6}{p+2}, \qquad p = 1, 2\cdots \qquad q_s = \frac{s}{2(p+2)} \tag{3.17}$$
$$\Delta_{n1}^{s} = \frac{(p+2-n)^2 - s^2 - 1}{4(p+2)} \qquad n = 0, 1, \cdots, p+1, \qquad |s| \le p-n+1$$
$$\Delta_{n0}^{s} = \frac{(n+|s|)^2 - s^2 - 1}{4(p+2)} \qquad n = 1, 3, 5 \qquad |s| + n \le p+1$$

and correspondingly one can introduce two kinds of NS superfields: N^s_{n1} which have a degeneracy at level n with zero relative charge and N^s_{n0} having level $n/2$ degeneracy with relative charge $\pm 1/2$. Due to the existence of the null vectors (which have zero norm) the corresponding LWR's are in general reducible ones. To find an irreducible representation one factorize the invariant subspace spanned by the null vectors, by imposing the covariant conditions

$$| \Delta^s_{n1} + n, q >= 0 \qquad or \qquad | \Delta^s_{n0} + \frac{n}{2}, q \pm \frac{1}{2} >= 0 \tag{3.18}$$

In the NS sector these conditions, together with the Ward identities (3.10) lead to super differential equations for the n-point functions of the corresponding fields. These equations allow one to find explicitely the fusion rules and the 4-point functions of all the fields of the model. In this sense the theories cointaing degenerate unitary fields only ("minimal models") are completely solvable.

The simplest null vectors (at level 1/2 and relative charge $\pm 1/2$) have the form

$$G^+_{-1/2} \mid \Delta, q >= 0, \qquad\qquad G^-_{-1/2} \mid \Delta, q >= 0$$

provided $\Delta = \pm q$. They generate the so called N=2 chiral superfields $N^+(z, \theta^+, \theta^-)$ and $N^-(z, \theta^+, \theta^-)$. Indeed combining the relation (3.9) with the condition coming from the first null vector (requiring $\Delta = q$) one can easily extract the corresponding consequences for the superfield N^q_Δ, namely:

$$\overline{\psi}^{q+1/2}_{\Delta+1/2} = 0, \qquad\qquad \overline{\varphi}^q_{\Delta+1} = \partial \varphi^q_\Delta$$

and therefore

$$N^{q+}_\Delta = \varphi^q_\Delta + \theta^+ \psi^{q-1/2}_{\Delta+1/2} + \theta^- \theta^+ \partial \varphi^q_\Delta, \qquad D^+ N^+ = 0 \tag{3.19}$$

The similar result holds for the other chiral field

$$N^{q-}_\Delta = \varphi^q_\Delta + \theta^- \overline{\psi}^{q+1/2}_{\Delta+1/2} + \theta^- \theta^+ \partial \varphi^q_\Delta, \qquad D^- N^- = 0 \tag{3.20}$$

The first nontrivial example is the level 1 (relative charge 0) null vector [68]

$$| \chi^0_1 >= (J_{-1} - \frac{1}{2(2\Delta + 1)} G^+_{-1/2} G^-_{-1/2} + \frac{1 - 2q}{2(2\Delta + 1)} L_{-1}) \mid \Delta, q >$$

if

$$\Delta = \frac{c - 12q^2}{2(3 - c)}$$

Considering the following zero 3-point function

$$< 0 \mid \varphi^{q_x}_{\Delta_x}(\infty) N^{q_2}_{\Delta_2}(z, \theta^+, \theta^-)(J_{-1} - \frac{1}{2(2\Delta + 1)} G^+_{-1/2} G^-_{-1/2} + \frac{1 - 2q}{2(2\Delta + 1)} L_{-1}) \mid \Delta, q >= 0$$

(or analogous zero 4-point function) one can derive superdifferential equations for the 3- and 4-point functions of the primary NS-fields and find in this way the corresponding NS FR's and the explicit form of the 4-point function, as it is done in ref. [69] for some simple cases. In general this procedure is rather complicated and we shall follow in the next sections the other certainly more powerful methods which are based on the Coulomb gas [68] and the Zamolodchkov-Fateev parafermionic constructions [33,34].

3.2.2 R-sector

In the R-sector we have in addition to L_0 and J_0 (which form the true Cartan subalgebra) the zero-modes $G_0^{\pm}$ of the supercurrents $G^{\pm}(z)$ with the properties:

$$\begin{aligned} [L_0, G_0^{\pm}] &= \{G_0^+, G_0^+\} = \{G_0^-, G_0^-\} = 0 \\ [J_0, G_0^{\pm}] &= \pm\frac{1}{2}G_0^{\pm}, \qquad \{G_0^+, G_0^-\} = 2L_0 - \frac{c}{24} \end{aligned} \tag{3.21}$$

As a consequence of eq. (3.21) for each $\Delta \neq c/24$ we have to consider two R-states $\mid \Delta, q >$ and $\mid \Delta, q + 1/2 >$ satisfying the following conditions:

$$\begin{aligned} G_0^+ \mid \Delta, q >&= \sqrt{2\Delta - \frac{c}{24}} \mid \Delta, q + \frac{1}{2} >, \qquad G_0^- \mid \Delta, q >= 0 \\ G_0^- \mid \Delta, q + \frac{1}{2} >&= \sqrt{2\Delta - \frac{c}{24}} \mid \Delta, q >, \qquad G_0^+ \mid \Delta, q + \frac{1}{2} >= 0 \end{aligned} \tag{3.22}$$

Thus we can introduce the corresponding R-fields R_Δ^q as fields creating the R-states from the NS-vacuum $\mid 0, 0 >$:

$$\mid \Delta, q >_R = R_\Delta^q(0) \mid 0, 0 >$$

Using this fact together with eq. (3.22) and the corresponding mode expansions of $G^{\pm}(z)$ we get the counterpart of the W.I. (3.9) for the R-sector:

$$\begin{aligned} G^+(z_1)R_\Delta^q(z_2) &= \frac{\sqrt{2\Delta - \frac{c}{24}}}{z_{12}^{3/2}} R_\Delta^{q+1/2}(z_2) + \cdots \\ G^-(z_1)R_\Delta^{q+1/2}(z_2) &= \frac{\sqrt{2\Delta - \frac{c}{24}}}{z_{12}^{3/2}} R_\Delta^q(z_2) + \cdots \\ G^+(z_1)R_\Delta^{q+1/2}(z_2) &= G^-(z_1)R_\Delta^q(z_2) = \mathcal{O}(\sqrt{z_{12}}) \end{aligned} \tag{3.23}$$

The structure of degenerate LWU representations in the R-sector is very similar to that one for the NS sector. The formula for the dimensions and charges of the degenerate fields is now

$$\begin{aligned} c &= 3 - \frac{6}{p+2}, \quad p = 1, 2\cdots \qquad q_s = \frac{s-r}{2(p+2)} + \frac{r}{4}, r = \pm 1 \\ \Delta_{n1}^{sr} &= \frac{(p+2-n)^2 - (s-r)^2 - 1}{4(p+2)} + \frac{1}{8} \qquad n = 1, \cdots, p+1; \qquad \mid s \mid \leq p - n + 1 \\ \Delta_{n0}^{sr} &= \frac{(n + \mid s - r \mid)^2 - (s-r)^2 - 1}{4(p+2)} + \frac{1}{8} \qquad n = 0, 2, 4, \cdots \qquad \mid s - r \mid + n \leq p \end{aligned} \tag{3.}$$

In the conformal family of the primary R-field $R_{n1}^{sr}(z)$ there is a degeneracy at level n with relative charge 0, while the null-field for the R^{sr}_{n0} is at level $n/2$ (which is an integer now) with relative charge $\pm 1/2$. Again as before in order to extract the irreducible part of the representation one has to impose the conditions (3.18). However because of the branch singularities in the WI's (2.20) we have to modify the null vector's method following the so called analytic method described in Sec.(2.3), for the R-sector of the N=1 models. Even though this method is effective only for the fields having degeneracy at the lowest levels, we present here few examples in order to demonstrate some specific features of this method. We postpone the general discussion of the FR's and the 4-point functions of the R-fields to Sec.(3.3) and (3.4).

Let us consider the FR's including fields that have degeneracy at 1^{st} level and relative charges 0 and $-1/2$. The corresponding null vectors are

$$\mid \chi_1^0 >= [L_{-1} - 4\frac{2\Delta + 3q}{c+4q}J_{-1} + (q - \frac{c(2\Delta+3q)}{(3\Delta - c/8)(c+4q)})G^-_{-1}G^+_0]R^q_\Delta(0) \mid 0 > \tag{3.25}$$

if

$$\Delta = \frac{3/8c - 6q(q+1/2)}{3-c}$$

and

$$\mid \chi_1^{-1/2} >= G^-_{-1}R^q_\Delta(0) \mid 0 > \tag{3.26}$$

if

$$\Delta = -2q - \frac{c}{8}$$

Then for the field $R^{q_1}_{\Delta_1}$ having degeneracy at 1^{st} level with relative charge $-1/2$ eqs. (3.18) and (3.26) imply

$$< 0 \mid N^{q_x}_{\Delta_x}(z_1)R^{q_2}_{\Delta_2}(z_2)G^-_{-1}R^{q_1}_{\Delta_1}(0) \mid 0 >= 0$$

Using the commutation relations (3.4) this equality can be transformed into

$$< 0 \mid N^{q_x}_{\Delta_x}(z_1)R^{q_2}_{\Delta_2}(z_2)(L_{-1} + J_{-1} - \frac{1}{2}G^-_{-1}G^+_0)R^{q_1}_{\Delta_1}(0) \mid 0 >= 0$$

It can be easily shown that

$$\begin{aligned} < N^{q_x}_{\Delta_x}(z_1)R^{q_2}_{\Delta_2}(z_2)L_{-1}R^{q_1}_{\Delta_1}(0) > &= -(\partial_{z_1} + \partial_{z_2}) < N^{q_x}_{\Delta_x}(z_1)R^{q_2}_{\Delta_2}(z_2)R^{q_1}_{\Delta_1}(0) > \\ < N^{q_x}_{\Delta_x}(z_1)R^{q_2}_{\Delta_2}(z_2)J_{-1}R^{q_1}_{\Delta_1}(0) > &= [(q_1+q_2)\frac{1}{z_2} - q_2\frac{1}{z_1}] < N^{q_x}_{\Delta_x}(z_1)R^{q_2}_{\Delta_2}(z_2)R^{q_1}_{\Delta_1}(0) > \end{aligned}$$

In order to find the differential equation for the 3-point function $< N_{\Delta_x}R_{\Delta_1}R_{\Delta_2} >$ we have to construct the function $< N^{q_x}_{\Delta_x}R^{q_2}_{\Delta_2}G^-_{-1}G^+_0R^{q_1}_{\Delta_1} >$ in terms of the 3-point function $< N_{\Delta_x}R_{\Delta_1}R_{\Delta_2} >$. To do this we have to start from an auxiliar 4-point function $< NRG(w)R >$ and use the analytic properties and the asymptotic behaviour of the currents $G^\pm(z)$ (similarly to what was done in section (2.3) to derive the function we were interested in). Omitting the straightforward but slightly tedious computations we present the final algebraic equation for the unknown dimension Δ_x ($q_x = -q_1 - q_2$ from the charge conservation):

$$D_x = \frac{1}{2}\Delta_1 + \Delta_2 + q_2 + \frac{c}{48}$$

Thus we have obtained the simplest FR's for the R-fields $R_{\Delta_{1,2}}$ having the first level degeneracy

$$R^{q_1}_{\Delta_1} R^{q_2}_{\Delta_2} \sim [N^{-q_1-q_2}_{\frac{1}{2}\Delta_1+\Delta_2+q_2+\frac{c}{48}}]$$

As a second example we will compute the simplest 4-point functions in the R- sector, namely those of the fields with dimensions $\Delta = c/24$ and U(1) charges $q = \pm c/12$ (the lowest dimensional fields in the R-sector). They have a zero level degeneracy:

$$G^{\pm}_0 R^{\pm\frac{c}{12}}_{\frac{c}{24}}(0) \mid 0 >= 0 \tag{3.27}$$

It follows from eq. (3.26) that $R^{-\frac{c}{12}}_{\frac{c}{24}}$ has also degeneracy at level one and relative charge $-1/2$ (similarly $R^{+\frac{c}{12}}_{\frac{c}{24}}$ has a degeneracy at 1^{st} level with relative charge $+1/2$). From (3.26) and (3.27) one can find

$$\{G^-_{-1}, G^+_0\} R^{-\frac{c}{12}}_{\frac{c}{24}} = 2(L_{-1} + J_{-1}) R^{-\frac{c}{12}}_{\frac{c}{24}} = 0 \tag{3.28}$$

Let denote the charge $-c/12$ with q. The charge conservation implies that the only nonzero functions are:

$$\begin{aligned} < R^q(z_1)R^q(z_2)R^{-q}(z_3)R^{-q}(z_4) > &= F_{++--}(z_1, \cdots, z_4) \\ < R^q(z_1)R^{-q}(z_2)R^q(z_3)R^{-q}(z_4) > &= F_{+-+-}(z_1, \cdots, z_4) \\ < R^q(z_1)R^{-q}(z_2)R^{-q}(z_3)R^q(z_4) > &= F_{+--+}(z_1, \cdots, z_4) \end{aligned} \tag{3.29}$$

(and also their mirror images which exactly coincide with the latter ones).

The equality (3.28) gives

$$< (L_{-1} + J_{-1})R^q(z_1)R^q(z_2)R^{-q}(z_3)R^{-q}(z_4) >= 0$$

Using the WI's:

$$\begin{aligned} < J_{-1}R^q(z_1)R^q(z_2)R^{-q}(z_3)R^{-q}(z_4) > &= q\left(\frac{1}{z_{12}} - \frac{1}{z_{13}} - \frac{1}{z_{14}}\right) F_{++--}(z_1, \cdots, z_4) \\ < L_{-1}R^q(z_1)R^q(z_2)R^{-q}(z_3)R^{-q}(z_4) > &= \partial_{z_1} F_{++--}(z_1, \cdots, z_4) \end{aligned}$$

we obtain the following differential equation for the function F_{++--}

$$\left[\partial_{z_1} + q\left(\frac{1}{z_{12}} - \frac{1}{z_{13}} - \frac{1}{z_{14}}\right)\right] F_{++--}(z_1 \cdots z_4) = 0 \tag{3.30}$$

The conformal invariance implies that

$$F_{++--}(z_1 \cdots z_4) = (z_{13}z_{24})^{-2\Delta} Y(x), \qquad x = \frac{z_{12}z_{34}}{z_{13}z_{24}} \tag{3.31}$$

Thus the eq. (3.30) transforms into the following ordinary differential equation for $Y(x)$:

$$x(1-x)\frac{dY}{dx} - 2\Delta Y = 0$$

and its general solution has the form:

$$Y(x) = c_1(x(1-x))^{2\Delta}x^{4\Delta}$$

where c_1 is an arbitrary constant. So the final result is

$$F_{++--}(z_1, \cdots z_4) = c_1(z_{13}z_{24})^{-2\Delta}(x(1-x))^{-2\Delta}x^{4\Delta} \tag{3.32}$$

In the same way we get for the other functions similar results:

$$F_{+-+-}(z_1, \cdots z_4) = c_2(z_{13}z_{24})^{-2\Delta}(x(1-x))^{-2\Delta} \tag{3.33}$$
$$F_{+--+}(z_1, \cdots z_4) = c_3(z_{13}z_{24})^{-2\Delta}(x(1-x))^{-2\Delta}(1-x)^{4\Delta} \tag{3.34}$$

and therefore the 4-point function of the fields with $\Delta = c/24$ is

$$< R_{\frac{c}{24}}(z_1)R_{\frac{c}{24}}(z_2)R_{\frac{c}{24}}(z_3)R_{\frac{c}{24}}(z_4) >= (z_{13}z_{24})^{-2\Delta}(x(1-x))^{-2\Delta}[c_2+c_1x^{4\Delta}+c_2(1-x)^{4\Delta}]$$

The implementation of locality is straightforward, here giving the following simple form of the crossing symmetric 2-D function:

$$< R_{\frac{c}{24}}(z_1)R_{\frac{c}{24}}(z_2)R_{\frac{c}{24}}(z_3)R_{\frac{c}{24}}(z_4) >= \frac{1}{2} \mid z_{13}z_{24} \mid^{-4\Delta} \Big\{\mid x(1-x) \mid^{-4\Delta} + \tag{3.35}$$
$$+ \left|\frac{x}{1-x}\right|^{4\Delta} + \left|\frac{1-x}{x}\right|^{4\Delta}\Big\}$$

3.2.3 Twisted sector

In T-sector because of the antiperiodic boundary condition of the U(1) current J(z) J_0 does not exist and hence the primary states are labeled by the value of the conformal dimension Δ only:

$$\begin{aligned} L_0 \mid \Delta >= & \ \Delta \mid \Delta > \\ L_n \mid \Delta >= & \ J_\alpha \mid \Delta >= G^1_\alpha \mid \Delta >= G^2_n \mid \Delta >= 0, \qquad n, \alpha > 0 \end{aligned} \tag{3.36}$$

Similarly to the R-sector of the N=1 SUSY $\mid \Delta >$ is doubly degenerated in the sense that to each primary state $\mid \Delta, + >$ it corresponds a state

$$\mid \Delta, - >\sim G^2_0 \mid \Delta, + >$$

with the same dimension Δ and which is again primary one. Due to the properties of G^2_0 we have for the corresponding primary fields:

$$G^2(z_1)T^{\pm}_{\Delta}(z_2) = \sqrt{\Delta - \frac{c}{24}}\frac{1}{z_{12}^{3/2}}T^{\mp}_{\Delta}(z_2) + \cdots \tag{3.37}$$

It is clear from the algebra (3.5) that the twisted N=2 multiplet has more complicated structure. At level 1/2 we have in general two independent descendents:

$$\begin{aligned} J_{-1/2} \mid \Delta, \pm > &= J_{-1/2}T^{\pm}_{\Delta}(0) \mid 0 > \equiv t^{\pm}_{\Delta+1/2}(0) \mid 0 > \\ G^1_{-1/2} \mid \Delta, \pm > &= T^{\pm}_{\Delta+1/2}(0) \mid 0 > \end{aligned} \tag{3.38}$$

which are not primary states of the full N=2 algebra. Considering only N=1 subalgebra of (3.5) (which is generated by the supercurrent $G^1(z)$) we can combine the fields $T^{\pm}_{\Delta}(z)$ and $T^{\pm}_{\Delta+1/2}(z)$ in one N=1 superfield

$$T^{\pm}_{\Delta}(z,\theta) = T^{\pm}_{\Delta}(z) + \theta T^{\pm}_{\Delta+1/2}(z)$$

The generators of the N=1 R-subalgebra of (3.5) mix the fields $t^{\pm}_{\Delta+1/2}(z)$ and $T^{\pm}_{\Delta+1/2}(z)$, for example

$$\begin{aligned} G^2(z_1)T^{\pm}_{\Delta+1/2}(z_2) &= -\frac{\sqrt{\Delta-\frac{c}{24}}}{z_{12}^{3/2}}T^{\mp}_{\Delta+1/2}(z_2) - \frac{1}{z_{12}^{3/2}}t^{\pm}_{\Delta+1/2}(z_2) \\ G^2(z_1)t^{\pm}_{\Delta+1/2}(z_2) &= \frac{1}{z_{12}^{3/2}}(-\frac{1}{2}T^{\pm}_{\Delta+1/2}(z_2) + \sqrt{\Delta-\frac{c}{24}}t^{\mp}_{\Delta+1/2}(z_2)) \end{aligned} \tag{3.39}$$

One has to introduce also the field $t^{\pm}_{\Delta+1}(z) = G^1_{-1/2}t^{\pm}_{\Delta+1/2}$. Then to obtain the full supercovariant FR's, i.e. to obtain in the RHS the whole NS and R multiplet, we must consider all the product of the fields $T^{\pm}_{\Delta}(z)$, $T^{\pm}_{\Delta+1/2}(z)$, $t^{\pm}_{\Delta+1/2}(z)$ and $t^{\pm}_{\Delta+1}(z)$. In order to be more complete we will present here some of the remaining OPE's of these fields with the generators (the most singular terms only):

$$\begin{aligned} J(z_1)T^{\pm}_{\Delta+1/2}(z_2) &= -\frac{1}{2}\frac{\sqrt{\Delta-\frac{c}{24}}}{z_{12}^{3/2}}T^{\mp}_{\Delta}(z_2) \\ J(z_1)t^{\pm}_{\Delta+1/2}(z_2) &= -\frac{c}{24}\frac{1}{z_{12}^{3/2}}T^{\pm}_{\Delta}(z_2) \\ G^1(z_1)t^{\pm}_{\Delta+1/2}(z_2) &= \frac{1}{2}\frac{\sqrt{\Delta-\frac{c}{24}}}{z_{12}^{3/2}}T^{\mp}_{\Delta}(z_2) + \frac{1}{z_{12}}t^{\pm}_{\Delta+1}(z_2) \end{aligned} \tag{3.40}$$

As in the NS and R sectors we are interested in the LWU representations only. The series of the dimensions of the degenerate primary fields is given by

$$\Delta_n = \frac{(\frac{p+2}{2}-n)^2-1}{4(p+2)} + \frac{1}{8}, \qquad n = 1,2,\cdots,p+2 \tag{3.41}$$

where the field T_n has degeneracy at level $n/2 (n \in Z_+)$.

The simplest null vector of the N=2 T-algebra (2.5) at level 1/2

$$(G^1_{-1/2}G^2_0 - 4\Delta J_{-1/2})T_1(0) \mid 0 >= 0$$

if

$$\Delta_1 = \frac{c}{8(3-c)}$$

allows to relate the fields with dimension $\Delta + 1/2$:

$$t^{\pm}_{\Delta+1/2}(z) = \frac{1}{4\Delta}\sqrt{\Delta - \frac{c}{24}} T^{\pm}_{\Delta+1/2}(z)$$

This simplifies the WI's of such field and makes it possible to work with the superfiels $T^{\pm}_{\Delta}(z,\theta)$ only.

In order to obtain the FR's and the 4-point functions of the N=2 twisted fields one may use in principle the method similar to that of the R sector. However the computations, even for the simplest 1/2 level, are more complicated and we will not present them here. The general formulas are derivated below using the Coulomb gas representation of the T-sector of the N=2 superconformal theory.

We will conclude this section with the qualitative description of the structure of the OPE algebras of the fields in N=2 m.m.. The full space F of the fields of a given m.m. can be splited into two sectors with respect to the Z_2 boundary conditions of the U(1)-current J(z):

1. "winding " sector$\equiv NS \oplus R$:

$$\begin{aligned} J(e^{2\pi i}z)\phi_k(0) &= J(z)\phi_k(0) \\ G^{\pm}(e^{2\pi i}z)\phi_k(0) &= e^{i\pi k}G^{\pm}(z)\phi_k(0) \end{aligned} \tag{3.42}$$

2. twisted sector :

$$\begin{aligned} J(e^{2\pi i}z)\phi_{tw}(0) &= -J(z)\phi_{tw}(0) \\ G^1(e^{2\pi i}z)\phi_{tw}(0) &= G^1(z)\phi_{tw}(0) \\ G^2(e^{2\pi i}z)\phi_{tw}(0) &= -G^2(z)\phi_{tw}(0) \end{aligned} \tag{3.43}$$

so that we have:

$$F = \bigoplus_{k=0,1} \{\phi_k\} \oplus \{\phi_{tw}\}$$

The NS, R and T fields belong to the $\{\phi_0\}, \{\phi_1\}$ and $\{\phi_{tw}\}$ subspaces respectively. They produce the corresponding boundary conditions for the N=2 generators $T, G^{\pm}, J$.

In the OPE's of these fields, one has to preserve this Z_2 structure and therefore we have:

$$\phi_{k_1}\phi_{k_2} \in \{\phi_{k_1+k_2(mod2)}\} \tag{3.44}$$

which gives

$$NN \sim N \qquad RN \sim R, \qquad RR \sim N$$

Since the product of two twisted fields produce "winding" boundary conditions for the J(z) we get

$$\phi_{tw}\phi_{tw} \in \bigoplus_{k=0,1} \{\phi_k\} \tag{3.45}$$

or

$$TT \sim N + R \tag{3.46}$$

The corresponding exact fusion rules will be described in Sec.(3.3.3)

3.3 N=2 extended Coulomb gas

The Coulomb gas representation of the N=2 extended supersymmetric models is sligtly different from that we have in the case of N=0 [10] and N=1 ones [13,17,18]. It is based on the theory of two N=2 NS chiral superfields [68]

$$\begin{aligned} S^{+}(z,\theta^{+},\theta^{-}) &= \varphi(z) + \sqrt{2}\theta^{-}\overline{\psi}(z) + \theta^{-}\theta^{+}\partial\varphi(z) \\ S^{-}(z,\theta^{+},\theta^{-}) &= \overline{\varphi}(z) + \sqrt{2}\theta^{+}\psi(z) - \theta^{-}\theta^{+}\partial\overline{\varphi}(z) \end{aligned} \tag{3.47}$$

where

$$\varphi = \frac{1}{2}(\varphi_1 + i\varphi_2), \qquad \psi = \frac{1}{2}(\psi_1 + i\psi_2)$$

are free complex scalar and spinor fields respectively. The chirality is guaranteed by the covariant conditions

$$D^{+}S^{-} = D^{-}S^{+} = 0$$

The free invariant action of the theory

$$A(S^{+},S^{-}) = \frac{1}{2}\int dzd\overline{z}\delta\theta^{+}d\theta^{-}d\overline{\theta}^{+}d\overline{\theta}^{-}(S^{+}+\overline{S}^{+})(S^{-}+\overline{S}^{-})$$

gives the following propagators of the chiral fields

$$< S^{\pm}(z_1,\theta_1^{+},\theta_1^{-})S^{\mp}(z_2,\theta_2^{+},\theta_2^{-}) >= -ln\frac{\hat{z}_{12}}{R} \mp \frac{\theta_{12}^{-}\theta_{12}^{+}}{\hat{z}_{12}} \tag{3.48}$$

or in terms of the component fields

$$< \psi(z_1)\overline{\psi}(z_2) >= -\frac{1}{z_{12}}, \qquad < \varphi(z_1)\overline{\varphi}(z_2) >= -ln\frac{z_{12}}{R} \tag{3.49}$$

(R is an infrared cut-off).

Following the analogy with the case of N=1 SUSY [13] one can define in a proper way the NS vertices:

$$N_{\alpha,\overline{\alpha}}(z,\theta^{+},\theta^{-}) =: e^{i(\alpha S^{-}(z,\theta^{+},\theta^{-})+\overline{\alpha}S^{+}(z,\theta^{+},\theta^{-}))}: \tag{3.50}$$

which are labeled by two real numbers$(\alpha, \overline{\alpha})$ called charges. The corresponding improved action:

$$\mathcal{A}(S^{\pm}, \hat{R}) = A(S^{+}, S^{-}) + \int d^2 z d\theta^{\pm} d\overline{\theta}^{\pm} \{2\beta \hat{R}(S^{+} + \overline{S}^{+}) + 2\overline{\beta}\hat{R}(S^{-} + \overline{S}^{-})\} \quad (3.51)$$

where

$$\hat{R} = \tau + \theta^{+}\chi^{-} + \theta^{-}\chi^{+}\overline{\theta}^{+}\overline{\chi}^{-} + \cdots \theta^{-}\theta^{+}\overline{\theta}^{-}\overline{\theta}^{+} R$$

leads effectively to the extra vertex at infinity with charges $(-2\beta, -2\overline{\beta})$. In what follows we have made a special choice of the charges at infinity, namely $\beta = \overline{\beta}$. This is due to the fact that we want to have a Coulomb gas description for the T-sector of the algebra too and as we shall see below this is impossible if $\beta \neq \overline{\beta}$.

Thus we get the following N-point function of the NS vertices

$$< \prod_{i=1}^{n} N_{\alpha_i, \overline{\alpha}_i}(z_i, \theta_i^{+}, \theta_i^{-}) >= \int \mathcal{D}S^{\pm} e^{-A(S^{\pm})} N_{-2\beta,-2\beta}(\infty, \theta^{\pm} = 0) \prod_{i=1}^{n} N_{\alpha_i, \overline{\alpha}_i}(i) = \quad (3.52)$$

$$= (\tfrac{a}{R})^{(\sum_{i=1}^{n} \alpha_i - 2\beta)(\sum_{i=1}^{n} \overline{\alpha}_i - 2\beta)} \prod_{i<j} (\hat{z}_{ij})^{\alpha_i \overline{\alpha}_j + \overline{\alpha}_i \alpha_j} exp\{\sum_{k<l}^{n} (\alpha_k \overline{\alpha}_l - \overline{\alpha}_k \alpha_l) \tfrac{\theta_{kl}^{-} \theta_{kl}^{+}}{\hat{z}_{kl}}\}$$

In order to eliminate the cut-off dependence of this function we must impose the neutrality conditions

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \overline{\alpha}_i = 2\beta \quad (3.53)$$

From the action (3.51) one can extract also the form of the improved generators of the N=2 superconformal symmetry:

$$\begin{aligned} J &= \ : \frac{1}{2}\overline{\psi}\psi - i\beta\partial\overline{\varphi} + i\beta\partial\varphi \\ G^{+} &= \ : -\sqrt{2}\overline{\psi}\partial\overline{\varphi} + 2\sqrt{2}i\beta\partial\overline{\psi} : \qquad G^{-} =: -\sqrt{2}\psi\partial\varphi + 2\sqrt{2}i\beta\partial\psi : \\ T &= \ : -\partial\varphi\partial\overline{\varphi} + \frac{1}{2}(\overline{\psi}\partial\psi + \psi\partial\overline{\psi}) + i\beta\partial^2\overline{\varphi} + i\beta\partial^2\varphi \end{aligned} \quad (3.54)$$

In the NS sector of the theory (periodic boundary conditions for all the currents (3.54)) they can be unify in N=2 superstress-energy tensor

$$\mathcal{W} = J + \frac{1}{2}\theta^{-}G^{+} - \frac{1}{2}\theta^{-}G^{+} + \theta^{-}\theta^{+}T$$

Then from (3.54) we have

$$\mathcal{W} =: \frac{1}{4}D^{+}S^{+}D^{-}S^{-} + i\beta\partial S^{+} - i\beta\partial S^{-} : \quad (3.55)$$

Using the Wick theorem and the free propagators (3.48) of the chiral fields $S^{\pm}(z, \theta^{+}, \theta^{-}$ one can easily compute its two point function

$$< \mathcal{W}(z_1, \theta_1^{+}, \theta_1^{-})\mathcal{W}(z_2, \theta_2^{+}, \theta_2^{-}) >= \frac{1}{z_{12}^2}(\frac{1}{4} - 2\beta^2) \quad (3.56)$$

Comparing (3.56) with the already known propagator of $\mathcal{W}$ (the first term in the OPE (3.10)) we find that the central charge of the theory is a function of the charge at infinity β:

$$c = 3 - 24\beta^2 \tag{3.57}$$

Thus, for the minimal models (3.17) it is quantized

$$\beta^2 = \frac{1}{4(p+2)}, p = 1, 2, 3 \cdots \tag{3.58}$$

3.3.1 Coulomb gas representation: NS sector

Using the explicit definitions of the NS vertex $N_{\alpha,\overline{\alpha}} \equiv N^{q_{\alpha,\overline{\alpha}}}_{\Delta_{\alpha,\overline{\alpha}}}$ (3.50) and the super-stress tensor (3.55) one can easily verify that the relations (3.11) are satisfied:

$$\begin{aligned} \mathcal{W}(z_1, \theta_1^+, \theta_1^-) N_{\alpha,\overline{\alpha}}(z_2, \theta_2^+, \theta_2^-) = &\left\{ \tfrac{1}{2\hat{z}_{12}} (\theta_{12}^- D^+ - \theta_{12}^+ D^-) + \tfrac{\theta_{12}^- \theta_{12}^+}{\hat{z}_{12}} \partial + \right. \\ &\left. + (\alpha\overline{\alpha} - \beta(\alpha + \overline{\alpha})) \tfrac{\theta_{12}^- \theta_{12}^+}{\hat{z}_{12}^2} - \beta(\alpha - \overline{\alpha}) \tfrac{1}{\hat{z}_{12}} \right\} N_{\alpha,\overline{\alpha}}(z_2, \theta_2^+, \theta_2^-) \end{aligned} \tag{3.59}$$

and therefore the vertex (3.50) represents NS primary superfield of dimension and U(1) charge given by

$$\begin{aligned} \Delta(\alpha, \overline{\alpha}) = &\ \alpha\overline{\alpha} - \beta(\alpha + \overline{\alpha}) \\ q(\alpha, \overline{\alpha}) = &\ \beta(\alpha - \overline{\alpha}) \end{aligned} \tag{3.60}$$

We want to note some symmetries of this formulas which will be useful below. For instance the change

$$\begin{aligned} \alpha &\rightarrow 2\beta - \overline{\alpha} \\ \overline{\alpha} &\rightarrow 2\beta - \alpha \end{aligned} \tag{3.61}$$

leads to the same dimension and charge (Δ, q) while the following two changes

$$\begin{aligned} \alpha &\rightarrow 2\beta - \alpha \qquad\qquad & \alpha \rightarrow \overline{\alpha} \\ \overline{\alpha} &\rightarrow 2\beta - \overline{\alpha} \qquad\qquad & \overline{\alpha} \rightarrow \overline{\alpha} \end{aligned} \tag{3.62}$$

give the same dimension Δ but the opposite U(1) charge $-q$.

Let us now introduce the following three NS vertices

$$\begin{aligned} V_{\alpha_+, \overline{\alpha}_+} = &\ : e^{i(\alpha_+ S^- + \overline{\alpha}_+ S^+)} : \\ V_{\alpha_-, 0} = &\ : e^{i(\alpha_- S^-)} : \\ V_{0, \overline{\alpha}_-} = &\ : e^{i(\overline{\alpha}_- S^+)} : \end{aligned} \tag{3.63}$$

with charges $(\alpha_\pm, \overline{\alpha}_\pm)$ chosen in such a way that the corresponding integrals (called screening operators (S.O.))

$$
\begin{aligned}
Q_+ &= \oint_{C_+} dz d\theta^\pm V_{\alpha_+,\overline{\alpha}_+} = \oint dz[i\partial(\overline{\alpha}_+\varphi - \alpha_+\overline{\varphi}) + 2\alpha_+\overline{\alpha}_+\overline{\psi}\psi]e^{i(\alpha_+\overline{\varphi}+\overline{\alpha}_+\varphi)} \\
Q_- &= \oint_{C_-} dz d\theta^+ V_{\alpha_-,0} = i\sqrt{2}\alpha_- \oint dz\psi(z)e^{i\alpha_-\overline{\varphi}} \qquad (3.64) \\
\overline{Q}_- &= \oint_{\overline{C}_-} dz d\theta^- V_{0,\overline{\alpha}_-} = i\sqrt{2}\overline{\alpha}_- \oint dz\overline{\psi}(z)e^{i\overline{\alpha}_-\varphi}
\end{aligned}
$$

(Q_+ are even S.O. whereas $Q_-, \overline{Q}_-$ are odd ones) are invariant under the N=2 superconformal transformations (i.e. have zero dimension and zero charge). The requirement for N=2 superconformal invariance determines the charges $(\alpha_\pm, \alpha_\pm)$ as a functions of the charge at infinity:

$$
\begin{aligned}
&\alpha_+ = 2\beta = \overline{\alpha}_+ \quad , \qquad \Delta(\alpha_+) = 0 = q(\alpha_+) \\
& \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (3.65) \\
&\alpha_- = -\tfrac{1}{2\beta} = \overline{\alpha}_- \quad , \qquad \Delta(\alpha_-) = \frac{1}{2} = \Delta(\overline{\alpha}_-), \qquad q(\alpha_-) = -\frac{1}{2} = -q(\overline{\alpha}_-)
\end{aligned}
$$

The OPE's of the vertices (3.63) with the generators have the remarkable property that all the singular terms are total derivatives:

$$
\begin{aligned}
J(z_1)Q_- &= \oint dz\{0 + nonsing.\} = J(z_1)Q_+ \\
G^\pm(z_1)Q_+ &= i\alpha_+\sqrt{2}\oint dz\{\partial_z(\frac{1}{z_{12}}\overline{\psi}e^{i\alpha_+\varphi}) + nonsing\} \\
G^\mp(z_1)\overline{Q}_- &= \sqrt{2}\oint dz\{\partial_z(\frac{1}{z_{12}}e^{i\alpha_-\overline{\varphi}}) + nonsing\} \\
T^\pm(z_1)\overline{Q}_- &= i\alpha_-\sqrt{2}\oint dz\{\partial_z(\frac{1}{z_{12}}\overline{\psi}e^{i\alpha_-\overline{\varphi}}) + nonsing\}
\end{aligned}
$$

which simply exbits the invariance of the screening operators $Q_+, \overline{Q}_-$. This property enables one to construct the null vectors at given level and relative charge in the family of $V_{\alpha,\alpha}$ using the S.O. and NS vertices according to the construction described in ref. [92]. Consider the contour integral

$$
Q_+N_{2\beta-\overline{\alpha}-\alpha_+,2\beta-\alpha-\alpha_+}(0) = \oint dzd\theta^\pm V_{\alpha_+,\overline{\alpha}_+}(z,\theta^+,\theta^-)N_{2\beta-\overline{\alpha}-\alpha_+,2\beta-\alpha-\alpha_+}(0)
$$

It is evident that this integral does not vanish and hence define nontrivial null vector (satisfying (3.6) if

$$
\alpha_+(4\beta - \alpha - \overline{\alpha} - 2\alpha_+) = -m
$$

with m =positive integer, i.e.

$$
\alpha + \overline{\alpha} = -m\alpha_-
$$

In the general case of n S.O.

$$\oint dz_n d\theta_n^{\pm} V_{\alpha_+,\overline{\alpha}_+}(z_n,\theta_n^+,\theta_n^-)\cdots\oint dz_1 d\theta_1^{\pm} V_{\alpha_+,\overline{\alpha}_+}(z_1,\theta_1^+,\theta_1^-)N_{2\beta-\overline{\alpha}-n\alpha_+,2\beta-\alpha-n\overline{\alpha}_+} \tag{3.66}$$

the similar consideration leads to the following quantization of the charges $(\alpha,\overline{\alpha})$:

$$\alpha+\overline{\alpha}=(1-n)\alpha_+ - m\alpha_- \tag{3.67}$$

Computing the commutator of the expression (3.66) with L_0, J_0 one can recognize also the level and the relative charge of the corresponding null vectors. In this way we have verified that eq. (3.66) indeed defines a null vector, provided that (3.67) holds and that we must have degeneracy at level $n\times m$ with relative charge zero. The condition (3.67) shows that only the sum of two charges is quantized. We have to use in addition the definition of the U(1) charge (3.60) in order to obtain separately α and $\overline{\alpha}$ in terms of $\alpha_\pm$ and q. As far as we are interested in LWR of N=2 superconformal algebra only we take the U(1) charge to be (3.17):

$$q=\beta(\alpha-\overline{\alpha})=\beta s\alpha_+ \tag{3.68}$$

and therefore

$$\begin{aligned}\alpha^s_{nm} &= \frac{1}{2}(1-n+s)\alpha_+ - \frac{m}{2}\alpha_- \\ \overline{\alpha}^s_{nm} &= \frac{1}{2}(1-n-s)\alpha_+ - \frac{m}{2}\alpha_-\end{aligned} \tag{3.69}$$

Comparing now the conformal dimension Δ computed using (3.69) with the already known one for the LWUR (3.17) we obtain that for the minimal models $m=1, n=0,1\cdots,p+1$. Thus, the fields N^s_{n1} given by $V_{\alpha,\alpha}$ with $\alpha=\alpha^s_{n1}$ (3.69) are indeed the primary degenerate fields of the series (3.17). In the similar way we can analyze the other possibilities of constructing null vector by $Q_-$and $\overline{Q}_-$:

$$\oint dz d\theta^+ V_{\alpha_-,0}(z,\theta^+,\theta^-)N_{2\beta-\overline{\alpha}-\alpha_-,2\beta-\alpha}(0) \tag{3.70}$$

and

$$\oint dz d\theta^- V_{0,\overline{\alpha}_-}(z,\theta^+,\theta^-)N_{2\beta-\overline{\alpha},2\beta-\alpha-\overline{\alpha}_-}(0) \tag{3.71}$$

We want to stress here that the application of more than one odd screening operator on the vertices gives identically zero and therefore does not give new nontrivial null vector. The eqs.(3.70)(3.71) obey (3.6) provided

$$\alpha=\frac{1}{2}(1-n)\alpha_+ \tag{3.72}$$

or

$$\overline{\alpha}=\frac{1}{2}(1-n)\alpha_+ \tag{3.73}$$

respectively. The degeneracy for (3.70) (if (3.72) holds) is at level $n/2$, relative charge $+1/2$, while for (3.71) (imposing (3.73)) we obtain $n/2, -1/2$. For the m.m. we have respectively:

$$\begin{aligned} \alpha^s_{n0} &= \frac{1}{2}(1-n)\alpha_+ \\ \overline{\alpha}^s_{n0} &= \frac{1}{2}(1-n-2s)\alpha_+, \qquad s>0 \end{aligned} \tag{3.74}$$

or

$$\begin{aligned} \alpha^s_{n0} &= \frac{1}{2}(1-n+2s)\alpha_+ \\ \overline{\alpha}^s_{n0} &= \frac{1}{2}(1-n)\alpha_+, \qquad s<0 \end{aligned} \tag{3.75}$$

The dimensions obtained from (3.74)(3.75) exactly coincide with those ones of the second discrete series Δ^s_{n0} given by (3.17). Note that the symmetry (3.61) expressed in terms of dimensions and charges becomes

$$(\Delta^s_{n,1}, q_s) \equiv (\Delta^s_{p+2-n-|s|,0}, q_s)$$

and therefore the field N^s_{n1} from the first discrete series exactly coincides with the field $N^s_{p+2-n-|s|,0}$ from the second one:

$$N^s_{n,1} = N^s_{p+2-n-|s|,0} \tag{3.76}$$

In the Coulomb gas picture the correlation functions of the fields $N^{q(\alpha,\overline{\alpha})}_{\Delta(\alpha,\overline{\alpha})}$ are obtained by inserting in the corresponding correlators of the vertices $N_{\alpha,\overline{\alpha}}$ a proper number of screening operators (3.64). They ensure that the neutrality condition (3.53) is satisfied. Also they do not destroy the N=2 superconformal covariance of the functions since they are N=2 superconformally invariant. This screening procedure in the case of 3-point functions allows one to recognize which of them do not vanish, or equivalently to obtain the corresponding FR's for the products of two arbitrary primary fields, following the analysis in ref. [17,18].

