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Multidimensional Poisson Vertex Algebras and Poisson cohomology of Hamiltonian structures of hydrodynamic type

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"Ma gli elettròlogi non ne vollero sapere d'una simile ipotesi, e sfoderarono delle equazioni differenziali: che pervennero anche ad integrare, con quale gioia del cav. Bertoloni si può presumere."

C. E. Gadda, La cognizione del dolore, 1963[1]

[1]see also G. Falqui, *Moduli spaces and geometrical aspects of two–dimensional conformal field theory.* PhD Thesis, Trieste 1990

Abstract

The Poisson brackets of hydrodynamic type, also called Dubrovin–Novikov brackets, constitute the Hamiltonian structure of a broad class of evolutionary PDEs, that are ubiquitous in the theory of Integrable Systems, ranging from Hopf equation to the principal hierarchy of a Frobenius manifold. They can be regarded as an analogue of the classical Poisson brackets, defined on an infinite dimensional space of maps $\Sigma \to M$ between two manifolds. Our main problem is the study of Poisson–Lichnerowicz cohomology of such space when $\dim \Sigma > 1$. We introduce the notion of multidimensional Poisson Vertex Algebras, generalizing and adapting the theory by A. Barakat, A. De Sole, and V. Kac [5]; within this framework we explicitly compute the first nontrivial cohomology groups for an arbitrary Poisson bracket of hydrodynamic type, in the case $\dim \Sigma = \dim M = 2$. For the case of the so-called scalar brackets, namely the ones for which $\dim M = 1$, we give a complete description on their Poisson–Lichnerowicz cohomology. From this computations it follows, already in the particular case $\dim \Sigma = 2$, that the cohomology is infinite dimensional.

Contents

Introduction

Evolutionary PDEs and Poisson brackets of hydrodynamic type

Evolutionary Partial Differential Equations are a broad class of equations

$$\frac{\partial u(x,t)}{\partial t} = A(u, \partial_\alpha u, \partial_{\alpha\beta} u, \ldots) \tag{1}$$

for smooth maps $u\colon \Sigma \times \mathbb{R} \to M$, $\dim \Sigma = D$, $\dim M = N$. $\partial_\alpha u$ is a shorthand notation for $\frac{\partial u(x,t)}{\partial x^\alpha}$, $\partial_{\alpha\beta} u$ for $\frac{\partial^2 u(x,t)}{\partial x^\alpha \partial x^\beta}$, etc. Such systems are usually called $(D+1)$-dimensional evolutionary PDEs, where D is the number of the (so-called) *space variables*. We regard evolutionary PDEs as the equations of motion for infinite dimensional dynamical systems.

The general framework for the study of these classes of equations under the algebraic and geometric point of view is the so-called *formal calculus of variations* introduced by Gel'fand and Dikii [35]. Their work provided a unifying language to understand the wide variety of infinite dimensional *integrable* systems that had been discovered since the late 60's, such as KdV and NLS equations with their hierarchies.

Consider the space of smooth maps $\mathcal{M} := \mathrm{Maps}(\Sigma \to M)$, that we regard as the configuration space of the dynamical system (1). Σ is a compact D-dimensional smooth manifold; we fix Σ to be a torus $(S^1)^D = T^D$ to avoid the problems arising from the integration.

Given a local chart $\{x^\alpha\}_{\alpha=1}^D$ on Σ and a chart $\{u^i\}_{i=1}^N$ on M, we represent a map $u(x) \in \mathcal{M}$ as a set of N functions in D variables. We borrow from the physics literature the name *fields* for the functions $u^i(x)$, as well as the nickname *source* and *target* manifolds for, respectively, Σ and M.

To endow the space $\mathcal{M}$ with the geometrical structures used to describe evolutionary PDEs, let us introduce the *jet coordinates*. Define the symbols $\{u^i_I\}$, $I \in \mathbb{Z}^D_{\geq 0}$, $i = 1, \ldots, N$, where $I = (i_1, i_2, \ldots, i_D)$ and

$$\begin{aligned} \boldsymbol{\partial}^I &:= \left(\frac{\partial}{\partial x^1}\right)^{i_1} \left(\frac{\partial}{\partial x^2}\right)^{i_2} \cdots \left(\frac{\partial}{\partial x^D}\right)^{i_D} \\ u^i_I &= \boldsymbol{\partial}^I u^i \qquad \text{for } I \neq \mathbf{0} := (0, 0, \ldots, 0). \end{aligned}$$

We call *generators* the dependent variables $\{u^i\} = \{u^i_{\mathbf{0}}\}$.

Definition 0.1 (Differential polynomial)**.** Let $U \subset M$ be a chart on M with coordinates $(u^1, \ldots, u^N)$. The space of *differential polynomials* $\mathcal{A} = \mathcal{A}(U)$ is the ring of polynomials in the jet variables $\{u^i_I\}$ with coefficients in $C^\infty(U)$, for $i = 1, \ldots, N$ and $I \in \mathbb{Z}^D_+$. A generic differential polynomial has the form

$$f(u; u_I) := \sum_{m\geq 0} f_{i_1 I_1;\ldots;i_m I_m}(u) u^{i_1}_{I_1} \ldots u^{i_m}_{I_m}. \tag{2}$$

We define on $\mathcal{A}$ a family of D commuting derivations $\{\partial_\alpha\}$, which correspond to the total derivatives with respect to each independent variable $\{x^\alpha\}$. According to the chain rule, they are of the form

$$\partial_\alpha f = \sum_{\substack{i=1,\ldots,N \\ I\in\mathbb{Z}^D_{\geq 0}}} u^i_{I+\xi_\alpha} \frac{\partial f}{\partial u^i_I}, \tag{3}$$

where we denote $\xi_\alpha = (0, \ldots, 0, \underbrace{1}_{\alpha}, 0, \ldots)$ the canonical basis vectors in $\mathbb{Z}^D$.

Definition 0.2 (Local functional)**.** A *local functional* is an element of the quotient space

$$\mathcal{F} := \frac{\mathcal{A}}{\partial_1\mathcal{A} + \partial_2\mathcal{A} + \cdots + \partial_D\mathcal{A}}. \tag{4}$$

We can regard a local functional $F[u] \in \mathcal{F}$, whose *density* is $f \in \mathcal{A}$, as the integral over Σ of the differential polynomial f. The idea is that a local functional is zero if and only if its density is a total derivative: this means that we are disregarding the boundary terms. According to this interpretation, we denote the projection operation $\mathcal{A} \to \mathcal{F}$ as $f \mapsto F = \int^D f \, \mathrm{d}^D x$. When there is no ambiguity about the dimension of the source manifold, we will often use the shorthand notation $F = \int f$.

We endow the space $\mathcal{F}$ with a Lie bracket $\{\cdot, \cdot\}$, which we call the *local Poisson bracket*, to be interpreted in analogy to the Poisson bracket used in classical mechanics. Such a bracket is defined in terms of a matrix-valued differential operator $\mathscr{P} = \left(P^{ij}\right)$, $i, j = 1, \ldots, N$

$$P^{ij} = \sum_{S\in\mathbb{Z}^D_{\geq 0}} P^{ij}_S(u; u_I)\boldsymbol{\partial}^S, \qquad\qquad P^{ij}_K \in \mathcal{A},$$

according to the formula

$$\{F, G\} := \int^D \frac{\delta F}{\delta u^i} P^{ij} \frac{\delta G}{\delta u^j} \mathrm{d}^D x. \tag{5}$$

Here and in the subsequent occurrences, sum over repeated indices is assumed unless otherwise stated. In (5), $\frac{\delta}{\delta u^i}$ is the standard variational derivative

$\sum_I (-1)^{|I|}\boldsymbol{\partial}^I \frac{\partial}{\partial u_I^i}$. In order for $\{\cdot,\cdot\}$ to be a Lie bracket, $\mathscr{P}$ must be a skewsymmetric operator and satisfy certain closure conditions imposed by the Jacobi identity; we call such an operator a *Hamiltonian operator* on $\mathcal{A}$.

Among the evolutionary PDEs we define the subclass constituted by *Hamiltonian PDEs*, namely the ones that can be written as

$$\frac{\partial u(x,t)}{\partial t} = \mathscr{P}\frac{\delta H}{\delta u}, \qquad\qquad H := \int^D h \, \mathrm{d}^D x \in \mathcal{F}, \tag{6}$$

with $\mathscr{P}$ a Hamiltonian operator. They are the infinite dimensional analogue of Hamiltonian systems in classical mechanics: such systems are characterized by the property of being conservative, and in particular they are the only ones for which a quantization can be defined.

The Poisson bracket (5) endows the space of local functionals $\mathcal{F}$ with the structure of a Lie bracket, in the same way as the Poisson brackets in classical mechanics does with the space of observables $C^\infty(M)$, where M is the phase space. The major difference is that the space of local functionals is not an algebra, hence the Poisson bracket in (5) is not a derivation, as opposed to the the finite dimensional setting.

Definition 0.3 (Grading on $\mathcal{A}$)**.** Let $\mathcal{A}$ be an algebra of differential polynomials. The *degree* of a differential polynomial f counts the number of derivatives acting on the monomials, according to the following basic grading for the generators and the jet variables

$$\deg u^i = 0 \qquad\qquad \deg u_I^i = |I| := i_1 + i_2 + \cdots + i_D.$$

We denote $\mathcal{A}_d$ the homogeneous component of $\mathcal{A}$ of degree d.

The grading on $\mathcal{A}$ can be used to define a grading for the operators. We say that the operator $\mathscr{Q}$ is a homogeneous operator of degree d if, for any homogeneous element $f \in \mathcal{A}$, $\deg \mathscr{Q}(f) = \deg f + k$. From the formula (3), it follows that the total derivatives ∂_α are homogeneous operators of degree 1.

An important class of local Poisson brackets are the *Poisson brackets of hydrodynamic type*, defined by Dubrovin and Novikov [23]. They are the Poisson brackets for which the Hamiltonian operator is homogeneous of degree 1. The first Hamiltonian structure of the KdV $P = \mathrm{d}/\mathrm{d}x$, as well as the one of the principal hierarchy of a Frobenius manifold [22], are just a few examples of such Poisson brackets. For $(1+1)$-dimensional systems, as the aforementioned ones, the Poisson brackets of hydrodynamic type have a clear geometric interpretation. A degree 1 differential operator

$$P^{ij} = g^{ij}(u)\frac{\mathrm{d}}{\mathrm{d}x} + b_k^{ij}(u)u_x^k \tag{7}$$

with $\det g \neq 0$ is a Hamiltonian structure if and only if the symmetric matrix g is a contravariant flat metric on the manifold M and b_k^{ij} are the corresponding contravariant Christoffel symbols of the Levi-Civita connection. This property implies that there exists a coordinate system $\{v^i(x)\}_{i=1}^N$ on M such that the Hamiltonian structure is constant; those coordinates are called the *flat coordinates* of the bracket.

Hydrodynamic Poisson brackets for maps $u(x)$, $x = \{x^1, \dots, x^D\}$ have been studied for several years [24, 43]. They are defined by a family of D matrix-valued functions $g^{ij\alpha}(u(x))$ and $b_k^{ij\alpha}(u(x))$; in term of a differential operator, they have the form

$$P^{ij} = \sum_{\alpha=1}^{D} g^{ij\alpha}(u)\frac{\mathrm{d}}{\mathrm{d}x^\alpha} + b_k^{ij\alpha}(u)u^k_{\xi_\alpha} \tag{8}$$

The main difference between the multidimensional brackets of hydrodynamic type and the one-dimensional ones is that, althought each metric g^α is flat (under the assumption $\det g^\alpha \neq 0$), there do not necessarily exist common flat coordinates for them: each metric becomes constant, in general, in a different system of coordinates $\{u^i\}_{i=1}^N$.

The search for a normal form of the Hamiltonian operators on an infinite dimensional manifold such as $\mathcal{M}$ is a formidable task for which only partial results are known even in the $D = 1$ case. We decided to consider the problem of deformations of the Hamiltonian operator of hydrodynamic type. This problem has been independently solved for $D = 1$ structures by several authors [36, 20, 25].

Let us briefly recall the setting of the problem. We extend the space $\mathcal{A}$ to $\tilde{\mathcal{A}} \subset \mathcal{A} \otimes \mathbb{R}[[\epsilon]]$. The formal indeterminate ϵ has degree -1, and an element of $\tilde{\mathcal{A}}$ is a formal infinite series

$$f(u; u_I; \epsilon) = \sum_{k=0}^{\infty} \epsilon^k f_k(u; u_I), \qquad f_k \in \mathcal{A}_k.$$

The same construction applies to $\mathcal{F}$ and give us $\tilde{\mathcal{F}}$; moreover, we define matrix-valued differential operators $\mathscr{P}$ acting on $\tilde{\mathcal{A}}$, that are of the form $\mathscr{P} = P_S^{ij}\partial^S$ with $P_S^{ij} \in \tilde{\mathcal{A}}$.

Let us consider the transformations

$$u^i \mapsto \tilde{u}^i = u^i + \sum_{k=1}^{\infty} \epsilon^k F_k^i(u; u_I), \qquad i = 1, \dots, N \tag{9}$$

on the space $\tilde{\mathcal{A}}$, where $F_k^i \in \mathcal{A}_k$. The transformations (9) form a subgroup in the *Miura group* [25] and they can be regarded as local diffeomorphisms on the infinite dimensional manifold.

The components of a differential operator $\mathscr{P} = (P)^{ij}$ change under the transformation law $\{u^i\} \mapsto \{\tilde{u}^i\}$. We define the linearized action of the Miura group on $\mathcal{A}$, also called the Fréchet derivative in this context [20], as the $N \times N$ differential operator

$$L^i_k := \frac{\partial \tilde{u}^i}{\partial u^k_S} \boldsymbol{\partial}^S.$$

and its adjoint as

$$L^{i*}_k := (-\boldsymbol{\partial})^S \circ \frac{\partial \tilde{u}^i}{\partial u^k_S}.$$

The components of $\mathscr{P}$ transform according to the rule

$$\tilde{P}^{ij} = L^i_l P^{lm} L^{j*}_m.$$

Definition 0.4. Given a Hamiltonian operator of hydrodynamic type $P^{ij}_{[0]}$, a n-th order infinitesimal *compatible deformation* of it is a skewsymmetric differential operator

$$P^{ij} = P^{ij}_{[0]} + \sum_{k=1}^{n} \epsilon^k P^{ij}_{[k]}$$

such that the bracket it defines by Equation (5) satisfies Jacobi identity up to terms in ϵ^{n+1}, and that the differential operators $P^{ij}_{[k]}(\partial)$ are homogeneous of degree $k+1$.

Definition 0.5. A deformation P^{ij} of $P^{ij}_{[0]}$ is said to be trivial if there exists an element of the Miura group such that

$$L^l_i P^{ij}_{[0]} L^{m*}_j = P^{lm} + O(\epsilon^{n+1})$$

for a deformed Hamiltonian operator of order n.

The theory of deformations of Poisson brackets of hydrodynamic type naturally fits within the study of the Poisson–Lichnerowicz cohomology they define. The Poisson–Lichnerowicz cohomology is the cohomology of a cochain complex of local multivectors, whose differential is defined by means of a natural extension of the brackets itself.

Definition 0.6 (Local multivectors)**.** A *local k-vector A* is a linear k-alternating map from $\mathcal{F} \times \mathcal{F} \times \cdots \times \mathcal{F}$ (k times) to $\mathcal{F}$ of the form

$$A(F_1, \dots, F_k) = \int^D A^{i_1,\dots,i_k}_{I_1,\dots,I_k} \partial^{I_1} \left(\frac{\delta F_1}{\delta u^{i_1}} \right) \cdots \partial^{I_k} \left(\frac{\delta F_k}{\delta u^{i_k}} \right) \, d^D x \qquad (10)$$

where $A^{i_1,\dots,i_k}_{I_1,\dots,I_k} \in \mathcal{A}$, for arbitrary $F_1, \dots, F_p \in \mathcal{F}$. We denote the space of local k-vectors by Λ^k.

In particular, an element of Λ^1 is a map $\mathcal{F} \to \mathcal{F}$ of the form

$$A(F) = \int^D A^i(u; u_L) \frac{\delta F}{\delta u^i} \mathrm{d}^D x = \int^D \tilde{A}^i_I(u; u_L) \frac{\partial F}{\partial u^i_I} \mathrm{d}^D x, \tag{11}$$

where the second identity is obtained integrating by parts and letting $\tilde{A}^i_I = \partial^I A^i$. We call an element of Λ^1 a *local vector field*. The commutator of two vector fields A and B is given by a vector field $\int [A, B]^i_I \partial F / \partial u^i_I \, \mathrm{d}^D x$ with

$$[A, B]^i_I := A^j_J \frac{\partial B^i_I}{\partial u^j_J} - B^j_J \frac{\partial A^i_I}{\partial u^j_J}. \tag{12}$$

On the space of local multivectors we can define an extension of the commutator of vector fields which is called the *Schouten bracket*. The Schouten bracket is a bilinear operation $\Lambda^l \times \Lambda^k \to \Lambda^{l+k-1}$ such that it coincides with the commutator of local vector fields for $l = k = 1$ and fulfills a graded Leibniz property for their exterior product $[A, X \wedge Y] = [A, X] \wedge Y + (-1)^{(a-1)x} X \wedge [A, Y]$ for $A \in \Lambda^a$ and $X \in \Lambda^x$.

Poisson brackets are defined by means of local bivectors, with $P^{ij}(\partial) = A^{ij}_{0,S} \partial^S$. Local bivectors whose density is a Hamiltonian operators are also called *Poisson bivectors*. They are characterized by the vanishing of the so-called *Schouten torsion* $[P, P] = 0$.

Lemma 0.1 (Poisson differential)**.** *Let $P = P^{ij}_S \boldsymbol{\partial}^S$ be a Hamiltonian operator. Then the adjoint action of P on Λ^p*

$$\begin{aligned} ad_P \colon \Lambda^p &\to \Lambda^{p+1} \\ A &\mapsto [P, A] \end{aligned}$$

squares to 0. We denote $ad_P =: \mathrm{d}_P$ and call it the Poisson differential.

Definition 0.7 (Poisson–Lichnerowicz cohomology)**.** Let $\Lambda^\bullet = \oplus_{p=0}^\infty$ be the space of local multivector fields. The Poisson differential defines the *Lichnerowicz cochain complex*

$$0 \to \Lambda^0 = \mathcal{F} \xrightarrow{\mathrm{d}_P} \Lambda^1 \xrightarrow{\mathrm{d}_P} \Lambda^2 \xrightarrow{\mathrm{d}_P} \cdots \xrightarrow{\mathrm{d}_P} \cdots$$

The cohomology of this complex is the *Poisson–Lichnerowicz cohomology* of P.

Getzler [36] proved that the Poisson–Lichnerowicz cohomology for a $D = 1$ constant Poisson bracket of hydrodynamic type is trivial, hence in particular the second cohomology group (as directly proved also in [20] and [25]) vanishes. The vanishing of the second cohomology group implies that all the deformations of a given Poisson bracket of hydrodynamic type are trivial. Then, relying on the Dubrovin and Novikov's result [23], for a certain coordinate system they must be trivial deformations of a *constant* Poisson bracket.

Such a system of coordinates can be regarded as the Darboux coordinates for the Poisson structure.

It must be stressed that this analogue of the Darboux theorem has been proved in the one-dimensional case *only.* The research work by E. Ferapontov and collaborators has produced a big outcome in the direction of classifying the integrable Hamiltonian equations of hydrodynamic type with D spatial variables and their deformations (for instance, in [31, 30]); such equations are Hamiltonian with respect to one of the Poisson brackets of hydrodynamic type. Moreover, several authors have studied the classification problem for degree 3 Hamiltonian operators, both for the $D = 1$ case [49, 3] and for $D > 1$ [28, 32]. The study of the lower cohomology groups, in particular of H^2, allows us to work towards a classification of higher degree Hamiltonian operators in terms of some operator of hydrodynamic type they are equivalent to. For both the cases $(D = 2, N = 2)$ and $(D > 1, N = 1)$ the cohomology is far from being trivial, as opposed to the $D = 1$ case studied in [36, 20, 25]; the corresponding Hamiltonian operators of higher degree, and in particular of degree 3, are not equivalent to the hydrodynamic type ones, but are parametrized in terms of the 2-cocycles.

Poisson Vertex Algebras

The theory of Poisson Vertex Algebras [5] provides a very effective framework to study Hamiltonian operators. The notion of a PVA, that can be seen as the semiclassical limit of Vertex Algebras [37], has been introduced in order to deal with evolutionary Hamiltonian PDEs in which the unknown functions depend on one spatial variable. It provides a good framework for the study of integrability of such a class of equations, and also gives some insights into the study of nonlocal Poisson structures [16]. Let us briefly remind the notion of a (one-dimensional) PVA.

A Poisson Vertex Algebra [5] is a differential algebra $(\mathcal{A}, \partial)$ endowed with a bilinear operation $\mathcal{A} \times \mathcal{A} \to \mathbb{R}[\lambda] \otimes \mathcal{A}$ called a λ-bracket and satisfying the set of properties

1. $\{f_\lambda \partial g\} = (\lambda + \partial)\{f_\lambda g\}$
2. $\{\partial f_\lambda g\} = -\lambda\{f_\lambda g\}$
3. $\{f_\lambda gh\} = \{f_\lambda g\}h + \{f_\lambda h\}g$
4. $\{fg_\lambda h\} = \{f_{\lambda+\partial} h\}g + \{g_{\lambda+\partial} h\}f$
5. $\{g_\lambda f\} = -_{\to}\{f_{-\lambda-\partial} g\}$
6. $\{f_\lambda\{g_\mu h\}\} - \{g_\mu\{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu} h\}$

Let us explain the notation used in 4. and 5. Expand $\{f_\lambda g\} = \sum C_n \lambda^n$ with $C_n \in \mathcal{A}$. Then for the first term of the RHS of equation 4 one has

$$\{f_{\lambda+\partial} g\} h := \sum_n C_n \left(\lambda + \partial\right)^n h,$$

and similarly for the second term. Notice that using this convention $\{f_{\lambda+\partial} g\} 1 = \{f_\lambda g\}$. The RHS of the fifth equation is defined by

$$_{\rightarrow}\{f_{-\lambda-\partial} g\} := \sum (-\lambda - \partial)^n C_n$$

The main theorem, on which all the theory of PVA in the framework of Hamiltonian PDEs is based, is that from a λ-bracket of a PVA we can get the Poisson bracket between local functionals as

$$\left\{ \int f, \int g \right\} = \int \{f_\lambda g\}\big|_{\lambda=0}.$$

Moreover, the Hamilton equation (6) is written

$$\frac{\partial u^i}{\partial t} = \{h_\lambda u^i\}\big|_{\lambda=0}$$

for $u = \{u^i\}_{i=1}^N$. Here h is the density of the Hamiltonian functional $H = \int h$.

Conversely, given a Poisson structure as a differential operator we can define a λ-bracket between the generators of a suitable differential algebra as the symbol of the differential operator; its extension to the full algebra is directly achieved by using the so called *master formula*. Chosen a set of generators $\{u^i\}_{i=1}^N$, the λ bracket between two elements $(f, g) \in \mathcal{A}$ is given by

$$\{f_\lambda g\} = \sum_{i,j=1}^N \sum_{l,n \geq 0} \frac{\partial g}{\partial u^j_{(n)}} \left(\lambda + \partial\right)^n \left\{u^i{}_{\lambda+\partial} u^j\right\} \left(-\lambda - \partial\right)^m \frac{\partial f}{\partial u^i_{(m)}}. \tag{13}$$

The effectiveness of the theory of PVAs is demonstrated by the number of results appeared in the recent years that use the notion to explore various aspects within the theory of integrable systems, such as the notion of nonlocal Poisson brackets [16], a Dirac-type reduction [19] or a generalization of Drinfeld–Sokolov's reduction [18]. The study of the Poisson–Lichnerowicz cohomology has been studied in detail in [17], where it is discussed using the PVA language and some results on the cohomology for more general spaces that the one studied by Getzler are provided; in particular, the case of the cohomology of a Hamiltonian operator which is not constant, as the normal form for $D = 1$ Poisson brackets of hydrodynamic type, but *quasiconstant*, namely it explicitly depends on the independent variables $\{x^\alpha\}$.

Summary of the results

Part of the original material that appears in this thesis is contained in the following two papers:

1. M. Casati. *On deformations of multidimensional Poisson brackets of hydrodynamic type.* Comm. Math. Phys., 335(2):851–894, 2015. arXiv:1312.1878

2. G. Carlet, M. Casati, and S. Shadrin. *Poisson cohomology of scalar multidimensional Dubrovin–Novikov brackets.* in preparation, 2015.

Definition of multidimensional Poisson Vertex Algebras

The theory of PVAs has been originally developed for one dimensional Hamiltonian operators (in the original version, for a differential algebra with one derivation); since we want to deal with higher dimensional operators, we extend the definitions and the main theorems of [5] introducing the so-called *multidimensional Poisson Vertex Algebras* (mPVAs).

Let A be a differential algebra endowed with D commuting derivations $\{\partial_\alpha\}_{\alpha=1}^{D}$. This class of differential algebras is also called Partial Differential Algebras [50].

Definition 0.8 (λ-bracket)**.** A λ-bracket of rank D on A is a $\mathbb{R}$-linear map

$$\begin{aligned}\{\cdot_{\boldsymbol{\lambda}}\cdot\}\colon A\times A&\to\mathbb{R}[\lambda_1,\dots,\lambda_D]\otimes A\\(f,g)&\mapsto\{f_{\boldsymbol{\lambda}}g\}\end{aligned}$$

which is *sesquilinear*, namely

$$\{\partial_\alpha f_{\boldsymbol{\lambda}}g\}=-\lambda_\alpha\{f_{\boldsymbol{\lambda}}g\} \tag{14a}$$

$$\{f_{\boldsymbol{\lambda}}\partial_\alpha g\}=(\partial_\alpha+\lambda_\alpha)\{f_{\boldsymbol{\lambda}}g\} \tag{14b}$$

and obeys, respectively, the *right* and *left Leibniz rule*

$$\{f_{\boldsymbol{\lambda}}gh\}=\{f_{\boldsymbol{\lambda}}g\}h+\{f_{\boldsymbol{\lambda}}h\}g \tag{15a}$$

$$\{fg_{\boldsymbol{\lambda}}h\}=\{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}g+\{g_{\boldsymbol{\lambda}+\boldsymbol{\partial}}h\}f \tag{15b}$$

Definition 0.9 (Multidimensional Poisson Vertex Algebra)**.** A (D-dimensional) *Poisson Vertex Algebra* is a partial differential algebra A endowed with a λ-bracket of rank D which is *skewsymmetric*

$$\{g_{\boldsymbol{\lambda}}f\}=-_{\to}\{f_{-\boldsymbol{\lambda}-\boldsymbol{\partial}}g\} \tag{16}$$

and satisfy the *PVA-Jacobi identity*

$$\{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}}h\}\}-\{g_{\boldsymbol{\mu}}\{f_{\boldsymbol{\lambda}}h\}\}=\{\{f_{\boldsymbol{\lambda}}g\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}}h\}. \tag{17}$$

The formulas are completely analogous to the ones for the standard Poisson Vertex Algebras, after the adoption of a tailored multiindex notation. We denote $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \cdots, \lambda_D)$ and write $\{f_{\boldsymbol{\lambda}} g\} = C(f,g)_{i_1,i_2,\dots,i_D} \lambda_1^{i_1} \lambda_2^{i_2} \cdots \lambda_D^{i_D} = C(f,g)_I \boldsymbol{\lambda}^I$. The terms in the RHS of (15b) are to be interpreted as

$$\{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}} h\} g = C(f,g)_I (\boldsymbol{\lambda}+\boldsymbol{\partial})^I = C(f,h)_{i_1,\dots,i_d} \left(\prod_{\alpha=1}^{D} (\lambda_\alpha + \partial_\alpha)^{i_\alpha} \right) g.$$

We take as the differential algebra A the algebra of differential polynomials $\mathcal{A}$, with the derivations $\partial_\alpha = \frac{\partial}{\partial x^\alpha} = \sum_{i,I} u^i_{I+\xi_\alpha} \frac{\partial}{\partial u^i_I}$. With the symbol ξ_α we denote the α-th element in the standard basis for $\mathbb{Z}^D$, namely the vector with 1 at the α-th entry and 0 elsewhere. The straightforward generalization of the master formula allows to extend, according to the axioms of the λ brackets, the bracket computed between the generators to the full algebra; the bracket then satisfies the skewsymmetry and the PVA–Jacobi identity if and only if the one between the generators does so. We have

$$\{f_{\boldsymbol{\lambda}} g\} = \sum_{\substack{i,j=1\dots,N \\ R,S\in\mathbb{Z}^D_{\geq 0}}} \frac{\partial g}{\partial u^j_S} (\boldsymbol{\lambda}+\boldsymbol{\partial})^S \{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}} u^j\} (-\boldsymbol{\lambda}-\boldsymbol{\partial})^R \frac{\partial f}{\partial u^i_R}. \tag{18}$$

The theory of the multidimensional Poisson Vertex Algebras provides a framework, alternative to the formal calculus of variations, to study the Poisson brackets on the space of local functionals $\mathcal{F}$ and on the space of densities $\mathcal{A}$. Let us consider the following bilinear bracket, defined in terms of a λ bracket on $\mathcal{A}$.

$$\{f,g\} = \{f_{\boldsymbol{\lambda}} g\}\big|_{\boldsymbol{\lambda}=0} \qquad f,g \in \mathcal{A}. \tag{19}$$

We generalize to the D-dimensional case the following theorems.

Theorem 0.2. *Let $\mathcal{A}$ be an algebra of differential polynomials with a λ-bracket and consider the bracket defined in* (19). *Then*

(a) The bracket (19) *induces a well-defined bracket on the quotient space $\mathcal{F} = \mathcal{A}/\sum_\alpha \partial_\alpha \mathcal{A}$;*

(b) If the λ-bracket satisfies the axioms of a PVA, then the induced bracket on $\mathcal{F}$ is a Lie bracket.

Theorem 0.3. *Given a local Hamiltonian operator on $\mathcal{A}$ of the form*

$$\sum_S P^{ij}_S \partial^S, \qquad P^{ij}_S \in \mathcal{A},$$

the λ-bracket defined on the generators as

$$\{u^i_{\boldsymbol{\lambda}} u^j\} := \sum_S P^{ji}_S \boldsymbol{\lambda}^S \tag{20}$$

and extended to $\mathcal{A}$ according to the master formula is the λ bracket of a multidimensional Poisson Vertex Algebra.

Theorem 0.2 and Theorem 0.3 establish a one-to-one correspondence between Poisson Vertex Algebras and local Poisson brackets. Hence, we can conduct our study in the – easier to handle – framework of mPVAs and claim that the results hold for the Poisson brackets.

Poisson–Lichnerowicz cohomology of two-dimensional brackets

The grading on the space $\mathcal{A}$ is extended to $\mathcal{A}[\boldsymbol{\lambda}]$ by setting $\deg \boldsymbol{\lambda}^I = |I| = i_1 + i_2 + \cdots + i_D$. Using the formalism of Poisson Vertex Algebras, and the notion of PVA cohomology [5, 15, 17], we study the symmetries – related to H^1 – and the deformations – related to H^2 – of the Poisson brackets of hydrodynamic type in the case $D = N = 2$, up to the third degree. Moreover, we computed the first degree component of the third cohomology group.

Poisson brackets of hydrodynamic type for $D = N = 2$ can be classified up to an arbitrary change of the dependent coordinates u^1, u^2 and a linear change of the independent variables x^1, x^2 [31]. If there do not exists constants (α, β) such that $\det(\alpha g^1 + \beta g^2) = 0$, namely if the Hamiltonian operator is nondegenerate [53], than there exist three normal forms of the brackets. They are, as λ brackets,

$$P_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{21}$$

$$P_2 = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \tag{22}$$

$$P_{LP} = -\begin{pmatrix} 2u^1 & u^2 \\ u^2 & 0 \end{pmatrix} \lambda_1 - \begin{pmatrix} 0 & u^1 \\ u^1 & 2u^2 \end{pmatrix} \lambda_2 - \begin{pmatrix} u^1_1 & u^2_1 \\ u^1_2 & u^2_2 \end{pmatrix}, \tag{23}$$

where we denote $\{u^i{}_{\boldsymbol{\lambda}} u^j\}_{1,2,LP} = (P_{1,2,LP})^{ij}(\boldsymbol{\lambda})$ and $u^i_\alpha = \partial_\alpha u^i$. The third structure P_{LP} deservess a specific interest, since it is the Lie–Poisson bracket associated to the algebra of vector fields on a 2-torus, and can be regarded as a direct generalization to the two-dimensional case of the Virasoro–Magri PVA with central charge 0, namely $\{u_\lambda u\} = 2u\lambda + u'$.

Theorem 0.4. *Let us denote $H^p_d(P)$ the component of degree d of the p-th cohomology group defined by the λ bracket $\{u^i_{\boldsymbol{\lambda}} u^j\} = (P)^{ij}$. For the three normal forms of the Poisson brackets of hydrodynamic type for $D = N = 2$*

we have:

$H^1_0(P_1) \cong \mathbb{R}^2$ $H^1_0(P_2) \cong \mathbb{R}^2$ $H^1_0(P_{LP}) \cong 0$	$H^1_1(P_1) \cong \mathbb{R}^2$ $H^1_1(P_2) \cong \mathbb{R}^2$ $H^1_1(P_{LP}) \cong 0$	$H^1_2(P_1) \cong 0$ $H^1_2(P_2) \cong 0$ $H^1_2(P_{LP}) \cong 0$
$H^2_1(P_1) \cong \mathbb{R}^2$ $H^2_1(P_2) \cong \mathbb{R}^3$ $H^2_1(P_{LP}) \cong 0$	$H^2_2(P_1) \cong 0$ $H^2_2(P_2) \cong 0$ $H^2_2(P_{LP}) \cong 0$	$H^2_3(P_1) \cong \mathbb{R}^4$ $H^2_3(P_2) \cong \mathbb{R}^5$ $H^2_3(P_{LP}) \cong \mathbb{R}^2$
$H^3_2(P_1) \cong 0$ $H^3_2(P_2) \cong \mathbb{R}$ $H^3_2(P_{LP}) \cong 0$		

(24)

In particular, notice that H^2 is not trivial for any of the Poisson brackets. We explicitly compute a basis for $H^2_3(P_{LP})$, that can be considered as a generalization of the Gel'fand–Fuchs cocycle [34], constituted by two cocycles in the classes (3.55) and (3.56).

Poisson–Lichnerowicz cohomology of scalar multidimensional brackets

The drawback of the approach we follow for the $D = N = 2$ case is that each cohomology group must be computed explicitly and independently from the others. As a matter of fact, this is not the case with Getzler's theorem, that is a statement valid for all but a finite number of the cohomology groups. The lack of a system of coordinates in which the Poisson bracket, in particular P_{LP}, is not constant, prevents us to adapt Getzler's approach to our problem. Nevertheless, if one considers the *scalar* multidimensional brackets, namely the case $N = 1$, then there exist a system of coordinates in which the λ bracket associated to a Poisson structure of hydrodynamic type has the constant form

$$\{u_{\boldsymbol{\lambda}} u\} = \sum_{\alpha=1}^{D} c_\alpha \lambda_\alpha. \tag{25}$$

A suitable language to address the problem of studying the Poisson–Lichnerowicz cohomology of this class of brackets is the θ calculus, in which the space of local multi-vector fields $\Lambda^\bullet$ is identified with the quotient space $\hat{\mathcal{F}} = \hat{\mathcal{A}}/\sum_\alpha \partial_\alpha \hat{\mathcal{A}}$, with

$$\hat{\mathcal{A}} := C^\infty(M)[[\{u^i_S, |S| > 0\} \cup \{\theta^S_i, |S| \geq 0\}]].$$

The symbols $\{\theta^S_i\}$ are Grassmann numbers and the associated jet variables, namely $\theta_i\theta_j = -\theta_j\theta_i$, $\theta^S_i\theta^T_j = -\theta^T_j\theta^S_i$, and $\theta^S_i = \boldsymbol{\partial}^S\theta_i$. On $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$ we introduce another grading $\deg_\theta$ by $\deg_\theta u^i_I = 0$ and $\deg_\theta \theta^S_i = 1$. We denote $\hat{\mathcal{A}}^p$ and $\hat{\mathcal{F}}^p$ the homogeneous components of $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$ with θ-degree p.

The identification is stated as follows.

Theorem 0.5. *The space of local multi-vectors* Λ^p *is isomorphic to* $\hat{\mathcal{F}}^p$ *for* $p \neq 1$. *Moreover*

$$\Lambda^1 \cong \frac{\hat{\mathcal{F}}^1}{\oplus_\alpha \mathbb{R} \int u^i_{\xi_\alpha} \theta_i} \cong \frac{\mathrm{Der}'(\mathcal{A})}{\oplus_\alpha \mathbb{R} \partial_\alpha}, \tag{26}$$

where $\mathrm{Der}'(\mathcal{A})$ *denotes the space of derivations of* $\mathcal{A}$ *that commute with* ∂_α, *for* $\alpha = 1, \dots, D$, *and* $\mathrm{Der}'(\mathcal{A}) \cong \hat{\mathcal{F}}^1$.

Let Θ be the polynomial ring $\mathbb{R}[\{\theta^S, S \in (\mathbb{Z}_{\geq 0})^{D-1}\}]$. Denote by Ξ^p_d the homogeneous component of bi-degree (p, d) of

$$\Xi = \frac{\Theta}{\partial_1 \Theta + \cdots + \partial_{D-1} \Theta}. \tag{27}$$

Theorem 0.6. *The Poisson cohomology of the Poisson bracket* (25) *in bi-degree* (p, d) *is isomorphic to the sum of vector spaces*

$$\Xi^p_d \oplus \Xi^{p+1}_d. \tag{28}$$

For small (p, d)'s, the Theorem allows us to compute the cohomology for any value of D.

Corollary 0.7. *We have that:*

$$H^0_d(\hat{\mathcal{F}}, P) \cong \begin{cases} \mathbb{R}^2 & d = 0, \\ 0 & d \geq 1 \end{cases} \tag{29}$$

$$H^1_d(\hat{\mathcal{F}}, P) \cong \begin{cases} \mathbb{R} & d = 0, \\ \mathbb{R}^{D-1} & d = 1, \\ 0 & d = 2 \end{cases} \tag{30}$$

For $D = 2$ we derive a general formula for $H^p_d(\hat{\mathcal{F}}, P)$. Define the partition function $P(n, k)$ as the number of ways of writing an integer n as sum of k positive integers [13].

Corollary 0.8. *For* $D = 2$, *the cohomology group* $H^p_d(\hat{\mathcal{F}}, P)$ *is isomorphic to* $\mathbb{R}^Q$ *with*

$$\begin{aligned} Q = {} & P\left(d + p - \binom{p}{2}, p\right) - P\left(d + p - \binom{p}{2} - 1, p\right) + \\ & + P\left(d + p - \binom{p+1}{2} + 1, p+1\right) - P\left(d + p - \binom{p+1}{2}, p+1\right) \end{aligned} \tag{31}$$

In particular and for any $k = 0, 1, 2, \dots$ we have

$$\dim H^2_d(\hat{\mathcal{F}}, P) = \begin{cases} 0 & d = 0 \\ 1 & d = 1 \\ 0 & d = 2 \\ \frac{1}{6}(d+3)+1 & d = 3+6k \\ \frac{1}{6}(d-4) & d = 4+6k \\ \frac{1}{6}(d+1)+1 & d = 5+6k \\ \frac{1}{6}d & d = 6+6k \\ \frac{1}{6}(d-1)+1 & d = 7+6k \\ \frac{1}{6}(d-2) & d = 8+6k \end{cases} \tag{32}$$

It should be noticed that $H^2_d(\hat{\mathcal{F}}, P) \neq 0$ for all $d > 4$, hence the full second cohomology group is infinite dimensional.

To prove Theorem 0.6 we have selected a particular case of the λ bracket (25), where $c_\alpha = \delta_{\alpha,D}$. Indeed, we have:

Lemma 0.9. *By a linear change of independent variables $(x^1, \dots, x^D)$ the bracket* (25) *can be brought to the form*

$$\{v_{\boldsymbol{\lambda}} v\} = \lambda_D. \tag{33}$$

If we denote P and $\tilde{P}$ the Hamiltonian operators defining the brackets (25) and (33), we can state

Lemma 0.10. *The cohomology groups $H^p_d(\hat{\mathcal{F}}, P)$ are isomoprhic to $H^p_d(\hat{\mathcal{F}}, \tilde{P})$.*

We prove Lemma 0.10 for the scalar case, but the proof does not change if we consider any value of N. This means that the Poisson cohomology groups are not only invariant with respect to any change of coordinates on M, but also with respect to a linear change of coordinates on Σ. This is a crucial fact, because Ferapontov and collaborators' classification of the Poisson brackets of hydrodynamic type for $D = N = 2$ has been obtained by allowing not only changes of the coordinates $\{u^i\}$, but also linear changes of the independent variables $\{x^\alpha\}$. Hence, we can state that the results we proved and listed in the table (24) are not valid only for the three brackets (21), (22), and (23),but for all the brackets of hydrodynamic type on the same space of maps.

The thesis is organized as follows

- In Chapter 1 we recall some preliminary and well established background material about Poisson geometry on both finite and infinite manifolds.

In particular, we introduce the notions of Poisson brackets and Poisson–Lichnerowicz cohomology in the classical finite dimensional approach; after revising the building block of the formal calculus of variations, mainly adapting [25, Chapter 2] to the D-dimensional case, we define the Poisson brackets of hydrodynamic type (also called Dubrovin–Novikov brackets) and discuss their classification and the Poisson cohomology for the $D = 1$ case.

- In Chapter 2 we define the notion of multidimensional Poisson Vertex Algebra (mPVAs) and prove the isomorphism between D-dimensional mPVAs with N generators and local Poisson structures on the space of maps from a D-dimensional to a N-dimensional manifold. We discuss the multidimensional version of the poly-λ-brackets and their identification with the space of multivectors, as well as the notion of the cohomology of Poisson Vertex Algebras. Moreover, we present the Mathematica package `MultiPVA`, with which we can implement the computation of the λ brackets.

- In Chapter 3 we apply the formalism of the previous chapter to explicitly compute the zeroth, first and second order symmetries and deformations for the three normal forms of the Poisson brackets of hydrodynamic type where $D = N = 2$. They corresponds, respectively, to H^1_0, H^1_1, H^1_2 and H^2_1, H^2_2, H^2_3. Moreover, we describe the technique to compute higher cohomology groups and, as an example, compute H^3_2.

- In Chapter 4 we describe the θ formalism and how it can be used to compute the Poisson cohomology for scalar multidimensional Dubrovin–Novikov brackets. After proving Theorem 0.6, we provide the explicit closed formula for the case $D = 2$. We also show how to derive a few of the same results within the framework of mPVA.

Section 2.1, as well as the computations in Chapter 3 for H^1_1 and H^2_2, have already been published in [12]. Chapter 4 presents the material that will be the content of [7].

1

Preliminaries

In this chapter we briefly recall some preliminary and well established material about the theory of infinite dimensional Poisson manifolds. This allows us to set the notation used in the forthcoming chapters and to start addressing the main problems we will deal in the study of multidimensional Poisson brackets of hydrodynamic type.

1.1 Basics of Poisson geometry

Poisson geometry, as we mean it nowadays, is quite a recent topic in differential geometry. Despite the ideas and the motivation for it have been laid down at the beginning of XIX century, the term *Poisson manifold* itself appeared first in the late '70s in a crucial paper by A. Lichnerowicz [39].

Because of the importance that Poisson Geometry has achieved in the realm of Mathematical Physics, the basics of the topic are usually taught as part of the master degree curricula in Mathematics. In this section we briefly present the general ideas of Poisson Geometry for *finite dimensional manifolds*, while the main point of this thesis is the study of Poisson geometry and the search for analogous patterns in infinite dimensional spaces.

1.1.1 Poisson manifolds

Definition 1.1 (Poisson manifold)**.** A *Poisson manifold* is a smooth manifold M, $\dim M = n$ endowed with an operation $\{\cdot,\cdot\}\colon C^\infty(M)\times C^\infty(M)\to C^\infty(M)$ called the *Poisson bracket* satisfying the following properties for all $(\alpha,\beta,\gamma)\in\mathbb{R}$ and $(f,g,h)\in C^\infty(M)$:

1. The bracket is $\mathbb{R}$-bilinear

$$\{\alpha f+\beta g,h\}=\alpha\{f,h\}+\beta\{g,h\}$$
$$\{f,\beta g+\gamma h\}=\beta\{f,g\}+\gamma\{f,h\}$$

2. The bracket is skewsymmetric

$$\{f,g\}=-\{g,f\}$$

3. The bracket is a derivation of the product
$$\{f, g \cdot h\} = \{f, g\}h + g\{f, h\}$$
4. The bracket is a derivation of itself
$$\{f, \{g, h\}\} = \{\{f, g\}, h\} + \{g, \{f, h\}\}$$

Property 4 is called the *Leibniz property* and Property 5 the *Jacobi identity*. By using the skewsymmetry, the Jacobi identity can be cast in the cyclic form
$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0. \tag{1.1}$$
Given a local chart $\{x^i\}_{i=1}^n$ on M, the Poisson bracket between two functions $f(x)$ and $g(x)$ is uniquely determined, according to the properties 1, 2, and 4, by the Poisson bracket between the coordinate function, as
$$\{f(x), g(x)\} = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial f}{\partial x^i}\{x^i, x^j\}\frac{\partial g}{\partial x^j} \tag{1.2}$$
The necessary and sufficient condition for $\{\cdot, \cdot\}$ to be a Poisson bracket is that the bracket between the coordinate function is skewsymmetric and satisfies Jacobi identity.

Definition 1.2 (Poisson bivector)**.** Let Π be a bivector on M, namely an element of $\Gamma(\Lambda^2 TM)$. A bivector Π is said to be a *Poisson bivector* if $[\Pi, \Pi] = 0$, where $[\cdot, \cdot]$ is the Schouten bracket.

Chosen a local chart $\{x^i\}$ on M, the Schouten bracket between two bivectors Π and Λ is a 3-vector with components
$$[\Pi, \Lambda]^{ijk} = \Pi^{il}\frac{\partial \Lambda^{jk}}{\partial x^l} + \Lambda^{il}\frac{\partial \Pi^{jk}}{\partial x^l} + \text{c.p}(i, j, k). \tag{1.3}$$
It is then obvious that a bivector which is constant in some coordinates is a Poisson bivector.

Proposition 1.1. There is a one-to-one correspondance between Poisson bivectors and Poisson brackets.
$$\{f, g\} = \Pi(\mathrm{d}f, \mathrm{d}g)$$
$$\Pi^{ij} = \{x^i, x^j\}$$

Proof. The only non-obvious part of the statement is that the vanishing of the Schouten torsion ($[\Pi, \Pi] = 0$) is equivalent to the Jacobi identity for the Poisson bracket between the coordinate functions. From the formula (1.3), $[\Pi, \Pi]^{ijk} = 2\sum_l \Pi^{il}\partial_l \Pi^{jk} + \text{c.p.}(i, j, k)$. From (1.2), $\{u^i, \{u^j, u^k\}\}$ coincides with the first summand in the Schouten relation and the following two terms of Jacobi identity are exactly obtained by the cyclic permutation of the indices. □

Because of the statement of Proposition 1.1, we will always denote a Poisson manifold by the pair (M, Π) of a smooth manifolds and a Poisson bivector. The solution of the Schouten property for a bivector with a constant $2r \leq n$ rank allows us, in a certain coordinate system, to write [39]

$$\Pi = \begin{pmatrix} h & 0 \\ 0 & 0 \end{pmatrix} \tag{1.4}$$

with h a constant nondegenerate skewsymmetric $2r \times 2r$ matrix. That means that there exist coordinates $y^1, \ldots, y^{2r}, c^1, \ldots c^{n-2r}$ such that

$$\{y^i, y^j\} = h^{ij} \qquad \{f(y,c), c^j\} = 0.$$

When $2r = n$ the Poisson manifold is indeed a symplectic manifold, with the symplectic form given by the inverse of Π. Darboux's theorem then guarantees the existence of a system of coordinates $(x^1, \ldots, x^r, y^1, \ldots, y^r)$ such that $\{x^i, x^j\} = \{y^i, y^j\} = 0$ and $\{x^i, y^j\} = \delta^{ij}$. If Π has constant rank $2r < n$, then the Poisson manifold admits a *symplectic foliation* of codimension $n - 2r$; each leaf is the level set of the functions $\{c^i\}$, which are called the *Casimirs* of the Poisson bivector.

More in general, an important theorem by A. Weinstein [57] provides a normal form for Poisson bivectors even in the case of nonconstant rank.

Theorem 1.1 (Splitting theorem)**.** *Let (M, Π) be a n-dimensional Poisson manifold. In a neighbourhood of any point $m \in M$ there exists a local coordinate system $(x^1, y^1, \ldots, x^r, y^r, z^1, \ldots, z^{n-2r})$ such that the Poisson bivector can be written as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial y^i} + \sum_{i,j=1}^{n-2r} f_{ij}(z) \frac{\partial}{\partial z^i} \wedge \frac{\partial}{\partial z^j}. \tag{1.5}$$

This means that any Poisson manifold is locally the product of a symplectic manifold of dimension $\operatorname{rk} \Pi_m$ and of a Poisson manifold with Poisson structure vanishing in the origin.

Definition 1.3 (Lie–Poisson bracket)**.** Let $\mathfrak{g}$ be a n-dimensional Lie algebra. The *Lie–Poisson bracket* on the space $\mathfrak{g}^*$ with coordinates $\{x^i\}$ is the bracket defined by the bivector

$$\Pi^{ij} = \{x^i, x^j\} = c^{ij}_k x^k. \tag{1.6}$$

where c^{ij}_k are the structure constants of the Lie algebra $\mathfrak{g}$. The Casimirs are the functions on $\mathfrak{g}^*$ invariants with respect to the co-adjoint action of the Lie group G, $\mathrm{Lie}(G) = \mathfrak{g}$ and the symplectic leaves are the orbits of the co-adjoint action of G.

1.1.2 Poisson geometry and classical mechanics

Historical remarks Poisson structures and mechanics are deeply related since the very beginning of the theory. The notion of Poisson brackets was introduced by S. D. Poisson [48] in the context of the so-called method of *variation of arbitrary constants*, used to find the general solution of non homogeneous linear differential equations and to deal with perturbations of mechanical systems. He noticed that, if a and b are functions constant in time, then the function

$$[a,b] = \frac{\mathrm{d}a}{\mathrm{d}u}\frac{\mathrm{d}b}{\mathrm{d}\phi} - \frac{\mathrm{d}a}{\mathrm{d}\phi}\frac{\mathrm{d}b}{\mathrm{d}u} + \frac{\mathrm{d}a}{\mathrm{d}v}\frac{\mathrm{d}b}{\mathrm{d}\psi} - \frac{\mathrm{d}a}{\mathrm{d}\psi}\frac{\mathrm{d}b}{\mathrm{d}v} + \cdots$$

where $(u, \phi), (v, \psi), \ldots$ are respectively the components of the moment and the position, is constant in time as well. The crucial importance of this structure for classical mechanics has been recognized and formalized by Jacobi [21], who rediscovered the formula for the brackets as the necessary and sufficient condition for the complete integrability of PDEs of form $H(x^i, p_i) = 0$, $p_i = \frac{\partial z}{\partial x^i}$, $i = 1, \ldots, n > 2$. Furthermore, he clarified that the brackets allow, given two first integrals of motion, to get a third one by simple differentiation, and then “une quatrième, une cinquième intégrale, et, en général, on parvient de cette manière à déduire des deux intégrals données toutes les intégrales, ou, ce qui revient au même, l'intégration complète du problème”.[1] The property later called *Jacobi identity*, that Poisson did not derive, is indeed the fundamental building block of Jacobi's method of integration.

We will give here a brief modern summary of notions in Poisson geometry that are tightly related to classical mechanics. However, it is well known that Poisson geometry provides a way (when one introduces more exotic structures as Lie and Poisson groupoids) to noncommutative geometry, quantum groups and so on [26, 38].

Poisson structures and mechanics A Poisson bracket defines an (anti) homomorphism of Lie algebras $C^\infty(M) \to \mathfrak{X}(M)$ that associates to a smooth function H of the manifold M a vector field X_H according to the formula

$$H \mapsto X_H = \{\cdot, H\} \tag{1.7}$$

We call the vector field X_H the *Hamiltonian vector field* of *Hamiltonian* H. The formula (1.7) defines an antihomomorphism because it easily follows from the Jacobi identity for the Poisson brackets that

$$[X_H, X_K] = -X_{\{H,K\}},$$

where $[\cdot,\cdot]$ is the commutator of vector fields.

[1] Jacobi, Sur un théorème de Poisson, *Comptes rendus de l'Académie des Sciences de Paris* (1840), cited in [21]

The notion of Hamiltonian vector field is the core of the application of Poisson geometry to classical mechanics. Given a system with n generalized coordinates $\{q^i\}$ and the corresponding momenta $\{p_i\}$, Hamilton's equations of motion read

$$\begin{cases} \frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \\ \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q_i} \end{cases}$$

for a Hamiltonian function $H(q,p)$ whose meaning is, as well known, the total energy of the system in the state $(q^1,\dots,q^n,p_1,\dots,p_n)$. The same equations can be written considering M the *phase space* of the system, endowed with the constant nondegenerate Poisson bivector

$$\Pi = \begin{pmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{pmatrix}$$

that is, of course, a nondegenerate Poisson bivector written in Darboux's coordinates. Hamilton's equations take then the form

$$\begin{cases} \frac{\mathrm{d}q^i}{\mathrm{d}t} = \{q^i, H\} = X_H(q^i) \\ \frac{\mathrm{d}p_i}{\mathrm{d}t} = \{p_i, H\} = X_H(p_i). \end{cases}$$

Comparing the chain rule for differentiation and the definition of Poisson brackets, the time evolution of any functions of the phase space $f(q,p)$ can be written in the so-called *Hamiltonian form* by

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \{f, H\} = X_H(f). \tag{1.8}$$

Remark 1.2. A function f that *Poisson commutes* with the Hamiltonian H, i.e. such that $\{f,H\}=0$, is a *first integral* of the system and it is a constant of motion. If f and g are two first integrals, then $\{f,g\}$ is a first integral as well. This remark reproduces Poisson's observation in modern language.

1.1.3 Poisson cohomology

Let us denote $\Lambda^\bullet = \oplus_{k=0}^{\dim M} \Lambda^k$ the space of multivectors on the manifold M. The extension to the space of multivectors of the commutator of vector fields is called the *Schouten–Nijenhuis bracket* we have already used in the previous paragraphs to write the Schouten property for a Poisson bivector. We start giving a brief summary of its main properties; then we introduce the notion of Poisson–Lichnerowicz cohomology for finite dimensional manifold and discuss its interpretation. Most of the content of this paragraph applies with just technical modifications to infinite dimensional Poisson manifolds; the details are provided in Paragraph 1.3.4.

Schouten–Nijenhuis bracket

Definition 1.4. The Schouten–Nijenhuis bracket is a bilinear pairing

$$\begin{aligned}[\cdot,\cdot]\colon \Lambda^k \times \Lambda^l &\to \Lambda^{k+l-1}\\ (a,b) &\mapsto [a,b]\end{aligned}$$

such that

1. $[b,a] = -(-1)^{(k-1)(l-1)}[a,b]$ if $a \in \Lambda^k$ and $b \in \Lambda^l$ (graded skewsymmetry)
2. $[c, a\wedge b] = [c,a]\wedge b + (-1)^{k(m-1)} a \wedge [c,b]$ if $a \in \Lambda^k$ and $c \in \Lambda^m$ (Leibniz property)
3. $[f,g] = 0$ if $f,g \in \Lambda^0 = C^\infty(M)$.
4. $[a,f] = a(f) = a^i \frac{\partial f}{\partial x^i}$ if $f \in \Lambda^0$ and $a \in \Lambda^1$
5. $[a,b] =$ commutator of vector fields if $a,b \in \Lambda^1$.

The operation defined by this set of properties is unique.

In particular, if $v \in \Lambda^1$ and $a \in \Lambda^\bullet$, $[v,a] = \mathcal{L}_v a$, where $\mathcal{L}_v$ denotes the Lie derivative along a vector field v. The Schouten–Nijenhuis bracket satisfies the *graded Jacobi identity*

$$[[a,b],c] = [a,[b,c]] - (-1)^{(k-1)(l-1)}[b,[a,c]] \tag{1.9}$$

if $a \in \Lambda^k$, $b \in \Lambda^l$, and $c \in \Lambda^m$. Using the graded skewsymmetry the Jacobi identity can be given in cyclic form

$$(-1)^{(k-1)(m-1)}[a,[b,c]] + (-1)^{(l-1)(k-1)}[b,[c,a]] + (-1)^{(m-1)(l-1)}[c,[a,b]] = 0.$$

Remark 1.3. In the previous paragraph we have shown that the bracket between two functions $\{f,g\}$ can be written as the action of the Hamiltonian vector field X_g on the function f. Using the Schouten brackets we can write $\{f,g\} = \mathcal{L}_{X_g}(f) = [f, X_g]$. Moreover, $X_g = [g,\Pi]$. This can be shown by applying the Leibniz and the skewsymmetry properties with Π written in normal form. Hence, we can write $\{f,g\} = [f,[g,\Pi]]$.

Poisson–Lichnerowicz cohomology From the properties of the Schouten bracket it follows that, if Π is a Poisson bivector, then $[\Pi,[\Pi,a]] = 0$ for any $a \in \Lambda^\bullet$. We can define on $\Lambda^\bullet$ a linear operator

$$\begin{aligned}\mathrm{d}_\Pi\colon \Lambda^k &\to \Lambda^{k+1}\\ a &\mapsto \mathrm{d}_\Pi a = [\Pi, a].\end{aligned}$$

The previous observation implies that $\mathrm{d}_\Pi^2 = 0$. The corresponding differential complex

$$0 \to \Lambda^0 = \mathcal{F} \xrightarrow{\mathrm{d}_\Pi} \Lambda^1 \xrightarrow{\mathrm{d}_\Pi} \Lambda^2 \xrightarrow{\mathrm{d}_\Pi} \cdots \xrightarrow{\mathrm{d}_\Pi} \Lambda^n \to 0$$

is called the *Lichnerowicz cochain complex* and its associated cohomology is the *Poisson–Lichnerowicz* (or simply Poisson) *cohomology*.

$$H^p(\mathrm{d}_\Pi, M) = \frac{\ker \mathrm{d}_\Pi \colon \Lambda^p \to \Lambda^{p+1}}{\mathrm{Im}\, \mathrm{d}_\Pi \colon \Lambda^{p-1} \to \Lambda^p}. \tag{1.10}$$

If the Poisson bivector Π is nondegenerate, then the Poisson cohomology is isomorphic to the De Rham cohomology of M [39].

Interpretation of the cohomology groups $H^0(\mathrm{d}_\Pi, M)$ coincides with the ring of Casimir functions: indeed, for a Casimir function c we have $X_c = 0$, and $X_c = [c, \Pi]$.

$H^1(\mathrm{d}_\Pi, M)$ is the quotient space of the symmetries of the Poisson bivector Π that are not Hamiltonian vector fields. A symmetry of Π is a vector field X such that $\mathcal{L}_X\Pi = 0$, that can be read in terms of Schouten brackets as $[X, \Pi] = 0$. Hence, the symmetries are 1-cocycles in the Lichnerowicz complex. Moreover, our previous remark about the way to write a Hamiltonian vector field in terms of the Schouten bracket allows us to identify 1-coboundaries with Hamiltonian vector fields.

$H^2(\mathrm{d}_\Pi, M)$ is the quotient space of the bivectors Ψ such that $[\Pi, \Psi] = 0$ modulo the bivectors of form $\Psi' = \mathcal{L}_X(f)$ for X a vector field. To understand the meaning of this space let us consider an *infinitesimal deformation* of the Poisson bivector Π. Introducing the formal parameter η, such a deformation has form $\Pi' = \Pi + \eta\Psi$, for $\Psi \in \Lambda^2$. We say that Π' is an infinitesimal deformation of Π if $[\Pi', \Pi'] = o(\eta)$. Expanding the computation gives us the cocycle condition for Ψ. On the other hand, if $\Pi' = \Pi + \eta[\Pi, X]$, then $\Pi' = \lim_{\eta\to 0} e^{\eta X}(\Pi)$, namely Π' and Π are equivalent up to an infinitesimal isomorphism. We call the deformed bivectors of this form *trivial deformations*.

$H^3(\mathrm{d}_\Pi, M)$ can be interpreted as the space of *obstructions to formal deformations* of Π. An infinitesimal deformation satisfies the Schouten property up to order η. In general, $\eta^2[\Psi, \Psi] \neq 0$. If we want to extend the deformation Π' to be a Poisson bivector up to order η^2, an easy computation shows that we have to introduce a further term in the expansion $\Pi' = \Pi + \eta\Psi + \eta^2\Psi_2$ and to solve

$$[\Pi, \Psi_2] + 2[\Psi, \Psi] = 0 \tag{1.11}$$

A solution to this equation can be always found only if $[\Psi, \Psi] \in \Lambda^3$ is of the form $\mathrm{d}_\Pi(\Psi_2)$ for Ψ_2 an unknown bivector. Moreover, one can prove using Jacobi identity that if $[\Pi, \Psi] = 0$, then $\mathrm{d}_\Pi[\Psi, \Psi] = 0$. This means that $[\Psi, \Psi]$ is a cocycle, but it must always be a coboundary to extend to the second order the deformation. The same computation is repeated when adding higher power terms in the formal deformation; this allows us to interpret the third cohomology group as the space of obstructions to the formal deformations: it must be trivial if we want to find further and further terms in the expansion.

1.2 The formal calculus of variations

The picture we have sketched in the previous section can be translated to develop a theory of *infinite dimensional Poisson manifolds*. Motivated by the discovery of a Hamiltonian formulation for KdV equation [33, 58], several authors have developed a formalism and techniques to formally deal with such kind of systems following the same ideas of Poisson geometry for finite dimensional mechanical systems [35, 46]. Roughly speaking, the Poisson manifold is replaced by a suitable space of maps, while the functions are replaced by local functionals; differential forms and multivectors are then consistently defined. In this section we will mainly follow the treatment of the topic presented in [25], with some adaptations due to the larger space of maps we are interested in.

1.2.1 The formal space of maps

Let M be a N-dimensional smooth manifold. We want to descrive a class of Poisson brackets on the space

$$\mathcal{M} = \mathrm{Maps}(\Sigma \to M)$$

where Σ is a compact D-dimensional smooth manifold. In order to avoid the problems arising from the integration, let us fix Σ to be $(S^1)^D = T^D$.

Let us define the formal map space $\mathcal{M}$ in terms of ring of functions on it.

Let $U \subset M$ be a chart on M with coordinates $(u^1, \dots, u^N)$ and denote $\bar{\mathcal{A}} = \bar{\mathcal{A}}(U)$ the space of polynomials in the independent variables u^i_I for $i = 1, \dots, N$ and $I \in \mathbb{Z}^D_+$ a multiindex (i.e., $I_\alpha = 1, 2, \dots$ with $\alpha = 1, \dots, d$)

$$f(x, u; u_I) := \sum_{m \geq 0} f_{i_1 I_1; \dots; i_m I_m}(x, u) u^{i_1}_{I_1} \dots u^{i_m}_{I_m} \tag{1.12}$$

with coefficients $f_{i_1 I_1; \dots; i_m I_m}(x, u)$ smooth functions on $\Sigma \times M$. Such an expression is called a *differential polynomial*. We can regard $\{u^i, u^i_I, x^\alpha\}$ as a set of local coordinates on the space $\mathcal{M}$, in the spirit of jet spaces. The space $\bar{\mathcal{A}}$, endowed with a family of operators

$$\begin{aligned} \partial_\alpha \colon \bar{\mathcal{A}} &\to \bar{\mathcal{A}} \\ f &\mapsto \frac{\partial f}{\partial x^\alpha} + u^i_{\xi_\alpha} \frac{\partial f}{\partial u^i} + u^i_{I+\xi_\alpha} \frac{\partial f}{\partial u^i_I} \end{aligned}$$

($\alpha = 1, \dots, D$ and $\xi_\alpha = (0, 0, \dots, \underbrace{1}_{\alpha}, 0, \dots, 0)$) satisfying the following commuting properties

$$
\begin{aligned}
&[\partial_\alpha, \partial_\beta] = 0 && \forall\, \alpha, \beta && \text{(1.13a)}\\
&\left[\frac{\partial}{\partial u^i_I}, \partial_\alpha\right] = \frac{\partial}{\partial u^i_{I-\xi\alpha}} && (= 0 \text{ if } I_\alpha = 0) && \text{(1.13b)}\\
&\left[\frac{\partial}{\partial u^i_I}, \frac{\partial}{\partial u^j_J}\right] = 0 && \forall\, (i, j, I, J) && \text{(1.13c)}
\end{aligned}
$$

form what in [5] is called an *algebra of differential polynomials*. Since we are interested in local (in the sense of [25]) structures on the space of maps, we do not have to take into account the explicit dependence on the points in Σ. This justifies the following definitions, where we will restrict ourselves to consider the space $\mathcal{A} \subset \bar{\mathcal{A}}$ of differential polynomials f that do not depend explicitly on x^α. The 'total derivatives' have thus the form

$$\partial_\alpha = \sum_{\substack{i=1,\dots,N\\ I\in\mathbb{Z}^D_{\geq 0}}} u^i_{I+\xi_\alpha} \frac{\partial}{\partial u^i_I} \tag{1.14}$$

and satisfy the same commutation relations as in (1.13).

The space of maps $\mathcal{M}$ being regarded as the "phase space", we associate the notion corresponding to $C^\infty(M)$ with the space $\mathcal{F}$ of *local functionals.* We recall that a (real-valued) local functional $F[u]$ is a mapping from $\mathcal{M}$ to $\mathbb{R}$ of form

$$F = \int f(x, u, u_J) \mathrm{d}x^1 \cdots \mathrm{d}x^D,$$

with $f \in \mathcal{A}$ is called the *density* of the local functional. When the space of functions we are dealing with does not allow the presence of boundary terms (as in the case on which we decided to focus, namely $\Sigma = T^D$) in the integrals, it is obvious that adding to f any total derivative $\partial_\alpha g$ does not change the result.

This observation (first due to Gel'fand and Dikii [35])allows us to define the *space of local functionals* $\mathcal{F}$ as a quotient space: we will call the projection map the D-dimensional *integral*

$$
\begin{aligned}
\int^D \mathrm{d}^D x \colon \mathcal{A} &\to \frac{\mathcal{A}}{\partial_1\mathcal{A} + \partial_2\mathcal{A} + \cdots \partial_D\mathcal{A}} =: \mathcal{F}\\
f(u, u_I) &\mapsto \int^D f(u, u_I) \mathrm{d}^D x.
\end{aligned} \tag{1.15}
$$

A more detailed description of the construction of the D-dimensional integrals will be provided in Chapter 4.

1.2.2 Variational derivatives and differential forms

Let us introduce an extension $\mathcal{A}^\Omega \supset \mathcal{A}$ of the algebra of differential polynomials, generated by the Grassmann generators δu^i_I, $i = 1, \dots, N$, $I \in \mathbb{Z}^D_{\geq 0}$. The commutation rules for the product $\mathcal{A}^\Omega$ are the following, with $(f, g) \in \mathcal{A}$:

$$\begin{aligned} f(u, u_I)g(u, u_I) &= g(u, u_I)f(u, u_I) \\ f(u, u_I)\,\delta u^j_J &= \delta u^j_J\, f(u, u_I) \\ \delta u^i_I \delta u^j_J &= -\delta u^j_J \delta u^i_I. \end{aligned}$$

Definition 1.5. A *density of variational p-form* is an element $\omega \in \mathcal{A}^\Omega$ represented as a finite sum

$$\omega = \frac{1}{k!}\omega^{I_1,\dots,I_k}_{i_1,\dots,i_k}\delta u^{i_1}_{I_1}\delta u^{i_2}_{I_2}\cdots\delta u^{i_k}_{I_k}, \tag{1.16}$$

where the coefficients $\omega^{I_1,\dots,I_k}_{i_1,\dots,i_k} \in \mathcal{A}$ are skewsymmetric with respect to the exchange of couples of indices $(i_j, I_j) \leftrightarrow (i_l, I_l)$.

The derivations $\{\partial_\alpha\}$ can be extended from $\mathcal{A}$ to $\mathcal{A}^\Omega$ by imposing the Leibniz property

$$\partial_\alpha\left(\omega_1\omega_2\right) = \left(\partial_\alpha\omega_1\right)\omega_2 + \omega_1\left(\partial_\alpha\omega_2\right), \tag{1.17}$$

and defining their action on the generators δu^i_I by

$$\partial_\alpha \delta u^i_I = \delta u^i_{I+\xi_\alpha}. \tag{1.18}$$

The sign rules we adopt for the Leibniz property is summarized by calling ∂ a *even derivation.* The elements of the quotient

$$\mathcal{F}^\Omega_k = \frac{\mathcal{A}^\Omega_k}{\partial_1\mathcal{A}^\Omega_k + \partial_2\mathcal{A}^\Omega_k + \cdots + \partial_D\mathcal{A}^\Omega_k}$$

are called *local k-forms* on $\mathcal{M}$. As for the local functionals, they can be written as integrals on Σ.

$$\int^D \omega \mathrm{d}^D x \in \mathcal{F}^\Omega_k, \quad \omega \in \mathcal{A}^\Omega_k. \tag{1.19}$$

From the definition of F^Ω as quotient space, it follows that there exists a unique representative for a 1-form $\int^D \omega \mathrm{d}^D x = \int^D \sum \omega^I_i \delta u^i_I\, \mathrm{d}^D x$ such that

$$\int^D \omega\, \mathrm{d}^D x = \int^D \tilde{\omega}\, \mathrm{d}^D x$$

and

$$\tilde{\omega} = \sum_i \tilde{\omega}_i \delta u^i = \sum_i \sum_I \left(\left(\prod_{\alpha=1}^D \left(-\partial_\alpha\right)^{I_\alpha}\right)\omega^I_i\right)\delta u^i \tag{1.20}$$

To avoid excessively long expressions, we'll adopt a multi-index notation and denote

$$\prod_{\alpha=1}^{D}(-\partial_\alpha)^{I_\alpha} =: (-\boldsymbol{\partial})^I .$$

Moreover, if $Z^D_{\geq 0} \ni I = (i_1, i_2, \dots, i_D)$, we will write the total order of the multindex as $|I| = i_1 + i_2 + \dots + i_D$. Analogue conventions will be adopted also for more general operators or expressions, always meaning that they must be regarded as "term by term" powers.

Note that a reduced form analogue to (1.20) can be given for generic local k-forms, discharging on the components $\omega^{I_1,\dots,I_k}_{i_1,\dots,i_k}$ the derivatives of the first generators $\delta u^{i_1}_{I_1}$. They are a straightforward generalisation to the D dimensional case of equation (2.2.16) of [25].

Let us introduce the *odd* derivation

$$\delta\colon \mathcal{A}^\Omega_k \to \mathcal{A}^\Omega_{k+1}$$

that we call *variational differential*.

For a monomial in $\mathcal{A}^\Omega_k$

$$\omega = f \delta u^{i_1}_{I_1} \dots \delta u^{i_k}_{I_k}$$

we define the action of δ as

$$\delta\omega = \sum_{\substack{j=1\dots N \\ J\in\mathbb{Z}^D_{\geq 0}}} \frac{\partial f}{\partial u^j_J} \delta u^j_J \delta u^{i_1}_{I_1} \dots \delta u^{i_k}_{I_k} = \sum \delta u^j_J \frac{\partial \omega}{\partial u^j_J}. \tag{1.21}$$

This definition implies $\delta(\delta u^i_I) = 0$ and $\delta\left(u^i_I\right) = \delta u^i_I$. The variational differential squares to 0: if we act on (1.21) with δ a second time we get

$$\delta^2\omega = \sum_{\substack{j_1=1\dots N \\ J_1\in\mathbb{Z}^D_{\geq 0}}} \sum_{\substack{j_2=1\dots N \\ J_2\in\mathbb{Z}^D_{\geq 0}}} \frac{\partial^2 f}{\partial u^{j_1}_{J_1} \partial u^{j_2}_{J_2}} \delta u^{j_1}_{J_1} \delta u^{j_2}_{J_2} \delta u^{i_1}_{I_1} \dots \delta u^{i_k}_{I_k} \tag{1.22}$$

where the derivative is symmetric in the exchange $(j_1, J_1) \leftrightarrow (j_2, J_2)$ and the product of δ's is skewsymmetric. Hence, the expression vanishes.

The variational differential δ commutes with the total derivatives $\{\partial_\alpha\}$, as one can check by a straightforward computation. This allows us to define a well-posed action of δ on the quotient space $\mathcal{F}^\Omega$. In particular, on F^Ω_0 we have

$$\begin{aligned} \delta \int^D f \mathrm{d}^D x &= \int^D \left(\sum_{i,I} \frac{\partial f}{\partial u^i_I} \delta u^i_I \right) \mathrm{d}^D x \\ &= \int^D \sum_i \left(\sum_I (-\boldsymbol{\partial})^I \frac{\partial f}{\partial u^i_I} \right) \delta u^i \mathrm{d}^D x \end{aligned} \tag{1.23}$$

The latest expression, obtained by integrating by parts, justifies the notation

$$\frac{\delta F}{\delta u^i} := \sum_I (-\boldsymbol{\partial})^I \frac{\partial f}{\partial u^i_I} \tag{1.24}$$

for the components of the variational 1-form (here F is short for $\int f$). We then adopt the formula (1.24) as definition of the *variational derivative.*

Proposition 1.2. If $f = \partial_\alpha g$ for some $g \in \mathcal{A}$, then $\delta f = 0$.

Proof. Since $\delta f = \frac{\delta f}{\delta u^i} \delta u^i$, we prove the statement by showing the stronger $\frac{\delta f}{\delta u^i} = 0$. From the defintion (1.24) we have

$$\frac{\delta}{\delta u^i} \partial_\alpha f = \sum_I (-\boldsymbol{\partial})^I \frac{\partial}{\partial u^i_I} (\partial_\alpha f) .$$

We apply (1.13b) and get

$$\frac{\delta}{\delta u^i} \partial_\alpha f = \sum_I \left((-1)^{|I|} \boldsymbol{\partial}^{I+\xi_\alpha} \frac{\partial f}{\partial u^i_I} + (-1)^{|I|} \boldsymbol{\partial}^I \frac{\partial f}{\partial u^i_{I-\xi_\alpha}} \right) .$$

Relabelling $I' = I - \xi_\alpha$ in the second term,

$$\begin{aligned} &= \sum_{I,I'} (-1)^{|I|} \boldsymbol{\partial}^{I+\xi_\alpha} \frac{\partial f}{\partial u^i_I} - (-1)^{|I'|} \boldsymbol{\partial}^{I'+\xi_\alpha} \frac{\partial f}{\partial u^i_{I'}} \\ &= 0. \end{aligned}$$

□

One can rely on this result to extend the formula (1.24) to elements of $\mathcal{A}$ (while it is originally defined for elements of $\mathcal{F}$). $\frac{\delta}{\delta u^i}$ turns out to be a projection map $\mathcal{A} \to \mathcal{A} / \sum_\alpha \partial_\alpha \mathcal{A} = \mathcal{F}$.

The converse statement of Proposition 1.2 is discussed and proved in Chapter 4. For a proof that relies on the notion of *variational bicomplex*, which we did not present here, see for instance [1].

1.3 Poisson brackets of hydrodynamic type

The notion of Poisson brackets of hydrodynamic type, or *Dubrovin–Novikov brackets*, has been introduced in [23] in order to characterize the Hamiltonian structure of a class of equations that describe systems such as ideal fluids with internal degrees of freedom. Poisson brackets (also called Poisson structures or Hamiltonian operators by different authors) of this form can be used to describe, for instance, Euler's equation [45]. The most important example of this class of Poisson brackets is the Hamiltonian structures of the KdV equation

$$u_t = u u_x + u_{xxx}$$

that, as well known and first shown by [35], can be written as

$$u_t = \frac{\partial}{\partial x}\frac{\delta}{\delta u}\int\left(\frac{u^3}{6}-\frac{u_x^2}{2}\right)\mathrm{d}x.$$

The total derivative ∂_x is the Hamiltonian structure, and the Poisson bracket it defines is a particular case of Poisson bracket of hydrodynamic type. In the remaining part of this chapter we will discuss the relation between Poisson brackets on the space of local functionals and the appropriate notion of vector fields; moreover, we will summarize the known results on the Poisson cohomology of such brackets, whose extension is the main topic of this thesis.

1.3.1 Vector fields and local k-vectors

Let us consider the space $\mathfrak{X}(\mathcal{A})$ of vector fields on the formal space of maps. These are formal infinite sums

$$X = \sum_{I\in\mathbb{Z}^d_{\geq 0}} X^i_I(u,u_J)\frac{\partial}{\partial u^i_I} \tag{1.25}$$

with $X^i_I \in \mathcal{A}$.

The derivative of a local functional $\int f = F \in \mathcal{F}$ along a vector field X is an element $\hat{X}(F)\in\mathcal{F}$ of form

$$\hat{X}(F) = \int^D \sum_{I\in\mathbb{Z}^D_{\geq 0}} X^i_I(u_J)\frac{\partial f}{\partial u^i_I}\mathrm{d}^D x. \tag{1.26}$$

The Lie bracket between two vector fields is the commutator of the derivations $\widehat{[X,Y]}(F) = \hat{X}(\hat{Y}(F)) - \hat{Y}(\hat{X}(F))$. With a slight abuse of notation and when there is not possibility of misunderstanding, we will denote X both the element of $\mathfrak{X}(\mathcal{A})$ that the linear mapping $\mathcal{F}\to\mathcal{F}$. The total derivatives ∂_α (1.14) can be regarded as vector fields with $X^i_I = u^i_{I+\xi_\alpha}$.

Definition 1.6. An *evolutionary vector field* is a derivation of $\mathcal{A}$ (hence, an element of $\mathfrak{X}(\mathcal{A})$) which commutes with all the total derivatives $\{\partial_\alpha\}$.

A simple computation shows that the condition imposes $X^i_I = \partial_\alpha X^i_{I-\xi_\alpha}$. Applying this relation reculsively we get that an evolutionary vector field has form

$$X = \sum_{\substack{i=1,\dots,n\\(i_1,i_2,\dots,i_D)=I\in\mathbb{Z}^D_{\geq 0}}} \prod_{\alpha=1}^{D}\left(\partial_\alpha^{i_\alpha}\right)\left(X^i(u_J)\right)\frac{\partial}{\partial u^i_I}. \tag{1.27}$$

Using the multiindex notation we have already introduced we can write

$$X = \left(\boldsymbol{\partial}^I X^i\right)\frac{\partial}{\partial u^i_I}.$$

Note that our definition of evolutionary vector field is slightly different from the one given in [25]. Dubrovin and Zhang, indeed, consider a larger class of vector fields (in our notation it would be $\mathfrak{X}(\bar{\mathcal{A}})$) and hence have to impose stricter conditions on it to give the same notion of evolutionary field.

Proposition 1.3. For any evolutionary vector field $X \in \mathfrak{X}(\mathcal{A})$ and for any $f \in \hat{\mathcal{A}}$, we have

$$\hat{X}(F) = \int X(f) = \int X^i \frac{\delta f}{\delta u^i}. \tag{1.28}$$

where, as usual, we denote $F = \int^D f \mathrm{d}^D x$.

The proof is straightforward and consists in the integration by parts of (1.27). Notice that this result can be interpreted, as it is well known in the finite dimensional case, as a duality relation between (evolutionary) vector fields and (variational) 1-forms. We can formally denote $X = X^i \frac{\delta}{\delta u^i}$ and $\delta f = \frac{\delta f}{\delta u^i} \delta u^i$; the pairing is then the D-dimensional integral of the product of densities.

Definition 1.7. A *local k-vector* A is a linear k-alternating map from $\mathcal{F}$ to itself of the form

$$A(F_1, \dots, F_k) = \int^D A^{i_1,\dots,i_k}_{I_1,\dots,I_k} \boldsymbol{\partial}^{I_1} \left(\frac{\delta F_1}{\delta u^{i_1}} \right) \cdots \boldsymbol{\partial}^{I_k} \left(\frac{\delta F_k}{\delta u^{i_k}} \right) d^D x \tag{1.29}$$

where $A^{i_1,\dots,i_k}_{I_1,\dots,I_k} \in \mathcal{A}$, for arbitrary $F_1, \dots, F_p \in \mathcal{F}$. We denote the space of local k-vectors by $\Lambda^k \subset \mathrm{Alt}^p(\mathcal{F}, \mathcal{F})$.

An alternative approach to the definition is to consider the formula (1.29) as the expression of the *evaluation* of a k-vector field on k variational 1-forms. With this picture in mind, we can give a formula for the density of a local k-vector field, that turns out to be

$$B^{i_1,\dots,i^k} = A^{i_1\dots i_k}_{I_1,\dots,I_k} \boldsymbol{\partial}^{I_1}_{x'} \delta(x' - x_1) \boldsymbol{\partial}^{I_2}_{x'} \delta(x' - x_2) \delta(x' - x_2) \cdots \boldsymbol{\partial}^{I_k}_{x'} \delta(x' - x_k). \tag{1.30}$$

The expression of the evaluated k-vector is then

$$\int \cdots \int^D B^{i_1,\dots,i_k}(x', x_1, \dots, x_k, u, u_J) \frac{\delta F_1}{\delta u^{i_1}(x_1)} \cdots \frac{\delta F_k}{\delta u^{i_k}(x_k)} \mathrm{d}^D x' \mathrm{d}^D x_1 \cdots \mathrm{d}^D x_k.$$

The delta functions and their derivatives are the D-dimensional ones and are defined by the usual formulae

$$\int^D f(y) \delta(x - y) \mathrm{d}^D y = f(x)$$

$$\int^D f(y) \boldsymbol{\partial}^I_x \delta(x - y) \mathrm{d}^D y = \int^D f(y) (-\boldsymbol{\partial}_y)^I \delta(x - y) \mathrm{d}^D y = \boldsymbol{\partial}^I f(x)$$

$$\int \cdots \int^D f(x_1, \dots, x_k) \boldsymbol{\partial}^{I_2}_{x_1} \delta(x_1 - x_2) \dots \boldsymbol{\partial}^{I_k}_{x_1} \delta(x_1 - x_k) \mathrm{d}^D x_2 \dots \mathrm{d}^D x_k =$$
$$= \boldsymbol{\partial}^{I_2}_{x_2} \dots \boldsymbol{\partial}^{I_k}_{x_k} f(x_1, \dots, x_k) \big|_{x_1 = x_2 = \dots = x_k}$$

Notice that $A^{i_1,\dots,i_k}_{I_1,\dots,I_k} \in \mathcal{A}$ but $B^{i_1,\dots,i^k}_{I_1,\dots,I_k} \notin \mathcal{A}$, since it explicitly depends on $k+1$ independent variables and it is not a differential polynomial because of the presence of Dirac's delta functions. In the next chapters we will discuss different formalisms that allow for an easier algebraic treatment of the polivector space. Nevertheless, we keep in this chapter the more traditional description in terms of Dirac's deltas or differential operators to facilitate the crossreference between the previously known results and the ones discussed in the thesis.

1.3.2 The Poisson bracket

A local bivector is an element of Λ^2. According to the definition (1.29), bivectors are written as

$$B(F_1, F_2) = \int^D \frac{1}{2} B^{ij}_{IJ}(u, u_L) \boldsymbol{\partial}^I \left(\frac{\delta F_1}{\delta u^i} \right) \boldsymbol{\partial}^J \left(\frac{\delta F_2}{\delta u^j} \right) \mathrm{d}^D x \tag{1.31}$$

and are antisymmetric with respect to the exchange $F_1 \leftrightarrow F_2$. It is possible to integrate by parts the expression (1.31) in such a way that no derivatives act on the first variational one form; without providing the formula for the conversion, we can equivalently define the same local bivector evaluated on the two variational 1-forms δF_1 and δF_2 by

$$B(F_1, F_2) = \int^D \frac{\delta F_1}{\delta u^i} \tilde{B}^{ij}_S \boldsymbol{\partial}^S \left(\frac{\delta F_2}{\delta u^j} \right) \mathrm{d}^D x. \tag{1.32}$$

Following the discussion of the previous paragraph and exploiting the properties of Dirac's delta function, the density of the bivector B has the form

$$C^{ij} = \sum_{S \in Z^D_{\geq 0}} \tilde{B}^{ij}_S (u_J(x)) \boldsymbol{\partial}^S \delta(x - y). \tag{1.33}$$

When we do not specify the variables on which the derivatives $\boldsymbol{\partial}$ act, we mean that they act on the first ones, namely – in (1.33) – on x.

The antisymmetry of the bivector imposes on the components

$$B^{ji}_S = \sum_{T \in \mathbb{Z}^D_{\geq 0}} (-1)^{|T|+1} \binom{T}{S} \boldsymbol{\partial}^{T-S} B^{ij}_T, \tag{1.34}$$

where we use the binomial coefficient for multi-indices

$$\binom{A}{B} = \binom{a_1}{b_1} \cdots \binom{a_d}{b_d} \tag{1.35}$$

and

$$\binom{a}{b} = \begin{cases} \frac{a!}{b!(a-b)!} & 0 \leq b \leq a \\ 0 & \text{otherwise.} \end{cases} \tag{1.36}$$

We will occasionally use also the multinomial coefficients

$$\binom{A}{B_1, \dots, B_n}, \qquad B_n = A - \textstyle\sum_{i=1}^{n-1} B_i$$

which definition is analogue to the one for binomial coefficients with multi-indices, given the usual multinomial coefficient

$$\binom{a}{b_1 \dots b_n} = \frac{a!}{b_1! b_2! \dots b_n!}. \tag{1.37}$$

From the componentwise expression (1.33) we see that each component can be interpreted as a differential operator acting on the Dirac's delta

$$B^{ij}(x, u(x), u(x)_I; \frac{\mathrm{d}}{\mathrm{d}x})\delta(x-y) \tag{1.38}$$

with

$$B^{ij}(x, u(x), u(x)_I; \frac{\mathrm{d}}{\mathrm{d}x}) = \sum B^{ij}_S \boldsymbol{\partial}^S.$$

The Lie bracket between vector fields and the wedge product allow us to define the *Schouten-Nijenhuis* bracket between k-vectors. It is defined by extending the Lie bracket of vector fields with respect to the product, imposing the Leibniz rule $[\alpha, \beta \wedge \gamma] = [\alpha, \beta] \wedge \gamma + (-1)^{(k-1)l} \beta \wedge [\alpha, \gamma]$ if $\alpha \in \Lambda^k$ and $\beta \in \Lambda^l$. The formulae for the Schouten-Nijenhuis bracket between generic k- and l- vectors are in general quite involved, and we are not providing them here. In Chapter 4 we will give an extremely useful formula, originally proposed by Soloviev [54], to compute the Schouten bracket in an easy way.

A *local Poisson structure* is a local bivector $P \in \Lambda^2$ satisfying the Schouten relation $[P, P] = 0$.

Given a Poisson bivector it is possible to define a bilinear operation (that we will call a bracket) on the space of local densities $\mathcal{A}$. It can be used to define a bracket on the space $\mathcal{F}$ of local functionals, which is usually called the *Poisson bracket* of functionals. This name is somehow confusing since the Poisson bracket of functionals is not the bracket on a Poisson algebra; indeed, it fails to be a derivation, because of the lack of a product in the space of functionals.

Given a Poisson structure P, we first define the bracket on $\mathcal{A}$ on the basis elements u^i; we will often refer to them as the *generators* of $\mathcal{A}$. We have

$$\{u^i(x), u^j(y)\} = \sum_S P^{ij}_S \left(u(x), u_I(x)\right) \boldsymbol{\partial}^S \delta(x-y). \tag{1.39}$$

This definition extends to two generic densities $f, g \in \mathcal{A}$ according to the formula

$$\{f(x), g(y)\} = \sum_{L,M} \frac{\partial f}{\partial u^i_L(x)} \frac{\partial g}{\partial u^j_M(y)} \boldsymbol{\partial}^L_x \boldsymbol{\partial}^M_y \{u^i(x), u^j(y)\}. \tag{1.40}$$

Such a bracket satisfies by definition the Leibniz rule, i.e. $\{f, gh\} = \{f, g\}h + g\{f, h\}$ and it is obviously bilinear. An important remark is that such a bracket does not satisfy neither the usual skewsymmetry property nor the Jacobi identity, thus it is not a Lie bracket and, *a fortiori*, not even a Poisson bracket. The reason why the two important properties do not hold is quite natural: we defined a Poisson bivector to be skewsymmetric in the sense (1.34), which means that the skewsymmetry makes sense only after the integration, i.e. on $\mathcal{F}$. On the other hand, in $\mathcal{F}$ the Leibniz property does not hold, but we can give a genuine Lie bracket.

Definition 1.8 (Poisson bracket)**.** A Poisson bracket $\{,\}$ in $\mathcal{F} = \mathcal{A}/\sum_\alpha \partial_\alpha \mathcal{A}$ is a bilinear operation

$$\begin{aligned}\{\cdot,\cdot\}\colon \mathcal{F}\times\mathcal{F} &\to \mathcal{F}\\ \left(\int f, \int g\right) &\mapsto \left\{\int f, \int g\right\}\end{aligned}$$

satisfying the following two fundamental properties:

1. Skewsymmetry: $\{\int f, \int g\} = -\{\int g, \int f\}$
2. Jacobi identity: $\{\int f, \{\int g, \int h\}\} - \{\int g, \{\int f, \int h\}\} = \{\{\int f, \int g\}, \int h\}$

Applying the skewsymmetry property, Jacobi identity can also be written as the vanishing of the expression $\{\{\int f, \int g\}, \int h\} + \text{cycl.}(f, g, h) = 0$ which is the most common way to denote it.

Given a Poisson bivector P of form (1.33), the Poisson bracket of two local functionals $\int^D f(u(x), u_I(x))\mathrm{d}^D x$ and $\int^D g(u(y), u_I(y))\mathrm{d}^D y$ is given by

$$\begin{aligned}\left\{\int f, \int g\right\} &= \iint^D \{f(x), g(y)\}\,\mathrm{d}^D x\mathrm{d}^D y\\ &= \iint^D \sum_{L,M} \frac{\partial f}{\partial u^i_L(x)}\frac{\partial g}{\partial u^j_M(y)}\boldsymbol{\partial}^L_x\boldsymbol{\partial}^M_y\{u^i(x), u^j(y)\}\mathrm{d}^D x\mathrm{d}^D y\\ &= \iint^D \frac{\delta f}{\delta u^i(x)}\frac{\delta g}{\delta u^j(y)}\{u^i(x), u^j(y)\}\mathrm{d}^D x\mathrm{d}^D y\\ &= \sum_{S\in\mathbb{Z}^D_{\geq 0}} \int^D \frac{\delta f}{\delta u^i(x)} P^{ij}_S(u(x), u_I(x))\boldsymbol{\partial}^S \frac{\delta g}{\delta u^j(x)}\mathrm{d}^D x\end{aligned} \tag{1.41}$$

where the second equality is given by (1.40), the third one is obtained by integrating by parts and transferring the total derivatives $\boldsymbol{\partial}$ on the partial derivatives of f and g respectively and the fourth one by performing the integration with respect to $\mathbf{y}$ for the Dirac's delta.

We do not prove here that the Schouten condition for P is the crucial requirement for the bracket to be Lie, namely to satisfy Jacobi identity. Although it is possible to get this result with the Dirac's delta formalism – or even by regarding the Poisson bracket of densities as a distribution and evaluating it on test functions – we are going to present in the next chapter two different formalisms that we deem more efficient in performing computations and in characterizing Poisson bivectors.

1.3.3 Poisson brackets of hydrodynamic type

Let us consider the space $\mathcal{M}$ of maps between a D dimensional manifold Σ and a N dimensional *target* manifold M. We associate to this space the suitable space of differential polynomials $\mathcal{A}$, which can be identified with the space of local densities, and the space of local functionals $\mathcal{F}$ as defined in the previous paragraphs.

Definition 1.9. A Poisson bracket of hydrodynamic type (or, equivalently, a Dubrovin-Novikov (DN) bracket) is a bracket defined by a Poisson bivector whose density is a first order homogeneous differential operator acting on Dirac's delta. Chosen a coordinate system $\{u^i\}_{i=1}^N$ on M and $\{x^\alpha\}_{\alpha=1}^D$ on Σ, the general form of the bracket is

$$P^{ij} = \left(\sum_{\alpha=1}^{D} g^{ij\alpha}(u(x)) \frac{\mathrm{d}}{\mathrm{d}x^\alpha} + b_k^{ij\alpha}(u(x)) u^k_{\xi_\alpha} \right) \delta(x-y). \tag{1.42}$$

The coefficients $g^{ij\alpha}$ and $b_k^{ij\alpha}$ must satisfy a certain set of condition for the expression (1.42) to define a Poisson bracket.

Such conditions have been studied and are still being studied for different values of (D, N) by several authors. They have been related to certain geometric structure, since the seminal paper by Dubrovin and Novikov [23]. A general set of equations valid for all D, N has been obtained by Mokhov [43]; a classification of the nondegenerate brackets for $D = 2$ exists for $N = 2$ [44] and $N = 3, 4$ [29], where it relies on the notion of Killing $(1, 1)$-tensors; in special cases a classification can be obtained for $D = 2$ and arbitrary N, or for $D = 3$, $N \leq 3$ [29], or again for arbitrary D and N [44]. Very recently some first results on the classification of degenerate brackets have appeared [52, 53].

We present here the results by Dubrovin and Novikov for nondegenerate $D = 1$ structures and Mokhov's ones for general (D, n) brackets.

Theorem 1.4 ([23])**.** *Let us consider a $D = 1$ dimensional Poisson bracket of hydrodynamic type*

$$P^{ij}(u(x)) = \left(g^{ij}(u(x)) \frac{\mathrm{d}}{\mathrm{d}x} + b_k^{ij}(u) u_x^k \right) \delta(x-y) \tag{1.43}$$

1. *Under local change of coordinates $u = u(w)$ on M, the coefficients $g^{ij}(u)$ transform like a $(2,0)$-tensor; if $\det g^{ij} \neq 0$, then the expression $b^{ij}_k(u) = g^{is}\Gamma^j_{sk}$ transforms in such a way that Γ^j_{sk} are the Christoffel symbols of a differential geometric connection. Equivalently, we say that b^{ij}_k are contravariant Christoffel symbols.*

2. *The bracket defined by* (1.43) *is skewsymmetric if and only if $g^{ij}(u)$ is symmetric and the connection b^{ij}_k is compatible with the metric, namely $\nabla_k g^{ij} = 0$. If $\det g^{ij} \neq 0$ we can say that g is a pseudo Riemannian metric.*

3. *The bracket satisfies Jacobi identity if and only Γ^i_{jk} is torsion free and the curvature tensor vanishes.*

Since in the nondegenerate case g^{ij} must be flat, then there exist a coordinate system $\{v^i\}_{i=1}^N$ on M such that g^{ij} is constant and b^{ij}_k vanish. We call such system the flat coordinates *or* Darboux coordinates *of the bracket.*

Multidimensional Poisson brackets of hydrodynamic type, namely the brackets of form (1.42), have been studied for several years [24, 43]. The complete set of axioms that the coefficients $(g^{ij\alpha}, b^{ij\alpha}_k)$ are provided by Mokhov [43].