As it was shown in Sec.(3.2) the general 3-point function of NS N=2 superfields contains one even (3.14) and two odd (3.15)(3.16) ingredients (with charges $\pm 1/2$ respectively) . This specific structure gives rise to three different FR's -one even and two odd- generated by the corresponding parts of this function. The even 3-point function is obtained here inserting an arbitrary number of even screening operators and an equal number of two kinds of odd ones (in order to have uncharged function). We will demonstrate this fact in the simplest example of the 3-point function without screenings:

$$\Gamma_{\Delta_1\Delta_2\Delta_3} = <\prod_{i=1}^{3} N^{q_i}_{\Delta_i}(z_i,\theta^+_i,\theta^-_i)> = \hat{z}_{12}^{\alpha_1\overline{\alpha}_2+\overline{\alpha}_1\alpha_2}\hat{z}_{13}^{\alpha_1\overline{\alpha}_3+\overline{\alpha}_1\alpha_3}\hat{z}_{23}^{\alpha_2\overline{\alpha}_3+\overline{\alpha}_2\alpha_3}$$
$$exp\{(\alpha_1\overline{\alpha}_2-\overline{\alpha}_1\alpha_2)\frac{\theta^-_{12}\theta^+_{12}}{\hat{z}_{12}} + (\alpha_1\overline{\alpha}_3-\overline{\alpha}_1\alpha_3)\frac{\theta^-_{13}\theta^+_{13}}{\hat{z}_{13}} + (\alpha_2\overline{\alpha}_3-\overline{\alpha}_2\alpha_3)\frac{\theta^-_{23}\theta^+_{23}}{\hat{z}_{23}}\}$$

For simplicity we choose $z \to \infty, z_3 \to 0, \theta_1^{\pm} = \theta_3^{\pm} = 0$ and thus

$$\Gamma_{\Delta_1\Delta_2\Delta_3} = z_1^{\alpha_1\overline{\alpha}_2+\overline{\alpha}_1\alpha_2+\alpha_1\overline{\alpha}_3+\overline{\alpha}_1\alpha_3} z_2^{\alpha_2\overline{\alpha}_3+\overline{\alpha}_2\alpha_3}\left[1+(\alpha_2\overline{\alpha}_3-\overline{\alpha}_2\alpha_3)\frac{\theta_2^-\theta_2^+}{z_2}\right]$$

The neutrality condition (3.53) gives in addition

$$\alpha_1+\alpha_2+\alpha_3 = 2\beta = \overline{\alpha}_1+\overline{\alpha}_2+\overline{\alpha}_3$$

and finally

$$\Gamma_{\Delta_1\Delta_2\Delta_3} = z_1^{-2\Delta} z_2^{\Delta_1-\Delta_2-\Delta_3}\left\{1+[\frac{\Delta_3}{q_3}(\Delta_1-\Delta_2-\Delta_3)-2q_3]\frac{\theta_2^-\theta_2^+}{z_2}\right\}$$

which exactly coincides with the corresponding limit of the even part of the 3-point function (3.14).

Let us now take an even 3-point function with arbitrary number of screenings:

$$\begin{aligned}\Gamma_{\Delta_1\Delta_2\Delta_3} &= <\prod_{i=1}^{k}\oint dz_i d\theta_i^{\pm} V_{\alpha_+,\overline{\alpha}_+}(i)\prod_{j=1}^{l}\oint dz_j d\theta_j^{+} V_{\alpha_-,0}(j)\prod_{s=1}^{l} dz_s d\theta_s^{-} V_{0,\overline{\alpha}_-}(s) \\ &\quad N_{\alpha_1,\overline{\alpha}_1}(\infty,0)N_{\alpha_2,\overline{\alpha}_2}(z,\theta^+,\theta^-)N_{2\beta-\overline{\alpha}_x,2\beta-\alpha_x}(0,0)>= \qquad (3.77)\end{aligned}$$

$$\equiv\{(\alpha_1,\overline{\alpha}_1),(\alpha_2,\overline{\alpha}_2),(2\beta-\overline{\alpha}_x,(2\beta-\alpha_x);k(\alpha_+,\overline{\alpha}_+),l(\alpha_-,0),l(0,\overline{\alpha}_-)\}$$

There exist at least three equivalent ways to do this, depending on which vertex is taken to have conjugate charges:

$$\begin{aligned}\Gamma_{\Delta_1\Delta_2\Delta_3} &= \{(\alpha_1,\overline{\alpha}_1),(2\beta-\overline{\alpha}_2,2\beta-\alpha_2),(\alpha_x,(\overline{\alpha}_x);m(\alpha_+,\overline{\alpha}_+),n(\alpha_-,0),n(0,\overline{\alpha}_-)\} = \\ &= \{(2\beta-\overline{\alpha}_1,2\beta-\alpha_1),(\alpha_2,\overline{\alpha}_2),(\alpha_x,(\overline{\alpha}_x);r(\alpha_+,\overline{\alpha}_+),s(\alpha_-,0),s(0,\overline{\alpha}_-)\}\end{aligned}$$

Now we have to impose the neutrality condition (3.53) and to take an intersection of the three equivalent solutions coming from (3.77). For instance when

$$\alpha_1 = \alpha_{n_1,1}^{s_1}, \qquad \alpha_2 = \alpha_{n_2,1}^{s_2}$$

we obtain

$$\begin{aligned}(\alpha_x+\overline{\alpha}_x)^I &= [1-(n_1+n_2-1-2k)]\alpha_+ + 2l\alpha_- \\ (\alpha_x+\overline{\alpha}_x)^{II} &= [1-(n_2-n_1+1+2m)]\alpha_+ - 2n\alpha_- \\ (\alpha_x+\overline{\alpha}_x)^{III} &= [1-(n_1-n_2+1+2r)]\alpha_+ - 2s\alpha_-\end{aligned}$$

(and always $q_x = -q_1 - q_2$) and therefore

$$N_{n_1,1}^{s_1} N_{n_2,1}^{s_2} = \sum_{k=|n_1-n_2|+1-|s_1+s_2|}^{n_1+n_2-1-|s_1+s_2|} [N_{k,0}^{s_1+s_2}] \qquad (3.78)$$

For the other possibilities: $\alpha_1 = \alpha^{s_1}_{n_1 1}, \alpha_2 = \alpha^{s_2}_{n_2 0}$ and $\alpha_1 = \alpha^{s_1}_{n_1 0}, \alpha_2 = \alpha^{s_2}_{n_2 0}$ the same procedure gives

$$N^{s_1}_{n_1,1} N^{s_2}_{n_2,0} = \sum_{k=|n_1-n_2-|s_2||+1}^{n_1+n_2-1+|s_2|} [N^{s_1+s_2}_{k,1}] \tag{3.79}$$

$$N^{s_1}_{n_1,0} N^{s_2}_{n_2,0} = \sum_{k=|n_1-n_2+|s_1|-|s_2||-|s_1+s_2|+1}^{n_1+n_2-1+|s_1|+|s_2|-|s_1+s_2|-1} [N^{s_1+s_2}_{k,0}] \tag{3.80}$$

Finally, in order to obtain the truncated FR's, we have to use the symmetries (3.76) and to take the intersection of the rules (3.77) for fixed $N^{q_1}_{\Delta_1}, N^{q_2}_{\Delta_2}$.

The odd 3-point function with charge $1/2(-1/2)$ can be obtained if we insert in the correlator an arbitrary number of even screening operators Q_+ but one more of the S.O. $Q_-(\overline{Q}_-)$ than $Q_-(\overline{Q}_-)$. Let us consider for example the correlation function with one inserted screening Q_-. In the limit $z_3 = 0, z_1 \to \infty, \theta^{\pm}_1 = \theta^{\pm}_3 = 0$ we have

$$\Gamma^{odd}_{\Delta_1\Delta_2\Delta_3} = \theta^+_2 z_1^{\alpha_1\overline{\alpha}_2+\alpha_2\overline{\alpha}_1+\alpha_1\overline{\alpha}_3+\alpha_3\overline{\alpha}_1+\alpha_-\overline{\alpha}_1} z_2^{\alpha_2\overline{\alpha}_3+\alpha_3\overline{\alpha}_2+(\overline{\alpha}_2+\overline{\alpha}_3)\alpha_-} \oint dv v^{\overline{\alpha}_3\alpha_-}(1-v)^{\overline{\alpha}_2\alpha_-}$$

Since the neutrality condition implies

$$\begin{aligned} \alpha_1\overline{\alpha}_2 + \overline{\alpha}_1\alpha_2 + \overline{\alpha}_1\alpha_3 + \overline{\alpha}_1\alpha_- &= -2\Delta_1 \\ \alpha_2\overline{\alpha}_3 + \overline{\alpha}_2\alpha_3 + (\overline{\alpha}_2 + \overline{\alpha}_3)\alpha_- &= \Delta_1 - \Delta_2 - \Delta_3 - 1/2 \end{aligned}$$

this 3-point function coincides with the corresponding limit of the odd 3-point function (3.15). The FR's generated by these odd functions can be obtained in the same way as in the even case, but one must keep attention for the right number of odd screening operators. Omitting the detailed calculations here we present the final result only:

$$\begin{aligned} N^{s_1}_{n_1,1} N^{s_2}_{n_2,1} &= \sum_{k=|n_1-n_2|+1}^{n_1+n_2-1} [N^{s_1+s_2\mp(p+2)}_{k,1}] \\ N^{s_1}_{n_1,1} N^{s_2}_{n_2,0} &= \sum_{k=|n_2-n_1+|s_2||+1+|s_1+s_2\mp(p+2)|}^{n_1+n_2-1+|s_2|-|s_1+s_2\mp(p+2)|} [N^{s_1+s_2\mp(p+2)}_{k,0}] \\ N^{s_1}_{n_1,1} N^{s_2}_{n_2,0} &= \sum_{k=|n_2-n_1+|s_2|-|s_1||+1}^{n_1+n_2-1+|s_1|+|s_2|} [N^{s_1+s_2\mp(p+2)}_{k,1}] \end{aligned} \tag{3.81}$$

As in the previous case in order to obtain the true (truncated) odd FR's we have to take the intersection of (3.81). Combining finally the even and odd FR's we are lead to the general ones, i.e. we obtain all the families of the primary fields N^q_Δ that contribute to the OPE of two given superfields.

The 4-point function of the NS fields can be constructed using the same screening procedure: the only difference with the case of 3-point function is that here we have an unique possibility to screen the extra charge. One can easily see from the charge quantization (3.67) that for the field N^s_{nm} we need (n-1) even S.O. Q_+ and m odd ones Q_- and Q_-:

$$
\begin{array}{c}
< N^q_\Delta(1) N^{-q}_\Delta(2) N^q_\Delta(3) N^{-q}_\Delta(4) >= \\
\prod_{i=1}^{n-1} \oint_{C_i^+} du_i d\theta_i^+ d\theta_i^- \prod_{j=1}^{m} \oint_{C_j^-} dv_j d\tilde{\theta}_j^+ \oint_{\overline{C}_j^-} dw_j d\tilde{\theta}_j^- \\
< N_{\alpha,\overline{\alpha}}(1) N_{2\beta-\alpha,2\beta-\overline{\alpha}}(2) N_{\alpha,\overline{\alpha}}(3) N_{\overline{\alpha},\alpha}(4) \\
\prod_{r=1}^{n-1} V_{\alpha_+,\alpha_+}(u_r, \theta_r^+, \theta_r^-) \prod_{s=1}^{m} V_{\alpha_-,0}(v_s, \tilde{\theta}_s^\pm) V_{0,\overline{\alpha}_-}(w_s, \tilde{\theta}_s^\pm) >
\end{array}
\qquad (3.82)
$$

3.3.2 FR's and 4-point functions in the R sector

In the R sector the generators of the N=2 superconformal algebra remain the same as in the NS sector (see eq. (3.54). The only difference is that in this case $G^\pm(z)$ are antiperiodic fields and therefore we have to impose antiperiodic boundary conditions for the fields $\psi(z), \overline{\psi}(z)$ too. Then the fields $\varphi(z), \overline{\psi}(z)$ and $\overline{\varphi}(z), \psi(z)$ are no more combined in the chiral supermultiplets $S^\pm(z, \theta^+, \theta^-)$. Similarly to the case of the N=1 SUSY we define the R vertices as follows:

$$
R^{q(\alpha,\overline{\alpha}),r}_{\Delta(\alpha,\overline{\alpha})}(z) = \sigma^r(z) : e^{i(\alpha\overline{\varphi}+\overline{\alpha}\varphi)(z)} :, \qquad r = \pm 1 \qquad (3.83)
$$

where the fields $\sigma^\pm(z)$ ($\Delta = 1/8, q = \pm 1/4$) correspond to the lowest energy states in the R sector of CAR algebra of the fields $\psi(z), \overline{\psi}(z)$ (see App.B) Using the explicit form (3.54) of $T(z)$ and $J(z)$ and the properties (B.6) of $\sigma^\pm(z)$ one can verify that the dimensions and the charges of R_Δ are:

$$
\begin{aligned}
\Delta(\alpha, \overline{\alpha}, r) &= \alpha\overline{\alpha} - \beta(\alpha + \overline{\alpha}) + \frac{1}{8} \\
q(\alpha, \overline{\alpha}, r) &= \beta(\alpha - \overline{\alpha}) + \frac{r}{4}
\end{aligned}
\qquad (3.84)
$$

The symmetries of these formulas are similar to the ones of the NS sector: for $(\Delta, q) \to (\Delta, q)$ we have

$$
\begin{aligned}
\alpha &\to 2\beta - \overline{\alpha} \\
\overline{\alpha} &\to 2\beta - \alpha \\
r &\to r
\end{aligned}
\qquad (3.85)
$$

and for $(\Delta, q) \to (\Delta, -q)$

$$
\begin{aligned}
\alpha &\to 2\beta - \alpha \\
\overline{\alpha} &\to 2\beta - \overline{\alpha} \\
r &\to -r
\end{aligned}
\qquad (3.86)
$$

$$
\begin{aligned}
\alpha &\to \overline{\alpha} \\
\overline{\alpha} &\to \alpha \\
r &\to -r
\end{aligned}
\qquad (3.87)
$$

The analogous computation gives for the $G^{\pm}(z)$:

$$\{G_0^-, G_0^+\}R^{\pm}_{\alpha,\overline{\alpha}} = 2(\alpha-\beta)(\overline{\alpha}-\beta)R^{\pm}_{\alpha,\overline{\alpha}}$$

which is exactly as we expect (since from eq. (3.84) and $c = 3 - 24\beta^2$ it follows that $2\Delta - c/12 = 2(\alpha-\beta)(\alpha-\beta)$). Therefore we can conclude that the vertices (3.83) form a representation of the N=2 Ramond algebra.

As in the N=1 case [17,18] we accept that the screening operators (3.64) are of the same form as in the NS sector. The only difference is that $\psi(z)$ and $\overline{\psi}(z)$ are now antiperiodic. So the null vectors in the R sector are constructed in the same way as before, i.e. in eqs. (3.66)(3.70)(3.71) NS vertices are substituted by R ones. Since for the LWUR of the Ramond algebra we have

$$q = \beta(\alpha-\overline{\alpha}) + \frac{r}{4} = \beta(s-r)\alpha_+ + \frac{r}{4} \tag{3.88}$$

the corresponding formulas (3.69)(3.74)(3.75) now become

$$\begin{aligned} \alpha^{sr}_{n1} &= \frac{1}{2}[1-n+(s-r)]\alpha_+ - \frac{1}{2}\alpha_- \\ \overline{\alpha}^{sr}_{n1} &= \frac{1}{2}[1-n-(s-r)]\alpha_+ - \frac{1}{2}\alpha_- \end{aligned} \tag{3.89}$$

for the fields $R^{sr}_m(z)$ and

$$\begin{aligned} \alpha^{sr}_{n0} &= \frac{1}{2}(1-n)\alpha_- \\ \overline{\alpha}^{sr}_{n0} &= \frac{1}{2}[1-n-2(s-r)]\alpha_+ \qquad s-r>0 \end{aligned} \tag{3.90}$$

or

$$\begin{aligned} \alpha^{sr}_{n0} &= \frac{1}{2}[1-n+2(s-r)]\alpha + \\ \overline{\alpha}^{sr}_{n0} &= \frac{1}{2}(1-n)\alpha_+ \qquad s-r<0 \end{aligned} \tag{3.91}$$

for the fields $R^{sr}_{n0}(z)$. These two cases turn out to give exactly the two discrete series of dimensions (3.24). Because of the symmetry (3.85) we have to make the identification

$$R^{sr}_{n1} \equiv R^{sr}_{p+2-n-|s-r|,0} \tag{3.92}$$

In order to obtain the FR's for the R fields we have to recognize which of the functions

$$< R^{q_1}_{\Delta_1} R^{q_2}_{\Delta_2} N^{-q_1-q_2}_{\Delta_x} > \tag{3.93}$$

(in accordance with the qualitative FR's (3.44) are different from zero. In the Coulomb gas picture this means to recognize how many different ways of screening the function (3.93) we have. Let us consider separately the two cases $R^r(z_1)R^{-r}(z_2)$ and $R^r(z_1)R^{+r}(z_2)$.

In the first case the two R fields under consideration are constructed by the help of different fields : $\sigma^r(z_1)$ and $\sigma^{-r}(z_2)$ respectively. Due to this fact the three point function

$$< e^{i(\alpha_1\overline{\varphi}+\overline{\alpha}_1\varphi)}e^{i(\alpha_2\overline{\varphi}+\overline{\alpha}_2\varphi)}e^{i(\alpha_x\overline{\varphi}+\overline{\alpha}_x\varphi)}\cdots >< \sigma^r\sigma^{-r}\overline{\psi}\psi\cdots > \tag{3.94}$$

is different from zero only if the number of S.O. Q_- is equal to the number of the $\overline{Q}_-$ (and hence the second function in eq.(3.94) is chargeless), the number of even S.O. Q_+ remains arbitrary. Therefore the screening procedure in this case is exactly the same as in the case of the even FR's in the NS sector. The only difference comes from the slightly different form of the charge quantization conditions (3.89)(3.90). Then the corresponding R-sector FR's can be written as follows:

$$R^{s_1,r}_{n_1,1}R^{s_2,-r}_{n_2,1} = \sum_{k=|n_1-n_2|+1-|s_1+s_2|}^{n_1+n_2-1-|s_1+s_2|}[N^{s_1+s_2}_{k,0}] \tag{3.95}$$

$$R^{s_1,r}_{n_1,1}N^{s_2,-r}_{n_2,0} = \sum_{k=|n_2-n_1-|s_2+r||+1}^{n_1+n_2-1+|s_2+r|-1}[N^{s_1+s_2}_{k,1}] \tag{3.96}$$

$$R^{s_1,r}_{n_1,1}R^{s_2,-r}_{n_2,0} = \sum_{k=|n_1-n_2+|s_2+r|+|s_1-r||+1-|s_1+s_2|}^{n_1+n_2-1+|s_1-r|+|s_2+r|-1-|s_1+s_2|}[N^{s_1+s_2}_{k,0}] \tag{3.97}$$

In the second case $(R^r(z_1)R^r(z_2))$ the corresponding 3-point function contains a function of the form:

$$< \sigma^r\sigma^r\psi^r\cdots > \tag{3.98}$$

where ψ^r is one of the fields $\psi,\overline{\psi}$ that can make the function (3.98) uncharged. Here we have two possibilities. The first one is to put one more S.O. Q_- or $\overline{Q}_-$ in the corresponding 3-point function (3.93) and hence to have the same procedure as in the case of odd FR's in the NS sector. The second possibility is to implement the fact that the second component of some NS field (which has the form $\psi e^{i(\overline{\alpha}\varphi+\alpha\overline{\varphi})}$) can contribute to the OPE R^rR^r. The corresponding 3-point function is screened then with equal number of Q_- and $\overline{Q}_-$. Therefore this possibility corresponds to the even screening procedure in the NS sector (with s replaced by (s-r) as was mentioned above). Finally these considerations lead to the following form of FR's in this case:

$$R^{s_1,r}_{n_1,1}R^{s_2,r}_{n_2,1} = \sum_{k=|n_1-n_2|+1}^{n_1+n_2-1-1}[N^{s_1+s_2+rp}_{k,1}] + \sum_{k=|n_1-n_2|+1-|s_1+s_2-2r}^{n_1+n_2-1-1-|s_1+s_2-2r|}[N^{s_1+s_2-2r}_{k,0}]^{II} \tag{3.99}$$

$$R^{s_1,r}_{n_1,1}N^{s_2,r}_{n_2,0} = \sum_{k=|n_1-n_2-|s_2-r||+1-|s_1+s_2+rp|}^{n_1+n_2-1+|s_2-r|-|s_1+s_2+rp|}[N^{s_1+s_2+rp}_{k,0}] + \sum_{k=|n_1-n_2-|s_2-r||+1}^{n_1+n_2-1+|s_2-r|-1}[N^{s_1+s_2-2r}_{k,1}]^{II} \tag{3.100}$$

$$R^{s_1,r}_{n_1,1} R^{s_2,r}_{n_2,0} = \sum_{k=|n_1-n_2+|s_2-r|+|s_1-r||+1}^{n_1+n_2-1+|s_1-r|+|s_2-r|-1} [N^{s_1+s_2+rp}_{k,1}] + \tag{3.101}$$

$$+ \sum_{k=|n_1-n_2-|s_2-r|+|s_1-r||-|s_1+s_2-2r|+1}^{n_1+n_2-1+|s_1-r|+|s_2-r|-1-|s_1+s_2-2r|} [N^{s_1+s_2-2r}_{k,0}]^{II}$$

(the notation $[N]^{II}$ stays for the second component of the primary superfield N). We want to stress once more that the corresponding covariant FR's can be obtained considering the product of whole R-multiplet $(R^q_\Delta, R^{q+1/2}_\Delta)$. In the next section we shall calculate explicitely the structure constants of the OPE algebra corresponding to the above FR's.

The same screening procedure gives for the 4-point functions:

$$< R^r_{\alpha\overline{\alpha}}(1) R^{-r}_{2\beta-\alpha,2\beta-\overline{\alpha}}(2) R^r_{\alpha,\overline{\alpha}}(3) R^{-r}_{\overline{\alpha}\alpha}(4) >=< \sigma^r(1) e^{i(\alpha\overline{\varphi}+\overline{\alpha}\varphi)} \sigma^{-r}(2)$$

$$e^{i[(2\beta-\alpha)\overline{\varphi}+(2\beta-\overline{\alpha})\varphi]} \sigma^r(3) e^{i(\alpha\overline{\varphi}+\overline{\alpha}\varphi)} \sigma^{-r}(4) e^{i(\overline{\alpha}\overline{\varphi}+\alpha\varphi)} \prod_{i=1}^{n-1} \oint du_i \tag{3.102}$$
$$(i\alpha_+\partial(\varphi-\overline{\varphi}) + 2\alpha_+^2 \overline{\psi}\psi)(u_i) e^{i\alpha_+(\varphi+\overline{\varphi})}$$

$$\prod_{j=1}^m \oint dv_j \oint dw_j \psi(v_j) \overline{\psi}(w_j) e^{i(\alpha_- \overline{\varphi}(v_j) + \overline{\alpha}_- \varphi(w_j))} >$$

It is possible to compute explicitely the integrand terms using the bosonization rules given in the Appendix B.

It is evident from (3.102) and the charge quantization (3.89) that in order to screen the extra charge $(\alpha+\overline{\alpha})$ we need in the case of the fields R^{sr}_{n1} $(n-1)$ screening Q_+, one operator Q_- and one $\overline{Q}_-$, i.e. at least two screenings (when n=1). In the case of the fields R^{sr}_{n0} we have

$$\alpha + \overline{\alpha} = (1-n)\alpha_+ - \mid s-r \mid \alpha_+ \tag{3.103}$$

and therefore to screen the extra charge in (3.102) we need

$$n + \mid s-r \mid -1$$

screening operators Q_+ only. The simplest case is when we have $n = s = 0$ which corresponds to the zero number of screenings, i.e. the function is simply powerlike. This is not surprising because according to (3.24) the corresponding fields R^{0r}_{00} are exactly those of dimension $\Delta = c/24$ and $q = \pm c/12$ and their functions were calculated in Sec.(3.2) (eq.(3.35). The first nontrivial example is that with one screening operator, i.e.

$$n + \mid s-r \mid -1 = 1$$

This gives $\mid s-r \mid = 2-n$ and since $n = 0, 2, \cdots$ we have two possibilities .

$$1) \quad n = 2 \rightarrow \mid s-r \mid = 0, \quad i.e. \quad s = r$$

2) $n = 0 \rightarrow \mid s - r \mid = 2$, $(containing \quad s = -r)$

Thus, it turns out that the simplest 4-point function are those of the fields R_{20}^{rr} (which have charge $\pm 1/4$) and $R_{00}^{sr}, with \mid s - r \mid = 2$.

Let us consider for example the case 1). Then eq. (3.102) takes the form

$$G_{20}^{r(-r)r(-r)}(z) =< R_{20}^{sr}(\infty) R_{20}^{s(-r)}(1) R_{20}^{sr}(z) R_{20}^{s(-r)}(0) > |_{s-r=0} = (1-z)^{-2(\alpha\overline{\alpha}-\beta(\alpha+\overline{\alpha}))}$$

$$z^{\alpha^2+\overline{\alpha}^2} \oint du (u(u-z))^{\alpha_+(\alpha+\overline{\alpha})} (u-1)^{\alpha_+(4\beta-\alpha-\overline{\alpha})} \{ i\alpha_+(\alpha-\overline{\alpha})[\tfrac{1}{u} + \tfrac{1}{u-1} - \tfrac{1}{u-z}]$$

$$< \sigma^r(\infty)\sigma^{-r}(1)\sigma^r(z)\sigma^{-r}(0) > -\alpha_+^2 < \sigma^r(\infty)\sigma^{-r}(1)\sigma^r(z) : \overline{\psi}\psi(u) : \sigma^{-r}(0) > \}$$

Taking into account the equality $\alpha - \overline{\alpha} = 0$ and eq.(B.10) for the function

$$< \sigma^r \sigma^{-r} \sigma^r \overline{\psi}\psi \sigma^{-r} >$$

we obtain the following result:

$$\begin{aligned} G_{20}^{r(-r)r(-r)}(z) &= z^{\frac{1}{2(p+2)}-5/4}(1-z)^{\frac{3}{2(p+2)}-1/4} \oint du (u(u-z))^{-\frac{1}{p+2}} \\ & (u-1)^{\frac{3}{p+2}} [\frac{u-z}{u(u-1)} + \frac{1}{u-1}(1-z)^2] \end{aligned} \tag{3.104}$$

The same procedure gives for the other two functions:

$$\begin{aligned} G_{20}^{r(-r)(-r)r}(z) &= z^{\frac{1}{2(p+2)}-5/4}(1-z)^{-\frac{3}{2(p+2)}+1/4} \oint du (u(u-z))^{-\frac{1}{p+2}} \\ & (u-1)^{\frac{3}{p+2}-1} [1 + \frac{u(1-z)}{u-z}] \end{aligned} \tag{3.105}$$

$$\begin{aligned} G_{20}^{rr(-r)(-r)}(z) &= z^{\frac{1}{2(p+2)}-3/4}(1-z)^{-\frac{3}{2(p+2)}-1/4} \oint du (u(u-z))^{-\frac{1}{p+2}} \\ & (u-1)^{\frac{3}{p+2}-1} [\frac{u-z)}{u} - \frac{u(1-z)}{u-z}] \end{aligned} \tag{3.106}$$

Then, the general 4-point function is obtained as a linear combination of the functions (3.104)(3.105)(3.106):

$$\begin{aligned} < R_{20}(\infty) R_{20}(1) R_{20}(z) R_{20}(0) > |_{s-r=0} &= c_1 G_{20}^{r(-r)r(-r)}(z) + c_2 G_{20}^{r(-r)(-r)r}(z) + \\ & + c_3 G_{20}^{rr(-r)(-r)}(z) \end{aligned} \tag{3.107}$$

We leave the explicit computation of the other 4-point functions of Ramond fields as well as the construction of the 2-dimensional crossing symmetric ones to the Sec. (3.4).

3.3.3 Twisted sector

In accordance with the preliminary discussion in Sec.(3.2) we redefine the supercurrents in the T sector as follows:

$$G^1(z) = \frac{1}{\sqrt{2}}(G^+ + G^-)(z) \qquad G^2(z) = -\frac{i}{\sqrt{2}}(G^+ - G^-)(z) \quad (3.108)$$

Then the generators of the N=2 superconformal symmetry become:

$$\begin{aligned} J(z) = & \ \frac{1}{4} : \psi_1\psi_2 + i\beta\partial\varphi_2 \\ G^1(z) = & \ : -\frac{1}{2}(\psi_1\partial\varphi_1 - \psi_2\partial\varphi_2) + 2i\beta\partial\psi_1 \\ G^2(z) = & \ : \frac{1}{2}(\psi_2\partial\varphi_1 + \psi_1\partial\varphi_2) - 2i\beta\partial\psi_2 \\ T(z) = & \ : -\frac{1}{4}[(\partial\varphi_1)^2 + (\partial\varphi_2)^2] + \frac{1}{4}(\psi_1\partial\psi_1 + \psi_2\partial\psi_2) + i\beta\partial^2\varphi_1 \end{aligned} \quad (3.109)$$

Since $J(z)$ is antiperiodic we must assume the same for $\varphi_2(z)$. The other bosonic field $\varphi_1(z)$ is periodic because T(z) is always periodic. This makes clear our choice $\beta = \overline{\beta}$ -otherwise T would have half-integer as well as integer modes in the corresponding Laurent expansion. Looking at (3.109) we have to choose $\psi_1(z)$ to be periodic and $\psi_2(z)$ antiperiodic field according to what we have chosen in Sec.(3.2).

Following the general idea we have to define the T-fields in terms of the lowest dimensional field in the twisted sector of the $\psi_1(z), \psi_2(z)$ theory, i.e. σ_0^ψ, the lowest twisted field of the U(1) current $\partial\varphi_2(z)$, i.e. σ_0^φ (see App.B) and the exponent of the free scalar field $\varphi_1(z)$, i.e.

$$T_{\Delta(\alpha)}(z) = \sigma_0^\psi(z)\sigma_0^\varphi(z) : e^{i\alpha\varphi_1(z)} : \quad (3.110)$$

The fields $\sigma_0^\psi(z), \sigma_0^\varphi(z)$ produce the right analytic behaviour of the ψ_1, ψ_2 and $\partial\varphi_2(z)$ respectively and hence of the currents (3.109) as well. One can check that all the properties of the T primary fields listed in Sec.(3.2) are satisfied by (3.110). For instance

$$G^2(z_1)T_{\Delta(\alpha)}(z_2) = \frac{\beta - \alpha}{z_{12}^3/2} T_{\Delta(\alpha)}(z_2) \quad (3.111)$$

Since the dimension of the vertices (3.110) (obtained by the OPE of the twisted vertex given by eq. (3.110) with stress-energy tensor from (3.109)) is

$$\Delta(\alpha) = \alpha^2 - 2\alpha\beta + \frac{1}{8} \quad (3.112)$$

we find that ($c = 3 - 24\beta^2$)

$$\Delta(\alpha) - \frac{c}{24} = (\alpha - \beta)^2 \tag{3.113}$$

and therefore eq. (2.32) is satisfied. The formula for the dimensions (3.113) has an obvious symmetry

$$\alpha \to 2\beta - \alpha, \qquad \Delta(\alpha) = \Delta(2\beta - \alpha) \tag{3.114}$$

(the same as in the case of the Virasoro [10] and N=1 superconformal algebra [13]). The screening operators remain invariant but correspondingly to eq. (3.108) we have to consider their linear combinations

$$\begin{aligned} Q_-^1 &= \frac{Q_- + \overline{Q}_-}{\sqrt{2}} = \oint dz(\psi_1 ch(\frac{\alpha}{2}\varphi_2) - i\psi_2 sh(\frac{\alpha}{2}\varphi_2))e^{\frac{i}{2}\alpha_- \varphi_1} \\ Q_-^2 &= \frac{Q_- - \overline{Q}_-}{i\sqrt{2}} = \oint dz(\psi_2 ch(\frac{\alpha}{2}\varphi_2) - i\psi_1 sh(\frac{\alpha}{2}\varphi_2))e^{\frac{i}{2}\alpha_- \varphi_1} \end{aligned} \tag{3.115}$$

The T sector null vectors can be constructed in the same way as the NS and R null vectors, i.e.

$$\chi^T = \prod_{i=1}^{n} \oint dz_i(\partial\varphi_2 + \alpha_+\psi_1\psi_2)e^{i\alpha_+\varphi_1(z_i)}\sigma_0^{\psi}(0)\sigma_0^{\varphi}(0)e^{i(2\beta-\alpha-n\alpha_+)\varphi_1(0)}$$

This expression represents a nontrivial null vector if

$$-\frac{1}{2} + (n-1)\alpha_+^2 + 2\alpha_+(2\beta - \alpha - n\alpha_+) = -m$$

(m=positive integer) and therefore

$$\alpha_{n,m} = \frac{1}{2}(1-n)\alpha_+ + \frac{1}{4}(1-2m)\alpha_- \tag{3.116}$$

Commuting now the T null vector χ^T with L_0 we obtain that the level of degeneracy is given by $N_{n,m} = n(m - \frac{1}{2})$. The dimensions, computed by using (3.113) and (3.116), coincide with (3.41) provided m=1. Thus for the LWUR we have

$$\alpha_n = \frac{1}{2}(1-n)\alpha_+ - \frac{1}{4}\alpha_- \tag{3.117}$$

and the field T_n has a degeneracy at level $n/2$.

Now we are going to obtain the fusion rules in the T sector following the same procedure as in the NS and R sectors. Firstly let us observe that we have the following FR's for the lowest twisted field σ_0^{ψ} of the ψ_1, ψ_2 algebra:

$$\sigma_0^{\psi}\sigma_0^{\psi} = [0] + [\sigma^{\pm}] + [\psi] \tag{3.118}$$

This gives nothing else but our qualitative FR's (3.45) namely that in the product of two twisted fields (3.110) both NS and R fields occur (due to our construction (3.50) and (3.83) of the NS and R vertices). Then for obtaining FR's for the twisted fields we have to impose the usual three ways for screening the charge in the 3-point functions:

$$< T_{\alpha_1} T_{\alpha_2} \phi_{2\beta-\overline{\alpha}_x, 2\beta-\alpha_x} \prod_{i=1}^{k} Q_+ \prod_{j=1}^{l} Q_-^1 Q_-^2 >=< T_{\alpha_1} T_{2\beta-\alpha_2} \phi_{\alpha_x \overline{\alpha}_x} \prod_{i=1}^{m} Q_+ \prod_{j=1}^{n} Q_-^1 Q_-^2 >= \tag{3.119}$$

$$=< T_{2\beta-al_1} T_{\alpha_2} \phi_{\overline{\alpha}_x, \alpha_x} \prod_{i=1}^{r} Q_+ \prod_{j=1}^{s} Q_-^1 Q_-^2 >$$

where $\phi_{\alpha,\alpha}$ stays for the $N_{\alpha,\alpha}$ or $R_{\alpha,\alpha}$. Let us observe that one of the 3-point functions contributing to (3.116) is of the following kind:

$$< \sigma_0^{\varphi} \sigma_0^{\varphi} e^{\frac{1}{2}(\alpha_x - \overline{\alpha}_x)\varphi_2} > \tag{3.120}$$

As it is shown in the Appendix B this function is different from zero provided $(\alpha_x - \overline{\alpha}_x)$ is quantized

$$\alpha_x - \overline{\alpha}_x = \frac{s}{\sqrt{p+2}} \qquad s = -p-2, \cdots, p+2 \tag{3.121}$$

This is exactly the charge quantization ($q = \beta(\alpha - \overline{\alpha}) = \frac{1}{2\sqrt{p+2}}(\alpha - \overline{\alpha})$) for the LWUR of the NS and R sectors, eqs (3.17) and (3.24). This means that the product of two T fields does not lead out of the minimal models. Finally in addition to (3.119) and (3.121) we must impose the conservation of the φ_1-charge α for the three cases (3.119). This gives:

$$\begin{aligned} \alpha_x + \overline{\alpha}_x &= 2(\alpha_1 + \alpha_2 + k\alpha_+ + \frac{l}{2}\alpha_-) \\ \alpha_x + \overline{\alpha}_x &= 2(\mid \alpha_1 - \alpha_2 \mid - m\alpha_+ - \frac{l}{2}\alpha_-) \end{aligned} \tag{3.122}$$

Taking the intersection of eqs. (3.122) we obtain the general formula for the FR's in the T sector of the N=2 superconformal algebra:

$$T_{n_1} T_{n_2} = \sum_{s,r} \{ \sum_{k=|n_1-n_2|+1}^{n_1+n_2-1} [N_{k_1}^s + R_{k1}^{sr}] + \sum_{k=|n_1-n_2|+1-|s|}^{n_1+n_2-1-|s|} [N_{k0}^s + R_{k0}^{rs}] \} \tag{3.123}$$

where s, r take all the relevant values for the given number k (the other possible values of s in (3.123) contribute to the descendents in the conformal family of the same fields N and R).

This screening procedure gives also the general formula for the 4-point functions of an arbitrary twisted field $T_\alpha(z)$:

$$< \prod_{i=1}^{4} T_{\Delta_n(\alpha)}(z_i) >=< \prod_i T_{\alpha_n}(z_i) \prod_{j=1}^{2n-1} Q_+ Q_-^1 Q_-^2 > \tag{3.124}$$

In fact this expression splits into a sum of products of functions that belong to different sectors of the "underlying theory" of φ_1, φ_2 and ψ_1, ψ_2. This functions can be expressed in terms of the functions $< \sigma^{\psi}_{k_1} \sigma^{\psi}_{k_2} \sigma^{\psi}_{k_3} \sigma^{\psi}_{k_4} >$ and $< \sigma^{\varphi}_{k_1} \sigma^{\varphi}_{k_2} \sigma^{\varphi}_{k_3} \sigma^{\varphi}_{k_4} >$ which are given in the Appendix B. We shall omit the final expression for the function (3.124) since it is cumbersome enough. In some particular case the 4-point function (3.124) become more simple and they are presented in Sec.(3.4) and (3.6).

3.4 Parafermionic construction

3.4.1 Introduction

The parafermionic construction (PF) of the N=2 superconformal theory is due to the observation of Zamolodchikov and Fateev [34] that the currents

$$\begin{aligned} T = &\ T_p + T_\varphi \\ J = &\ \frac{i}{2} \frac{p}{\sqrt{2p(p+2)}} \partial\varphi \\ G^+ = &\ \sqrt{\frac{2p}{p+2}} \psi_1 : exp \left\{ i \frac{p+2}{\sqrt{2p(p+2)}} \varphi \right\} : \\ G^- = &\ \sqrt{\frac{2p}{p+2}} \psi_1^\dagger : exp \left\{ -i \frac{p+2}{\sqrt{2p(p+2)}} \varphi \right\} : \end{aligned} \tag{3.125}$$

obey the algebra of N=2 SUSY (3.4). Here ψ_1 and $\psi_1^\dagger$ are parafermionic currents in D_{2p} theories with Z_p charge 1 and p-1 respectively, and the reflection C acts as follows: $\psi_1 \rightarrow \psi_1^\dagger$. T_p is the stress-energy tensor of the parafermionic system. In eq.(3.125) φ denotes a free scalar field with propagator

$$< \varphi(z_1)\varphi(z_2) >= -2ln(z_{12}) \tag{3.126}$$

and central charge $c = 1$.

The parafermionic currents $\psi_k, k = 1, \cdots p$ together with the stress energy tensor close the following OPE algebra [33]

$$\begin{aligned} \psi_l(z_1)\psi_{l'}(z_2) = &\ c_{l,l'} z_{12}^{\Delta_{l+l'} - \Delta_l - \Delta_{l'}} \psi_{l+l'}(z_2) + \cdots \\ \psi_l(z_1)\psi_l^\dagger(z_2) = &\ z_{12}^{-2\Delta_l}[1 + \frac{2\Delta_l}{c_p} z_{12}^2 T_p(z_2) + \cdots] \end{aligned} \tag{3.127}$$

where the dimensions of the latter are given by

$$\Delta_k = \frac{k(p-k)}{p}$$

The central charge c_p of such theories can be determined by the OPE's (3.127) and it takes the following values:

$$c_p = 2\frac{p-1}{p+2} \qquad p = 1, 2\cdots \tag{3.128}$$

The stress energy tensor of the composite theory (3.125) is of course a sum of the stress energy tensor of the D_{2p} PF theory T_p and of the free field stress-energy tensor:

$$T_\varphi = -\frac{1}{4} : (\partial\varphi)^2 : \tag{3.129}$$

Therefore the central charge of the algebra (3.125) is

$$c = 1 + c_p = \frac{3p}{p+2}, \qquad p = 1, 2\cdots \tag{3.130}$$

which coincides with the series (3.17).