Theorem 1.5. *Let P be a differential operator of form* (1.42). *The bracket among local functionals of density f, g defined by*

$$\left\{ \int f, \int g \right\} := \iint^D \frac{\delta f}{\delta u^i(x)} P^{ij} \frac{\delta g}{\delta u^j(y)} \mathrm{d}^D x \mathrm{d}^D y$$

is a Poisson bracket if and only if

$$g^{ij\alpha} = g^{ji\alpha} \tag{1.44a}$$

$$\frac{\partial g^{ij\alpha}}{\partial u^k} = b^{ij\alpha}_k + b^{ji\alpha}_k \tag{1.44b}$$

$$\sum_{(\alpha,\beta)} \left(g^{ai\alpha} b^{jk\beta}_a - g^{aj\beta} b^{ik\alpha}_a \right) = 0 \tag{1.44c}$$

$$\sum_{(i,j,k)} \left(g^{ai\alpha} b^{jk\beta}_a - g^{aj\beta} b^{ik\alpha}_a \right) = 0 \tag{1.44d}$$

$$\sum_{(\alpha,\beta)} \left[g^{ai\alpha} \left(\frac{\partial b^{jk\beta}_a}{\partial u^r} - \frac{\partial b^{jk\beta}_r}{\partial u^a} \right) + b^{ij\alpha}_a b^{ak\beta}_r - b^{ik\alpha}_a b^{aj\beta}_r \right] = 0 \tag{1.44e}$$

$$g^{ai\beta} \frac{\partial b^{jk\alpha}_r}{\partial u^a} - b^{ij\beta}_a b^{ak\alpha}_r - b^{ik\beta}_a b^{ja\alpha}_r = g^{aj\alpha} \frac{\partial b^{ik\beta}_r}{\partial u^a} - b^{ja\alpha}_a b^{ak\beta}_r - b^{jk\alpha}_a b^{ia\beta}_r \tag{1.44f}$$

$$\begin{aligned}
&\frac{\partial}{\partial u^s}\left[g^{ai\alpha}\left(\frac{\partial b_a^{jk\beta}}{\partial u^r}-\frac{\partial b_r^{jk\beta}}{\partial u^a}\right)+b_a^{ij\alpha}b_r^{ak\beta}-b_a^{ik\alpha}b_r^{aj\beta}\right]\\
&+\frac{\partial}{\partial u^r}\left[g^{ai\beta}\left(\frac{\partial b_a^{jk\alpha}}{\partial u^s}-\frac{\partial b_s^{jk\alpha}}{\partial u^a}\right)+b_a^{ij\beta}b_s^{ak\alpha}-b_a^{ik\beta}b_s^{aj\alpha}\right]\\
&+\sum_{(i,j,k)}\left[b_r^{ai\beta}\left(\frac{\partial b_s^{jk\alpha}}{\partial u^a}-\frac{\partial b_a^{jk\alpha}}{\partial u^s}\right)\right]+\sum_{(i,j,k)}\left[b_s^{ai\alpha}\left(\frac{\partial b_r^{jk\beta}}{\partial u^a}-\frac{\partial b_a^{jk\beta}}{\partial u^r}\right)\right]=0
\end{aligned}\tag{1.44g}$$

The notation $\sum_{(a,b)}$ *means the cyclic summation over the indices. Conditions* (1.44a) – (1.44b) *are equivalent to the skewsymmetry of the bracket, while the other ones are equivalent to the fulfilling of the Jacobi identity.*

In case $\det(g^\alpha)\neq 0$ the bracket is said to be *nondegenerate*. In this case the geometric meaning of the set of conditions is made clearer. Indeed, (1.44b) allows us to write $b_k^{ij\alpha}=-g^{ia\alpha}\Gamma_{ak}^{j\alpha}$ where Γ^α are the (standard) Christoffel symbols of the Levi-Civita connection of the metric g^α . Defining for convenience the *obstruction tensors*

$$\begin{aligned}
T_{jk}^{i\alpha\beta}&=\Gamma_{jk}^{i\beta}-\Gamma_{jk}^{i\alpha}\\
T^{ijk\alpha\beta}&=g^{ia\alpha}g^{kb\beta}T_{ab}^{j\alpha\beta}
\end{aligned}$$

and recalling the definition for the Riemann tensor

$$R_{ijk}^{l\alpha}=\partial_j\Gamma_{ik}^{l\alpha}-\partial_k\Gamma_{ij}^{l\alpha}+\Gamma_{ki}^{m\alpha}\Gamma_{mj}^{l\alpha}-\Gamma_{ij}^{m\alpha}\Gamma_{mk}^{l\alpha}$$

the set of condition (1.44c) – (1.44g) can be replaced by the following ones:

$$\Gamma_{jk}^{i\alpha}=\Gamma_{kj}^{i\alpha}\tag{1.45a}$$

$$R_{ijk}^{l\alpha}=0\tag{1.45b}$$

$$T^{ijk\alpha\beta}=T^{kji\alpha\beta}\tag{1.45c}$$

$$\sum_{(i,j,k)}T^{ijk\alpha\beta}=0\tag{1.45d}$$

$$T^{ijl\alpha\beta}T_{lr}^{k\alpha\beta}=T^{ikl\alpha\beta}T_{lr}^{j\alpha\beta}\tag{1.45e}$$

$$\nabla_r^\alpha T^{ijk\alpha\beta}=0\tag{1.45f}$$

where by ∇^α we denote the covariant derivative compatible with the metric g^α – and thus defined in terms of the Christoffels Γ^α.

The tensors T are called obstruction tensors because their vanishing is the necessary and sufficient conditions for the existence of a system of coordinates $\{u^i\}_{i=1}^n$ on M such that the Poisson brackets can be written in constant form. Despite each metric is flat, indeed, there do not necessarily exist Darboux coordinates: however, if the obstruction tensors vanish it means that in any coordinate system the Christoffel's symbols are equal, hence they are equal and null when one of (and all) the metrics are constant.

Lie–Poisson brackets of hydrodynamic type The notion of *Lie–Poisson bracket* we have introduced in Section 1.1 exists for Poisson brackets of hydrodynamic type as well.

We consider the Lie algebra $\mathfrak{g} = \mathfrak{X}(\Sigma)$ of the vector fields on a manifold, let us say a D-dimensional torus. It has been known for long time that this algebra is tightly related to the Euler's equation for ideal fluids ([2]), in particular when restricted to the divergence–free vector fields. In some coordinates such vector fields can be written as $X(x) = \sum X^i(x)\partial_i$, $i = 1 \dots D$; the components of their commutator are $[X,Y]^i(x) = \sum X^j(x)\partial_j Y^i(x) - Y^j(x)\partial_j X^i(x)$. This implies that the structure functions of $\mathfrak{g}$ must have the form $C^i_{jk}(x,y,z) = \delta^i_j \delta(z-x)\partial_k \delta(y-z) - \delta^i_k \delta(y-x)\partial_j \delta(z-y)$. Given a Lie algebra $\mathfrak{g}$, we endow its dual space $\mathfrak{g}^*$ with a Poisson bracket called the *Lie–Poisson bracket.* In this setting, the coordinates on $\mathfrak{g}^*$ are a set of functions $p_i(x)$ such that

$$\int^D p_i(x) v^i(x) \mathrm{d}^D x$$

behaves as a scalar under change of variables. Here, $v^i(x)$ are the components of a vector field. This means that $p_i(x)$ are densities of 1-forms. The Lie–Poisson bracket is linear in the coordinates and defined by the structure functions as

$$\{p_j(y), p_k(z)\} = \int^D C^i_{jk}(x,y,z) p_i(x) \mathrm{d}^D x. \tag{1.46}$$

The Poisson brackets is then defined by the density

$$P_{ij}(p(x)) = \left(p_i(x) \frac{\partial}{\partial x^j} + p_j(x) \frac{\partial}{\partial x^i} + \frac{\partial p_j(x)}{\partial x^i} \right) \delta(x-y). \tag{1.47}$$

Notice that, in this case $D = n$ and we use the same indices in lower case latin letters both for the independent variables $\{x^i\}$ and the dependent ones $\{p^i\}$. This construction is due to Novikov [45].

1.3.4 Cohomology of 1-dimensional Poisson brackets

There exist Hamiltonian operators on the space $\mathcal{M}$ which are of order greater than one: one of the first examples to be discovered and probably the most celebrated one is the second Hamiltonian structure of KdV equation [42]

$$P_2(u(x)) = \left(\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 4u \frac{\mathrm{d}}{\mathrm{d}x} + 2u_x \right) \delta(x-y). \tag{1.48}$$

The search for a canonical form of the Hamiltonian operators in the infinite dimensional manifold $\mathcal{M}$ has been independently completed by several authors for 1-dimensional case, namely where $D = 1$ [36, 20, 25]. In [25] and [20], this problem takes the name of *triviality problem*, namely the problem of classificating all the deformations of a Poisson structures compatible with a chosed one of hydrodynamic type. This problem is tightly related to the notion of the Poisson–Lichnerowicz cohomology on the class of infinite dimensional manifold we are interested in. As for a finite dimensional symplectic manifold the triviality of the Poisson cohomology (or better, the isomorphism between the Poisson and the De Rham cohomology) has as a consequence the validity of Darboux's theorem, the triviality of the second and third cohomology groups of the Poisson bracket of hydrodynamic type allows to give a statement about the equivalence, up to a certain class of transformations, of all the Poisson brackets on that space.

In this paragraph we first recall the main points of the theory of deformations of Poisson structures, then we define the Poisson–Lichnerowicz cohomology stating the known results.

Deformations of Poisson brackets We introduce a gradation on the space $\mathcal{A}$; it is given by $\deg u^i_I = |I|$. We have $\deg ab = \deg a + \deg b$ for $a, b \in \mathcal{A}$. It easily follows from the definition of total derivatives that $\deg \partial_\alpha f = \deg f + 1$. Let us denote $\mathcal{A}_k$ the homogeneous component of $\mathcal{A}$ of degree k.

Let us consider the transformations

$$u^i \mapsto \tilde{u}^i = \sum_{k=0}^{\infty} \epsilon^k F^i_k(u; u_I),\ i = 1, \dots, N \tag{1.49}$$

on the space $\mathcal{A}$, where $F^i_k \in \mathcal{A}_k$ and

$$\det\left(\frac{\partial F^i_0(u)}{\partial u^j}\right) \neq 0.$$

The transformations (1.49) form a group who is called the *Miura group* [25]. It can be regarded as the group of local diffeomorphisms on the space $\mathcal{A}$, whose Lie algebra is the algebra of the (evolutionary) vector fields on $\mathcal{A}$. The transformation of the 0-th order coordinates u^i is then lifted to the higher order jet variables u^i_I. An important subclass of Miura transformations, that plays a central role in the theory of the deformations of DN brackets, are the so-called *second kind Miura deformations* [40], for which $F^i_0 = u^i$.

Definition 1.10. Given a Poisson bivector $P_0 \in \Lambda^2$, a n-th order infinitesimal *compatible deformation* of P_0 is a bivector $P = P_0 + \sum_{k=1}^{n} \epsilon^k P_k$ such that $[P, P] = O(\epsilon^{n+1})$ and $\deg P_k = \deg P_0 + k$. The bracket associated to a deformed bivector is then

$$\{,\}^{\sim} = \{,\}^0 + \sum_{k=1}^{n} \epsilon^k \{,\}^k. \tag{1.50}$$

In the hydrodynamic type case, where $\deg P_0 = 1$, the degree of each deformation $\deg P_k$ is $k + 1$.

Definition 1.11. A deformation of P_0 is said to be trivial if there exists an element ϕ of the Miura group such that $\phi_* P = P_0$. From Definition 1.10, this implies that ϕ must be of second kind. Equivalently, an infinitesimal deformation is trivial if there exist an evolutionary vector field X such that $[X, P_0] = P$. This is equivalent to say that $\{\phi(u(x)), \phi(u(y))\}_0 = \phi\left(\{u(x), u(y)\}^{\sim}\right) + O(\epsilon^{n+1})$ for a deformed bracket of degree $n+1$.

Example For the case $n = 1$, the constant Poisson structure of hydrodynamic type is the total derivative with respect to the spatial variable x. The second Hamiltonian structure of the KdV equation can be obtained after a change of coordinates in the space $\mathcal{M}$

$$v = v(u) = u^2 + iu'$$

where $\{u(x), u(y)\} = \delta'(x-y)$ and $u'(x) = u_x(x)$. Let us compute $\{v(x), v(y)\}$ with the help of the formula (1.40)

$$\{f(u(x)), g(u(y))\} = \sum_{m,n\in\mathbb{Z}_{\geq 0}} \frac{\partial f}{\partial u^{(m)}} \frac{\partial g}{\partial u^{(n)}} \partial_x^m \partial_y^n \{u(x), u(y)\}$$

and recalling that $f(y)\delta^{(p)}(x-y) = \sum_{q=0}^{p} f^{(q)}(x)\delta^{(p-q)}(x-y)$ and $\partial_x \delta(x-y) = -\partial_y \delta(x-y)$. We get

$$\begin{aligned}
\{v(x), v(y)\} &= \{u^2(x) + iu'(x), u^2(y) + iu'(y)\} \\
&= 4u(x)u(y)\delta'(x-y) + 2iu(y)\partial_x \delta'(x-y) + \\
&\quad + 2iu(x)\partial_y \delta'(x-y) - \partial_x \partial_y \delta'(x-y) \\
&= \delta'''(x-y) + 4(u^2(x) + iu'(x))\delta'(x-y) + \\
&\quad + 2\partial_x (u^2(x) + iu'(x))\delta(x-y) \\
&= \left(\frac{\mathrm{d}^3}{\mathrm{d}x^3} + 4v\frac{\mathrm{d}}{\mathrm{d}x} + 2v' \right) \delta(x-y),
\end{aligned}$$

namely the operator (1.48) acting on the Dirac's delta.

The Poisson–Lichnerowicz complex and its cohomology The Schouten-Nijenhuis bracket we have defined in Paragraph 1.3.2 is a mapping $\Lambda^k \times \Lambda^l \to \Lambda^{k+l-1}$. As in the finite dimensional case, the adjoint action of the Poisson bivector $\mathrm{d}_P = [P, \cdot]$ is a map $\Lambda^k \to \Lambda^{k+1}$ which squares to 0 because of the graded Jacobi identity and the Schouten relation $[P, P] = 0$. This allows to define the Lichnerowicz cochain complex for local k-vector

$$0 \to \Lambda^0 = \mathcal{F} \xrightarrow{\mathrm{d}_P} \Lambda^1 \xrightarrow{\mathrm{d}_P} \Lambda^2 \xrightarrow{\mathrm{d}_P} \cdots \xrightarrow{\mathrm{d}_P} \Lambda^k \xrightarrow{\mathrm{d}_P} \cdots$$

and the cohomology groups as usual

$$H^p(\mathrm{d}_P, \mathcal{F}) = \frac{\ker \mathrm{d}_P \colon \Lambda^p \to \Lambda^{p+1}}{\operatorname{Im} \mathrm{d}_P \colon \Lambda^{p-1} \to \Lambda^p}. \tag{1.51}$$

Remark 1.6. The formula (1.32) that gives the Poisson bracket of two local functionals in terms of the Poisson bivector can be expressed using the Schouten bracket as $\{F, G\} = [[P, F], G] = \hat{X}_F(G)$.

In Paragraph 1.1.3 we have described the geometric meaning of the lower order cohomology groups. Despite the setting was finite dimensional, the formalism of the Schouten brackets and the previous remark allow us to repeat without any difference most of the discussion we have already performed. In particular, the 0-th cohomology group is constituted by the Casimir functionals of the bracket, the elements of H^1 are the symmetries of the Poisson bivector that are not Hamiltonian evolutionary vector fields, and we would like to state that the cocycles in H^2 parametrize the not Miura-equivalent infinitesimal deformations of P_0 while H^3 represents the obstruction to pass from infinitesimal to finite deformations.

The presence of the gradation we introduced earlier in this paragraph changes the picture, in such a way that we need to make clearer statements about H^2 and H^3. First we need to extend the gradation to the k-vectors; this will be done in the following chapters according to a different formalism, but for now let us consider the Dirac's delta functions in the densities of the multivectors and fix

$$\deg \delta(x-y) = 0 \qquad\qquad \deg \boldsymbol{\partial}^I \delta(x-y) = |I|.$$

We will consider the homogeneous component of the cohomology groups according to this gradation, which we denote H^p_k. If we consider first order infinitesimal deformations, namely $P = P_0 + \epsilon P_1$, then the argument of Paragraph 1.1.3 applies – H^2_2 is then the space of compatible deformations that are not Miura equivalent to the undeformed bracket, and the elements of H^3_4 are the obstructions to extend the deformation to the next order.

Let us consider, on the other hand, a k-th order infinitesimal deformation according to Definition 1.10

$$P = P_0 + \epsilon P_1 + \epsilon^2 P_2 + \epsilon^3 P_3 + \cdots + \epsilon^k P_k.$$

The compatibility condition becomes, collecting the coefficients of ϵ^n with $n = 1, \dots, k$, the set of equation

$$\sum_{\substack{l,m \\ l+m=n}} [P_l, P_m] = 0 \tag{1.52}$$

that, explicitly, has form

$$[P_0, P_1] = 0 \tag{1.53a}$$
$$2[P_0, P_2] + [P_1, P_1] = 0 \tag{1.53b}$$
$$[P_0, P_3] + [P_1, P_2] = 0 \tag{1.53c}$$
$$\cdots\cdots$$

It should be noticed that only the first equation has the standard cohomological meaning for infinitesimal deformations, stating $P_1 \in H^1$; the following ones are, indeed, conditions on elements of Λ^3. However, they should not be confused with requirements of triviality of H^3, as it would be in the finite dimensional setting: the deformations P_m ($m > 1$) are given, hence the existence of, for instance, P_1 and P_2 such that (1.53b) holds true does not say anything about H^3_4.

Proposition 1.4. If $H^2_m = 0$ for all $2 \leq m < n$, then there exist a sequence of Miura transformations that can be applied to P in order to reduce the compatible bracket to the form

$$\tilde{P} = P_0 + \epsilon^n \tilde{P}_n + \epsilon^{n+1} \tilde{P}_{n+1} + \cdots + \epsilon^{k'} \tilde{P}_{k'}.$$

Moreover, $\tilde{P}_m \in H^2_{m+1}$ for all $n \leq m \leq 2n-1$.

The first part of the proposition can be proved by induction; in particular and as a basic example let us consider $n = 3$. The hypothesis of Proposition 1.4 implies that $P_1 = [P_0, X]$ for a certain evolutionary vector field X; the Miura transformation is simply $\phi = \exp \mathrm{ad}_{\epsilon X}$, as one easily checks. The second statement is a straightforward consequence of (1.52). Proposition 1.4, in particular, implies that, if $H^2 = 0$, then any compatible deformation of P_0 is trivial.

For $D = 1$ the Poisson cohomology of hydrodynamic type Poisson brackets has been computed, provided that De Rham's cohomology of M is trivial. Degiovanni, Magri and Sciacca [20] and Dubrovin and Zhang [25] directly proved that $H^1 = H^2 = 0$. The more powerful result got by Getzler in an earlier paper provides the computation of all the positive cohomology groups

Theorem 1.7 ([36]). *Given a local Poisson bivector P on a space of maps $\mathcal{M} = \mathrm{Maps}(\Sigma \to M)$, $\dim \Sigma = 1$ and $H^*(\mathrm{d}_{DR}, M) = 0$, the Poisson–Lichnerowicz cohomology vanishes in the positive degrees:*

$$H^p(\mathrm{d}_P) = 0 \quad p > 0. \tag{1.54}$$

The vanishing of the second cohomology group implies that all the deformations of a given Poisson bracket of hydrodynamic type are trivial. Then, relying on Dubrovin and Novikov's result [23], for a certain coordinate system they must be trivial deformations of a *constant* Poisson bracket. Conversely, there locally exists a system of coordinates in the space of maps for which the Poisson bracket is constant; such a system of coordinates can be regarded as the infinite dimensional analogue of the Darboux coordinates on finite dimensional symplectic manifolds.

2

Multidimensional Poisson Vertex Algebras

To effectively deal with the theory of deformations of $D > 1$ Poisson brackets of hydrodynamic type, we recall that the standard formalism to work with would be the formal variatonal calculus introduced in Chapter 1. In the last fifteen years an alternative but equivalent formalism has been used quite extensively, namely what we will call θ *calculus* and that can be traced back to [54]. It has been extensively used starting from [36] until today [4, 40, 41]: we will present it in Chapter 4.

In this chapter we will deal with a newer instrument, the notion of *Poisson Vertex Algebra* [5] (PVA), which has been shown to provide a very effective framework to study Hamiltonian operators. The notion of a PVA, that can be seen as the semiclassical limit of Vertex Algebras [37], has been introduced in order to deal with evolutionary Hamiltonian PDEs in which the unknown functions depend on only one spatial variable (namely, for the case $D = 1$). It provides a good framework for the study of integrability of such a class of equations, and also gives some insights into the study of nonlocal Poisson structures [16].

Let us first briefly introduce the notion of a (one-dimensional) PVA, following [5].

A Poisson Vertex Algebra is a differential algebra $(\mathcal{A}, \partial)$ endowed with a bilinear operation $\mathcal{A} \times \mathcal{A} \to \mathbb{R}[\lambda] \otimes \mathcal{A}$ called a λ-bracket and satisfying the set of properties

1. $\{f_\lambda \partial g\} = (\lambda + \partial)\{f_\lambda g\}$

2. $\{\partial f_\lambda g\} = -\lambda\{f_\lambda g\}$

3. $\{f_\lambda gh\} = \{f_\lambda g\}h + \{f_\lambda h\}g$

4. $\{fg_\lambda h\} = \{f_{\lambda+\partial} h\}g + \{g_{\lambda+\partial} h\}f$

5. $\{g_\lambda f\} = -_{\to}\{f_{-\lambda-\partial} g\}$

6. $\{f_\lambda\{g_\mu h\}\} - \{g_\mu\{f_\lambda h\}\} = \{\{f_\lambda g\}_{\lambda+\mu} h\}$

Let us explain the notation used in 4. and 5. Expand $\{f_\lambda g\} = \sum c_n \lambda^n$ with $c_n \in A$. Then in each term of the RHS of equation 4 one has

$$\{f_{\lambda+\partial} g\} h := \sum_n c_n (\lambda + \partial)^n h$$

Notice that using this convention $\{f_{\lambda+\partial} g\} = \{f_{\lambda+\partial} g\} 1 = \{f_\lambda g\}$. The RHS of the fifth equation is defined by

$$_{\to}\{f_{-\lambda-\partial} g\} := \sum (-\lambda - \partial)^n c_n$$

The main theorem, on which all the theory of PVA in the framework of Hamiltonian PDEs is based, is that from a λ-bracket of a PVA we can get the Poisson bracket between local functionals as

$$\left\{ \int f, \int g \right\} = \int \{f_\lambda g\}\big|_{\lambda=0}.$$

Conversely, given a Poisson structure as a differential operator we can define a λ-bracket between the generators of a suitable differential algebra as the symbol of the differential operator; its extension to the full algebra is directly achieved by using the so called *master formula*.

In the original paper and even in the more recent literature [18] the theory of PVAs has been developed only for one dimensional Hamiltonian operators (in the original language, for a differential algebra with one derivation); since we want to deal with higher dimensional operators, we extend the definitions and the main theorems of [5] introducing so-called *multidimensional Poisson Vertex Algebras*, where the algebra A is endowed with D commuting derivations. For a suitable A, modelled on the algebra of differential polynomials of several variables, we show that the same axioms of a standard PVA, conveniently rephrased, can be used to characterize Poisson structures on this more general space of maps. As an example, we will prove in this setting Theorem 1.5.

The content of this chapter, in particular of Section 2.1, is the core of the published paper [12].

2.1 Poisson Vertex Algebras

Let A be a differential algebra with D commuting derivations. Usually, we consider the algebra of differential polynomials or an extension thereof.

Definition 2.1 (λ-bracket)**.** A λ-bracket (of rank D) on A is a $\mathbb{R}$-linear map

$$\begin{aligned} \{\cdot_{\boldsymbol\lambda}\cdot\}\colon A \times A &\to \mathbb{R}[\lambda_1, \dots, \lambda_D] \otimes A \\ (f, g) &\mapsto \{f_{\boldsymbol\lambda} g\} \end{aligned}$$

which is *sesquilinear*, namely

$$\{\partial_\alpha f_{\boldsymbol{\lambda}} g\} = -\lambda_\alpha \{f_{\boldsymbol{\lambda}} g\} \tag{2.1a}$$

$$\{f_{\boldsymbol{\lambda}} \partial_\alpha g\} = (\partial_\alpha + \lambda_\alpha)\,\{f_{\boldsymbol{\lambda}} g\} \tag{2.1b}$$

and obeys, respectively, the *right* and *left Leibniz rule*

$$\{f_{\boldsymbol{\lambda}} gh\} = \{f_{\boldsymbol{\lambda}} g\}h + \{f_{\boldsymbol{\lambda}} h\}g \tag{2.2a}$$

$$\{fg_{\boldsymbol{\lambda}} h\} = \{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}} h\}g + \{g_{\boldsymbol{\lambda}+\boldsymbol{\partial}} h\}f \tag{2.2b}$$

By definition, the λ-bracket of two elements in A is a polynomial in $\lambda_1, \dots, \lambda_d$ (we will often refer to the collection of λ_α as $\boldsymbol{\lambda}$) with coefficients in A. In general, we can write $\{f_{\boldsymbol{\lambda}} g\} = a(f,g)_{i_1,\dots,i_D} \lambda_1^{i_1} \dots \lambda_D^{i_D}$ which, using the usual multiindex notation, is equivalent to writing $a(f,g)_I \boldsymbol{\lambda}^{\boldsymbol{I}}$. When, as in (2.2b), we write $\{f_{\boldsymbol{\lambda}+\boldsymbol{\partial}} g\}$ it means that the λ product is $a(f,g)_I\,(\boldsymbol{\lambda}+\boldsymbol{\partial})^I$, with the derivation acting on the right (if nothing is written on the right, it is equivalent to the derivatives acting on 1 and thus the only term not vanishing is $\boldsymbol{\lambda}^I$).

Definition 2.2 (Multidimensional Poisson Vertex Algebra)**.** A (D-dimensional) *Poisson Vertex Algebra* is a differential algebra A endowed with a λ-bracket of rank D which is *skewsymmetric*

$$\{g_{\boldsymbol{\lambda}} f\} = -_{\to}\{f_{-\boldsymbol{\lambda}-\boldsymbol{\partial}} g\} \tag{2.3}$$

and satisfy the *PVA-Jacobi identity*

$$\{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}} h\}\} - \{g_{\boldsymbol{\mu}}\{f_{\boldsymbol{\lambda}} h\}\} = \{\{f_{\boldsymbol{\lambda}} g\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}} h\}. \tag{2.4}$$

The notation used in (2.3) means that the differential operators $(-\boldsymbol{\lambda}-\boldsymbol{\partial})$ must be regarded as acting on the coefficient of the bracket, too; namely $_{\to}\{f_{-\boldsymbol{\lambda}-\boldsymbol{\partial}} g\} = (-\boldsymbol{\lambda}-\boldsymbol{\partial})^I a(f,g)_I$.

Theorem 2.1 (Master formula)**.** *Let $A = \mathcal{A}$ be the algebra of differential polynomials as defined in Section 1.2.1. Given two elements $(f,g) \in \mathcal{A}$, their λ-bracket can be expressed in terms of the λ-bracket between the so-called* generators *of $\mathcal{A}$, $\{u^i\}_{i=1,\dots,N}$. We have*

$$\{f_{\boldsymbol{\lambda}} g\} = \sum_{\substack{i,j=1\dots,N \\ R,S\in\mathbb{Z}^D_{\geq 0}}} \frac{\partial g}{\partial u^j_S} (\boldsymbol{\lambda}+\boldsymbol{\partial})^S \{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}} u^j\} (-\boldsymbol{\lambda}-\boldsymbol{\partial})^R \frac{\partial f}{\partial u^i_R}. \tag{2.5}$$

In particular, the skewsymmetry and the PVA-Jacobi property hold if and only if the same properties for the generators hold.

We give here only a sketch of the proof of the theorem. The complete – rather cumbersome – proof extends to the D-dimensional case the Theorem 1.15 of [5] and follows the same ideas, without major technical issues. Our aim is to prove that the master formula provides the unique bilinear opeation satisfying the properties of a PVA for any two elements of $\mathcal{A}$. From sesquilinearity of the bracket between two generators we have that $\{u^i_R{}_{\boldsymbol{\lambda}} u^j_S\} = (\boldsymbol{\lambda}+\boldsymbol{\partial})^S(-\boldsymbol{\lambda})^R\{u^i{}_{\boldsymbol{\lambda}} u^j\}$. Moreover, from the right Leibniz property (2.2a) follows that $\{f_{\boldsymbol{\lambda}}\cdot\}$ is a derivation of $\mathcal{A}$, thus it acts on g only by its derivatives $\frac{\partial g}{\partial u^j_S}$. We get

$$\{f_{\boldsymbol{\lambda}} g\} = \sum \{f_{\boldsymbol{\lambda}} u^j_S\}\frac{\partial g}{\partial u^j_S}. \tag{2.6}$$

Applying the sesquilinearity (2.1b) we thus obtain

$$\{f_{\boldsymbol{\lambda}} g\} = \sum \frac{\partial g}{\partial u^j_S}(\boldsymbol{\lambda}+\boldsymbol{\partial})^S\{f_{\boldsymbol{\lambda}} u^j\} \tag{2.7}$$

where the partial derivatives of g have been put on the left to denote that the total derivatives in the parenthesis act only on the λ-bracket itself, according to the right Leibniz rule (2.2a).

The way in which the derivatives of the first function f enter into the master formula, conversely, is dictated by the left Leibniz rule and the sesquilinearity for the first entry of the bracket. We have

$$\{f_{\boldsymbol{\lambda}} g\} = \sum \{u^i{}_{\boldsymbol{\lambda}+\boldsymbol{\partial}} g\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^R\frac{\partial f}{\partial u^i_R}. \tag{2.8}$$

Note that in (2.8) the total derivatives act also on the partial derivatives of f, as imposed by the left Leibniz rule (2.2b). One can then prove that the skewsymmetry and the PVA-Jacobi identity for the brackets between the generators are the only conditions needed for the corresponding properties between generic elements of $\mathcal{A}$.

In Section 1.3.2 we have noticed that there is a remarkable difference between the bracket defined by the same Poisson bivector in the space of local densities and the one among local functionals. In short, while the former is not a Lie bracket but it is a derivation, the latter – despite being an actual Lie bracket – fails at being the bracket of a Poisson algebra. The main discovery which establishes a relation between the theory of Hamiltonian PDEs and Poisson Vertex Algebras has originally been proved in [5] for a PVA of rank 1, namely that the Poisson bracket (strictly speaking, the bracket defined by a Poisson bivector) among local densities is related to a λ-bracket by the relation

$$\{f, g\} = \{f_{\boldsymbol{\lambda}} g\}\big|_{\boldsymbol{\lambda}=0} \qquad f, g \in \mathcal{A}. \tag{2.9}$$

Its extension to the more general case we are dealing with is straightforward. This fact is summarized by the following

Theorem 2.2. *Let $\mathcal{A}$ be an algebra of differential polynomials with a λ-bracket and consider the bracket on $\mathcal{A}$ defined in* (2.9). *Then*

(a) The bracket (2.9) *induces a well-defined bracket on the quotient space $\mathcal{F}$;*

(b) If the λ-bracket satisfies the axioms of a PVA, then the induced bracket on $\mathcal{F}$ is a Lie bracket.

Proof. Part (a). From the property of sesquilinearity we have that, for any $\alpha = 1, \dots, d$,

$$\{f + \partial_\alpha h, g\} = (\{f_{\boldsymbol{\lambda}} g\} - \lambda_\alpha \{h_{\boldsymbol{\lambda}} g\})\,\big|_{\boldsymbol{\lambda}=0} = \{f, g\} \tag{2.10}$$

$$\begin{aligned} \{f, g + \partial_\alpha h\} &= (\{f_{\boldsymbol{\lambda}} g\} + (\lambda_\alpha + \partial_\alpha)\{f_{\boldsymbol{\lambda}} h\})\,\big|_{\boldsymbol{\lambda}=0} \\ &= \{f, g\} + \partial_\alpha \{f, h\} \sim \{f, g\}. \end{aligned} \tag{2.11}$$

Part (b). The Jacobi property for the bracket follows immediately by setting $\boldsymbol{\lambda} = \boldsymbol{\mu} = 0$ in PVA-Jacobi, while the skewsymmetry is a consequence of the skewsymmetry for the λ-bracket. First, we introduce a notation widely used in [5], namely

$$\left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}} u\right) f(\boldsymbol{\lambda}) = f(\boldsymbol{\lambda} + \boldsymbol{\partial}) u. \tag{2.12}$$

In words, we use the convention that the $\boldsymbol{\partial}$ in the exponent acts only on what is inside the parentheses. This notation is justified by the Taylor expansion of the exponential, which turns out to be equivalent to the RHS; the most important part is to always keep track of the terms on which the derivations are acting on.

We have

$$\begin{aligned} \{g, f\} &= \{g_{\boldsymbol{\lambda}} f\}\big|_{\boldsymbol{\lambda}=0} \\ &= -{}_{\to}\{f_{-\boldsymbol{\lambda}-\boldsymbol{\partial}} g\}\big|_{\boldsymbol{\lambda}=0} \qquad \text{(skewsymmetry)} \\ &= -\left(e^{\boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}}} \{f_{-\boldsymbol{\lambda}} g\}\right)\big|_{\boldsymbol{\lambda}=0} \qquad \text{using (2.12)} \\ &= -\left(1 + \boldsymbol{\partial} \frac{\mathrm{d}}{\mathrm{d}\boldsymbol{\lambda}} + \cdots\right) \{f_{-\boldsymbol{\lambda}} g\}\big|_{\boldsymbol{\lambda}=0} \\ &\sim -\{f, g\}. \end{aligned} \tag{2.13}$$

□

Conversely, given a Poisson bracket among local densities, the corresponding λ-bracket is its formal Fourier transform. The aim of this paragraph is to show that the Fourier transform of the bracket of local densities is indeed a λ-bracket, which satisfies the PVA axioms if and only if the bracket is defined by a local Poisson bivector. This result is very important because working with the λ-brackets we do not deal with differential operators on a quotient space, but with simple differential polynomials.

Definition 2.3 (Formal Fourier transform)**.** Given a $\mathcal{A}$ valued formal distribution $E(x,y)$ (with $x,y\in M$, $\dim M=D$), its *formal Fourier transform* is the linear map

$$E(x,y)\mapsto \int^D e^{\boldsymbol{\lambda}\cdot(x-y)}E(x,y)\mathrm{d}^Dx =: \mathrm{F}E(y,\boldsymbol{\lambda})$$

with values in $\mathcal{A}[\lambda_1,\dots,\lambda_D]$. It is equivalent to the one introduced, in a different context, by Kac and De Sole in [14]. The symbol of the integral $\int^D \mathrm{d}^Dx$ must be regarded as the quotient operator with respect to $\sum_\alpha \partial_{x^\alpha}$, as already discussed in Chapter 1.

Lemma 2.3. *Let us consider a differential operator acting on a Dirac's delta*

$$P(u(x),\boldsymbol{\partial}_x)\delta(x-y)=\sum_S P(u(x))_S\boldsymbol{\partial}_x^S\delta(x-y).$$

Its formal Fourier transform is the symbol of the operator itself, namely

$$\sum_S P(u(x))_S\boldsymbol{\lambda}^S. \tag{2.14}$$

Proof. Expanding the multiindex notation and keeping the sum implicit we have

$$\begin{aligned}\mathrm{F}P(u(y),\boldsymbol{\lambda}) &= \int^D e^{\boldsymbol{\lambda}\cdot(x-y)}P_{s_1\dots s_D}(u(y))\partial_{y^1}^{s_1}\dots\partial_{y^D}^{s_D}\delta(x-y)\mathrm{d}^Dx\\ &= \int^D e^{\boldsymbol{\lambda}\cdot(x-y)}P_{s_1\dots s_D}(u(y))(-\partial_{x^1}^{s_1})\dots(-\partial_{x^D}^{s_D})\delta(x-y)\mathrm{d}^Dx\end{aligned}$$

integrating by parts

$$\begin{aligned}&= \int^D \partial_{x^1}^{s_1}\dots\partial_{x^D}^{s_D}e^{\boldsymbol{\lambda}\cdot(x-y)}P_{s_1\dots s_D}(u(y))\delta(x-y)\mathrm{d}^Dx\\ &= \lambda_1^{s_1}\cdots\lambda_D^{s_D}P_{s_1\dots s_D}(u(y))\end{aligned}$$

which, using the usual multiindex notation, is

$$= P_S(u(y))\boldsymbol{\lambda}^S.$$

□

In order to prove our claim that the Fourier transform of a Poisson bracket of densities is a λ-bracket, we proceed as follows: first, we will prove that the skewsymmetry and the Jacobi property of the bracket among the generators, i.e. the coordinate functions, imply the skewsymmetry (2.3) and the PVA-Jacobi identity (2.4) for λ-bracket. Then we will compute the Fourier transform of the Poisson bracket between two generic densities and we will prove that it is expressed in terms of the Fourier transform of the bracket of generators by the master formula. Hence, the Fourier transform of the Poisson bracket is a λ-bracket.

The Poisson bracket of two coordinate functions $u^i(x)$ and $u^j(y)$ is given by

$$\{u^i(x), u^j(y)\} = P^{ji}(u(y))_S \boldsymbol{\partial}_y^S \delta(x-y)$$

where $P_S^{ji}\boldsymbol{\partial}^S$ are the components of the Poisson bivector defining the bracket. From the lemma 2.3, its Fourier transform is

$$\{u^i_{\boldsymbol{\lambda}} u^j\}(y) = P^{ji}(u(y))_S \boldsymbol{\lambda}^S. \tag{2.15}$$

Lemma 2.4. *The Lie bracket* (2.15) *is skewsymmetric in the sense of* (2.3).

Proof. From the form of the Poisson brackets of generators we have that

$$\begin{aligned}\{u^i(x), u^j(y)\} &= P_S^{ji}(u(y))\boldsymbol{\partial}_y^S \delta(x-y)\\ \{u^j(y), u^i(x)\} &= P_S^{ij}(u(x))\boldsymbol{\partial}_x^S \delta(y-x).\end{aligned}$$

We recall the skewsymmetry relation of the Poisson bivector (1.34), which gives

$$P_S^{ji}(y) = -\sum_T (-1)^{|T|}\binom{T}{S}\boldsymbol{\partial}^{T-S}P_S^{ij}(x) \tag{2.16}$$

and apply it within the Fourier transform. We get

$$\begin{aligned}\{u^i_{\boldsymbol{\lambda}} u^j\} &= \int^D e^{\boldsymbol{\lambda}\cdot(x-y)} P_S^{ji}(u(y))\boldsymbol{\partial}_y^S\delta(x-y)\mathrm{d}^D x\\ &= -\int^D e^{\boldsymbol{\lambda}\cdot(x-y)}(-1)^{|T|}\binom{T}{S}\boldsymbol{\partial}^{T-S}(P_S^{ij}(u(x)))(-\boldsymbol{\partial}_x^S)\delta(x-y)\mathrm{d}^D x\\ &= -\int^D (-1)^{|T|}\binom{T}{S}\boldsymbol{\partial}_x^S\left[e^{\boldsymbol{\lambda}\cdot(x-y)}\boldsymbol{\partial}_x^{T-S}P_T^{ij}(u(x))\right]\delta(x-y)\mathrm{d}^D x\\ &= -\int^D (-1)^{|T|}\binom{T}{S}\binom{S}{L}\boldsymbol{\lambda}^L\boldsymbol{\partial}^{S-L+T-S}P_T^{ij}(u(x))\delta(x-y)\mathrm{d}^D x\\ &= -(-\boldsymbol{\lambda}-\boldsymbol{\partial})^T P_T^{ij}(u(y))\\ &= -_{\rightarrow}\{u^j_{-\boldsymbol{\lambda}-\boldsymbol{\partial}} u^i\}.\end{aligned}$$

□

Lemma 2.5. *The Lie bracket* (2.15) *satisfies the PVA-Jacobi identity, namely*

$$\{u^i_{\boldsymbol{\lambda}}\{u^j_{\boldsymbol{\mu}} u^k\}\} - \{u^j_{\boldsymbol{\mu}}\{u^i_{\boldsymbol{\lambda}} u^k\}\} = \{\{u^i_{\boldsymbol{\lambda}} u^j\}_{\boldsymbol{\mu}+\boldsymbol{\lambda}} u^k\}.$$

The proof of the lemma is a lenghty computation of the double formal Fourier transform with respect to $e^{\boldsymbol{\lambda}\cdot(x-y)}e^{\boldsymbol{\mu}\cdot(y-z)}$ for the three terms of the Jacobi identity, where the dependency of the coordinate functions $u^i(x)$, $u^j(y)$ and $u^k(z)$ on different independent variables plays a crucial role. The detailed account of the computations is left to Paragraph 2.1.1.

To conclude this discussion, we want to show that taking the Fourier transform of the Poisson bracket between two densities gives a formula which coincides with the master formula for a λ-bracket. The computation is rather lengthy but in a sense straightforward. We want to compute

$$\int^D e^{\boldsymbol{\lambda}\cdot(x-y)}\{f(x),g(y)\}\mathrm{d}^D x.$$

We expand the Poisson bracket and get

$$\int^D e^{\boldsymbol{\lambda}\cdot(x-y)}\frac{\partial f(x)}{\partial u^i_M}\frac{\partial g(y)}{\partial u^j_N}\boldsymbol{\partial}_x^M\boldsymbol{\partial}_y^N\left(P^{ji}_S(y)\boldsymbol{\partial}_y^S\delta(x-y)\right)\mathrm{d}^D x.$$

The derivatives respect to x do not act on the coefficients P^{ji}_S because they depend on functions of y by definition. Inside the bracket, moreover, we can trade the derivatives respect to y with the ones respect to x exploiting the properties of Dirac's delta, thus obtaining

$$(-1)^{|M|}\int^D e^{\boldsymbol{\lambda}\cdot(x-y)}\frac{\partial f(x)}{\partial u^i_M}\frac{\partial g(y)}{\partial u^j_N}\boldsymbol{\partial}_y^N\left(P^{ji}_S(y)(-\boldsymbol{\partial}_x)^{M+S}\delta(x-y)\right)\mathrm{d}^D x.$$

Then we perform the derivatives $\boldsymbol{\partial}_y^N$ and use the same trick

$$(-1)^{|M|}\binom{N}{T}\int^D e^{\boldsymbol{\lambda}\cdot(x-y)}\frac{\partial f(x)}{\partial u^i_M}\frac{\partial g(y)}{\partial u^j_N}\boldsymbol{\partial}_y^T(P^{ji}_S(y)(-\boldsymbol{\partial}_x)^{N-T+M+S}\delta(x-y)\mathrm{d}^D x.$$

We integrate by parts and let the $\boldsymbol{\partial}_x$ act properly. Then, we can finally integrate the Dirac's delta and get

$$(-1)^{|M|}\frac{\partial g}{\partial u^j_N}\binom{N}{T}\binom{N-T+M+S}{R}\boldsymbol{\lambda}^{N-T+M+S-R}\boldsymbol{\partial}^T(P^{ji}_S)\boldsymbol{\partial}^R\frac{\partial f}{\partial u^i_M}.$$

By applying the Newton's binomial,

$$\begin{aligned}(-1)^{|M|}\frac{\partial g}{\partial u^j_N}\binom{N}{T}\boldsymbol{\lambda}^{N-T}\boldsymbol{\partial}^T(P^{ji}_S)(\boldsymbol{\lambda}+\boldsymbol{\partial})^{M+S}\frac{\partial f}{\partial u^i_M}\\
=(-1)^{|M|}\frac{\partial g}{\partial u^j_N}(\boldsymbol{\lambda}+\boldsymbol{\partial})^N(P^{ji}_S(\boldsymbol{\lambda}+\boldsymbol{\partial})^{M+S}\frac{\partial f}{\partial u^i_M})\\
=\frac{\partial g}{\partial u^j_N}(\boldsymbol{\lambda}+\boldsymbol{\partial})^N(P^{ji}_S(\boldsymbol{\lambda}+\boldsymbol{\partial})^S(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\frac{\partial f}{\partial u^i_M}).\end{aligned}$$

Recalling the form of the λ-bracket between the generators, the last expression is

$$\frac{\partial g}{\partial u^j_N}(\boldsymbol{\lambda}+\boldsymbol{\partial})^N\{u^i_{\boldsymbol{\lambda}+\boldsymbol{\partial}}u^j\}(-\boldsymbol{\lambda}-\boldsymbol{\partial})^M\frac{\partial f}{\partial u^i_M}),$$

namely the master formula.

We have thus proved the following theorem

Theorem 2.6. *Given a local Poisson bivector P on the space of maps* $\mathrm{Map}(\Sigma, M) \cong \mathcal{A}$*, the Fourier transform of the bracket induced by the bivector is the λ-bracket of a Poisson Vertex Algebra on $\mathcal{A}$.*

$$\{f_{\boldsymbol{\lambda}}g\}(y) := \int^D e^{\boldsymbol{\lambda}\cdot(x-y)}\{f(x), g(y)\}\mathrm{d}^D x. \tag{2.17}$$

2.1.1 Proof of Lemma 2.5

Let us consider the three generators $u^i(x)$, $u^j(y)$ and $u^k(z)$. Let us consider the double Fourier-like transform

$$\iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{u^i(x),\{u^j(y),u^k(z)\}\}\mathrm{d}^D x\mathrm{d}^D y \tag{2.18}$$

The first step is to expand the outer bracket , which gives

$$\begin{aligned}&\iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\left(\boldsymbol{\partial}^L_z\{u^i(x),u^l(z)\}\right)\frac{\partial\{u^j(y),u^k(z)\}}{\partial u^l_L}\mathrm{d}^D x\mathrm{d}^D y\\ &=\iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}\frac{\partial}{\partial u^l_L}\left(e^{\boldsymbol{\mu}\cdot(y-z)}\{u^j(y),u^k(z)\}\right)\left(\boldsymbol{\partial}^L_z\{u^i(x),u^l(z)\}\right)\mathrm{d}^D x\mathrm{d}^D y.\end{aligned}$$

If we perform the integration with respect to y, which appears only in the first parenthesis, we get by definition the λ-bracket (with parameter $\boldsymbol{\mu}$) of the two generators u^j and u^k. Thus, we have got the partial result

$$\begin{aligned}&\iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{u^i(x),\{u^j(y),u^k(z)\}\}\mathrm{d}^D x\mathrm{d}^D y\\ &\qquad\qquad=\int^D e^{\boldsymbol{\lambda}\cdot(x-z)}\{u^i(x),\{u^j_{\boldsymbol{\mu}}u^k\}(z)\}\mathrm{d}^D x.\end{aligned}$$

Let us for simplicity denote $\{u^j_{\boldsymbol{\mu}}u^k\}(z) = g(z)$. A step backwards in the computation brings us back to

$$\begin{aligned}&\int^D e^{\boldsymbol{\lambda}\cdot(x-z)}\frac{\partial g(z)}{\partial u^l_L}\left(\boldsymbol{\partial}^L_z\{u^i(x),u^l(z)\}\right)\mathrm{d}^D x\\ &\qquad=\binom{L}{T}\int^D\frac{\partial g(z)}{\partial u^l_L}e^{\boldsymbol{\lambda}\cdot(x-z)}\left(\boldsymbol{\partial}^T_z P^{li}_S(z)\right)\boldsymbol{\partial}^{L-T+S}_z\delta(x-z)\mathrm{d}^D x\end{aligned}$$

where the second line is obtained by simply expanding the derivatives of the bracket. By substituting as usual $\boldsymbol{\partial}_z\delta(x-z)$ with $(-\boldsymbol{\partial}_x)\delta(x-z)$ and integrating by parts we get

$$\binom{L}{T}\frac{\partial g(z)}{\partial u^l_L}\boldsymbol{\lambda}^{L-T+S}\left(\boldsymbol{\partial}_z^T P^{li}_S(z)\right) = \frac{\partial g(z)}{\partial u^l_L}(\boldsymbol{\lambda}+\boldsymbol{\partial}_z)^T P^{li}_S(z)\boldsymbol{\lambda}^S = \{u^i_{\boldsymbol{\lambda}} g\}$$

where the last equality is given by (2.7). Summarizing, we have

$$\iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{u^i(x),\{u^j(y),u^k(z)\}\}\mathrm{d}^Dx\mathrm{d}^Dy = \{u^i_{\boldsymbol{\lambda}}\{u^j_{\boldsymbol{\mu}}u^k\}\}. \quad (2.19)$$

The second term for the Jacobi identity among three coordinate functions is the same with $u^i(x)$ replaced by $u^j(y)$. The same computations hold provided the switching, and this gives as second term of the Fourier transform of the Jacobi identity

$$\iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{u^j(y),\{u^i(x),u^k(z)\}\}\mathrm{d}^Dx\mathrm{d}^Dy = \{u^j_{\boldsymbol{\mu}}\{u^i_{\boldsymbol{\lambda}}u^k\}\}. \quad (2.20)$$

The RHS term of the PVA-Jacobi identity is more complicated to achieve. As before, let us start from expanding the usual formula for the Poisson bracket

$$\begin{aligned}
&\iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\{\{u^i(x),u^j(y)\},u^k(z)\}\mathrm{d}^Dx\mathrm{d}^Dy \\
&= \iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\frac{\partial\{u^i(x),u^j(y)\}}{\partial u^l_L(y)}\boldsymbol{\partial}_y^L\{u^l(y),u^k(z)\}\mathrm{d}^Dx\mathrm{d}^Dy \\
&= \iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\frac{\partial\{u^i(x),u^j(y)\}}{\partial u^l_L(y)}\boldsymbol{\partial}_y^L\left(P^{kl}_M(z)\boldsymbol{\partial}_z^M\delta(y-z)\right)\mathrm{d}^Dx\mathrm{d}^Dy \\
&= \iint^D e^{\boldsymbol{\lambda}\cdot(x-z)}e^{\boldsymbol{\mu}\cdot(y-z)}\frac{\partial\{u^i(x),u^j(y)\}}{\partial u^l_L(y)}P^{kl}_M(z)\boldsymbol{\partial}_y^L\boldsymbol{\partial}_z^M\delta(y-z)\mathrm{d}^Dx\mathrm{d}^Dy
\end{aligned}$$

The derivative with respect to y in the third does not act on $P^{kl}_M(z)$, so we could move it further. Moreover, for convenience we can trade $\boldsymbol{\partial}_y^L\boldsymbol{\partial}_z^M\delta(y-z)$ for $(-1)^{|L|}(-\boldsymbol{\partial}_y)^{M+L}\delta(y-z)$ exchanging two times the variables respect to which we derive the Dirac's delta. It allows us to integrate by parts the delta's derivatives, in order to get

$$
\begin{aligned}
&= (-1)^{|L|} \iint^D \boldsymbol{\partial}_y^{L+M} \left(e^{\boldsymbol{\mu}\cdot(y-z)} \frac{\partial\{u^i(x), u^j(y)\}}{\partial u^l_L(y)} \right) e^{\boldsymbol{\lambda}\cdot(x-z)} P^{kl}_M(z)\delta(y-z)\mathrm{d}^D x \mathrm{d}^D y \\
&= (-1)^{|L|} \binom{L+M}{T} \iint^D \boldsymbol{\partial}_y^{T} \left(\frac{\partial\{u^i(x), u^j(y)\}}{\partial u^l_L(y)} \right) \boldsymbol{\mu}^{L+M-T} e^{\boldsymbol{\mu}\cdot(y-z)} e^{\boldsymbol{\lambda}\cdot(x-z)} \cdot \\
&\qquad \cdot P^{kl}_M(z)\delta(y-z)\mathrm{d}^D x \mathrm{d}^D y \\
&= (-1)^{|L|} \binom{L+M}{T} \int^D \boldsymbol{\partial}_z^{T} \left(\frac{\partial\{u^i(x), u^j(z)\}}{\partial u^l_L(z)} \right) \boldsymbol{\mu}^{L+M-T} e^{\boldsymbol{\lambda}\cdot(x-z)} P^{kl}_M(z)\mathrm{d}^D x.
\end{aligned}
$$

From the form for $\{u^i(x), u^j(z)\}$ we see that the partial derivatives act only on the coefficients P^{ji}_N. So, we get that our expression is equal to

$$
\begin{aligned}
&(-1)^{|L|} \binom{L+M}{T} \boldsymbol{\mu}^{L+M-T} \cdot \\
&\qquad \cdot \int^D \boldsymbol{\partial}_z^{T} \left(\frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)} \boldsymbol{\partial}_z^N \delta(x-z) \right) e^{\boldsymbol{\lambda}\cdot(x-z)} P^{kl}_M(z)\mathrm{d}^D x
\end{aligned}
$$

Basically we repeat the computation applying the same rules for multiderivatives of product and the integration by parts of the Dirac's delta and we end, after the integration, with

$$
(-1)^{|L|} \binom{L+M}{T} \binom{T}{R} P^{kl}_M(z) \boldsymbol{\mu}^{L+M-T} \boldsymbol{\lambda}^{T-R+N} \boldsymbol{\partial}_z^R \frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)}.
$$

The rules for the product of binomials hold also in the multiindices case, since they are only a product of ordinary binomials. It means, by slightly abusing the notation, that

$$
\binom{A}{B}\binom{B}{C} = \frac{A!}{B!(A-B)!}\frac{B!}{C!(B-C)!} = \binom{A}{A-B, C}
$$

In our case, calling $L+M-T=Q$, we get

$$
\begin{aligned}
&(-1)^{|L|} \binom{L+M}{Q, R} P^{kl}_M(z) \boldsymbol{\mu}^{Q} \boldsymbol{\lambda}^{L+M-Q-R} \boldsymbol{\partial}^R \frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)} \boldsymbol{\lambda}^N \\
&\qquad = (-1)^{|L|} P^{kl}_M(z) (\boldsymbol{\lambda}+\boldsymbol{\mu}+\boldsymbol{\partial})^{L+M} \frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)} \boldsymbol{\lambda}^N.
\end{aligned}
$$

Finally, the sign in front of the expression can be replaced by writing

$$
P^{kl}_M(z) (\boldsymbol{\lambda}+\boldsymbol{\mu}+\boldsymbol{\partial})^{M} (-\boldsymbol{\lambda}-\boldsymbol{\mu}-\boldsymbol{\partial})^{L} \frac{\partial P^{ji}_N(z)}{\partial u^l_L(z)} \boldsymbol{\lambda}^N
$$

which is clearly the expression in terms of (2.8) of $\{u^i_{\boldsymbol{\lambda}} u^j\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}} u^k\}$, the RHS of the PVA-Jacobi identity. We have finally proved that taking the double Fourier transform with respect to $e^{\boldsymbol{\lambda}\cdot(x-z)+\boldsymbol{\mu}\cdot(y-z)}$ of the Jacobi identity for the Poisson bracket of the generators gives the PVA-Jacobi identity among them.

2.1.2 Poisson brackets of hydrodynamic type and Poisson Vertex Algebras

In this paragraph we exploit the formalism we have just introduced to provide a new proof of Theorem 1.5.

The Poisson bracket is defined on the space $\mathcal{A}$ of differential polynomials without explicit dependance on the independent variable. Let us consider N dependent fields u^i and D independent variables x^α. A generic Poisson bracket of hydrodynamic type (1.42), in terms of λ-bracket among the generators of $\mathcal{A}$, has the form

$$\{u^i_{\boldsymbol{\lambda}} u^j\} = \sum_{\alpha=1}^{D} g^{ij\alpha}(u)\lambda_\alpha + \sum_{\substack{\alpha=1\dots D\\ k=1\dots N}} b^{ij\alpha}_k(u)\partial_\alpha u^k. \tag{2.21}$$

Since the order of derivatives we will deal with is not very high, it is easier to switch back to a single-index notation, namely $\boldsymbol{\lambda}^{\xi_\alpha} = \lambda_\alpha$ and (for instance) $u_{\xi_\alpha+\xi_\beta} = \boldsymbol{\partial}_{\alpha\beta} u = u_{\alpha\beta}$.

To get the conditions of Theorem 1.5, we explicitly impose the skewsymmetry condition (2.3) and the PVA-Jacobi identity (2.4) for the bracket (2.21) among three generators of $\mathcal{A}$. The vanishing of the first degree terms in λ_α for (2.3) are the conditions (1.44a), while the vanishing of the coefficients of u^k_α are (1.44b).

We then use the master formula to compute (2.4). It gives a degree 2 differential polynomial in the λ's and the μ's. The remaining conditions are the vanishing of the coefficients for, respectively, $\lambda_\alpha\lambda_\beta$, $\lambda_\alpha\mu_\beta$ (the coefficients for $\mu \leftrightarrow \lambda$ are equivalent, provided the skewsymmetry), $u^r_{\alpha\beta}$, $u^r_\alpha\lambda_\beta$, and $u^r_\alpha u^s_\beta$.

2.2 Poly-lambda brackets and the cohomology of Poisson Vertex Algebras

The identification we established in the previous Section between bivectors and λ-brackets, and in particular the bijection between Poisson bivectors and λ-brackets of a PVA, can be extended to other objects that can be identified with higher polyvector fields. These ideas are presented, for a $D = 1$ Poisson Vertex Algebra, in [5] following the notions introduced in [15]. In [17] the same structure is defined in an equivalent but at sight very different fashion.

The identification is much more natural in the original approach, that we will revise in this Section for the multidimensional case. Then, in the same way that a Poisson bivector can be used to define the Poisson–Lichnerowicz cohomology of polyvector fields, we will define the cohomology of Poisson Vertex Algebras. However, we will briefly comment on the definition of poly-λ-brackets used in [16] in Section 3.4, since it can be effectively used to speed up computations.

2.2.1 Poly-λ brackets

Definition 2.4. A k-λ *bracket* on a (D-dimensional) Poisson Vertex Algebra $(A, \{\cdot_{\boldsymbol{\lambda}}\cdot\})$ is a $\mathbb{R}$-linear map

$$\begin{aligned} A^{\otimes k} &\to A \otimes \mathbb{R}[\boldsymbol{\lambda}_1, \cdots, \boldsymbol{\lambda}_{k-1}] \\ f_1 \otimes f_2 \otimes \cdots \otimes f_k &\mapsto \{f_1{}_{\boldsymbol{\lambda}_1} f_2{}_{\boldsymbol{\lambda}_2} \cdots f_{k-1}{}_{\boldsymbol{\lambda}_{k-1}} f_k\} \end{aligned}$$

satisfying the following conditions

1. $\{f_1{}_{\boldsymbol{\lambda}_1} f_2{}_{\boldsymbol{\lambda}_2} \cdots {}_{\boldsymbol{\lambda}_{i-1}} \partial_\alpha(f_i)_{\boldsymbol{\lambda}_i} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\} = -\lambda_{i,\alpha}\{f_1{}_{\boldsymbol{\lambda}_1} f_2{}_{\boldsymbol{\lambda}_2} \cdots {}_{\boldsymbol{\lambda}_{i-1}} f_i{}_{\boldsymbol{\lambda}_i} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\}$ for $i \leq 1 < k$; we denote $(\boldsymbol{\lambda}_i)_{\xi_\alpha} = \lambda_{i,\alpha}$;

2. $\{f_1{}_{\boldsymbol{\lambda}_1} f_2{}_{\boldsymbol{\lambda}_2} \cdots {}_{\boldsymbol{\lambda}_{k-1}} \partial_\alpha(f_k)\} = (\lambda_{1,\alpha} + \cdots + \lambda_{k-1,\alpha} + \partial_\alpha)\{f_1{}_{\boldsymbol{\lambda}_1} f_2{}_{\boldsymbol{\lambda}_2} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\}$.

3. The bracket is skewsymmetric in the simultaneous exchange of f_i's and $\boldsymbol{\lambda}_i$'s. Let us call σ the permutation of the indices $\{1, \ldots, k\}$. We have

   $$\{f_1{}_{\boldsymbol{\lambda}_1} f_2{}_{\boldsymbol{\lambda}_2} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\} = \operatorname{sgn}(\sigma)\{f_{\sigma(1)}{}_{\boldsymbol{\lambda}_{\sigma(1)}} f_{\sigma(2)}{}_{\boldsymbol{\lambda}_{\sigma(2)}} \cdots {}_{\boldsymbol{\lambda}_{\sigma(k-1)}} f_{\sigma(k)}\}|_{\boldsymbol{\lambda}_k = \boldsymbol{\lambda}'_k}$$

   where we define $\boldsymbol{\lambda}'_k = -\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 - \cdots - \boldsymbol{\partial}$ and $\boldsymbol{\partial}$ acts on the coefficients of the bracket itself;

4. $\{f_1{}_{\boldsymbol{\lambda}_1} \cdots f_i g_i{}_{\boldsymbol{\lambda}_i} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\} = \{f_1{}_{\boldsymbol{\lambda}_1} \cdots f_i{}_{\boldsymbol{\lambda}_i + \boldsymbol{\partial}} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\} g_i +$ $+ \{f_1{}_{\boldsymbol{\lambda}_1} \cdots g_i{}_{\boldsymbol{\lambda}_i + \boldsymbol{\partial}} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\} f_i$ for $1 \leq i < k$;

5. $\{f_1{}_{\boldsymbol{\lambda}_1} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k g_k\} = \{f_1{}_{\boldsymbol{\lambda}_1} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\} g_k + \{f_1{}_{\boldsymbol{\lambda}_1} \cdots {}_{\boldsymbol{\lambda}_{k-1}} g_k\} f_k$.

It is easy to see that for $k = 2$ the properties listed above are respectively sesquilinearity, skewsymmetry and Leibniz properties of a λ-bracket.

Proposition 2.1. Any k-λ-brackets on the algebra of differential polynomials $\mathcal{A}$ is in one-to-one correspondance to a local k-vector on $\mathcal{F}$. The mapping is

$$B(F_1, F_2, \ldots, F_k) = \int^D \{f_1{}_{\boldsymbol{\lambda}_1} f_2{}_{\boldsymbol{\lambda}_2} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\}|_{\boldsymbol{\lambda}_1 = \boldsymbol{\lambda}_2 = \cdots = \boldsymbol{\lambda}_{k-1} = 0} \mathrm{d}^D x \quad (2.22)$$

where the k-vector B is defined according to Definition 1.7 and $F_i = \int^D f_i$.

The identification is achieved generalizing to $(k-1)$ formal indeterminates $\{\boldsymbol{\lambda}_i\}$ the procedure we used in Theorem 2.3 and Theorem 2.6. The only further remark we need is to notice that, given a k-vector A as defined in (1.29), we can write it as a polydifferential operator acting on $(k-1)$ Dirac's delta functions evaluated on k one-forms, as we did for a bivector in (1.32). The symbol of such an operator turns out to be a k-λ-bracket by a straightforward generalization of Theorem 2.3; the k-vector field corresponding to a poly-λ-bracket of form $\{u^1{}_{\boldsymbol{\lambda}_1} u^2{}_{\boldsymbol{\lambda}_2} \cdots {}_{\boldsymbol{\lambda}_{k-1}} u^k\} = f^{i_1,i_2,\dots,i_k}_{I_1,I_2,\cdots,I_{k-1}} \boldsymbol{\lambda}_1^{I_1} \boldsymbol{\lambda}_2^{I_2} \cdots \boldsymbol{\lambda}_{k-1}^{I_{k-1}}$ is

$$B(F_1,F_2,\dots,F_k) = \int^D f^{i_1,i_2,\dots,i_k}_{I_1,I_2,\cdots,I_{k-1}} \boldsymbol{\partial}^{I_1}\left(\frac{\delta F_1}{\delta u^{i_1}}\right) \cdots \boldsymbol{\partial}^{I_{k-1}}\left(\frac{\delta F_{k-1}}{\delta u^{i_{k-1}}}\right) \frac{\delta F_k}{\delta u^k} \mathrm{d}^D x, \tag{2.23}$$

namely we integrate by parts in the definition of k-vector fields to drop all the total derivatives acting on the last entry.

It is possible to establish an isomorphism between the space of k-vector fields (with some care for the cases $k=0$ and $k=1$) and k-λ-brackets. The mapping between the spaces is the content of Proposition 2.1; the missing piece in our discussion is the operation between k-λ-brackets that is mapped to the Schouten-Nijenhuis brackets between multivectors. A full account of it is given in [17] for the case of $D=1$, but it is easily generalized to the generic multidimensional case. Let us call $W_k(\mathcal{A})$ or, for short W_k the space of k-λ-brackets, isomorphic to Λ^k.

2.2.2 Cohomology of Poisson Vertex Algebras

In Section 2.1 we have proved the correspondence between local Poisson bivectors and PVAs. We have discussed in Paragraph 1.1.3 the notion of Poisson–Lichnerowicz cohomology and in Paragraph 1.3.4 we have considered the case of Poisson brackets of hydrodynamic type.

The main remark is that, if P is a Poisson bivector, namely a bivector such that $[P,P]=0$, then $\mathrm{d}_P = [P,\cdot]$ is a coboundary operator. It is quite natural to repeat the construction in the context of Poisson Vertex Algebras, closely following [17].

Let us consider an element $X \in W_k(A)$. We will write alternatively $X_{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\dots,\boldsymbol{\lambda}_k}(f_1,\dots,f_k) = \{f_1{}_{\boldsymbol{\lambda}_1} \cdots {}_{\boldsymbol{\lambda}_{k-1}} f_k\}$, $\boldsymbol{\lambda}_k = -\boldsymbol{\lambda}_1 - \boldsymbol{\lambda}_2 - \cdots - \boldsymbol{\lambda}_{k-1} - \boldsymbol{\partial}$. Using the first notation $X_{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\dots,\boldsymbol{\lambda}_k}(f_1,\dots,f_k)$ the skewsymmetry of the k-λ-bracket is equivalent to the total skewsymmetry of X in the simultaneous exchange $(f^i,\boldsymbol{\lambda}_i) \leftrightarrow (f^j,\boldsymbol{\lambda}_j)$.

Given a Poisson bivector P, whose associated λ-bracket we denote $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_P$, we extend to the multidimensional PVA the definition of [17] for the PVA differential and we get

$$
\begin{aligned}
&(\mathrm{d}_P X)_{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2,\dots,\boldsymbol{\lambda}_{k+1}}(f^1,\dots,f^{k+1}) = \\
&\qquad(-1)^{k+1}\left(\sum_{i=1}^{k+1}(-1)^{i-1}\{f^i{}_{\boldsymbol{\lambda}_i} X_{\boldsymbol{\lambda}_1,\overset{i}{\check{\dots}},\boldsymbol{\lambda}_{k+1}}(f^1,\overset{i}{\check{\dots}},f^{k+1})\}_P + \right.\\
&+\sum_{1\le i<j\le k+1}(-1)^{i+j} X_{\boldsymbol{\lambda}_i+\boldsymbol{\lambda}_j,\boldsymbol{\lambda}_1,\overset{i}{\check{\dots}}\overset{j}{\check{\dots}},\boldsymbol{\lambda}_{k+1}}\left(\{f^i_{\boldsymbol{\lambda}_i} f^j\}_P, f^1,\overset{i}{\check{\dots}},\overset{j}{\check{\dots}},f^{k+1}\right)\Bigg)
\end{aligned}
\tag{2.24}
$$

The differential $\mathrm{d}_P\colon W_{k-1}\to W_k$ is in one-to-one correspondence with $[P,\cdot]$ the ordinary coboundary operator of the Poisson–Lichnerowicz cohomology, and it squares to 0 thanks to the PVA-Jacobi identity.

We provide the special formulae for $k=0,1,2$, respectively for ($h\in W_0, \boldsymbol{\lambda}_1=-\boldsymbol{\partial}$), ($X\in W_1, \boldsymbol{\lambda}_2=-\boldsymbol{\lambda}_1-\boldsymbol{\partial}$), and ($Q\in W_2,\boldsymbol{\lambda}_3=-\boldsymbol{\lambda}-\boldsymbol{\mu}-\boldsymbol{\partial}$).

$$(\mathrm{d}_P h)_{\boldsymbol{\lambda}_1}(f) = \{h_{\boldsymbol{\lambda}} f\}_P|_{\boldsymbol{\lambda}=0} \tag{2.25}$$

$$(\mathrm{d}_P X)_{\boldsymbol{\lambda}_1,\boldsymbol{\lambda}_2}(f,g) = \{f_{\boldsymbol{\lambda}_1} X(g)\} + \{X(f)_{\boldsymbol{\lambda}_1} g\} - X(\{f_{\boldsymbol{\lambda}_1} g\}) \tag{2.26}$$

$$
\begin{aligned}
(\mathrm{d}_P Q)_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\lambda}_3}(f,g,h) ={}& \{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}} h\}_Q\}_P + \{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}} h\}_P\}_Q - \{g_{\boldsymbol{\mu}}\{f_{\boldsymbol{\lambda}} h\}_Q\}_P \\
&- \{g_{\boldsymbol{\mu}}\{g_{\boldsymbol{\lambda}} h\}_P\}_Q - \{\{f_{\boldsymbol{\lambda}} g\}_{Q\,\boldsymbol{\lambda}+\boldsymbol{\mu}} h\}_P - \{\{f_{\boldsymbol{\lambda}} g\}_{P\,\boldsymbol{\lambda}+\boldsymbol{\mu}} h\}_Q
\end{aligned}
\tag{2.27}
$$

The space $W=\bigoplus_{k\ge 0} W_k$ together with the differential d_P form a cochain complex, whose cohomology we define as always and denote $H^\bullet(\mathcal{A},\mathrm{d}_P)$. The interpretation of the cohomology groups is the same as in Paragraph 1.3.4; this follows immediately from the identification of W_k with Λ^k.

Notice that $Z^0=H^0$, $Z^1=\ker \mathrm{d}_P(W_1)$ and Z^2 are immediately characterized by formulae (2.25), (2.26), and (2.27) as the Casimir functions of the bracket, the symmetries of the Poisson bivector, and the λ-brackets compatible with the structure P respectively.

We use the same grading defined in Paragraph 1.3.4 also when dealing with the cohomology of PVAs. Since in this picture the role of the derivatives of Dirac's delta function is replaced by the formal indeterminates $\boldsymbol{\lambda}$s, we assign to $\boldsymbol{\lambda}^I$ the degree $|I|$. Any λ-bracket can be decomposed, according to this grading, into homogenous parts. A rigorous way to obtain such a decomposition is to introduce a formal indeterminate ε of degree -1 and rescale the bracket in such a way that $\sum_{k=0}^\infty \varepsilon^k\{u^i_{\boldsymbol{\lambda}} u^j\}^k$ is of degree 0.

We can use this grading to decompose each cohomology group

$$H^k(\mathcal{A},\mathrm{d}_P) = \bigoplus_d H^k_d(\mathcal{A},\mathrm{d}_P).$$

Notice that the degree of the composition is consisten with the one introduced in the previous chapter; on the other hand, it should not be confused with the notion of *order* we introduced in the same place and that we will discuss more extensively in the next chapter, where we explicitly compute some of the cohomology groups H^k_d for the brackets of type $(N=2, D=2)$.

2.3 A Mathematica package for computing λ-brackets

We have already claimed that the formalism of multidimensional PVAs provides a very effective way to perform the computations in the context of Poisson structures on infinite dimensional spaces. In particular, we are interested in writing down the equations that define Z^k and B^k, the spaces of cocycles and coboundaries of d_P. The theorems we proved in the previous Sections allow to work in the context of PVAs rather than in the standard formal calculus of variations; the reason why it is easier to work in this new framework is twofold.

On one hand, the objects (brackets, multivectors, etc...) defined in Chapter 1 need the integration, or equivalently the projection to the quotient spaces $\mathcal{F}$ and Λ; this is particularly complicated in the case of multivectors, since the integration of the Dirac's delta functions, in particular of the product of some of them, relies on a series of properties as the ones given in [25, Equations (2.3.13) and (2.3.14)] that are not easy to handle. As opposite, the axioms of PVA allow to characterize the skewsymmetry and the fulfillment of Jacobi identity of the bracket between local functionals in terms of λ-bracket between densities, without any further integration. This does not hold true only for the bracket itself, as stated in Equation (2.9) and Theorem 2.6, but for instance also for Hamiltonian vector fields, for which the following relation holds (see [5])

$$X_h(f) = \{h_{\boldsymbol{\lambda}} f\}|_{\boldsymbol{\lambda}=0}. \tag{2.28}$$

This can be easily shown by writing the master formula for the RHS, dropping $\boldsymbol{\lambda}$ and keeping only $\boldsymbol{\partial}$; for the LHS it is sufficient to remind that $X(f) = \boldsymbol{\partial}^S X^j \frac{\partial f}{\partial u^i_S}$ and that the component of the characteristic of a Hamiltonian vector fields are $X^j = P^{ji}(\boldsymbol{\partial})\frac{\delta h}{\delta u^i}$, as can be easily read by Equation (1.32).

On the other hand, the existence of a direct formula (2.5) for the computation of the λ-bracket (and the fact that the action d_P can be written using the λ-bracket itself, see (2.26) and (2.27)) that contains only derivations allows to write a code and let a computer algebra system write down the equations. Together with D. Valeri we wrote a Mathematica package, called `MasterPVA`, specialized to this aim. It is based on J. Ekstrand's `Lambda` [27], which is much larger in scope and deals with standard vertex algebras. The source code is given and briefly commented in Appendix A. The main functions that the package provides are `PVASkew`, `JacobiCheck`, `EvVField` and `LambdaB`; the differential polynomials that the package uses must be written considering the independent variables `var[[i]]`, $i = 1, \dots$ `r` and the generators `gen[var][[i]]`, $i = 1, \dots,$ `d`. General parameters that must be initializated are `d` the number of generators ($= N$), `r` the rank of the bracket ($= D$) and the maximum differential order up to which the program will compute derivatives `maxO`, that can be adjusted according to the computation to be performed to save machine time.

The λ-brackets between the generators must be provided as a $N \times N$ array corresponding to $P^{ji}(\boldsymbol{\lambda}) = \{u^i{}_{\boldsymbol{\lambda}} u^j\}$. The indeterminate $\boldsymbol{\lambda}$ in the definition of the bivector is replaced by the protected vector `\[Beta]\[Beta]`. In the following paragraphs we briefly describe the use of the aforementioned main functions.

PVASkew

```
PVASkew[P]
```

The input is a $N \times N$ array representing the components of the λ-bracket between the generators, with `\[Beta]\[Beta]` as formal indeterminate. The output is the $N \times N$ array $\{u^i_{\boldsymbol{\lambda}} u^j\} + _{\to}\{u^j{}_{-\boldsymbol{\lambda}-\boldsymbol{\partial}} u^i\}$, that vanishes if the array P are the components of a skewsymmetric λ-bracket.

JacobiCheck

```
JacobiCheck[P]
```

The input is a $N \times N$ array representing the components of the λ-bracket between the generators, with `\[Beta]\[Beta]` as formal indeterminate. The output is the $N \times N \times N$ array representing the PVA-Jacobi identity (2.4) for all $(i, j, k) = 1, \ldots, N$. It vanishes if and only if P defines the λ-bracket of a Poisson Vertex Algebra.

EvVField

```
EvVField[X,f]
```

The input are a N dimensional vector X representing the components of the characteristic of an evolutionary vector field and a differential polynomial f. The output is the differential polynomial

$$\left(\boldsymbol{\partial}^I X^i\right) \frac{\partial f}{\partial u^i_I},$$

namely the action of an evolutionary vector field on the density of a local functional

LambdaB

```
LambdaB[f,g,P,\[Lambda]]
```

This is the most important function provided by the package. It computes the λ bracket defined by P as in the argument of `PVASkew` and `JacobiCheck`, between two differential polynomials f and g, and expresses it as a polynomial in the formal indeterminate `\[Lambda]` (a vector of length D) to be provided by the user.

3

Cohomology of $D = 2$ Poisson brackets of hydrodynamic type

In the previous chapter we have discussed the language of the Poisson Vertex Algebras and how it provides an effective approach to the study of the Poisson brackets on the spaces of maps. In this chapter we focus on the study of the Poisson brackets of hydrodynamic type in the case $D = N = 2$ and on its cohomology. The results about the first and second cohomology group at the first order have been presented in [12]. We provide here the first and second cohomology group up to the second order, and an example of the procedure needed to compute the higher cohomology groups, as the third one.

The summary of the results of this chapter is provided in the following table. Let us denote for short $H^p_d(P_{1,2,LP})$ the d-th degree component of the p-th cohomology group for the Poisson bracket respectively defined by P_1, P_2, and P_{LP} as in (3.1), (3.2), (3.3). The rows of the table correspond to the cohomology groups, while the columns correspond to the 0-th, 1−st and 2−nd *order* of the dispersive expansion. Note the degree shift between the order of the expansion and the degree of the p-vectors.

$H^1_0(P_1) \cong \mathbb{R}^2$	$H^1_1(P_1) \cong \mathbb{R}^2$	$H^1_2(P_1) \cong 0$
$H^1_0(P_2) \cong \mathbb{R}^2$	$H^1_1(P_2) \cong \mathbb{R}^2$	$H^1_2(P_2) \cong 0$
$H^1_0(P_{LP}) \cong 0$	$H^1_1(P_{LP}) \cong 0$	$H^1_2(P_{LP}) \cong 0$
$H^2_1(P_1) \cong \mathbb{R}^2$	$H^2_2(P_1) \cong 0$	$H^2_3(P_1) \cong \mathbb{R}^4$
$H^2_1(P_2) \cong \mathbb{R}^3$	$H^2_2(P_2) \cong 0$	$H^2_3(P_2) \cong \mathbb{R}^5$
$H^2_1(P_{LP}) \cong 0$	$H^2_2(P_{LP}) \cong 0$	$H^2_3(P_{LP}) \cong \mathbb{R}^2$
$H^3_2(P_1) \cong 0$		
$H^3_2(P_2) \cong \mathbb{R}$		
$H^3_2(P_{LP}) \cong 0$		

3.1 Classification of 2-dimensional Poisson brackets of hydrodynamic type

Ferapontov and collaborators provide in [31] a classification, based on Mokhov's results [44], of all the undeformed Poisson structures on such a space up to Miura transformations and linear change of the independent variables. We remark that the Miura transformations (1.49) are the natural diffeomorphisms on the space of differential polynomials $\mathcal{A}$, hence the cohomology groups are invariant with respect to them.

On the other hand, we can prove that a linear change of the independent variables does not affect the cohomology groups as well or, more precisely, that the cohomology groups are mapped by the transformation to isomorphic cohomology groups. This theorem is stated and proved in Paragraph ch:4-sec:2-ssec:indep for the special case of $N = 1$, but the number of the dependent variable is irrelevant for the proof.

Proposition 3.1. Let us denote the generators of the algebra of differential polynomials $\mathcal{A}$ as $(p_1 \equiv p, p_2 \equiv q)$ and the independent variables as $(x^1 \equiv x, x^2 \equiv y)$. Any $D = N = 2$ λ-bracket associated to a Poisson bracket of hydrodynamic type, namely a solution of the system (1.44), can be written, after a Miura transformation and a linear change of the independent variables, in one of the following three forms.

$$P_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \tag{3.1}$$

$$P_2 = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_1 & \lambda_2 \end{pmatrix} \tag{3.2}$$

$$P_{LP} = -\begin{pmatrix} 2p & q \\ q & 0 \end{pmatrix} \lambda_1 - \begin{pmatrix} 0 & p \\ p & 2q \end{pmatrix} \lambda_2 - \begin{pmatrix} p_x & q_x \\ p_y & q_y \end{pmatrix}, \tag{3.3}$$

where we denote $\{p_{i\boldsymbol{\lambda}} p_j\}_{1,2,LP} = \left(P_{1,2,LP}\right)_{ij}(\boldsymbol{\lambda})$

Remark 3.1. The λ-bracket (3.3) is the (λ-bracket associated to the) Lie–Poisson bracket for the algebra of the vector fields on a $2-$torus.

3.2 Symmetries of the brackets

Definition 3.1. A *n-th order symmetry* of a PVA $(\hat{\mathcal{A}}, \{\cdot_{\boldsymbol{\lambda}}\cdot\})$ is an evolutionary vector field $X \in \mathrm{Der}(\mathcal{A})$, $[\xi, \partial_\alpha] = 0\ \forall \alpha$, with the following properties:

1. The characteristics X^i of the vector field are homogeneous differential polynomials of order n;
2. $X\left(\{f_{\boldsymbol{\lambda}} g\}\right) = \{X(f)_{\boldsymbol{\lambda}} g\} + \{f_{\boldsymbol{\lambda}} X(g)\}$

Definition 3.2. A *Hamiltonian vector field* in the context of PVA [5] is an evolutionary vector field X whose characteristics X^i are

$$X^i(u_J) = X_H(u^i) = \{H_{\boldsymbol{\lambda}} u^i\}|_{\boldsymbol{\lambda}=0} \tag{3.4}$$

for $H \in \mathcal{A}$.

In terms of PVA cohomology, a symmetry is a cocycle in $\Omega^1(\mathcal{A}, \{\cdot_{\boldsymbol{\lambda}}\cdot\})$ and a Hamiltonian vector field is a coboundary in the same space.

The fact that a Hamiltonian vector field, in the terms we defined it, is a symmetry of the λ-bracket is easily obtained from the PVA-Jacobi (2.4) identity after setting $\boldsymbol{\lambda} = 0$

$$\begin{aligned} \{H_{\boldsymbol{\lambda}}\{f_{\boldsymbol{\mu}}g\}\}|_{\boldsymbol{\lambda}=0} &= (\{f_{\boldsymbol{\mu}}\{H_{\boldsymbol{\lambda}}g\}\} + \{\{H_{\boldsymbol{\lambda}}f\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}}g\})\,|_{\boldsymbol{\lambda}=0} \\ X_H(\{f_{\boldsymbol{\mu}}g\}) &= \{f_{\boldsymbol{\mu}}X_H(g)\} + \{X_H(f)_{\boldsymbol{\mu}}g\} \end{aligned} \tag{3.5}$$

On the other hand, the classification of the symmetries of the λ-bracket allows us to characterize the first PVA-cohomology group and hence the Poisson-Lichnerowicz cohomology of the associated Poisson bracket.

Let us consider each of the normal forms for the λ-bracket of hydrodynamic type. We will compute the action of an evolutionary vector field on the brackets between two generators in order to characterize the conditions it must satisfy in order to be a symmetry of the brackets themselves. According to Definition 3.1 we will consider separately evolutionary vector fields of degree 0, 1 and 2. In each case, the conditions for the vector fields X to be a symmetry can be directly computed and will be reported in the following paragraphs.

We shift then into considering the form of the Hamiltonian vector fields for each normal form of the λ-bracket. From Definition 3.2 we can explicitly compute the coefficients of the Hamiltonian vector fields, provided that the Hamiltonian h is a function of (p, q) only. With abuse of notation, we will often denote $X(p) = X(p, q)$. Hamiltonian functions homogeneous of differential degree k produce, for brackets of hydrodynamic type, vector fields of order $k + 1$. Apart from the case of 0-th order symmetries that cannot be Hamiltonian, we then compare the condition of symmetry with the general form of Hamiltonian vector fields; this allows us to characterize the first cohomology group as the space of symmetries of the brackets that are not Hamiltonian.

3.2.1 0th order symmetries

The generic vector field of 0-th differential degree has characteristic

$$X(p_i) = X_i(p) \tag{3.6}$$

for $i = 1, 2$. Since there do not exist Hamiltonian functions that can give a 0-th degree vector field by a bracket of degree 1, the first cohomology groups (of each of the three brackets) at the 0-th order are constituted by all the symmetries of the brackets themselves. The equation that must be satisfied for each of the brackets is

$$\{p_{i\boldsymbol{\lambda}}X_j\} + \{X_{i\boldsymbol{\lambda}}p_j\} = X\left(\{p_{i\boldsymbol{\lambda}}p_j\}\right)$$

where, obviously, the RHS vanishes for the brackets (3.1) and (3.2). The computations are straightforward and can be very easily performed using the package `MasterPVA` we have described in Section 2.3. The reduced systems of equations for the symmetries of (3.1) and (3.2) are the same, namely

$$\frac{\partial X_1}{\partial p} = 0 \qquad \frac{\partial X_2}{\partial p} = 0$$
$$\frac{\partial X_1}{\partial q} = 0 \qquad \frac{\partial X_2}{\partial q} = 0.$$

Hence, the solution is

$$\begin{aligned} X_1(p, q) &= c_1 \\ X_2(p, q) &= c_2. \end{aligned} \tag{3.7}$$

for c_1 and c_2 arbitrary constant.

The condition for the existence of 0-th order symmetries of (3.3), on the other hand, admits only the trivial solution

$$X_1(p, q) = 0 \qquad X_2(p, q) = 0. \tag{3.8}$$

We have proved

Theorem 3.2. *The 0-th order component of the first cohomology group of* $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_1$ *and* $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_2$ *are isomorphic to* $\mathbb{R}^2$*. The 0-th order component of the first cohomology group for the Poisson Vertex Algebra* $(\mathcal{A}, \{\cdot_{\boldsymbol{\lambda}}\cdot\}_{LP})$ *is trivial.*

3.2.2 1st order symmetries

A vector fields whose characteristic is a homogeneous first degree differential polynomial has the form

$$X(p_i) = X_i(p, p_I) = A_i^{ab}(p)\partial_a p_b, \tag{3.9}$$

where each index runs from 1 to $D = N = 2$ and we follow the Einstein convention for the sum over repeated indices.

The conditions for X to be a symmetry can be directly computed and are summarized in the following lemmas.

Lemma 3.3. *An evolutionary vector field of the form* (3.9) *is a first order symmetry of the bracket* (3.1) *if and only if the following conditions hold:*

$$A_j^{ab}\delta_i^b + A_j^{ba}\delta_i^a - A_i^{ab}\delta_j^b - A_i^{ba}\delta_j^a = 0 \tag{3.10a}$$

$$\frac{\partial A_i^{bl}}{\partial p_j}\delta_j^a - \frac{\partial A_i^{aj}}{\partial p_l}\delta_j^b - \frac{\partial A_i^{bj}}{\partial p_l}\delta_j^a + \frac{\partial A_j^{bl}}{\partial p_i}\delta_i^a = 0 \tag{3.10b}$$

$$\frac{\partial^2 A_i^{al}}{\partial p_j \partial p_m}\delta_j^b + \frac{\partial^2 A_i^{bm}}{\partial p_j \partial p_l}\delta_j^a - \frac{\partial^2 A_i^{aj}}{\partial p_l \partial p_m}\delta_j^b - \frac{\partial^2 A_i^{bj}}{\partial p_l \partial p_m}\delta_j^a = 0 \tag{3.10c}$$

$$\frac{\partial A_i^{al}}{\partial p_j}\delta_j^b + \frac{\partial A_i^{bl}}{\partial p_j}\delta_j^a - \frac{\partial A_i^{aj}}{\partial p_l}\delta_j^b - \frac{\partial A_i^{bj}}{\partial p_l}\delta_j^a = 0 \tag{3.10d}$$

In particular, for the case $D = N = 2$, the solutions of (3.10) *for $A_i^{ab}(p,q)$ are*

$$\begin{aligned} A_1^{11} &= \frac{\partial^2 K}{\partial p^2} & A_2^{11} &= 0 \\ A_1^{12} &= \frac{\partial^2 K}{\partial p \partial q} & A_2^{12} &= c_1 \\ A_1^{21} &= c_2 & A_2^{21} &= \frac{\partial^2 K}{\partial p \partial q} \\ A_1^{22} &= 0 & A_2^{22} &= \frac{\partial^2 K}{\partial q^2} \end{aligned} \tag{3.11}$$

where c_1 and c_2 are constants and $K = K(p,q)$ is a generic function of the coordinates.

Proof. Recalling that $X(p_i) = A_i^{ab}\partial_a p_b$ and $X(f) = \boldsymbol{\partial}^I \left(A_i^{ab}\partial_a p_b\right) \partial f/\partial p_{iI}$ we compute $X(\{p_i {}_{\boldsymbol{\lambda}} p_j\} - \{A_i^{ab}\partial_a p_b {}_{\boldsymbol{\lambda}} p_j\} - \{p_i {}_{\boldsymbol{\lambda}} A_j^{ab}\partial_a p_b\}$ and set to 0 the coefficients of the four terms $\lambda_a\lambda_b$, $\lambda_a\partial_b p_l$, $\partial_a p_l \partial_b p_m$, and $\partial_{ab} p_l$. This procedure gives the set of equations (3.10); for the case $D = N = 2$ we explicitly write down the algebraic equations (3.10a) that imply $A_2^{11} = 0$, $A_1^{22} = 0$, and $A_1^{12} = A_2^{21}$. The complete solution is then easily found using (3.10b). Indeed, equations (3.10d) turn out to be equivalent to (3.10b) and (3.10c) are differential consequences of that. □

Lemma 3.4. *An evolutionary vector field of the form* (3.9) *is a first order symmetry of the bracket* (3.2) *if and only if the following conditions hold:*

$$\begin{aligned} A_2^{12} &= A_1^{11} & A_1^{22} &= 0 \\ A_1^{12} + A_1^{21} &= A_2^{22} & & \\ A_2^{21} - A_1^{11} &= c_1 & A_1^{12} - A_2^{22} &= c_2 \\ \frac{\partial A_2^{11}}{\partial q} - \frac{\partial A_2^{21}}{\partial p} &= 0 & \frac{\partial A_2^{21}}{\partial q} - \frac{\partial A_2^{22}}{\partial p} &= 0 \end{aligned} \tag{3.12}$$

Proof. The particular form of the bracket (3.2) makes explicitly computing the symmetry condition in the case $D = N = 2$ the most effective approach to the problem. As in Lemma 3.3, the coefficients of the terms $\lambda_a\lambda_b$ are algebraic equations, while the coefficients of the other terms are linear PDEs. Hence we can first reduce the number of unknowns for the differential equations. In order to simplify the set of the remaining equations, we relied to a powerful computational tool which is called a *Janet basis* for the linear system of PDEs [47]. It provides the normal form for the system, which is unique up to the ordering of variables, and can be computed using the Maple package `Janet` [6]. □

With the same technique used to prove Lemma 3.3, and then computing the Janet basis for the explicit formula of the symmetry conditions as we did in Lemma 3.4, we can characterize the symmetries of (3.3). The results are summarized in the following lemma.

Lemma 3.5. *An evolutionary vector field of the form* (3.9) *is a first order symmetry of the bracket* (3.3) *if and only if the following conditions hold:*

$$A_j^{al}\frac{\partial}{\partial p_b}(p_ip_l) - A_i^{al}\frac{\partial}{\partial p_b}(p_jp_l) + A_j^{bl}\frac{\partial}{\partial p_a}(p_ip_l) - A_i^{bl}\frac{\partial}{\partial p_a}(p_jp_l) = 0 \tag{3.13a}$$

$$\begin{aligned} &A_j^{ba}\delta_i^l + A_j^{al}\delta_i^b - A_i^{bl}\delta_j^a - A_i^{ab}\delta_j^l + \frac{\partial A_i^{bl}}{\partial p_m}\frac{\partial}{\partial p_a}(p_mp_j) + \\ &-\frac{\partial A_i^{am}}{\partial p_l}\frac{\partial}{\partial p_b}(p_mp_j) - \frac{\partial A_i^{bm}}{\partial p_l}\frac{\partial}{\partial p_a}(p_mp_j) - \frac{\partial A_j^{bl}}{\partial p_m}\frac{\partial}{\partial p_a}(p_ip_m) = 0 \end{aligned} \tag{3.13b}$$

$$\left(\frac{\partial^2 A_i^{al}}{\partial p_s\partial p_m} - \frac{\partial^2 A_i^{as}}{\partial p_l\partial p_m}\right)p_s\delta_j^b + \left(\frac{\partial^2 A_i^{bm}}{\partial p_s\partial p_l} - \frac{\partial^2 A_i^{bs}}{\partial p_l\partial p_m}\right)p_s\delta_j^a + \tag{3.13c}$$

$$\left(\frac{\partial^2 A_i^{bm}}{\partial p_a\partial p_l} - \frac{\partial^2 A_i^{ab}}{\partial p_l\partial p_m}\right)p_j + \left(\frac{\partial^2 A_i^{al}}{\partial p_b\partial p_m} - \frac{\partial^2 A_i^{ba}}{\partial p_l\partial p_m}\right)p_j + \tag{3.13d}$$

$$+\left(\frac{\partial A_i^{al}}{\partial p_b} - \frac{\partial A_i^{ab}}{\partial p_l}\right)\delta_j^m + \left(\frac{\partial A_i^{bm}}{\partial p_a} - \frac{\partial A_i^{ba}}{\partial p_m}\right)\delta_j^l = 0$$

$$\left(\frac{\partial A_i^{al}}{p_s} - \frac{\partial A_i^{as}}{\partial p_l}\right)\frac{\partial}{\partial p_b}(p_sp_j) + \left(\frac{\partial A_i^{bl}}{p_s} - \frac{\partial A_i^{bs}}{\partial p_l}\right)\frac{\partial}{\partial p_a}(p_sp_j) = 0$$

For the case $D = N = 2$, the algebraic equations allow us to express A_1^{21}, A_2^{12}, and A_2^{21} in terms of the remaining five functions; we then compute the Janet basis of the remaining set of linear PDEs. After this procedure, (3.13) *in the two-dimensional case is reduced to the system*

$$A_1^{21} = A_2^{22} - \frac{2q}{p}A_1^{22} \qquad A_2^{12} = A_1^{11} - \frac{2p}{q}A_2^{11}$$

$$A_2^{21} = \frac{p^3A_2^{11} + pq^2A_1^{12} - q^3A_1^{22}}{p^2q}$$

$$2\frac{\partial A_2^{22}}{\partial q}p + \frac{\partial A_1^{12}}{\partial q}p + 3\frac{\partial A_1^{22}}{\partial q}q + 5A_1^{22} = 0 \tag{3.14a}$$

$$\frac{\partial A_1^{22}}{\partial p}p + 2\frac{\partial A_1^{22}}{\partial q}q + 2A_1^{22} - \frac{\partial A_2^{22}}{\partial q}p = 0 \tag{3.14b}$$

$$\frac{\partial A_1^{11}}{\partial p}q - 2\frac{\partial A_2^{11}}{\partial p}p - \frac{\partial A_2^{11}}{\partial q}q - 2A_2^{11} = 0 \tag{3.14c}$$

$$\frac{\partial A_1^{11}}{\partial q}p^2q^2 - 2\frac{\partial A_2^{11}}{\partial q}p^3q + 2\frac{\partial A_1^{22}}{\partial q}q^4 - \frac{\partial A_2^{22}}{\partial q}pq^3 + \\ 2A_2^{11}p^3 + 4A_1^{22}q^3 - A_1^{12}pq^2 = 0 \tag{3.14d}$$

$$4\frac{\partial A_1^{22}}{\partial q}q^4 - \frac{\partial A_2^{11}}{\partial q}p^3q + \frac{\partial A_2^{22}}{\partial p}p^2q^2 - 2\frac{\partial A_2^{22}}{\partial q}pq^3 + \\ A_2^{11}p^3 - A_1^{12}pq^2 + 7A_1^{22}q^3 = 0 \tag{3.14e}$$

$$\frac{\partial A_1^{12}}{\partial p}p^2q^2 - 2\frac{\partial A_2^{11}}{\partial q}p^3q + 2\frac{\partial A_1^{22}}{\partial q}p^4 - \frac{\partial A_2^{22}}{\partial q}pq^3 + \\ 2A_2^{11}p^3 - A_1^{12}pq^2 + 4A_1^{22}q^3 = 0 \tag{3.14f}$$

The Hamiltonian vector fields of the bracket (3.1) can be immediately compared with (3.11), since for this easy case we have explicitly solved the symmetry conditions. For a Hamiltonian $h(p,q)$, the associated Hamiltonian vector field has the form

$$\xi_h(p_i) = A_i^{ab}\partial_a p_b = \frac{\partial^2 h}{\partial p_i \partial p_b}\delta_i^a \partial_a p_b. \tag{3.15}$$

In particular, this means that the first cohomology group for $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_1$ is not trivial, as opposite as what is known for 1-dimensional Poisson brackets of hydrodynamic type [36]. Indeed, there exists a family of non Hamiltonian symmetry depending on two arbitrary constants c_1 and c_2.

The Hamiltonian vector fields of the bracket (3.2) can be easily computed, too. Their components are given in terms of the coefficients A_i^{ab}, which for an Hamiltonian $h(p,q)$ are

$$\begin{aligned} A_1^{11} = A_2^{12} = A_2^{21} &= \frac{\partial^2 h}{\partial p \partial q} \\ A_1^{21} = A_1^{22} &= 0 \\ A_2^{11} &= \frac{\partial^2 h}{\partial p^2} \\ A_1^{12} = A_2^{22} &= \frac{\partial^2 h}{\partial q^2} \end{aligned} \tag{3.16}$$

There exist solutions of (3.12) with $c_1, c_2 \neq 0$, but such solutions are not Hamiltonian vector fields. For example, the vector field whose components are $A_1^{11} = A_2^{11} = A_2^{12} = A_1^{22} = A_2^{22} = 0$, $A_2^{21} = c_1$, and $A_1^{12} = -A_1^{21} = c_2$ is a symmetry but cannot be a Hamiltonian vector field. Thus, the first cohomology group of $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_2$ is not trivial.