In the following sections, after a short discussion of the parafermionic current algebra which has interesting features in its own, we use these exactly solvable models to compute the correlation functions and the structure constants of the N=2 superconformal minimal models.

3.4.2 Parafermionic current algebra

A statistical system with Z_p symmetry is described at the critical point by $2p-1$ parafermionic fields $\psi_l, \overline{\psi}_l (l = 0, 1, \cdots p-1)$ which implement this symmetry. They are holomorphic (antiholomorphic) fluctuation fields

$$\partial_{\bar{z}}\psi_l = \partial_z\overline{\psi}_l = 0 \tag{3.131}$$

The identity operator is given by $\psi_0, \overline{\psi}_0$ and the hermiticity relation holds

$$\psi_l^\dagger = \psi_{k-l}$$

Z_p symmetric systems possess order-disorder duality [85]. We restrict our considerations to the Z_p systems which are self-dual. This implies that at the critical point the symmetry is enlarged to the group $Z_p \times \tilde{Z}_p$, where the second group comes from the dual variables. The fields are characterized by their $Z_p \times \tilde{Z}_p$ charges. Two fields with charges $(r, q), (r', q')$ have a mutual locality exponent given by

$$\gamma = -\frac{rq' + r'q}{p} \tag{3.132}$$

This means that the correlation function pick a phase $e^{2\pi i\gamma}$ when we circle one field around the other.

The operatorial algebra of the parafermionic fields is

$$\psi_l(z)\psi_{l'}(w) = c_{l,l'}(z-w)^{\Delta_{l+l'}-\Delta_l-\Delta_{l'}} \sum_{n=0}^{\infty} \psi_{l+l'}^n(w) \tag{3.133}$$

where $c_{l,l'}$ are the structure constants determined by the associativity condition of the algebra and the fields $\psi^n_{l+l'}$ have charges $(l+l', l+l')$. The field $\psi^0_{l+l'}$ coincides with the parafermionic current $\psi_{l+l'}$. The mutual locality exponent of ψ_l and $\psi_{l'}$ is $\gamma = -2\frac{ll'}{p}$ so we have the following equation for the dimensions Δ_l

$$\Delta_{l+l'} - \Delta_l - \Delta_{l'} = \frac{2ll'}{p} \qquad \mathrm{mod}\ Z \tag{3.134}$$

The simplest solution of this monodromy relation is

$$\Delta_l = \Delta_{p-l} = \frac{l(p-l)}{p} \tag{3.135}$$

Specializing the OPE to the case of conjugate fields

$$\psi_l(z)\psi_l^\dagger(z') = (z-z')^{-\frac{2l(p-l)}{p}}[1 + \frac{2\Delta_l}{c_p}(z-z')^2 T_p(z') + \cdots] \tag{3.136}$$

we can define the stres-energy tensor of the parafermionic system and by its two-point function we can recover the quantized values of the central charge

$$c = \frac{2(p-1)}{p+2} \tag{3.137}$$

Conformal invariance requires that the stress-energy tensor T_p will be local with respect to all the fields in the field space of the Z_p theory. Thus we can classify the fields according to the representations of the Virasoro algebra. Alternatively we can decompose $\mathcal{H}$ according to the transformation properties under $Z_p \times \tilde{Z}_p$. The field space $\mathcal{H}$ splits into a direct sum of subspaces with specified $(m, \overline{m})$ charges

$$\mathcal{H} = \frac{1}{2} \bigoplus_{p \geq m, \overline{m} \geq 1-p} \mathcal{H}_{m,\overline{m}} \qquad m - \overline{m} \in 2Z$$

We have introduced the $Z_{2p} \times \tilde{Z}_{2p}$ charges $(m, \overline{m})$ defined by

$$(m, \overline{m}) = (l + l', l - l')$$

The right and left charges $m, \overline{m}$ are defined modulo $2p$ and take arbitrary integer values such that $m + \overline{m}$ is even. Each value of the original Z_p charge occurs twice which accounts for the factor $\frac{1}{2}$ in front. In this notation $\psi_l[\overline{\psi}_l]$ has $Z_{2p} \times \tilde{Z}_{2p}$ charges $(2l, 0)[(0, 2l)]$ and the mutual locality exponent of two fields with charges $(q, \overline{q})$ and $(s, \overline{s})$ is

$$\gamma = -\frac{1}{2p}(qs - \overline{qs})$$

The various fields in $\mathcal{H}$ may be generated by applying the parafermionic fields, in particular only ψ_1 and $\psi_1^\dagger$. If $\phi_{l,\bar{l}} \in \mathcal{H}_{l,\bar{l}}$ and its conformal dimensions are $(h, \overline{h})$ we have

$$\psi_1(z)\phi_{l,\bar{l}}(0,0) = \sum_{m=-\infty}^{\infty} z^{-l/p+m-1} A_{l/p-m}\phi_{l,\bar{l}}(0,0) \tag{3.138}$$

$$\psi_1^{\dagger(z)}\phi_{l,\bar{l}}(0,0) = \sum_{m=-\infty}^{\infty} z^{l/p+m-1} A^\dagger_{-l/p-m}\phi_{l,\bar{l}}(0,0) \tag{3.139}$$

The power of z is determined by the mutual locality of the two fields. The action of the operators A and $A^\dagger$ so defined is

$$A_{l/p-m} \quad \phi_{l,\bar{l}} \in \mathcal{H}_{l+2,\bar{l}} \qquad\qquad (h+m-\frac{l+1}{p},\bar{h})$$
$$A^\dagger_{-l/p-m} \quad \phi_{l,\bar{l}} \in \mathcal{H}_{l-2,\bar{l}} \qquad\qquad (h+m-\frac{1-l}{p},\bar{h})$$

One can derive generalized commutation relations that are obeyed by A and $A^\dagger$ [33]. This algebra is known in mathematics as a Z algebra.

The primary fields of the parafermionic system satisfy

$$A_{l/p+n} \quad \phi^{l,\bar{l}}_{l,\bar{l}} = A^\dagger_{-l/p+n+1}\phi^{l,\bar{l}}_{l,\bar{l}} = 0 \qquad\qquad n \geq 0$$
$$\overline{A}_{\bar{l}/p+n} \quad \phi^{l,\bar{l}}_{l,\bar{l}} = \overline{A}_{-\bar{l}/p+n+1}\phi^{l,\bar{l}}_{l,\bar{l}} = 0 \qquad\qquad l,\bar{l} = 0,1\cdots p-1$$

Their dimensions are

$$h_l = \frac{l(p-l)}{2p(p+2)} \qquad\qquad \bar{h}_{\bar{l}} = \frac{\bar{l}(p-\bar{l})}{2p(p+2)} \tag{3.140}$$

For the moment we neglect the $\overline{m}$ dependence of $\mathcal{H}$ and write $\mathcal{H}_{m,\overline{m}}$ as $\mathcal{H}_m$. All the fields in the space $\mathcal{H}$ can be obtained by applying the operators A and $A^\dagger$ to ϕ^l_l. The independent fields obtained generate a highest weight representation of the parafermionic current algebra. Each field ϕ^l_l generates a series of fields $\phi^l_m(m=-l,-l+2,\cdots 2p-l-2)$ defined by

$$\phi^l_{l+2n} = A_{\frac{l+2n-2}{p}-1}A_{\frac{l+2n-4}{p}-1}\cdots A_{\frac{l}{p}-1}\phi^l_l \tag{3.141}$$
$$n = 0,1,\cdots p-l \qquad (h = h_l + \frac{n(p-n-l)}{p})$$
$$\phi^l_{l-2n} = A^\dagger_{-\frac{l-2n+2}{p}}A^\dagger_{-\frac{l-2n+4}{p}}\cdots A_{-\frac{l}{p}}\phi^l_l \tag{3.142}$$
$$n = 0,1,\cdots l \qquad (h = h_l + \frac{n(l-n)}{p})$$

We define also $\phi^l_{m+2p} = \phi^l_m$. The fields ϕ^l_m has the conformal dimension

$$h^l_m = h_l + \frac{(l-m)(l+m)}{4p} \qquad -l \leq m \leq l$$
$$h^l_m = h_l + \frac{2p-l-m)(m-l)}{4p} \qquad l \leq m \leq 2p-l-2 \tag{3.143}$$

The rest of the conformal fields in $\mathcal{H}_m$ have conformal dimensions which differ by an integer from h^l_m. In general $\mathcal{H}_m$ contains infinitely many primary fields. The exceptions occur when $p=2$ or 3: this cases are respectively the Ising model and the Z_3 Potts model which belong to the discrete series of the Virasoro algebra [7,8]. To familiarize the reader with the formalism of the parafermionic system we discuss briefly the Ising model in this formalism.

We have one primary field σ with dimension $\Delta = \frac{1}{16}$ and $Z_2 \times \tilde{Z}_2$ charges $(1,0)$. Correspondingly the $Z_4 \times \tilde{Z}_4$ charges are $(1,1)$. σ is the magnetization order parameter. In addition there is the disorder operator μ with the same dimension and $Z_2 \times \tilde{Z}_2$ charges $(0,1)$. The $Z_4 \times \tilde{Z}_4$ charges are $(0,-1)$. Applying $A_{\frac{1}{2}-1}$ to σ we have

$$\phi_{3,1} = A_{\frac{1}{2}-1}\sigma$$

This field has dimension $\Delta = \frac{1}{16}$ and

$$\begin{aligned} \{Z_4 \times \tilde{Z}_4\} &= (3,1) \\ \{Z_2 \times \tilde{Z}_2\} &= (2,1) \equiv (0,1) \end{aligned}$$

So we see that this field has the quantum number of μ and

$$\psi_(z)\sigma(0) = \frac{1}{\sqrt{z}}\mu(0) + \cdots$$

Similarly for μ, i.e.

$$\psi(z)\mu(0) = \frac{1}{\sqrt{z}}\sigma(0) + \cdots$$

The parafermionic system can be put in correspondence with the SU(2) WZW theory. We introduce a free massless Bose field

$$\begin{gathered} \phi(z,\overline{z}) = \phi(z) + \overline{\phi}(\overline{z}) \\ < \phi(z)\phi(0) >= -2lnz; \qquad < \overline{\phi}(\overline{z})\overline{\phi}(0) >= -2ln\overline{z} \\ T(z) = -\tfrac{1}{4}(\partial_z\phi)^2 \end{gathered} \tag{3.144}$$

and consider the following fields with dimensions $(1,0)$

$$\begin{aligned} J^+(z) = &\ \sqrt{2p}\psi_1(z) : exp\left\{i\frac{\phi(z)}{\sqrt{p}}\right\} : \\ J^-(z) = &\ \sqrt{2p}\psi_1^\dagger(z) : exp\left\{-i\frac{\phi(z)}{\sqrt{p}}\right\} : \\ J^0(z) = &\ \sqrt{\frac{p}{2}}\partial\phi(z) \end{aligned} \tag{3.145}$$

Their OPE is

$$J^a(z)J^b(z') = \frac{pq^{ab}}{(z-z')^2} + \frac{f^{abc}J^c}{z-z'} \tag{3.146}$$

where f^{abc} are the structure constants of the SU(2) algebra and q^{ab} are related to the Casimir operator

$$f^{0++} = -f^{+0+} = -f^{0--} = f^{-0-} = \frac{1}{2}f^{+-0} = -\frac{1}{2}f^{-+0} = 1$$

$$q^{00} = \frac{1}{2}q^{+-} = \frac{1}{2}q^{-+} = 1$$

The field space of this SU(2) Kac-Moody algebra contains the primary fields $G^{l,\bar{l}}_{m,\overline{m}}$ which are the m and $\overline{m}$ components of the isospin l and $\bar{l}$ [35,67]

$$\begin{array}{ll}
[J^a_n, J^b_m] = f^{abc}J^c_{n+m} + \frac{1}{2}pnq^{ab}\delta_{n+m,0} & \\
J^a_n G^{l,\bar{l}}_{m,\overline{m}} = \overline{J}^a_n G^{l,\bar{l}}_{m,\overline{m}} = 0 & n > 0 \\
J^+_0 G^{l,\bar{l}}_{m,\overline{m}} = \overline{J}^+_0 G^{l,\bar{l}}_{m,\overline{m}} = 0 & \\
(J^0_0 - m)G^{l,\bar{l}}_{m,\overline{m}} = (\overline{J}^0_0 - \overline{m})G^{l,\bar{l}}_{m,\overline{m}} = 0 & \\
-l \leq m \leq l & l = 0, 1, \cdots p
\end{array} \tag{3.147}$$

The fields $G^{l,\bar{l}}_{m,\overline{m}}$ have dimensions

$$\Delta = \frac{l(l+2)}{4(p+2)} \qquad \overline{\Delta} = \frac{\bar{l}(\bar{l}+2)}{4(p+2)} \tag{3.148}$$

The relation between the primary fields of the WZW theory and those of the Z_p theory is given by [33]

$$G^{l,\bar{l}}_{m,\overline{m}}(z) = \phi^{l,\bar{l}}_{m,\overline{m}}(z) : e^{(i\frac{m.\phi(z)}{2\sqrt{p}} + i\frac{\overline{m}.\overline{\phi}(\bar{z})}{2\sqrt{p}})} : \tag{3.149}$$

3.4.3 Construction of the primary fields in the various sectors

As it was shown in last section the lowest dimensional fields $\phi^{(l)}_m$ of the D_{2p} theory have conformal dimensions

$$d^l_m = d_l + \frac{l^2 - m^2}{4p} \qquad -l \leq m \leq l \tag{3.150}$$

where

$$d_l = \frac{l(p-l)}{2p(p+2)}, \qquad l = 0, 1, \cdots, p \tag{3.151}$$

is the dimension of the corresponding order parameter fields $\sigma_l = \phi^{(l)}_l$. The formula (3.150) has the following symmetries

$$\begin{array}{ll}
l \to \; l, m & \to -m \\
 & \\
l \to \; p - l, m & \to p \pm m
\end{array} \tag{3.152}$$

where we identify $m \equiv m + 2p$, since m is the Z_{2p} charge of the field $\phi^{(l)}_m$.

It is clear that the primary fields in the N=2 theories are constructed, in accordance with (3.125), from the lowest fields of the D_{2p} PF theories and exponentials of the free scalar field φ. This must be done in such a way that the so obtained fields have a correct OPE's with the generators (3.125).

For the NS sector this gives:

$$N^l_m(z) = \phi^l_m(z) : exp\left\{ i\frac{m}{\sqrt{2p(p+2)}}\varphi(z)\right\} :$$

$$l = 0, 1, \cdots, p; \qquad m = -l, -l+2, \cdots l \tag{3.153}$$

Now it is straightforward to compute the U(1) charge and the dimensions. Combining (3.125)and (3.153) we have

$$J(z_1)N^l_m(z_2) = \frac{m}{2(p+2)}\frac{1}{z_{12}}N^l_m(z_2) + \cdots$$

and therefore

$$q^l_m = \frac{m}{2(p+2)} \tag{3.154}$$

The dimension of N^l_m is simply the sum of the dimensions of the two ingredients:

$$\Delta^l_m = d^l_m + \frac{m^2}{2p(p+2)} = \frac{l(l+2) - m^2}{4(p+2)} \tag{3.155}$$

To make a connection with the Coulomb gas representation of the NS sector we note that the first series (3.17) is obtained from (3.155) if

$$s = m \qquad n = p - l + 1 \tag{3.156}$$

and the second one if

$$s = m, \qquad n = l - s + 1 \tag{3.157}$$

For the products with the supercurrents we have

$$G^{\pm}(z_1)N^l_m(z_2) = \sqrt{\frac{2p}{p+2}}\frac{1}{z_{12}}\phi^l_{m\pm 2}(z_2) : exp\left\{ i\frac{m \pm (p+2)}{\sqrt{2p(p+2)}}\varphi(z_2)\right\} : \tag{3.158}$$

It is useful to note here that due to the symmetries (3.152) the second component of the field N^l_m has the form

$$(N^l_m)^{II\pm} \sim \phi^{p-l}_{m\pm(p+2)} : exp\left\{ i\frac{m \pm (p+2)}{\sqrt{2p(p+2)}}\varphi\right\} : \tag{3.159}$$

and therefore it looks just as $N^{p-l}_{m\pm(p+2)}$ but does not satisfy the conditions (3.153).

The primary fields of the R sector are represented in a similar way

$$R^{l}_{m,\alpha} = \phi^{l}_{m} : exp\left\{ i\frac{m-\frac{\alpha p}{2}}{\sqrt{2p(p+2)}}\varphi \right\} :$$

$$l = 0,1,\cdots,p; \qquad m = -l,-l+2,\cdots,l; \alpha = \pm 1 \tag{3.160}$$

The U(1) charge now is given by

$$q^{l}_{m,\alpha} = \frac{2m-\alpha p}{4(p+2)} \tag{3.161}$$

and the conformal dimensions are

$$\Delta^{l}_{m,\alpha} = \frac{l(l+2)-(m+\alpha)^2}{4(p+2)} + \frac{1}{8} \tag{3.162}$$

The fields (3.160) produce the right analytic behaviour of the supercurrents. For example

$$G^{+}(z_1)R^{l}_{m,\alpha}(z_2) = \sqrt{\frac{2}{p+2}}z_{12}^{-(1+\frac{\alpha}{2})}\phi^{l}_{m+2}(z_2) : exp\left\{ i\frac{m+2+(2-\alpha)\frac{p}{2}}{\sqrt{2p(p+2)}}\varphi \right\} : \tag{3.163}$$

This means that for $\alpha = 1$ the second component of the '$R^{l}_{m,\alpha}$ multiplet' is the field $R^{(l)}_{m+2,-\alpha}$ and if $\alpha = -1$ we have

$$G^{+}(z_1)R^{l}_{m,-1}(z_2) = \mathcal{O}(z_{12})$$

in accordance with eq.(3.23). The similar result holds for the OPE with $G^{-}(z)$. The formulas (3.161)(3.162) exactly coincide with (3.17) with the following identification:

$$m = s, r = -\alpha, n = p-l+1 \qquad \text{for the first series } \Delta^{rs}_{n1}$$

$$\tag{3.164}$$

$$m = s, r = -\alpha, n = l-\mid m \mid +1 \qquad \text{for the second one } \Delta^{rs}_{n0}$$

The expressions (3.161)(3.163) convince us that (3.160) indeed represents a primary R field.

The D_{2p} PF theories have in addition to the 'even sector' (3.150) (3.151) also an odd one -containing the C-disorder fields $\varphi^{(s)}(z)$. The analytic behaviour of the PF currents here is given by the following OPE [34]

$$\psi_1(z)\varphi^{(s)}(0) = z^{-\Delta_1}\sum_{n\in Z} z^{-\frac{n}{2}}A^{(1)}_{\frac{n}{2}}\varphi^{(s)}(0)$$

According to ref. [34] the dimensions of the C-disorder fields have the values

$$\Delta^{(s)} = \frac{p-2+(p-2s)^2}{16(p+2)}, \qquad s = 0,1\cdots \leq \frac{p}{2} \tag{3.165}$$

These fields together with the U(1) twisted fields of the current $\partial\varphi(z)$ can be used in the construction of the primary fields of the twisted sector of the N=2 superconformal algebra:

$$T_s(z) = \varphi^{(s)}(z)\sigma_0^{\varphi}(z) \tag{3.166}$$

As it can be easily seen these fields indeed produce the correct branch behaviour of the U(1) current (3.125):

$$J(z_1)T_s(z_2) = \frac{1}{\sqrt{z_{12}}}\sqrt{\frac{p}{2(p+2)}}\varphi^{(s)}(z_2)\sigma_1^{\varphi}(z_2) \tag{3.167}$$

Therefore the field t_m^s of eq.(3.38) is proportional to the product $\varphi^s(z)\sigma_1^{\varphi}(z)$. From eq.(3.166) one can extract the dimensions of the T-fields as well

$$\Delta_s = \frac{(p-2s)^2-4}{16(p+2)} + \frac{1}{8} \qquad s = 0,1,\cdots \leq \frac{s}{2} \tag{3.168}$$

The last formula (3.168) reproduces the discrete series (3.41) if one takes n to be $s+1$.

Finally we want to note that for each theory with central charge

$$c = \frac{3p}{p+2}, \qquad p = 1,\cdots$$

the fields

$$N_k(z) = N^p_{-p+2k+2}\;, \qquad N_k^{\dagger}(z) = N^p_{p-2k+2}, \qquad k = 0,1,\cdots p \tag{3.169}$$

with dimensions

$$\Delta_k = \frac{k(p+2-k)}{p+2} - \frac{1}{2}$$

close an OPE algebra of Z_{p+2} PF type. This very surprising fact exhibits in more explicit form the observation of ref. [56] that N=2 superconformal theories with given p possess Z_{p+2} symmetry. The detailed discussion of the theories with parafermionic currents (3.169) is presented in the Sec.(3.5)

3.4.4 FR's

For obtaining the FR's of NS and R fields we need the corresponding FR's of the PF fields $\phi_m^{(l)}$. The latter can be obtained using the relation (3.149) between PF fields and the primary fields of the SU(2) WZW theories For the SU(2) fields G_m^j the FR's are found in [70] and are as follows:

$$G^{j_1}_{m_1}G^{j_2}_{m_2} = \sum_{j=|j_1-j_2|}^{min(j_1+j_2,p-j_1-j_2)} [G^j_{m_1+m_2}] \tag{3.170}$$

From (4.26) it is easy now to obtain the corresponding FR's for $\phi_m^{(l)}$ using (3.175). The result is

$$\phi_{m_1}^{l_1}\phi_{m_2}^{l_2} = \sum_{l=|l_1-l_2|}^{L} [\phi_{m_1+m_2}^{l}] \tag{3.171}$$

$$L = min(l_1+l_2, 2p-l_1-l_2)$$

Investigating the FR's in the NS sector one must keep attention that they have more complicated structure. Namely the full superfield FR's are found combining both even and odd ones. The meaning of the odd FR's in terms of component fields is that in the product of two first components of given superfields the second component of the RHS superfield occurs. Due to our observation (3.159) we know that this happens when the quantum number l, m of the RHS field N_m^l does not satisfy the condition (3.153). Taking into account all these peculiarities of the NS sector and using (3.153),(3.171) we are lead to the following FR's:

$$N_{m_1}^{l_1}N_{m_2}^{l_2} = \sum_{l=|l_1-l_2|}^{L} [\phi_m^l] \tag{3.172}$$

where

$$(\phi_m^l) = \begin{cases} (N_{m_1+m_2}^l)^{even} & \text{if } | m_1+m_2 |\leq l \\ (N_{m_1+m_2\pm(p+2)}^{p-l})^{odd} & \text{if } | m_1+m_2 |> l \end{cases}$$

This is the most compact form of the NS FR's. They exactly coincide with those found in the Coulomb gas representation (3.31)(3.32) if we take into account (3.156)(3.157). Concrete examples of (3.172) for some models will be given below.

In the case of the R sector we proceed exactly in the same way. The final result is

$$\begin{aligned} R_{m_1,\alpha}^{l_1}R_{m_2,-\alpha}^{l_2} &= \sum_{l=|l_1-l_2|}^{L} [\phi_{m_1+m_2}^l] \\ R_{m_1,\alpha}^{l_1}R_{m_2,\alpha}^{l_2} &= \sum_{l=|l_1-l_2|}^{L} [\phi_{m_1+m_2-\alpha p}^{p-l}] \end{aligned} \tag{3.173}$$

where:

$$(\phi_m^l) = \begin{cases} (N_m^l) & \text{if } | m |\leq l \\ (N_{m\pm(p+2)}^{p-l})^{II,\pm} & \text{if } | m |> l \end{cases}$$

In the T sector the situation is more complicated. Again as above the product $\sigma_0^\phi\sigma_0^\phi$ reproduces the exponents

$$exp\left\{ i\frac{m}{\sqrt{2p(p+2)}}\varphi \right\}$$

with all the relevant $\partial\phi$ charges $\frac{m}{\sqrt{2p(p+2)}}$. The product of C-disorder fields gives the fields of the even sector of the PF theory but unfortunately the exact FR's are not known. However here we can reverse the above procedure and using the known FR's of the twisted fields to recognize the corresponding ones for the C-disorder fields. This gives

$$\varphi^{(s_1)}\varphi^{(s_2)} = \sum_{l=|s_1-s_2|}^{s_1+s_2} [\sigma_l] \tag{3.174}$$

(the fields $\phi_m^{(l)}$ are simply descendants of the order parameter σ_l and therefore are included in the conformal family $[\sigma_l]$).

3.4.5 2D-correlation functions and structure constants of the N=2 superconformal OPE algebra

The above mentioned connection between PF fields $\phi_{m,m}^{(l)}$ and SU(2) WZW primary fields $G_{m,m}^{j}(z,z)$ gives a simple way of calculating the 4-point functions and structure constants of the OPE algebra. Therefore the 4-point function of the PF fields can be expressed in terms of the 4-point function of the corresponding SU(2) fields:

$$< \prod_{i=1}^{4} G_{m_i,\overline{m}_i}^{l_i}(z_i,\overline{z}_i) >= \prod_{i<j}^{4} z_{ij}^{-\frac{m_i m_j}{2p}} \overline{z}_{ij}^{-\frac{\overline{m}_i\overline{m}_j}{2p}} < \prod_{s=1}^{4} \phi^{(\frac{l_s}{2})}_{\frac{m_s}{2},\frac{\overline{m}_s}{2}}(z_s,\overline{z}_s) > \tag{3.175}$$

The most general correlation function of the fields $G_{m,\overline{m}}^{j}$ is calculated in [70,71] and it has the following form

$$\begin{aligned}
&< G_{m_1\overline{m}_1}^{j_1} G_{m_2\overline{m}_2}^{j_2} G_{m_3\overline{m}_3}^{j_3} G_{m_4\overline{m}_4}^{j_4} >=| z_{14} |^{-4\Delta_1}| z_{24} |^{2\nu_2}| z_{34} |^{2\nu_3}| z_{32} |^{2\nu_4} \\
&\prod_{k=1}^{4} \frac{\partial^{j_k+m_k}}{\partial x_k^{j_k+m_k}} \frac{\partial^{j_k+\overline{m}_k}}{\partial \overline{x}_k^{j_k+\overline{m}_k}} A_k \mid x_{14} |^{4j_1}| x_{24} |^{2(-j_1+j_2-j_3+j_4)}| x_{34} |^{2(-j_1-j_2+j_3+j_4)} \\
&| x_{32} |^{2(j_1+j_2+j_3-j_4)} U_{j_1 j_2 j_3 j_4}(x,\overline{x};z,\overline{z}) |_{x=\overline{x}=0}
\end{aligned} \tag{3.176}$$

where

$$\begin{aligned}
\nu_2 &= \Delta_1 - \Delta_2 + \Delta_3 - \Delta_4 \\
\nu_3 &= \Delta_1 + \Delta_2 - \Delta_3 - \Delta_4 \\
\nu_4 &= -\Delta_1 - \Delta_2 - \Delta_3 + \Delta_4
\end{aligned}$$

$$\Delta_k = \frac{j_k(j_k+1)}{p+2}, \qquad x = \frac{x_{12}x_{34}}{x_{14}x_{32}}, \qquad z = \frac{z_{12}z_{34}}{z_{14}z_{32}}$$

$$A_k = \frac{(C_{2j_k}^{j_k+m_k} C_{2j_k}^{j_k+\overline{m}_k})^{-1/2}}{(j_k+m_k)!(j_k+\overline{m}_k)!} \qquad C_{2j}^{j=m} = \begin{pmatrix} 2j \\ j+m \end{pmatrix}$$

and the function $U(x,x;z,z)$ has the form

$$\begin{aligned}
U_{j_1\cdots j_4} \quad = \quad & | z |^{\frac{4j_1j_2}{p+2}}| 1-z |^{\frac{4j_1j_3}{p+2}} N(j_1,\cdots j_4) \int \prod_{l=1}^{2j_1} dt_l d\overline{t}_l \mid t_l - z |^{-\frac{2\beta_1}{p+2}} \\
& | t |^{-\frac{2\beta_2}{p+2}}| t_l - 1 |^{-\frac{2\beta_3}{p+2}}| x - t_l |^2 \prod_{i<j} | t_i - t_j |^{\frac{4}{p+2}}
\end{aligned} \tag{3.177}$$

$$\beta_1 = j_1 + j_2 + j_3 + j_4 + 1$$
$$\beta_2 = p + j_1 + j_2 - j_3 - j_4 + 1$$
$$\beta_1 = p + j_1 + j_3 - j_2 - j_4 + 1$$

The constant $N(j_1, \cdots, j_4)$ are given in ref. [70]

$$N^2(j_1 \cdots j_4) = \left[\frac{\Gamma(\frac{1}{p+2})}{\Gamma(\frac{p-1}{p+2})}\right]^{4j_1+2} \frac{\Gamma(1 - \frac{2j_1+1}{p+2})P^2(j_1 + j_2 + j_3 + j_4 + 1)}{\Gamma(\frac{2j_1+1}{p+2})P^2(2j_1)}$$
$$\prod_{n=2}^{4} \frac{\Gamma(1 - \frac{2j_n+1}{p+2})P^2(j_2 + j_3 + j_4 - j_1 - 2j_n)}{\Gamma(\frac{2j_n+1}{p+2})P^2(2j_n)} \qquad (3.178)$$
$$P(j) = \prod_{n=1}^{j} \frac{\Gamma(\frac{n}{p+2})}{\Gamma(1 - \frac{n}{p+2})}$$

Now using the PF construction of NS and R fields we can write down their most general 4-point functions:

$$< \prod_{i=1}^{4} N^{l_i}_{m_i \overline{m}_i}(z_i, \overline{z}_i) >= \prod_{i<j}^{4} z_{ij}^{-\frac{m_i m_j}{2(p+2)}} \overline{z}_{ij}^{-\frac{\overline{m}_i \overline{m}_j}{2(p+2)}} < \prod_{s=1}^{4} \phi^{(\frac{l_s}{2})}_{\frac{m_s}{2}, \frac{\overline{m}_s}{2}} > \qquad (3.179)$$

for the NS case and

$$< \prod_{i=1}^{4} R^{l_i}_{m_i,\alpha_i;\overline{m}_i,\overline{\alpha}_i} >= \prod_{i<j}^{4} (z_{ij})^{-\frac{2(m_i m_j + \alpha_i m_j + \alpha_j m_i) - \alpha_i \alpha_j p}{4(p+2)}}$$
$$(\overline{z}_{ij})^{-\frac{2(\overline{m}_i \overline{m}_j + \overline{\alpha}_i \overline{m}_j + \overline{\alpha}_j \overline{m}_i) - \overline{\alpha}_i \overline{\alpha}_j p}{4(p+2)}} < \prod_{s=1}^{4} \phi^{(\frac{l_s}{2})}_{\frac{m_s}{2} \frac{\overline{m}_s}{2}} > \qquad (3.180)$$

for the R fields. In two cases, $j_1 = 1/2$ and $j_4 = \frac{p-1}{2}$, the integral (3.177) can be expressed in terms of hypergeometric functions. For example for $j_1 = 1/2$ we have

$$U_{j_1 \cdots j_4} =| z |^{\frac{2j_2}{p+2}} | 1 - z |^{\frac{2j_3}{p+2}} N(\frac{1}{2}, \cdots j_4) \sum_{q,\overline{q}=0,1} \sum_{\sigma=\mp} x^q \overline{x}^{\overline{q}} h^{(\sigma)} F^{(\sigma)}_q(z) F^{(\sigma)}_{\overline{q}}(\overline{z}) \qquad (3.181)$$

where

$$F^{(-)}_1(z) = z^{-\frac{2j_2+1}{p+2}} F(a, b, c; z)$$
$$F^{(+)}_1(z) = F(b - c + 1, a - c + 1, 2 - c; z) \qquad (3.182)$$
$$F^{(-)}_0(z) = -\frac{b}{c} z^{1-\frac{2j_2+1}{p+2}} F(a, b + 1, c + 1; z)$$
$$F^{(+)}_0(z) = \frac{1 - c}{c - a} F(b - c + 1, a - c, 1 - c; z)$$

Here

$$(p + 2)a = -j_2 + j_3 - j_4 + p + \frac{3}{2}$$

$$
\begin{aligned}
(p+2)b &= -j_2+j_3+j_4++\frac{1}{2} \\
(p+2)c &= -2j_2+p+1 \\
h^{(-)} &= \frac{\Gamma(b)\Gamma(c-b)\Gamma(1-c)}{\Gamma(c)\Gamma(1-b)\Gamma(1+b-c)} \\
h^{(+)} &= \frac{\Gamma(a-c+1)\Gamma(1-a)\Gamma(c-1)}{\Gamma(2-c)\Gamma(a)\Gamma(c-a)}
\end{aligned} \tag{3.183}
$$

All these expressions become more simple if we consider some particular 4-point functions. For example the correlator of two R fields of dimension $\Delta^1_{1,\alpha}$ and two of dimension $\Delta^k_{k,\alpha}$ have the form

$$
< R^1_{1,\alpha}(z_1)R^1_{-1,\alpha}(z_2)R^k_{k,\alpha}(z_3)R^k_{-k,-\alpha}(z_4) >=| z_{12} |^{\frac{2+4\alpha-p}{2(p+2)}} | z_{34} |^{\frac{4k(\alpha-1)-p+6}{2(p+2)}} | z_{14}z_{23} |^{-\frac{3}{p+2}} \tag{3.184}
$$

$$
| \frac{z_{14}z_{23}}{z_{13}z_{24}} |^{\frac{2(k+\alpha+k\alpha)}{2(p+2)}} | z |^{\frac{1}{p+2}} | 1-z |^{\frac{k}{p+2}} N(\tfrac{1}{2},\tfrac{1}{2},\tfrac{k}{2},\tfrac{k}{2}) \textstyle\sum_{q,\bar{q}=0,1} \sum_{\sigma=\mp} h^{(\sigma)} F^{(\sigma)}_q(z) F^{(\sigma)}_{\bar{q}}(\bar{z})
$$

and the parameters (3.183) become here

$$
\begin{aligned}
a &= \frac{p+1}{p+2}, \qquad b=\frac{k}{p+2}, \qquad c=\frac{p}{p+2} \\
h^- &= \frac{\Gamma(\frac{k}{p+2})\Gamma(\frac{p-k}{p+2})\Gamma(\frac{2}{p+2})}{\Gamma(\frac{p}{p+2})\Gamma(\frac{p+2-k}{p+2})\Gamma(\frac{k+2}{p+2})} \qquad h^+ = \frac{\Gamma(\frac{p+3}{p+2})\Gamma(\frac{1}{p+2})\Gamma(-\frac{2}{p+2})}{\Gamma(\frac{p+4}{p+2})\Gamma(\frac{p+1}{p+2})\Gamma(-\frac{1}{p+2})}
\end{aligned} \tag{3.185}
$$

The normalization constant $N(\frac{1}{2},\frac{1}{2},\frac{k}{2},\frac{k}{2})$ also has more simple form

$$
N(\frac{1}{2},\frac{1}{2},\frac{k}{2},\frac{k}{2}) = \frac{\Gamma(\frac{p}{p+2})\Gamma(\frac{k+2}{p+2})\Gamma(\frac{p-k+2}{p+2})}{\Gamma(\frac{2}{p+2})\Gamma(\frac{p-k}{p+2})\Gamma(\frac{k}{p+2})} \tag{3.186}
$$

From the 4-point functions (3.179)(3.180) we can extract also the structure constants of the OPE algebra in NS and R sectors. They appear in the explicit form of the FR's (3.172)

$$
N^{l_1}_{m_1,\overline{m}_1}(z_1,\bar{z}_1)N^{l_2}_{m_2,\overline{m}_2}(z_2,\overline{m}_2) = \sum_l \sum_{m,\overline{m}=-l}^{l} C \begin{pmatrix} l & m & \overline{m} \\ l_1 & m_1 & \overline{m}_1 \\ l_2 & m_2 & \overline{m}_2 \end{pmatrix} | z_{12} |^{2(\Delta_l-\Delta_1-\Delta_2)} N^l_{m\overline{m}} \tag{3.187}
$$

Since the 2-point function of the PF fields is normalized in such a way that the coefficient in front is one, the same holds also for NS and R fields. Then the constant C in (3.187) exactly coincides with the constant of the 3-point function. Due to the construction (3.153) and using (3.175) one can express all the 3-point functions in terms of the 3-point function of the SU(2) WZW fields $G^{l/2}_{m/2,m/2}$ and finally it turns out that the structure constants we looked for are exactly equal to the corresponding ones in SU(2)WZW theory. The latter were obtained in ref. [70] and their expression is

$$
C \begin{pmatrix} l & m & \overline{m} \\ l_1 & m_1 & \overline{m}_1 \\ l_2 & m_2 & \overline{m}_2 \end{pmatrix} = \begin{bmatrix} \frac{l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ \frac{m_1}{2} & \frac{m_2}{2} & \frac{m_3}{2} \end{bmatrix} \begin{bmatrix} \frac{l_1}{2} & \frac{l_2}{2} & \frac{l_3}{2} \\ \frac{\overline{m}_1}{2} & \frac{\overline{m}_2}{2} & \frac{\overline{m}_3}{2} \end{bmatrix} \rho(\frac{l_1}{2},\frac{l_2}{2},\frac{l_3}{2}) \tag{3.188}
$$

where the first two coefficients are $3j$ Wigner symbols and

$$\begin{aligned}
\frac{\rho^2}{(l_1+1)(l_2+1)(l_3+1)} &= \frac{\Gamma(\frac{p+3}{p+2})}{\Gamma(\frac{p+1}{p+2})}\prod_{k=1}^{3}\frac{\Gamma(1-\frac{l_k+1}{p+2})}{\Gamma(1+\frac{l_k+1}{p+2})} \\
\tilde{P}^2(\frac{l_1+l_2+l_3}{2}+1)\prod_{k=1}^{3}&\frac{\tilde{P}^2(\frac{l_1+l_2+l_3}{2}-l_k)}{\tilde{P}^2(l_k)} \\
\tilde{P}(l) = \prod_{k=1}^{l}\Gamma(1+&\frac{k}{p+2})\Gamma^{-1}(1-\frac{k}{p+2})
\end{aligned} \tag{3.189}$$

Exactly the same procedure goes also for the R sector. The structure constants are defined here by

$$R^{l_1}_{m_1,\alpha}(z_1)R^{l_2}_{m_2,-\alpha} = \sum_l \sum_{m,\overline{m}=-l}^{l} C\begin{pmatrix} l & m & \overline{m} \\ l_1 & m_1 & \overline{m}_1 \\ l_2 & m_2 & \overline{m}_2 \end{pmatrix} \mid z_{12}\mid^{2(\Delta_l-\Delta_1-\Delta_2)} \Phi^l_{m\overline{m}} \tag{3.190}$$

$$R^{l_1}_{m_1,\alpha}(z_1)R^{l_2}_{m_2,\alpha} = \sum_l \sum_{m,\overline{m}=-l}^{l} C\begin{pmatrix} l & m & \overline{m} \\ l_1 & m_1 & \overline{m}_1 \\ l_2 & m_2 & \overline{m}_2 \end{pmatrix} \mid z_{12}\mid^{2(\Delta_l-\Delta_1-\Delta_2)} \Phi^{p-l}_{m-\alpha p,\overline{m}-\overline{\alpha}p}$$

and coincide with the expressions (3.188) (we use here the notation of (3.173)). Thus formally the structure constants of both NS and R sector of the N=2 superconformal OPE algebra are the same as in SU(2) WZW theories. But here they have a different sense and different explicit values corresponding to the above obtained FR's (3.166) and (3.173).

The knowledge of these structure constants makes possible the analysis of renormalization group (RG) flow properties of the N=2 superconformal models following the strategy of the ref.[93,94] for the Virasoro m.m. and the Refs. [96,97,95] in the case of N=1 superconformal m.m. . The preliminary analysis of this problem shows that there exist marginal operators acting in the N=2 supersymmetric directions as follows: $c(p+2) \rightarrow c(p)$ [96]. The problem of finding the relevant marginal operators in the nonsupersymmetric directions is still open. The full understanding of the RG-flow picture for the N=2 m.m. (i.e. the knowledge of all allowed RG-flow directions) can have an important influence on the proof of the relations between Gepner's compactifications (which represent a special point in the moduli space of certain Calabi-Yau spaces) and the corresponding Calabi-Yau nonlinear sigma models.

Examples

The first model of the series (3.150) has a central charge c=1. From the PF point of view it is trivial since it is based on the group Z_1 which contains only the identity. Therefore all the fields in NS and R sectors are represented by exponents of the free scalar field φ (as we expect since c=1). Their 4-point functions are simply powerlike. This model turns out to coincide with the corresponding one in the N=1 superconformal series. The next model has a central charge $c = 3/2$ and will be considered in Sec.6.

The first nontrivial model we consider here is with $c = 9/5$ (p=3). It has the important peculiarity that the corresponding D_6 PF theory ($c = 4/5$) coincides with the well-known 3-state Potts model. This property helps us to express all the 4-point functions in terms of hypergeometric ones. In the NS sector there are 5 superfields:

$$N^1_{\pm 1} = (\tfrac{1}{10}, \pm\tfrac{1}{10}) \qquad N^2_0 = (\frac{2}{5}, 0), \qquad N^2_{\pm 2} = (\frac{1}{5} \pm \frac{1}{5}) \quad (3.191)$$

$$N^3_{\pm 1} = (\tfrac{7}{10}, \pm\tfrac{1}{10}) \quad , \qquad N^3_{\pm 3} = (\frac{3}{10}, \pm\frac{3}{10}) \quad (3.192)$$

The explicit superfield FR's (3.172) in this case take the form:

$$\begin{array}{ll} N^1_{\pm 1}N^1_{\mp 1} = [0]^{even} + [N^2_0]^{even} & N^3_{\pm 3}N^1_{\pm 1} = [N^1_{\mp 1}]^{odd} \\ N^1_{\pm 1}N^1_{\pm 1} = [N^2_{\pm 2}]^{even} + [N^3_{\mp 3}]^{odd} & N^3_{\pm 1}N^1_{\pm 1} = [N^2_{\pm 2}]^{even} \\ N^1_{\pm 1}N^2_0 = [N^1_{\pm 1}]^{even} + [N^3_{\pm 1}]^{even} & \\ N^2_0 N^2_0 = [0]^{even} + [N^2_0]^{even} & \\ N^3_{\pm 1}N^3_{\pm 1} = [N^3_{\mp 3}]^{odd} & \\ N^3_{\pm 3}N^3_{\pm 3} = [N^3_{\pm 1}]^{odd} & N^3_{\pm 3}N^3_{\pm 1} = [N^3_{\mp 1}]^{odd} \end{array} \quad (3.193)$$

In the last three relations are coded the FR's of the N=2-Z_5 PF currents, which are represented here by the superfields $N^3_{\pm 1}, N^3_{\pm 3}$. They are very similar to those ones in the usual PF theories. In agreement with what we said above, the 4-point functions of the fields $N^1_{\pm 1}, N^2_{\pm 2}, N^2_0$ can be expressed in terms of hypergeometric functions (since p=3 here). The functions of the fields $N^3_{\pm 1}$ and $N^3_{\pm 3}$ are simply powerlike, as it must be since they represent N=2 Z_5 SPF currents (see also Sec.(3.5). For example the 4-point function of the field $N^1_{\pm 1}$ follows directly from (3.179) and has the following form:

$$< N^1_1(1)N^1_{-1}(2)N^1_1(3)N^1_{-1}(4) >_{p=3} = \left| \frac{z_{12}z_{34}}{z_{13}z_{24}} \right|^{\frac{1}{5}} | z_{14}z_{32} |^{-\frac{2}{5}} | z(1-z) |^{\frac{1}{5}} \quad (3.194)$$

$$\frac{\Gamma^2(\frac{3}{5})\Gamma(\frac{4}{5})}{\Gamma^2(\frac{2}{5})\Gamma(\frac{1}{5})} \sum_{q,\bar{q}=0,1} \sum_{\sigma=\mp} h^{(\sigma)} F^{(\sigma)}_q(z) F^{(\sigma)}_{\bar{q}}(\bar{z})$$

where

$$h^- = \frac{\Gamma(\frac{1}{5})\Gamma^2(\frac{2}{5})}{\Gamma(\frac{4}{5})\Gamma^2(\frac{3}{5})} \qquad h^+ = \frac{\Gamma(\frac{1}{5})\Gamma(-\frac{2}{5})\Gamma(\frac{6}{5})}{\Gamma(\frac{4}{5})\Gamma(\frac{7}{5})\Gamma(-\frac{1}{5})}$$

and the functions $F^{\pm}_{q,(q)}$ are given by (3.182) with $a = \frac{4}{5}, b = \frac{1}{5}, c = \frac{3}{5}$

In the R sector there are 4 fields with lowest dimension $\Delta = c/24 = 3/40$ and U(1) charges $\pm 3/20, \pm 1/20$. There are also two fields with charges $\pm 1/4$ and dimensions 11/40 and 7/8, two with dimensions 19/40 (charges $\pm 3/20, \pm 7/20$) and two with dimension 27/40 ($\pm 1/20, \pm 9/20$). We shall write down some of the FR's which are direct consequence of eq.(3.173):

$$(\frac{3}{40},\pm\frac{3}{20})\times(\frac{3}{40},\pm\frac{3}{20})=(\frac{3}{10},\pm\frac{3}{10})$$
$$(\frac{3}{40},\pm\frac{3}{20})\times(\frac{3}{40},\pm\frac{1}{20})=(\frac{1}{5},\pm\frac{1}{5})$$
$$(\frac{3}{40},\pm\frac{3}{20})\times(\frac{11}{40},\pm\frac{1}{4})=(\frac{1}{10},\mp\frac{1}{10})^{II,\pm}$$
$$(\frac{3}{40},\pm\frac{3}{20})\times(\frac{7}{8},\pm\frac{1}{4})=(\frac{7}{10},\mp\frac{1}{10})^{II,\pm}$$
$$(\frac{3}{40},\pm\frac{1}{20})\times(\frac{3}{40},\pm\frac{1}{20})=(\frac{7}{10},\pm\frac{1}{10})+(\frac{1}{10},\pm\frac{1}{10})$$

The 4-point function of the fields $(3/40,\pm 3/20)$ is powerlike (they are pure exponents of φ) in accordance with (3.35). The correlator of the other fields of dimension $3/40$ $(\pm 1/20)$ is given by (3.184):

$$< R^1_{1,1}R^1_{-1,-1}R^1_{1,1}R^1_{1,-1} >\sim \mid z_{12}z_{34}\mid^{\frac{1}{10}}\mid \frac{z_{14}z_{32}}{z_{13}z_{24}}^{\frac{1}{10}}\mid z_{14}z_{32}\mid^{-\frac{3}{5}}$$
$$\mid z(1-z)\mid^{\frac{1}{5}}\sum_{q,\bar{q}=0,1}\sum_{\sigma=\mp} h^{(\sigma)}F^{(\sigma)}_q(z)F^{(\sigma)}_{\bar{q}}(\bar{z})$$

where the h^σ and F's are the same as in (3.194). The other functions in NS and R sectors can also be represented as hypergeometric ones. Moreover it turns out that also the twisted field $T_0(z)$ with dimension 3/16 has the same peculiarity. This is due to the above mentioned coincidence between D_6 PF and Potts model. According to our construction (3.166) we have

$$T_0(z)=\varphi^{(0)}(z)\sigma_0^{\varphi}(z) \tag{3.195}$$

The field $\varphi^{(0)}(z)$ has dimension 1/8 and coincides with the field $\phi_{2,1}$ in the N=0 theory with $c=4/5$. Taking this into account we have

$$< \prod_{i=1}^{4} T_0(z_i,\bar{z}_i) >=\frac{1}{4}\{2\mid I_1(z)\mid^2+\mid I_2(z)\mid^2\}$$
$$\mid z_{13}z_{24}\mid^{-\frac{1}{2}}\mid z(1-z)\mid^{\frac{5}{6}}< \prod_{i=1}^{4}\sigma_0^{\varphi}(z_i,\bar{z}_i) > \tag{3.196}$$

where

$$I_1(z) = \frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{6})}{\Gamma(\frac{5}{3})}F(\frac{5}{6},\frac{3}{2},\frac{5}{3};z)$$
$$I_2(z) = \frac{\Gamma^2(\frac{1}{6})}{\Gamma(\frac{1}{3})}z^{-\frac{2}{3}}F(\frac{5}{6},\frac{1}{6},\frac{1}{3};z)$$

and the function $< \sigma_0\sigma_0\sigma_0\sigma_0 >$ is computed in the Appendix B. The other twisted field $T_1(z)$ (dimension 7/80) is expressed as a two-dimensional integral up to a function $< \sigma_0 \ldots \sigma_0 >$ since the corresponding C-disorder field $\varphi^{(1)}$ coincide with the field $\phi_{2,2}$ in the Virasoro m.m.

The next model corresponds to p=4 (c=2) and therefore is built up on the D_8 PF theory ($c = 1$). The even sector of this theory can be viewed as a particular case of the theory of a free scalar field. Thus, all the functions of NS and R fields here are powerlike.

We shall consider here also the model with p=6 ($c = 9/4$). The corresponding D_{12} PF model coincides with the N=1 superconformal m.m. with $c = 5/4$. As before this property helps us to compute explicitely the 4-point function of the twisted field $T_0(z)$ of dimension 3/8. The C-disorder field $\varphi^{(0)}(z)$ (dimension 5/16) is the same as the field $R_{2,1}(z)$ in the Ramond sector of the N=1 theory. Therefore we have [17,18]

$$< \prod_{i=1}^{4} T_0(z_i\bar{z}_i) >=| z_{13}z_{24} |^{-\frac{5}{4}} | z(1-z) |^{\frac{1}{8}} < \prod_{i=1}^{4} \sigma_0^{\varphi}(z_i\bar{z}_i) >$$

$$[| Y_1 |^2 + | Y_3 |^2 + (\frac{1}{8} cos^2\frac{\pi}{8} - 1)(| Y_2 |^2 + | Y_4 |^2)] < \prod_{i=1}^{4} \sigma_0^{\varphi}(z_i,\bar{z}_i) > \quad (3.197)$$

where

$$\begin{aligned}
Y_1(z) &= \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(\frac{1}{8},\frac{9}{8})\sqrt{z}F(\frac{7}{8},\frac{1}{8},\frac{5}{4};z) + \\
&\quad - B(\frac{1}{8},\frac{1}{8})F(\frac{7}{8},\frac{1}{8},\frac{1}{4};z)] \\
Y_2(z) &= \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(\frac{5}{8},\frac{1}{8})z^{\frac{1}{4}}F(-\frac{1}{8},\frac{5}{8},\frac{3}{4};z) + \\
&\quad - B(\frac{13}{8},\frac{1}{8})\sqrt{z-1}z^{\frac{3}{4}}F(\frac{7}{8},\frac{13}{8},\frac{7}{4};z)] \\
Y_3(z) &= \sqrt{1-\sqrt{1-\frac{1}{z}}}[B(\frac{1}{8},\frac{9}{8})\sqrt{z}F(\frac{7}{8},\frac{1}{8},\frac{5}{4};z) + \\
&\quad B(\frac{1}{8},\frac{1}{8})F(\frac{7}{8},\frac{1}{8},\frac{1}{4};z)] \\
Y_4(z) &= \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(\frac{5}{8},\frac{1}{8})z^{\frac{1}{4}}F(-\frac{1}{8},\frac{5}{8},\frac{3}{4};z) + \\
&\quad B(\frac{13}{8},\frac{1}{8})\sqrt{z-1}z^{\frac{3}{4}}F(\frac{7}{8},\frac{13}{8},\frac{7}{4};z)]
\end{aligned}$$

Similarly the functions of the other fields $T_1(\Delta = 7/32), T_2(\Delta = 1/8), T_3(\Delta = 3/32)$ can be expressed in terms of multidimensional integrals (times the function $< \sigma_0 \ldots \sigma_0 >$)

In the NS and R sectors there are a lot of fields. In each particular case we can apply the general formulas in order to obtain the corresponding FR's, 4-point functions and structure constants. The above mentioned connection with N=1 models have to be taken into account as well. For instance there is a field in the R sector $R^3_{\pm 3,\pm 1}$ (and for each even p as well) with dimension $\Delta = c/24 = 3/32$ and zero U(1) charge. It coincides exactly with the field σ_3 in PF theory and also with the field $R_{1,2}$ in the R sector of the N=1 theory.

For the 4-point function of this field we have [17,18]

$$
\begin{aligned}
< R^3_{3,1} R^3_{-3,-1} R^3_{3,1} R^3_{-3,-1} > \; &= \; | z_{13} z_{24} |^{-\frac{3}{4}} | z |^{-\frac{3}{4}} | 1-z |^{\frac{5}{12}} \\
&\quad [| W_1 |^2 + | W_3 |^2 + 2(| W_2 |^2 + | W_4 |^2)]
\end{aligned}
$$

$$
\begin{aligned}
W_1(z) &= \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(\frac{5}{6},\frac{1}{6})\sqrt{z}F(\frac{7}{6},-\frac{1}{6},\frac{2}{3};z)+ \\
&\quad - B(-\frac{1}{6},-\frac{1}{6})\sqrt{z-1}F(\frac{7}{6},-\frac{1}{6},-\frac{1}{3};z)] \\
W_2(z) &= \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(\frac{3}{2},-\frac{1}{6})z^{\frac{5}{6}}F(\frac{1}{6},\frac{7}{6},\frac{4}{3};z)+ \\
&\quad B(\frac{5}{2},-\frac{1}{6})\sqrt{z-1}z^{-\frac{7}{3}}F(\frac{7}{6},\frac{5}{2},\frac{7}{3};z)] \\
W_3(z) &= \sqrt{1-\sqrt{1-\frac{1}{z}}}[B(\frac{5}{6},\frac{1}{6})\sqrt{z}F(\frac{7}{6},-\frac{1}{6},\frac{2}{3};z)+ \\
&\quad B(-\frac{1}{6},-\frac{1}{6})\sqrt{z-1}F(\frac{7}{6},-\frac{1}{6},-\frac{1}{3};z)] \\
W_4(z) &= \sqrt{1+\sqrt{1-\frac{1}{z}}}[B(\frac{3}{2},-\frac{1}{6})z^{\frac{5}{6}}F(\frac{1}{6},\frac{7}{6},\frac{4}{3};z)+ \\
&\quad - B(\frac{5}{2},-\frac{1}{6})\sqrt{z-1}z^{-\frac{7}{3}}F(\frac{7}{6},\frac{5}{2},\frac{7}{3};z)]
\end{aligned}
$$

3.5 Z_{p+2} symmetry and N=2 superparafermionic models

We are going now to discuss in more details the discrete symmetries of the N=2 m.m.. One can expect, due to the explicit construction in terms of PF fields, that they possess the same Z_p symmetry. However because of the presence of the free bosonic field φ (which generate a bigger U(1) symmetry), this symmetry is lost. Nevertheless as it was mentioned by Gepner [55], a bigger Z_{p+2} discrete symmetry appears. In our considerations the origin of this surprising fact can be understood as follows.

Let consider the fields N^p_{-m} having principal quantum number p. They form a closed OPE algebra as it can be seen from the FR's obtained above:

$$N^p_{-m_1}N^p_{-m_2} = [N^p_{p+2-(m_1+m_2)}]^{odd} \tag{3.198}$$

Denote all the fields of this type by $N^p_{-(p+2)+2k}k = 1,2\cdots p+1$. Then the FR's (3.198) can be reinterpreted as a multiplication law of some abelian discrete group, whose charges are of the form $\frac{k}{p+2}$. Therefore the above fields generate a PF type symmetry of the N=2 minimal models.

To make this statement more clear we shall briefly describe here the N=2 PF theory with Z_{p+2} discrete symmetry. It contains in addition to the N=2 superstress-energy tensor $\mathcal{W}(z,\theta^+,\theta^-)$ PF supercurrents $\Psi_k(z,\theta^+,\theta^-)$, carring Z_{p+2} charge $(k = 1,2,\cdots p+1)$ with dimension Δ_k and charge q_k. According to the Z_{p+2} symmetry they have the following OPE algebra:

$$\Psi_{k_1}(z_1,\theta_1^+,\theta_1^-)\Psi_{k_2}(z_2,\theta_2^+,\theta_2^-) = C_{k_1,k_2}\hat{z}_{12}^{\Delta_{k_1+k_2}-\Delta_{k_1}-\Delta_{k_2}-\frac{1}{2}}\mathcal{D}_1\Psi_{k_1+k_2}(z_2,\theta_2^+,\theta_2^-)+\cdots \tag{3.19}$$

$$\Psi_{k_1}(z_1,\theta_1^+,\theta_1^-)\Psi^\dagger_{k_2}(z_2,\theta_2^+,\theta_2^-) = \hat{z}_{12}^{-2\Delta_k}[1+2q_k\frac{\theta_{12}^-\theta_{12}^+}{\hat{z}_{12}}+12\frac{q_k}{c_p}\hat{z}_{12}\mathcal{D}_2\mathcal{W}(z_2,\theta_2^+,\theta_2^-)+\cdot$$

where

$$\begin{aligned}
\mathcal{D}_1 &= \hat{z}_{12}D^- + (\Delta_{k_1+k_2}+\frac{1}{2})\theta_{12}^-(1+\frac{1}{2}\theta_{12}^+D^-) - \frac{1}{2}\theta_{12}^-\hat{z}_{12}(D^-D^+-\partial)\\
\mathcal{D}_2 &= \frac{\theta_{12}^-\theta_{12}^+}{\hat{z}_{12}} + \frac{1}{2}(\theta_{12}^-D^+ - \theta_{12}^+D^-) + \theta_{12}^-\theta_{12}^+[\partial + \frac{1}{3}q_k(D^-D^+-\partial)] +\\
&\quad + \frac{1}{6}\hat{z}_{12}(D^-D^+-\partial)
\end{aligned}$$

and $\Psi^\dagger_k$ is the conjugate of Ψ_k, having Z_{p+2} charge $p+2-k$, dimension Δ_k and U(1) charge $-q_k$.

Following ref. [33] we conclude that the superconformal OPE (3.199) of the superfields Ψ_k is consistent with their Z_{p+2} properties if the dimensions Δ_k obey the following monodromy condition:

$$\Delta_{k_1+k_2} + \Delta_{k_1+k_3} + \Delta_{k_2+k_3} = \Delta_{k_1+k_2+k_3} + \Delta_{k_1} + \Delta_{k_2} + \Delta_{k_3} - \frac{1}{2} \tag{3.200}$$

The most general solution of (3.200) is

$$\Delta_k(M) = M\frac{k(p+2-k)}{p+2} + M_k - \frac{1}{2} \tag{3.201}$$

where M and M_k are arbitrary integers and $M_k = M_{p+2-k}$. We shall consider here the simplest example $M=1, M_k=0$ only. Therefore the superparafermionic currents have dimensions:

$$\Delta_k = \frac{k(p+2-k)}{p+2} - \frac{1}{2} \tag{3.202}$$

In addition we must render the Z_{p+2} symmetry consistent with the bigger U(1) symmetry coming from N=2 SUSY algebra. This imposes a condition on the possible U(1) charges of the parafermions:

$$q_{k_1} + q_{k_2} = q_{k_1+k_2} - \frac{1}{2} \qquad (3.203)$$

The solution of (3.203), if we take into account also the condition $q_k = -q_{p+2-k}$ is

$$q_k = N\frac{k}{p+2} - \frac{1}{2} \qquad (3.204)$$

Again as before choosing N=1 we get

$$q_k = \frac{k}{p+2} - \frac{1}{2} \qquad (3.205)$$

The key observation is now that the dimensions and charges of the parafermions given by eq. (3.202) and (3.205) respectively, coincide with those of the fields $N^p_{-(p+2)+2k}$ in the N=2 models. Thus we are led to the suspect that indeed these fields generate a Z_{p+2} parafermionic symmetry in the p'th minimal N=2 superconformal model. To complete our discussion we should compute also the central charge of the N=2 PF theories. To do this we follow the idea of ref. [33]. Consider for simplicity the OPE of the first components N_1 of the PF supercurrent Ψ_1. According to (3.199) we have

$$N_1(z_1)N_1^\dagger(z_2) = z_{12}^{-1+\frac{2}{p+2}}\{1 + 12\frac{q_1}{c}z_{12}J(z_2) + 6\frac{q_1}{c}z_{12}^2\partial J(z_2) + \cdots\} \qquad (3.206)$$

It is easy to compute the 4-point function of the field N_1 using the OPE (3.206) and the WI's (3.11)

$$< N_1(1)N_1(2)N_1^\dagger(3)N_1^\dagger(4) >= (z_{13}z_{24})^{-1+\frac{2}{p+2}}(\frac{1-x}{x})^{-1+\frac{2}{p+2}} \qquad (3.207)$$

where

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}$$

(Note that this function is the same as the 4-point function of the field N^p_{-p} obtained above).

Now the point is that the OPE (3.206) (and therefore the function (3.207)) is consistent with the initial N=2 SUSY:

$$\begin{aligned} J(z_1)J(z_2) &= \frac{c}{12}\frac{1}{z_{12}^2} + \cdots \\ J(z_1)N_1(z_2) &= \frac{q_1}{z_{12}}N_1(z_2) + \cdots \end{aligned}$$

providing that the central charge is equal to

$$c = \frac{3p}{p+2}$$

i.e. the central charge c of these superparafermionic models coincides with the central charge of the p'th minimal models of the N=2 series. This completes the proof of the equivalence between Z_{p+2} N=2 superparafermionic theories and p'th N=2 minimal models.

We can look now for the spectrum of these theories with enlarged symmetry. The first observation is that they should have Ramond and NS type order parameters corresponding to the different choices of boundary conditions for the supercurrents $G^{\pm}(z)$. The OPE's with PF currents are given by

$$\begin{aligned}
\psi_1(z)N_k(0) &= \sum_{n\in Z} z^{-\frac{k}{p+2}+n-1} A_{\frac{1+k}{p+2}+\frac{1}{2}-n} N_k(0) \\
\psi_1^\dagger(z)N_k(0) &= \sum_{n\in Z} z^{\frac{k}{p+2}+n-1} A^\dagger_{\frac{1-k}{p+2}+\frac{1}{2}-n} N_k(0)
\end{aligned}$$

for the NS order parameters and

$$\begin{aligned}
\psi_1(z)R_k(0) &= \sum_{n\in Z} z^{-\frac{k}{p+2}+n-\frac{1}{2}} B_{\frac{1+k}{p+2}-n} R_k(0) \\
\psi_1^\dagger(z)R_k(0) &= \sum_{n\in Z} z^{\frac{k}{p+2}+n-\frac{1}{2}} B^\dagger_{\frac{1-k}{p+2}-n} R_k(0)
\end{aligned}$$

for the R ones. The operators A_k, B_k are defined by these equations and

$$\begin{aligned}
A_{\frac{1+k}{p+2}-n} N_k &= N_{k+2} \qquad & B_{\frac{1+k}{p+2}-n} R_k &= R_{k+2} \\
A^\dagger_{\frac{1-k}{p+2}-n} N_k &= N_{k-2} \qquad & B^\dagger_{\frac{1-k}{p+2}-n} R_k &= R_{k-2}
\end{aligned}$$

Using our explicit construction we can easily recognize the corresponding fields in the minimal models. In the case of NS fields we find

$$\begin{aligned}
N^p_{-p}(z)N^k_k(0) &= z^{-\frac{k}{p+2}} (N^k_{k+2}(0))^{II} \\
N^p_{-p}(z)N^k_{-k}(0) &= z^{\frac{k}{p+2}} (N^k_{-k+2}(0))^{II}
\end{aligned}$$

which shows us that the chiral superfields N^k_k and N^k_{-k} are the NS order parameters with dimensions and charges

$$\begin{aligned}
d_k &= \frac{k}{p+2} = q_k \qquad & for \qquad N^k_k &= \sigma_k^{NS} \\
d_k &= \frac{k}{p+2} = -q_k \qquad & for \qquad N^k_{-k} &= \sigma_k^{\dagger NS}
\end{aligned} \tag{3.208}$$

The analogous calculations in the R sector lead to

$$\begin{aligned}
N^p_{-p}(z)R^{k-1}_{k-1,1}(0) &= z^{-\frac{k}{p+2}+\frac{1}{2}} \left(R^{k-1}_{k+1,1}(0)\right)^{II} \\
N^p_{p}(z)R^{k-1}_{k-1,1}(0) &= z^{\frac{k}{p+2}-\frac{1}{2}} \left(R^{k-1}_{k-3,1}(0)\right)^{II}
\end{aligned}$$

Therefore the fields which represent the order parameters of Z_{p+2} charge k are given in the minimal models by $R^{k-1}_{k-1,1}$, $k = 1, 2, \cdots p+1$. These are in fact all the Ramond fields with lowest dimension and U(1) charge

$$d_k = \frac{c}{24} \quad q_k = \frac{k}{2(p+2)} - \frac{1}{4} \tag{3.209}$$

Thus we find that the spectrum of the N=2 PF theories is given by (3.208) in NS and by (3.209) in R sector. The other primary fields in the minimal models correspond to the descendents of the above primary fields of the PF theories, i.e. they are obtained by acting with $A^\dagger_\alpha$ and $B^\dagger_\alpha$ on the fields $N^k_{\pm k}$ and $R^{k-1}_{k-1,1}$. Finally we note that the fields in the twisted sector correspond to the C- disorder sector of the N=2 superparafermionic theories, which also coincides with the twisted sector for their U(1) current as it can be seen from (3.206).

3.6 Models with c=3/2 and c=3

3.6.1 Series of models with $c = 3/2$

In this section we consider the theory of a free scalar superfield

$$\phi(z,\theta) = \varphi(z) + \theta\psi(z) \tag{3.210}$$

of dimension zero. The invariant 2-D action of such theory is given by

$$A = \frac{2}{\pi}\int dz d\bar{z}(\frac{1}{2}\partial\varphi\overline{\partial\varphi} - \psi\overline{\partial\varphi}) \tag{3.211}$$

As follows from this expression, the propagator of the superfield (3.210) is

$$< \phi(z_1,\theta_1)\phi(z_2,\theta_2) >= -ln\frac{\tilde{z}_{12}}{R} \qquad \tilde{z}_{12} = z_{12} - \theta_1\theta_2$$

(R is an infrared cut-off) or in terms of components

$$< \varphi(z_1)\varphi(z_2) >= -ln\frac{z_{12}}{R}, \qquad < \psi(z_1)\psi(z_2) >= \frac{1}{z_{12}} \tag{3.212}$$

From (3.211) we can extract also the corresponding expressions for the generators of the N=1 SUSY, the stress-energy tensor T(z) and supercurrent G(z):

$$T(z) = -\frac{1}{2}[(\partial\varphi)^2 - \psi\partial\psi] \qquad G = i\psi\partial\varphi \tag{3.213}$$

They satisfy the well-known N=1 OPE's with central charge $c = 3/2$. In addition to (3.213) for each theory there exists two additional "currents": the U(1) current $\partial\varphi(z)$ (with $\Delta = 1$) and the fermionic current $\psi(z)$ ($\Delta = 1/2$).

The various fields of the theory are combined in different sectors which realize different boundary conditions of these currents. The NS superfields

$$N^\pm_\alpha(z,\theta) =: e^{\pm i\alpha\phi(z,\theta)} : \tag{3.214}$$

evidently produce periodic boundary conditions for both currents. The first and second component of the superfield (3.214) are : $e^{\pm i\alpha\varphi(z)}$: and $\psi(z)$: $e^{\pm i\alpha\varphi(z)}$: respectively. Using the explicit form of the current (3.213) and the propagators (3.212) it is easy to compute the dimension and the U(1) charge of the NS field (3.214)

$$\Delta(\pm\alpha) = \alpha^2, \qquad q(\pm\alpha) = \pm\alpha \tag{3.215}$$

The fields belonging to the R sector of the theory produce antiperiodic boundary conditions for the supercurrent G(z). Therefore they can be constructed by the Ising model lowest energy field $\sigma^I(z)$ of dimension 1/16 (see [18] and the Appendix B) since it produces antiperiodic boundary conditions for $\psi(z)$. Thus we define

$$R^{\pm}_{\alpha}(z) = \sigma^{\pm}(z) : e^{\pm i\alpha\varphi(z)} : \tag{3.216}$$

The dimension and charge of R fields are

$$\Delta^{\pm}(\alpha) = \frac{\alpha^2}{2} + \frac{1}{16}, \qquad q^{\pm}(\alpha) = \pm\alpha \tag{3.217}$$

There is one more possibility here of choosing also antiperiodic boundary conditions for the U(1) current $\partial\varphi(z)$. They are realized by the fields σ^{φ}_0 introduced in the Appendix B. In fact we have two twisted fields in each theory:

$$T_0(z) = \sigma^{\varphi}_0(z), \qquad T_1(z) = \sigma^I\sigma^{\varphi}_0 \tag{3.218}$$

which correspond to periodic and antiperiodic boundary conditions of $\psi(z)$ respectively. The dimensions of the fields (3.218) are

$$\Delta_0 = \frac{1}{16}; \qquad \Delta_1 = \frac{1}{8} \tag{3.219}$$

for $T_0(z)$ and $T_1(z)$ respectively. Following ref.[98] we shall classify the $c = 3/2$ models corresponding to their symmetries larger than the one spanned by T(z),G(z), $\partial\varphi(z), \psi(z)$. The additional currents $S(z)$ we introduce belong to the NS sector and therefore have the form:

$$S^{\pm}_g(z) =: e^{\pm ig\varphi(z)} :, \qquad (S^{\pm}_g(z))^{II} = g\psi(z) : e^{\pm ig\varphi(z)} : \tag{3.220}$$

The symmetry algebra is defined by the OPE's of these currents:

$$\begin{aligned} S^+_g(z_1)S^-_g(z_2) = &\quad z_{12}^{-g^2}[1 \pm igz_{12}\partial\varphi(z_2) + \cdots] \\ S^+_g(z_1)(S^-_g)^{II}(z_2) = &\quad gz_{12}^{-g^2}\psi(z) + \cdots \end{aligned}$$

Since we want to consider local theories only ($S^{\pm}_g$ being bosonic or fermionic fields) we must impose the following condition

$$g^2 = n, \qquad n = 1, 2, \cdots \tag{3.221}$$

Therefore each theory is labeled by one integer number n and has a symmetry generated by the currents T,G,$\partial\varphi, \psi$ and $S^{\pm}_n$ (dimension $\frac{n}{2}$),$(S^{\pm}_n)^{II}$ (dimension $\frac{n+1}{2}$).

Let us consider the product of the currents S(z) with the NS fields of the theory:

$$S_g^{\pm}(z_1)N_{\alpha}^{\mp}(z_2) = z_{12}^{-g\alpha} : e^{\pm i(g-\alpha)\varphi(z_2)} : \tag{3.222}$$

Similar results hold also for the R fields and for the product with the currents $(S_g^{\pm})^{II}$. From (3.222) it follows that NS and R fields realize periodic and antiperiodic boundary conditions also for the additional currents provided

$$g\alpha = \frac{k}{2} \tag{3.223}$$

Thus the implementation of locality here, together with the condition (3.221), impose the following quantization of the charges of the fields:

$$\alpha_k = \frac{k}{2\sqrt{n}} \qquad k = 0, 1 \cdots n \qquad n = 0, 1, \cdots \tag{3.224}$$

The corresponding dimensions are:

$$\Delta_{NS}(\pm\alpha_k) = \frac{k^2}{8n} \qquad k = 0, 1, \cdots n \tag{3.225}$$

in the NS sector and

$$\Delta_R(\pm\alpha_k) = \frac{k^2}{8n} + \frac{1}{16} \tag{3.226}$$

in the R sector (the other dimensions differ by (half)integers and are descendents of the latter ones). In addition to these we have in each theory also twisted sector, which contains always two fields of dimension 1/16 and 1/8 (see eqs. (3.218)(3.219)).

The FR's of the fields in such theories can be obtained in a straightforward way. The result is:

$$\begin{aligned} N_{\alpha_1}^{\pm}N_{\alpha_2}^{\pm} &= (N_{\alpha_1+\alpha_2}^{\pm}) & N_{\alpha_1}^{\pm}N_{\alpha_2}^{\mp} &= (N_{\alpha_1-\alpha_2}^{\pm}) \\ R_{\alpha_1}^{\pm}R_{\alpha_2}^{\pm} &= (N_{\alpha_1+\alpha_2}^{\pm}) + (N_{\alpha_1+\alpha_2}^{\pm})^{II} & R_{\alpha_1}^{\pm}R_{\alpha_2}^{\mp} &= (N_{\alpha_1-\alpha_2}^{\pm}) + (N_{\alpha_1-\alpha_2}^{\pm})^{II} \end{aligned} \tag{3.227}$$

and for the T sector

$$\begin{aligned} T_0T_0 &= T_1T_1\sum_{k=0}^{n} N_k \\ T_0T_1 &= \sum_{k=0}^{n} R_k \end{aligned} \tag{3.228}$$

where we denote $N_k = N(\alpha_k), R_k = R(\alpha_k)$.

The N-point functions of the NS fields are simply powerlike and are given by

$$< \prod_{i=1}^{n} N_{\alpha_i}(z_i\theta_i) >= \prod_{i<j}^{n}(\tilde{z}_{ij})^{\alpha_i\alpha_j} \tag{3.229}$$

In the R sector this function contains also the N-point function of the Ising order parameter σ^I:

$$< \prod_{i=1}^{n} R_{\alpha_i}(z_i) >= \prod_{i<j}(z_{ij})^{\alpha_i\alpha_j} < \prod_{l=1}^{n} \sigma_l(z_l) > \tag{3.230}$$

Due to the Z_2 symmetry of the Ising model the latter are different from zero only if N is even.

The 4-point function of the fields T_0 was computed in ref. [99,100,37] (see Appendix B)

$$< T_0T_0T_0T_0 >=< \sigma_0^\varphi\sigma_0^\varphi\sigma_0^\varphi\sigma_0^\varphi > \tag{3.231}$$

and for $T_1 = \sigma_0^\varphi\sigma^I = T_0\sigma^I$ we have:

$$< \prod_{i=1}^{4} T_1(z_i) >= \left(A_1\sqrt{1+\sqrt{1-x}} + A_2\sqrt{1-\sqrt{1-x}}\right)(z_{13}z_{24})^{-\frac{1}{8}} < \prod_{i=1}^{4} \sigma_0^\varphi(z_i) > \tag{3.232}$$

The first example is the theory with n=1. The additional currents (3.221) in this case are

$$\psi^\pm(z) = \sqrt{2} : e^{\pm i\varphi} : \qquad\qquad \psi(z) \tag{3.233}$$

with dimension 1/2 and

$$J^\pm = \sqrt{2}\psi : e^{\pm i\varphi} : \qquad\qquad J^0 = i\partial\varphi \tag{3.234}$$

with dimension 1. They generate the so called super SU(2) Kac-Moody algebra (SSU(2)) which is defined by the singular terms in the following OPE's [101]

$$\begin{aligned}
J^0(z_1)J^0(z_2) &= \frac{1}{z_{12}^2} + \cdots \\
J^0(z_1)J^\pm(z_2) &= \pm\frac{1}{z_{12}}J^\pm(z_2) \\
J^\pm(z_1)J^\mp(z_2) &= \frac{2}{z_{12}^2} \pm \frac{2}{z_{12}}J^0(z_2) \\
J^\pm(z_1)\psi(z_2) &= \frac{1}{z_{12}}\psi^\pm(z_2) + \cdots \\
J^\pm(z_1)\psi^\mp(z_2) &= \frac{2}{z_{12}}\psi(z_2) + \cdots \\
\psi^\pm(z_1)\psi^\mp(z_2) &= \frac{2}{z_{12}} \pm 2J^0(z_2) + \cdots
\end{aligned} \tag{3.235}$$

In the NS sector of the theory there are two fields of dimension 1/8

$$N_1^\pm(z) =: e^{\pm\frac{i}{2}\varphi(z)} : \tag{3.236}$$

The R fields are

$$\begin{aligned}
R_0^\pm &= \sigma^\pm(z) & \Delta_0 &= \frac{1}{16} \\
R_1^\pm &= \sigma^\pm(z) : e^{\pm\frac{i}{2}\varphi} : & \Delta_1 &= \frac{3}{16}
\end{aligned} \tag{3.237}$$

For their FR's we have from (3.227):

$$(\frac{1}{16})^{\pm} \times (\frac{1}{16})^{\pm} = [\psi] \qquad (\frac{1}{16})^{\pm} \times (\frac{1}{16})^{\mp} = [0]$$
$$(\frac{3}{16})^{\pm} \times (\frac{1}{16})^{\mp} = [\frac{1}{8}] \qquad (\frac{3}{16})^{\pm} \times (\frac{1}{16})^{\pm} = [\frac{1}{8}]^{\pm, II}$$
$$(\frac{3}{16})^{\pm} \times (\frac{3}{16})^{\pm} = [J^{\pm}] \qquad (\frac{3}{16})^{\pm} \times (\frac{3}{16})^{\mp} = [0] + [J^0]$$

In accordance with the results of the Appendix B the twisted field T_0 satisfies the FR's:

$$T_0 \times T_0 = [0] + [\frac{1}{8}]^{+} + [\frac{1}{8}]^{-} \qquad (3.238)$$

The other FR's follow from this formula.

The next theory is labeled by n=2. Let us take here linear combinations of the currents (3.221) as follows:

$$J_3 =: cos\sqrt{2}\varphi : \qquad J_2 = - : sin\sqrt{2}\varphi : \qquad J_1 = i\partial\varphi$$
$$G_2 = -i\sqrt{2}\psi : sin\sqrt{2}\varphi : \qquad G_3 = \sqrt{2}\psi : cos\sqrt{2}\varphi : \qquad G_1 = i\psi\partial\varphi \qquad (3.239)$$

The symmetry algebra can be obtained from the OPE's:

$$\begin{aligned}
J^i(z_1)J^j(z_2) &= \frac{i\epsilon^{ijk}}{z_{12}}J^k(z_2) + \frac{1}{2z_{12}^2}\delta^{ij} \\
J^i(z_1)G^j(z_2) &= \frac{i\epsilon^{ijk}}{z_{12}}G^k(z_2) + \frac{1}{z_{12}^2}\delta^{ij}\psi(z_2) \\
G^i(z_1)G^j(z_2) &= \frac{\delta^{ij}}{2z_{12}^3} + i\epsilon^{ijk}(\frac{2}{z_{12}^2}J^k + \frac{1}{z_{12}}\partial J^k(z_2)) + \frac{2\delta^{ij}}{z_{12}}T(z) \\
G^i(z_1)\psi(z_2) &= \frac{1}{z_{12}}J^i(z_2) + \cdots
\end{aligned} \qquad (3.240)$$

In terms of Laurent coefficients

$$G^i(z) = \sum_{\alpha\in Z+\frac{1}{2}} \frac{G^i_\alpha}{z^{\alpha+\frac{3}{2}}}$$
$$J^i(z) = \sum_{n\in Z} \frac{J^i_n}{z^{n+1}}$$
$$\psi(z) = \sum_{\alpha\in Z+\frac{1}{2}} \frac{\psi_\alpha}{z^{\alpha+\frac{1}{2}}}$$

the algebra (3.235) turns out to be the well-known [91] SO(3) N=3 extended superconformal algebra in 2D (for $c = 3/2$):

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{1}{8}n(n^2-1)\delta_{n+m,0}$$

$$
\begin{aligned}
&[L_n, G^i_\alpha] = (\frac{n}{2} - \alpha)G^i_{n+\alpha}\\
&[L_n, J^i_m] = -mJ^i_{n+m}\\
&[L_n, \psi_\alpha] = -(\frac{n}{2} + \alpha)\psi_\alpha\\
&[J^i_n, J^j_m] = i\epsilon^{ijk}J^k_{n+m} + \frac{1}{2}n\delta^{ij}\psi_{n+\alpha}\\
&[J^i_n, G^j_\alpha] = i\epsilon^{ijk}G^k_{n+\alpha} + n\delta^{ij}\psi_{n+\alpha}\\
&\{G^i_\alpha, G^j_\beta\} = 2\delta^{ij}L_{\alpha+\beta} + i\epsilon^{ijk}(\alpha-\beta)J^k_{\alpha+\beta} + \frac{1}{2}\delta^{ij}(\alpha^2 - \frac{1}{4})\delta_{\alpha+\beta,0}\\
&\{G^i_\alpha, \psi_\beta\} = J^i_{\alpha+\beta}\\
&\{\psi_\alpha, \psi_\beta\} = \frac{1}{2}\delta_{\alpha+\beta}\\
&[J^i_n, \psi_\alpha] = 0
\end{aligned}
\tag{3.241}
$$

The supercurrents $G^i(z)$ obviously form a triplet (i.e. they belong to adjoint representation) of SO(3). So the full symmetry of this model is given by N=3 SUSY algebra. As far as the initial N=1 symmetry is concerned we can see from (3.225)(3.226) that there are two fields in the NS sector:

$$
\begin{aligned}
N_1^\pm &= \frac{1}{2}\left(e^{\frac{i}{2\sqrt{2}}\varphi} \pm e^{-\frac{i}{2\sqrt{2}}\varphi}\right)\\
N_2^\pm &= \frac{1}{2}\left(e^{\frac{i}{\sqrt{2}}\varphi} \pm e^{-\frac{i}{\sqrt{2}}\varphi}\right)
\end{aligned}
\tag{3.242}
$$

with dimension 1/16 and 1/4 respectively, and three fields in the R sector:

$$
\begin{aligned}
R_1^\pm &= \frac{1}{2}\sigma^\pm\left(e^{\frac{i}{2\sqrt{2}}\varphi} \pm e^{-\frac{i}{2\sqrt{2}}\varphi}\right)\\
R_0^\pm &= \sigma^\pm\\
R_2^\pm &= \frac{1}{2}\sigma^\pm\left(e^{\frac{i}{\sqrt{2}}\varphi} \pm e^{-\frac{i}{\sqrt{2}}\varphi}\right)
\end{aligned}
\tag{3.243}
$$

with dimensions $\frac{1}{16}, \frac{1}{8}, \frac{5}{16}$. As usually there are two fields in T sector

$$
\begin{aligned}
T_0 &= \sigma_0^\varphi & \Delta_0 &= \frac{1}{16}\\
\theta_1 &= \sigma^\pm\sigma_0^\varphi & \Delta_1 &= \frac{1}{8}
\end{aligned}
\tag{3.244}
$$

All these fields form also a representation of the whole N=3 algebra. Similarly to the N=2 case, the N=3 superconformal algebra has NS, R and T sectors. The difference between NS and R sectors is that they produce different analytic behaviour of the currents $G^i(z)$, or equivalently the algebra (3.241) is the NS algebra when $\alpha, \beta \in Z + 1/2$ and the R algebra if $\alpha, \beta \in Z$. The twisted N=3 algebra is recovered when two of the SO(3) currents, say $J^{1,2}(z)$, have half-integer modes, and the remaining J^3 integer ones, i.e. we have twisted SU(2) algebra in this case. Then the algebra (3.241) imposes the integer and half-integer modes for the supercurrents $G^i(z)$ as well. There are two nonequivalent ways to do this. Firstly we can choose the two corresponding G(z)'s -$G^{1,2}(z)$- to have half-integer modes and G^3 integer ones. As it can be seen from (3.241) this means that ψ has also integer modes. The other possibility is to take integer modes for $G^{1,2}(z)$ and half-integer for G^3. In this case ψ has half-integer modes. In other words we have two twisted sectors in N=3 superconformal theories -T_1 for the first possibility and T_2 for the second-one. To recognize to which sector belongs each of the fields (3.242)(3.244) it is necessary to recognize what kind of analytic behaviour it produces for the different currents (3.239). We do this using the explicit construction of the fields and currents and the propagators (3.212). It turns out that there is only one field in the N=3 NS sector -the field $N_2^{\pm}$ with dimension 1/4, which form a SU(2) doublet. The singlet (1/16 dimensional field R_0) and the doublet ($R_2^{\pm}$ with dimension 5/16) belong to the N=3 R sector. The fields $R_1^{\pm}$ and T_1 (both with dimension 1/8) produce antiperiodic boundary conditions for $\psi(z)$ and hence are in T_1 sector, while the fields $N_1^{\pm}$ and T_0 (dimension 1/16) remain in T_2.