In order to characterize the first cohomology group for $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_{LP}$ we choose to proceed in a different way. The algebraic equations in the set of conditions (3.13) allowed us to express the linear PDEs (3.14a) in terms of 5 out of the 8 coefficients A_i^{ab}. We compute these 5 coefficients for a Hamiltonian vector field. They are

$$A_1^{11} = -\left(\frac{\partial h}{\partial p} + 2p\frac{\partial^2 h}{\partial p^2} + q\frac{\partial^2 h}{\partial p \partial q}\right) \tag{3.17a}$$

$$A_2^{11} = -q\frac{\partial^2 h}{\partial p^2} \tag{3.17b}$$

$$A_1^{12} = -\left(2p\frac{\partial^2 h}{\partial p \partial q} + q\frac{\partial^2 h}{\partial q^2}\right) \tag{3.17c}$$

$$A_1^{22} = -p\frac{\partial^2 h}{\partial q^2} \tag{3.17d}$$

$$A_2^{22} = -\left(\frac{\partial h}{\partial q} + 2q\frac{\partial^2 h}{\partial q^2} + \frac{\partial^2 h}{\partial p \partial q}\right) \tag{3.17e}$$

We can regard the five equations (3.17) as an overdetermined system of inhomogeneous linear PDEs for the unknown function h. The compatibility conditions for the functions A_i^{ab} are the conditions that a symmetry of the λ-bracket must satisfy in order to be Hamiltonian. Indeed, they guarantee that a solution (i.e., a Hamiltonian) exists for a generic vector field expressed in terms of the same coefficients. The compatibility conditions may or may not have the same solution as the conditions for a vector field to be a symmetry. Of course, all the solutions of the compatibility conditions are symmetries: they are components of a Hamiltonian vector field. The converse is in general not true, namely the solutions of the symmetry conditions may not be solutions of the compatibility ones. That would mean that there exist non Hamiltonian symmetries.

The compatibility conditions among the parameters in the LHS of the system (3.17) can be found using the tools of `Janet` package. We compute the Janet basis for them, getting exactly the set of equations (3.14a). That means that all the first order symmetries are Hamiltonian vector fields.

We have proved the following theorem:

Theorem 3.6. *The first order compontent of the first cohomology groups of* $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_1$ *and* $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_2$ *are isomorphic to* $\mathbb{R}^2$*. The first order component of the first cohomology group for the Poisson Vertex Algebra* $(\mathcal{A}, \{\cdot_{\boldsymbol{\lambda}}\cdot\}_{LP})$ *is trivial.*

3.2.3 2nd order symmetries

Let us denote a vector fields whose characteristic is a homogeneous second degree differential polynomial as

$$X(p_i) = X_i(p, p_I) = A_i^{al,bm}(p)\partial_a p_l \partial_b p_m + B_i^{abl}\partial_{ab} p_l. \tag{3.18}$$

From the definition it immediately follows that the coefficients $A_i^{al,bm}$ and $B_i^{ab,l}$ must be respectively symmetric in the simultaneous exchange $(a, l \leftrightarrow b, m)$ and in the exchange $(a \leftrightarrow b)$. The commas in the indices are used, here and in the following paragraphs, as a bookkeeping device and do not have any further meaning.

Lemma 3.7. *An evolutionary vector field of the form* (3.18) *is a symmetry of the bracket* (3.1) *if and only if the following system of equations holds true:*

$$
\begin{array}{llll}
A_1^{11,11} = 0 & A_2^{11,11} = 0 & A_1^{11,12} = -\frac{\partial B_1}{\partial p} & A_2^{11,12} = 0 \\
A_1^{11,21} = 0 & A_2^{11,21} = \frac{\partial B_1}{\partial p} & A_1^{11,22} = \frac{\partial B_2}{\partial p} & A_2^{11,22} = \frac{\partial B_1}{\partial q} \\
A_1^{12,12} = -2\frac{\partial B_1}{\partial q} & A_2^{12,12} = 0 & A_1^{12,21} = 0 & A_2^{12,21} = 0 \\
A_1^{12,22} = \frac{\partial B_2}{\partial q} & A_2^{12,22} = 0 & A_1^{21,21} = 0 & A_2^{21,21} = -2\frac{\partial B_2}{\partial q} \\
A_1^{21,22} = 0 & A_2^{21,22} = -\frac{\partial B_2}{\partial q} & A_1^{22,22} = 0 & A_2^{22,22} = 0
\end{array}
$$

$$
\begin{array}{llll}
B_1^{111} = 0 & B_2^{111} = 0 & B_1^{121} = 0 & B_2^{121} = B_1(p,q) \\
B_1^{221} = 0 & B_2^{221} = -2B_2(p,q) & B_1^{112} = -2B_1(p,q) & B_2^{112} = 0 \\
B_1^{122} = B_2(p,q) & B_2^{122} = 0 & B_1^{222} = 0 & B_2^{222} = 0
\end{array}
$$

Lemma 3.8. *An evolutionary vector field of the form* (3.18) *is a symmetry of the bracket* (3.2) *if and only if the following system of equations holds true:*

$$
\begin{array}{llll}
A_1^{11,11} = 0 & A_2^{11,11} = 0 & A_1^{11,12} = -\frac{\partial B_1}{\partial p} & A_2^{11,12} = 0 \\
A_1^{11,21} = 0 & A_2^{11,21} = \frac{\partial B_1}{\partial p} & A_1^{11,22} = \frac{\partial B_2}{\partial p} & A_2^{11,22} = \frac{\partial B_1}{\partial q} \\
A_1^{12,12} = -2\frac{\partial B_1}{\partial q} & A_2^{12,12} = 0 & A_1^{12,21} = 0 & A_2^{12,21} = 0 \\
A_1^{12,22} = \frac{\partial B_2}{\partial q} & A_2^{12,22} = 0 & A_1^{21,21} = 0 & A_2^{21,21} = -2\frac{\partial B_2}{\partial q} \\
A_1^{21,22} = 0 & A_2^{21,22} = -\frac{\partial B_2}{\partial q} & A_1^{22,22} = 0 & A_2^{22,22} = 0
\end{array}
$$

$$
\begin{array}{llll}
B_1^{111} = B_1(p,q) & B_2^{111} = 0 & B_2^{121} = 0 & B_2^{121} = \frac{1}{2}B_1(p,q) \\
B_1^{221} = 0 & B_2^{221} = 2B_2(p,q) & B_1^{112} = 0 & B_2^{112} = -B_1(p,q) \\
B_1^{122} = 0 & B_2^{122} = -B_2(p,q) & B_1^{222} = 0 & B_2^{222} = 0
\end{array}
$$

Lemma 3.9. *An evolutionary vector field of the form* (3.18) *is a symmetry of the bracket* (3.3) *if and only if the following system of equations holds true:*

$$
\begin{array}{ll}
A_1^{11,11} = \frac{\partial B_1}{\partial p} & A_2^{11,11} = 0 \\
A_1^{11,12} = -\frac{1}{q}B_1 + \frac{1}{2}\frac{\partial B_1}{\partial q} - \frac{p}{q}\frac{\partial B_1}{\partial p} & A_2^{11,12} = -\frac{1}{2}\frac{\partial B_1}{\partial p} \\
A_1^{11,21} = \frac{1}{2q}B_1 + \frac{q}{2p^2}B_2 + \frac{p}{2q}\frac{\partial B_1}{\partial p} - \frac{q}{2p}\frac{\partial B_2}{\partial p} & A_2^{11,21} = \frac{\partial B_1}{\partial p} \\
A_1^{11,22} = \frac{1}{2}\left(-\frac{p}{q^2}B_1 - \frac{1}{p}B_2 + \frac{p}{q}\frac{\partial B_1}{\partial q} + 2\frac{\partial B_2}{\partial p}\right) & \\
A_2^{11,22} = \frac{1}{2}\left(-\frac{q}{p^2}B_2 - \frac{1}{q}B_1 + \frac{q}{p}\frac{\partial B_2}{\partial p} + 2\frac{\partial B_1}{\partial q}\right) &
\end{array}
$$

$$
\begin{aligned}
A_1^{12,12} &= \tfrac{2p}{q^2}\left(B_1 - q\tfrac{\partial B_1}{\partial q}\right) & A_2^{12,12} &= -\tfrac{\partial B_1}{\partial q}\\
A_1^{12,21} &= -\tfrac{q}{2p}\tfrac{\partial B_2}{\partial q} & A_2^{12,21} &= -\tfrac{p}{2q}\tfrac{\partial B_1}{\partial p}\\
A_1^{12,22} &= \tfrac{\partial B_2}{\partial q} & A_2^{12,22} &= \tfrac{1}{2p}B_2 + \tfrac{p}{2q^2}B_1 + \tfrac{q}{2p}\tfrac{\partial B_2}{\partial q} - \tfrac{p}{2q}\tfrac{\partial B_1}{\partial q}\\
A_1^{21,21} &= -\tfrac{\partial B_2}{\partial p} & A_2^{21,21} &= \tfrac{2q}{p^2}\left(B_2 - p\tfrac{\partial B_2}{\partial p}\right)\\
A_1^{21,22} &= -\tfrac{1}{2}\tfrac{\partial B_2}{\partial q} & A_2^{21,22} &= -\tfrac{1}{p}B_2 + \tfrac{1}{2}\tfrac{\partial B_2}{\partial p} - \tfrac{q}{p}\tfrac{\partial B_2}{\partial q}\\
A_1^{22,22} &= 0 & A_2^{22,22} &= \tfrac{\partial B_2}{\partial q}
\end{aligned}
$$

$$
\begin{aligned}
B_1^{111} &= B_1(p,q) & B_2^{111} &= 0\\
B_2^{121} &= \tfrac{1}{2}\left(\tfrac{p}{q}B_1 - \tfrac{q}{p}B_2\right) & B_2^{121} &= B_1\\
B_1^{221} &= -B_2(p,q) & B_2^{221} &= -2\tfrac{q}{p}B_2\\
B_1^{112} &= -2\tfrac{p}{q}B_1 & B_2^{112} &= -B_1\\
B_1^{122} &= B_2(p,q) & B_2^{122} &= -\tfrac{1}{2}\left(\tfrac{p}{q}B_1 - \tfrac{q}{p}B_2\right)\\
B_1^{222} &= 0 & B_2^{222} &= B_2
\end{aligned}
$$

As it can be easily seen, all the three families of symmetries of the λ-brackets (3.1). (3.2), and (3.3) depend only on two arbitrary functions $B_1(p,q)$ and $B_2(p,q)$. In the same way as we did for the first order case, we consider now a generic Hamiltonian, that must be a differential polynomial of degree 1 to give a second order vector field, and compare the outcome of Lemmas 3.7, 3.8, and 3.9 with the three families of Hamiltonian vector fields. The generic Hamiltonian has the form

$$H = \sum_{a,b=1}^{N} h^{ab}(p,q)\partial_a p_b \tag{3.19}$$

and, hence, it depends on four arbitrary functions h^{ab}. The functional freedom we have for the Hamiltonian functions, in other words, is bigger than the one allowed for the symmetries of the brackets: one could imagine that this implies that, given a symmetry of the λ-brackets, one can always find a Hamiltonian of the form (3.19) for which it is a Hamiltonian vector field.

We can show it by explicitly computing B_1 and B_2 in terms of H, for each of the three families of symmetries. We have, respectively for (3.1), (3.2), and (3.3),

$$B_1 = \frac{1}{2}\left(\frac{\partial h^{11}}{\partial q} - \frac{\partial h^{12}}{\partial p}\right) \qquad B_2 = \frac{1}{2}\left(\frac{\partial h^{22}}{\partial p} - \frac{\partial h^{21}}{\partial q}\right) \tag{3.20}$$

$$B_1 = \frac{1}{2}\left(\frac{\partial h^{11}}{\partial q} - \frac{\partial h^{12}}{\partial p}\right) \qquad B_2 = \frac{1}{2}\left(\frac{\partial h^{22}}{\partial p} - \frac{\partial h^{21}}{\partial q}\right) \tag{3.21}$$

$$B_1 = q\left(\frac{\partial h^{11}}{\partial q} - \frac{\partial h^{12}}{\partial p}\right) \qquad B_2 = p\left(\frac{\partial h^{22}}{\partial p} - \frac{\partial h^{21}}{\partial q}\right) \tag{3.22}$$

It is obvious that, given the freedom we have in the choice of H, any arbitrary function B_i can be obtained by a particular choice of h^{ab}. For instance, for any B_1 as in (3.22), $B_1 = q\left(\partial_q h^{11} - \partial_p h^{12}\right)$, a suitable Hamiltonian function is given by $h^{12} = 0$, $h^{11} = \int B_1/q \mathrm{d}q$. The same easy choice can be made for all the three families of symmetries; this allows us to give the following theorem

Theorem 3.10. *The second order component of the first cohomology groups for the $D = N = 2$ Poisson brackets of hydrodynamic type is trivial.*

3.3 Deformations of the brackets

We restate Definition 1.10 and Definition 1.11 in the context of Poisson Vertex Algebras, simply translating the notions already given in Paragraph 1.3.4. As already discussed, the space of the infinitesimal compatible deformations of a Poisson bracket that are not trivial, namely that are not obtained by the action of an evolutionary vector fields, constitute the second cohomology group.

Definition 3.3. A *n-th order deformation* of a PVA $(\mathcal{A}, \{\cdot_{\boldsymbol{\lambda}}\cdot\}_0)$ is a PVA defined by a deformed λ-bracket

$$\{\cdot_{\boldsymbol{\lambda}}\cdot\} = \{\cdot_{\boldsymbol{\lambda}}\cdot\}_0 + \sum_{k=1}^{n} \epsilon^k \{\cdot_{\boldsymbol{\lambda}}\cdot\}_{[k]} \tag{3.23}$$

such that $\{\cdot_{\boldsymbol{\lambda}}\cdot\}$ is PVA-skewsymmetric and the PVA-Jacobi identity holds up to order n, namely

$$\{f_{\boldsymbol{\lambda}}\{g_{\boldsymbol{\mu}}h\}\} - \{g_{\boldsymbol{\mu}}\{f_{\boldsymbol{\lambda}}h\}\} - \{\{f_{\boldsymbol{\lambda}}g\}_{\boldsymbol{\lambda}+\boldsymbol{\mu}}h\} = O(\epsilon^{n+1}).$$

Definition 3.4. A deformation is said to be *trivial* if there exists an element ϕ_ϵ of the Miura group (see Equation (1.49)) which pulls back $\{\cdot_{\boldsymbol{\lambda}}\cdot\}$ to $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_0$,

$$\{\phi_\epsilon(a)_{\boldsymbol{\lambda}}\phi_\epsilon(b)\}_0 = \phi_\epsilon\left(\{a_{\boldsymbol{\lambda}}b\}\right), \qquad \forall a, b \in \hat{\mathcal{A}}.$$

The Lie algebra of the Miura group, as already discussed, is the algebra of evolutionary vector fields.

To compute the cohomology groups, we will first consider a generic λ-bracket of the selected degree – respectively 1, 2, and 3. We impose the skewsymmetry condition, that reduces the number of free parameters in the brackets, and then the fulfilling of the PVA-Jacobi identity, by setting equal to zero the coefficients of the powers of $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$, and of the jets variables $\partial_I p_i$. We get a huge overdetermined system of algebraic-differential equations for the coefficients that define the deformation we need to simplify. Most of the parameters can be algebraically solved for in terms of a much smaller subset of the parameters themselves; we are still left with a system of overdetermined linear PDEs. We can algorithmically compute the Janet basis of such a system by using the Maple package `Janet`, hence getting a minimal set of equations that the parameters must fulfill in order for the bracket to be a compatible deformation. We call those equations the *cocycle condition*.

Then, we consider the trivial deformations of the brackets for which we want to compute the cohomology, namely the deformed brackets given by performing a general Miura transformation (1.49) of the suitable order to the undeformed brackets. In particular, we look for the expression of the unknown coefficients of the cocycle conditions in terms of the parameters of the Miura transformation. Indeed, we can regard each set of such equations as an inhomogeneous linear system of PDEs for the unknown parameters. A solution of the system, if there exists, is the set of parameters of a Miura transformation which produces a given compatible bracket. The existence of the solution is granted if and only if the coefficients of the bracket, that are the affine terms of the system, satisfy certain compatibility conditions that can be found with the tools of `Janet` package [51]. We call those sets the *coboundary condition*, because they give the relation that the coefficients of the bracket must satisfy to be obtained from a Miura transformation.

We can compare the (Janet basis of the) coboundary condition with the cocycle one. When the coefficients satisfy the coboundary condition, they satisfy *a fortiori* the cocycle one – all the coboundaries are cocycles – but the converse is not always true. If it is true, that means that all the cocycles are coboundaries, and hence the cohomology group we are computing is trivial; if it is false, we can look for solutions of the latter that are not solutions of the former equations, hence characterizing the cohomology.

In the following paragraphs, we report the computations done for the second cohomology group in degrees 1, 2, and 3. Most of the results are left to Appendix B to improve the clarity of the statements.

3.3.1 0th order deformations

A zeroth order deformation of (3.1), (3.2), or (3.3) must be a first order homogeneous bracket of the general form

$$\{p_{i\boldsymbol{\lambda}}p_j\}_{[0]} = A^a_{ij}(p)\lambda_a + B^{al}_{ij}(p)\partial_b p_l \tag{3.24}$$

The conditions of skewsymmetry, easily obtained by setting to 0 the coefficients of $\boldsymbol{\lambda}$ and ∂p in $\{p_{i\boldsymbol{\lambda}}p_j\}_{[0]} +_{\to} \left\{p_{j\,-\boldsymbol{\lambda}-\boldsymbol{\partial}}p_i\right\}_{[0]}$, are the same that are found in (1.44a) and (1.44b). For the particular case $N = D = 2$ we are dealing with, this implies that the coefficients A's and B's can be split in their symmetric and antisymmetric part with respect to the lower indices and we have

$$A^a_{(A)} = \frac{1}{2}\left(A^a_{12} - A^a_{21}\right) = 0 \tag{3.25a}$$

$$A^a_{ij(S)} = \frac{1}{2}\left(A^a_{ij} + A^a_{ji}\right) \tag{3.25b}$$

$$B^{al}_{ij(S)} = \frac{1}{2}\left(B^{al}_{ij} + B^{al}_{ji}\right) = \frac{1}{2}\frac{\partial A^a_{ij}}{\partial p_l} \tag{3.25c}$$

$$B^{al}_{(A)} = \frac{1}{2}\left(B^{al}_{12} - B^{al}_{21}\right) \tag{3.25d}$$

The number of free coefficients, that used to be 24 (each index in the definition (3.24) can assume the values $1, 2$ and there are not built-in symmetries), is in this way reduced to 10 (the components of $A_{(S)}$ amd $B_{(A)}$.

It should be noticed that the notion of zeroth order deformation does not fit into Definition 3.3, since the dispersive terms start with $k = 1$! Nevertheless, since we are interested in computing the PVA cohomology of the brackets, the groups H^2_1 should be taken into account. For doing so, we acknowledge that the cocycle condition, that can be written as (2.27)$= 0$ with P and Q the two brackets $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_0$ and $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_{[0]}$, is formally equivalent to considering a formal first order deformation where $\deg P_1 = \deg P_0$.

Lemma 3.11. *A homogeneous λ-bracket of degree 1 as defined in* (3.24) *and* (3.25) *is a zeroth order deformation of the bracket* (3.1) *if and only if the following conditions hold:*

$$\frac{\partial A^2_{12}}{\partial q} - 2B^{22}_{(A)} = 0 \tag{3.26a}$$

$$\frac{\partial A^2_{11}}{\partial q} = 0 \tag{3.26b}$$

$$\frac{\partial A^1_{22}}{\partial q} = 0 \tag{3.26c}$$

$$2B^{12}_{(A)} + \frac{\partial A^1_{12}}{\partial q} = 0 \tag{3.26d}$$

$$\frac{\partial A^1_{11}}{\partial q} - 4B^{21}_{(A)} = 0 \tag{3.26e}$$

$$-\frac{\partial B^{21}_{(A)}}{\partial q} + \frac{\partial B^{22}_{(A)}}{\partial p} = 0 \tag{3.26f}$$

$$-\frac{\partial B^{11}_{(A)}}{\partial q} + \frac{\partial B^{12}_{(A)}}{\partial p} = 0 \tag{3.26g}$$

$$\frac{\partial A^2_{22}}{\partial p} + 4B^{12}_{(A)} = 0 \tag{3.26h}$$

$$\frac{\partial A^2_{12}}{\partial p} - 2B^{21}_{(A)} = 0 \tag{3.26i}$$

$$\frac{\partial A^2_{11}}{\partial p} = 0 \tag{3.26j}$$

$$\frac{\partial A^1_{22}}{\partial p} = 0 \tag{3.26k}$$

$$2B^{11}_{(A)} + \frac{\partial A^1_{12}}{\partial p} = 0 \tag{3.26l}$$

Proof. We compute the cocycle condition by setting (2.27)= 0, where the bracket defined by the first bivector P is (3.1) and the one defined by Q is of the general form for a skewsymmetric first degree bracket, hence in terms of the 10 coefficients A's and B's. The cocycle condition is homogeneous of second degree, and we compute the conditions (3.26) by setting to 0 the coefficients of λ^2, $\lambda\mu$, $\lambda\partial_a p_l$, $\partial_a p_l \partial_b p_m$, and $\partial_{ab} p_l$. The overdetermined system of PDEs does not allow to algebraically solve for some of the coefficients, hence we compute the Janet basis of the full system. This basis is made by the LHS of the condition (3.26). □

We follow the same approach for the compatible deformations of (3.2) and (3.3). The results are summarized in the following lemmas

Lemma 3.12. *A homogeneous λ-bracket of degree 1 as defined in* (3.24) *and* (3.25) *is a zeroth order deformation of the bracket* (3.2) *if and only if the following conditions hold:*

$$\frac{\partial}{\partial q} A^2_{12} - 2\, B^{22}_{(A)} = 0 \tag{3.27a}$$

$$\frac{\partial}{\partial q} A^2_{11} = 0 \tag{3.27b}$$

$$\frac{\partial}{\partial q} A^1_{12} - 3/4\, \frac{\partial}{\partial q} A^2_{22} - B^{12}_{(A)} - B^{21}_{(A)} = 0 \tag{3.27c}$$

$$\frac{\partial}{\partial q} A^1_{11} - 4\, B^{22}_{(A)} = 0 \tag{3.27d}$$

$$-\frac{\partial}{\partial q} B^{21}_{(A)} + \frac{\partial}{\partial p} B^{22}_{(A)} = 0 \tag{3.27e}$$

$$-\frac{\partial}{\partial q} B^{11}_{(A)} + \frac{\partial}{\partial p} B^{12}_{(A)} = 0 \tag{3.27f}$$

$$-\frac{\partial}{\partial q} A^1_{22} + \frac{\partial}{\partial p} A^2_{22} = 0 \tag{3.27g}$$

$$\frac{\partial}{\partial p} A^2_{12} - 1/4\, \frac{\partial}{\partial q} A^2_{22} - B^{12}_{(A)} - B^{21}_{(A)} = 0 \tag{3.27h}$$

$$\frac{\partial}{\partial p} A^2_{11} = 0 \tag{3.27i}$$

$$\frac{\partial}{\partial p} A^1_{12} - \frac{\partial}{\partial q} A^1_{22} - 2\, B^{11}_{(A)} = 0 \tag{3.27j}$$

$$\frac{\partial}{\partial p}A^1_{11} - 3/4\,\frac{\partial}{\partial q}A^2_{22} - 3\,B^{12}_{(A)} - B^{21}_{(A)} = 0 \tag{3.27k}$$

$$\frac{\partial^2}{\partial q^2}A^2_{22} + 4\,\frac{\partial}{\partial q}B^{12}_{(A)} - 4\,\frac{\partial}{\partial q}B^{21}_{(A)} = 0 \tag{3.27l}$$

$$4\,\frac{\partial}{\partial q}B^{11}_{(A)} + \frac{\partial^2}{\partial q^2}A^1_{22} - 4\,\frac{\partial}{\partial p}B^{21}_{(A)} = 0 \tag{3.27m}$$

Lemma 3.13. *A homogeneous* λ*-bracket of degree 1 of the form* (3.24) *and* (3.25) *is a zeroth order deformation of the bracket* (3.3) *if and only if the following conditions hold:*

$$A^1_{11} - 2\,\frac{pA^1_{12}}{q} + \frac{p^2A^1_{22}}{q^2} + \frac{qA^2_{11}}{p} - 2\,A^2_{12} + \frac{pA^2_{22}}{q} = 0 \tag{3.28a}$$

$$-2/3\,A^1_{12} - 2/3\,\frac{pA^1_{22}}{q} + p\frac{\partial}{\partial q}A^1_{22} + 1/3\,\frac{q^2A^2_{11}}{p^2} + 2/3\,\frac{q^3\frac{\partial}{\partial q}A^2_{11}}{p^2} \tag{3.28b}$$

$$+2/3\,\frac{q^2\frac{\partial}{\partial q}A^2_{12}}{p} + 1/3\,A^2_{22} - 1/3\,q\frac{\partial}{\partial q}A^2_{22} + 4/3\,qB^{12}_{(A)} - 8/3\,\frac{q^2B^{22}_{(A)}}{p} = 0$$

$$-4/3\,\frac{pA^1_{12}}{q} + p\frac{\partial}{\partial q}A^1_{12} + 2/3\,\frac{p^2A^1_{22}}{q^2} + 2/3\,\frac{qA^2_{11}}{p} + 4/3\,\frac{q^2\frac{\partial}{\partial q}A^2_{11}}{p} + \tag{3.28c}$$

$$+1/3\,q\frac{\partial}{\partial q}A^2_{12} + 2/3\,\frac{pA^2_{22}}{q} - 2/3\,p\frac{\partial}{\partial q}A^2_{22} + 2/3\,pB^{12}_{(A)} - 10/3\,qB^{22}_{(A)} = 0$$

$$-\frac{\partial}{\partial q}B^{21}_{(A)} + \frac{\partial}{\partial p}B^{22}_{(A)} = 0 \tag{3.28d}$$

$$-\frac{\partial}{\partial q}B^{11}_{(A)} + \frac{\partial}{\partial p}B^{12}_{(A)} = 0 \tag{3.28e}$$

$$-A^1_{12} + \frac{pA^1_{22}}{q} - 1/2\,\frac{q^2A^2_{11}}{p^2} - \frac{q^3\frac{\partial}{\partial q}A^2_{11}}{p^2} + 3\,\frac{q^2\frac{\partial}{\partial q}A^2_{12}}{p} + 1/2\,A^2_{22} + \tag{3.28f}$$

$$+p\frac{\partial}{\partial p}A^2_{22} - q\frac{\partial}{\partial q}A^2_{22} + 2\,pB^{11}_{(A)} + 2\,qB^{12}_{(A)} - 2\,qB^{21}_{(A)} - 4\,\frac{q^2B^{22}_{(A)}}{p} = 0$$

$$p\frac{\partial}{\partial p}A^2_{12} + 2\,q\frac{\partial}{\partial q}A^2_{12} - p\frac{\partial}{\partial q}A^2_{22} - 2\,pB^{21}_{(A)} - 4\,qB^{22}_{(A)} = 0 \tag{3.28g}$$

$$p\frac{\partial}{\partial p}A^2_{11} + 2\,q\frac{\partial}{\partial q}A^2_{11} - p\frac{\partial}{\partial q}A^2_{12} - 2\,pB^{22}_{(A)} = 0 \tag{3.28h}$$

$$1/3\,\frac{qA^1_{12}}{p} + 1/3\,A^1_{22} + p\frac{\partial}{\partial p}A^1_{22} - 7/6\,\frac{q^3A^2_{11}}{p^3} - 7/3\,\frac{q^4\frac{\partial}{\partial q}A^2_{11}}{p^3} + 5/3\,\frac{q^3\frac{\partial}{\partial q}A^2_{12}}{p^2} - \tag{3.28i}$$

$$-1/6\,\frac{qA^2_{22}}{p} - 1/3\,\frac{q^2\frac{\partial}{\partial q}A^2_{22}}{p} + 2\,qB^{11}_{(A)} - 2/3\,\frac{q^2B^{12}_{(A)}}{p} - 2\,\frac{q^2B^{21}_{(A)}}{p} + 4/3\,\frac{q^3B^{22}_{(A)}}{p^2} = 0$$

$$\frac{\partial}{\partial p}A^1_{12} - 2\,\frac{q^2 A^2_{11}}{p^3} - 4\,\frac{q^3 \frac{\partial}{\partial q}A^2_{11}}{p^3} + 4\,\frac{q^2 \frac{\partial}{\partial q}A^2_{12}}{p^2} - \frac{q \frac{\partial}{\partial q}A^2_{22}}{p} + 2\,B^{11}_{(A)} - 4\,\frac{qB^{21}_{(A)}}{p} = 0 \tag{3.28j}$$

$$\frac{A^2_{11}}{p} + 7/2\,\frac{q \frac{\partial}{\partial q}A^2_{11}}{p} + \frac{q^2 \frac{\partial^2}{\partial q^2}A^2_{11}}{p} - 7/2\,\frac{\partial}{\partial q}A^2_{12} - 2\,q\frac{\partial^2}{\partial q^2}A^2_{12} + p\frac{\partial^2}{\partial q^2}A^2_{22} + \tag{3.28k}$$

$$+2\,p\frac{\partial}{\partial q}B^{12}_{(A)} + 2\,q\frac{\partial}{\partial q}B^{22}_{(A)} + 3\,B^{22}_{(A)} = 0$$

As described in general, we consider a Miura transformation of the generators $\{p_i\}$ of the form $p_i \mapsto P_i = p_i + F_i(p,q)$ and compute the trivial deformed bracket, respectively for (3.1), (3.2), and (3.3). The expression of the ten coefficients As and Bs in terms of the two parameters of the Miura transformation are given in Appendix B.1. For each bracket, the compatibility conditions that allow to find the parameters of the transformation given the 10 coefficients A's and B's are, respectively, given as the following set of equations (3.29), (3.30), and (3.31).

$$A^2_{11} = 0 \tag{3.29a}$$

$$A^1_{22} = 0 \tag{3.29b}$$

$$\frac{\partial}{\partial q}A^2_{12} - 2\,B^{22}_{(A)} = 0 \tag{3.29c}$$

$$2\,B^{12}_{(A)} + \frac{\partial}{\partial q}A^1_{12} = 0 \tag{3.29d}$$

$$\frac{\partial}{\partial q}A^1_{11} - 4\,B^{21}_{(A)} = 0 \tag{3.29e}$$

$$-\frac{\partial}{\partial q}B^{21}_{(A)} + \frac{\partial}{\partial p}B^{22}_{(A)} = 0 \tag{3.29f}$$

$$-\frac{\partial}{\partial q}B^{11}_{(A)} + \frac{\partial}{\partial p}B^{12}_{(A)} = 0 \tag{3.29g}$$

$$\frac{\partial}{\partial p}A^2_{22} + 4\,B^{12}_{(A)} = 0 \tag{3.29h}$$

$$\frac{\partial}{\partial p}A^2_{12} - 2\,B^{21}_{(A)} = 0 \tag{3.29i}$$

$$2\,B^{11}_{(A)} + \frac{\partial}{\partial p}A^1_{12} = 0 \tag{3.29j}$$

$$A^2_{11} = 0 \tag{3.30a}$$

$$-2\,A^2_{12} + A^1_{11} = 0 \tag{3.30b}$$

$$\frac{\partial}{\partial q}A^2_{22} + 4\,B^{12}_{(A)} - 4\,B^{21}_{(A)} = 0 \tag{3.30c}$$

$$\frac{\partial}{\partial q}A^2_{12} - 2\,B^{22}_{(A)} = 0 \tag{3.30d}$$

$$2\,B^{12}_{(A)} + \frac{\partial}{\partial q}A^1_{12} - 4\,B^{21}_{(A)} = 0 \tag{3.30e}$$

$$-\frac{\partial}{\partial q}B^{21}_{(A)} + \frac{\partial}{\partial p}B^{22}_{(A)} = 0 \tag{3.30f}$$

$$-\frac{\partial}{\partial q}B^{11}_{(A)} + \frac{\partial}{\partial p}B^{12}_{(A)} = 0 \tag{3.30g}$$

$$-\frac{\partial}{\partial q}A^1_{22} + \frac{\partial}{\partial p}A^2_{22} = 0 \tag{3.30h}$$

$$\frac{\partial}{\partial p}A^2_{12} - 2\,B^{21}_{(A)} = 0 \tag{3.30i}$$

$$\frac{\partial}{\partial p}A^1_{12} - \frac{\partial}{\partial q}A^1_{22} - 2\,B^{11}_{(A)} = 0 \tag{3.30j}$$

$$4\,\frac{\partial}{\partial q}B^{11}_{(A)} + \frac{\partial^2}{\partial q^2}A^1_{22} - 4\,\frac{\partial}{\partial p}B^{21}_{(A)} \tag{3.30k}$$

$$A^1_{11}pq^2 - 2\,A^1_{12}p^2q + p^3A^1_{22} + A^2_{11}q^3 - 2\,A^2_{12}pq^2 + A^2_{22}p^2q = 0 \tag{3.31a}$$

$$-2/3\,\frac{p^2A^1_{12}}{q^2} - 2/3\,\frac{p^3A^1_{22}}{q^3} + \frac{p^3\frac{\partial}{\partial q}A^1_{22}}{q^2} + 1/3\,A^2_{11} + 2/3\,q\frac{\partial}{\partial q}A^2_{11} + \tag{3.31b}$$

$$+2/3\,p\frac{\partial}{\partial q}A^2_{12} + 1/3\,\frac{p^2A^2_{22}}{q^2} - 1/3\,\frac{p^2\frac{\partial}{\partial q}A^2_{22}}{q} + 4/3\,\frac{p^2B^{12}_{(A)}}{q} - 8/3\,pB^{22}_{(A)} = 0$$

$$-4/3\,\frac{p^2A^1_{12}}{q^2} + \frac{p^2\frac{\partial}{\partial q}A^1_{12}}{q} + 2/3\,\frac{p^3A^1_{22}}{q^3} + 2/3\,A^2_{11} + 4/3\,q\frac{\partial}{\partial q}A^2_{11} + \tag{3.31c}$$

$$+1/3\,p\frac{\partial}{\partial q}A^2_{12} + 2/3\,\frac{p^2A^2_{22}}{q^2} - 2/3\,\frac{p^2\frac{\partial}{\partial q}A^2_{22}}{q} + 2/3\,\frac{p^2B^{12}_{(A)}}{q} - 10/3\,pB^{22}_{(A)} = 0$$

$$-\frac{\partial}{\partial q}B^{21}_{(A)} + \frac{\partial}{\partial p}B^{22}_{(A)} = 0 \tag{3.31d}$$

$$-\frac{\partial}{\partial q}B^{11}_{(A)} + \frac{\partial}{\partial p}B^{12}_{(A)} = 0 \tag{3.31e}$$

$$-A^1_{12}p^2q + p^3A^1_{22} - 1/2\,A^2_{11}q^3 - \left(\frac{\partial}{\partial q}A^2_{11}\right)q^4 + 3\,pq^3\frac{\partial}{\partial q}A^2_{12} + \tag{3.31f}$$

$$+1/2\,A^2_{22}p^2q + \left(\frac{\partial}{\partial p}A^2_{22}\right)p^3q - \left(\frac{\partial}{\partial q}A^2_{22}\right)p^2q^2 + 2\,B^{11}_{(A)}p^3q + 2\,B^{12}_{(A)}p^2q^2 -$$

$$-2\,q^2B^{21}_{(A)}p^2 - 4\,B^{22}_{(A)}pq^3 = 0$$

$$p\frac{\partial}{\partial p}A^2_{12} + 2\,q\frac{\partial}{\partial q}A^2_{12} - p\frac{\partial}{\partial q}A^2_{22} - 2\,pB^{21}_{(A)} - 4\,qB^{22}_{(A)} = 0 \tag{3.31g}$$

$$p\frac{\partial}{\partial p}A^2_{11} + 2\,q\frac{\partial}{\partial q}A^2_{11} - p\frac{\partial}{\partial q}A^2_{12} - 2\,pB^{22}_{(A)} = 0 \tag{3.31h}$$

$$1/3\,\frac{p^2A^1_{12}}{q} + 1/3\,\frac{p^3A^1_{22}}{q^2} + \frac{p^4\frac{\partial}{\partial p}A^1_{22}}{q^2} - 7/6\,A^2_{11}q - 7/3\left(\frac{\partial}{\partial q}A^2_{11}\right)q^2 +$$
$$+5/3\,pq\frac{\partial}{\partial q}A^2_{12} - 1/6\,\frac{p^2A^2_{22}}{q} - 1/3\,p^2\frac{\partial}{\partial q}A^2_{22} + 2\,\frac{B^{11}_{(A)}p^3}{q} - 2/3\,p^2B^{12}_{(A)} -$$
$$-2\,p^2B^{21}_{(A)} + 4/3\,B^{22}_{(A)}pq = 0 \tag{3.31i}$$

$$\frac{p^3\frac{\partial}{\partial p}A^1_{12}}{q} - 4\left(\frac{\partial}{\partial q}A^2_{11}\right)q^2 - 2\,A^2_{11}q + 4\,pq\frac{\partial}{\partial q}A^2_{12} - \tag{3.31j}$$
$$-p^2\frac{\partial}{\partial q}A^2_{22} + 2\,\frac{B^{11}_{(A)}p^3}{q} - 4\,p^2B^{21}_{(A)} = 0$$

$$\frac{A^2_{11}}{p} + 7/2\,\frac{q\frac{\partial}{\partial q}A^2_{11}}{p} + \frac{q^2\frac{\partial^2}{\partial q^2}A^2_{11}}{p} - 7/2\,\frac{\partial}{\partial q}A^2_{12} - 2\,q\frac{\partial^2}{\partial q^2}A^2_{12} + p\frac{\partial^2}{\partial q^2}A^2_{22} + \tag{3.31k}$$
$$+2\,p\frac{\partial}{\partial q}B^{12}_{(A)} + 2\,q\frac{\partial}{\partial q}B^{22}_{(A)} + 3\,B^{22}_{(A)} = 0$$

The cohomology group $H^2_1(P_1)$ can be easily computed. It is easy to see that (3.29b) and (3.29a) imply, respectively, ((3.26c), (3.26k)) and ((3.26b), (3.26j)). In the other direction, however, $A^1_{22} = c_1$ and $A^2_{11} = c_2$ constants satisfy the cocycle condition between being coboundaries. Hence, $H^2_1(P_1) \cong \mathbb{R}^2$.

The cohomology group $H^2_1(P_2)$ gives us more troubles. We can compare, as before, the cocycle conditions (3.27) with the coboundary conditions (3.30). Six of the conditions coincide, for example (3.27a) and (3.30d); for the remaining ones, we can exploit the built-in procedures in the package `Janet` to reduce the coboundary conditions and their differential consequences in terms of the cocycle ones. This allows to generalize and make powerful the procedure we described for $H^2_1(P_1)$: there are coboundary conditions that are not true for all the cocycles, but for instance we can check that their first (second) derivatives are 0 for all the cocycles. That means that a coefficient, or a particular combination of the coefficients must be 0 for the coboundaries but can be a constant (a linear function) for the cocycles: hence, we can characterize the cohomology groups. Performing this procedure we explained in abstract for the coboundary conditions (3.30) we get that $\partial_q A^2_{22} + 4B^{12}_{(A)} - 4B^{22}_{(A)}$, $A^1_{11} - A^2_{12}$, and A^2_{11} must vanish for the coboundaries, but for a cocycle can be respectively c_1, $c_1p + c_2$, and c_3 with (c_1, c_2, c_3) constants. Hence, $H^2_1(P_2) \cong \mathbb{R}^3$.

Finally, the cohomology group $H^2_1(P_{LP})$ is easily got by finding that the cocycle conditions imply the coboundary ones; since the converse is always true, we can conclude that all the cocycles are coboundaries, and hence $H^2_1(P_{LP}) = 0$.

Remark 3.14. A representative in the class $H^2_1(P_1)$ is a bracket of the form

$$\{p_i{}_{\boldsymbol{\lambda}}p_j\} = \begin{pmatrix} 0 & 0 \\ 0 & c_1 \end{pmatrix} \lambda_1 + \begin{pmatrix} c_2 & 0 \\ 0 & 0 \end{pmatrix} \lambda_2. \tag{3.32}$$

This bracket is itself the λ-bracket of a PVA of hydrodynamic type, in the same class as (3.1). On the other hand, a representative in the class $H^2_1(P_2)$ is a bracket of the form

$$\begin{aligned} \{p_i{}_{\boldsymbol{\lambda}}p_j\} = &\begin{pmatrix} c_1 p + c_2 & 0 \\ 0 & 0 \end{pmatrix} \lambda_1 + \begin{pmatrix} c_3 & 0 \\ 0 & 0 \end{pmatrix} \lambda_2 \\ &+ \begin{pmatrix} \frac{1}{2}c_1 & 0 \\ 0 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & \frac{1}{4}c_1 \\ -\frac{1}{4}c_1 & 0 \end{pmatrix} q_x. \end{aligned} \tag{3.33}$$

Such a bracket *is not* the λ-bracket of a PVA, as it can be easily checked by computing the PVA-Jacobi identity.

3.3.2 1st order deformations

A first order deformation of (3.1), (3.2) or (3.3) is a second degree homogeneous bracket. In general, such a bracket is of the form

$$\begin{aligned} \{p_i{}_{\boldsymbol{\lambda}}p_j\}_{[1]} = A^{ab}_{ij}(p)\lambda_a\lambda_b + B^{a,bl}_{ij}(p)\partial_b p_l \lambda_a + \\ + C^{al,bm}_{ij}(p)\partial_a p_l \partial_b p_m + D^{ab,l}(p)\partial_{ab} p_l \end{aligned} \tag{3.34}$$

Here, A, B, C and D are arbitrary functions of the p's only. It should be apparent from the definition that A^{ab}_{ij} and $D^{ab,l}_{ij}$ are symmetric in the exchange of a and b while $C^{al,bm}_{ij}$ must be symmetric in the simultaneous exchange of (a,l) with (b,m). The deformation depends on 108 parameters for $D = N = 2$. The formalism of the Poisson Vertex Algebras makes finding the conditions on A, B, C and D for the bracket $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_{[1]}$ to the first order deformation of (3.1), (3.2), and (3.3) relatively simple, and anyhow straightforward. We will prove the following

Theorem 3.15. *The first order second cohomology group for the Poisson Vertex Algebra associated to a multidimensional Poisson bracket of hydrodynamic type of the form* (3.1), (3.2), *or* (3.3) *for* $D = N = 2$ *is trivial.*

While the condition of skewsymmetry is independent from the particular form of the undeformed bracket, both the trivial deformations and the PVA–Jacobi identities must be computed for each undeformed bracket. Hence, the full proof of the theorem is split in several lemmas.

Lemma 3.16. *A first order deformation of the λ-bracket of a PVA for $D = N$ is skewsymmetric if and only if the following conditions hold:*

$$A_{ij}^{ab} = -A_{ji}^{ab} \tag{3.35a}$$

$$\frac{\partial A_{ij}^{ab}}{\partial p_l} = \frac{1}{2}\left(B_{ij}^{a,bl} - B_{ji}^{a,bl}\right) = \frac{1}{2}\left(B_{ij}^{b,al} - B_{ji}^{b,al}\right) \tag{3.35b}$$

$$B_{ij}^{a,bl} + B_{ji}^{b,al} = B_{ij}^{b,al} + B_{ji}^{a,bl} = 2D_{ij}^{ab,l} + 2D_{ji}^{ab,l} \tag{3.35c}$$

$$\frac{\partial B_{ij}^{a,bm}}{\partial p_l} + \frac{\partial B_{ji}^{b,al}}{\partial p_m} = \frac{\partial B_{ij}^{b,al}}{\partial p_m} + \frac{\partial B_{ji}^{a,bm}}{\partial p_l} = 2C_{ij}^{al,bm} + 2C_{ji}^{al,bm} \tag{3.35d}$$

Proof. We compute $\{p_i{}_{\boldsymbol{\lambda}} p_j\}_{[1]} + {}_{\rightarrow}\{p_j{}_{-\boldsymbol{\partial}-\boldsymbol{\lambda}} p_i\}_{[1]}$ and set equal to zero respectively the coefficients of $\lambda_a\lambda_b$, $\lambda_a\partial_b p_l$, $\partial_a p_l \partial_b p_m$ and $\partial_{ab} p_l$. □

In particular, for the case $D = N = 2$, the condition of skewsymmetry is equivalent to impose the following form for the parameters of (3.34):

$$A_{ij(S)}^{ab} = \frac{1}{2}\left(A_{ij}^{ab} + A_{ji}^{ab}\right) \overset{(3.35a)}{=} 0 \tag{3.36a}$$

$$A_{12(A)}^{ab} = \frac{1}{2}\left(A_{12}^{ab} - A_{21}^{ab}\right) = \tilde{A}^{ab} \tag{3.36b}$$

$$B_{ij(S)}^{a,bl} = \frac{1}{2}\left(B_{ij}^{a,bl} + B_{ji}^{ab,l}\right) = \tilde{B}_{ij}^{a,bl} \tag{3.36c}$$

$$B_{12(A)}^{a,bl} = \frac{1}{2}\left(B_{12}^{a,bl} - B_{21}^{ab,l}\right) \overset{(3.35b)}{=} \frac{\partial \tilde{A}^{ab}}{\partial p_l} \tag{3.36d}$$

$$\begin{aligned} C_{ij(S)}^{al,bm} &= \frac{1}{2}\left(C_{ij}^{al,bm} + C_{ji}^{al,bm}\right) \\ &\overset{(3.35d)}{=} \frac{1}{4}\left(\frac{\partial\left(\tilde{B}_{ij}^{a,bm} + B_{ij(A)}^{a,bm}\right)}{\partial p_l} + \frac{\partial\left(\tilde{B}_{ij}^{b,al} - B_{ij(A)}^{b,al}\right)}{\partial p_m}\right) \end{aligned} \tag{3.36e}$$

$$C_{12(A)}^{al,bm} = \frac{1}{2}\left(C_{12}^{al,bm} - C_{21}^{al,bm}\right) = \tilde{C}^{al,bm} \tag{3.36f}$$

$$\begin{aligned} D_{ij(S)}^{ab,l} &= \frac{1}{2}\left(D_{ij}^{ab,l} + D_{ji}^{ab,l}\right) \\ &\overset{(3.35c)}{=} \frac{1}{4}\left(\tilde{B}_{ij}^{a,bl} + B_{ij(A)}^{a,bl} + \tilde{B}_{ij}^{b,al} - B_{ij(A)}^{b,al}\right) \end{aligned} \tag{3.36g}$$

$$D_{12(A)}^{ab,l} = \frac{1}{2}\left(D_{12}^{ab,l} - D_{21}^{ab,l}\right) = \tilde{D}^{ab,l}. \tag{3.36h}$$

Imposing the skewsymmetry condition reduces the number of free parameters (now they are the functions denoted with the tilde) to 43.