Finally we want to note that this model coincides with the $p = 2$ ($c = 3/2$) N=2 superconformal one and correspondingly all the 4-point functions in the NS and R sectors are simply powerlike.

3.6.2 Series of models with c=3

We want to consider now the theories which have a central charge c=3. As it was explained in Sec.(3.3) this is the case in the theory of two free N=2 chiral superfields $S^{\pm}$ with $\beta = 0$, i.e. the N=2 superstress-tensor has the form

$$\mathcal{W} = J + \frac{1}{2}\theta^- G^+ - \frac{1}{2}\theta^+ G^- + \theta^-\theta^+ T = \frac{1}{4} : D^+\phi D^-\phi : \tag{3.245}$$

We have combined these chiral superfields in a full N=2 supermultiplet

$$\phi = S^+ + S^- = \varphi_1 + \sqrt{2}\theta^-\overline{\psi} + \sqrt{2}\theta^+\psi + i\theta^-\theta^+\partial\varphi_2 \tag{3.246}$$

whose derivatives are exactly equal to those of $S^{\pm}$

$$D^+\phi = D^+S^+, \qquad\qquad D^-\phi = D^-S^-$$

Using eq. (3.48), for the 2-point function of $\mathcal{W}$ we have:

$$< \mathcal{W}(1)\mathcal{W}(2) >= \frac{1}{4\hat{z}_{12}^2}$$

and therefore the central charge of the theory is c=3. In addition to $\mathcal{W}(z,\theta^+,\theta^-)$ in each such theory we have two more currents of dimension $\frac{1}{2}$ and U(1) charges $\pm 1/2$:

$$\begin{aligned} D^+\phi &= \sqrt{2}\overline{\psi} + 2\theta^+\partial\varphi - \sqrt{2}\theta^-\theta^+\partial\overline{\psi} \\ D^-\phi &= \sqrt{2}\psi + 2\theta^-\partial\overline{\varphi} + \sqrt{2}\theta^-\theta^+\partial\psi \end{aligned} \tag{3.247}$$

Similarly to the previous section we define also some additional currents

$$Q_g^{\pm}(z,\theta^+,\theta^-) =: e^{\pm i(gS^- + \bar{g}S^+)} : \tag{3.248}$$

which span a symmetry algebra larger than the initial N=2 SUSY. It is defined by the following OPE's

$$\begin{aligned} Q_g^+(1)Q_g^-(2) &= \hat{z}_{12}^{-2g\bar{g}}(1+\cdots) \\ Q_g^{\pm}(1)Q_g^{\pm}(2) &= \hat{z}_{12}^{2g\bar{g}} : e^{\pm 2i(gS^- + \bar{g}S^+)} : \end{aligned} \tag{3.249}$$

Therefore the condition of locality leads to the restriction:

$$g\bar{g} = \frac{n}{2}, \qquad n = 1,2,\cdots \tag{3.250}$$

and again we obtain an infinite series of models labeled by the integer n.

The NS superfields are defined as usually as pure exponents of the kind (3.248):

$$N_\alpha^{\pm}(z,\theta^+,\theta^-) =: e^{\pm i(\alpha S^- + \bar{\alpha}S^+)} : \tag{3.251}$$

while in the R vertices we have to put the lowest fields $\sigma^{\pm}$ (dimension 1/8, U(1) charge $\pm 1/4$) of the ψ, ψ algebra:

$$R_\alpha^{\pm}(z) = \sigma^{\pm} : e^{\pm i(\alpha\bar{\varphi} + \bar{\alpha}\varphi)} : \tag{3.252}$$

Similarly to the Coulomb gas representation of N=2 m.m., the T fields give a right analytic behaviour for the generators if they have the form

$$T(z) = \sigma_0^{\psi}\sigma_0^{\varphi} : e^{\pm i\alpha\varphi_1} : \tag{3.253}$$

We already know that the vertices (3.251) represent indeed a NS field with dimension $\Delta = \alpha\alpha$ and U(1) charge 0 (since $\beta = 0$ here). Let us consider also the product with the additional currents:

$$Q_g^+(1)N_\alpha^-(2) = \hat{z}_{12}^{-(g\bar{\alpha}+\bar{g}\alpha)} : e^{i(g-\alpha)S^- + (\bar{g}-\bar{\alpha})S^+} : + \cdots \tag{3.254}$$

As before from here we have

$$g\bar{\alpha} + \bar{g}\alpha = \frac{k}{2} \tag{3.255}$$

Since the U(1) charge is always zero in our theories we have no other condition to define separately α and $\bar{\alpha}$. We restrict ourself to consider the simplest choice, namely we impose the condition $\alpha = \bar{\alpha}$ (and of course $g = \bar{g}$ as well). Then the eqs. (3.250),(3.255) become

$$g^2 = \frac{n}{2}, \qquad n = 1,2,\cdots \tag{3.256}$$

$$\alpha = \frac{k}{2\sqrt{2n}}, \qquad k = 0,\cdots,n \tag{3.257}$$

Therefore the dimension and charge of the NS field N_k are

$$\Delta_k^N = \frac{k^2}{8n}, \qquad q = 0 \tag{3.258}$$

and correspondingly for $R_k^{\pm}$

$$\begin{aligned} \Delta_k^R &= \frac{k^2}{8n} + \frac{1}{8} \\ q &= \pm\frac{1}{4} \end{aligned} \tag{3.259}$$

Similarly to the m.m. the R fields are combined in "multiplets" with equal dimensions and charges $\pm 1/4$. For example

$$\begin{aligned} G^+(1)R^+(2) &= \mathcal{O}(\frac{1}{\sqrt{z_{12}}}) \\ G^+(1)R^-(2) &= \frac{\alpha}{(z_{12})^{\frac{3}{2}}}\sigma^+ e^{-\alpha\varphi_1} + \cdots \end{aligned}$$

The T fields produce antiperiodic boundary conditions for the U(1) current so there are no charges here, and the dimensions are given by

$$\Delta_k^T = \frac{k^2}{8n} + \frac{1}{8} \tag{3.260}$$

The structure of the T multiplet is the same as in the m.m.

The n-point functions of the NS fields can be obtained directly using the Wick theorem and eq. (3.48)

$$< \prod_{i=1}^{n} N_{\alpha_i}(z_i, \theta^+, \theta^-) >= \prod_{i<j}^{n} (\hat{z}_{12})^{2\alpha_i\alpha_j} \tag{3.261}$$

Similarly, for the R and T fields we obtain:

$$< R^+_{\alpha_1} R^-_{\alpha_2} R^+_{\alpha_3} R^-_{\alpha_4} >= \prod_{i<j}^{4} z_{ij}^{2\alpha_i\alpha_j} < \sigma^+\sigma^-\sigma^+\sigma^- > \tag{3.262}$$

(different from zero if $\sum_i \alpha_i = 0$) and

$$< \prod_{i=1}^{4} T_{\alpha_i}(z_i) >= \prod_{i<j}^{4} z_{ij}^{2\alpha_i\alpha_j} \prod_{s=1}^{4} < \sigma_0^{\psi}(z_s) >< \sigma_0^{\varphi}(z_s) > \tag{3.263}$$

Let us consider the case n=1. There exist three currents of dimension 1/2 here in addition to $\mathcal{W}$:

$$\begin{aligned} \overline{\psi} &= \frac{i}{\sqrt{2}} D^+\phi = \overline{\psi} + 2\theta^+ J - \theta^-\theta^+\partial\overline{\psi} \\ \psi &= \frac{i}{\sqrt{2}} D^-\phi = \psi + 2\theta^-\overline{J} + \theta^-\theta^+\partial\psi \\ \psi^{\pm} &= \frac{1}{\sqrt{2}}; (e^{\frac{i}{\sqrt{2}}\phi} \pm e^{-\frac{i}{\sqrt{2}}\phi}) := \psi^{\pm} + \theta^- J^{\mp} + \theta^+\overline{J}^{\mp} + \theta^-\theta^+\tilde{G}^{\pm} \end{aligned} \tag{3.264}$$

Their algebra is as follows:

$$
\begin{aligned}
\overline{\psi}(1)\psi(2) &= \frac{1}{\hat{z}_{12}}(1+\frac{\theta_{12}^{-}\theta_{12}^{+}}{\hat{z}_{12}}) \\
\overline{\psi}(1)\psi^{\pm}(2) &= \frac{\theta_{12}^{+}}{\hat{z}_{12}}\psi^{\mp}(2)+\cdots \\
\psi(1)\psi^{\pm}(2) &= \frac{\theta_{12}^{-}}{\hat{z}_{12}}\psi^{\mp}(2) \\
\psi^{\pm}(1)\psi^{\pm}(2) &= \pm[\frac{1}{\hat{z}_{12}}+2\frac{\theta_{12}^{-}\theta_{12}^{+}}{\hat{z}_{12}}\mathcal{W}(2)] \\
\psi^{\pm}(1)\psi^{\mp}(2) &= \mp[\frac{\theta_{12}^{-}}{\hat{z}_{12}}\overline{\psi}(2)+\frac{\theta_{12}^{-}}{\hat{z}_{12}}\psi(2)]+\cdots
\end{aligned}
\tag{3.265}
$$

This algebra contains four supercurrent $G^{\pm}, \tilde{G}^{\pm}$ and has a complicated current algebra part, spaned by the J-currents of dimension 1: $\tilde{J}, J, \overline{J}, J^{\pm}, \overline{J}^{\pm}$. We present here also the commutation relations of their Laurent coefficients, obtained directly from (3.265):

$$
\begin{aligned}
&\{\overline{\psi}_{\alpha}, \psi_{\beta}\} = \delta_{\alpha+\beta,0} \\
&\{\psi_{\alpha}^{\pm}, \psi_{\beta}^{\pm}\} = \pm\delta_{\alpha+\beta,0} \\
&[\tilde{J}_n, \tilde{J}_m] = \frac{1}{4}n\delta_{n+m,0} \\
&\{G_{\alpha}^{+}, G_{\beta}^{-}\} = 2L_{\alpha+\beta} + 2(\alpha-\beta)\tilde{J}_{\alpha+\beta} + (\alpha^2-\frac{1}{4})\delta_{\alpha+\beta,0} \\
&\{G_{\alpha}^{\pm}, \tilde{G}_{\beta}^{\pm}\} = (\alpha-\beta)\overline{J}_{\alpha+\beta}^{\mp} \\
&\{\tilde{G}_{\alpha}^{\pm}, \tilde{G}_{\beta}^{\pm}\} = \pm[2L_{\alpha+\beta} + (\alpha^2-\frac{1}{4})\delta_{\alpha+\beta,0}] \\
&\{\tilde{G}_{\alpha}^{\pm}, \tilde{G}_{\beta}^{\mp}\} = \mp(\alpha-\beta)(J_{\alpha+\beta}+\overline{J}_{\alpha+\beta}) \\
&[\tilde{J}_n, \overline{J}_m^{\pm}] = \pm\frac{1}{2}\overline{J}_{n+m}^{\pm} \\
&[\overline{J}_n, \overline{J}_m^{\pm}] = \frac{1}{2}\overline{J}_{n+m}^{\mp} \\
&[J_n, \overline{J}_m] = \frac{1}{2}n\delta_{n+m,0} \\
&[\overline{J}_n, J_m^{\pm}] = \frac{1}{2}J_{n+m}^{\mp} \\
&[J_n^{\pm}, \overline{J}_m^{\mp}] = \pm(J_{n+m}+\overline{J}_{n+m}) \\
&[J_n^{\pm}, \overline{J}_m^{\pm}] = \mp(2\tilde{J}_{n+m}+n\delta_{n+m,0}) \\
&[J_n, \overline{J}_m] = \frac{1}{2}n\delta_{n+m,0} \\
&[\overline{J}_n, J_m^{\pm}] = \frac{1}{2}J_{m+n}^{\mp}
\end{aligned}
\tag{3.266}
$$

This big algebra has many interesting subalgebras. For example, as can be seen from (3.266), the currents T, $G^{\pm}, \tilde{G}^{\pm}, \tilde{J}, J^{-}, \overline{J}^{-}, \psi^{+}$ span a N=3 superconformal algebra with c=3.

The spectrum of the fields in different sectors is as follows:

$$N_1^\pm = \left(\frac{1}{8}, 0\right)^\pm, \qquad R_0^\pm = \sigma^\pm \qquad R_1^\pm = \left(\frac{1}{4}, \pm\frac{1}{4}\right)$$
$$T_0 \quad = \sigma_0^\varphi \sigma_0^\psi = \left(\frac{1}{8}\right) \qquad T_1^\pm = (\frac{1}{4})^\pm$$

In the theory with n=2 the additional currents

$$Q^\pm(z,\theta^+,\theta^-) = \frac{1}{2} : (e^{i\phi} \pm e^{-i\phi}) := J^\pm + \theta^- \tilde{G}_1^\mp + \theta^+ G_2^\mp + \theta^-\theta^+ T^\pm \tag{3.267}$$

have dimension 1 and the algebra is determined by the following superfield OPE's

$$\begin{aligned}
\overline{\psi}(1)Q^\pm(2) &= \frac{\theta_{12}^\pm}{\hat{z}_{12}} Q^\mp \\
Q^\pm(1)Q^\pm(2) &= \{\frac{1}{2\hat{z}_{12}^2} + 2\frac{\theta_{12}^-\theta_{12}^+}{\hat{z}_{12}^2} + \frac{1}{\hat{z}_{12}}(\theta_{12}^- D^+ - \theta_{12}^+ D^-)\mathcal{W}(2) + \\
&\quad 2\frac{\theta_{12}^-\theta_{12}^+}{\hat{z}_{12}}\partial\mathcal{W}(2)\} + \cdots \\
Q^\pm(1)Q^\mp(2) &= \mp\{(\frac{\theta_{12}^-}{2\hat{z}_{12}^2} + \frac{\theta_{12}^-\theta_{12}^+}{2\hat{z}_{12}^2} D^- + \frac{1}{2\hat{z}_{12}} D^-)\overline{\psi}(2) + \\
&\quad [\frac{\theta_{12}^+}{\hat{z}_{12}^2} + \frac{\theta_{12}^-\theta_{12}^+}{2\hat{z}_{12}^2} D^+ + \frac{1}{2\hat{z}_{12}} D^+]\psi(2)\} + \cdots
\end{aligned} \tag{3.268}$$

It is clear from (3.268) that $\mathcal{W}$ and $Q^+(Q^-)$ span a semidirect product of two N=2 superconformal algebras and Ψ gives rise to some internal symmetries of the theory.

The discrete symmetry algebra, corresponding to (3.267) is

$$\begin{aligned}
&\{G_\alpha^+, G_{2\beta}^\pm\} = 2L_{\alpha+\beta}^\pm + (\alpha-\beta)J_{\alpha+\beta}^\mp \\
&\{G_\alpha^-, G_{1\beta}^\pm\} = L_{\alpha+\beta}^\pm + (\alpha-\beta)\overline{J}_{\alpha+\beta}^\mp \\
&\{G_\alpha^\pm, G_\beta^\mp\} = \pm(\alpha+\frac{1}{2})(J_{\alpha+\beta} + \overline{J}_{\alpha+\beta}) \\
&\{G_\alpha^{1\pm}, G_{2\beta}^\mp\} = \mp[(\alpha-\beta)\tilde{J}_{\alpha+\beta} + \frac{1}{2}(\alpha^2 - \frac{1}{4})\delta_{\alpha+\beta,0}] \\
&[J_n^\pm, \overline{J}_m^\pm] = -\frac{1}{2}J_{n+m}^\mp \\
&[L_n^\pm, L_n^\pm] = \pm[(n-m)L_{n+m} + \frac{1}{2}n(n^2-1)\delta_{n+m,0}] \\
&[L_n^\pm, L_n^\mp] = \pm(n+1)(n+m+1)(J_{n+m} + \overline{J}_{n+m}) \\
&[J_n^\pm, J_m^\mp] = \mp(J_{n+m} + \overline{J}_{n+m}) \\
&[J_n^\pm, J_m^\pm] = \pm\frac{1}{2}n\delta_{n+m,0}
\end{aligned} \tag{3.269}$$

There are two fields in the NS sector:

$$N_1^\pm = \left(\frac{1}{16}, 0\right)^\pm \qquad N_2^\pm = \left(\frac{1}{4}, 0\right)^\pm$$

three ones in the R sector

$$R_0^{\pm} = \sigma^{\pm}, \qquad R^{\pm} = \left(\frac{3}{16}, \pm\frac{1}{4}\right), \qquad R_2^{\pm} = \left(\frac{3}{8}, \pm\frac{1}{4}\right)$$

and three twisted fields:

$$T_0 = \sigma_0^{\psi}\sigma_0^{\varphi}, \qquad T_1^{\pm} = \left(\frac{3}{16}\right)^{\pm} \qquad T_2^{\pm} = \left(\frac{3}{8}\right)^{\pm}$$

Their 4-point functions follow directly from (3.261)(3.263).

We will conclude our rather uncomplete discussion of the series of local N=2 superconformal models with central charge c=3 noting that the strategy of the recent paper by Dixon, Ginsparg and Harvey [102] for the series of $c = 3/2$ models can be easily extended to the models with c=3 and it seems to be certainly more consistent and powerful in the problem of classification of all the models with c=3.

Chapter 4

Compactification of string theory by superconformal minimal models

In this last chapter we are going to apply our previous study on the N=2 superconformal minimal models to the problem of compactification in string theory.

Superstrings provide the basis for an elegant unification of the fundamental forces including gravity. Moreover only in ten dimensions superstring propagation in flat Minkowski space is free from anomaly. In order to construct viable models in four flat dimensions it is necessary to consider string propagation on a manifold $M \times K$ where M is four-dimensional Minkowski space and K is a six-dimensional internal manifold whose size is of order M_{Planck}^{-1}. The internal manifold K would give rise to the observable forces and the particular choice for it has a deep influence on the physical predictions of the theory, as the values of masses and coupling constants of the effective low-energy Lagrangian. Despite to the uniqueness of the theory in ten dimensions (we restrict our attention to the $E_8 \times E_8$ heterotic string [38]), the moment one discusses compactification down to four dimensions one immediately faces the problem of huge degeneration of vacua. Many possibilities have been considered including Calabi-Yau [3], orbifold [37] and the so-called '4-dimensional string' construction [41]. For the selection of the true vacuum non perturbative effects are usually invoked as a *deus ex machina* that will somehow achieve this, but this remains wishful thinking at present. What usually one does is to develop some tools for connecting spacetime properties of these vacua with properties of the underlying world-sheet conformal field theory. For instance, in the case of the heterotic string if one impose space-time supersymmetry on the classical vacuum one gets that the $(0,1)$ superconformal invariance is promoted to a $(0,2)$ superconformal invariance [44,42]. Finally if one insists on having not just $(0,1)$ but $(2,2)$ invariance, one can characterize the vacuum much more fully.

On this direction recently Gepner has been developed a particularly interesting construction of the internal manifold K (with central charge $c = 9$) by tensoring products of minimal N=2 superconformal field theories [55,56,57,58]. Remarkably the massless spectrum and discrete symmetry of some of the first such models studied closed match those of known complete intersection Calabi-Yau [49,50] leading Gepner to the conjecture that all models constructed in such way correspond to string propagation on Calabi-Yau manifolds or some their limit.

This new idea has opened new perspectives in the computation of the scattering amplitudes and the Yukawa couplings of the low-energy effective action since these quantities are deeply related to the N=2 minimal models involved which are completely solvable.

We do not pretend to give an exaustive treatment on the compactification problem which covers now a large literature : after a brief discussion on Calabi-Yau manifold (the basis mathematical notions are collected in App.C) we apply the construction developed by Gepner to compute the Yukawa couplings of the fermionic families belonging to the 27 of E_6. To do this we enlight the role of the N=2 supersymmetry and its realizations by minimal models. Analysing the characters of the superconformal blocks we recover the Z_{p+2} symmetry of the p-th N=2 minimal model previously discussed.

4.1 String theory on Calabi-Yau manifold

The spectrum of the $E_8 \times E_8$ heterotic string at the massless level contains the N=1, D=10 gravitational supermultiplet

g_{mn}	$\rightarrow$	graviton (symmetric traceless tensor)
B_{mn}	$\rightarrow$	antisymmetric tensor
ϕ	$\rightarrow$	dilaton (scalar)
ψ_m	$\rightarrow$	gravitinos (Rarita-Schwinger fields)
χ	$\rightarrow$	dilatino (Majorana-Weyl spinor)

and the super Yang-Mills multiplets

A^a_m	$\rightarrow$	gauge vector bosons
λ^a	$\rightarrow$	gauge fermions
	$a = 1, 2, \cdots 496$	

These are the only states which can be seen at low energy ($E \ll M_{Pl}$), and enter in the effective Lagrangian [40]. A compactification is given by a solution of the equation of motion for the metric g_{mn} describing a space-time that a scale larger then M_{Pl}^{-1} looks like a flat four dimensional Minkowski space. The simplest solution is given by $M \times K$ where K is a compact six-dimensional manifold. Additional constraints on K come from other requirements, the main ones being:

1. space-time supersymmetry (which will eventually be broken)
2. a gauge symmetry at least equal to $SU(3)_c \otimes SU(2)_L \otimes U(1)_Y$
3. presence of Higgs fields, i.e. scalar fields non singlet under the gauge symmetry

In the pioneering work by Candelas et al [3] these authours enforce those conditions by putting to zero the background values of the supersymmetry variations of the various fields. The conditions that follow from imposing N=1 supersymmetry are the following

$$\delta\psi_M = D_M\eta = 0 \tag{4.1}$$

$$\delta\chi^a = \Gamma^{ij}F^a_{ij}\eta = 0 \tag{4.2}$$

$$\frac{1}{30}F^a_{[mn}F^a_{pq]} = R^{rs}_{[mn}R_{pq]rs} \tag{4.3}$$

where η is the supersymmetry transformation parameter (a Majorana-Weyl spinor).

The first equation says that on K there exists a covariantly constant spinor. Moreover one can show that this implies that the manifold K is complex: for a general six-dimensional complex manifold the holonomy group H is $S0(6) \sim SU(4)$. But eq.(4.1) expresses the fact that there is one direction in the tangent space which remains untouched by the action of H. Since η is a Majorana-Weyl spinor with (say) positive helicity, it is therefore in the 4_+ of $SU(4)$. The holonomy group H is the subgroup of $SU(4)$ which leaves invariant a 4_+, hence $H = SU(3)$. This is an equivalent definition of a Calabi-Yau manifold (CY), i.e. a complex Kahler manifold with a metric of $SU(3)$ holonomy (see also App.C). An example of CY manifold, denoted by $Y_{4,5}$ is

$$Q = \{Z = (z_1, \cdots, z_5) \in CP^5 : \sum_{i=1}^{5} z_i^5 = 0\} \tag{4.4}$$

Other examples of CY spaces and the investigation of a class of them which can be represented as a complete intersection of polynomials in a product of projective spaces can be found in the recent references [49,50,51].

Because of this non-trivial $SU(3)$ spin connection eq.(4.3) implies that the gauge field has to assume a non-trivial background value. It is natural to solve eq.(4.3) simply by identifying some of the gauge fields A^a_M with the spin connection [3]. Choose one E_8 group (the other will be denoted E_8'): decompose it into $E_6 \times SU(3)$ and set

$$A_M = A^a_M T^a = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma^i_{Mj} \end{pmatrix} \tag{4.5}$$

where the first row is the E_6 field vacuum expectation value and the second one is the $SU(3)$ corresponding quantities: Γ^i_{Mj} is the $SU(3)$ spin connection. For $M = 5, \cdots 10$ the $(A_M)^i_j$ are 4-dimensional scalar fields in the adjoint of $SU(3)$ with a nonzero vacuum expectation value, since if K in not flat Γ^i_{Mj} cannot be set to be zero globally. They therefore break E_8 gauge symmetry to E_6, given an interesting grand unification group at energy scales of M_{Pl}. The space-time supersymmetry allows us to discuss massless spectrum analysing only the zero-modes of the Dirac and Rarita-Schwinger operators on K. Take the example of 10 dimensional gauginos which transform under $E_8 \supset E_6 \times SU(3)$ and the transverse Lorentz group $SO(8) \supset SO(2) \times SU(3)_H$ as

$$248 = (78,1) + (1,8) + (27,3) + (\overline{27},\overline{3}) \tag{4.6}$$
$$8_s = (h = \pm\frac{1}{2}, 1) + (h = \frac{1}{2}, 3) + (h = -\frac{1}{2}, \overline{3})$$

A trivial way of finding a zero-mode of the Dirac operator on K is to take a K -scalar which also commutes with the action of the SU(3) gauge group. Only the $(78,1)$ component survives which describe the gauginos of E_6. Additional zero-modes can be found by looking at the four dimensional spinors which transfor under both the spin and gauge connections. In this case one can expect the spin rotation to be compensate for by a gauge transformation induced by the gauge background. This actually happens for the states transforming as $(27,3)$ and $(\overline{27},\overline{3})$ under $E_6 \times SU(3)_{YM}$. But to find the actual number of these zero-modes one must use the relevant index theorem: the number of chiral families, i.e. the difference between the number of massless 27's and $\overline{27}$'s, is expressed in terms of the Euler characteristic $\chi(K)$ of the manifold [3]

$$n_{gen} = \mid n^{27} - n^{\overline{27}} \mid = \frac{1}{2} \mid \chi(k) \mid \tag{4.7}$$

a rather striking result, since it relates a fundamental low energy parameter to a topological invariant of the compact manifold.

Moreover in complex geometry we can predict separately the number of 'families' and 'antifamilies'[3]: they are given by the Hodge number $h^{2,1}$ and $h^{1,1}$ respectively (see App.C). If $\chi(K) \neq 0$ the four dimensional theory is chiral. Since $\chi(K) \neq 0$ is a rather common property of CY manifolds, one of the most important feature of the low energy world (the chirality of the fermionic spectrum) is easily recovered by the superstring compactification.

Even though the success to obtain a reasonable gauge group, E_6, with fermions in the proper representation, the 27, one still is disturbed by some glaring faults. Simple constructions of CY manifolds give far too many fermion generations. For instance the quintic hypersurface $Y_{4,5}$ in CP^4 gives 100 generations. Also while E_6 ia an interesting group for grand unification, phenomenology imposes that E_6 must be broken at low energy.

There is a way to break E_6 and at the same time to lower the Euler characteristic of the manifold. The gauge symmetry is not broken by the vacuum expectation values of a Higgs field but by the Wilson loops, via a non trivial first homotopy group $\pi_1(K)$ [52,54]. In this case non contractible loops exist and it is possible to introduce non integrable phase factors, i.e. non trivial gauge configurations with field strength equal to zero. This resembles the Bohm-Ahoronov effect in Quantum-Electrodynamics. The Wilson loops

$$U_\gamma = \mathcal{P}exp\{i \oint_\gamma A_n dx^n\} \tag{4.8}$$

realize an isomorphism between $\pi_1(K)$ and a subgroup $\mathcal{G}$ of E_6. Even though $F_{mn} = 0$, in order to do not spoil the embedding of the $SU(3)$ spin connection in one of the two E_8's, U_γ does not need to be trivial if γ is anon contractible loop. The presence of these non integrable phase factors breaks E_6 down to the group commuting with $\mathcal{G}$. We will see some examples later, after discussion on the symmetry group F acting freely on K. F will automatically be a discrete symmetry group since a CY does not have continous symmetries. If the group act freely on K, i.e. without fixed points, we can introduce the quotient space

$$\tilde{K} = K/F \tag{4.9}$$

and correspondingly

$$\chi(\tilde{K}) = \chi(K)/ordF \tag{4.10}$$

Example 1. Consider the quintic hypersurface $Y_{4,5}$ in CP^4, eq.(4.4). Take

$$\begin{aligned} A \ &: \ z_i \longrightarrow z_{i+1} \\ & z_{i+5} \equiv z_i \end{aligned} \tag{4.11}$$

Since $A^5 = 1$, the group $A = \{A^n; 0 \leq n \leq 4\}$ is isomorphic to Z_5. Similarly for $B = \{B^n; 0 \leq n \leq 4\}$ where $B : z_i \rightarrow \alpha^i z_i$ ($\alpha = e^{2\pi i/5}$). The group $G_1 = A \times B$ is isomorphic to $Z_5 \times Z_5$ (ord $G_1 = 25$) and has no fixed point in $Y_{4,5}$. Hence $\tilde{K} = Y_{4,5}/G_1$ has a Euler characteristic

$$\chi(\tilde{K}) = -\frac{200}{25} = -8$$

Then this is an example of a CY manifold with 4 generations. We can determine the number of families counting the deformation of the complex structure of $Y_{4,5}$, coded in $h^{2,1}$ (see also App.C). These are ($Z_5 \times Z_5$) invariant perturbations in the quintic polynomial (4.4) that cannot be absorbed in a change of variables. A part from P itself there are five independent ($Z_5 \times Z_5$) invariant quintic polynomials:

(a) $z_1^4 z_2 z_5 + z_2^3 z_3 z_1 + z_3^3 z_4 z_2 + z_4^3 z_5 z_3 + z_5^3 z_1 z_4$
(b) $z_1^3 z_3 z_4 +$ cycl.perm.
(c) $z_1^2 z_2 z_3^2 +$ cycl.perm.
(d) $z_1^2 z_2^2 z_4 +$ cycl.perm.
(e) $z_1 z_2 z_3 z_4 z_5$

Since there is only one (1,1) form, the Kahler form, $h^{1,1} = 1$ then using eq.(4.7) we have

$$n_{gen} = \frac{1}{2} \mid \chi \mid = \mid h^{1,1} - h^{2,1} \mid = 4$$

Example 2. Take F acting on $CP^3 \times CP^3$ as

$$f : (x_1 \cdots, x_4)(y_1, \cdots, y_4) \rightarrow (x_1, \beta x_2, \beta x_3, \beta x_4)(y_1, \beta y_2, \beta^2 y_3, \beta^2 y_4)$$

($\beta = e^{2\pi i/3}$). Since $f^3 = 1$, $F \equiv Z_3$. This group has no fixed points on the manifold K described by these algebraic equations

$$\begin{aligned} \textstyle\sum_{i=1}^4 x_i^3 = & \sum_{i=1}^4 y_i^3 = 0 \\ \textstyle\sum_{i=1}^4 x_i y_i = & \ 0 \end{aligned} \tag{4.12}$$

($\chi(K) = -18$). Then $\tilde{K} = K/F$ has Euler characteristic equal to

$$\chi(\tilde{K}) = -\frac{18}{3} = -6$$

This example is of special interest because it is the only explicit candidate for the minimal number of 3 generations.

Now we come back to the gauge groups which arises after breaking of E_6 by topological Wilson loops. To derive them it is easier to consider the maximal subgroup of $E_6 : SU(3)_c \otimes SU(3)_L \otimes SU(3)_R$ and to write U as

$$U = U_L \otimes U_L \otimes U_R \tag{4.13}$$

Since the effective gauge group S in four dimensions commutes with $\mathcal{G}$ and S contains at least $SU(3)_c \otimes SU(2)_R \otimes U(1)_Y$ we have constraints on the possible form of U.

U_c in (4.13) is a SU(3) matrix and in order to commute with the $SU(3)_c$ group it must be an element of the center

$$U_c = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta \end{pmatrix} \qquad \eta^3 = 1 \tag{4.14}$$

For simplicity we will take $\eta = 1$.

The embedding of $SU(2)_L$ in $SU(3)_L$ is realized by

$$SU(2)_L = 1 \otimes \begin{pmatrix} SU(2)_L & 0 \\ 0 & 1 \end{pmatrix} \otimes 1 \tag{4.15}$$

In order for U to commute with $SU(2)_L$ U_L must be of the form

$$U_L = \begin{pmatrix} \beta & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & \beta^{-2} \end{pmatrix} \tag{4.16}$$

Finally, to define the hypercharge Y we use the relation $Q = T_3 + \frac{Y}{2}$ for the quarks contained in a 27, where Q is the charge operator and T_3 is the diagonal generator of $SU(2)_L$

$$T_3 = 0 \oplus \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix} \oplus 0 \tag{4.17}$$

Then $U(1)_Y$ is generated by

$$Y = 0 \otimes \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}_L \oplus \begin{pmatrix} \frac{4}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix}_R \tag{4.18}$$

Having found the embedding of $U(1)_Y$ in $SU(3)_c \times SU(3)_L \times SU(3)_R$ we can derive the last constraint on U_R, i.e. to commute with $U(1)_Y$ it must be

$$U_R = \begin{pmatrix} \gamma & 0 \\ 0 & A \end{pmatrix} \tag{4.19}$$

where A is a 2×2 matrix and $\gamma det\ A = 1$.

The matrix U has to satisfy of course the algebra of $\mathcal{G}$, in particular if $\mathcal{G} = Z_n$ then $U^n = 1$. Moreover if $\mathcal{G}$ is abelian A can be diagonalized and the final form of U_R is in this case

$$U_R = \begin{pmatrix} \gamma & 0 & 0 \\ 0 & \delta & 0 \\ 0 & 0 & \eta \end{pmatrix} \qquad \gamma\delta\eta = 1 \tag{4.20}$$

The general discussion of possible symmetry breaking is out of our scope and the reader interested can find it in the original literature [54,40]. We only note that the gauge symmetry is broken by the Hosotani mechanism at most to $SU(3)_c \times SU(2)_L \times U(1)_{Y_L} \times U(1)_{Y_R}$, since any matrix that commutes with Y will commute also with the two matrices

$$\begin{aligned} [Y_L] &= 0 \oplus \begin{pmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \oplus 0 \\ [Y_R] &= 0 \oplus 0 \oplus \begin{pmatrix} \frac{4}{3} & 0 & 0 \\ 0 & -\frac{2}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{pmatrix} \end{aligned}$$

i.e. it appears an extra U(1). The optimistic answer about the origin of this extra U(1) is that this result is not so bad as it may look at first sight, because an ordinary Higgs mechanism at lower energy can always occur so it remains to see wheter this extra U(1) will be unbroken down to the low energy scales.

With this last observation we finish our general introduction to the compactification problem and in the following sections we concer with the construction of the internal space by N=2 superconformal theories.

4.2 Spacetime and world-sheet supersymmetry

The importance of having N=1 spacetime supersymmetry in a classical superstring vacuum state is based on several reasons, for instance to solve the hierarchy problem or the vanishing of cosmological constant, as long as supersymmetry remain unbroken. Can we formulate simple criterium on the two-dimensional field theory to check four dimensional N=1 supersymmetry? A necessary and sufficient condition is the existence of an N=2 supersymmetry on a world-sheet plus a charge quantization on the physical vertex operators with respect the U(1) current contained in this algebra [42].

Preliminary result was obtained by Hull and Witten [44]: they showed that in any spavetime supersymmetric classical vacuum of the heterotic string, described by a non linear sigma model, the local N=1 superconformal invariance extends to a global N=2 superconformal one. The reason is very simple and depends on the existence of a complex structure onthe internal manifold K (see App.C): denoted by $\Phi^i = \varphi^i + \theta\lambda^i$ the superfield living on the internal manifold K with action

$$A = -i \int d^2x d^2\theta [g_{ij}(\Phi) + b_{ij}(\Phi)] D\Phi^i \frac{\partial}{\partial x^-} \Phi^j \tag{4.21}$$

where g_{ij} is the symmetric metric tensor whereas b_{ij} is antisymmetric and

$$D = \frac{\partial}{\partial\theta} + i\theta\frac{\partial}{\partial x^+}$$

with an obvious meaning for the light cone variables $x^{\pm}$.

The action (4.21) is invariant under the supersymmetric transformation

$$\delta\varphi^i = \epsilon\lambda^i \qquad\qquad \delta\lambda^i = -i\epsilon\partial_+\varphi^i \tag{4.22}$$

But if A is invariant under this chiral symmetry

$$\delta\lambda^i = J^i_j(\Phi)\lambda^j \qquad\qquad \delta\varphi^i = 0 \tag{4.23}$$

where $J^i_j(\Phi)$ is a tensor field on K, then A is also invariant under a second supersymmetry

$$\delta\varphi^i = \epsilon J^i_j(\Phi)\lambda^j \qquad\qquad \delta(J^i_j\lambda^j) = -i\epsilon\partial_+\varphi^i \tag{4.24}$$

The conditions for this new supersymmetry to anticommute with the previous one are just the equations of existence of complex structure (see App.C) which we rewrite here

$$\begin{aligned} J^i_j J^j_k &= -\delta^i_k \\ N^k_{ij} &= J^l_i(\partial_l J^k_j - \partial_j J^k_l) - J^l_j(\partial_l J^k_i - \partial_i J^k_l) = 0 \end{aligned} \tag{4.25}$$

This result can be extended to more general SUSY vacua of the heterotic string [42]. Spacetime SUSY implies the existence of the two dimensional currents V^α, V^α, whose form in the $(-\frac{1}{2})$ picture is [15]

$$\begin{aligned} V^\alpha_{-\frac{1}{2}} &= e^{-\phi/2} S^\alpha \Sigma(z) \\ V^\alpha_{-\frac{1}{2}} &= e^{-\phi/2} S^\alpha \Sigma^\dagger(z) \end{aligned} \tag{4.26}$$

where $e^{-\phi/2}$ is a spin field for the (β, γ) superconformal ghosts, S^α and S^α are spin fields for the free world-sheet fermions ψ^μ with indeces in the 4-dimensional Minkowski space and $\Sigma, \Sigma^\dagger$ are Ramond fields of the internal N=1 superconformal field theory with dimension 3/8. The supersymmetry charges are

$$\begin{aligned} Q^\alpha &= \oint dz V^\alpha_{-\frac{1}{2}}(z) \\ Q^\alpha &= \oint dz V^\alpha_{-\frac{1}{2}}(z) \end{aligned} \tag{4.27}$$

From the closure of the algebra

$$\begin{aligned} \{Q^\alpha, Q^\beta\} &= (\sigma^\mu)^{\alpha,\beta} P_\mu \\ \{Q^\alpha, Q^\beta\} &= 0 \end{aligned} \tag{4.28}$$

and the OPE's of the spin fields

$$\begin{aligned} e^{q_1\phi(z_1)}e^{q_2\phi(z_2)} &= z_{12}^{-q_1q_2} e^{(q_1+q_2)\phi(z_2)} + \cdots \\ S_\alpha(z_1)S_\beta(z_2) &= \sigma^\mu_{\alpha\beta}\psi_\mu(z_2) + \cdots \\ S_\alpha(z_1)S_\beta(z_2) &= z_{12}^{-\frac{1}{2}}\eta_{\alpha\beta}1 + \cdots \end{aligned} \tag{4.29}$$

we recover the corresponding OPE's for Σ and $\Sigma^\dagger$

$$\begin{aligned} \Sigma(z_1)\Sigma^\dagger(z_2) &= z_{12}^{-\frac{3}{4}}1 + \cdots \\ \Sigma(z_1)\Sigma(z_2) &= z_{12}^{\frac{3}{4}}\mathcal{O}(z_2) + \cdots \\ \Sigma^\dagger(z_1)\Sigma^\dagger(z_2) &= z_{12}^{\frac{3}{4}}\mathcal{O}^\dagger(z_2) + \cdots \end{aligned} \tag{4.30}$$

$\mathcal{O}$(z) is an operator with dimension 3/2.