Lemma 3.17. *A homogeneous λ-bracket of degree 2 of the form* (3.34) *is a first order deformation of the bracket* (3.1) *if and only if the following conditions hold:*

$$\sum_{\sigma(a,b,c)}\left[D^{ab,i}_{jk}\delta^c_i-D^{ab,k}_{ij}\delta^c_k+\frac{1}{2}\left(B^{a,bk}_{ij}+B^{a,bk}_{ji}\right)\delta^c_k\right]=0 \tag{3.37a}$$

$$B^{c,bi}_{jk}\delta^a_i+B^{c,ai}_{jk}\delta^b_i-\left(B^{a,bj}_{ik}-B^{a,bj}_{ki}\right)\delta^c_j+\left(B^{a,ck}_{ij}-2D^{ac,k}_{ij}\right)\delta^b_k+$$
$$+\left(B^{b,ck}_{ij}-2D^{bc,k}_{ij}\right)\delta^a_k+2D^{ab,k}_{ji}\delta^c_k=0 \tag{3.37b}$$

$$\sum_{\sigma(a,b,c)}\left(2C^{ak,bl}_{ij}-\frac{\partial D^{ab,l}_{ij}}{\partial p_k}-\partial\partial D^{ab,k}_{ij}\partial p_l\right)\delta^c_k=0 \tag{3.37c}$$

$$2C^{ai,cl}_{jk}\delta^b_i+2C^{bi,cl}_{jk}\delta^a_i+\left(\frac{\partial B^{a,ck}_{ij}}{\partial p_l}-\frac{\partial B^{a,cl}_{ij}}{\partial p_k}-2\frac{\partial D^{ac,k}_{ij}}{\partial p_l}+2C^{ak,cl}_{ij}\right)\delta^b_k+$$
$$+\left(\frac{\partial B^{b,ck}_{ij}}{\partial p_l}-\frac{\partial B^{b,cl}_{ij}}{\partial p_k}-2\frac{\partial D^{bc,k}_{ij}}{\partial p_l}+2C^{bk,cl}_{ij}\right)\delta^a_k+2\frac{\partial D^{ab,k}_{ji}}{\partial p_l}\delta^c_k=0 \tag{3.37d}$$

$$\frac{\partial B^{b,cl}_{jk}}{\partial p_i}\delta^a_i-\frac{\partial B^{a,cl}_{ik}}{\partial p_j}\delta^b_j+2\left(C^{bk,cl}_{ij}-\frac{\partial D^{bc,k}_{ij}}{\partial p_l}\right)\delta^a_k-2\left(C^{ak,cl}_{ji}-\frac{\partial D^{ac,k}_{ji}}{\partial p_l}\right)\delta^b_k+$$
$$+\frac{\partial}{\partial p_l}\left(B^{ab,k}_{ij}-2D^{ab,k}_{ij}\right)\delta^c_k=0 \tag{3.37e}$$

$$2\frac{\partial D^{bc,l}_{jk}}{\partial p_i}\delta^a_i+\left(\frac{\partial B^{a,bk}_{ij}}{\partial p_l}-\frac{\partial B^{a,bl}_{ij}}{\partial p_k}+2C^{ak,bl}_{ij}-2\frac{\partial D^{ab,k}_{ij}}{\partial p_l}\right)\delta^c_k+$$
$$+\left(\frac{\partial B^{a,ck}_{ij}}{\partial p_l}-\frac{\partial B^{a,cl}_{ij}}{\partial p_k}+2C^{ak,cl}_{ij}-2\frac{\partial D^{ac,k}_{ij}}{\partial p_l}\right)\delta^b_k+$$
$$+2\left(C^{bk,cl}_{ij}+C^{ck,bl}_{ij}-\frac{\partial D^{bc,l}_{ij}}{\partial p_k}-\frac{\partial D^{bc,k}_{ij}}{\partial p_l}\right)\delta^a_k=0 \tag{3.37f}$$

$$\sum_{\sigma(al,bm,cn)}\left(2\frac{\partial^2 C^{ak,bm}_{ij}}{\partial p_l\partial p_n}-\frac{\partial^2 C^{al,bm}_{ij}}{\partial p_k\partial p_n}-\frac{\partial^3 D^{ab,k}_{ij}}{\partial p_l\partial p_m\partial p_n}\right)\delta^c_k=0 \tag{3.37g}$$

$$2\frac{\partial C^{bl,cm}_{ij}}{\partial p_i}\delta^a_i+\left(2\frac{\partial C^{bl,ck}_{ij}}{\partial p_m}+\frac{\partial C^{cm,bk}_{ij}}{\partial p_l}-2\frac{\partial^2 D^{bc,k}_{ij}}{\partial p_l\partial p_m}-2\frac{\partial C^{bl,cm}_{ij}}{\partial p_k}\right)\delta^a_k+$$
$$+\left(\frac{\partial^2 B^{a,bk}_{ij}}{\partial p_l\partial p_m}-\frac{\partial^2 B^{a,bl}_{ij}}{\partial p_k\partial p_m}+2\frac{\partial C^{ak,bl}_{ij}}{\partial p_m}-2\frac{\partial^2 D^{ab,k}_{ij}}{\partial p_l\partial p_m}\right)\delta^c_k+$$
$$+\left(\frac{\partial^2 B^{a,ck}_{ij}}{\partial p_l\partial p_m}-\frac{\partial^2 B^{a,cl}_{ij}}{\partial p_k\partial p_m}+2\frac{\partial C^{ak,cl}_{ij}}{\partial p_m}-2\frac{\partial^2 D^{ac,k}_{ij}}{\partial p_l\partial p_m}\right)\delta^b_k=0 \tag{3.37h}$$

$$\left(\frac{\partial C^{ak,bl}_{ij}}{\partial p_m} - \frac{\partial^2 D^{ab,l}_{ij}}{\partial p_m \partial p_k} + \frac{\partial C^{bk,al}_{ij}}{\partial p_m} - \frac{\partial^2 D^{ab,k}_{ij}}{\partial p_m \partial p_l}\right)\delta^c_k +$$
$$+\left(\frac{\partial C^{ck,al}_{ij}}{\partial p_m} + \frac{\partial C^{ak,cm}_{ij}}{\partial p_l} - \frac{\partial C^{al,cm}_{ij}}{\partial p_k} - \frac{\partial^2 D^{ac,k}_{ij}}{\partial p_m \partial p_l}\right)\delta^b_k +$$
$$+\left(\frac{\partial C^{ck,bl}_{ij}}{\partial p_m} + \frac{\partial C^{bk,cm}_{ij}}{\partial p_l} - \frac{\partial C^{bl,cm}_{ij}}{\partial p_k} - \frac{\partial^2 D^{bc,k}_{ij}}{\partial p_m \partial p_l}\right)\delta^a_k = 0 \qquad (3.37\text{i})$$

Repeated indices are summated according to Einstein's rule; $\sum_{\sigma(a,b,c)}$ *means the complete symmetrization with respect to the listed indices (or couples of indices).*

Proof. When computing the PVA-Jacobi identity for $\{\cdot_{\boldsymbol{\lambda}}\cdot\}$, we end up with a degree 0 term in ϵ which is the PVA-Jacobi identity for the undeformed bracket $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_1$, plus a degree 1 term which reads

$$\epsilon\Bigg(\left\{p_{i\,\boldsymbol{\lambda}}\left\{p_{j\,\boldsymbol{\mu}}p_k\right\}_1\right\}_{[1]} + \left\{p_{i\,\boldsymbol{\lambda}}\left\{p_{j\,\boldsymbol{\mu}}p_k\right\}_{[1]}\right\}_1 +$$
$$-\left\{p_{j\,\boldsymbol{\mu}}\left\{p_{i\,\boldsymbol{\lambda}}p_k\right\}_1\right\}_{[1]} - \left\{p_{j\,\boldsymbol{\mu}}\left\{p_{i\,\boldsymbol{\lambda}}p_k\right\}_{[1]}\right\}_1 +$$
$$-\left\{\left\{p_{i\,\boldsymbol{\lambda}}p_j\right\}_{[1]\,\boldsymbol{\lambda}+\boldsymbol{\mu}}p_k\right\}_1 - \left\{\left\{p_{i\,\boldsymbol{\lambda}}p_j\right\}_{1\,\boldsymbol{\lambda}+\boldsymbol{\mu}}p_k\right\}_{[1]}\Bigg) \qquad (3.38)$$

and terms of higher order that are discharged. The sets of equations (3.37) are then obtained collecting the homogeneous terms in λ, μ and derivatives of p, up to the third degree. □

We apply to (3.37) the skewsymmetry conditions (3.36); all the algebraic relations, that can be found by direct inspection, among the 43 parameters are given in Appendix B.2.1. There are still 9 functions (according to our choice, $\tilde{A}^{11}$, $\tilde{A}^{12}$, $\tilde{A}^{22}$, $\tilde{B}^{1,12}_{11}$, $\tilde{B}^{1,22}_{11}$, $\tilde{B}^{2,11}_{11}$, $\tilde{B}^{2,21}_{22}$, $\tilde{B}^{2,11}_{22}$, and $\tilde{B}^{1,22}_{22}$) which must satisfy the following set of linear PDEs in order to be the components of the first order deformed bracket.

$$\frac{\partial \tilde{B}^{2,11}_{11}}{\partial q} = \frac{\partial \tilde{B}^{1,22}_{11}}{\partial p} + 2\frac{\partial^2 \tilde{A}^{22}}{\partial p^2} \qquad (3.39\text{a})$$
$$\frac{\partial \tilde{B}^{1,22}_{22}}{\partial p} = \frac{\partial \tilde{B}^{2,11}_{22}}{\partial q} - 2\frac{\partial^2 \tilde{A}^{11}}{\partial q^2}. \qquad (3.39\text{b})$$

The same procedure can be repeated for the deformations of (3.2). In this case we do not start looking for the general set of conditions for any D, but compute explictly the PVA–Jacobi identity at the first order (3.38) for $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_2$ and $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_{[1]}$, having imposed the form (3.36) to the parameters of the deformation. All the 43 parameters can be expressed as linear combinations and derivatives of just 9 of them, namely $\tilde{A}^{11}$, $\tilde{A}^{12}$, $\tilde{A}^{22}$, $\tilde{B}^{1,11}_{11}$, $\tilde{B}^{1,21}_{11}$, $\tilde{B}^{1,12}_{22}$,

$\tilde{B}_{22}^{2,11}$, $\tilde{B}_{22}^{2,12}$, and $\tilde{B}_{22}^{2,21}$. The formulas for the remaining can be found solving linear algebraic equations and are explicitly given in Appendix B.2.2. The nine parameters we are left with must satisfy the following pair of linear PDEs:

$$\frac{\partial \tilde{B}_{22}^{1,12}}{\partial q} + \frac{\partial \tilde{B}_{22}^{2,11}}{\partial q} = \frac{\partial \tilde{B}_{22}^{2,12}}{\partial p} \tag{3.40a}$$

$$\frac{\partial \tilde{B}_{11}^{1,11}}{\partial q} + 2\frac{\partial^2 \tilde{A}^{11}}{\partial q^2} + \frac{\partial^2 \tilde{A}^{22}}{\partial p^2} = 2\frac{\partial \tilde{B}_{22}^{2,12}}{\partial q} + \frac{\partial \tilde{B}_{22}^{2,21}}{\partial q} + \frac{\partial \tilde{B}_{11}^{1,21}}{\partial p} + 4\frac{\partial^2 \tilde{A}^{12}}{\partial p \partial q}. \tag{3.40b}$$

Finally, we perform the same computation of Lemma 3.17 for the third class of λ-brackets.

Lemma 3.18. *A homogeneous λ-bracket of degree 2 of the form* (3.34) *is a first order deformation of the bracket* (3.3) *if and only if the following conditions hold:*

$$D_{ji}^{ab,c} p_k + D_{jk}^{ab,c} p_i + \left(D_{ji}^{ab,l} \delta_k^c + D_{jk}^{ab,l} \delta_i^c \right) p_l + \circlearrowleft (a,b,c) = 0 \tag{3.41a}$$

$$\begin{aligned} &2\left(A_{ik}^{bc}\delta_j^a + A_{ik}^{ac}\delta_j^b \right) + 2\left(A_{ji}^{ab}\delta_k^c - A_{ik}^{ab}\delta_j^c + A_{kj}^{ab}\delta_i^c \right) + \\ &- \left(B_{ki}^{a,bc} - B_{ik}^{a,bc} \right) p_j - \left(B_{jk}^{c,ab} - B_{jk}^{c,ba} \right) p_i + \\ &- \left[\left(B_{ki}^{a,bl} - B_{ik}^{a,bl} \right) \delta_j^c + B_{jk}^{c,bl}\delta_i^a + B_{jk}^{c,al}\delta_i^b \right] p_l + \\ &- \left[\left(2D_{ji}^{bc,l} - B_{ji}^{c,bl} \right) \delta_k^a + \left(2D_{ji}^{ca,l} - B_{ji}^{c,al} \right) \delta_k^b + 2D_{ji}^{ab,l}\delta_k^c \right] p_l + \\ &- \left(2D_{ji}^{cb,a} - B_{ji}^{c,ba} + 2D_{ji}^{ca,b} - B_{ji}^{c,ab} + 2D_{ji}^{ab,c} \right) p_k = 0 \end{aligned} \tag{3.41b}$$

$$\begin{aligned} &\sum_{\sigma(a,b,c)} \left[\left(\frac{\partial D_{ij}^{ab,l}}{\partial p_m} + \frac{\partial D_{ij}^{ab,m}}{\partial p_l} - 2C_{ij}^{am,bl} \right) p_m \delta_k^c + \right. \\ &\left. + \left(\frac{\partial D_{ij}^{ab,l}}{\partial p_c} + \frac{\partial D_{ij}^{ab,c}}{\partial p_l} - 2C_{ij}^{ac,bl} \right) \right] = 0 \end{aligned} \tag{3.41c}$$

$$\begin{aligned} &\left(C_{ji}^{ba,cl} + C_{ji}^{ab,cl} \right) p_k - \left(C_{jk}^{ba,cl} + C_{jk}^{ab,cl} \right) p_i - \frac{\partial}{\partial p_l}\left(D_{ji}^{ab,c} + D_{ji}^{bc,a} + D_{ji}^{ca,b} \right) p_k + \\ &- \left(D_{jk}^{bc,a} + D_{jk}^{ac,b} \right) \delta_i^l - D_{ji}^{ab,c}\delta_k^l - \left(D_{jk}^{bc,l}\delta_i^a + D_{jk}^{ca,l}\delta_i^b + D_{jk}^{ab,l}\delta_i^c \right) + \\ &+ \left[\left(C_{ji}^{bm,cl}\delta_k^a + C_{ji}^{am,cl}\delta_k^b \right) - \left(C_{jk}^{bm,cl}\delta_i^a + C_{jk}^{am,cl}\delta_i^b \right) + \right. \\ &\left. - \frac{\partial}{\partial p_l}\left(D_{ji}^{ab,m}\delta_k^c + D_{ji}^{bc,m}\delta_k^a + D_{ji}^{ca,m}\delta_k^b \right) \right] p_m = 0 \end{aligned} \tag{3.41d}$$

$$
\begin{gathered}
B^{b,cl}_{ik}\delta^a_j - B^{a,cl}_{jk}\delta^b_i + B^{b,cl}\delta^a_k - B^{a,cl}_{ij}\delta^b_k + B^{b,al}_{jk}\delta^c_i + \\
+B^{a,cb}_{ik}\delta^l_j - B^{b,ca}_{jk}\delta^l_i + \left(2D^{ab,c}_{ij} - B^{a,bc}_{ij}\right) + + \frac{\partial B^{a,cl}_{ik}}{\partial p_b}p_j - \frac{\partial B^{b,cl}_{jk}}{\partial p_a}p_i + \\
+\left[2C^{ab,cl}_{ji} - 2C^{ba,cl}_{ij} + 2\frac{\partial D^{cb,a}_{ij}}{\partial p_l} - 2\frac{\partial D^{ca,b}_{ji}}{\partial p_l} + \frac{\partial}{\partial p_l}\left(2D^{ab,c}_{ij} - B^{a,bc}_{ij}\right)\right] p_k + \\
+\left[\frac{\partial B^{a,cl}_{ik}}{\partial p_m}\delta^b_j - \frac{\partial B^{b,cl}_{jk}}{\partial p_m}\delta^a_i + 2\left(\frac{\partial D^{cb,m}_{ij}}{\partial p_l} - C^{bm,cl}_{ij}\right)\delta^a_k + \right. \\
\left. -2\left(\frac{\partial D^{ab,m}_{ji}}{\partial p_l} - C^{am,cl}_{ji}\right)\delta^b_k + \frac{\partial}{\partial p_l}\left(2D^{ab,m}_{ij} - B^{a,bm}_{ij}\right)\delta^c_k\right] = 0
\end{gathered} \tag{3.41e}
$$

$$
\begin{gathered}
\left(C^{ac,bl}_{ji} + C^{ab,cl}_{ji} - \frac{\partial D^{ab,c}_{ji}}{\partial p_l} - \frac{\partial D^{ac,b}_{ji}}{\partial p_l} + \frac{\partial D^{bc,a}_{ij}}{\partial p_l} + \frac{\partial D^{bc,l}_{ij}}{\partial p_a} - C^{ba,cl}_{ij} - C^{ca,bl}_{ij}\right) p_k + \\
-\frac{\partial D^{bc,l}_{jk}}{\partial p_a}p_i - D^{bc,a}_{jk}\delta^l_i - D^{ab,l}_{jk}\delta^c_i - D^{ac,l}_{jk}\delta^b_i + D^{bc,l}_{ji}\delta^a_k + \\
+\left[-\frac{\partial D^{bc,l}_{jk}}{p_m}\delta^a_i + \left(\frac{\partial D^{bc,l}_{ij}}{\partial p_m} - \frac{\partial D^{bc,m}_{ij}}{\partial p_l} - C^{bm,cl}_{ij} - C^{cm,bl}_{ij}\right)\delta^a_k + \right. \\
\left. +\left(C^{am,bl}_{ji} - \frac{\partial D^{ab,m}_{ji}}{\partial p_l}\right)\delta^c_k + \left(C^{am,cl}_{ji} - \frac{\partial D^{ac,m}_{ji}}{\partial p_l}\right)\delta^b_k\right] p_m = 0
\end{gathered} \tag{3.41f}
$$

$$
\begin{gathered}
-2\frac{\partial}{\partial p_a}\left(C^{bl,cm}_{jk}p_i\right) + 2\frac{\partial C^{bl,cm}_{ij}}{\partial p_a}p_k + 2C^{bl,cm}_{ji}\delta^a_k \\
+\frac{\partial}{\partial p_m}\left(2C^{ac,bl}_{ji} - 2C^{ca,bl}_{ij} + 2\frac{\partial D^{bc,a}_{ij}}{\partial p_l} - 2\frac{\partial D^{ab,c}_{ji}}{\partial p_l}\right) p_k + \\
+\frac{\partial}{\partial p_l}\left(2C^{ab,cm}_{ji} - 2C^{ba,cm}_{ij} + 2\frac{\partial D^{bc,a}_{ij}}{\partial p_m} - 2\frac{\partial D^{ac,b}_{ji}}{\partial p_m}\right) p_k + \\
-2C^{bl,am}_{jk}\delta^c_i - 2C^{cm,al}_{jk}\delta^b_i - 2C^{bl,ca}_{jk}\delta^m_i - 2C^{cm,ba}_{jk}\delta^l_i + \\
+2\left(C^{ac,bl}_{ji} - \frac{\partial D^{ab,c}_{ji}}{\partial p_l}\right)\delta^m_k + 2\left(C^{ab,cm}_{ji} - \frac{\partial D^{ac,b}_{ji}}{\partial p_m}\right)\delta^l_k = 0
\end{gathered} \tag{3.41g}
$$

$$
\begin{gathered}
\sum_{\sigma(al,bm,cn)}\left[\frac{\partial}{\partial p_n}\left(\frac{\partial C^{al,bm}_{ij}}{\partial p_c}p_k - 2\frac{\partial C^{ac,bm}_{ij}}{\partial p_l}p_k + \frac{\partial^2 D^{ab,c}_{ij}}{\partial p_m \partial p_l}p_k\right) + \right. \\
\left. +p_s\frac{\partial}{\partial p_n}\left(\frac{\partial C^{al,bm}_{ij}}{\partial p_s} - 2\frac{\partial C^{as,bm}_{ij}}{\partial p_l} + \frac{\partial^2 D^{ab,s}_{ij}}{\partial p_m \partial p_l}\right)\delta^c_k\right] = 0
\end{gathered} \tag{3.41h}
$$

$$\frac{\partial}{\partial p_m}\left[\left(\frac{\partial D^{ab,l}_{ij}}{\partial p_c}+\frac{\partial D^{ab,c}_{ij}}{\partial p_l}-C^{ac,bl}_{ij}-C^{bc,al}_{ij}\right)p_k\right]+ \tag{3.41i}$$

$$+\left(\frac{\partial C^{al,cm}_{ij}}{\partial p_b}+\frac{\partial C^{bl,cm}_{ij}}{\partial p_a}-\frac{\partial C^{ab,cm}_{ij}}{\partial p_l}-\frac{\partial C^{ba,cm}_{ij}}{\partial p_l}+\right. \tag{3.41j}$$

$$\left.-\frac{\partial C^{al,cb}_{ij}}{\partial p_m}-\frac{\partial C^{bl,ca}_{ij}}{\partial p_m}+\frac{\partial^2 D^{ac,b}_{ij}}{\partial p_l\partial p_m}+\frac{\partial^2 D^{bc,a}_{ij}}{\partial p_l\partial p_m}\right)p_k+ \tag{3.41k}$$

$$+p_n\left[\left(\frac{\partial^2 D^{ab,l}_{ij}}{\partial p_m\partial p_n}+\frac{\partial^2 D^{ab,n}_{ij}}{\partial p_l\partial p_m}-\frac{\partial C^{an,bl}_{ij}}{\partial p_m}-\frac{\partial C^{bn,al}_{ij}}{\partial p_m}\right)\delta^c_k+\right. \tag{3.41l}$$

$$\left(\frac{\partial^2 D^{ac,n}_{ij}}{\partial p_l\partial p_m}+\frac{\partial C^{al,cm}_{ij}}{\partial p_n}-\frac{\partial C^{an,cm}_{ij}}{\partial p_l}-\frac{\partial C^{cn,al}_{ij}}{\partial p_m}\right)\delta^b_k+ \tag{3.41m}$$

$$\left(\frac{\partial^2 D^{ac,n}_{ij}}{\partial p_l\partial p_m}+\frac{\partial C^{al,cm}_{ij}}{\partial p_n}-\frac{\partial C^{an,cm}_{ij}}{\partial p_l}-\frac{\partial C^{cn,al}_{ij}}{\partial p_m}\right)\delta^b_k+ \tag{3.41n}$$

$$\left.\left(\frac{\partial^2 D^{bc,n}_{ij}}{\partial p_l\partial p_m}+\frac{\partial C^{bl,cm}_{ij}}{\partial p_n}-\frac{\partial C^{bn,cm}_{ij}}{\partial p_l}-\frac{\partial C^{cn,bl}_{ij}}{\partial p_m}\right)\delta^a_k\right]=0 \tag{3.41o}$$

The notation $\circlearrowleft (a,b,c)$ *means cyclic permutations of the indices* (a,b,c)*.*

Remark 3.19. Let us consider the trivial case $D=1$. The undeformed bracket reads $\{p_\lambda p\}_{LP}=-2p\lambda-p'$ (the prime means the only derivative of p, namely wrt x), which is the so-called *Virasoro-Magri PVA* with central charge 0 (see Ex. 1.18 in [5]). We easily get the well known result, shown for instance in [20], that such deformations do not exist in the scalar case. From the skewsymmetry conditions we get (now the indices have become useless, but notice that here D does not denote the number of the independent variables but rather the coefficient in front of p'') $A=0$, $2D=B$, $2C=B'$; moreover, (3.41a) is enough to get $D=0$, hence $B=C=0$.

We follow the same approach as for the bracket $\{\cdot_{\boldsymbol\lambda}\cdot\}_1$, setting $D=2$ and expressing the parameters of the deformation according to (3.36). We simplify the condition and get an overdetermined system of 45 equations for 9 unknown functions $\tilde{A}^{11}$, $\tilde{A}^{12}$, $\tilde{A}^{22}$, $\tilde{B}^{1,22}_{11}$, $\tilde{B}^{2,11}_{11}$, $\tilde{B}^{1,21}_{11}$, $\tilde{B}^{2,11}_{22}$, $\tilde{B}^{1,22}_{22}$, and $\tilde{B}^{2,12}_{22}$. The expressions for the remaining ones in terms of these nine are left to the Appendix B.2.3. The Janet basis of the system of PDEs according which the deformed λ-bracket is a PVA up to the first order are the following two equations

$$
\begin{aligned}
&-\frac{3}{4}p^2\tilde{A}^{11}+\frac{3}{4}q^2\tilde{A}^{22}-\frac{3}{4}q^3\tilde{B}^{1,22}_{11}-\frac{3}{4}p^3\tilde{B}^{2,11}_{22}-\frac{3}{8}q^3\frac{\partial\tilde{A}^{22}}{\partial q}+\frac{3}{8}p^3\frac{\partial\tilde{A}^{11}}{\partial p}+\\
&-\frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{2,11}_{11}}{\partial p}-\frac{1}{2}p^3q\frac{\partial\tilde{B}^{1,22}_{22}}{\partial p}-\frac{1}{2}pq^2\tilde{B}^{2,11}_{11}-\frac{1}{4}pq^2\tilde{B}^{1,21}_{11}-\frac{1}{4}p^2q\tilde{B}^{2,12}_{22}+\\
&-\frac{1}{2}p^2q\tilde{B}^{1,22}_{22}-p^2q^2\frac{\partial^2\tilde{A}^{11}}{\partial q^2}-\frac{1}{2}p^3q\frac{\partial^2\tilde{A}^{11}}{\partial p\partial q}+p^2q^2\frac{\partial^2\tilde{A}^{22}}{\partial p^2}+\frac{1}{2}pq^3\frac{\partial^2\tilde{A}^{22}}{\partial p\partial q}+\\
&+\frac{5}{8}p^2q\frac{\partial\tilde{A}^{11}}{\partial q}+\frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{2,12}_{22}}{\partial q}-\frac{1}{4}pq^2\frac{\partial\tilde{A}^{12}}{\partial q}-\frac{1}{2}pq^3\frac{\partial\tilde{B}^{2,11}_{11}}{\partial q}+\frac{1}{2}p^3q\frac{\partial\tilde{B}^{2,11}_{22}}{\partial q}+\\
&+\frac{1}{2}pq^3\frac{\partial\tilde{B}^{1,22}_{11}}{\partial p}+\frac{1}{4}p^2q\frac{\partial\tilde{A}^{12}}{\partial p}-\frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{1,22}_{22}}{\partial q}+\frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{1,21}_{11}}{\partial p}-\frac{5}{8}pq^2\frac{\partial\tilde{A}^{22}}{\partial p}=\\
&=0 \qquad (3.42a)
\end{aligned}
$$

and

$$
\begin{aligned}
&-5p^2q^2\frac{\partial^2\tilde{A}^{12}}{\partial p\partial q}-2p^3q\frac{\partial^2\tilde{A}^{12}}{\partial p^2}+6p^2q^2\frac{\partial^2\tilde{A}^{11}}{\partial q^2}+5p^3q\frac{\partial^2\tilde{A}^{11}}{\partial p\partial q}+2pq^3\frac{\partial^2\tilde{A}^{22}}{\partial p\partial q}+\\
&+\frac{11}{4}p^2\tilde{A}^{11}-\frac{7}{4}q^2\tilde{A}^{22}+\frac{15}{4}q^3\tilde{B}^{1,22}_{11}+\frac{3}{4}p^3\tilde{B}^{2,11}_{22}-\frac{1}{8}q^3\frac{\partial\tilde{A}^{22}}{\partial q}-\frac{19}{8}p^3\frac{\partial\tilde{A}^{11}}{\partial p}+\\
&+p^4\frac{\partial^2\tilde{A}^{11}}{\partial p^2}+q^4\frac{\partial^2\tilde{A}^{22}}{\partial q^2}+\frac{5}{4}p^2q^2\frac{\partial\tilde{B}^{2,11}_{11}}{\partial p}+\frac{1}{2}p^3q\frac{\partial\tilde{B}^{1,22}_{22}}{\partial p}+\frac{5}{2}pq^2\tilde{B}^{2,11}_{11}+\frac{9}{4}pq^2\tilde{B}^{1,21}_{11}+\\
&-\frac{3}{4}p^2q\tilde{B}^{2,12}_{22}+\frac{1}{2}p^2q\tilde{B}^{1,22}_{22}+2q^4\frac{\partial\tilde{B}^{1,22}_{11}}{\partial q}+2pq^3\frac{\partial\tilde{B}^{1,21}_{11}}{\partial q}-2p^3q\frac{\partial\tilde{B}^{2,12}_{22}}{\partial p}-2p^4\frac{\partial\tilde{B}^{2,11}_{22}}{\partial p}+\\
&-2pq^3\frac{\partial^2\tilde{A}^{12}}{\partial q^2}-\frac{17}{8}p^2q\frac{\partial\tilde{A}^{11}}{\partial q}-\frac{9}{4}p^2q^2\frac{\partial\tilde{B}^{2,12}_{22}}{\partial q}+\frac{1}{4}pq^2\frac{\partial\tilde{A}^{12}}{\partial q}+\frac{5}{2}pq^3\frac{\partial\tilde{B}^{2,11}_{11}}{\partial q}+\\
&-\frac{7}{2}p^3q\frac{\partial\tilde{B}^{2,11}_{22}}{\partial q}+\frac{1}{2}pq^3\frac{\partial\tilde{B}^{1,22}_{11}}{\partial p}-\frac{5}{4}p^2q\frac{\partial\tilde{A}^{12}}{\partial p}+\frac{1}{4}p^2q^2\frac{\partial\tilde{B}^{1,22}_{22}}{\partial q}+\frac{3}{4}p^2q^2\frac{\partial\tilde{B}^{1,21}_{11}}{\partial p}+\\
&+\frac{13}{8}pq^2\frac{\partial\tilde{A}^{22}}{\partial p}=0. \qquad (3.42b)
\end{aligned}
$$

Now, let us consider the trivial deformations of (3.1), (3.2), and (3.3), namely the deformed brackets given by performing a general Miura transformation (1.49) of the first order to the undeformed brackets. Such a change of coordinates will have the form

$$p_i \mapsto P_i = p_i + \sum_{j,k=1,2} \epsilon F_i^{jk}(p,q)\partial_j p_k$$

and thus depends on 8 arbitrary functions of ($p_1 \equiv p, p_2 \equiv q$). We compute $\{P_i(p)_{\boldsymbol{\lambda}} P_j(p)\}_{1,2,LP} = \{P_i(p)_{\boldsymbol{\lambda}} P_j(p)\}_a$, which is in all three cases very straightforward. We start by the expansion to the order ϵ,

$$\{P_i{}_{\boldsymbol{\lambda}}P_j\}_a = \{p_i{}_{\boldsymbol{\lambda}}p_j\}_a + \epsilon\left(\left\{F_i^{al}\partial_a p_l{}_{\boldsymbol{\lambda}}p_j\right\}_a + \left\{p_i{}_{\boldsymbol{\lambda}}F_j^{al}\partial_a p_l\right\}_a\right) + O(\epsilon^2)$$

and then we use the master formula (2.5) for the two latter brackets. The expression we get is written in terms of the 'old' coordinates; up to the first order, we can invert the transformation by $p_i = P_i - \epsilon F_i^{al}(P)\partial_a P_l$, getting the formula for the deformed bracket $\{P_i(p){}_{\boldsymbol{\lambda}}P_j(p)\}_{1,2,LP} = \{P_i{}_{\boldsymbol{\lambda}}P_j\}_{1,2,LP} + \epsilon\{P_i{}_{\boldsymbol{\lambda}}P_j\}_{[1]}$. We need only the 9 parameters of each first order deformed bracket we have chosen to express all the other ones in terms of. For deformations of (3.1) we have:

$$\tilde{A}^{11} = F_2^{11} \tag{3.43a}$$
$$2\tilde{A}^{12} = F_2^{21} - F_1^{12} \tag{3.43b}$$
$$\tilde{A}^{22} = -F_1^{22} \tag{3.43c}$$
$$B_{11}^{1,12} = 2\frac{\partial F_1^{12}}{\partial p} - 2\frac{\partial F_1^{11}}{\partial q} \tag{3.43d}$$
$$B_{11}^{1,22} = 2\frac{\partial F_1^{22}}{\partial p} - \frac{\partial F_1^{21}}{\partial q} \tag{3.43e}$$
$$B_{11}^{2,11} = -\frac{\partial F_1^{21}}{\partial p} \tag{3.43f}$$
$$B_{22}^{2,21} = 2\frac{\partial F_2^{11}}{\partial q} - 2\frac{\partial F_2^{22}}{\partial p} \tag{3.43g}$$
$$B_{22}^{2,11} = 2\frac{\partial F_2^{11}}{\partial q} - \frac{\partial F_2^{12}}{\partial p} \tag{3.43h}$$
$$B_{22}^{2,12} = -\frac{\partial F_2^{12}}{\partial q} \tag{3.43i}$$

Following the same procedure, the parameters we chose for the deformations of (3.2) are

$$\tilde{A}^{11} = -F_1^{11} + F_2^{12} \tag{3.44a}$$
$$2\tilde{A}^{12} = F_2^{22} - F_1^{21} - F_1^{12} \tag{3.44b}$$
$$\tilde{A}^{22} = -F_1^{22} \tag{3.44c}$$
$$B_{11}^{1,11} = 2\frac{\partial F_1^{11}}{\partial q} - 2\frac{\partial F_1^{12}}{\partial p} \tag{3.44d}$$
$$B_{11}^{1,21} = 2\frac{\partial F_1^{21}}{\partial q} - \frac{\partial F_1^{22}}{\partial p} \tag{3.44e}$$
$$B_{22}^{1,12} = 2\frac{\partial F_2^{12}}{\partial p} - 2\frac{\partial F_2^{11}}{\partial q} \tag{3.44f}$$
$$B_{22}^{2,11} = 2\frac{\partial F_2^{11}}{\partial q} - \frac{\partial F_2^{12}}{\partial p} - \frac{\partial F_2^{21}}{\partial p} \tag{3.44g}$$

$$B_{22}^{2,12} = \frac{\partial F_2^{12}}{\partial q} - \frac{\partial F_2^{21}}{\partial q} \tag{3.44h}$$

$$B_{22}^{2,21} = 2\frac{\partial F_2^{21}}{\partial q} - 2\frac{\partial F_2^{22}}{\partial p} \tag{3.44i}$$

Finally, for the trivial deformations of (3.3) we get

$$\tilde{A}^{11} = qF_1^{11} - 2pF_2^{11} - qF_2^{12} \tag{3.45a}$$

$$2\tilde{A}^{12} = pF_1^{11} + 2qF_1^{12} - pF_2^{12} + qF_2^{21} - 2pF_2^{21} - qF_2^{22} \tag{3.45b}$$

$$\tilde{A}^{22} = pF_1^{21} + 2qF_1^{22} - pF_2^{22} \tag{3.45c}$$

$$B_{11}^{1,21} = F_1^{12} - 2q\frac{\partial F_1^{21}}{\partial q} + p\frac{\partial F_1^{12}}{\partial p} - 2p\frac{\partial F_1^{21}}{\partial p} + q\frac{\partial F_1^{22}}{\partial p} \tag{3.45d}$$

$$B_{11}^{12,2} = F_1^{22} + p\frac{\partial F_1^{12}}{\partial q} + 2p\frac{\partial F_1^{21}}{\partial q} - q\frac{\partial F_1^{22}}{\partial q} - 4p\frac{\partial F_1^{22}}{\partial p} \tag{3.45e}$$

$$B_{11}^{2,11} = -F_1^{12} - 2p\frac{\partial F_1^{11}}{\partial q} + p\frac{\partial F_1^{12}}{\partial p} + 2p\frac{\partial F_1^{21}}{\partial p} + q\frac{\partial F_1^{22}}{\partial p} \tag{3.45f}$$

$$B_{22}^{2,12} = F_2^{21} + p\frac{\partial F_2^{11}}{\partial q} - 2q\frac{\partial F_2^{12}}{\partial q} + q\frac{\partial F_2^{21}}{\partial q} - 2p\frac{\partial F_2^{12}}{\partial p} \tag{3.45g}$$

$$B_{22}^{2,11} = F_2^{11} + q\frac{\partial F_2^{21}}{\partial p} + 2q\frac{\partial F_2^{12}}{\partial p} - p\frac{\partial F_2^{11}}{\partial p} - 4q\frac{\partial F_2^{11}}{\partial q} \tag{3.45h}$$

$$B_{22}^{2,12} = -F_2^{21} - 2q\frac{\partial F_2^{22}}{\partial p} + q\frac{\partial F_2^{21}}{\partial q} + 2q\frac{\partial F_2^{12}}{\partial q} + p\frac{\partial F_2^{11}}{\partial q} \tag{3.45i}$$

In the three sets of equations (3.43), (3.44), and (3.45) we have dropped the tilde from the parameters B's because, by definition (3.36), we have $\tilde{B}_{ii}^{a,bc} = B_{ii}^{a,bc}$. Since the three brackets we have just defined are the Miura transformed of the undeformed ones, they are a first order deformation of a PVA bracket; the sets of coefficients satisfy, as it can be easily checked, the corresponding PVA–Jacobi identities up to the first order.

We can regard each set of equations (3.43), (3.44), and (3.45) as an inhomogeneous linear system of 9 PDEs for the 8 unknown functions F's. A solution of the system, if there exists, is the set of the eight parameters of a Miura transformation which produces a given coboundary.

The compatibility conditions for (3.43) are (3.39); the compatibility conditions for (3.44) are (3.40). Computing the compatibility conditions for (3.45) we get a system of two second order differential equations, whose Janet basis is exactly (3.42a) and (3.42b).

That means that a generic first order cocycle, i.e. a first order deformed bracket, can be written in terms of the nine parameters if and only if they satisfy the corresponding pair of linear PDEs (3.39), (3.40) or (3.42). On the other hand, the same conditions allow to find the eight parameters of a Miura transformations for which we get that cocycle. It follows that every cocycle in

$W_{2,2}$ is a coboundary, so that $H^2_2(\mathcal{A}, \{\cdot_{\boldsymbol{\lambda}}\cdot\}) = 0$, for $(\mathcal{A}, \{\cdot_{\boldsymbol{\lambda}}\cdot\})$ a 2-dimensional Poisson Vertex Algebra of hydrodynamic type of rank 2.

3.3.3 2nd order deformations

A second order deformation of (3.1), (3.2), or (3.3) must be a third order homogeneous bracket of the general form

$$\begin{aligned}\{p_i{}_{\boldsymbol{\lambda}}p_j\}_{[2]} = A^{abc}_{ij}(p)\lambda_a\lambda_b\lambda_c + B^{ab,cl}_{ij}(p)\lambda_a\lambda_b\partial_c p_l + C^{a,bl,cm}(p)\lambda_a\partial_b p_l\partial_c p_m +\\ + D^{a,bcl}(p)\lambda_a\partial_{bc}p_l + E^{abcl}(p)\partial_{abc}p_l +\\ + F^{abl,cm}(p)\partial_{ab}p_l\partial_c p_m + G^{al,bm,cn}(p)\partial_a p_l\partial_b p_m\partial_c p_n \quad (3.46)\end{aligned}$$

From the definition, it follows that the coefficients A's must be totally symmetric in the exchange of the indices (a, b, c), the coefficients B's are symmetric in the exchange of (a, b), C's for the simultaneous exchange of $(b, l \leftrightarrow c, m)$, D's symmetric in (b, c), E's in (a, b, c), F's in (a, b) and, finally, G's are completely symmetric for the simultaneous exchange of $(a, l \leftrightarrow b, m \leftrightarrow c, n)$. Taking into account the symmetries built in in the definition, a second order deformations depends on 400 coefficients. First of all, we proceed to impose the skewsymmetry conditions to reduce the number of free parameters. These conditions fix the skewsymmetric part of coefficients of the odd degree parts in λ, namely A's, C's, and D's, and the symmetric part of the even degree part B's, E's, F's and G's. They can be easily computed, and we get

$$A^{abc}_{(A)} = 0 \quad (3.47a)$$

$$B^{ab,cl}_{ij(S)} = \frac{3}{2}\frac{\partial A^{abc}_{ij}}{\partial p_l} \quad (3.47b)$$

$$C^{a,bl,cm}_{(A)} = \frac{1}{2}\left(3\frac{\partial^2 A^{abc}_{ij}}{\partial p_l\partial p_m} - \frac{\partial B^{ab,cm}_{ji}}{\partial p_l} - \frac{\partial B^{ac,bl}_{ji}}{\partial p_m}\right) \quad (3.47c)$$

$$D^{a,bcl}_{(A)} = \frac{1}{2}\left(3\frac{\partial A^{abc}_{ij}}{\partial p_l} - B^{ab,cl}_{ji} - B^{ac,bl}_{ji}\right) \quad (3.47d)$$

$$E^{abcl}_{ij(S)} = \frac{1}{6}\left(D^{a,bcl}_{ji} + D^{b,acl}_{ji} + D^{c,abl}_{ji} - B^{ab,cl}_{ji} - B^{ac,bl}_{ji} - B^{bc,al}_{ji} + \frac{\partial A^{abc}_{ji}}{\partial p_l}\right) \quad (3.47e)$$

$$F^{abl,cm}_{ij(S)} = \frac{1}{2}\left(3\frac{\partial^2 A^{abc}_{ij}}{\partial p_l\partial p_m} - \frac{\partial B^{ab,cm}_{ji}}{\partial p_l} - \frac{\partial B^{ac,bm}_{ji}}{\partial p_l} - \frac{\partial B^{bc,al}_{ji}}{\partial p_m} + C^{c,al,bm}_{ji} + C^{b,al,cm}_{ji} + \frac{\partial D^{a,bcm}_{ji}}{\partial p_l}\right) \quad (3.47f)$$

$$G_{ij(S)}^{al,bm,cn} = \frac{1}{6}\left(\frac{\partial C_{ji}^{a,bm,cn}}{\partial p_l} + \frac{\partial C_{ji}^{b,al,cn}}{\partial p_m} + \frac{\partial C_{ji}^{c,al,bm}}{\partial p_n}\right. \\ \left. - \frac{\partial^2 B_{ji}^{ab,cn}}{\partial p_l \partial p_m} - \frac{\partial^2 B_{ji}^{ac,bm}}{\partial p_l \partial p_n} - \frac{\partial^2 B_{ji}^{bc,al}}{\partial p_m \partial p_n} + 3\frac{\partial^3 A_{ji}^{abc}}{\partial p_l \partial p_m \partial p_n}\right) \tag{3.47g}$$

The number of free coefficients is reduced to 172; only writing down the set of algebraic and differential equations that they must satisfy is a formidable task. To this aim, the Mathematica code we have developed and described in Section 2.3 is very slow. A redesign of the main code, that exploits the properties of skewsymmetry of the λ-brackets, makes the procedure fast enough to be completed in a few hours; more details will be provided in the next Section 3.4 since that is the only way one can perform computations for higher cohomology groups. However, it is possible to perform the analogous computation for the cocycle conditions $[P_{1,2,LP}, P_{[2]}] = 0$ using the `CDE` package [56, 55] for the REDUCE computer algebra system; in fact, it is much faster and it allows to get the system of equations in a few minutes.

The three sets of equations we get, respectively, for brackets compatible with (3.1), (3.2), and (3.3), count 1700, 1879 , and 3723 algebraic differential equations. In principle, the procedure we would like to follow is the same already experimented for 0-th and 1-st order deformations of the bracket. In practice, this can be immediately done for P_1 and P_2, namely we can compute the Janet basis of the two systems of cocycle conditions and compare it with the coboundary conditions. The Janet basis of the cocycle condition for a bracket compatible with P_1 consists of 193 equations, the one for the bracket compatible with P_2 consists of 212 equations. We provide in Appendix B.3.1 the former system just as an example.

The Miura transformation that produces trivial deformed brackets of the second order has the general form

$$p_i \mapsto P_i = p_i + f_i^{abl} \partial_{ab} p_l + g_i^{al,bm} \partial_a p_l \partial_b p_m. \tag{3.48}$$

As in the previous cases, we consider the compatibility conditions to solve for f's and g's in terms of the generic coefficients and find the coboundary conditions. They are respectively a set of 189 and of 207 equations. To compute the cohomology group we proceed as in Section 3.3.1 by looking for the coboundary conditions that are not equivalent to or a consequence of the cocycle ones.

For the deformations of (3.1), we have the following identities:

Coboundary	**Cocycle**	
$D_{11}^{2,221} = 0$	$\dfrac{\partial D_{11}^{2,221}}{\partial p} = 0$	$\dfrac{\partial D_{11}^{2,221}}{\partial q} = 0$
$D_{22}^{1,112} = 0$	$\dfrac{\partial D_{22}^{1,112}}{\partial p} = 0$	$\dfrac{\partial D_{22}^{1,112}}{\partial q} = 0$
$A_{11}^{222} = 0$	$\dfrac{\partial A_{11}^{222}}{\partial p} = -\dfrac{1}{3}c_1$	$\dfrac{\partial A_{11}^{222}}{\partial q} = 0$
$A_{22}^{111} = 0$	$\dfrac{\partial A_{22}^{111}}{\partial p} = 0$	$\dfrac{\partial A_{22}^{111}}{\partial q} = -\dfrac{1}{3}c_2.$

We identify a representative for the cohomology class $H_3^2(P_1)$ as a solution of the cocycle conditions that is not a solution of the coboundary ones, such as the following:

$$\begin{aligned} A_{22}^{111} &= -\frac{1}{3}c_2 q + c_4 & \qquad D_{22}^{1,112} &= c_2 \\ A_{11}^{222} &= -\frac{1}{3}c_1 p + c_3 & \qquad D_{11}^{2,221} &= c_1. \end{aligned} \tag{3.49}$$

The coefficients for which we do not provide the form are set equal to 0 for the representative of the cohomology class. We conclude that $H_3^2(P_1) \cong \mathbb{R}^4$.

The picture is more involved for the deformations of (3.2), since the elements of the Janet basis of the coboundary conditions are no more independent when one imposes the fulfilling of the cocycle conditions. Luckily, they become a set of ordinary polynomials in the coefficients and they are, in particular, linear. This easily allows to look for a basis of their span. As in the previous case, such a basis consists of expression that vanish for coboundary, but in general do not vanish for a generic cocycle; their first or higher derivatives do, thus allowing us to compute the cohomology. We have

Coboundary

$$\begin{aligned} b_1 &:= A_{11}^{122} - \frac{2}{3}A_{12}^{222} = 0 \\ b_2 &:= D_{12}^{1,222} + D_{11}^{2,112} + 10D_{11}^{2,121} - 2D_{12}^{2,122} - 5D_{12}^{2,221} + 6E_{12}^{1222} + 6E_{12}^{2221} = 0 \\ b_3 &:= D_{11}^{1,221} + 2D_{11}^{2,121} - 2D_{12}^{2,221} = 0 \\ b_4 &:= 2B_{12}^{22,11} + D_{11}^{1,112} + 2D_{11}^{1,121} - 4D_{12}^{1,122} + D_{22}^{1,222} - D_{11}^{2,111} - 2D_{12}^{2,112} + \\ &\quad + 2D_{22}^{2,122} - D_{22}^{2,221} - 6E_{12}^{1221} = 0 \\ b_5 &:= A_{11}^{222} = 0, \end{aligned}$$

where we denote $(b_1, b_2, b_3, b_4, b_5)$ the five conditions, for further reference, and

Cocycle

$$\frac{\partial b_1}{\partial p} = \frac{1}{27}\left(2b_3 - b_2\right) \qquad \frac{\partial b_1}{\partial q} = 0$$
$$\frac{\partial b_2}{\partial p} = 0 \qquad \frac{\partial b_2}{\partial q} = 0$$
$$\frac{\partial b_3}{\partial p} = 0 \qquad \frac{\partial b_3}{\partial q} = 0$$
$$\frac{\partial b_4}{\partial p} = 0 \qquad \frac{\partial b_4}{\partial q} = 0$$
$$\frac{\partial b_5}{\partial p} = 0 \qquad \frac{\partial b_5}{\partial q} = 0.$$

This result is enough to state that $H^2_3(P_2) \cong \mathbb{R}^5$. Indeed, a solution of the cocycle conditions that is not a solution of coboundary conditions depend on five constants, since we have

$$b_1 = \frac{1}{27}\left(2c_3 - c_2\right)p + c_1 \tag{3.50}$$
$$b_2 = c_2 \qquad b_3 = c_3 \tag{3.51}$$
$$b_4 = c_4 \qquad b_5 = c_5 \tag{3.52}$$

with $(c_1, \dots, c_5)$ constants.

When solving the cocycle conditions for (3.3), one immediately realizes that it is much more complicated than for the previous two cases. Indeed, it is quite different to deal with a nonconstant structure as P_{LP}, since it is for instance necessary to compute, for the PVA-Jacobi identity, terms as $\{p_{i\boldsymbol{\lambda}}\{p_{j\boldsymbol{\mu}}p_k\}_{LP}\}_{[2]}$ that vanish for a constant structure such as P_1 and P_2. The complexity of the equations and their number (they are more than twice the ones we had to simplify in the previous case) do not allow us to directly feed the system to the computer and get the Janet basis. We have to carefully inspect all the equations to look for the ones that can be algebraically solved for some of the coefficients, solve them and substitute back in the system; this has been done using Mathematica, despite it still requires a direct intervention and, however, took weeks on several running computers.

We reduced the cocycle condition to algebraic and differential identities for 128 coefficients of the deformed bracket and to a system of 1706 linear PDEs for 34 variables, namely $A^{abc}_{ij(S)}$, $B^{ab,cl}_{12}$, and $D^{2,111}_{22}$, $D^{1,222}_{22}$, $D^{1,111}_{22}$, $D^{2,222}_{12}$, $D^{2,111}_{12}$, $D^{1,221}_{12}$, $D^{1,111}_{12}$, $D^{2,222}_{11}$, $D^{2,111}_{11}$, and $D^{1,222}_{11}$. We compute the Janet basis of this system, which turns out to be constituted by 6 equations, together with the coboundary conditions, a set of 4 equations. We provide, as an example, the coboundary conditions in Appendix B.3.2.

Comparing the coboundary and the cocycle conditions, we notice that (B.1) and (B.2) – two of the four coboundary conditions – are implied by the cocycle conditions. By taking successive derivatives of the remaining two coboundary conditions, we observe the following

Coboundary	**Cocycle**
(B.3) $= 0$	(B.3) $= c_1 p^2 q^3 + c_2 p^3 q^2$
(B.4) $= 0$	(B.4) $= -\frac{33}{5} c_1 p^2 q^3 - c_2 p^3 q^2$

Since the solution of the cocycle conditions that are not solution of the coboundary ones depends on two arbitrary constants, the cohomology group $H^2_3(P_{LP})$ has dimension 2. As an illustration of the form of the cocycles, we choose two solutions for the system of 34 coefficients, meaning that the ones for which we do not provide an expression vanish. We have, respectively,

$$\begin{cases} A^{111}_{11} &= 1 \\ A^{112}_{12} &= 1 \\ A^{122}_{22} &= 1 \end{cases} \tag{3.53}$$

and

$$B^{22,12}_{12} = \frac{p}{q^2} \tag{3.54}$$

To get the full form of the two cocycles we need to use the identities for the remaining 128 coefficients of the skewsymmetric bracket, and then use (3.47) for the remaining 228. The full form for a basis of $H^2_3(P_{LP})$ is, hence,

$$P_{11} = -\frac{3q_x^3}{2q^3} + \frac{p_y q_x^2}{2q^3} - \frac{3pq_y q_x^2}{2q^4} + \frac{2q_{xx}q_x}{q^2} + \frac{pq_{xy}q_x}{q^3} + \lambda_1^3 + \frac{pq_{xx}q_y}{q^3} + \tag{3.55a}$$
$$+ \left(\frac{3q_x^2}{2q^2} + \frac{pq_y q_x}{q^3} - \frac{p_y q_x}{2q^2} - \frac{q_{xx}}{q} - \frac{pq_{xy}}{2q^2}\right)\lambda_1 + \left(\frac{p_x q_x}{2q^2} - \frac{pq_{xx}}{2q^2}\right)\lambda_2 - \frac{q_{xxx}}{2q} - \frac{p_y q_{xx}}{2q^2} - \frac{pq_{xxy}}{2q^2}$$

$$P_{12} = -\frac{p_x p_y^2}{2p^2 q} - \frac{q_y p_y^2}{2q^3} + \frac{2q_x q_y p_y}{q^3} + \frac{p_{xy}p_y}{2pq} - \frac{q_{xy}p_y}{q^2} + \frac{p_{yy}p_y}{2q^2} - \frac{p_x q_y p_y}{2pq^2} + \tag{3.55b}$$
$$+ \frac{3q_x q_{xy}}{q^2} + \frac{q_{xx}q_y}{q^2} + \left(\frac{3q_x q_y}{2q^2} - \frac{q_{xy}}{q}\right)\lambda_1 + \lambda_1^2\lambda_2 +$$
$$+ \left(\frac{p_y^2}{4q^2} + \frac{p_x p_y}{2pq} - \frac{q_x p_y}{q^2} + \frac{5q_x^2}{4q^2} - \frac{q_{xx}}{q}\right)\lambda_2 - \frac{q_{xxy}}{q} + \frac{p_x p_{yy}}{2pq} - \frac{p_{yy}q_x}{q^2} - \frac{3q_x^2 q_y}{q^3}$$

$$P_{21} = -\frac{5q_y q_x^2}{2q^3} + \frac{2q_{xy}q_x}{q^2} + \frac{3q_{xx}q_y}{2q^2} + \left(\frac{3q_x q_y}{2q^2} - \frac{q_{xy}}{q}\right)\lambda_1 + \lambda_1^2\lambda_2 + \tag{3.55c}$$
$$+ \left(\frac{p_y^2}{4q^2} + \frac{p_x p_y}{2pq} - \frac{q_x p_y}{q^2} + \frac{5q_x^2}{4q^2} - \frac{q_{xx}}{q}\right)\lambda_2 - \frac{q_{xxy}}{q}$$

$$P_{22} = -\frac{p_x p_y^2}{2p^3} + \frac{q_y^2 p_y}{q^3} + \frac{p_{xy}p_y}{2p^2} + \frac{q_{xy}p_y}{2pq} - \frac{p_x q_y p_y}{2p^2 q} - \frac{p_{yy}p_y}{2q^2} - \frac{q_x q_y p_y}{2pq^2} + \frac{p_x p_{yy}}{2p^2} + \tag{3.55d}$$
$$+ \frac{p_{yy}q_x}{q^2} + \frac{2q_{xy}q_y}{q^2} + \frac{p_{xy}q_y}{2pq} + \left(-\frac{p_{xy}}{2p} + \frac{p_x p_y}{2p^2} + \frac{2q_x q_y}{q^2} + \frac{p_{yy}}{q} - \frac{2q_{xy}}{q} - \frac{p_y q_y}{q^2}\right)\lambda_2 +$$
$$+ \lambda_1\left(\lambda_2^2 + \frac{p_y q_y}{pq} - \frac{p_{yy}}{2p}\right) - \frac{p_{xyy}}{2p} - \frac{q_{xyy}}{q} + \frac{p_{yyy}}{2q} - \frac{p_{yy}q_y}{q^2} - \frac{2q_x q_y^2}{q^3}$$

and

$$\begin{aligned}
P_{11} = {} & \frac{10p_y p_x^2}{p^3} - \frac{5q_x p_x^2}{p^3} + \frac{2q_y^2 p_x}{q^3} + \frac{5p_y q_x p_x}{2p^2 q} + \frac{5q_{xx} p_x}{p^2} + \frac{5q_x q_y p_x}{2pq^2} - \frac{10p_{xy} p_x}{p^2} + \\
& + \frac{5p_{yy} p_x}{pq} - \frac{5q_{xy} p_x}{2pq} - \frac{5p_y^2 p_x}{p^2 q} - \frac{p_{yy} p_x}{q^2} - \frac{5p_y q_y p_x}{pq^2} + \frac{5p_{xxy}}{p} + \frac{5p_{xx} q_x}{2p^2} + \\
& + \frac{13q_x q_{xy}}{2q^2} + \frac{2pq_{xyy}}{q^2} + \frac{4p_y q_x q_y}{q^3} + \frac{q_{xx} q_y}{2q^2} + \frac{3p_{xy} q_y}{q^2} + \frac{2pp_{yy} q_y}{q^3} + \\
& + \left(\frac{10p_{xy}}{p} + \frac{5p_x q_x}{p^2} + \frac{9q_x q_y}{2q^2} + \frac{p_y q_y}{q^2} - \frac{5q_{xx}}{p} - \frac{10p_x p_y}{p^2} - \frac{p_{yy}}{q} - \frac{9q_{xy}}{2q} \right) \lambda_1 + \\
& + \left(\frac{2p_y^2}{q^2} + \frac{10p_x p_y}{pq} - \frac{5q_x p_y}{q^2} + \frac{2q_x^2}{q^2} + \frac{7q_{xx}}{2q} + \frac{4pq_{xy}}{q^2} - \frac{7p_{xy}}{q} - \frac{5p_x q_x}{pq} - \right. \\
& \left. - \frac{2pp_{yy}}{q^2} - \frac{2p_x q_y}{q^2} \right) \lambda_2 - \frac{5q_{xxx}}{2p} - \frac{5p_{xx} p_y}{p^2} - \frac{4p_{xyy}}{q} - \frac{q_{xxy}}{2q} + \frac{5p_{xy} p_y}{pq} - \\
& - \frac{5p_{xy} q_x}{2pq} + \frac{p_y p_{yy}}{q^2} - \frac{pp_{yyy}}{q^2} - \frac{2p_{yy} q_x}{q^2} - \frac{2p_y^2 q_y}{q^3} - \frac{4pq_{xy} q_y}{q^3} - \frac{13q_x^2 q_y}{2q^3}
\end{aligned} \tag{3.56a}$$

$$P_{12} = \frac{pq_y \lambda_2^2}{q^2} + \left(-\frac{5p_y^2}{p^2} + \frac{5q_x p_y}{p^2} + \frac{5q_y p_y}{pq} - \frac{5q_x^2}{4p^2} - \frac{5q_x q_y}{2pq} - \frac{q_y^2}{q^2} \right) \lambda_1 \tag{3.56b}$$

$$\begin{aligned}
P_{21} = {} & -\frac{6pq_y^3}{q^4} + \frac{2q_x q_y^2}{q^3} + \frac{4p_y q_y^2}{q^3} + \frac{5q_x^2 q_y}{2pq^2} + \frac{5p_x q_x q_y}{2p^2 q} + \frac{5p_{xy} q_y}{pq} - \frac{5q_{xx} q_y}{2pq} - \\
& - \frac{5p_x p_y q_y}{p^2 q} - \frac{p\lambda_2^2 q_y}{q^2} - \frac{p_{yy} q_y}{q^2} - \frac{2q_{xy} q_y}{q^2} - \frac{5p_y q_x q_y}{pq^2} + \frac{6pp_{yy} q_y}{q^3} + \frac{10p_x p_y^2}{p^3} + \frac{5p_x q_x^2}{2p^3} + \\
& + \frac{5p_{xy} q_x}{p^2} + \frac{5p_y q_{xx}}{p^2} + \frac{5p_y q_{xy}}{pq} + \left(-\frac{5p_y^2}{p^2} + \frac{5q_x p_y}{p^2} + \frac{5q_y p_y}{pq} - \frac{5q_x^2}{4p^2} - \frac{5q_x q_y}{2pq} - \frac{q_y^2}{q^2} \right) \lambda_1 + \\
& + \left(\frac{4pq_y^2}{q^3} - \frac{2p_y q_y}{q^2} - \frac{2pp_{yy}}{q^2} \right) \lambda_2 - \frac{10p_{xy} p_y}{p^2} - \frac{5q_x q_{xx}}{2p^2} - \frac{10p_x p_y q_x}{p^3} - \frac{5q_x q_{xy}}{2pq} - \\
& - \frac{2p_y p_{yy}}{q^2} - \frac{pq_{yyy}}{q^2}
\end{aligned} \tag{3.56c}$$

$$P_{22} = 0. \tag{3.56d}$$

3.4 Computation of higher cohomology groups

Apart from the technical difficulties, the computation of higher cohomology groups for a PVA can be done following the same ideas we have presented for H^1 and H^2. We recall the correspondence between local polyvector fields and poly-λ-bracket we established in Section 2.2, and the general form for the analogue of the Poisson differential (2.24). Let us first describe in general the procedure we follow – we had to devise a more efficient computational tecnique – and then, as an example, we will compute H^3_2 for the three PVA structures.

3.4.1 The general procedure

In Section 2.2 we have described an element of W_k, that corresponds to a local k-vector, in two equivalent ways. We can regard it as a bracket depending on k arguments and $k-1$ formal indeterminates $\boldsymbol{\lambda}_1, \dots \boldsymbol{\lambda}_{k-1}$ – in the same way as a bivector is represented by a λ-bracket in the indeterminate $\boldsymbol{\lambda}$ –, satisfying a generalization of the skewsymmetry property for ordinary λ-brackets, according to Definition 2.4. Alternatively, we can consider it as a *totally* skewsymmetric linear mapping from k differential polynomials $f_1, \dots, f_k$ to an $\mathcal{A}$-valued polynomial in $\boldsymbol{k}$ formal indeterminates $\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k$, provided that we identify $\boldsymbol{\lambda}_k$ with $-\boldsymbol{\lambda}_1 - \cdots - \boldsymbol{\lambda}_{k-1} - \boldsymbol{\partial}$ acting on the left. In this second formalism, the sesquilinearity property assumes the simple form

$$X_{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_k}(f_1, \dots, \partial_\alpha f_i, \dots, f_k) = -\lambda_{i,\alpha} X_{\boldsymbol{\lambda}_1,\dots,\boldsymbol{\lambda}_k}(f_1, \dots, f_k) \tag{3.57}$$

for all $i = 1, \dots, k$.

Our general strategy is to write the generic form for X a k-λ-bracket, for which we will impose the cocycle condition $\mathrm{d}_P X = 0$. The condition is a system of algebraic and differential equations; solving the former ones allows us to reduce the size of the linear system of PDEs and to compute its Janet basis. Then, given Y a generic $(k-1)$-λ-bracket, we compute its image by the Poisson differential

$$\tilde{X} = \mathrm{d}_P Y. \tag{3.58}$$

Among the equations (3.58) for the coefficients of $\tilde{X}$, we select the ones for the unknown of the cocycle condition and regard them as inhomogeneous linear PDEs for the unknown generic coefficients of Y. We identify the coefficients of $\tilde{X}$ with the ones of a coboundary in W_k if and only if they satisfy the compatibility conditions for the solvability of (the reduced version of) (3.58). For the sake of brevity, we call them *coboundary conditions*. A comparison of the cocycle conditions with the coboundary ones, then, allows to characterize the cohomology group H^k, by looking for solutions of the former that are not solutions of the latter.

An important remark is that the “identification” of $\boldsymbol{\lambda}_k$ with $-\boldsymbol{\lambda}_1 - \cdots - \boldsymbol{\lambda}_{k-1} - \boldsymbol{\partial}$ is actually a projection from the space $W_k := \mathcal{A} \otimes \mathbb{R}[\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_k]$ to the space $\tilde{W}_k := \mathcal{A} \otimes \mathbb{R}[\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_{k-1}]$, and in particular it is not injective [17]. Hence, there are different choices of the coefficients for X or Y that give the same element in W_k or W_{k-1}. It is convenient, when giving the generic form of the poly-λ-brackets in W_k , adopting a choice of coefficients which depends on the same number of parameters as the bracket in $\tilde{W}_k$. This choice is k-dependent, so we will discuss it for the particular cases needed to compute H^3.

3.4.2 Examples for H^3

We compute H^3_2 for the three PVA structures (3.1), (3.2), and (3.3). A generic element in W_3 has the form

$$\begin{aligned} X_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}(p_i,p_j,p_k) = &A^{ab}_{ijk}\lambda_a\lambda_b + A^{ab}_{jki}\mu_a\mu_b + A^{ab}_{kij}\nu_a\nu_b + \\ &+ B^{a,b}_{ijk}\lambda_a\mu_b + B^{a,b}_{jki}\mu_a\nu_b + B^{a,b}_{kij}\nu_a\lambda_b + \\ &+ C^{a,bl}_{ijk}\lambda_a\partial_b p_l + C^{a,bl}_{jki}\mu_a\partial_b p_l + C^{a,bl}_{kij}\nu_a\partial_b p_l \end{aligned} \tag{3.59}$$

where the coefficients A's, B's, and C's satisfy the following identities

$$A^{ab}_{ijk} = A^{ba}_{ijk} \qquad A^{ab}_{ijk} = -A^{ab}_{ikj} \tag{3.60a}$$

$$B^{a,b}_{ijk} = -B^{b,a}_{ijk} \qquad B^{a,b}_{ijk} = B^{a,b}_{ikj} = B^{a,b}_{jki} \tag{3.60b}$$

$$C^{a,bl}_{ijk} = -C^{a,bl}_{ikj}. \tag{3.60c}$$

Hence, the number of free parameters for a 3-$\boldsymbol{\lambda}$-bracket is 26; we compute the set of equations (2.24) with respect to the three PVA $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_1$, $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_2$, and $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_{LP}$; after looking for the algebraic relations among the coefficients we are left with respectively 10, 11, and 11 coefficients. The cocycle conditions for the three cases, however, do not impose constraints for all of them. The picture for the three structures is summarized in the following table

	Coefficients	**Unknowns for cocycle condition**
P_1	A^{11}_{112}, A^{11}_{212}, A^{12}_{112}, A^{12}_{212}, A^{22}_{112}, A^{22}_{212}, $B^{1,2}_{111}$,$B^{1,2}_{112}$, $B^{1,2}_{122}$, $B^{1,2}_{222}$	A^{11}_{212}, A^{22}_{112}, $B^{1,2}_{111}$, $B^{1,2}_{222}$
P_2	A^{11}_{112}, A^{11}_{212}, A^{12}_{112}, A^{12}_{212}, A^{22}_{112}, A^{22}_{212}, $B^{1,2}_{111}$,$B^{1,2}_{112}$, $B^{1,2}_{122}$, $B^{1,2}_{222}$, $C^{1,11}_{112}$	All but A^{11}_{112} and A^{12}_{212}
P_{LP}	A^{11}_{112}, A^{11}_{212}, A^{12}_{112}, A^{12}_{212}, A^{22}_{112}, A^{22}_{212}, $B^{1,2}_{111}$,$B^{1,2}_{112}$, $B^{1,2}_{122}$, $B^{1,2}_{222}$, $C^{1,11}_{112}$	All

The cocycle condition for P_1 is

$$\begin{cases} \frac{\partial A^{11}_{212}}{\partial q} + \frac{\partial B^{1,2}_{222}}{\partial p} &= 0 \\ \frac{\partial A^{22}_{112}}{\partial p} + \frac{\partial B^{1,2}_{111}}{\partial q} &= 0. \end{cases} \tag{3.61}$$

The cocycle condition for P_2 is

$$\begin{cases} J1 := \frac{\partial}{\partial q}A^{22}_{112} + \frac{\partial}{\partial q}B^{1,2}_{111} &= 0 \\ J2 := \frac{\partial}{\partial q}A^{12}_{112} - \frac{1}{2}\frac{\partial}{\partial q}A^{22}_{212} + \frac{\partial}{\partial p}B^{1,2}_{111} - \frac{3}{2}\frac{\partial}{\partial q}B^{1,2}_{112} &= 0 \\ J3 := -\frac{\partial}{\partial q}A^{12}_{112} + \frac{\partial}{\partial p}A^{22}_{112} + \frac{1}{2}\frac{\partial}{\partial q}A^{22}_{212} + \frac{3}{2}\frac{\partial}{\partial q}B^{1,2}_{112} &= 0 \\ J4 := \frac{\partial^2}{\partial q^2}A^{11}_{212} + \frac{\partial^2}{\partial p^2}A^{12}_{112} - \frac{1}{2}\frac{\partial^2}{\partial p^2}A^{22}_{212} + \frac{3}{2}\frac{\partial^2}{\partial p^2}B^{1,2}_{112} - \\ \quad -3\frac{\partial^2}{\partial q \partial p}B^{1,2}_{122} + \frac{\partial^2}{\partial q^2}B^{1,2}_{222} - \frac{\partial}{\partial q}C^{1,11}_{112} &= 0 \end{cases} \tag{3.62}$$

where we have denoted $J1, \dots, J4$ the four expressions for further reference.

The cocycle condition for P_{LP} is left to Appendix C since it is much longer.

Now let us compute the coboundary conditions. In principle, it is sufficient to compute (2.27), where $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_Q$ is the λ-bracket defined by Y. However, let us denote $P_{\boldsymbol{\lambda},\boldsymbol{\mu}}(p_i, p_j)$ a preimage of $\{p_i{}_{\boldsymbol{\lambda}} p_j\}_P = g^{\alpha}_{ij}\lambda_\alpha + b^{al}_{ij}\partial_a p_l$ defined as follows:

$$P_{\boldsymbol{\lambda},\boldsymbol{\mu}}(p_i, p_j) = \frac{1}{2} g^{\alpha}_{ij}\left(\lambda_\alpha - \mu_\alpha\right) + \frac{1}{2}\left(b^{al}_{ij} - b^{al}_{ji}\right)\partial_a p_l. \tag{3.63}$$

The coboundary $\tilde{X} = \mathrm{d}_P Y$ can be computed using the formula

$$\begin{aligned}(\mathrm{d}_P Y)_{\boldsymbol{\lambda},\boldsymbol{\mu},\boldsymbol{\nu}}(p_i, p_j, p_k) = {} & \{p_i{}_{\boldsymbol{\lambda}} Y_{\boldsymbol{\mu},\boldsymbol{\nu}}(p_j, p_k)\}_P - \{p_j{}_{\boldsymbol{\mu}} Y_{\boldsymbol{\lambda},\boldsymbol{\nu}}(p_i, p_k)\}_P + \\ & + \{p_k{}_{\boldsymbol{\nu}} Y_{\boldsymbol{\lambda},\boldsymbol{\mu}}(p_i, p_j)\}_P + \{p_i{}_{\boldsymbol{\lambda}} P_{\boldsymbol{\mu},\boldsymbol{\nu}}(p_j, p_k)\}_Y - \\ & - \{p_j{}_{\boldsymbol{\mu}} P_{\boldsymbol{\lambda},\boldsymbol{\nu}}(p_i, p_k)\}_Y + \{p_k{}_{\boldsymbol{\nu}} P_{\boldsymbol{\lambda},\boldsymbol{\mu}}(p_i, p_j)\}_Y,\end{aligned} \tag{3.64}$$

which is equivalent both to (2.27) and to (2.24) for $k = 2$ but it runs much faster. We identify the coefficients of $\tilde{X}$ corresponding to the unknown for the cocycle condition and use them to compute the coboundary condition as already explained several times.

The cocycle and the coboundary conditions for P_1 and P_{LP} coincide, so that we can conclude that $H^3_2(P_1) = H^3_2(P_{LP}) = 0$. On the other hand, the coboundary condition for P_2 is constituted by the three equations

$$\begin{cases} K1 := B^{1,2}_{111} + A^{22}_{112} & = 0 \\ K2 := \frac{\partial}{\partial q} A^{12}_{112} - \frac{1}{2}\frac{\partial}{\partial q} A^{22}_{212} + \frac{\partial}{\partial p} B^{1,2}_{111} - \frac{3}{2}\frac{\partial}{\partial q} B^{1,2}_{112} & = 0 \\ K3 := \frac{\partial^2}{\partial q^2} A^{11}_{212} + \frac{\partial^2}{\partial p^2} A^{12}_{112} - \frac{1}{2}\frac{\partial^2}{\partial p^2} A^{22}_{212} + \frac{3}{2}\frac{\partial^2}{\partial p^2} B^{1,2}_{112} - & \\ \quad -3\frac{\partial^2}{\partial q \partial p} B^{1,2}_{122} + \frac{\partial^2}{\partial q^2} B^{1,2}_{222} - \frac{\partial}{\partial q} C^{1,11}_{112} & = 0 \end{cases} \tag{3.65}$$

The package `Janet` allows us to compute the relations between the two sets of equation (3.62) and (3.65). As expected by the cochain structure, the coboundary conditions $K = 0$ imply the cocycle ones $J = 0$, and in particular

$$J1 = \frac{\partial K1}{\partial q} \qquad J2 = K2$$

$$J3 = \frac{\partial K1}{\partial p} - K2 \qquad J4 = K3.$$

Conversely, $K1 = 0$ is not a consequence of the system $J = 0$ and hence it is a condition satisfied by coboundaries and not by cocycles. However, it is straightforward to conclude that $K1 = c_1$ a constant is the only allowed form of the expression that is a solution J, and hence that holds true for all the cocycles. Indeed, $\partial_q K1 = J1 = 0$ and $\partial_p K1 = J2 + J3 = 0$. We can finally conclude that $H^3_2(P_2) \cong \mathbb{R}$.

As a final remark for this chapter, we recall that in Paragraph 3.3.3 we have mentioned a redesign of the code in `MasterPVA` that makes the computation faster; despite it has not been implemented in the package yet, the main idea is to trade the expression for a generic λ-bracket such as (3.46), that belongs to $\tilde{W}_2$, with its preimage in W_2. The idea is to introduce an element of W_2 that is mapped by $\boldsymbol{\mu} \mapsto -\boldsymbol{\lambda} - \boldsymbol{\partial}$ to a λ-bracket, generalizing the formula (3.63) to a higher degree homogeneous expression. The equation $\mathrm{d}_P Y = 0$, with $\mathrm{d}_P Y$ defined in (3.64) contains differential polynomials only in the second entry of the two λ-brackets $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_P$ and $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_Y$, that according to the master formula are never acted on by total derivatives. This fact dramatically speeds up the computations; the projection from the space W to $\tilde{W}$ is done only once as the final step.

4

Poisson cohomology of scalar multidimensional brackets

In Chapter 3 we have explicitly computed a few componentes of the zeroth, first, and second cohomology group for the $D = N = 2$ Poisson brackets of hydrodynamic type. Using the formalism of Poisson Vertex Algebra we are able, in principle, to explicitly compute any fixed degree component of any cohomology group, but we cannot make statements for the full Poisson cohomology apart from the negative ones (i.e., whether it is not trivial).

Motivated by the spectral sequence techniques exploited in [8, 10, 9] we have studied the $N = 1$ case for the Poisson brackets of hydrodynamic type, or Dubrovin–Novikov (DN) brackets. The content of this chapter is a joint work with G. Carlet and S. Shadrin in an almost final version [7].

The general form of a multidimensional DN type Poisson bracket is [43]

$$\{u(x), u(y)\} = g(u(x))c^\alpha \frac{\partial}{\partial x^\alpha}\delta(x-y) + \frac{1}{2}g'(u(x))c^\alpha \frac{\partial u}{\partial x^\alpha}(x)\delta(x-y) \quad (4.1)$$

where $g(u)$ is a non-vanishing function and c^α are constants, with $\alpha = 1, \dots, D$.

Our main result is the computation of the full Poisson cohomology of the Poisson bracket (4.1), in a quite implicit form, see Theorem 4.5. As a consequence of this result we find the explicit description of the cohomology groups of low degree, which are relevant for the deformation theory, see Corollary 4.7. For doing so, we adopt the so-called θ formalism we discuss in Section 4.1.

Finally, we explicitly compute some of the cohomology groups using PVAs theory, showing the agreement with the expected results.

4.1 The θ calculus formalism

In this Section we introduce the θ formalism, useful to describe the space of multivector fields, the starting building block of the Poisson–Lichnerowicz cohomology. We will define the Schouten bracket of multivector fields in this language, as anticipated in Paragraph 1.3.2, and discuss the Poisson differential.

Before introducing the new formalism, we state a preliminary Lemma that is a direct generalization of Lemma 2.1.7 of [40].

Lemma 4.1. *Let $P^i \in \mathcal{A}$. If*

$$\int^D \sum_i P^i \frac{\delta I}{\delta u^i} \mathrm{d}^D x = 0 \tag{4.2}$$

for all $I \in \mathcal{A}$, then $P^i = \sum_{\alpha=1}^D c_\alpha u^i_{\xi_\alpha}$ for some $c_\alpha \in \mathbb{R}$.

4.1.1 Local multivectors

Let $\hat{\mathcal{A}}$ be the algebra of formal power series in the commutative variables $\{u^i_S\}_{i=1}^N$, $|S| > 0$ and anticommutative variables $\{\theta_i^S\}$, $|S| \geq 0$ with coefficients given by smooth functions of the variables $\{u^i\} \in U$, i.e.,

$$\hat{\mathcal{A}} := C^\infty(U)[[\{u^i_S, |S| > 0\} \cup \{\theta_i^S, |S| \geq 0\}]]. \tag{4.3}$$

The standard gradation deg and the super gradation $\deg_\theta$ of $\hat{\mathcal{A}}$ are defined by setting

$$\deg u^i_S = \deg \theta_i^S = |S|, \qquad \deg_\theta u^i_S = 0, \qquad \deg_\theta \theta_i^S = 1. \tag{4.4}$$

We denote $\hat{\mathcal{A}}_d$, resp. $\hat{\mathcal{A}}^p$, the homogeneous components of standard degree d, resp. super degree p, while $\hat{\mathcal{A}}^p_d := \hat{\mathcal{A}}_d \cap \hat{\mathcal{A}}^p$. Clearly $\hat{\mathcal{A}}^0 = \mathcal{A}$.

The commuting derivations ∂_α defined in Paragraph 1.2.1 for $\alpha = 1, \dots, D$ are extended to $\hat{\mathcal{A}}$ by

$$\partial_\alpha = \sum_{i,S} \left(u^i_{S+\xi_\alpha} \frac{\partial}{\partial u^i_S} + \theta_i^{S+\xi_\alpha} \frac{\partial}{\partial \theta_i^S} \right). \tag{4.5}$$

Note that the kernel of ∂_α on $\hat{\mathcal{A}}$ is $\mathbb{R}$.

We denote by $\hat{\mathcal{F}}$ the quotient of $\hat{\mathcal{A}}$ by the subspace $\partial_1 \hat{\mathcal{A}} + \cdots + \partial_D \hat{\mathcal{A}}$, and by a multiple integral $\int^D \cdot \, d^D x$ the projection map from $\hat{\mathcal{A}}$ to $\hat{\mathcal{F}}$. Since the derivations ∂_α are homogeneous, i.e., $\deg \partial_\alpha = 1$ and $\deg_\theta \partial_\alpha = 0$, $\hat{\mathcal{F}}$ inherits both gradations of $\hat{\mathcal{A}}$.

Equations (1.13) and Proposition 1.2 hold and, similarly,

$$\left[\frac{\partial}{\partial \theta_i^S}, \partial_\beta \right] = \frac{\partial}{\partial \theta_i^{S-\xi_\beta}}, \text{ if } S = \sum_\alpha s_\alpha \xi_i \text{ with } s_\beta > 0, \tag{4.6}$$

and is equal to zero otherwise. It follows that the variational derivative

$$\frac{\delta}{\delta \theta_i} = \sum_S (-1)^{|S|} \boldsymbol{\partial}^S \frac{\partial}{\partial \theta_i^S} \tag{4.7}$$

satisfies

$$\frac{\delta}{\delta \theta_i} \partial_\alpha = 0. \tag{4.8}$$

Hence both variational derivatives (1.24) and (4.7) define maps from $\hat{\mathcal{F}}$ to $\hat{\mathcal{A}}$.

Proposition 4.1. The space of local multi-vectors Λ^p is isomorphic to $\hat{\mathcal{F}}^p$ for $p \neq 1$. Moreover

$$\Lambda^1 \cong \frac{\hat{\mathcal{F}}^1}{\oplus_\alpha \mathbb{R} \int u^i_{\xi_\alpha} \theta_i} \cong \frac{\mathrm{Der}'(\mathcal{A})}{\oplus_\alpha \mathbb{R} \partial_\alpha}, \tag{4.9}$$

where $\mathrm{Der}'(\mathcal{A})$ denotes the space of derivations of $\mathcal{A}$ that commute with ∂_α, for $\alpha = 1, \dots, D$, and $\mathrm{Der}'(\mathcal{A}) \cong \hat{\mathcal{F}}^1$.

Proof. The proof given in [40] can be easily adapted to the present case. We give it here for completeness.

For $p = 0$, the isomorphism is trivial, since $\hat{\mathcal{F}}^0 = \mathcal{F} = \Lambda^0$.

Let $p \geq 1$. Given $P \in \hat{\mathcal{F}}^p$, and arbitrary $I_1, \dots, I_p \in \mathcal{F}$, let

$$\iota(P)(I_1, \dots, I_p) = \frac{\partial}{\partial \theta^{S_p}_{i_p}} \cdots \frac{\partial}{\partial \theta^{S_1}_{i_1}} P \cdot \boldsymbol{\partial}^{S_1} \left(\frac{\delta I_1}{\delta u^{i_1}} \right) \cdots \boldsymbol{\partial}^{S_p} \left(\frac{\delta I_p}{\delta u^{i_p}} \right). \tag{4.10}$$

Clearly $\iota(P)$ is an p-alternating map from $\mathcal{F}$ to $\mathcal{A}$, and it satisfies

$$\iota(\partial_\alpha P)(I_1, \dots, I_p) = \partial_\alpha(\iota(P)(I_1, \dots, I_p)). \tag{4.11}$$

The desired map $\tilde{\iota}$ from $\hat{\mathcal{F}}^p$ to Λ^p is then defined by

$$\tilde{\iota}\left(\int^D P \mathrm{d}^D x \right) = \int^D \iota(P) \mathrm{d}^D x. \tag{4.12}$$

A local p-vector (1.29) is the image through ι of

$$P = \frac{1}{p!} P^{i_1, \dots, i_p}_{S_1, \dots, S_p} \theta^{S_1}_{i_1} \cdots \theta^{S_p}_{i_p}, \tag{4.13}$$

hence $\tilde{\iota}$ is surjective.

Consider the case $p = 1$. An element P of $\hat{\mathcal{F}}^1$ can be written uniquely as

$$P = \int^D P^i \theta_i \, d^D x \tag{4.14}$$

for $P^i \in \mathcal{A}$. The map that sends P to

$$\hat{P} = \sum_S \boldsymbol{\partial}^S P^i \frac{\partial}{\partial u^i_S} \tag{4.15}$$

defines an isomorphism $\hat{\mathcal{F}}^1 \cong \mathrm{Der}'(\mathcal{A})$. Notice that

$$\tilde{\iota}(P)(I) = \int^D \hat{P}(I) \mathrm{d}^D x. \tag{4.16}$$

Clearly, for each α, the derivation associated with $P = \int^D \sum_i u^i_{\xi_\alpha} \theta_i \mathrm{d}^D x$ corresponds to $\hat{P} = \partial_\alpha$, therefore is in the kernel of $\tilde{\iota}$. From Lemma 4.1 it follows that the kernel of $\tilde{\iota}$ is indeed generated by these elements.

Let us consider the case $p \geq 2$. An element P of $\hat{\mathcal{F}}^p$ can be uniquely written as $\frac{1}{p}\theta_i \frac{\delta P}{\delta \theta_i}$, recalling the so–called *normalization operator* introduced by A. Barakat [4]. Let us assume $\tilde{\iota}(P)(I_1, \dots, I_p) = 0 \ \forall I_1, \dots, I_p$; we need to prove that $P = \partial_\alpha Q_\alpha$, hence it is 0 in $\hat{\mathcal{F}}$. We will denote the equivalence in $\hat{\mathcal{F}}^p$ of two elements $A, B \in \hat{\mathcal{F}}^a$ as $A \sim B$. From definition (4.10), we have that

$$\begin{aligned} \frac{1}{p}\iota\left(\theta_i \frac{\delta P}{\delta \theta_i}\right)(I_1, \dots, I_p) &= \frac{1}{p}\sum_{j=1}^{p}(-1)^{p+1}\iota\left(\frac{\delta P}{\delta \theta_i}\right)(I_1, \overset{j}{\hat{\dots}}, I_p)\frac{\delta I_j}{\delta u^i} \\ &\sim \iota\left(\frac{\delta P}{\delta \theta_i}\right)(I_2, \dots, I_p)\frac{\delta I_1}{\delta u^i} \sim 0, \end{aligned} \tag{4.17}$$

where the two last equalities follow from the skewsymmetry of P and the hypothesis. Lemma 4.1 implies that $\iota(\delta P/\delta \theta_i)(I_2, \dots, I_p) = \sum_\alpha c_\alpha(I_2, \dots, I_p) u^i_{\xi_\alpha}$, with $c_\alpha(I_1, \dots, I_p)$ constants. From Lemma 2.1.4 of [40] we conclude that $\iota(\delta P)$ is zero, since we can see each of the c_α's as differential operators acting on $\mathcal{A}^{\otimes(p-1)}$ whose image is in $\mathbb{R}$. By choosing suitable differential polynomials $I_2, \dots, I_P$, it is possible to show that δP, that in principle belongs to $\hat{\mathcal{F}}^{p-1}$, does not contain any monomial in $\theta^{S_2}_{i_2} \cdots \theta^{S_p}_{i_p}$, namely it is 0. The thesis follows from (4.8).

□

4.1.2 The Schouten-Nijenhuis bracket

The Schouten-Nijenhuis bracket

$$[\cdot, \cdot] : \hat{\mathcal{F}}^p \times \hat{\mathcal{F}}^q \to \hat{\mathcal{F}}^{p+q-1} \tag{4.18}$$

is defined as

$$[P, Q] = \int^D \left(\frac{\delta P}{\delta \theta_i}\frac{\delta Q}{\delta u^i} + (-1)^p \frac{\delta P}{\delta u^i}\frac{\delta Q}{\delta \theta_i}\right) \mathrm{d}^D x. \tag{4.19}$$

It is a bilinear map that satisfies the graded symmetry

$$[P, Q] = (-1)^{pq}[Q, P] \tag{4.20}$$

and the graded Jacobi identity

$$(-1)^{pr}[[P, Q], R] + (-1)^{qp}[[Q, R], P] + (-1)^{rq}[[R, P], Q] = 0 \tag{4.21}$$

for arbitrary $P \in \hat{\mathcal{F}}^p$, $Q \in \hat{\mathcal{F}}^q$ and $r \in \hat{\mathcal{F}}^r$.

A bivector $P \in \hat{\mathcal{F}}^2$ is a Poisson structure when $[P, P] = 0$. In such case $\mathrm{d}_P := ad_P = [P, \cdot]$ squares to zero, as a consequence of the graded Jacobi identity, and the cohomology of the complex $(\hat{\mathcal{F}}, \mathrm{d}_P)$ is called Poisson–Lichnerowicz cohomology of P.

4.1.3 The differential on $\hat{\mathcal{A}}$

Given an element $P \in \hat{\mathcal{F}}^p$ we define the following differential operator on $\hat{\mathcal{A}}$

$$D_P = \sum_S \left(\boldsymbol{\partial}^S \left(\frac{\delta P}{\delta \theta_i} \right) \frac{\partial}{\partial u^i_S} + (-1)^p \boldsymbol{\partial}^S \left(\frac{\delta P}{\delta u^i} \right) \frac{\partial}{\partial \theta_i^S} \right). \tag{4.22}$$

Since $[D_P, \partial_\alpha] = 0$ for all $i = 1, \ldots, D$, the operator D_P descends to an operator on $\hat{\mathcal{F}}$ which is given by the adjoint action $ad_P = [P, \cdot]$ of P on $\hat{\mathcal{F}}$ via the Schouten-Nijenhuis bracket, i.e.,

$$ad_P \left(\int^D Q \mathrm{d}^D x \right) = \int^D D_P(Q) \mathrm{d}^D x, \tag{4.23}$$

for $Q \in \hat{\mathcal{A}}$. This can be easily checked by integration by parts.

Lemma 4.2. *If $P \in \hat{\mathcal{F}}^2$ is such that $[P, P] = 0$, then $D_P^2 = 0$.*

Remark 4.3. The Lemma simply follows from the identity

$$D_{[P,Q]} = (-1)^{p+1} [D_P, D_Q] \tag{4.24}$$

which holds for $P \in \hat{\mathcal{F}}^p$ and $Q \in \hat{\mathcal{F}}$, where the brackets on the righthand-side represent the graded commutator that induces a graded Lie algebra structure on the space of graded derivations to which D_P belongs, see [41]. We will not prove the identity (4.24) here, since in our case the fact that $D_P^2 = 0$ simply follows from a trivial computation, see next Section.

4.1.4 Partial integrations

We have previously defined the projection map from the space of the local densities $\hat{\mathcal{A}}$ to the space of local p-vector fields $\hat{\mathcal{F}}$ as the quotient by the images of the total differentials $\{\partial_\alpha\}$. For our construction, we need to define the notion of *partial integration* with respect of some independent variable x^β.

A crucial role is played by the following Lemma.

Lemma 4.4. *Let us denote*

$$\hat{\mathcal{F}}_\alpha = \frac{\hat{\mathcal{A}}}{\partial_1 \hat{\mathcal{A}} + \cdots + \partial_\alpha \hat{\mathcal{A}}} \tag{4.25}$$

and $\int \mathrm{d}x^\alpha \colon W \to W/\partial_\alpha W$ for each $\alpha = 1, \ldots, d$ and any vector space W.

The sequences

$$\begin{array}{ccccccccc}
0 & \to & \hat{\mathcal{A}}/\mathbb{R} & \xrightarrow{\partial_1} & \hat{\mathcal{A}} & \xrightarrow{\int \mathrm{d}x^1} & \hat{\mathcal{F}}_1 & \to & 0 \\
0 & \to & \hat{\mathcal{F}}_1/\mathbb{R} & \xrightarrow{\partial_2} & \hat{\mathcal{F}}_1 & \xrightarrow{\int \mathrm{d}x^2} & \hat{\mathcal{F}}_2 & \to & 0 \\
0 & \to & \hat{\mathcal{F}}_2/\mathbb{R} & \xrightarrow{\partial_3} & \hat{\mathcal{F}}_2 & \xrightarrow{\int \mathrm{d}x^3} & \hat{\mathcal{F}}_3 & \to & 0 \\
 & & \vdots & & \vdots & & \vdots & & \\
0 & \to & \hat{\mathcal{F}}_{D-1}/\mathbb{R} & \xrightarrow{\partial_D} & \hat{\mathcal{F}}_{D-1} & \xrightarrow{\int \mathrm{d}x^D} & \hat{\mathcal{F}}_D & \to & 0
\end{array} \tag{4.26}$$

are exact.

Proof. The exactness of the first line is obvious: by quotienting out the kernel of ∂_1, which is given by constants, we have injectivity on the left side, while $\int \mathrm{d}x^1$ just denotes the projection to the quotient $\hat{\mathcal{F}}_1$, which is indeed surjective.

Let us consider the second line. The derivation ∂_2 on $\hat{\mathcal{A}}$ commutes with ∂_1, hence defines a map on $\hat{\mathcal{F}}_1$ which we denote with the same symbol. Notice that

$$\frac{\hat{\mathcal{F}}_1}{\partial_1 \hat{\mathcal{F}}_1} = \frac{\hat{\mathcal{A}}}{\partial_1 \hat{\mathcal{A}} + \partial_2 \hat{\mathcal{A}}} = \hat{\mathcal{F}}_2, \tag{4.27}$$

therefore surjectivity is guaranteed.

The same argument works for the rest of the lines, since it is easy to check that

$$\frac{\hat{\mathcal{F}}_\alpha}{\partial_{\alpha+1} \hat{\mathcal{F}}_\alpha} = \frac{\hat{\mathcal{A}}}{\partial_1 \hat{\mathcal{A}} + \cdots + \partial_{\alpha+1} \hat{\mathcal{A}}} = \hat{\mathcal{F}}_{\alpha+1}. \tag{4.28}$$

It remains to be proved that the map induced by ∂_β on $\hat{\mathcal{F}}_{\beta-1}$ has kernel given by $\mathbb{R}$, for $\beta = 2, \dots, D$. In order to do this we reformulate this property as the vanishing of the cohomology of some auxiliary complex, and then we construct an explicit homotopy contraction for that auxiliary complex that implies the vanishing of its cohomology.

The map induced by ∂_β on $\hat{\mathcal{F}}_{\beta-1}$ has only constants in the kernel if and only if for any function $f \in \hat{\mathcal{A}}$ the conditions $\partial_\beta f \in \partial_1 \hat{\mathcal{A}} + \cdots + \partial_{\beta-1} \hat{\mathcal{A}}$ and f is non-constant implies that $f \in \partial_1 \hat{\mathcal{A}} + \cdots + \partial_{\beta-1} \hat{\mathcal{A}}$. Consider the following complex:

$$0 \to \Omega^0 \xrightarrow{d_H} \Omega^1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \Omega^{\beta-1} \xrightarrow{d_H} \Omega^\beta, \tag{4.29}$$

where Ω^α, $0 \le \alpha \le \beta$, is the space of local differential α-forms with coefficients in $\hat{\mathcal{A}}/\mathbb{R}$, that is,

$$\Omega^\alpha := \bigoplus_{\substack{I \subset \{1,\dots,\beta\}, |I| = \beta \\ I = \{\beta_1 < \cdots < \beta_\alpha\}}} \hat{\mathcal{A}}/\mathbb{R} \cdot \mathrm{d}x^{\beta_1} \wedge \cdots \wedge \mathrm{d}x^{\beta_\alpha}, \tag{4.30}$$

and the differential d_H is equal to $\sum_{\alpha=1}^{\beta} \mathrm{d}x^\alpha \wedge \partial_\alpha$. In terms of this complex, the condition $\partial_\beta f = \partial_1 g_1 + \cdots + \partial_{\beta-1} g_{\beta-1}$ is the same as $d_H \omega = 0$, where $\omega \in \Omega^{\beta-1}$ is given by

$$\omega := \left[f \frac{\partial}{\partial(\mathrm{d}x^\beta)} - \sum_{\alpha=1}^{\beta-1} g_\alpha \frac{\partial}{\partial(\mathrm{d}x^\alpha)} \right] \mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^\beta. \tag{4.31}$$

The property that $f \in \partial_1 \hat{\mathcal{A}} + \cdots + \partial_{\beta-1} \hat{\mathcal{A}}$ (and the similar statements for g_α, $\alpha = 1, \dots, \beta - 1$, obtained by relabeling of the independent variables) is equivalent to $\omega \in d_H \Omega^{\beta-2}$. That is, we have to prove that the auxiliary complex (4.29) is acyclic in cohomological degree $\beta - 1$.

In order to do this, we revisit the argument of Anderson, cf. [1, Proposition 4.2 and 4.3]. While the horizontal rows of the variational bicomplex that is considered in [1] are, in general, quite different from (4.29) (the meaning of the symbol θ is completely different), the combinatorics of the differential d_H is literally the same, which allows us to adapt the formulas of Anderson.

The first thing we need to do is to refine the standard gradation. Namely, we represent $S \in \mathbb{Z}^D_{\geq 0}$, $|S| > 0$, as the sum $S = S' + S''$, where $S' = \sum_{\alpha=1}^{\beta} s_\alpha \xi_\alpha$ and $S'' = \sum_{\alpha=\beta+1}^{D} s_\alpha \xi_\alpha$. The same representation is inherited by $\deg f$, that can be written as the sum $\deg' f + \deg'' f$, where $\deg' = |S'|$ and $\deg'' = |S''|$. Observe that both $\deg''$ and the super gradation $\deg_\theta$ are preserved by the differential d_H. This means that we can split the complex (4.29) into the direct summands

$$0 \to \left(\Omega^p_{d''}\right)^0 \xrightarrow{d_H} \left(\Omega^p_{d''}\right)^1 \xrightarrow{d_H} \cdots \xrightarrow{d_H} \left(\Omega^p_{d''}\right)^{\beta-1} \xrightarrow{d_H} \left(\Omega^p_{d''}\right)^{\beta}, \tag{4.32}$$

where the coefficients have $\deg'' = d''$ and $\deg_\theta = p$.

There are two different cases, $p = 0$ and $p \neq 0$. In the case $p = 0$, we have no θ's. Also, we consider the variables u^i_S as $u^i_{S'',S'}$, that is, we introduce a new pair of indices (i, S'') for the dependent variables, and we take into account only the dependence on the independent variables $x^1, \dots, x^\beta$. In other words, we redefine the algebra $\mathcal{A}$ to be

$$C^\infty(U)[[u^i_{S''}, i = 1, \dots, N, |S''| > 0]][[u^i_{S'',S'}, |S''| > 0]],$$

where $\{u^i_{S''}\}$ is the new set of dependent variables, $\{x^1, \dots, x^\beta\}$ is the new set of independent variables, and $u^i_{S'',S'} = \boldsymbol{\partial}^{S'} u^i_{S''}$. If we fix the degree $\deg''$, we still have a finite number of dependent variables, and the only difference with the standard case is that we require our functions to be homogeneous polynomials in some of them (as opposed to just smooth functions). This modification and also a minor difference that we do not consider explicit dependence on independent variables are the only differences that we have between the complex (4.32) for $p = 0$ and the complex considered in the first half of [1, Proposition 4.3]. It is then straightforward to see that the argument of Anderson proves that this complex is acyclic (up to cohomological degree $(\beta - 1)$).

It remains to prove that the complex (4.32) is acyclic in cohomological degree $\leq (\beta - 1)$ for $p > 0$. For that we introduce an explicit homotopy contraction operator $h^{p,\alpha}\colon \left(\Omega^p_{d''}\right)^\alpha \to \left(\Omega^p_{d''}\right)^{\alpha-1}$. We have:

$$h^{p,\alpha}(\omega) := \frac{1}{p} \sum_{\gamma=1}^{\beta} \sum_{I,J \in \mathbb{Z}^\beta_{\geq 0}} \sum_{(i,S'')} \frac{|I| + 1}{\beta - \gamma + |I| + 1} \cdot \tag{4.33}$$

$$\boldsymbol{\partial}^I \left(\theta_i^{S''} \wedge \binom{|I + J + \xi_\gamma|}{|J|} (-\boldsymbol{\partial})^J \frac{\partial}{\partial \theta_i^{I+J+\xi_\gamma+S''}} \frac{\partial \omega}{\partial (\mathrm{d}x^\gamma)} \right)$$

This is a version of Anderson's homotopy contraction operator from [1, Proof of Proposition 4.2] adapted for our case. Repeating mutatis mutandis the arguments of [1, Lemma 4.4 and Proof of Proposition 4.2], we prove that $(h^{p,\alpha+1}d_H + d_H h^{p,\alpha})\omega = \omega$ for $\omega \in \left(\Omega^p_{d''}\right)^\alpha$, $p > 0$ and $\alpha < \beta$. This implies that for $p > 0$ the cohomology of the chain complex (4.32) in cohomological degree $\leq (\beta - 1)$ is equal to zero.