Of course the Ramond fields $\Sigma, \Sigma^\dagger$ are nonlocal with respect the fermionic component G(z) of the N=1 superstress energy tensor

$$\begin{aligned} G(z)\Sigma(w) &\sim (z-w)^{-\frac{1}{2}} \\ G(z)\Sigma^\dagger(w) &\sim (z-w)^{-\frac{1}{2}} \end{aligned} \tag{4.31}$$

The N=1 superconformal algebra which we discussed in chapter 2 can be enlarged to N=2 if we find an U(1) current J(z) which complete the N=2 superstress energy tensor $\mathcal{W} = J - 1/2\theta^+ G^- + 1/2\theta^- G^+ + \theta^+\theta^- T$ where we split the N=1 supercurrent G by

$$G = \frac{1}{\sqrt{2}}(G^+ = G^-) \tag{4.32}$$

and $G^\pm$ has charge $\pm\frac{1}{2}$ under J

$$J(z)G^\pm(w) = \pm\frac{1}{2}\frac{G^\pm(w)}{z-w} \tag{4.33}$$

The current J occurs in the OPE of the Σ fields. The most convenient way to discover it is to compute the following 4-point function

$$\Gamma(z) = <\Sigma(\infty)\Sigma^\dagger(1)\Sigma(z)\Sigma^\dagger(0)> \tag{4.34}$$

Its expression is fixed by the analytic properties (1.30) and by conformal invariance to be

$$\Gamma(z) = <\Sigma(\infty)\Sigma^\dagger(1)\Sigma(z)\Sigma^\dagger(0)> = [z(1-z)]^{-\frac{3}{4}} \tag{4.35}$$

Looking at the expansion $z \to 0$ we get

$$\Gamma(z) \sim z^{-\frac{3}{4}}[1 + \frac{3}{4}z + \cdots] \tag{4.36}$$

The second term is the signal for the presence of a dimension 1 field in the OPE $\Sigma(z)\Sigma^{\dagger}(0)$, which we identify with J(z). Normalizing

$$J(z)J(w) = \frac{c}{12}\frac{1}{(z-w)^2} \tag{4.37}$$

in the case c=9 (central charge of the internal superconformal theory) we have

$$< \Sigma(z_1)\Sigma^{\dagger}(z_2)J(z_3) >= \frac{3}{4}z_{12}^{1/4}(z_{13}z_{23})^{-1} \tag{4.38}$$

So finally we have the following OPE's

$$\begin{aligned}
\Sigma(z_1)\Sigma^{\dagger}(z_2) &= z_{12}^{-\frac{3}{4}}[1 + z_{12}J(z_2) + \cdots] \\
J(z_1)\Sigma(z_2) &= \frac{3}{4}\frac{\Sigma(z_2)}{z_1 - z_2} + \cdots \qquad (4.39) \\
J(z_1)\Sigma^{\dagger}(z_2) &= -\frac{3}{4}\frac{\Sigma^{\dagger}(z_2)}{z_1 - z_2} + \cdots
\end{aligned}$$

(4.40)

The fields T, $G^{\pm}$ and J generate the N=2 superconformal algebra (3.1). This N=2 algebra is to be distinguished from the N=1 local algebra generated by T and $G = 1/\sqrt{2}(G^+ + G^-)$ because there are no ghosts for these extra global conformal symmetries, i.e. they are not gauge symmetries of the physical states.

One of the interesting features of this extended superconformal algebra is that the current J is basically decoupled from the rest of the conformal fields, i.e. it can be expressed in terms of free scalar field H(z) by

$$\begin{aligned}
J(z) &= \frac{i\sqrt{3}}{2}\partial H(z) \\
< H(z)H(w) > &= -ln(z-w)
\end{aligned} \tag{4.41}$$

and the fields in the different charge sectors are all of the form

$$\phi =: exp[\frac{2iq}{\sqrt{3}}H] : \tilde{\phi} \tag{4.42}$$

where $\tilde{\phi}$ commutes with the current J. The bosonization formula for the Ramond fields $\Sigma, \Sigma^{\dagger}$ is

$$\begin{aligned}
\Sigma(z) &=: exp[\frac{i\sqrt{3}}{2}H] : \\
\Sigma^{\dagger}(z) &=: exp[\frac{-i\sqrt{3}}{2}H] :
\end{aligned} \tag{4.43}$$

G(z) does not have a definite charge but can be decompose as

$$G(z) = \sum_q e^{\frac{2iq}{\sqrt{3}}H} \tilde{G}_q \tag{4.44}$$

Using this formula, together with (4.43), in (4.31) we find that only the charges $q = \pm 1/2$ can be present in the above expansion of G(z). Then the decomposition (4.32) is just a split of G(z) into the charge eigenstate. This completes the proof of the existence of N=2 superconformal symmetry. The final forma of the spacetime supersymmetry charges is

$$\begin{aligned} Q^\alpha &= e^{-\phi/2} S^\alpha e^{\frac{i\sqrt{3}}{2}H(z)} \\ Q^\alpha &= e^{-\phi/2} S^\alpha e^{\frac{-i\sqrt{3}}{2}H(z)} \end{aligned} \tag{4.45}$$

In order to have physical vertex operators local with respect to the SUSY charges Q^α, Q^α the U(1) charge of the internal part of the vertex operators must be quantized [42,43]. Since the unitary representations of the N=2 algebra have been determined [31,32] we can read what are the allowed fields in the NS and R sector when c=9. In particular, requiring the norm of $g^{\pm}_{-1/2} \mid \Delta, q >$ to be non- negative and using

$$\{G^+_{1/2}, G^-_{-1/2}\} = 2L_0 + J_0$$

we find

$$\frac{1}{2} \mid q \mid \leq \Delta \tag{4.46}$$

The HWV with $\Delta = \pm 1/2q$ correspond to the null-vector states

$$G^+_{-1/2} \mid \Delta = 1/2, q >= 0 = G^-_{-1/2} \mid \Delta = 1/2, q > \tag{4.47}$$

Massless NS states have $\Delta = 1/2$ so they must have $\mid q \mid \leq 1$. But the only values compatible with locality is $q = \pm 1$.

Similarly the only possibility in the R sector is $q = \pm 1/2$ and the corresponding spin fields have dimension $\Delta = 3/8$.

Having emphasizing the role of the N=2 superalgebra in the compactification problem we can now discuss specific compactification scheme by N=2 superconformal minimal models.

4.3 Exactly solvable string compactification

A new approach to the compactification problem was proposed by Gepner [55] and there is now a keen interest on the subject [56,57,58,59,60,61,62,63]. The basic strategy is to take for the internal space any solvable conformal field theory with N=2 world-sheet supersymmetry: N=2 minimal models are the simplest yet non trivial theories.

	$1\cdot5\cdot46\cdot334$	$1\cdot5\cdot47\cdot292$	$1\cdot5\cdot49\cdot236$	$1\cdot5\cdot52\cdot187$
	$1\cdot5\cdot54\cdot166$	$1\cdot5\cdot58\cdot138$	$1\cdot5\cdot61\cdot124$	$1\cdot5\cdot68\cdot103$
	$1\cdot5\cdot76\cdot89$	$1\cdot5\cdot82\cdot82$	$1\cdot6\cdot23\cdot598$	$1\cdot6\cdot24\cdot310$
	$1\cdot6\cdot25\cdot214$	$1\cdot6\cdot26\cdot166$	$1\cdot6\cdot28\cdot118$	$1\cdot6\cdot30\cdot94$
	$1\cdot6\cdot31\cdot86$	$1\cdot6\cdot34\cdot70$	$1\cdot6\cdot38\cdot58$	$1\cdot6\cdot40\cdot54$
	$1\cdot6\cdot46\cdot46$	$1\cdot7\cdot17\cdot340$	$1\cdot7\cdot18\cdot178$	$1\cdot7\cdot19\cdot124$
$r=4$	$1\cdot7\cdot20\cdot97$	$1\cdot7\cdot22\cdot70$	$1\cdot7\cdot25\cdot52$	$1\cdot7\cdot28\cdot43$
	$1\cdot7\cdot34\cdot34$	$1\cdot8\cdot14\cdot238$	$1\cdot8\cdot16\cdot88$	$1\cdot8\cdot18\cdot58$
	$1\cdot8\cdot22\cdot38$	$1\cdot8\cdot28\cdot28$	$1\cdot9\cdot12\cdot229$	$1\cdot9\cdot13\cdot108$
	$1\cdot9\cdot20\cdot31$	$1\cdot10\cdot11\cdot154$	$1\cdot10\cdot12\cdot82$	$1\cdot10\cdot13\cdot58$
	$1\cdot10\cdot14\cdot46$	$1\cdot10\cdot16\cdot34$	$1\cdot10\cdot18\cdot28$	$1\cdot10\cdot19\cdot26$
	$1\cdot10\cdot22^2$	$1\cdot11^2\cdot76$	$1\cdot12^2\cdot40$	$1\cdot12\cdot13\cdot33$
	$1\cdot12\cdot19^2$	$1\cdot13^2\cdot28$	$1\cdot13\cdot18^2$	$1\cdot14^2\cdot22$
	$1\cdot16^3$	$2\cdot3\cdot19\cdot418$	$2\cdot3\cdot20\cdot218$	$2\cdot3\cdot22\cdot118$
	$2\cdot3\cdot23\cdot98$	$2\cdot3\cdot26\cdot68$	$2\cdot3\cdot28\cdot58$	$2\cdot3\cdot34\cdot43$
	$2\cdot3\cdot38^2$	$2\cdot4\cdot11\cdot154$	$2\cdot4\cdot12\cdot82$	$2\cdot4\cdot13\cdot58$
	$2\cdot4\cdot14\cdot46$	$2\cdot4\cdot16\cdot34$	$2\cdot4\cdot18\cdot28$	$2\cdot4\cdot19\cdot26$
	$2\cdot4\cdot22^2$	$2\cdot5\cdot8\cdot138$	$2\cdot5\cdot10\cdot40$	$2\cdot5\cdot12\cdot26$
	$2\cdot6\cdot7\cdot70$	$2\cdot6\cdot8\cdot38$	$2\cdot6\cdot10\cdot22$	$2\cdot6\cdot14^2$
	$2\cdot7^2\cdot34$	$2\cdot7\cdot10\cdot16$	$2\cdot8^2\cdot18$	$2\cdot8\cdot10\cdot13$
	$2\cdot10^3$	$3^2\cdot9\cdot108$	$3^2\cdot10\cdot58$	$3^2\cdot12\cdot33$
	$3^2\cdot13\cdot28$	$3^2\cdot18^2$	$3\cdot4\cdot6\cdot118$	$3\cdot4\cdot7\cdot43$
	$3\cdot4\cdot8\cdot28$	$3\cdot4\cdot10\cdot18$	$3\cdot4\cdot13^2$	$3\cdot5^2\cdot68$
	$3\cdot6^2\cdot18$	$3\cdot8^3$	$4^2\cdot5\cdot40$	$4^2\cdot6\cdot22$
	$4^2\cdot7\cdot16$	$4^2\cdot8\cdot13$	$4^2\cdot10^2$	$4\cdot5^2\cdot19$
	$4\cdot6^2\cdot10$	$4\cdot7^3$	$5^3\cdot12$	6^4

$r=5$	$1^2\cdot2\cdot11\cdot154$	$1^2\cdot2\cdot12\cdot82$	$1^2\cdot2\cdot13\cdot58$	$1^2\cdot2\cdot14\cdot46$
	$1^2\cdot2\cdot16\cdot34$	$1^2\cdot2\cdot18\cdot28$	$1^2\cdot2\cdot19\cdot26$	$1^2\cdot2\cdot22^2$
	$1^2\cdot3\cdot6\cdot118$	$1^2\cdot3\cdot7\cdot43$	$1^2\cdot3\cdot8\cdot28$	$1^2\cdot3\cdot10\cdot18$
	$1^2\cdot3\cdot13^2$	$1^2\cdot4\cdot5\cdot40$	$1^2\cdot4\cdot6\cdot22$	$1^2\cdot4\cdot7\cdot16$
	$1^2\cdot4\cdot8\cdot13$	$1^2\cdot4\cdot10^2$	$1^2\cdot5^2\cdot19$	$1^2\cdot6^2\cdot10$
	$1^2\cdot7^3$	$1\cdot2^2\cdot5\cdot40$	$1\cdot2^2\cdot6\cdot22$	$1\cdot2^2\cdot7\cdot16$
	$1\cdot2^2\cdot8\cdot13$	$1\cdot2^2\cdot10^2$	$1\cdot2\cdot3^2\cdot58$	$1\cdot2\cdot3\cdot4\cdot18$
	$1\cdot2\cdot4^2\cdot10$	$1\cdot2\cdot4\cdot6^2$	$1\cdot3^3\cdot13$	$1\cdot3^2\cdot4\cdot8$
	$1\cdot4^4$	$2^3\cdot3\cdot18$	$2^3\cdot4\cdot10$	$2^3\cdot6^2$
	$2^2\cdot3^2\cdot8$	$2^2\cdot4^3$	3^5	
$r=6$	$1^4\cdot5\cdot40$	$1^4\cdot6\cdot22$	$1^4\cdot7\cdot16$	$1^4\cdot8\cdot13$
	$1^4\cdot10^2$	$1^3\cdot2\cdot3\cdot18$	$1^3\cdot2\cdot4\cdot10$	$1^3\cdot2\cdot6^2$
	$1^3\cdot3^2\cdot8$	$1^3\cdot4^3$	$1^2\cdot2^3\cdot10$	$1^2\cdot2^2\cdot4^2$
	$1\cdot2^4\cdot4$	2^6		
$r=7$	$1^5\cdot2\cdot10$	$1^5\cdot4^2$	$1^4\cdot2^2\cdot4$	$1^3\cdot2^4$
$r=8$	$1^7\cdot4$	$1^6\cdot2^2$		
$r=9$	1^9			

Figure 4.1:

At the first stage any possible geometrical meaning is ignored, but the final result will be equivalent to a Calabi-Yau compactification [56,57], with extra bonus to have a completely solvable theory, i.e. all correlation functions and partition function may be computed exactly and by these quantities one can extract Yukawa couplings of the superpotential involving the 27 ($\overline{27}$) of E_6 [58,60,61,59].

The first requirement is to saturate the necessary central charge

$$c_D = 15 - \frac{3}{2}D \tag{4.48}$$

for a compactification from a 10 to a D-dimensional spacetime by tensoring minimal models with central charge $c = 3k/(k+2)$

$$c_D = 15 - \frac{3}{2}D = \sum_{i=1}^{r} \frac{3p_i}{p_i + 2} \tag{4.49}$$

In the case of physical interesting case we have D=4 and in the simplest way to compose the minimal models there are 168 possibilities, with range between p=1 and p=1804 [63] as it is shown in the fig.4.1

But this is not the only condition: one has to solve the stringent constraints that string theory must obey, first the modular invariance of the theory. To achieve easily modular invariance we assume that the left and the right compactifying spaces are identical, i.e. we consider (2,2) compactification scheme. We have to mention that the more realistic case (2,0) is now reconsidered since the first suggestions about its instability produced by world-sheet instantons were not necessarily correct [64]. But the problem to construct a modular invariant partition function for these theories is not yet solved, so we discuss the simplest case (2,2).

As we saw in the last section to have space-time supersymmetry we must eliminate all states with U(1) charge that is not odd integer. Then it arises the techical problem to implement charge integrality compatible with modular invariance. The difficulty consists in the fact that modular transformations mix all possible representations so one must select some invariant combinations which realize the charge integrality condition. To do this we consider the characters of N=2 minimal models and their transformation properties under modular group.

We label the N=2 primary fields by $\phi^{l,\bar{l}}_{q,s;\bar{q},\bar{s}}$ extending our previous notation. The quantum numbers satisfy the conditions

$$\begin{array}{ll} 0 \leq l \leq p & \\ \mid q \mid \leq l & (mod\ p) \\ s = 0,2 & NS\ sector \\ s = \pm 1 & R\ sector \\ s \quad mod\ 4 & \end{array} \tag{4.50}$$

and the conformal dimension and U(1) charge are given by [1].

$$\begin{aligned} \Delta &= \frac{l(l+2)-\tilde{q}^2}{4(p+2)} + \frac{s^2}{8} \\ Q &= \frac{\tilde{q}}{p+2} - \frac{s}{2} \\ \tilde{q} &= \begin{cases} q+s & \text{R sector} \\ q & \text{NS sector} \end{cases} \end{aligned} \tag{4.51}$$

The necessity to introduce this extra index s is that to construct massless states one must include non-primary fields which are conveniently labeled by s=2 in the NS sector, while in the R sector $s = \pm 1$ parametrizes the two helicity states. Using the parafermionic construction fields with indices not in the range (4.50) may be re-expressed in terms of these labels using the identity

$$\phi^{l,\bar{q}}_{q,s;\bar{q},\bar{s}} = \phi^{p-l,\bar{p}-\bar{l}}_{q+p+2,s+2;\bar{q}+p+2,\bar{s}+2} \tag{4.52}$$

The characters of the superconformal blocks are computed in terms of Hecke indefinite modular form $c^l_m(\tau)$ of the parafermionic system and the theta functions of SU(2) $\Theta_{n,m}(\tau, z, u)$ [55]

$$\chi^{l(s)}_q(\tau, z) = tr\ exp[2\pi i\tau(L_0 - \frac{c}{24}) + 2\pi i z J_0] \mid_{l,q,s} =$$

[1]Note that we have changed our normalization of the U(1) current $J \to 2J$ and consequently the charge values Q are twice those we considered in chapter 3

$$= \sum_{j \; mod \; p} c^l_{q+4j-s}(\tau)\Theta_{2q+(4j-s)(p+2),2p(p+2)}(\tau, pz, 0) \tag{4.53}$$

The definition of the classical theta functions of SU(2) at level m is

$$\Theta_{n,m} = e^{-2\pi i u} \sum_{j \in Z+\frac{n}{2m}} e^{2\pi i \tau m j^2 + 2\pi j z} \tag{4.54}$$

and the *string function* c^l_m obeys [67]

$$\begin{aligned} c^l_m &= c^l_{-m} = c^{p-l}_{m+p} \\ c^l_m &= 0 \qquad l-n \neq 0 \; (\text{mod } 2) \end{aligned} \tag{4.55}$$

The characters $\chi^{l(s)}_q(\tau, z)$ satisfy the relation

$$\sum_{q \; mod \; 2(p+2)} \chi^l_{q,s}(\tau, z)\Theta_{q,p+2}(\tau, -z) = A^l(\tau, 0, 0)\Theta_{s,2}(\tau, z, 0) \tag{4.56}$$

where A^l is the affine character at level p

$$A^l = \frac{\Theta_{l+1,p+2} - \Theta_{-l-1,p+2}}{\Theta_{1,2} - \Theta_{-1,2}} \tag{4.57}$$

Eq.(4.56) is very powerful to discuss modular transformations of the $\chi^l_{q,s}$. Under modular group the three indices of $\chi^l_{q,s}$ transform independently, where l transforms like an affine character, q like a level (-(p+2)) theta function (i.e. the complex conjugate of the level (p+2) theta function) and s like a level 2-theta function. In view of this factorization a modular invariant solution for the N=2 superconformal models can be construct by the modular invariant solutions of the A^l system [81,82] and the invariant solutions for the level (p+2) and 2 theta system [67]. So if the following partition functions are modular invariant

$$\begin{aligned} Z_A &= \sum_{l,\bar{l}} N_{l,\bar{l}} A_l A^*_{\bar{l}} \\ Z_{p+2} &= \sum_{n,\bar{n}} L_{n,\bar{n}} \Theta_{n,p+2} \Theta^*_{\bar{n},p+2} \\ Z_2 &= \sum_{s,\bar{s}} S_{s,\bar{s}} \Theta_{s,2} \Theta^*_{\bar{s},2} \end{aligned} \tag{4.58}$$

a modular invariant solution for the N=2 system is given by

$$\begin{aligned} Z &= \frac{1}{2} \sum_{\substack{l,\; q,\; s \\ \bar{l},\; \bar{q},\; \bar{s}}} M_{l,q,s;\bar{l},\bar{q},\bar{s}} \chi^l_{q,s} \chi^{*\bar{l}}_{\bar{q},\bar{s}} \\ M_{l,q,s;\bar{l},\bar{q},\bar{s}} &= N_{l,\bar{l}} L_{n,\bar{n}} S_{s,\bar{s}} \end{aligned} \tag{4.59}$$

The transformation law of the q index signals that the theory having central charge $c = 3p/(p+2)$ is invariant under a Z_{p+2} [56]. We discovered this fact in our discussion about N=2 superparafermionic system (Sec.(3.5)) and now this result is recovered analysing the transformation properties of the characters. To see this, we write the partition function on a torus with twisted boundary conditions, i.e. with boundary conditions in the world-sheet space and time directions given by some arbitrary elements (r,s) of Z_{p+2}.

The partition function $Z(0,s)$ is simple: its formal expression in terms of transfer matrix is

$$Z(0,s) = tr\ Qe^{-\beta H} \tag{4.60}$$

where Q is an operator which measures the Z_{p+2} charge of the fields, i.e. on a state with Z_{p+2} charge p the operator takes the value $exp(2\pi i/(p+2))$. If

$$\begin{aligned} Z(0,0) = &\ \mid \eta \mid^{-2} \sum_n \Theta_{n,p+2}\Theta^*_{n,p+2} \\ \eta(\tau) = &\ e^{\frac{2\pi i \tau}{24}} \prod_{n=1}^{\infty}(1 - e^{2\pi i n \tau}) \end{aligned} \tag{4.61}$$

is the diagonal modular invariant solution for the (p+2) level theta system with untwisted boundary conditions, then the partition function for the (0,s) sector is given by

$$Z(0,s) = \mid \eta \mid^{-2} \sum_m e^{\frac{2\pi i m s}{p+2}} \Theta_{m,p+2}\Theta^*_{m,p+2} \tag{4.62}$$

The guess for the partition function for the (r,s) sector is

$$Z(r,s) = \mid \eta \mid^{-2} e^{-\frac{2\pi i r s}{p+2}} \sum_m e^{\frac{2\pi i s m}{p+2}} \Theta_{m,p+2}\Theta^*_{m-2r,p+2} \tag{4.63}$$

It reproduces eq.(4.62) when $r = 0$. We have to show that under modular transformation

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d}$$

the partition function for the (r,s) sector transforms correctly to the partition function for the (ar+bs,cr+ds) sector [37]. It is sufficient to consider the action of the generators T and S

$$\begin{aligned} T: &\ \tau \longrightarrow \tau + 1 \\ S: &\ \tau \longrightarrow -\frac{1}{\tau} \end{aligned}$$

Using the following transformation laws of the theta functions

$$\begin{aligned} T &: \Theta_{m,l}(\tau + 1, z, u) = e^{\frac{\pi i m^2}{2l}} \Theta_{m,l}(\tau, z, u) \\ S &: \Theta_{m,l}(-\frac{1}{\tau}, -\frac{z}{\tau}, u - \frac{z^2}{2\tau}) = \frac{1}{\sqrt{2l}}(-i\tau)^{1/2} \sum_{n=-l+1}^{l} e^{\frac{i\pi n m}{l}} \Theta_{n,l}(\tau, z, u) \end{aligned} \tag{4.64}$$

It is easy to prove that

$$\begin{aligned} Z(r,s)\mid_T = & \ Z(r+s,s) \\ Z(r,s)\mid_S = & \ Z(-s,r) \end{aligned} \tag{4.65}$$

These relations prove that $Z(r,s)$ is the partition function of the sector twisted by (r,s). Moreover $Z(r,s)$ depends on r,s correctly only modulo (p+2). This means that if we do a modular transformation A which leaves the boundary conditions invariant

$$\begin{aligned} ar + bs = r \quad & \bmod (p+2) \\ cr + ds = s \quad & \bmod (p+2) \end{aligned}$$

then $Z(r,s)$ is also invariant

$$Z(r,s)\mid_A = Z(r,s) \tag{4.66}$$

Since there is no anomaly in this transformation, Z_{p+2} is a true symmetry of the system and its operatorial implementation was discussed in the chapter 4.

Let us consider now the problem of constructing space-time supersymmetric superstring compactifications by tensor product of N=2 minimal models. A second stage will be to convert superstring compactification to a heterotic one.

In light cone gauge we can organize the two transverse fermions and bosons of the uncompactified part into complex fermions and bosons of a N=2 supermultiplet. The corresponding N=2 super-stress energy tensor is

$$\mathcal{W} = \overline{\psi}_i\psi_i + \theta\psi_i\partial_z\phi_i + \overline{\theta\psi}_i\partial_z\overline{\phi}_i + \theta\overline{\theta}(-\overline{\psi}_i\partial_z\psi_i - \frac{1}{2}\overline{\phi}_i\partial\phi_i) \tag{4.67}$$

(i=1,2). and the contribution to the partition function is given by a level 2-theta function

$$Z_s = \frac{\Theta_{s,2}(\tau,z,0)}{\eta^3} \tag{4.68}$$

The total super-stress energy tensor of the N=2 world-sheet SUSY is then

$$\mathcal{W} = \sum_{i=1}^{2}\mathcal{W}_i + \sum_{i=1}^{r}\mathcal{W}_i^{int} \tag{4.69}$$

To implement the charge integrality in a modular invariant way we need to worry only about the theta functions. This comes from the factorization property (4.56) and the fact that the affine A^l part does not have any charge. There are $r+1$ level-2 theta functions coming from the r-copies of the minimal models plus that one of the transverse degree of freedom, eq.(4.68), and r-level (p_i+2) theta functions of the r-minimal models. A generic term of the partition function has the expression

$$Z_{r_1,\cdots,r_j} = \Theta_{r_1,n_1}\cdots\Theta_{r_j,n_j} \equiv \Theta_{\vec{r}} \tag{4.70}$$

How to select the correct combination having charge integrality? The β method by Gepner [55] consists in the following: first we introduce a scalar product between two terms of the partition function

$$\vec{r}\vec{m} = \frac{r_1 m_1}{2n_1} + \frac{r_2 m_2}{2n_2} + \cdots \frac{r_j m_j}{2n_j} \tag{4.71}$$

which resembles the factor in (4.54) which takes in account the charge of the theta function. Then having a vector $\vec{\beta}$ which obeys $\vec{\beta}\cdot\vec{\beta} = integer$ we can define these functions

$$\Theta_{\vec{\mu}}^{(s,t)} = \sum_{n=0}^{\alpha-1}(-1)^{\beta^2 ns}\Theta_{\vec{\mu}+n\vec{\beta}} \tag{4.72}$$

where

$$\begin{aligned} \vec{\beta}\cdot\vec{\mu} + \tfrac{1}{2}t\vec{\beta}^2 &= \quad integer \\ (s,t) \in N \quad & \begin{cases} (s,t)\ mod\ 2\text{if } \vec{\beta}^2 = odd \\ (s,t) = (1,1) \text{ if } \vec{\beta}^2 = even \end{cases} \end{aligned} \tag{4.73}$$

and α is the least integer such that

$$\frac{\beta_i \alpha}{2n_i} = integer \qquad i = 1, \cdots, j \tag{4.74}$$

The remarkable fact is that under modular transformations these functions transform into themselves in a unitary representation of the modular group

$$\begin{aligned} \Theta_{\vec{\mu}}^{(s,t)} \mid_S = & \frac{1}{\sqrt{n_1 n_2 \cdots n_j}}\left(\frac{-i\tau}{2}\right)^{j/2}\sum_{\vec{\nu}} e^{-2\pi i\vec{\mu}\cdot\vec{\nu}}\Theta_{\vec{\nu}}^{(t,s)} \\ \Theta_{\vec{\mu}}^{(s,t)} \mid_T = & \ e^{\pi i\vec{\mu}^2}\Theta_{\vec{\mu}}^{(t+s+1,t)} \end{aligned} \tag{4.75}$$

In this way we can select those representations which satisfy the integrality of the U(1) charge given by

$$Q_{\vec{\beta}}(\vec{\mu}) = \delta\vec{\beta}\cdot\vec{\mu} \tag{4.76}$$

where $\delta = 1$ for even $\vec{\beta}^2$ and $\delta = 2$ for odd $\vec{\beta}^2$.

The transformation properties (4.75) implies that for $\vec{\beta}^2 = odd$ the sectors given by $(s,t) = \{(0,0),(1,0),(0,1)\}$ transform into themselves, while the sector $(s,t) = (1,1)$ transforms into itself. This fact resembles the analogous mechanism of the spin-structure in string theory and we can define a bosonic-type partition function

$$Z_B = \mid \eta \mid^{-2} \sum_{\vec{\mu},s,t|(s,t)\neq(1,1)} \Theta_{\vec{\mu}}^{(s,t)}\Theta_{\vec{\mu}}^{*(s,t)} \tag{4.77}$$

in which all the multiplicities are non negative (the negative terms in $\Theta_{\vec{\mu}}^{(1,0)}\Theta_{\vec{\mu}}^{*(1,0)}$ are cancelled by the positive terms in $\Theta_{\vec{\mu}}^{(0,0)}\Theta_{\vec{\mu}}^{*(0,0)}$ piece) and a fermionic modular invariant partition function given by

$$Z_F = \mid \eta \mid^{-2} \sum_{\vec{\mu}} \Theta_{\vec{\mu}}^{(1,1)}\Theta_{\vec{\mu}}^{*(1,1)} \tag{4.78}$$

in which the multiplicities are either positive or negative integers. The negative contribution are associated with the space-time fermions. At this point it is perhaps useful to discuss a toy example to see how the β method works. Let us consider

$$\Theta_{\vec{r}} = \Theta_{r_1,4}\Theta_{r_2,9} \tag{4.79}$$

and the vector $\vec{\beta} = (2,3)$

$$\vec{\beta}^2 = \frac{4}{8} + \frac{9}{18} = 1 \tag{4.80}$$

We have to solve the diophantic equation

$$\vec{\beta}\cdot\vec{\mu} = \frac{4\mu_1}{8} + \frac{3\mu_2}{18} = k \qquad k \in N \tag{4.81}$$

for t=0 and

$$\vec{\beta}\cdot\vec{\mu} = \frac{4\mu_1}{8} + \frac{3\mu_2}{18} = k \qquad k \in N \tag{4.82}$$

for t=1.

Taking in account the range of μ_i (they are defined mod $2n_i$), eq.(4.81) has solutions for $k \leq 7$; for instance, if $k = 2$ the possible $\vec{\mu}$ vectors are given by

$$\vec{\mu} = \begin{cases} (0,12) \\ (1,9) \\ (2,6) \\ (3,3) \\ (4,0) \end{cases} \tag{4.83}$$

In correspondance of each solution $\vec{\mu}$ we can construct the associate $\Theta_{\vec{\mu}}^{(s,0)}$. Analogously the equation (4.82) has solution for $n \leq 6$ and the corresponding theta function are $\Theta_{\vec{\mu}}^{(s,1)}$.

Coming back to the superstring compactification, it is easy to see that it exists a vector $\vec{\beta}$ with $\vec{\beta}^2 = 1$. This has all components equal to one, both the (r+1) level 2-theta functions as well as the r level (-(p_i+2))theta functions

$$\beta^2 = \frac{1}{4} + \sum_{i=1}^{r}\left(\frac{1}{4} - \frac{1}{2(p_i+2)}\right) = \frac{1}{4} + \frac{1}{12}\sum_{i=1}^{r}\frac{3p_i}{p_i+2} = 9 \tag{4.84}$$

(we have used eq.(4.49) for D=4)

By this vector we can construct the space-time SUSY operator: the internal part is made by tensor product of the fields with $l_i = 0$ and $\vec{\beta}$ for the theta functions

$$\phi^{l_i}_{q_i,s_i;\bar{q}_i,\bar{s}_i} = \phi^{0}_{1,1;0,0} = Iexp[\frac{1}{2}\sqrt{\frac{p_i}{p_i+2}}\varphi(z)] \tag{4.85}$$

I is the identity operator of the parafermionic theory. This state is the Ramond vacuum, with dimension $\Delta = \frac{c_i}{24}$ and charge $q = \frac{c_i}{6}$. The tensor product of such fields creates a field with dimension $\Delta = \frac{c_{tot}}{24} = \frac{3}{8}$ and charge $q = \frac{c_{tot}}{6} = \frac{3}{2}$, i.e. the field Σ of the section (3.5).

Of course to respect the N=1 superconformal gauge condition of the superstring all the fields in the tensor product should be either in the Ramond sector of all the minimal models involved or in the NS sector. This constraint can be implemented by additional β_i vectors ($i = 1, 2 \cdots, r$) where β_i is zero in all the level $(p_i + 2)$ theta functions and all the level-2 theta functions, except at the i-th and the last level-2 theta functions where its value is 2 [55]. Since $\beta_i^2 = 2$ the condition on the possible μ's is

$$\vec{\beta}_i \cdot \vec{\mu} = integer \tag{4.86}$$

If s_j is the index of μ in the j-place of the level-2 theta functions eq.(4.86) reads

$$s_j = s_{l+1} \qquad (mod\ 2) \tag{4.87}$$

Since s_j is even in the NS sector whereas is odd in the R one, the set of equations coming from the $\vec{\beta}_i$ vectors assure that the fields construct by the ones in the minimal models are either in the NS sector or in the R sector.

So, finally we obtain a modular invariant partition function for the superstring

$$Z = \sum_{\lambda, \bar{\lambda}} B_\lambda(\tau) B^*_{\bar{\lambda}}(\bar{\tau}) Z_{\lambda,\bar{\lambda}}(\tau, \bar{\tau}) \tag{4.88}$$

where $Z_{\lambda,\bar{\lambda}}$ correspond to the partition function of the internal theory (the compactified one) and B_λ is the contribution of the transverse fermions of the Minkowski space. They realize a level one representation of SO(2) then the index λ spans the singlet (0), the vector (v) and the spinors ($s, \bar{s}$) representations.

In the heterotic string the left sector of the theory has a N=1 gauge condition whereas the right one is bosonic-like. We have to convert the superstring compactification to a heterotic one, i.e. we have to replace the B_λ in the right sector by another set of characters which transform in the same way as B_λ do but realized the gauge group degree of freedom of the heterotic string. A surprising result is that the character of SO(d) transform in the same way of the characters of $SO(10) \times E_8$ or SO(26) if we interchange the singlet and vector of SO(2) with the vector and singlet of $SO(10) \times E_8$ or $SO(26)$ and flip the sign of SO(2) spinors with the corresponding $SO(10) \times E_8$ or SO(26) spinors (taking the singlet representation of E_8 for the first case) [2] These are the only two possibilities: these groups correspond to the heterotic internal degree of freedom. If we denote the characters of SO(2) weights by this vector $\vec{B} = (B_0, B_v, B_s, B_{\bar{s}})$ the matrix M which implements the above mechanism is

$$\begin{pmatrix} B_0 \\ B_v \\ B_s \\ B_{\bar{s}} \end{pmatrix}^{SO(2)} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} B_0 \\ B_v \\ B_s \\ B_{\bar{s}} \end{pmatrix}^{SO(26)} \tag{4.89}$$

[2] The generalization of this formula to the case of d transverse uncompactified dimensions is given by the replacement of SO(2) with SO(d) and the corresponding internal groups are $SO(8+d) \times E_8$ and $SO(d+24)$

(The same for the first case $SO(10) \times E_8$) Then a modular invariant partition function for the heterotic string is

$$Z_{het} = \sum_{\lambda,\bar{\lambda}} B_{\lambda}^{SO(2)}(\tau)(MB^{SO(26)})^*_{\bar{\lambda}}(\bar{\tau})Z_{\lambda,\bar{\lambda}}(\tau,\bar{\tau}) \tag{4.90}$$

The explicit expression is

$$\begin{aligned} Z_{het} &= \sum_{v,\bar{v},l,\bar{l}} \pm A_{l,\bar{l}} Z^l_{\vec{v}} Z^{\bar{l}*}_{\vec{\mu}+\vec{\bar{v}}} \\ &\vec{v} - \vec{\bar{v}} \in \Lambda \\ &\vec{v}\cdot\vec{\beta}_0, \vec{\bar{v}}\cdot\vec{\beta} \in \frac{1}{2} + Z \\ &\vec{v}\cdot\vec{\beta}_i, \vec{\bar{v}}\cdot\vec{\beta}_i \in Z \end{aligned} \tag{4.91}$$

The quantity $\vec{v}$ is a vector (a,b,c) where a is a weight of the algebra SO(2) in the left sector, or SO(26) or $SO(10) \times E_8$ in the right sector. The other component b corresponds to the indices of the r level-2 theta functions and c to the r level $p_i + 2$ theta functions. β_0 is the vector which assures charge integrality

$$\vec{\beta}_0 = (\bar{s}, 1, \cdots, 1) \tag{4.92}$$

and β_i have the form

$$\vec{\beta}_i = (v, 0, \cdots, 0, 2, 0, \cdots, 0) \qquad i = 2, \cdots r + 1 \tag{4.93}$$

where v is the weight of the vector representation. The lattice spanned by $\vec{\beta}_0$ and $\vec{\beta}_i$ over the integer is Λ. The condition $\vec{v} - \vec{\bar{v}} \in \Lambda$ in (4.91) restrict the sum of states to include right-left symmetric states and those connected by either the supersymmetry generator $\vec{\beta}_0$ or the vector $\vec{\beta}_i$ which span the NS and R sector. The condition $\vec{v}\cdot\vec{\beta}_0, \vec{\bar{v}}\cdot\vec{\beta}_0 \in \frac{1}{2} + Z$ projects onto integer supercharge states whereas the condition $\vec{v}\cdot\vec{\beta}_i, \vec{\bar{v}}\cdot\vec{\beta}_i \in Z$ projects onto the NS or R sectors. $\mu = (v, 0, \cdots, 0)$ realizes the heterotic construction and $A_{l,\bar{l}}$ represents aproduct of the multiplicities of the A^l invariants, which does not carry any charge and it can be combined in any modular invariant way according to the classification found in [81,82]

$$A_{l,\bar{l}} = \prod_{i=1}^{r} A^{p_i}_{l_i,\bar{l}_i} \tag{4.94}$$

In this scheme, we start to analyse the massless spectrum recovering first tha gauge group and supergravity multiplets. The fermionic left sector component for these fields has s for the weight of SO(2). This weight has U(1) charge equal to $-\frac{1}{2}$ (in our normalization $\bar{s} = 1, s = 1, v = 2, O = 0$ are the weight of SO(2) representations whose charge is just half of their values) so to have total charge equal to 1 the contribution of the internal Ramond fields has to be $\frac{3}{2}$. Using (4.52 we have the equation

$$\sum_{i=1}^{r} \left(\frac{k_i + 1}{p_i + 2} - \frac{1}{2}\right) = \frac{3}{2} \tag{4.95}$$

whose solution is

$$k_i = p_i \qquad i = 1, \cdots, r \tag{4.96}$$

The corresponding vertex is

$$V_{sp}(z) = \lambda^s \prod_{i=1}^{r} R^{p_i}_{p_i,1}(z) \tag{4.97}$$

where $R^k_{k,1}(z)$ is the superparafermionic order parameter and λ^s is the spinor of SO(2). The antispinor vertex is of course

$$V_{antisp}(z) = \lambda^{\bar{s}} \prod_{i=1}^{r} R^{p_i}_{-p_i,-1}(z) \tag{4.98}$$

The left sector vectorial component is just given by the identity operators in the internal part multiplied by the vector state of SO(2)

$$V_{vect}(z) = \lambda^v 1 \tag{4.99}$$

Using this set of left sector states we can construct the right sector by applying the lattice shift operator μ and those ones belonging to Λ. The spinor part has the same expression of eq.(4.97) but λ^s is now the spinor of SO(10) whereas we have to do the permutation of vector representation with the scalar one. [3] For denoting all these states we use the compact notation of the N=2 superconformal field $\phi^l_{q,s} = (l, q, s)$ plus additional index for the weight representation of SO(2) or SO(10). So eqs. (4.97)(4.98) (4.99) become

$$\begin{array}{c} (s)_{SO(2)} \prod_{i=1}^{r}(p_i, p_i, 1) \\ (\bar{s})_{SO(2)} \prod_{i=1}^{r}(p_i, -p_i, -1) \\ (v)_{SO(2)} \prod_{i=1}^{r}(0, 0, 0) \end{array} \tag{4.100}$$

The corresponding right parts are

$$\begin{array}{c} (s)_{SO(10)} \prod_{i=1}^{r}(p_i, p_i, 1) \\ (\bar{s})_{SO(10)} \prod_{i=1}^{r}(p_i, -p_i, -1) \\ (0)_{SO(10)\times E_8'} \prod_{i=1}^{r}(0, 0, 0) \end{array} \tag{4.101}$$

If we take the scalar component in the right sector with any of two transverse bosons excited (which are not included in the partition function (4.91)) and compose it with any state (4.100) of the left sector we obtain the usual supergravity multiplet.

The gauge group is $E_8 \times E_6 \times [U(1)]^{r-1}$. To see this we organize the fields of SO(10) into the 78 of E_6: in fact we have in the right part the adjoint of SO(10) with dimension 45 (with the adjoint of E_8) in the vector

$$(0)_{SO(10)\times E_8} \prod_{i=1}^{r}(0, 0, 0)$$

[3] There is also the adjoint representation in the game: in SO(2) it coincides with the scalar one and in $SO(10) \times E_8$ we include it into the same notation of scalar one

and two spinor representations, the 16,$\overline{16}$ of SO(10) from the first and second line of (4.101) respectively. To complete the decomposition

$$\begin{aligned} E_6 &\rightarrow SO(10) \times U(1) \\ 78 &= 45 + 16 + \overline{16} + 1 \end{aligned} \tag{4.102}$$

we need a scalar representation which comes from the total U(1) internal boson J

$$J = J_1 + J_2 + \cdots J_r \tag{4.103}$$

The remaining (r-1) U(1) internal bosons implements the additional group $[U(1)]^{r-1}$.

Taking the vector representation in the left sector we get the corresponding gauge bosons, whereas their superpartner are obtained taking in the left part spinor or antispinor.

The construction of supergravity and gauge multiplets is general mechanism in any way of tensoring N=2 minimal models. To discuss the matter supermultiplet is better to have in mind specific examples, which will be the subject of the next section.

4.4 Massless matter fields and Yukawa couplings

In this last section we analyse some models constructed by tensor product of N=2 minimal models. In particular we compare the massless matter spectrum so obtained with the corresponding spectrum coming from a Calabi-Yau compactification and their match together with the match of the automorphic symmetry group indicates that these models correspond to string propagation on Calabi-Yau manifold, in particular they are solvable Calabi-Yau string theory.

Moreover, denoting by S a four dimensional chiral superfield

$$S = \varphi + \theta\psi + \theta^2 F \tag{4.104}$$

(φ is the spin zero component, ψ is the spinor and F is the spin one component) and looking at the effective superpotential

$$V \sim S^3$$

in particular at the term

$$\psi^i \psi^j \varphi^k$$

which involves two spinors and one scalar, we are able to extract the corresponding Yukawa couplings of the low-energy theory. The problem consists in finding a vertex representation of these fields, in determining the number of generations and in computing their 3-point functions.

The massless matter fields appear in the $\overline{27}$ (in the case of generations) or in the 27 (for the antigenerations) of E_6. In fact we can use the decomposition $E_6 \rightarrow SO(10) \times U(1)$ to construct the $\overline{27}$ by

$$\overline{27} = \overline{16}_{-\frac{1}{2}} + 10_1 + 1_2 \tag{4.105}$$

where the sub-index is the U(1) charge.

In a supersymmetric theory the couplings between different multiplets $S_i = (\varphi_i, \psi_i)$ are described by the superpotential $W(S_i)$ of mass dimension 3 and analytic in the fields. From $W(S_i)$ one obtains the scalar potential, given by

$$
\begin{aligned}
V(\varphi_i) &= \sum_i \left(\frac{\partial W}{\partial \varphi_i}\right)\left(\frac{\partial W^*}{\partial \varphi_i^*}\right) + \frac{1}{2}\sum_a (D^a)^2 \\
D^a &= g\varphi_i^* T_{ij}^a \varphi_j
\end{aligned}
\tag{4.106}
$$

(T^a's are the generators of gauge group) and the Yukawa interaction terms

$$
\mathcal{L}_Y = \frac{\partial^2 W}{\partial\varphi_i \partial\varphi_j}\psi_i\psi_j + \quad h.c. \tag{4.107}
$$

Using the product of representations of E_6

$$
\begin{aligned}
27 \times 27 &= \overline{27} + 135_a + 351_s \\
27 \times \overline{27} &= 1 + 78 + 650
\end{aligned}
\tag{4.108}
$$

we see that no term of the form 27×27 can appear in W since it is not a singlet of E_6, but a term like $27 \times \overline{27}$ is a priori possible. Actually we discard this term since it is a quadratic term contributing to the mass term in the potential (4.106) and we are considering only the massless spectrum. So, it remains only a term 27^3 or $\overline{27}^3$ and the most general renormalizable form of the superpotential is

$$
W = a_{ijk} S^i S^j S^k + b_{lmn} \overline{S}^l \overline{S}^m \overline{S}^n \tag{4.109}
$$

where S^i's are fields in the 27 and $\overline{a}^i$'s belong to the $\overline{27}$. It is useful to decompose the representation 27 of E_6 under $SU(3)_c \otimes SU(3)_l \otimes SU(3)_R$ (fig.4.2), i.e. in terms of the fields $(Q, D, u^c, d^c, D^c, H_u, H_d, L, e^c, \nu^c, N)$. Of course the decomposition of $\overline{27}$ is with opposite quantum numbers. Then we can write the superpotential in the following way

$$
\begin{aligned}
V \;=\; & \lambda^1_{ijk} H_{u_i} Q_j u^c_k + \lambda^2_{ijk} H_{d_i} Q_j d^c_k + \lambda^3_{ijk} H_{d_i} L_j e^c_k + \\
& + \lambda^4_{ijk} H_{u_i} H_{d_j} N_k + \lambda^5_{ijk} D_i D^c_j N_k + \lambda^6_{ijk} H_{u_i} L_j \nu^c_k + \\
& + \lambda^7_{ijk} Q_i L_j D^c_k + \lambda^8_{ijk} u^c_i e^c_i D_k + \lambda^9_{ijk} d^c_i \mu^c_j D_k + \\
& + \lambda^{10}_{ijk} Q_i Q_j D_k + \lambda^{11}_{ijk} d^c_i u^c_j D^c_k + \overline{\mu}^1_{lmn} \overline{H}_{u_l} \overline{Q}_m \overline{u}^c_n + \cdots
\end{aligned}
\tag{4.110}
$$

With E_6 exact all these coupling constants are equals

$$
\begin{aligned}
\lambda^1_{ijk} &= \lambda^2_{ijk} = \cdots = \lambda^{11}_{ijk} \\
\overline{\mu}^1_{lmn} &= \overline{\mu}^2_{lmn} = \cdots \overline{\mu}^{11}_{lmn}
\end{aligned}
\tag{4.111}
$$

$SU(3)_C \otimes SU(3)_L \otimes SU(3)_R$		$SU(3)_C \otimes SU(2)_L$	Y_L	Y_R	Y'_R	$Y = Y_L + Y_R$
(3, 3, 1)	$Q = \begin{pmatrix} u \\ d \end{pmatrix}$	(3,2)	1/3	0	0	1/3
	D	(3,1)	-2/3	0	0	-2/3
$(\bar{3}, 1, \bar{3})$	u^c	$(\bar{3},1)$	0	-4/3	-1/3	-4/3
	d^c	$(\bar{3},1)$	0	2/3	-1/3	2/3
	D^c	$(\bar{3},1)$	0	2/3	2/3	2/3
$(1, \bar{3}, 3)$	$H_u = \begin{pmatrix} H_u^+ \\ H_u^0 \end{pmatrix}$	(1,2)	-1/3	4/3	1/3	1
	$H_d = \begin{pmatrix} H_d^0 \\ H_d^- \end{pmatrix}$	(1,2)	-1/3	-2/3	1/3	-1
	$L = \begin{pmatrix} \nu \\ e^- \end{pmatrix}$	(1,2)	-1/3	-2/3	-2/3	-1
	e^c	(1,1)	2/3	4/3	1/3	2
	ν^c	(1,1)	2/3	-2/3	1/3	0
	N	(1,1)	2/3	-2/3	-2/3	0

Figure 4.2:

We note that H_u is the scalar field which couples to the u quark then by spontaneous symmetry breaking its vacuum expectation value determines the mass of the up quark, while H_d couples to the d-quark and also determines the mass of the electron. On the other hand, some of these couplings must vanish. If all the five terms

$$QQD + d^c u^c D^c + QLD^c + d^c v^c D + u^c e^c D \tag{4.112}$$

are present, the low energy theory does not have baryon number symmetry. If this symmetry is absent, the proton is unstable and decays at an unacceptable large rate [39]. If the Yukawa couplings had respected the E_6 invariance, these couplings could not have vanished when those which provide the quarks and leptons with masses are not zero. So E_6 symmetry must be broken: this is what the Wilson loops do in the case of superstring compactification.

After this brief digression about the N=1 supersymmetric four dimensional theory we come back to the problem of the vertex representation of the various fields. We use the decomposition (4.105) to construct the 27 of E_6 since the SO(10) group appears in the construction of modular invariant solution. The vertex operator of a space-time fermion in the $-\frac{1}{2}$ picture is

$$\mathcal{V}_{-\frac{1}{2}} = \xi S_a V S \tag{4.113}$$

where V is a primary field of the tensor product of the N=2 part, i.e. is a product of primary fields from the minimal sub-theories $p_1, p_2, \cdots p_r$

$$V = \prod_{i=1}^{r} \phi^{l_i, \bar{l}_i}_{q_i, s_i; \bar{q}_i, \bar{s}_i} \tag{4.114}$$

$\mathcal{S}_a$ is the spin field for the four dimensional uncompactified spinors, ξ is the ghost wave function and $\mathcal{S}$ is a vertex operator for a primary field of the internal SO(10) current algebra. The possible SO(10) representations are four, i.e. vector, spinor, antispinor and the singlet. Their expression can be written using the bosonization rules on the root lattice [53].

The vertex operator for a space-time scalar is given by

$$\mathcal{V}_{-1} = V\mathcal{S} \tag{4.115}$$

where V ia a N=2 primary field and $\mathcal{S}$ is the SO(10) term.

In this scheme the Yukawa couplings are given by the structure constants

$$g_{ijk} = < V^i_{-\frac{1}{2}} V^j_{-\frac{1}{2}} V^k_{-1} > \tag{4.116}$$

To get a E_6 singlet it is particular useful to take two fields in the vector representation of SO(10) and the third in the singlet of SO(10). From eq. (4.116) we note that since correlation functions of exponential parts give a coefficient equal to 1 the non trivial value of g_{ijk} (as well as their possible vanishing!) it is coded into the N=2 compactified part of the massless string vertices. This is constructed in terms of the lowest energy states of the N=2 minimal models, i.e. the superparafermionic order parameters (Sect.(3.5)) which we rewrite here

$$\begin{aligned} NS \;&:\quad \sigma^{NS}_k = N^k_{k,p} \quad \Delta_k = \frac{k}{2(p+2)} \quad q_k = \frac{k}{p+2} \quad Z_{p+2} \text{ charge } k \\ R \;&:\quad \sigma^{R}_{k,p} = R^k_{k,p} \quad \Delta_k = \frac{c}{24} \quad q_k = \frac{k+1}{p+2} - \frac{1}{2} \quad Z_{p+2} \text{ charge } k+1 \end{aligned} \tag{4.117}$$

A Ramond ground state in the compactified $c = 9$ theory is a product of Ramond ground states in each of the $1, \cdots r$ theories

$$R = \prod_{i=1}^{r} R^{k_i}_{k_i, p_i} \tag{4.118}$$

The U(1) charge of this field in order to survive the charge projection must be $\pm\frac{1}{2}$, depending on the space-time helicity. Assuming the case of $-\frac{1}{2}$ (generation case) we must have

$$\sum_{i=1}^{r} \frac{k_i}{p_i + 2} = 1 \tag{4.119}$$

For any set of integers $\{k_i\}$ satisfying this equation (together with $0 \leq k_i \leq p_i$) there is a massless field in the $\overline{27}$ of E_6 and thus the generations are labelled by these integers.

To be more explicit let us consider the thery made out of 5 copies of the $p = 3$ thoery, $(3)^5$. Eq.(4.119) becomes

$$\sum_{i=1}^{5} k_i = 5 \tag{4.120}$$

and we have the following solutions (with their multiplicities)

k_1	k_2	k_3	k_4	k_5	#
1	1	1	1	1	(1)
2	1	1	1	0	(20)
2	2	1	0	0	(30)
3	1	1	0	0	(30)
3	2	0	0	0	(20)

We have 101 generations. We can recover a model with four generations (namely five generation minus one anti-generation) if we identify fields up a Z_5 transformation (cyclic permutation of the five copies)

$$A: \quad i \to i+1$$

(i is the i'th copy) and require neutrality respect the $Z_{p+2} = Z_5$ internal discrete group of each model

$$\sum_{i}^{5} n_i = 0 \mod 5$$

(n_i is the $Z_{p+2} = z_5$ charge of each submodel).

In this case of four generations, the corresponding left part of the vertices of the massless space-time spinor component χ_a of the supermultiplet S_a (where $\chi_a = \eta_a(z)\xi_a(\bar{z})$) are the following [60]

$$\begin{aligned}
\eta_0 &= \sigma^-(R_1^1)^5(z) \\
\eta_1 &= \sigma^- R_1^1 R_3^3 R_1^1 (R_0^0)^2 + \text{cycl.perm} \\
\eta_1^\dagger &= \sigma^-(R_1^1)^2 R_0^0 R_3^3 R_0^0 + \text{cycl.perm} \\
\eta_2 &= \sigma^- R_2^2 R_1^1 R_2^2 (R_0^0)^2 + \text{cycl.perm} \\
\eta_2^\dagger &= \sigma^-(R_2^2)^2 R_0^0 R_1^1 R_0^0 + \text{cycl.perm}
\end{aligned} \tag{4.121}$$

(the field σ^- $(\Delta = 1/8; q = -1/2)$ represents the spin-field of the uncompactified transversal spinor ψ^μ)

Each of them are associated with a right moving part in a way dictated by the modular invariance. For instance, taking the singlet of SO(10) and the left-right symmetric solution of the modular invariance for the l-indices, the associated right field of η_0 is

$$\xi_0 = (N_{-2}^2)^5(\bar{z}) \tag{4.122}$$

In general case, the expression of the vertex χ_a in the singlet represntation of SO(10) is

$$\chi_a = \prod_{i=1}^{5} R_{k_i}^{k_i}(z) N_{-l_i}^{l_i}(\bar{z}) \tag{4.123}$$

where

$$k_i - (-l_i) = \text{const} = p_i = 3 \tag{4.124}$$

One can determine by these equations all the corresponding right part of the fields in eq.(4.122).

To complete the $\overline{27}$ of E_6 we need also the vector ($\underline{10}$) and the antispinor ($\overline{16}$) representations of SO(10). So the right part of η_0 has the following expression

$$\xi_0(\bar{z}) = [0,(N^2_{-2})^5] \oplus [\bar{s},(R^1_1)^5] \oplus [v,(N^1_1)^5] \tag{4.125}$$

where $0, \bar{s}, v$ label the $\underline{1}, \overline{16}$ and $\underline{10}$ of SO(10). The right part of the other η's fields is easily determined.

The space-time scalar superpartner h of χ_a can be obtained acting on χ_a with the space-time supercharge Q^α and using the 2-D OPE. For instance the scalar superpartner h_0 of η_0 has the following left part vertex

$$h_0 = (N^1_{-1})^5 \tag{4.126}$$

For computing the antigeneration vertex we take in account the decomposition

$$27 = 16_{\frac{1}{2}} + 10_{-1} + 1_2$$

In the case of singlet representation of SO(10), the general expression of the vertex is now

$$\chi_a(z,\bar{z}) = \prod_{i=1}^{5} R^{k_i}_{k_i}(z) N^{l_i}_{l_i}(\bar{z}) \tag{4.127}$$

with

$$k_i - l_i = \text{const} = -1 \tag{4.128}$$

In this case the only solution is

$$\overline{\chi}_a(z,\bar{z}) = (R^1_1(z))^5 \times [0,(N^2_2)^5(\bar{z})] \tag{4.129}$$

Completing the remaining part we have

$$\begin{aligned}\overline{\chi}(z,\bar{z}) \quad &= \quad (R^1_1(z))^5 \times ([0,(N^2_2)^5(\bar{z})] \oplus [\bar{s},(R^2_2)^5(\bar{z})] + \\ &\oplus [v,(N^1_{-1}(\bar{z})])\end{aligned} \tag{4.130}$$

We see that the $(p=3)^5$ theory has 101 generations and 1 anti-generation, and $\frac{(p=3)^5}{Z_5 \times Z_5}$ has 5 generations and one anti-generation. These numbers match with the corresponding numbers of the string theory propagating on Calabi-Yau manifold $Y_{4,5}$ and $Y_{4,5}/(Z_5 \times Z_5)$. Now the crucial observation is about the automorphic symmetries of these theories, i.e. the theory constructed in terms of tensor product and the theory describing string propagation on Calabi-Yau manifold. Are these symmetries equal?

The hypersurface $Y_{4,5}$ described by

$$z_1^5 + z_2^5 + z_3^5 + z_4^5 + z_5^5 = 0$$
$$(z_1, \cdots, z_5) \in CP^4 \tag{4.131}$$

enjoys the glogal symmetry group

$$G = \frac{(S_5 \times (Z_5)^5)}{Z_5} \tag{4.132}$$

where S_5 is the permutation group of the five variables and the n'th Z_5 is generated by

$$z_n \to exp(\frac{i\pi s_n}{5})z_n \tag{4.133}$$

Since an overall phase is irrelavant in CP^4, the transformation $\{s_i\}$ and $\{s_i + 1\}$ are identified and we quotient correctly by Z_5. The group G has 75000 elements. Some of them do not commute with supersymmetry and are called R symmetries: they are given by $\{s\}$ where $\sum_i s_i \neq 0 \text{mod } 5$ [40,56].

Let analyse the symmetries of the $(p = 3)^5$ theory. There are the $(p_i + 2)$ discrete symmetries of each sub-theories

$$\tilde{G} = Z_5 \times Z_5 \times Z_5 \times Z_5 \times Z_5 \tag{4.134}$$

plus the permutation symmetries among the five identical sub-theories

$$G_1 = S_5 \times (Z_5)^5 \tag{4.135}$$

But this is not the whole story since the summation over the vector β_0, eq.(4.72), implies that elements of G_1 related by $4\beta_0$ should be identified (the factor 4 comes from the theta functions at level-2). If we denote the generic element of G_1 by $(s_1, s_2, \cdots, s_5)$, the group symmetry generated by β_0 is $H = (2,2,2,2,2)$ and the final symmetry group of $(p = 3)^5$ theory is

$$G = \left(S_5 \times (Z_5)^5\right)/Z_5 \tag{4.136}$$

Also in this case not all the elements of G commute with supersymmetry and the condition for the element $(s_1, \cdots, s_5)$ to commute with supersymmetry is [56]

$$\sum_{i=1}^{5} s_i = 0 \quad \text{mod } 5 \tag{4.137}$$

So finally we see that not only the spectrum matches but also the automorphic symmetries. One can further analyse these correspondance to conclude that $(p = 3)^5$ theory describes a string propagation on $Y_{4,5}$ Calabi-Yau [56].

In the field theory approximation the Yukawa couplings of three 27's has a remarkable expression in terms of a quasi topological formula [65,66,40], i.e.

$$\int_K \Omega_{\mu\nu\rho}\Omega \wedge \omega^\mu \wedge \omega^\nu \wedge \omega^\rho \tag{4.138}$$

where Ω is the holomorphic (3,0) form and ω^μ is anti-holomorphic one form with value in the tangent bundle. This expression is particular useful when the compact space is given by an intersection of polynomials, since it can be computed in terms of polynomial deformations. In the case of $Y_{4,5}$ the general expression of a deformation polynomial is

$$P(n_1, \cdots, n_5) = z_1^{n_1} z_2^{n_2} z_3^{n_3} z_4^{n_4} z_5^{n_5} \tag{4.139}$$

$$\sum_{i=1}^{5} n_i = 5 \qquad\qquad 0 \leq n_i \leq 3 \tag{4.140}$$

The product of three such kind of polynomial has degree 15. Any such polynomial which contains z_i^4 corresponds to an exact form and then gives a vanishing Yukawa coupling. The only non trivial polynomial is

$$P = z_1^3 z_2^3 z_3^3 z_4^5 z_5^3 \tag{4.141}$$

The Yukawa coupling is the coefficient of P in the product of the appropriate three polynomials which represent the generations. Using the correspondance between generations and polynomials (see Appendix C) we denote the i'th generation as $\psi^i = P(n_1^i, n_2^i, n_3^i, n_4^i, n_5^i)$. Then

$$g_{ijl} = \prod_{r=1}^{5} \delta(n_r^i + n_r^j + n_r^l - 3) \tag{4.142}$$

In this normalization g_{ijl} is either zero or one.

Actually from $Y_{4,5}$ Calabi-Yau manifold we can obtain a four generation model identifying the points under the action of two Z_5 groups

$$\begin{aligned} A \ &: \ z_i \rightarrow z_{i+1} \\ & \quad\ z_{i+5} \equiv z_i \\ B \ &: \ z_k \rightarrow \alpha^k z_k \\ & \quad\ \alpha = exp\{\frac{2\pi i}{5}\} \end{aligned} \tag{4.143}$$

The corresponding polynomials describing the generation fermions are the quintic polynomials invariant under $Z_5 \times Z_5$

$$\begin{aligned} \psi_2 &= z_1^2 z_2 z_3^2 + z_2^2 z_3 z_4^2 + z_3^2 z_4 z_5^2 + z_4^2 z_5 z_1^2 + z_5^2 z_1 z_2^2 \\ \psi_{-2} &= z_1^2 z_2^2 z_4 + z_2^2 z_3^2 z_5 + z_3^2 z_4^2 z_1 + z_4^2 z_5^2 z_2 + z_5^2 z_1^2 z_3 \\ \psi_1 &= z_1^3 z_2 z_5 + z_2^3 z_3 z_1 + z_3^3 z_4 z_2 + z_4^3 z_5 z_3 + z_5^3 z_1 z_4 \\ \psi_{-1} &= z_1^3 z_3 z_4 + z_2^3 z_4 z_5 + z_3^3 z_5 z_1 + z_4^3 z_1 z_2 + z_5^3 z_2 z_3 \\ \psi_0 &= z_1 z_2 z_3 z_4 z_5 \end{aligned} \tag{4.144}$$

Applying eq.(4.142) we see that the allowed Yukawa couplings are

$$\psi_0^3, \psi_0\psi_2\psi_{-2}, \psi_2\psi_{-1}\psi_{-1}, \\ \psi_{-2}\psi_1\psi_1, \psi_2\psi_2\psi_1, \psi_{-2}\psi_{-2}\psi_{-1} \tag{4.145}$$

but

$$\psi_0\psi_{-1}\psi_{-1} \tag{4.146}$$

is absent [40].

To compute the analogous quantities using the vertex representation of the fields we consider the three point function

$$A_{ijl}f(z,\bar{z}) = < \psi_v^i(z_1,\bar{z}_1)\psi_v^j(z_2,\bar{z}_2)\phi_0^l(z_3,\bar{z}_3) > \tag{4.147}$$

where ψ_v^i is the vertex operator for a fermion in the vector represntation of SO(10) and ϕ_0^l is the vertex operator for the scalar in the singlet representation of SO(10). $f(z,\bar{z})$ is the standard function fixed by SL(2,C) invariance

$$f(z,\bar{z})\prod_{i<j}(z_i - z_j)^{\Delta_k-\Delta_i-\Delta_j}(\bar{z}_i - \bar{z}_j)^{\overline{\Delta}_k-\overline{\Delta}_i-\overline{\Delta}_j} \tag{4.148}$$

We see that the non trivial values of A_{ijl} come from the correlation function of the spin field σ_n of the parafermionic system in terms of which the field (4.117) are written

$$\begin{aligned} A_{ijl} &= \prod_{r=1}^{5} \delta(n_r^i + n_r^j + n_r^l - 3)B(n_r^i, n_r^j, n_r^k) \\ B(n_r^i, n_r^j, n_r^k) &= < \sigma_{n_r^i}\sigma_{n_r^j}\sigma_{n_r^k} > \end{aligned} \tag{4.149}$$

The coefficient $B(i,j,l)$ for $i+j+l=3$ are given by

$$B(n_1,n_2,n_3) = C(n_1,n_2,n_1+n_2) \tag{4.150}$$

where $C(n_1,n_2,n_3)$ are given in eq.(3.188).

In our case the only non trivial structure constant that appears is

$$B(1,1,1) = k^2 = \frac{\Gamma(\frac{1}{5})\Gamma(\frac{3}{5})^3}{\Gamma(\frac{4}{5})\Gamma(\frac{2}{5})^3} \tag{4.151}$$

Hence the allowed Yukawa couplings are direct consequences of the FR's [60]

$$\begin{aligned} S_0^3 &\rightarrow K^5 \\ S_0S_2S_2^\dagger &\rightarrow K \\ S_2S_1^\dagger S_1^\dagger &\rightarrow K \\ S_2^\dagger S_1S_1 &\rightarrow K \\ S_2S_2S_1 &\rightarrow 1 \\ S_2^\dagger S_2^\dagger S_1^\dagger &\rightarrow 1 \end{aligned} \tag{4.152}$$

The others are absent. We see that the terms which appear coincide with those obtained for the corresponding Calabi-Yau non-linear sigma model but their values is different from those in eq.(4.145). Actually we can absorbed these value in the normalization of fields. In fact, normalizing

$$\begin{aligned} S_0 &\rightarrow K^{5/3} S_0 \\ S_2 &\rightarrow K^{-1/3} S_2 \\ S_2^\dagger &\rightarrow K^{-1/3} S_2^\dagger \\ S_1 &\rightarrow K^{2/3} S_1 \\ S_1^\dagger &\rightarrow K^{2/3} S_2^\dagger \end{aligned} \tag{4.153}$$

The field theory and string theory Yukawa couplings are seen to be identical. The same can be done for the case of 100 generations (namely 101 generations minus 1 anti-generation) and the non-triviality of the result consists in this case in the fact that there are fourteen non vanishing Yukawa couplings and five possible normalizations so these ones have to obey a consisten equation which is indeed satisfied [59].

The same analysis can be done for the more realistic case of three generation model [57,60,61]. Let start with the Calabi-Yau manifold described by the zeros of the following polynomials

$$\begin{aligned} &P_1 = z_0^3 + z_1^3 + z_2^3 + z_3^3 = 0 \\ &P_2 = z_1 x_1^3 + z_2 x_2^3 + z_3 x_3^3 = 0 \\ &\{z_i\} \in CP^3, \quad \{x_i\} \in CP^2 \end{aligned} \tag{4.154}$$

The Euler characteristic is $\chi = -54$. The discrete symmetries are

1. S_3 permutation group of the indices i=1,2,3

$$\begin{aligned} z_i &\rightarrow z_{p(i)} \\ x_i &\rightarrow x_{p(i)} \end{aligned} \tag{4.155}$$

2. $Z_3 \times Z_9^3$ given by

$$\begin{aligned} &z_0 \rightarrow e^{\frac{2\pi i r_0}{3}} z_0 \\ &z_i \rightarrow e^{\frac{2\pi i r_i}{3}} z_i \quad , \quad x_i \rightarrow e^{-\frac{2\pi i r_i}{9}} x_i \end{aligned} \tag{4.156}$$

A generic element of this group is denoted by $g_0 = (r_0, r_1, r_2, r_3)$. As before the irrelevance of an overall phase requires that the element $g_0 = (1,1,1,1)$ acts trivially then the full discrete automorphic group of this CY manifold is

$$G = S_3 \times Z_3 \times Z_3^9 / Z_9 \quad \text{ord G} = 1458 \tag{4.157}$$

One can count the number of generations finding the Hodge number $h^{2,1}$ which gives the number of deformations of complex structure. The possible deformations come as follows

Families	Polynomials ($\begin{matrix}\delta P_1\\ \delta P_2\end{matrix}$)	$Z_3 \times Z_9^3$ charges m	#
1	$\begin{pmatrix}0\\ z_0x_1^3\end{pmatrix}$	(1,6,0,0)	3
2	$\begin{pmatrix}0\\ z_0x_1^2x_2\end{pmatrix}$	(1,-2,-1,0)	6
3	$\begin{pmatrix}z_0z_1z_2\\ 0\end{pmatrix}$	(1,3,3,0)	3
4	$\begin{pmatrix}0\\ z_0x_1x_2x_3\end{pmatrix}$	(1,-1,-1,-1)	1
5	$\begin{pmatrix}0\\ z_1x_2^3\end{pmatrix}$	(0,3,6,0)	6
6	$\begin{pmatrix}0\\ z_1x_2^2x_1\end{pmatrix}$	(0,2,-2,0)	6
7	$\begin{pmatrix}0\\ z_1x_1x_2x_3\end{pmatrix}$	(0,2,-1,-1)	3
8	$\begin{pmatrix}0\\ z_1x_2^2x_3\end{pmatrix}$	(0,3,-2,-1)	6
9	$\begin{pmatrix}z_1z_2z_3\\ 0\end{pmatrix}$	(0,3,3,3)	1

(m_0, m_1, m_2, m_3) denotes the charges, i.e. under a $Z_3 \times Z_9^3$ transformation a component with charge m transforms as

$$v \rightarrow exp\{2\pi i\left(\frac{r_0m_0}{3} + \frac{r_1m_1 + r_2m_2 + r_3m_3}{9}\right)\} \tag{4.158}$$

Since $g = (1, 1, 1, 1)$ is equivalent to (0,0,0,0) we must have

$$3m_0 + m_1 + m_2 + m_3 = 0 \quad \mod 9 \tag{4.159}$$

The number of deformations is $h^{21} = 35$ and from

$$\chi = 2(h^{11} - h^{21})$$

we find $h^{11} = 8$.

So the hypersurface (4.154) describes a model with 27 families (35 generations minus 8 anti-generations).

There exists a model in the tensor product of N=2 minimal models which has the same number of families and enjoys the same automorphic symmetry group, namely $(p = 1) \times (p = 16)^3$ [57].

The discrete symmetries of the theory $1^1 16^3$ are easily determined. Since the p'th minimal model has a Z_{p+2} discrete symmetry, we have a $Z_3 \times Z_{18}^3$ symmetry. In addition we can permute the three identical $p = 16$ sub-theories then there is a S_3 permutation group. It is possible to see from the massless spectrum that only fields with even Z_{18} charges appear [57,61] and therefore the effective discrete group becomes $S_3 \times Z_3 \times Z_9$. Finally the condition of integrality for the U(1) charge of the composite fields implies that the element $g_0 = (1,1,1,1) \in Z_3 \times Z_9^3$ acts trivially. Therefore the symmetry group of the $1^1 16^3$ theory is

$$G = S_3 \times Z_3 \times Z_9^3 / Z_9 \tag{4.160}$$

We see that the theory $1^1 16^3$ has the same symmetry group of the hyrsurface (4.154).

Let us discuss first the generations. We present here the compactified part of the vertices omitting their free field (space-time and gauge) parts. The vertex operator for the space-time spinor component of the matter superfields S_a in $\underline{1}$ of $SO(10) \times U(1)$ is given by

$$V_{\underline{1}}(z,\bar{z}) = \left(R_k^k \prod_{i=1}^{3} R_{l_i}^{l_i} \right)(z) \left(N_{-m}^m \prod_{i=1}^{3} N_{-n_i}^{n_i} \right)(\bar{z}) \tag{4.161}$$

satisfying the following conditions [57,61]

1. $6k + \sum l_i = 18$ for having U(1) charge $q = -1/2$
2. $6m + \sum n_i = 36$ for having U(1) charge $q = -2$
3. $(m + k) \mod 3 = (n_i + l_i) \mod 9 \quad i = 1,2,3$
4. the left-right 2-D construction are restricted by the exceptional modular invariance at level k=16 of the affine part [81]

The corresponding spinor components of S_a in $\overline{16}_{-1/2}$ and $\underline{10}_1$ have a form similar to (4.161)

$$\begin{aligned} V_{\overline{16}}(z,\bar{z}) &= \left(R_1^1 (R_{16}^{16})^3 \right)(\bar{z}) V_1(z,\bar{z}) \\ V_{10}(z,\bar{z}) &= \left(R_1^1 (R_{16}^{16})^3 \right)(\bar{z}) V_{16}(z,\bar{z}) \end{aligned} \tag{4.162}$$

In order to compare the geometrical description of the model given by the table of the polynomial deformations with the algebraic one given by the construction of the tensor product $1^1 16^3$ we must make a correspondance between the vertices and the polynomials comparing their $Z_3 \times Z_9^3$ charges. As argued in [57] the charges $\{Q_k\}$ to be compared are those of the scalar $V_{\underline{10}}^{scalar}$ normalized as follows

$$m_0 = Q_0 \mod 3, \qquad m_i = 2Q_i \mod 9 \quad i = 1,2,3 \tag{4.163}$$

In the next table the nine "families" of polynomial deformations are represented by the spinor vertices and the relevant charges of $V_{\underline{10}}^{scalar}(z,\bar{z})$:

Family	$V_1(z,\bar{z})$	$Z_3 \times Z_9^3$ charges of $V_{\underline{10}}^{scalar}$
1	$R(1,12,0,0)N_-(0,4,16,16)$	(1,6,0,0)
2	$R(1,8,4,0)N_-(0,8,12,16)$	(1,-2,-1,0)
3	$R(1,6,6,0)N_-(0,10,10,16)$	(1,3,3,0)
4	$R(1,4,4,4)N_-(0,12,12,12)$	(1,-1,-1,-1)

5	$R(0,12,6,0)N_-(1,4,10,16)$	(0,3,6,0)
6	$R(0,10,8,0)N_-(1,6,8,16)$	(0,2,-2,0)
7	$R(0,10,4,4)N_-(1,6,12,12)$	(0,2,-1,-1)
8	$R(0,8,6,4)N_-(1,8,10.12)$	(0,3,-2,-1)
9	$R(0,6,6,6)N_-(1,10,10,10)$	(0,3,3,3)

where we have used the notation

$$R(0,6,6,6)N_\pm(1,10,10,10) \equiv (R_0^0(R_6^6)^3)(z)(N_\pm^1(N_{\pm 10}^{10})^3)(\bar{z})$$

etc. As before the corresponding Yukawa couplings for the superfields belonging to the $\overline{27}$ of E_6, i.e. $S^i S^j S^k$ coincide with these of $\psi^i_{\overline{16}}\psi^j_{\overline{16}}\varphi^k_{10}$. The latter can be expressed as products of the 2-D OPE structure constants of the N=2 fields R^k_k from the compactified part of the vertices:

$$\lambda_{ijk} = N_{ijk}\delta(Q_i^0 + Q_j^0 + Q_k^0 - 1)\prod_{r=1}^{3}\delta(Q_i^r + Q_j^r + Q_k^r - 16) \tag{4.164}$$

$$N_{ijk} = \prod_{l=1}^{4} < \sigma^i_{n_1^l}(\infty)\sigma^j_{n_2^l}(1)(\sigma^k_{n_1^l+n_2^l})^\dagger(0) >$$

The δ function part is a direct consequence of the N=2 fusion rules previously discussed. The nonvanishing Yukawa couplings are the following

$$< 994 >= k_1^3 \qquad < 973 >= k_1^2 =< 883 > \qquad < 882 >= k_1 k_2$$
$$< 884 >= k_1 k_2^2 \qquad < 872 >= k_2^2 \qquad < 862 >= k_2 \tag{4.165}$$
$$< 861 >=< 751 >=< 663 >=< 652 >= 1$$
$$< 852 >=< 753 >=< 554 >= k_1$$

(4.166)

where

$$k_1^2 = \frac{\Gamma(\frac{1}{18})\Gamma(\frac{13}{18})\Gamma^2(\frac{11}{18})}{\Gamma(\frac{17}{18})\Gamma(\frac{5}{18})\Gamma^2(\frac{7}{18})} \tag{4.167}$$

$$k_2^2 = \frac{\Gamma(\frac{1}{18})\Gamma^2(\frac{13}{18})}{\Gamma(\frac{17}{18})\Gamma^2(\frac{5}{18})}$$

To compare with the quasi-topological ones, which are zero or one, one has to show that it is possible to absorbe the constants (4.165) in the normalization of the corresponding polynomials. However, having 9 families and 14 non zero couplings the normalization has to safisfy a nontrivial consistency condition. It turns out that in this case this condition is satisfied and the proper normalization can be chosen in the form:

$$9^G = k_1^{4/3}k_2^{-2/3}\ 9^{CY} \qquad 5^G = k_1^{1/3}k_2^{-2/3}\ 5^{CY}$$
$$8^G = k_1^{1/3}k_2^{1/3}\ 8^{CY} \qquad 4^G = k_1^{1/3}k_2^{4/3}\ 4^{CY}$$
$$7^G = k_1^{-2/3}k_2^{4/3}\ 7^{CY} \qquad 3^G = k_1^{4/3}k_2^{-2/3}\ 3^{CY} \tag{4.168}$$
$$6^G = k_1^{-2/3}k_2^{1/3}\ 6^{CY} \qquad 2^G = k_1^{1/3}k_2^{1/3}\ 2^{CY}$$
$$1^G = k_1^{1/3}k_2^{-2/3}\ 1^{CY}$$

In the case of antigenerations the vertex representing the spinor component in $\underline{1}_2$ of $SO(10) \times U(1)$ can be taken in the form

$$\overline{V}_1(z,\bar{z}) == \left(R_k^k \prod_{i=1}^{3} R_{l_i}^{l_i} \right)(z) \left(N_m^m \prod_{i=1}^{3} N_{n_i}^{n_i} \right)(\bar{z}) \tag{4.169}$$

The conditions 1),2),4) for the vertex (4.161) remain the same but 3) becomes now

$$2') \quad m-k \mod 3 = n_i - l_i \mod 9, \quad i=1,2,3 \tag{4.170}$$

The connection between scalar and spinor components of the chiral superfield S_a^i is as above and the corresponding spinors $\overline{V}_{16}$ and $\overline{V}_{10}$ are realized in terms of $\overline{V}_1$ as follows:

$$\begin{aligned} \overline{V}_{16}(z,\bar{z}) &= \left(R_0^0 (R_0^0)^3 \right)(\bar{z}) \overline{V}_1(z,\bar{z}) \\ \overline{V}_{10}(z,\bar{z}) &= \left(R_0^0 (R_0^0)^3 \right)(\bar{z}) V_{16}(z,\bar{z}) \end{aligned} \tag{4.171}$$

All these constructions allow to represent the 8 anti-generations in the form [57]

Family	$\overline{V}_1(z,\bar{z})$	$Z_3 \times Z_9^3$ charges of $V_{\underline{10}}^{scalar}$	#
$\overline{1}$	$R(1,8,2,2)N_+(1,14,8,8)$	(2,6,3,3)	3
$\overline{2}$	$R(0,4,4,4)N_+(0,12,12,12)$	(0,0,0,0)	1
$\overline{3}$	$R(0,8,8,2)N_+(0,14,14,8)$	(1,6,6,3)	3
$\overline{4}$	$R(0,6,6,6)N_+(1,10,10,10)$	(0,0,0,0)	1

Applying the N=2 Fusion Rules the only two allowed Yukawa couplings for the above "families" of antigenerations are

$$\begin{aligned} &< \overline{442} >= k_1^3 \\ &< \overline{431} >= i k_3^3 \end{aligned} \tag{4.172}$$

where

$$k_3^2 = \frac{\Gamma(\frac{1}{18})\Gamma(\frac{11}{18})\Gamma(\frac{5}{6})}{\Gamma(\frac{17}{18})\Gamma(\frac{7}{18})\Gamma(\frac{1}{6})} \tag{4.173}$$

Up to now the discussion has been on the Yukawa couplings for the 27 generations CY models defined by (4.154) and the corresponding tensor product Gepner model. To get the three generation model we have to consider the $Z_3 \times Z_3$ subgroup of G generated by

$$\begin{aligned} h \ : \ & z_i \longrightarrow z_{i+1} \qquad i=1,2,3 \\ & z_{i+3} \equiv z_i \\ & \\ & x_i \longrightarrow x_{i+1} \\ & x_{i+3} \equiv x_i \end{aligned}$$

and by $g = (0,3,6,0)$. Factorizing respect to this subgroup we obtain a Calabi-Yau manifold with $\chi = -6$, then a three generation model (9 generations minus 6 anti-generations). The corresponding tensor product is denoted by

$$\frac{(p=1)\times(p=16)^3}{Z_3 \times Z_3}$$

The detailed analysis of this model is almost technical and it can be found in Refs.[57,61]. Here we give only the expression of the vertices of the spinor fields for the generations and antigenerations [60,61]

$$\begin{aligned}
L_1 &= R(1,12,0,0)N_-(0,4,16,16) + cycl.per.\\
L_2 &= R(1,6,6,0)N_-(0,10,10,16) + cycl.perm.\\
L_3 &= R(1,4,4,4)N_-(0,12,12,12)\\
L_4 &= R(0,12,6,0)N_-(1,4,10,16) + cycl.perm.\\
L_5 &= R(0,6,12,0)N_-(1,10,4,16) + cycl.perm.\\
L_6 &= R(0,10,4,4)N_-(1,6,12,12) + cycl.perm.\\
L_7 &= R(0,6,6,6)N_-(1,10,10,10)\\
L_8^t &= R(0,8,8,2)N_-(1,8,14,8) + cycl.perm.\\
L_9^t &= R(0,8,8,2)N_-(1,14,8,8) + cycl.perm.
\end{aligned} \tag{4.174}$$

and

$$\begin{aligned}
\overline{L}_1 &= R(1,8,2,2)N_+(1,14,8,8) + cycl.perm.\\
\overline{L}_2 &= R(1,4,4,4)N_+(0,12,12,12)\\
\overline{L}_3 &= R(0,8,8,2)N_+(0,14,14,8) + cycl.per.\\
\overline{L}_4 &= R(0,6,6,6)N_+(1,10,10,10)\\
\overline{L}_5^t &= R(0,12,0,6)N_+(1,4,16,10) + cycl.perm.\\
\overline{L}_6^t &= R(0,12,6,0)N_+(1,4,10,16) + cycl.perm.
\end{aligned} \tag{4.175}$$

The index t denotes twisted fields [61].

We note that these vertices correspond to the trivial embedding of the subgroup $H = Z_3 \times Z_3$ in E_6 which leaves it unbroken, i.e. the entire 27 ($\overline{27}$) contribute to the massless spectrum. One can identify properly quarks and leptons if one chooses a nontrivial embedding of the discrete subgroup in E_6: in this case only part of 27 ($\overline{27}$) survives the compactification. This problem is discussed in ref.[61] where E_6 is broken to $SU(3)_c \times SU(3)_L \times SU(3)_R$. In addition to the vertices written above there are quark and antiquark vertices

$$\begin{aligned}
Q_1 &= R(1,4,8,0)N_-(0,12,8,16) + cycl.perm.\\
Q_2 &= R(0,10,8,0)N_-(1,6,8,16) + cycl.perm.\\
Q_3 &= R(0,8,6,4)N_-(1,8,10,12) + cycl.perm.\\
\overline{Q}_1 &= R(1,8,4,0)N_-(0,8,12,16) + cycl.perm.\\
\overline{Q}_2 &= R(0,8,10,0)N_-(1,8,6,16) + cycl.perm.\\
\overline{Q}_3 &= R(0,6,8,4)N_-(1,10,8,12) + cycl.perm.
\end{aligned} \tag{4.176}$$

The explicit construction of the vertices together with the exact values of the Yukawa couplings lead to the following general expression for the low-energy cubic superpotential [60,61]

$$W = \sum_{ijk}[\lambda_{ijk}L_iL_jL_k + \overline{\lambda}_{ijk}\overline{L}_i\overline{L}_j\overline{L}_k + (\mu_{ijk}Q_iQ_jQ_k + h.c.) + \rho_{ijk}Q_i\overline{Q}_jL_k] \quad (4.177)$$

The coupling constants in the various sectors are

1. Leptons and Higgs bosons

   - generations

$$\lambda_{773} = k_1^3; \qquad \lambda_{641} = \lambda_{651} = 1$$

$$\lambda_{762} = k_1^2; \qquad \lambda_{652} = \lambda_{642} = \frac{1}{3}\lambda_{543} = k_1$$

   - antigenerations

$$\overline{\lambda}_{652} = 3k_1 \qquad \overline{\lambda}_{442} = k_1^3 \qquad \overline{\lambda}_{431} = ik_3^3$$

2. Quarks

$$\mu_{133} = \mu_{\overline{133}} = k_1k_2; \qquad \mu_{123} = \mu_{\overline{123}} = k_2$$

3. Quarks-antiquarks-Higgs bosons

$$\begin{aligned}
&\rho_{3\bar{3}2} = k_1^2 && \rho_{3\bar{3}3} = 3k_1k_2^2 \\
&\rho_{3\bar{1}5} = k_1 && \rho_{1\bar{3}4} = k_1 \\
&\rho_{3\bar{1}6} = k_2^2 && \rho_{1\bar{3}6} = k_2^2 \\
&\rho_{2\bar{3}1} = 1 && \rho_{\bar{2}31} = 1 \\
&\rho_{2\bar{2}2} = \rho_{1\bar{2}5} = \rho_{\bar{1}24} = 1
\end{aligned}$$

Specific features of this model are the absence of quark-antigenerations, few parameters of quark selcouplings and the absence of the Yukawa interactions for the twisted generations of leptons-Higgses L_8^t, L_9^t [61]

Conclusions

At the end of this work, looking back we have gained more familiarity with the conformal field theory in 2-dimensions, in particular with their supersymmetric extensions, but still we can say that it remains a largely unexplored universe. In these short notes we wish to summarize the principal results obtained and to give freely some items about open problems, regard the opinion of the author.

The first result concerns the description of the Ramond sector of the superconformal field theories and the corresponding computations of the important quantities as the N-point functions and the structure constants of the operatorial algebra. The key concept is the realization of the Ramond algebra and states in terms of modified Coulomb gas involving in so deep way the Ising fluctuation variables.

In our study of the N=2 superconformal minimal models different original results are scattered through the chapter 3; among these, the complete fusion rules in all the three sectors of theory, many explicit expressions of correlation functions of the fields involved, the discussion of the twisted sector and our attempts to classify the series of models arised in the limit points of the N=1 series ($c = 3/2$) and the N=2 series ($c = 3$) using additional symmetries present in these models.

More important is our definition of N=2 superparafermionic systems. The construction of the p'th model of the N=2 superconformal series in terms of Z_p parafermionic theory could suggest that the p'th model shares the same discrete symmetry of the underlying parafermionic structure: actually the discrete symmetry is more large, i.e. Z_{p+2}, so our aim was to understand its origin. How it is realized this new symmetry? How the primary fields organize themselves in the representations of this discrete abelian group? The answers to these questions led us to the discovery of N=2 superparafermionic systems, i.e. a conformal field theory defined by the OPE of field generators having the N=2 quantum numbers plus additional discrete charge under the Z_{p+2} group. The discrete infinite series of the N=2 superparafermions has a new important feature of Ramond as well as Neveu-Schwarz order parameters. These fields play an important role in our construction of the vertices of the massless states in the heterotic string compactification, which is the subject of our last chapter.

Our original contributions in the compactification problem are the expression of the vertices and the computation of the allowed Yukawa couplings of the low energy Lagrangian, in the case of four and three generations, using the fusion reules precedently found.

The computation of the four dimensional low energy parameters as the Yukawa couplings by using methods and results of the two dimensional conformal field theory is remarkable and it has the attractive feature to combine them with the geometrical beauty of the Calabi-Yau description. The analysis of all the phenomenological consequences of these constructions is in progress and now there is a keen interest on the subject.

About the open problems, the first consists in the unexplored correspondance of the N=1 and N=2 minimal models with exactly solvable statistical models, or in more general words, with the universality classes of the 2-dimensional critical phenomena. Even though an exact comparison was done for the minimal models of the Virasoro series (among these one discoveres the Ising model, the 3-state Potts model and their corresponding tricritical models and also the so called RSOS models) little progress is in the analogous identification of the supersymmetric minimal models. The only sure identification is the $c = 1$ model with the Gaussian one. The rich phase structure of this model requires an analysis involving also techniques of the renormalization group.

In this respect, one can apply the renormalization group methods and the celebrated *c-theorem* of Zamolodchikov to clarify also the whole structure of the minimal models involved in each series. This problem for the Virasoro as well as N=1 minimal models is still discussed in the literature. In our opinion it appears to be particular interesting to develop analogous analysis for the parafermionic system and consequntly for the N=2 superconformal theory. Without going in more details, the main reason to study the renormalization group flow for the parafermions consists in the presence of large number of quasi marginal operator having the clear physical interpretation of density energy operators which fix the thermal exponents for a self-dual critical point of the Z_p theory. How they mix? What is the flow directions? Such questions are still unanswered. For the N=2 minimal models the most important problem is to determine the flow directions which break the supersymmetry. We wish to stress that the starting point of these investigations is based on the knowledge of the structure constants of the corresponding operatorial algebra and this is sufficient to justify our efforts to compute them.

Beyond these specific topics, more serious open problem consists in the complete classification of all conformal field theories. As it was clarified in the introduction they are deeply related to the structure of vacuum in string theory. Is their number exausted by all possible symmetry principles? Are there some topological reasons which fixe their appearance? Perhaps these questions will find a solution in a next future. The physical picture as well as the mathematical structure involved give us a lively challenge to pursue the reserch in this direction.

Appendix A

Hypergeometric functions and their monodromy matrices

In order to construct the crossing symmetric 4-point functions of the R-fields $R^{(p)}_{12}(z), \overline{R}^{(p)}_{12}(\overline{z}), R^{(p)}_{2,1}(z)\overline{R}^{(p)}_{21}(\overline{z})$ we have to find the monodromy matrices for the 4-point 1-D functions $\{W_i\}$ given by eqs. (2.75),(2.78). Since W_i are expressed in terms of products of hypergeometric functions and root factors coming from the 4-point functions of the order-parameter field $\sigma(z)$, the monodromy matrices in these cases are products of the hypergeometric monodromy matrices.

It is technically simpler to introduce g_0-diagonal basis in the case of $G^p_{1,2}(z,\overline{z})$:

$$F_{1,2} = W_1 \pm W_3 \qquad F_{3,4} = W_2 \pm W_4$$

Using the well known monodromy matrices of the hypergeometric equation [103]

$$g_0 : \begin{cases} u_1 & \to u_1 \\ u_2 & \to e^{-2\pi i c} u_2 \end{cases} \qquad \begin{cases} u_1(z) & = F(a,b,c,z) \\ u_2(z) & = z^{1-c} F(a-c+1, b-c+1, 2-c, z) \end{cases}$$

$$\begin{aligned} (g_1)_{11} &= 1 - 2ie^{i\pi(c-a-b)} \frac{s(a)s(b)}{s(c)} \\ (g_1)_{12} &= 2i\pi e^{i\pi(c-a-b)} \frac{\Gamma(c)\Gamma(c-1)}{\Gamma(c-a)\Gamma(c-b)\Gamma(a)\Gamma(b)} \\ (g_1)_{21} &= 2i\pi e^{i\pi(c-a-b)} \frac{\Gamma(2-c)\Gamma(1-c)}{\Gamma(1-a)\Gamma(1-b)\Gamma(1+a-c)\Gamma(1+b-c)} \\ (g_1)_{22} &= 1 + 2ie^{i\pi(c-a-b)} \frac{s(a)s(c-b)}{s(c)} \end{aligned}$$

and g_∞ (which is not independent)

$$\begin{aligned} (g_\infty)_{11} &= \frac{s(b)s(c-a)e^{2\pi i a} - s(a)s(c-b)e^{2\pi i b}}{s(c)s(a-b)} \\ (g_\infty)_{12} &= \frac{\pi}{c-1} \frac{e^{2\pi i b} - e^{2\pi i a}}{s(a-b)} \frac{\Gamma^2(c)}{\Gamma(a)\Gamma(c-a)\Gamma(b)\Gamma(c-b)} \end{aligned}$$

$$(g_\infty)_{21} = \frac{\pi}{c-1}\frac{e^{2\pi ia}-e^{2\pi ib}}{s(a-b)}\frac{\Gamma^2(2-c)}{\Gamma(1-a)\Gamma(1+a-c)\Gamma(1-b)\Gamma(1+b-c)}$$

$$(g_\infty)_{22} = \frac{s(b)s(a)e^{2\pi ib}-s(a)s(c-b)e^{2\pi ia}}{s(c)s(a-b)}$$

we get the following expressions for the monodromy matrices corresponding to the function W_i (2.75) ($h \equiv \frac{1}{p}$):

$$g_0 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -y & 0 \\ 0 & 0 & 0 & y \end{pmatrix} \qquad y = e^{-4\pi ih}$$

$$g_1 = \begin{pmatrix} r & 0 & t & 0 \\ 0 & -r & 0 & -t \\ s & 0 & u & 0 \\ 0 & -s & 0 & -u \end{pmatrix}$$

where

$$r = 1 - ie^{-2\pi ih}tg(\pi h) \qquad t = 2ie^{-2\pi ih}\frac{s(3h)s(h)}{s(2h)}$$

$$s = -2ie^{-2\pi ih}\frac{s^2(h)}{s(2h)} \qquad u = 1-2ie^{-2\pi ih}\frac{s(3h)s(h)}{s(2h)}$$

and finally

$$g_\infty = \begin{pmatrix} 0 & a & 0 & c \\ a & 0 & c & 0 \\ 0 & b & 0 & d \\ b & 0 & d & 0 \end{pmatrix}$$

$$a = -\frac{s(h)s(3h)e^{2\pi ih}+s^2(h)e^{-2\pi ih}}{s^2(2h)}$$

$$b = 2i\frac{s^3(h)}{s^2(2h)}$$

$$c = -2is(3h)[\frac{s(h)}{s(2h)}]^2$$

$$d = -\frac{s(h)s(3h)e^{-2\pi ih}+s^2(h)e^{2\pi ih}}{s^2(2h)}$$

The invariance under g_0 restricts the matrix I_{ij} (in the diagonal basis) to be diagonal

$$I_{ij} = \begin{pmatrix} x_1 & 0 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & x_3 & 0 \\ 0 & 0 & 0 & x_4 \end{pmatrix}$$

Imposing invariance under the g_1 matrix we have the equations

$$\begin{array}{ll} \mid r \mid^2 x_1 + \mid s \mid^2 x_3 = x_1 & r\bar{t}x_1 + s\bar{u}x_3 = 0 \\ \mid r \mid^2 x_2 + \mid s \mid^2 x_4 = x_2 & r\bar{t}x_2 + s\bar{u}x_4 = 0 \\ \mid u \mid^2 x_3 + \mid t \mid^2 x_1 = x_3 & \bar{r}tx_1 + \bar{s}ux_3 = 0 \\ \mid u \mid^2 x_4 + \mid t \mid^2 x_2 = x_4 & \bar{r}tx_2 + \bar{s}ux_4 = 0 \end{array}$$

Consequently x_i are real and it holds the following consistency equations

$$\frac{x_1}{x_3} = \frac{x_2}{x_4} = A = \frac{\mid s \mid^2}{1- \mid r \mid^2} = \frac{1- \mid u \mid^2}{\mid t \mid^2} =$$
$$-\frac{s\bar{u}}{r\bar{t}} = \frac{\Gamma(3h)\Gamma(1-3h)}{\Gamma(h)\Gamma(1-h)} = \frac{1}{4\cos^2(\pi h) - 1}$$

Finally the last ratios

$$\frac{x_1}{x_2} = \frac{x_3}{x_4}$$

are fixed by the invariance under g_∞. The resulting equations are

$$\begin{array}{ll} \mid a \mid^2 Ax_4 + \mid b \mid^2 x_4 & = Ax_3 \\ \mid a \mid^2 Ax_3 + \mid b \mid^2 x_3 & = Ax_4 \\ \mid c \mid^2 Ax_3 + \mid d \mid^2 x_3 & = Ax_4 \\ \mid c \mid^2 Ax_4 + \mid d \mid^2 x_4 & = Ax_3 \\ \bar{a}cA + \bar{b}d = 0 & \end{array}$$

Thus we have

$$\frac{x_1}{x_2} = \frac{x_3}{x_4} = 1 \tag{A.1}$$

with the consistency conditions

$$A = \frac{\mid b \mid^2}{1- \mid a \mid^2} = \frac{1- \mid d \mid^2}{\mid c \mid^2} = -\frac{\bar{b}d}{\bar{a}c} \tag{A.2}$$

The direct inspection convince us that eq. (A.2) are satisfied identically. The analogous calculations in the case of $G^p_{2,1}(z,\bar{z})$ lead to the crossing symmetric function (2.86).

In the case of the mixed 4-point function $G^p_{(1,2)(2,1)}(z,\bar{z})$ it is sufficient to find the matrices g_0 and g_1 only. Since we work in g_0-diagonal basis, g_0 has the usual form while for g_1 we get:

$$r = 1 - 2ie^{-i\pi h}sin(\pi h), \qquad t = -2ie^{-i\pi h}sin(2\pi h), \qquad s = 0, \qquad u = 1$$

Then the condition of monodromy invariance (...) requires

$$x_1 = x_2 = 0 \qquad\qquad x_3 = x_4 \neq 0$$

and finally we get the monodromy invariant 4-point function (2.87).

Appendix B

Fermionic system

Let us consider together with the OPE algebra of the Majorana fermions ψ_1, ψ_2 (or in the complex basis $\psi = \frac{1}{\sqrt{2}}(\psi_1 + i\psi_2)$, $\overline{\psi} = \frac{1}{\sqrt{2}}(\psi_1 - i\psi_2)$):

$$\begin{aligned} \overline{\psi}(z_1)\psi(z_2) &= \frac{1}{z_{12}} - 2I(z_2) + z_{12}(T(z_2) - \partial I(z_2)) \\ \psi(z_1)\psi(z_2) &= \overline{\psi}(z_1)\overline{\psi}(z_2) = \mathcal{O}(z_{12}) \end{aligned} \qquad (B.1)$$

the corresponding OPE algebras of their stress-energy tensor $T(z) = \frac{1}{2} : (\psi\partial\psi + \psi\partial\psi) :$ and U(1)-current $I(z) =: \frac{1}{2}\psi\psi :$

$$\begin{aligned} T(z_1)T(z_2) &= \frac{1}{2z_{12}^4} + \frac{2}{z_{12}^2}T(z_2) + \frac{1}{z_{12}}\partial T(z_2) \\ I(z_1)I(z_2) &= \frac{1}{4z_{12}^2} + \cdots \\ T(z_1)\overline{\psi}(z_2) &= \frac{1}{2z_{12}^2}\overline{\psi}(z_2) + \frac{1}{z_{12}}\partial\overline{\psi}(z_2) + \cdots \\ I(z_1)\overline{\psi}(z_2) &= \pm\frac{1}{2z_{12}}\overline{\psi}(z_2) \end{aligned} \qquad (B.2)$$

In this normalization ψ and $\overline{\psi}$ have dimensions $\frac{1}{2}$ and charges $\pm\frac{1}{2}$. We have three possibilities to choose boundary conditions for $\psi_{1,2}$:

$$\begin{aligned} &\psi_{1,2}(e^{2\pi i}z) = \psi_{1,2}(z) && I(e^{2\pi i}z) = I(z) \\ &\psi_{1,2}(e^{2\pi i}z) = -\psi_{1,2}(z) && I(e^{2\pi i}z) = I(z) \\ &\psi_k(e^{2\pi i}z) = e^{i\pi(1-k)}\psi_k(z) && I(e^{2\pi i}z) = -I(z) \end{aligned}$$

and correspondingly we have the NS, R and twisted sector. Then the space of the LW states of the algebra (B.1)(B.2) splits into three sectors. Introduce the standard mode expansions of ψ and ψ for the NS and the R sectors:

$$\begin{aligned} \psi(z)\sigma_k(0) &= \sum_{n\in Z} z^{-n-k}\psi_{n+k-\frac{1}{2}}\sigma_k(0) \\ \overline{\psi}(z)\sigma_k(0) &= \sum_{n\in Z} z^{-n-k}\overline{\psi}_{n+k+\frac{1}{2}}\sigma_k(0) \end{aligned}$$

where $k = 0$ in the NS sector and $k = \frac{1}{2}$ in the R sector, and $\sigma_0(z)$ and $\sigma_{1/2}(z)$ are the fields representing LW states in the corresponding sectors. Then the OPE's (B.1)(B.2) lead to the well known canonical anticommutation relation algebra:

$$\{\psi_{n+k+\frac{1}{2}}, \overline{\psi}_{-m+k+\frac{1}{2}}\}\sigma_k = \delta_{n+m,0}\sigma \tag{B.3}$$

and to the Sugawara formulas:

$$\begin{aligned}(2I_n + k\delta_{n,0})\sigma_k &= \sum_{s=0}^{\infty}(\psi_{n-s-\frac{1}{2}+k}\overline{\psi}_{-s-\frac{1}{2}+k} - \overline{\psi}_{s+k+\frac{1}{2}}\psi_{n+s+k+\frac{1}{2}})\sigma_k \\ L_n + (n-1+2k)I_n &= \sum_{s=0}^{\infty}(s+1)\left(\overline{\psi}_{s+\frac{1}{2}+k}\psi_{n+s+\frac{1}{2}+k} - \psi_{n-s+k-\frac{3}{2}}\overline{\psi}_{-s+k-\frac{3}{2}}\right) + \\ &\quad \frac{1}{2}k(1-k)\delta_{n,0}\end{aligned} \tag{B.4}$$

where $I_n = \oint I(z)z^n dz$ and $L_n = \oint T(z)z^{n+1}dz$. Since the usual definition of LWR's of the algebra (B.1)(B.2) requires

$$\overline{\psi}_{-\alpha}\sigma_k = \psi_\alpha\sigma_k = 0 \qquad \alpha > 0, k = 0, \frac{1}{2} \tag{B.5}$$

we get (for the LW of the NS sector)

$$I_0\sigma_0 = 0$$

and therefore LW state is chargeless. In the R sector we have:

$$(I_0 + \frac{1}{4})\sigma_{\frac{1}{2}} = \frac{1}{2}\psi_0\overline{\psi}_0\sigma_{\frac{1}{2}} \tag{B.6}$$

Taking into account eq.(B.3) which gives

$$\overline{\psi}_0^2 = \psi_0^2 = 0 \qquad \overline{\psi}_0\psi_0 + \psi_0\overline{\psi}_0 = 1$$

we can realize ψ_0 and ψ_0 as follows

$$\psi_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \overline{\psi}_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

Then eq.(B.6) takes the form:

$$I_0\begin{pmatrix} \sigma^+ \\ \sigma^- \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} \sigma^+ \\ \sigma^- \end{pmatrix}$$

which means that LW states $\sigma^\pm$ have charges $\pm\frac{1}{4}$. Eqs.(B.4) give the dimensions for the LW states:

$$L_0\sigma_k = [(1-2k)I_0 + \frac{1}{2}k(1-k)]\sigma_k$$

i.e. $\Delta_0 = 0$ and therefore $\sigma_0 \sim [0]$ corresponding to the vacuum $\mid \Delta, q >=\mid 0, 0 >$. The LW states in the R sector have dimensions $\Delta^{\pm} = \frac{1}{8}$ (and charges $q^{\pm} = \pm\frac{1}{4}$). The mode expansions (B.3) together with the eq.(B.5) lead to the following OPE's of the R-fields $\sigma^{\pm}_{1/8}$:

$$\begin{aligned} \psi(z_1)\sigma^+(z_2) &= \frac{1}{\sqrt{z_{12}}}\sigma^-(z_2) + \sqrt{z_{12}}\psi_{-1}\sigma^+(z_2) + \cdots \\ \psi(z_1)\sigma^-(z_2) &= \overline{\psi}(z_1)\sigma^+(z_2) = \mathcal{O}(\sqrt{z_{12}}) \\ \overline{\psi}(z_1)\sigma^-(z_2) &= \frac{1}{\sqrt{z_{12}}}\sigma^+(z_2) + \sqrt{z_{12}}\overline{\psi}_1\sigma^-(z_2) + \cdots \end{aligned} \tag{B.7}$$

Following Friedan et al. [15] we can bosonize all these fields in terms of free dimensionless scalar field $\chi(z)$ (such that $< \chi(z_1)\chi(z_2) >= -ln z_{12}$):

$$\psi = e^{i\chi}, \qquad \overline{\psi} = e^{-i\chi}$$
$$\sigma^{\pm} = e^{\pm\frac{i}{2}\chi} \qquad I = \frac{i}{2}\partial\chi$$

Then using the Wick theorem we can easily calculate the n-point functions of the ψ, ψ and $\sigma^{\pm}$ which we use in the Ramond Coulomb gas representation in Sect.(3.3):

$$< \prod_i^n \sigma^{q_i}(z_i) >= \prod_{i<j}^n (z_{ij})^{\frac{1}{4}q_iq_j} \qquad q_i = \pm 1, \ \sum_{i=1}^n q_i = 0 \tag{B.8}$$

For example:

$$< \sigma^-(1)\sigma^+(2)\sigma^+(3)\sigma^-(4) >= (z_{13}z_{24})^{-\frac{1}{4}}(x(1-x))^{-\frac{1}{4}}\sqrt{1-x}$$

where

$$x = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad 1 - x = \frac{z_{23}z_{14}}{z_{13}z_{24}}$$

and finally:

$$\begin{aligned} &< \prod_{i=1}^4 \sigma^{q_i}(z_i) \prod_{l=1}^n \psi(u_l)\overline{\psi}(v_l) >= \prod_{i<j}^4 (z_{ij})^{\frac{1}{4}q_iq_j} \prod_{l,k=1}^4 (\frac{z_k - u_l}{z_k - v_l})^{\frac{q_i}{2}} \\ &\prod_{s<t=1}^n u_{st}v_{st} \prod_{p<r}^n (v_p - u_p)^{-1} \end{aligned} \tag{B.9}$$

One can use this formula to express eq.(3.102) in more explicit way. Now we consider the T-sector of the U(1) current $\partial\varphi_1$ algebra.Using a method due to Al. Zamolodchikov [99] we compute 4-point function of the lowest energy twisted state σ^t. In the presence of σ^t the mode expansion of $I(z)$ is

$$I(z) = \sum \frac{I_{n-\frac{1}{2}}}{z^{n+\frac{1}{2}}}$$

which gives the commutation relations

$$[I_{n-\frac{1}{2}}, I_{m+\frac{1}{2}}] = \frac{1}{4}(n - \frac{1}{2})\delta_{n+m,0} \tag{B.10}$$

and the Sugawara formula

$$L_0 = \frac{1}{16} + 2\sum_{m=1}^{\infty} : I_{-m+\frac{1}{2}} I_{m-\frac{1}{2}} :$$

$$L_n = \sum_{m=-\infty}^{\infty} I_{m-\frac{1}{2}} I_{n-m+\frac{1}{2}} \qquad n \neq 0 \tag{B.11}$$

It follows from eq. (B.11) that the lowest energy state of the T algebra has dimension $\Delta = \frac{1}{16}$. We denote it by σ_0^t. The twisted algebra has the property that the representations constructed by the lowest energy state σ_0^t is reducible with respect to the conformal algebra. The first example is given by the field

$$\sigma_1^t = 2I_{-\frac{1}{2}}\sigma_0^t$$

which is an invariant conformal field with dimension $\frac{9}{16}$. The analysis done in [100] shows that there exists a whole series of conformal fields $\sigma_k^t, k = 0, 1, \cdots$ with dimensions $\Delta_k = \frac{1}{16}(2k+1)^2$ which are conformal invariant. For the computation of the 4-point function of the field σ_0^t it is sufficient to consider only the field σ_1^t and the OPE

$$I(z_1)\sigma_0^t(0) = \frac{1}{2\sqrt{z}}\sigma_1^t(0)$$

$$I(z_1)\sigma_1^t(0) = \frac{1}{2z^{\frac{3}{2}}}\sigma_0^t(0) + \frac{2}{\sqrt{z}}\partial\sigma_0^t(0) + \cdots \tag{B.12}$$

Since the product of two fields $\sigma_0^t(z_1)\sigma_0^t(z_2)$ is single valued respect the current I(z), in their OPE appears the charge eigenvector V_q of the NS and R sector with the corresponding values of q. For example in the case when I(z) is built by $\psi_1, \psi_2 : \quad I = 2i\psi_1\psi_2$ we have

$$\sigma^t\sigma^t \sim I + \sigma_{1/8}^{\pm} + \psi$$

We consider the 4-point function of σ_0^t's field with defined charge in the 2-point channel:

$$G(x_1, \cdots x_4) = < \sigma_0^t(1)\sigma_0^t(2) \mid_q \sigma_0^t(3)\sigma_0^t(4) > \tag{B.13}$$

This implies

$$\oint_C \Gamma(z; \{x_i\})\frac{dz}{2\pi i} = qG(x_1, \cdots x_4) \tag{B.14}$$

where

$$\Gamma(z; \{x_i\}) = < I(z)\sigma_0^t(1)\sigma_0^t(2) \mid_q \sigma_0^t(3)\sigma_0^t(4) > \tag{B.15}$$

and C is a contour enclosing the points x_1 and x_2. We can fix as usual $x_1 = 0, x_3 = 1, x_4 = \infty$ and $x_2 \equiv x$. Using the analytic behaviour of the current I(z) in the presence of the field σ_0^t we have

$$\Gamma(z; \{x_i\}) = [z(z-x)(z-1)]^{-\frac{1}{2}}A(x)$$

and inserting into (B.14) we get

$$A(x)K(x) = 2\pi i q G(x) \tag{B.16}$$

where

$$K(x) = 2\int_0^1 [t(1-t)(1-xt)]^{-\frac{1}{2}} dt$$

is the complete elliptic integral of the first kind, or equivalently it is expressed in terms of hypergeometric function

$$K(x) = 2B(\frac{1}{2},\frac{1}{2})F(\frac{1}{2},\frac{1}{2},1;x)$$

Using the OPE (B.12) and eq.(B.16) one gets the following differential equation for $G(x)$ [99]

$$K^{3/2}\frac{d}{dx}[(x(1-x))^{1/8}K^{1/2}(x)G(x)] = -(2\pi q)^2(x(1-x))^{-7/8}G(x) \tag{B.17}$$

Since we have:

$$G(x) = (x(1-x))^{-1/8}K^{-1/2}(x)g(x)$$

eq.(B.17) reduces to the following differential equation for g(x)

$$\frac{dg}{dx} = \frac{(2\pi q)^2}{x(1-x)K^2(x)}g(x) \tag{B.18}$$

whose solution is

$$g(x) = exp(i\pi q^2\tau(x))$$

$$\tau = i\frac{K(1-x)}{K(x)} \tag{B.19}$$

The 4-point function of the σ_0^ψ field is

$$< \sigma_0^\psi(\infty) < \sigma_0^\psi(x) < \sigma_0^\psi(1) < \sigma_0^\psi(0) >= (x(1-x))^{-1/8}K^{-1/2}(x)\sum_q e^{i\pi q^2\tau(x)} \tag{B.20}$$

where $q = 0, \pm\frac{1}{4}, \pm\frac{1}{2}$.

Finally we want to point out the quantization of charge used to recover the fusion rules in the T sector by Coulomb gas method. This is a simple consequence of the OPE made on the expression of the 4-point function of the T-fields.

$$< \prod_{i=1}^4 T_{\Delta_n}(z_i) >\sim \oint dvdw < \prod_{i=1}^4 \sigma_0^\varphi(z_i)e^{\pm\frac{\alpha_-}{2}\varphi_2(v)\pm\frac{\alpha_-}{2}\varphi_2(w)}\partial\varphi_2 > + \cdots$$

In fact, considering the term with the σ_0^φ fields, we have an integer number of screening operators: taking the limits $z_1 \to z_2$ and $z_3 \to z_4$ in the OPE of the σ_0^φ fields appear the charge eigenvectors $e^{ip\varphi_2}$. The neutrality condition for the correlation functions of the exponential terms gives us the quantization of the charge eigenvalues p in terms of α_- charge, see eq.(3.121).

Appendix C

Calabi-Yau Manifolds

Here it is presented a brief review of the relevant geometry and topology of Calabi-Yau manifolds, useful in our considerations in chapter 4. For the understanding of this appendix it is enough to know some elementary facts about real algebraic geometry. For more extensive treatments on the whole subject, see [40,45].

A *Calabi-Yau* (CY) space is a compact n-dimensional complex manifold with Ricci-flat Kahler metric. In the case of superstring compactification $n = 3$, i.e. 6 real dimensions. Another definition of CY spaces is given in terms of holonomy group: a CY space is a complex space with $SU(n)$ holonomy group.

The aim of these notes is to clarify this terminology. We start with the definition of complex manifold: this is a real even dimensional real manifold on which is defined a *complex structure* by a tensor field J^μ_ν satisfying the condition

$$J^\mu_\rho J^\rho_\nu = -\delta^\mu_\nu \tag{C.1}$$

and the integrability constraint

$$N^k_{ij} = 0 \tag{C.2}$$

where N^k_{ij} (Nijenhuis tensor) is

$$N^k_{ij} = J^l_i(\partial_l J^k_j - \partial_j J^k_l) - J^l_j(\partial_l J^k_i - \partial_i J^k_l) \tag{C.3}$$

If we introduce complex coordinates

$$\begin{aligned} z_j &= x_j + i x_{j+n} \\ \overline{z}_j &= x_j - i x_{j+n} \end{aligned} \qquad j = 1, 2 \cdots n \tag{C.4}$$

the complex structure J takes its standard form

$$J^a_b = i\delta^a_b, \qquad J^{\overline{a}}_{\overline{b}} = -i\delta^{\overline{a}}_{\overline{b}} \tag{C.5}$$

other components zero and the transition functions between coordinate patches are holomorphic (antiholomorphic) ones. One should note that there are many complex structure possible on a manifold: some of them can be continously deform each other, but there is also the possibility to define inequivalent complex structures related to the moduli space of the complex manifold [46].

In general it is not true that a real even dimensional manifold is a complex one and it is a difficult mathematical problem to decide whether a real manifold is complex manifold or not. For example it is now known that the only sphere S^n which does admit a complex structure is S^2, the Riemann sphere or CP^1. The simplest example of a complex manifold is the flat C^n, whereas other interesting examples for our aims are CP^n or hypersurfaces in CP^n.

Important topological information about a complex manifold is contained in the complex cohomology group $H^{p,q}_{\overline{\partial}}(M,C)$ defined by

$$H^{p,q}_{\overline{\partial}} = \frac{\overline{\partial} \text{ closed (p,q) form}}{\overline{\partial} \text{ exact (p,q) form}} \tag{C.6}$$

where a (p,q) form is an antisymmetric tensor field on M with p holomorphic and q antiholomorphic indeces and the real exterior derivative d verifies

$$d = \frac{1}{2}(\partial + \overline{\partial}) \tag{C.7}$$

For each (p,q) $H^{p,q}_{\overline{\partial}}$ is a complex vector space.

The operator ∂ maps (p,q) forms into $(p+1,q)$ forms whereas $\overline{\partial}$ maps the space of (p,q) forms into the space of $(p,q+1)$ ones.

If the manifold M has a hermitian metric one can define the adjoints of the operator ∂ and $\overline{\partial}$ and the corresponding Laplacians

$$\begin{aligned} \Delta_{\partial} &= \partial\partial^{\dagger} + \partial^{\dagger}\partial \\ \Delta_{\overline{\partial}} &= \overline{\partial}\overline{\partial}^{\dagger} + \overline{\partial}^{\dagger}\overline{\partial} \end{aligned} \tag{C.8}$$

Then a consequence of the complex Hodge theorem is that there exists an unique harmonic form $\alpha_{p,q}$ which is a representative for each cohomology class $H^{p,q}_{\overline{\partial}}(m)$. Harmonic form means

$$\Delta_{\overline{\partial}}\alpha_{p,q} = 0 \tag{C.9}$$

The complex dimension of $H^{p,q}_{\overline{\partial}}(M,C)$ (i.e. the linear independent solutions of (C.9)) is $h^{p,q}$ (Hodge numbers). These numbers $h^{p,q}$ satisfy some important relations which we shall discuss after having introduce the Kahler manifolds. To define these manifolds we first note that on a complex hermitian manifold one can always consider the following 2-form (actually $(1,1)$ form) called Kahler form

$$K = -ig_{a\overline{b}}dz^a \wedge d\overline{z}^{\overline{b}} \tag{C.10}$$

K is real since g is hermitian.

If $dK = 0$, i.e. K is closed then g is a Kahler metric and M is called Kahler manifold.

One of the nice properties of Kahler manifold is that on them the complex structure J is covariantly constant with respect to the connection defined by the Kahler metric. Hence the Riemann structure is globally compatible with the complex structure.

There are a great many simplifications on Kahler manifolds. One of these is that all three Laplacians are equal

$$2\Delta_d = \Delta_\partial = \Delta_{\overline{\partial}} \tag{C.11}$$

A consequence of this is that a holomorphic p form ($\overline{\partial}\alpha = 0$) on a Kahler manifold is harmonic. Conversely, $\Delta_d \alpha = 0$ implies $\overline{\partial}\alpha = 0$ so every harmonic $(p, 0)$ form is holomorphic.

Now we can come back to the properties of the Hodge numbers. They are summarized in the following three points:

1. complex conjugation maps (p, q) form into (q, p) forms. Since $\Delta_\partial = \Delta_{\overline{\partial}}$ we know that ω is harmonic if and only if $\overline{\omega}$ is. Hence $h^{p,q} = h^{q,p}$.

2. the volume form is a (n, n) form. Since the dual of a harmonic form is also harmonic, $h^{p,q} = h^{n-p,n-q}$

3. one can put in correspondance the complex cohomology groups with the real ones. The dimension of the real cohomology space $H^p(M, R)$ (the vector space of the p real forms closed but not exact respect to the operator d) is given by the Betti number b_p. The connection between these cohomology groups is given by the following equation

$$b_n = \sum_{p+q=n} h^{p,q} \tag{C.12}$$

A convenient way to present the Hodge numbers is in terms of the *Hodge diamond* (drawn in the case n=3)

$$\begin{array}{ccccccc} & & & h^{33} & & & \\ & & h^{32} & & h^{23} & & \\ & h^{31} & & h^{22} & & h^{13} & \\ h^{30} & & h^{21} & & h^{12} & & h^{03} \\ & h^{20} & & h^{11} & & h^{02} & \\ & & h^{10} & & h^{01} & & \\ & & & h^{00} & & & \end{array}$$

This figure is symmetric about the verticle line and about the center point. The sum of the Hodge numbers along the n^{th} row is the n^{th} Betti number.

We finally come to Ricci-flat Kahler manifolds (Calabi-Yau). The Ricci tensor of a Kahler manifold can be expressed in terms of metric by

$$R_{j\overline{k}} = -\frac{\partial^2[ln\mathcal{D}et(g_{s\overline{t}})]}{\partial z_j \partial \overline{z}_k} \tag{C.13}$$

The Ricci form is given by

$$R = iR_{j\overline{k}} dz^j \wedge d\overline{z}^k = -i\partial\overline{\partial}[ln\mathcal{D}et(g_{s\overline{t}})] \tag{C.14}$$

It is easy to see that the real $(1,1)$ form R is closed, i.e.

$$dR = 0 \tag{C.15}$$

on a Kahler manifold. Then $R \in H^{1,1}(m)$ and this cohomology class depends only on the complex structure of M. What is especially significant here is that $H^{1,1}(m)$ is equal to the first Chern class c_1 of the manifold, i.e.

$$\begin{aligned} c_1 = & \ \frac{i}{2\pi} tr\Omega = \frac{1}{2\pi} R \\ \Omega = & \ R^i_{jk\bar{l}} dz^k \wedge d\bar{z}^{\bar{l}} \end{aligned} \tag{C.16}$$

It is important to note that although we have expressed c_1 in terms of local coordinates, the characteristic classes deal with the global, topological properties of the manifold. If $c_1 \neq 0$, i.e. the Ricci form is not exact, there cannot exist a Ricci flat metric. But if $c_1 = 0$ Calabi has shown the uniqueness [47] and Yau has proved the existence [48] of a metric which is Ricci-flat. An important point to stress is that this metric depending on the cohomology class of J and on the original complex structure chosen.

This theorem is a big improvement in one's ability to construct such manifold since it is easier to construct spaces with $c_1 = 0$ rather than to find metric with $R_{j\bar{k}} = 0$. Indeed it is still not known how to write such a metric in closed form. CY manifolds are very difficult spaces to visualize: they are Ricci-flat but not actually flat and they have no continous isometries, i.e. there are no Killing vectors.

Of use in constructing CY manifolds is the fact that $c_1 = 0$ is equivalent to the existence of a nowhere vanishing holomorphic $(n, 0)$ form. This leads to an additional symmetry on the Hodge numbers, namely $h^{p,0} = h^{0,n-p}$. For our $n = 3$ case, this reduces the six independent Hodge numbers we had for a Kahler manifold to only four for a CY manifold. It is convenient to take $h^{0,0}, h^{1,0}, h^{1,1}, h^{2,1}$. However $h^{0,0}$ is the dimension of the space of constant functions, so $h^{0,0} = 1$. Furthemore, it turns out that for Euler characteristic χ to be non-zero $h^{1,0}$ must be zero, $h^{1,0} = 0$.

Hence we are left with $h^{1,1}$ and $h^{2,1}$ as the only non trivial Hodge numbers which characterize a CY with $\chi \neq 0$ and the Euler characteristic is given by

$$\chi = \sum_{k=0}^{n} (-1)^k b_k = \sum_{k=0}^{n} \sum_{p+q=k} (-1)^k h^{p,q} = 2(h^{1,1} - h^{1,2}) \tag{C.17}$$

It can be shown that $h^{1,1}$ is the number of real parameters which fix the radius and 'shape' of CY. Also $h^{2,1}$ is the complex dimension of the space of complex structures which can be placed on M. Physically they are related to the number of generations of the fermion massless families.

Many of the examples of CY manifolds are based on algebraic varieties. Let us consider CP^n. Although these spaces are Kahler manifolds, it turns out that $h^{n,0} = 0$, so they have no holomorphic n-form and then $c_1 \neq 0$. But we can consider algebraic varieties defined by intersection of homogeneous polynomials $P_1, \cdots P_j$

with degree k_i:

$$P_1(z_1, \cdots z_n) \quad = 0$$
$$\cdots$$
$$P_j(z_1, \cdots z_n) \quad = 0$$

There is a simple criterium to define in such a way a CY manifold

$$\sum_{i=1}^{j} k_j = n + 1 \tag{C.18}$$

A celebrated example is the space called $Y_{4,5}$, the submanifold of CP^4 defined by

$$Q = \{Z = (z_1, \cdots z_5) : \sum_{i=1}^{5} z_i^5 = 0\} \tag{C.19}$$

or consider $CP^3 \times CP^3 = \{x_1, \cdots x_4\} \times \{y_1, \cdots y_4\}$ and the algebraic curve defined by

$$\begin{aligned} \sum_{i=1}^{4} x_i^3 = \sum_{i=1}^{4} y_i^3 = 0 \\ \sum_{i=1}^{4} x_i y_i = 0 \end{aligned} \tag{C.20}$$

As a sample calculation we compute the Euler characteristic of the $Y_{4,5}$ manifold determining the Hodge numbers $h^{2,1}$ and $h^{1,1}$. On Q there is precisely one harmonic $(1,1)$ form, namely the Kahler form, so $h^{1,1} = 1$. The elements of $H^{2,1}$ correspond to perturturbation of the quintic polynomial P that cannot be absorbed in linear changes of coordinates, since the choice of P determines the complex structure of Q

$$P \to P + \delta P \tag{C.21}$$

and δP must be at most cubic in any one z_k. They are

$$\begin{array}{rr} z_1^3 z_2^2 + permutations & (20) \\ z_1^3 z_2 z_3 + permutations & (30) \\ z_1^2 z_2^2 z_3 + permutations & (30) \\ z_1^2 z_2 z_3 z_4 + permutations & (20) \\ z_1 z_2 z_3 z_4 z_5 & (1) \end{array}$$

Adding the numbers of different possibilities (in parenthesis) we see that the complex stuctures of Q depends on 101 complex parameters, so

$$\chi = 2(h^{1,1} - h^{2,1}) = -200 \tag{C.22}$$

To obtain realistic number of massless fermions one can reduce this number by dividing out by discrete symmetry groups which act freely on Q. In the case of $Y_{4,5}$ two such symmetries are

$$\begin{aligned} A \ &: \ z_i \rightarrow z_{i+1} \\ & \quad\ z_{i+5} \equiv z_i \\ B \ &: \ z_k \rightarrow \alpha^k z_k \\ & \quad\ \alpha = e^{\frac{2\pi i}{5}} \end{aligned} \tag{C.23}$$

Since $A^5 = B^5 = 1$ they are isomorphic to Z_5. It can be shown that they act freely on Q so the Euler characteristic of the manifold obtained by $Y_{4,5}$ with the identification of the points by A or B is

$$\tilde{\chi} = \frac{\chi}{ord(Z_5 \times Z_5)} = -\frac{200}{25} = -8 \tag{C.24}$$

This is a model for a four generation theory.

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