This completes the proof of the Lemma. □

4.2 The main theorem

The notion of Poisson–Lichnerowicz cohomology defined in Chapter 1 stays the same, after the replacement of the space of local multivectors $\Lambda^\bullet$ with $\hat{\mathcal{F}}$.

Let us introduce an auxiliary cochain complex $(\hat{\mathcal{A}}, D_P)$. If $P \in \hat{\mathcal{F}}^2$ and $[P,P] = 0$, then D_P defined in (4.22) squares to 0. That means that it is possible to define

$$0 \to \hat{\mathcal{A}}^0 \xrightarrow{D_P} \hat{\mathcal{A}}^1 \xrightarrow{D_P} \hat{\mathcal{A}}^2 \xrightarrow{D_P} \cdots \tag{4.34}$$

and its cohomology; moreover, since D_P commutes with all the ∂_α the complex and the cohomology groups pass to the quotient space $\hat{\mathcal{F}}$.

We denote

$$H^p(\hat{\mathcal{A}}) = \frac{\ker\left(D_P \colon \hat{\mathcal{A}}^p \to \hat{\mathcal{A}}^{p+1}\right)}{\operatorname{Im}\left(D_P \colon \hat{\mathcal{A}}^{p-1} \to \hat{\mathcal{A}}^p\right)} \tag{4.35}$$

and

$$H^p(\hat{\mathcal{F}}) = \frac{H^p(\hat{\mathcal{A}})}{\partial_1\hat{\mathcal{A}} + \cdots + \partial_D\hat{\mathcal{A}}} = \frac{\ker\left(d_P \colon \hat{\mathcal{F}}^p \to \hat{\mathcal{F}}^{p+1}\right)}{\operatorname{Im}\left(d_P \colon \hat{\mathcal{F}}^{p-1} \to \hat{\mathcal{F}}^p\right)}. \tag{4.36}$$

The groups $H^\bullet(\hat{\mathcal{F}}, d_P)$ constitute the standard Poisson–Lichnerowicz cohomology, in the θ calculus formalism. As usual, we identify the first cohomology group H^1 with the symmetries of the Poisson bivector P that are not Hamiltonian, and the second cohomology group H^2 with the infinitesimal *compatible deformations* of the Poisson bracket defined by the bivector P that are not *trivial*. Recalling the definition of d_P, a symmetry X is an (evolutionary) vector field such that $[P,X] = 0$, a Hamiltonian vector field is a vector field of form $X_H = [P,H]$ for $H \in \hat{\mathcal{F}}^0$ a local functional, a compatible bivector P' is a bivector such that $[P,P'] = 0$, and a trivial compatible bivector is such that $P' = [P,Y]$ for some vector field $Y \in \hat{\mathcal{F}}^1$.

The gradation on $\hat{\mathcal{A}}$ defined in Paragraph 4.1.1 can be used to decompose the cohomology groups both on $\hat{\mathcal{A}}$ and $\hat{\mathcal{F}}$. We will denote, for instance, $H^2_2(\hat{\mathcal{F}})$ the first order nontrivial compatible deformations of a Poisson bivector of degree 1. Indeed, if the original Poisson bivector is of degree 1 as in the Dubrovin–Novikov case, the k-th deformation has degree $k+1$.

Despite in the previous Section we have established a general formalism valid for any D and N, in the following part we focus on $N = 1$, namely on scalar brackets.

4.2.1 Main result

Our main result can be formulated as follows. Let Θ be the polynomial ring $\mathbb{R}[\{\theta^S, S \in (\mathbb{Z}_{\geq 0})^{D-1}\}]$. The derivations ∂_α, $\alpha = 1, \dots, D-1$ act on Θ in the obvious way. Denote by Ξ^p_d the homogeneous component of bi-degree (p, d) of

$$\Xi = \frac{\Theta}{\partial_1 \Theta + \cdots + \partial_{D-1}\Theta}. \tag{4.37}$$

Theorem 4.5. *The Poisson cohomology of the Poisson bracket* (4.1) *in bi-degree* (p, d) *is isomorphic to the sum of vector spaces*

$$\Xi^p_d \oplus \Xi^{p+1}_d. \tag{4.38}$$

The proof of this Theorem will be given in the following paragraphs. The strategy is to compute first the cohomology of a particular Poisson bracket, and then to show that the cohomology does not change under linear changes of the independent variables that allow us to extend the result to the whole class of Poisson brackets (4.1).

Let us first derive some consequences of Theorem 4.5. Let's start by an explicit description of the spaces Ξ^p_d for small p:

Lemma 4.6. *We have that*

$$\Xi^0_d \cong \Xi^1_d \cong \begin{cases} \mathbb{R} & d = 0, \\ 0 & d \geq 1, \end{cases} \tag{4.39}$$

$$\Xi^2_d \cong \begin{cases} 0 & d = 0, \\ \mathbb{R}^{D-1} & d = 1, \\ 0 & d = 2 \end{cases} \tag{4.40}$$

Proof. For $p = 0, 1$ the statement is trivial.

For $p = d = 2$ a generic element in Θ^2_2

$$\sum_{\alpha,\beta=1}^{D-1} \left(a_{\alpha\beta} \theta^{\xi_\alpha} \theta^{\xi_\beta} + b_{\alpha\beta} \theta \theta^{\xi_\alpha + \xi_\beta} \right) \tag{4.41}$$

can always be cancelled by the quotient map for elements in $\sum_{\alpha=1}^{D-1} \partial_\alpha \Theta^2_1$. Indeed, one can write (4.41) as

$$\sum_{\alpha=1}^{D-1} \partial_\alpha \left(\sum_{\beta=1}^{D-1} c_{\alpha\beta} \theta \theta^{\xi_\beta} \right) = \sum_{\alpha,\beta=1}^{D-1} c_{\alpha\beta} \left(\theta^{\xi_\alpha} \theta^{\xi_\beta} + \theta \theta^{\xi_\alpha + \xi_\beta} \right), \tag{4.42}$$

From (4.41) it follows by definition that $a_{\alpha\beta}$ is skewsymmetric in the exchange of the indices and $b_{\alpha\beta}$ is symmetric. The commutation rules of the odd variables θ's let us to choose $c_{\alpha\beta}$ in (4.42) such that its skewsymmetric and symmetric parts are respectively $a_{\alpha\beta}$ and $b_{\alpha\beta}$. □

A straightforward application of Theorem 4.5 gives us the explicit form of the Poisson cohomology groups in low degree:

Corollary 4.7. *We have that:*

$$H^0_d(\hat{\mathcal{F}}) \cong \begin{cases} \mathbb{R}^2 & d=0, \\ 0 & d \geq 1 \end{cases} \tag{4.43}$$

$$H^1_d(\hat{\mathcal{F}}) \cong \begin{cases} \mathbb{R} & d=0, \\ \mathbb{R}^{D-1} & d=1, \\ 0 & d=2 \end{cases} \tag{4.44}$$

Remark 4.8. For $D=1$ case Theorem 4.5 implies $H^p_d(\hat{\mathcal{F}})$ is isomorphic to $\mathbb{R}^2$ for $p=d=0$, to $\mathbb{R}$ for $p=1$, $d=0$, and vanishes otherwise, as in the scalar case of Getzler's result [36].

Remark 4.9. For $D=2$ we can provide a general formula for $H^p_d(\hat{\mathcal{F}})$. Indeed, in this case $\Theta = \mathbb{R}[\theta^{(\alpha,0)}]$ and in particular Θ^p_d is generated by the monomials $\theta^{(\alpha_1,0)}\theta^{(\alpha_2,0)}\dots\theta^{(\alpha_p,0)}$ such that $\alpha_1+\alpha_2+\dots+\alpha_p = d$, $\alpha_k \geq 0$. The dimension of Θ^p_d is given by the number of ways of writing d as the sum of p distinct nonnegative integers, regardless of the order. The result, in terms of the partition function $P(n,k)$ giving the number of ways of writing n as sum of k positive addends, is [13]

$$\dim \Theta^p_d = P\left(d+p-\binom{p}{2}, p\right). \tag{4.45}$$

Since $\ker \partial_1 = \mathbb{R} \neq 0$ only in Θ^0_0, the dimension of $\partial_1\Theta^p_{d-1}$ for $(p,d) \neq (0,1)$ is given by the same formula (4.45) with $d' = d-1$. For $(p,d)=(0,1)$, on the other hand, we have $H^0_1(D) = \Theta^0_1 = 0 = P(1,0) - P(0,0)$, hence the same formula applies. Then Theorem 4.5 for $D=2$ is $H^p_d(\hat{\mathcal{F}}) \cong \mathbb{R}^Q$ with

$$\begin{aligned} Q = {} & P\left(d+p-\binom{p}{2}, p\right) - P\left(d+p-\binom{p}{2}-1, p\right) + \\ & + P\left(d+p-\binom{p+1}{2}+1, p+1\right) - P\left(d+p-\binom{p+1}{2}, p+1\right) \end{aligned} \tag{4.46}$$

Moreover, from the explicit form of $P(n,k)$ for $k=2,3$ we can improve the explicit results of Lemma 4.6

$$\dim \Xi^2_d(D=2) = \begin{cases} 0 & d = 2k \\ 1 & d = 2k+1 \end{cases} \tag{4.47}$$

$$\dim \Xi^3_d(D=2) = \begin{cases} 0 & d < 3 \\ \frac{1}{6}(d+3) & d = 3+6k \\ \frac{1}{6}(d-4) & d = 4+6k \\ \frac{1}{6}(d+1) & d = 5+6k \\ \frac{1}{6}d & d = 6+6k \\ \frac{1}{6}(d-1) & d = 7+6k \\ \frac{1}{6}(d-2) & d = 8+6k \end{cases} \tag{4.48}$$

Combining the two previous results we can explicitly give the dimensions of $H^2_p(\hat{\mathcal{F}})$. In particular it should be noticed that $H^2_d(\hat{\mathcal{F}}) \neq 0$ for all $d > 4$. Finally, we exhibit the dimension of some cohomology groups $H^2_p(\hat{\mathcal{F}})$ that can be obtained by formula (4.46).

d	0	1	2	3	4	5	6	7	8
$\dim H^2_d(\hat{\mathcal{F}})$	0	1	0	2	0	2	1	2	1

4.2.2 Cohomology in a special case

Let us consider the DN brackets with one dependent variable and D independent variables

$$\{u(x), u(y)\}_{\hat{P}} = \partial_D \delta(x-y), \tag{4.49}$$

which, in the θ formalism, corresponds to the bivector

$$\hat{P} = \frac{1}{2} \int^D \theta \theta^{\xi_D} \mathrm{d}^D x \in \hat{\mathcal{F}}^2. \tag{4.50}$$

In this Paragraph we compute the Poisson cohomology of the bracket $\{,\}_{\hat{P}}$, i.e., the cohomology of the complex $(\hat{\mathcal{F}}, d_{\hat{P}})$, where $d_{\hat{P}} = [\hat{P}, \cdot]$.

Let

$$D_{\hat{P}} = \sum_S \theta^{S+\xi_D} \frac{\partial}{\partial u^S}, \tag{4.51}$$

be the differential operator on $\hat{\mathcal{A}}$ associated with $\hat{P}$ defined in (4.22). Clearly, $D_{\hat{P}}$ squares to zero.

Moreover, this operator commutes with $\partial_1, \dots, \partial_D$, therefore it induces a differential on each of the spaces $\hat{\mathcal{F}}_1, \dots, \hat{\mathcal{F}}_D$. In particular, it is obvious that on $\hat{\mathcal{F}}_D$ this operator coincides with (4.19).

With the differentials induced by $D_{\hat{P}}$ on $\hat{\mathcal{A}}, \hat{\mathcal{F}}_1, \dots, \hat{\mathcal{F}}_D$, the short exact sequences of (4.26) become the short exact sequences of complexes. So, we can use the associated long exact sequences in order to compute the cohomology of $\hat{\mathcal{F}}_1, \dots, \hat{\mathcal{F}}_D$.

All the cohomology groups that we consider are to be understood with respect to the differential induced by $D_{\hat{P}}$, so we will generally refrain from indicating it all the time.

Recall that we denote by Z_D the sub-semiring of Z that consists of multi-indices of the form $S = \sum_\alpha s_\alpha \xi_\alpha$ with $s_D = 0$.

Lemma 4.10. *We have that* $H(\hat{\mathcal{A}}) = \mathbb{R}[\{\theta^S, S \in Z_D\}]$.

Proof. Under the identification $\theta^{S+\xi_D} \leftrightarrow du^S$, $S \in Z$, the operator $D_{\hat{P}}$ turns into the de Rham differential on the differential forms in u^S, $S \in Z$, hence its cohomology is given by the degree 0 forms constant in u^S, $S \in Z$, that is, by all elements of $\hat{\mathcal{A}}$ independent of u^S and $\theta^{S+\xi_D}$, $S \in Z$. These are the polynomials in θ^S, $S \in Z_D$. □

For $D = 1$ we have $H(\hat{\mathcal{A}}) = \mathbb{R} \oplus \mathbb{R}\theta$, as in [40, 41].

We denote the polynomial ring $\mathbb{R}[\{\theta^S, S \in Z_D\}]$ by Θ. It is important to stress that the polynomials in θ^S, $S \in Z_D$, are indeed representatives of cohomology classes in $H(\hat{\mathcal{A}})$.

Proposition 4.2. We have:

$$H(\hat{\mathcal{F}}_\alpha) = \frac{\Theta}{\partial_1\Theta + \cdots + \partial_\alpha\Theta} \tag{4.52}$$

for $\alpha = 1, \dots, D-1$.

Proof. Let us start by considering the short exact sequence

$$0 \to \hat{\mathcal{A}}/\mathbb{R} \xrightarrow{\partial_1} \hat{\mathcal{A}} \xrightarrow{\int dx^1} \hat{\mathcal{F}}_1 \to 0, \tag{4.53}$$

which induces a long exact sequence in cohomology

$$H^p_{d-1}(\hat{\mathcal{A}}/\mathbb{R}) \xrightarrow{\partial_1} H^p_d(\hat{\mathcal{A}}) \xrightarrow{\int dx^1} H^p_d(\hat{\mathcal{F}}_1) \to H^{p+1}_d(\hat{\mathcal{A}}/\mathbb{R}) \xrightarrow{\partial_1} H^{p+1}_{d+1}(\hat{\mathcal{A}}). \tag{4.54}$$

By Lemma 4.10, the cohomology classes in $H(\hat{\mathcal{A}})$, and of course in $H(\hat{\mathcal{A}}/\mathbb{R})$, are represented by elements of Θ, i.e., by polynomials in θ^S with $S \in Z_D$. The derivations $\partial_1, \dots, \partial_{D-1}$ map such polynomials to polynomials of the same form, hence their action on the cohomology coincides with their natural action on Θ. It clearly follows that the kernel of $H^{p+1}_d(\hat{\mathcal{A}}/\mathbb{R}) \xrightarrow{\partial_1} H^{p+1}_{d+1}(\hat{\mathcal{A}})$ is equal to zero.

The Bockstein homomorphism $H^p_d(\hat{\mathcal{F}}_1) \to H^{p+1}_d(\hat{\mathcal{A}}/\mathbb{R})$ is therefore the zero map, and we can conclude that $H^p_d(\hat{\mathcal{F}}_1)$ is the quotient of $H^p_d(\hat{\mathcal{A}})$ by the image of ∂_1. So, we have

$$H^p_d(\hat{\mathcal{F}}_1) \cong \frac{H^p_d(\hat{\mathcal{A}})}{\partial_1 H^p_{d-1}(\hat{\mathcal{A}})} \cong \frac{\Theta^p_d}{\partial_1\Theta^p_{d-1}}, \tag{4.55}$$

which is precisely the assertion of the Proposition for $\hat{\mathcal{F}}_1$.

Let us now prove the same statement for $\hat{\mathcal{F}}_\alpha$, $\alpha = 2, \dots, D-1$, by induction. The short exact sequence

$$0 \to \hat{\mathcal{F}}_{\alpha-1}/\mathbb{R} \xrightarrow{\partial_\alpha} \hat{\mathcal{F}}_{\alpha-1} \xrightarrow{\int dx^\alpha} \hat{\mathcal{F}}_\alpha \to 0 \tag{4.56}$$

induces the long exact sequence in cohomology

$$H^p_{d-1}(\hat{\mathcal{F}}_{\alpha-1}/\mathbb{R}) \xrightarrow{\partial_\alpha} H^p_d(\hat{\mathcal{F}}_{\alpha-1}) \xrightarrow{\int dx^\alpha} H^p_d(\hat{\mathcal{F}}_\alpha) \to H^{p+1}_d(\hat{\mathcal{F}}_{\alpha-1}/\mathbb{R}) \xrightarrow{\partial_\alpha} H^{p+1}_{d+1}(\hat{\mathcal{F}}_{\alpha-1}). \tag{4.57}$$

Notice that the map $\partial_\alpha \colon H(\hat{\mathcal{F}}_{\alpha-1}/\mathbb{R}) \to H(\hat{\mathcal{F}}_{\alpha-1})$ is given, by inductive assumption, by

$$\partial_\alpha \colon \frac{\Theta}{\partial_1\Theta + \cdots + \partial_{\alpha-1}\Theta + \mathbb{R}} \to \frac{\Theta}{\partial_1\Theta + \cdots + \partial_{\alpha-1}\Theta}. \tag{4.58}$$

Lemma 4.11. *The kernel of the map* (4.58) *is equal to zero.*

Proof. We can follow the proof of Lemma 4.4. By the same argument as there, we reduce the statement of the lemma to the vanishing of the cohomology of a certain complex, where we have an explicit homotopy contraction. □

Since the kernel of the map (4.58) is equal to zero, the Bockstein homomorphism vanishes, and therefore we can conclude that

$$H(\hat{\mathcal{F}}_\alpha) \cong \frac{\frac{\Theta}{\partial_1\Theta+\cdots+\partial_{\alpha-1}\Theta}}{\partial_\alpha\left(\frac{\Theta}{\partial_1\Theta+\cdots+\partial_{\alpha-1}\Theta}\right)} \cong \frac{\Theta}{\partial_1\Theta + \cdots + \partial_\alpha\Theta}. \tag{4.59}$$

□

In particular, the previous Proposition implies that $H^p_d(\hat{\mathcal{F}}_{D-1})$ is a finite dimensional vector space, whose dimension one can compute explicitly for each choice of p, d, D. Due to its importance we denote it by Ξ^p_d, which is then the degree (p, d) component of

$$\frac{\Theta}{\partial_1\Theta + \cdots + \partial_{D-1}\Theta}. \tag{4.60}$$

Finally we can compute the Poisson cohomology in the scalar case:

Theorem 4.12. *We have that* $H^p_d(\hat{\mathcal{F}}) \simeq \Xi^p_d \oplus \Xi^{p+1}_d$.

Proof. The short exact sequence

$$0 \to \hat{\mathcal{F}}_{D-1}/\mathbb{R} \xrightarrow{\partial_D} \hat{\mathcal{F}}_{D-1} \xrightarrow{\int dx^D} \hat{\mathcal{F}}_D \to 0 \tag{4.61}$$

induces in cohomology the exact sequence

$$H^p_{d-1}(\hat{\mathcal{F}}_{D-1}/\mathbb{R}) \xrightarrow{\partial_D} H^p_d(\hat{\mathcal{F}}_{D-1}) \xrightarrow{\int dx^D} H^p_d(\hat{\mathcal{F}}_D) \to H^{p+1}_d(\hat{\mathcal{F}}_{D-1}/\mathbb{R}) \xrightarrow{\partial_D} H^{p+1}_{d+1}(\hat{\mathcal{F}}_{D-1}). \tag{4.62}$$

It is easy to check that, for $a \in \Theta$,

$$\partial_D a = D_{\hat{P}} \sum_S u^S \frac{\partial a}{\partial \theta^S}, \tag{4.63}$$

hence the map ∂_D sends any element of Θ, considered as a subspace of $\hat{\mathcal{A}}$, to a Δ-exact element of $\hat{\mathcal{A}}$. This implies that the operator induces by ∂_D on the cohomology of $\hat{\mathcal{A}}$ is equal to zero, and therefore it is also equal to zero on the cohomology of $\hat{\mathcal{F}}_\alpha$, $\alpha = 1, \ldots, D-1$. The long exact sequence (4.62) is then equivalent to the collection of short exact sequences

$$0 \to H^p_d(\hat{\mathcal{F}}_{D-1}) \to H^p_d(\hat{\mathcal{F}}_D) \to H^{p+1}_d(\hat{\mathcal{F}}_{D-1}/\mathbb{R}) \to 0, \tag{4.64}$$

for $p \geq 0$, $d \geq 0$. This implies that $H^p_d(\hat{\mathcal{F}}_D) \simeq H^p_d(\hat{\mathcal{F}}_{D-1}) \oplus H^{p+1}_d(\hat{\mathcal{F}}_{D-1})$. □

4.2.3 Change of independent variables

In Section 4.2.2 we have proved a theorem about the Poisson cohomology for the bracket $\{u(x), u(y)\} = \partial_D \delta(x-y)$. On the other hand, the generic nondegenerate Poisson bracket (4.1) has the form, when written in normalized flat coordinates (for which $g(u) = \text{const} = 1$),

$$\{u(x), u(y)\} = \sum_{\alpha=1}^{D} c^\alpha \frac{\partial}{\partial x^\alpha} \delta(x-y). \tag{4.65}$$

Let us denote Δ the Poisson differential associated to the bracket (4.65).

In this Section we prove that the cohomology groups for the bracket (4.65) are isomorphic to the ones we computed in the previous one, and hence that Theorem 4.5 holds for all the nondegenerate scalar DN brackets.

Lemma 4.13. *By a linear change of independent variables* $(x^1, \ldots, x^D)$ *the bracket* (4.65) *can be brought to the form*

$$\{v(\tilde{x}), v(\tilde{y})\} = \partial_D \delta(\tilde{x} - \tilde{y}). \tag{4.66}$$

Proof. Let us denote ∂_{x^α} the total derivative ∂_α computed with respect to the independent variables $\{x^\alpha\}_{\alpha=1}^D$ and $\partial_{\tilde{x}^\alpha}$ the one computed with respect to the new independent variables $\{\tilde{x}^\alpha\}$. Under a linear change of coordinates $x \mapsto \tilde{x} = Jx$, with J a constant $D \times D$ matrix, the derivations $\boldsymbol{\partial}_x = \{\partial_{x^\alpha}\}$ change according to $\boldsymbol{\partial}_x \mapsto \boldsymbol{\partial}_{\tilde{x}} = (J^{-1})^T \boldsymbol{\partial}_x$. For the D-dimensional Dirac's delta function to be invariant, we require in addition that $\det J = 1$. Introducing the D-dimensional vector $C = (c_1, \ldots, c_D)$ we have to solve the algebraic set of equations $J \cdot C = \xi_D$. We get D equations for $D^2 - 1$ entries of the matrix J. Hence, the solution is not unique but it always exists. □

We will denote $\tilde{\Delta}$ the Poisson differential of the bracket (4.66).

Lemma 4.14. *The cohomology groups* $H^p_d(\hat{\mathcal{F}}, \Delta)$ *are isomoprhic to* $H^p_d(\hat{\mathcal{F}}, \tilde{\Delta})$.

Proof. The same transformation law $\boldsymbol{\partial}_x \mapsto \boldsymbol{\partial}_{\tilde{x}} = (J^{-1})^T \boldsymbol{\partial}_x$ applies to the variables $u^{\xi_\alpha} = \partial_{x^\alpha} u$ and θ^{ξ_α}. Since J is constant, the higher order derivatives leave it unaffected, in such a way that the transformation law is tensorial

$$u^I \mapsto \tilde{u}^I = \left((J^{-1})^T\right)^{|I|} u^{I'} \tag{4.67}$$

where $|I| = k_1 + \ldots + k_D$ and $|I| = |I'|$. More precisely, we have

$$\tilde{u}^{(k_1,\ldots,k_D)} = \left((J^{-1})^T\right)_1^{\alpha_1} \cdots \left((J^{-1})^T\right)_1^{\alpha_{k_1}} \left((J^{-1})^T\right)_2^{\alpha_{k_1+1}} \ldots \left((J^{-1})^T\right)_D^{\alpha_{|I|}} \frac{\partial^{|I|}}{\partial x^{\alpha_1} \ldots \partial x^{\alpha_{|I|}}} u. \tag{4.68}$$

Moreover, the partial derivatives with respect to the jets transform with J^T, so that the differential Δ is transformed, being homogeneous of differential order 1, as $(J^{-1})^T \Delta$. Under the change of independent variables, $\hat{\mathcal{A}}^p_d \mapsto \hat{\mathcal{A}}^p_d$ for any component of bidegree (p, d).

Since J is invertible and the change of coordinates does not change the differential order of the jets, there exists an isomorphism between the kernels and the images of $\Delta\colon \hat{\mathcal{A}}^p_d \to \hat{\mathcal{A}}^p_{d+1}$ and $\tilde{\Delta}\colon \hat{\mathcal{A}}^p_d \to \hat{\mathcal{A}}^p_{d+1}$. Hence the quotient spaces $H^p_d(\Delta, \hat{\mathcal{A}})$ and $H^p_d(\tilde{\Delta}, \hat{\mathcal{A}})$ are isomorphic.

Moreover, the change of independent variable leaves the space $\hat{\mathcal{F}}$ invariant. Let us consider the quotient operation

$$\hat{\mathcal{F}} = \frac{\hat{\mathcal{A}}}{\partial_{x^1} \hat{\mathcal{A}} + \cdots + \partial_{x^D} \hat{\mathcal{A}}}.$$

The change of independent coordinates maps each partial derivative to a linear combination of all the D derivatives

$$\partial_{x^\alpha} \mapsto \left((J^{-1})^T \boldsymbol{\partial}_x\right)_\alpha = \sum_\beta \frac{\partial x^\beta}{\partial \tilde{x}^\alpha} \partial_{x^\beta}.$$

Since J is nondegenerate, however, $\operatorname{Span}(\partial_{\tilde{x}} \hat{\mathcal{A}})$ is the same as $\operatorname{Span}(\partial_x \hat{\mathcal{A}})$, which proves the claim.

Equation (4.36) reads

$$H^p_d(\hat{\mathcal{F}}) \cong \frac{\Theta^p_d}{\partial_{x^1} \Theta^p_{d-1} + \cdots + \partial_{x^D} \Theta^p_{d-1}}$$

where $\Theta^p_d = H^p_d(\hat{\mathcal{A}})$. The linear change of independent coordinates, as already discussed, maps the numerator of the quotient to an isomorphic space. For the denominator we have

$$\partial_{x^1}\Theta^p_{d-1} + \cdots \partial_{x^D}\Theta^p_{d-1} \mapsto$$
$$\partial_{\tilde{x}^1}\tilde{\Theta}^p_{d-1} + \cdots \partial_{\tilde{x}^D}\tilde{\Theta}^p_{d-1} = \partial_{x^1}\tilde{\Theta}^p_{d-1} + \cdots \partial_{x^D}\tilde{\Theta}^p_{d-1} \cong \partial_{x^1}\Theta^p_{d-1} + \cdots \partial_{x^D}\Theta^p_{d-1}.$$

Since both the numerator and the denominator of the quotient $H^p_d(\hat{\mathcal{F}})$ are mapped to isomorphic spaces, such is the quotient itself. This allows us to extend the Theorem 4.5 to all the nondegenerate scalar DN brackets. □

Theorem 4.12, which holds for the bracket (4.65), and Lemma 4.14 constitute the proof of Theorem 4.5.

4.3 Some explicit examples

In this section we show how one can explicitly compute some cohomology groups $H^p_q(\hat{\mathcal{F}}, \Delta)$ using the formalism of Poisson Vertex Algebras introduced in Chapter 2.

4.3.1 Symmetries of the bracket

The λ bracket equivalent to the Poisson bracket defined by the bivector $\hat{P} = \frac{1}{2}\theta\theta^{\xi_D}$ is $\{u_{\boldsymbol{\lambda}}u\} = \lambda_D$. The elements of the first cohomology group $H^1(\hat{\mathcal{F}}, \mathrm{d}_{\hat{P}})$ are the symmetries of the bracket that are not Hamiltonian, namely the evolutionary vector fields

$$X = \sum_I \boldsymbol{\partial}^I X(u; u_L)\frac{\partial}{\partial u_I} \tag{4.69}$$

satisfying

$$X(\{u_{\boldsymbol{\lambda}}u\}) = \{X(u)_{\boldsymbol{\lambda}}u\} + \{u_{\boldsymbol{\lambda}}X(u)\} \tag{4.70}$$

of form different than

$$X(u) = \{h_{\boldsymbol{\lambda}}u\}|_{\boldsymbol{\lambda}=0}. \tag{4.71}$$

Let us compute H^1_0, H^1_1, and H^1_2 for generic D, showing the agreement with the results of Corollary 4.7.

For H^1_0 we want to consider evolutionary vector field whose component X depends only on u. Since the bracket between the generators u is constant, the LHS of (4.70) vanishes; computing the RHS with the help of the master formula (2.5) and setting it equal to 0 immediately gives that X must be a constant. Given the form of the Poisson bivector, we cannot have any constant Hamiltonian vector fields; thus, $H^1_0 = \mathbb{R}$.

To compute H^1_1 we are interested in vector fields of first degree, namely of form $X = \sum_{\alpha=1}^D X^\alpha(u)u_{\xi_\alpha}$. Imposing the condition of symmetry does not give any condition on X^D but forces X^α for $\alpha = 1, \dots, D-1$ to be constants. On the other hand, if we take a generic Hamiltonian density of degree 0, i.e. a function $h(u)$, its Hamiltonian vector field will be $X_h = h'(u)u_{\xi_D}$, where $X^D = h'$ can be arbitrary. This means that the $(D-1)$ constants are all and only the elements of $H^1_1 = \mathbb{R}^{(D-1)}$.

In principle, computing any component of the first cohomology group means performing the same computations as the ones for the first two. However, they become more and more involved with the growing of the degree (and hence of the differential order), hence requiring more computational effort even if we resort to computer algebra systems. Computing (4.70) for second degree evolutionary vector field, whose component is $X = \sum_{\alpha,\beta=1}^{D} X_1^{\alpha\beta} u_{\xi_\alpha} u_{\xi_\beta} + X_2^{\alpha\beta} u_{\xi_\alpha+\xi_\beta}$, we get that the necessary condition is that $X_1 = X_2 = 0$ for all (a, b). This means that $H^1_2 = 0$.

4.3.2 Deformations of the bracket

he second cohomology group is the class of the compatible infinitesimal deformations of the λ bracket associated to the Poisson bivector $\hat{P}$ that cannot be obtained by a Miura transformation. We will demonstrate a few results for the case $D = 2$. Since $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_{\hat{P}}$ is constant, it is enough to impose the skewsymmetry of the deformation $\{\cdot_{\boldsymbol{\lambda}}\cdot\}^\sim$ and the cocycle condition is reduced to

$$\{u_{\boldsymbol{\lambda}}\{u_{\boldsymbol{\mu}}u\}^\sim\}_{\hat{P}} + \{u_{\boldsymbol{\mu}}\{u_{\boldsymbol{\lambda}}u\}^\sim\}_{\hat{P}} = \{\{u_{\boldsymbol{\lambda}}u\}^\sim_{\boldsymbol{\lambda}+\boldsymbol{\mu}}u\}_{\hat{P}}. \tag{4.72}$$

The form of the deformed bracket we choose depends on which component of the second cohomology group we are interested in: when computing H^2_d we will consider homogeneous λ brackets of degree d (as for the gradation on $\hat{\mathcal{A}}$, we consider $\deg \boldsymbol{\lambda}^I = |I|$ and $\deg\{u_{\boldsymbol{\lambda}}u\} = \deg(B(u; u_L)_I\boldsymbol{\lambda}^I) = \deg B + |I|$). The skewsymmetry of the deformation prevents the existence of elements in H^2_0, since there cannot be skewsymmetric brackets of form $\{u_{\boldsymbol{\lambda}}u\} = A(u)$.

For H^2_1 we consider brackets of form $\{u_{\boldsymbol{\lambda}}u\}^\sim = \sum_{\alpha=1}^{2} 2A^\alpha(u)\lambda_\alpha + A^{\alpha\prime}u_{\xi_\alpha}$ — we have already implemented the skewsymmetry of the deformation. Computing the condition (4.72) for this bracket does not give any constraint on A^2 but imposes $A^1 = C$ constant. Let us consider a generic Miura transformation that gives rise to a bracket of first degree: since $\hat{P}$ is of first degree as well, the transformation must be of degree 0, so $u \mapsto U = u + \epsilon F(u)$. We get

$$\{U_{\boldsymbol{\lambda}}U\}_{\hat{P}} = \{u_{\boldsymbol{\lambda}}u\}_{\hat{P}} + \epsilon\left(2F'(u)\lambda_2 + F''(u)u_{\xi_2}\right) + O(\epsilon^2). \tag{4.73}$$

This computation shows that we can never get from a Miura transformation a bracket of form $\{u_{\boldsymbol{\lambda}}u\} = \lambda_1$, which is compatible with $\{\cdot_{\boldsymbol{\lambda}}\cdot\}_{\hat{P}}$. This means that $H^2_1 = \mathbb{R}$.

We can proceed similarly for H_2^2; in this case a general deformed bracket of degree 2 will be

$$\{u_{\boldsymbol{\lambda}} u\}^{\sim} = \sum_{\alpha,\beta} A^{\alpha\beta}\lambda_\alpha\lambda_\beta + B^{\alpha\beta}\lambda_\alpha u_{\xi_\beta} + C^{\alpha\beta} u_{\xi_\alpha} u_{\xi_\beta} + D^{\alpha} u_{\xi_\alpha+\xi_\beta}. \tag{4.74}$$

Note that, by definition, A, C, and D are symmetric in the indices (α, β). Imposing the skewsymmetry we find the relations

$$A^{\alpha} = 0 \tag{4.75}$$

$$C^{\alpha\beta} = \frac{1}{4}\left(B^{\alpha\beta\prime} + B^{\beta\alpha\prime}\right) \tag{4.76}$$

$$D^{\alpha\beta} = \frac{1}{4}\left(B^{\alpha\beta} + B^{\beta\alpha}\right). \tag{4.77}$$

We now impose the condition (4.72) to the bracket and find that the parameters B must satisfy the set of equations

$$B^{11} = 0 \tag{4.78}$$

$$B^{12} + B^{21} = 0 \tag{4.79}$$

$$B^{22} = 0 \tag{4.80}$$

In other words, all the compatible infinitesimal deformation of $\hat{P}$ of degree 2 are parametrized by a single function $B(u) = B^{12}$ according to the given prescriptions. Any of these compatible deformations can be obtained by $\hat{P}$ after the Miura transformation

$$u \mapsto U = u - \epsilon u_{\xi_1} \int^u B(s)\mathrm{d}s \tag{4.81}$$

that exists for any function B of a single variable. This means that $H_2^2 = 0$, in agreement with the formula (4.46).

5

Conclusive remarks

In this thesis we have addressed the problem of the Poisson–Lichnerowicz cohomology of multidimensional Poisson brackets of hydrodynamic type following two different approaches. As we have pointed out several times in the previous chapters, Getzler's result about the one-dimensional case is tightly related to the existence of flat coordinates for the brackets; the existence of essentially one constant bracket allows to establish the isomorphism with the De Rham cohomology on the target manifold.

For higher dimensional brackets, the solutions of Mokhov's conditions (1.44) do not entail the existence of such flat coordinates; this implies that one has, in general, to find some alternative strategies.

In Chapter 3 we have attacked the problem explicitly, writing down the cocycle and coboundary conditions and reducing them to get informations about some homogeneous components of the cohomology groups. To this aim, we have extended to the multidimensional case the theory of Poisson Vertex Algebras, that proved to be an efficient framework for this kind of computations. Its main advantage is the (apparent) lack of integrations to be performed, since the axioms of the λ brackets encode the properties of skewsymmetry and Jacobi identity, as stated in Theorem 2.2. The master formula, which is the fundamental tool to compute the value of the brackets when we work with differential polynomials, can easily been coded, to dramatically speed up the computations by performing most of them with a computer. We have written a Mathematica package that contains all the needed procedures.

The second Poisson–Lichnerowicz cohomology contains the compatible deformations of the DN brackets that are not trivial. Having proved that the group is not trivial for $D = N = 2$, we could explicit provide a basis of its. Because of the huge freedom in the choice of the representatives classes in H^2_3, however, we still cannot conclude whether there exists another basis with a clearer interpretation.

In Chapter 4, for the particular case of scalar brackets, we could rely on the existence of a constant form of the Poisson structure to find an isomoprhism with the De Rham differential on a suitable space. This allows to find general results on all the cohomology groups, thanks to well known facts in differential geometry. The ideas exploited there, deeply rooted in the formal calculus of the variations, require notwithstanding a lot of technical lemmas to be properly applied. Most of the work has been completed by S.-Q. Liu and Y. Zhang [40, 41], but a careful review of it in light of the multidimensional setting seems to be necessary. Part of it is presented in the aforementioned chapter; a more general review might be useful, too.

Further developments The work presented in this thesis provides a few answers to the original problem we have addressed. According to the two main frameworks we have worked in, namely the mPVAs approach to the explicit computations and the search for techniques of homological algebra to give general results on the Poisson cohomology, we deem that the following list of open problems may be the natural prosecutions of the work done so far.

- The Mathematica package `MultiPVA` can be an useful tool for all the community of people working in Hamiltonian PDEs. Some improvements, in particular in the execution speed, and the publishing of a well-documented version of it will be a priority;
- The basis of cocycles we have found for $H^2_3(P_{LP})$ may be made simpler to interpret, and give rise to a new class of Hamiltonian PDEs;
- The theory of mPVA can be applied to a broader class of Hamiltonian and Integrable PDEs than the Dubrovin–Novikov one. In [11] we have already studied the Hamiltonian structure of Euler's equations as a reduction of P_{LP}. A broader study on the Hamiltonian structure of Vlasov's equations as a reduced bracket, related to 2D Toda, is currently in progress together with A. Raimondo and D. Valeri;
- The ideas of Chapter 4 can be useful to study, even for $N > 1$, the class of Poisson brackets that admits a coordinate systems in which they are constant. Such brackets, indeed, exists for any N as a special case. An extension of Theorem 4.5 may be easily obtained;
- The quite implicit statement of Theorem 4.5 should lead, at least for the low degrees, to a more explicit characterization of the cohomology groups for $D > 2$, in similar fashion to Formula (4.46).

A

Source code of MasterPVA

The package should be loaded at the beginning of your `Mathematica` notebook; in this way it is easier to avoid conflicts with the variables' names.

To load the package in the file, the standard code is

```
<< "path/MasterPVA.m"
```

We give the full raw transcript of the source code, already mildly commented. Some supplementary comments are given in Section 2.3.

```
 (* ::Package:: *)

BeginPackage["MasterPVAmulti`"];

Print["MasterPVAmulti: a Mathematica package for computing the
   lambda bracket of a Poisson Vertex Algebra of arbitrary rank
   "];
Print["2013. M. Casati & D. Valeri (SISSA,Trieste)"];

Clear["MasterPVAmulti`*"];
Clear["MasterPVAmulti`Private`*"];




(*Explain meaning of the functions*)
d::usage="d is the parameter for the number of generators of the
     PVA. Default=1";
r::usage="d is the rank of the algebra, i.e. the number of
   variables for the generators. Default=1"
maxO::usage="maxO is the parameter for the max order of
   derivatives to which to compute the bracket. Default=5";
var::usage="var is the array of the variables";
gen::usage="gen is the array of the generators";
LambdaB::usage="LambdaB[f,g,P,\[Lambda]] is the function which
   computes the lambda bracket of the functions f and g with
   respect to the structure P, written as a polynomial in the
```

```
    indeterminates \!\(\*SubscriptBox[\(\[Beta]\), \(i\)]\),
    with respect to the indeterminates \!\(\*SubscriptBox[\(\[
    Lambda]\), \(i\)]\) of the vector \[Lambda]";
\[Beta]\[Beta]::usage="formalparameter";
PVASkew::usage="PVASkew computes the condition of skewsymmetry
    for a lambda bracket and returns a matrix";
PrintPVASkew::usage="PrintPVASkew computes the condition of
    skewsymmetry for a lambda bracket and returns a table with
    the list of result for the single generators. Useful for
    checking the properties of a generic bracket";
JacobiCheck::usage="Jacobi check computes the terms of the PVA-
    Jacobi identity for all the triples of generators and
    returns an array of results";
PrintJacobiCheck::usage="PrintJacobiCheck computes the terms of
    PVA-Jacobi identity and returns a table with the list of
    results for each triple of generators. Useful for checking
    the properties of a generic bracket";
EvVField::usage="Apply the evolutionary vector field of given
    characteristic to a function";


Begin["‘Private‘"];
d=1;
r=1;
\[Beta]\[Beta]:=Array[Subscript[\[Beta], #]&,r];
maxO=5;


myD[{arg_,MInd_},j_]:={Nest[D[#,var[[j]]]&,arg,MInd[[j]]],MInd};
MultiD[arg_,MInd_]:=Fold[myD,{arg,MInd},Range[r]][[1]];
DLambda[{arg_,\[Lambda]_,MInd_},j_]:={Nest[\[Lambda][[j]] #+D[#,
    var[[j]]]&,arg,MInd[[j]]],\[Lambda],MInd};
DLambdamin[{arg_,\[Lambda]_,MInd_},j_]:={Nest[-\[Lambda][[j]]
    *#-D[#,var[[j]]]&,arg,MInd[[j]]],\[Lambda],MInd};
MultiDLambda[arg_,\[Lambda]_,MInd_]:=Fold[DLambda,{arg,\[Lambda
    ],MInd},Range[r]][[1]];
MultiDLambdamin[arg_,\[Lambda]_,MInd_]:=Fold[DLambdamin,{arg,\[
    Lambda],MInd},Range[r]][[1]];


Term3D[f_,MInd_,i_,\[Alpha]_]:=MultiDLambdamin[D[f,MultiD[gen[[i
    ]],MInd]],\[Alpha],MInd];
Term2D[f_,P_,i_,j_,\[Alpha]_]:=Module[{ListIndex=
    CoefficientRules[MonomialList[P[[j,i]]],\[Beta]\[Beta]]/.
```

```
    Rule->List},If[ListIndex=={{}},0,Total[Table[Flatten[
    ListIndex[[k]],1][[2]]*MultiDLambda[f,\[Alpha],Flatten[
    ListIndex[[k]],1][[1]]],{k,Length[MonomialList[P[[j,i
    ]]]]}]]]];
Term1D[f_,MInd_,g_,j_,\[Alpha]_]:=D[g,MultiD[gen[[j]],MInd]]*
    MultiDLambda[f,\[Alpha],MInd];
Lambdamnij[f_,g_,Multi1_,Multi2_,i_,j_,P_,\[Alpha]_]:=Term1D[
    Term2D[Term3D[f,Multi1,i,\[Alpha]],P,i,j,\[Alpha]],Multi2,g,
    j,\[Alpha]];
ListaMultiIndici:=Module[{IndexFamily=Array[Subscript[i, #]&,r],
    Mi=Table[0,{l,r}]},Flatten[Table[For[k=0,k<r,k++,Mi[[k+1]]=
    IndexFamily[[k+1]]];Mi,##]&@@({IndexFamily[[#]],0,maxO} &/
    @Range[r]),r-1]];
LambdaB[f_,g_,P_,\[Alpha]_]:=Sum[Lambdamnij[f,g,ListaMultiIndici
    [[m]],ListaMultiIndici[[n]],i,j,P,\[Alpha]],{m,1,(maxO+1)^r
    },{n,1,(maxO+1)^r},{i,1,d},{j,1,d}];
EvVField[X_,f_]:=Sum[MultiD[X[[i]],ListaMultiIndici[[m]]]D[f,
    MultiD[gen[[i]],ListaMultiIndici[[m]]]],{m,1,(maxO+1)^r},{i,
    d}];


(*Skewsymmetry*)
Skew[P_,i_,j_]:=Module[{dummy=P[[j,i]]},For[k=0,k<r,k++,dummy=
    dummy/.{Times[\[Beta]\[Beta][[k+1]]^n_, e_] :>Nest[-\[Beta
    ]\[Beta][[k+1]]*#-D[#,var[[k+1]]]&,e,n],Times[\[Beta]\[Beta
    ][[k+1]],e_]:>Times[-\[Beta]\[Beta][[k+1]],e]-D[e,var[[k
    +1]]],\[Beta]\[Beta][[k+1]]:>-\[Beta]\[Beta][[k+1]]}];dummy
    ];
PVASkew[P_]:=Table[Simplify[P[[i,j]]+Skew[P,i,j]],{i,d},{j,d}];
PrintPVASkew[P_]:=TableForm[Partition[Flatten[Table[{HoldForm[i
    ]==i,HoldForm[j]==j,Simplify[P[[i,j]]+Skew[P,j,i]]},{j,1,d
    },{i,1,j}]],3],TableSpacing->{2,2}];


(*Jacobi identity*)
Jacobi[i_,j_,k_,P_]:=Module[{\[Lambda]\[Lambda]=Array[Subscript
    [\[Lambda], #]&,r],\[Mu]\[Mu]=Array[Subscript[\[Mu], #]&,r
    ]},LambdaB[gen[[i]],LambdaB[gen[[j]],gen[[k]],P,\[Mu]\[Mu]],
    P,\[Lambda]\[Lambda]]-LambdaB[gen[[j]],LambdaB[gen[[i]],gen
    [[k]],P,\[Lambda]\[Lambda]],P,\[Mu]\[Mu]]-LambdaB[LambdaB[
    gen[[i]],gen[[j]],P,\[Lambda]\[Lambda]],gen[[k]],P,\[Lambda
    ]\[Lambda]+\[Mu]\[Mu]]];
JacobiCheck[P_]:=Table[Simplify[Jacobi[i,j,k,P]],{i,1,d},{j,1,d
    },{k,1,d}];
```

```
PrintJacobiCheck[P_]:=TableForm[Partition[Flatten[Table[{
    HoldForm[i]==i,HoldForm[j]==j,HoldForm[k]==k,Simplify[Jacobi
    [i,j,k,P]]},{k,1,d},{j,1,d},{i,1,d}]],4],TableSpacing
    ->{2,2}];


End[ ];
var:=Array[Subscript[x, #]&,r];
gen:=Array[Subscript[u, #][var]&,d];
EndPackage[];
```

B

Computations for H^2

B.1 0-th order deformations

We provide here the form of the 10 coefficients of the deformed brackets in terms of the two parameters of the Miura transformation (F_1, F_2)

B.1.1 Trivial deformations of (3.1)

$$A^1_{11} = 2\,\frac{\partial}{\partial p}F1 \qquad A^1_{12} = \frac{\partial}{\partial p}F_2 \qquad A^1_{22} = 0$$

$$A^2_{11} = 0 \qquad A^2_{12} = \frac{\partial}{\partial q}F_1 \qquad A^2_{22} = 2\,\frac{\partial}{\partial q}F_2$$

$$B^{11}_{(A)} = \frac{1}{2}\left(-\frac{\partial^2}{\partial p^2}F_2\right) \qquad B^{12}_{(A)} = \frac{1}{2}\left(-\frac{\partial^2}{\partial q\partial p}F_2\right)$$

$$B^{21}_{(A)} = \frac{1}{2}\left(\frac{\partial^2}{\partial q\partial p}F_1\right) \qquad B^{22}_{(A)} = \frac{1}{2}\left(\frac{\partial^2}{\partial q^2}F_1\right)$$

B.1.2 Trivial deformations of (3.2)

$$A^1_{11} = 2\,\frac{\partial}{\partial q}F1 \qquad A^1_{12} = \frac{\partial}{\partial q}F_2 + \frac{\partial}{\partial p}F_1 \qquad A^1_{22} = 2\,\frac{\partial}{\partial p}F_2$$

$$A^2_{11} = 0 \qquad A^2_{12} = \frac{\partial}{\partial q}F_1 \qquad A^2_{22} = 2\,\frac{\partial}{\partial q}F_2$$

$$B^{11}_{(A)} = \frac{1}{2}\left(-\frac{\partial^2}{\partial q\partial p}F_2 + \frac{\partial^2}{\partial p^2}F_1\right) \qquad B^{12}_{(A)} = \frac{1}{2}\left(-\frac{\partial^2}{\partial q^2}F_2 + \frac{\partial^2}{\partial q\partial p}F_1\right)$$

$$B^{21}_{(A)} = \frac{1}{2}\left(+\frac{\partial^2}{\partial q\partial p}F_1\right) \qquad B^{22}_{(A)} = \frac{1}{2}\left(\frac{\partial^2}{\partial q^2}F_1\right)$$

B.1.3 Trivial deformations of (3.3)

$$
\begin{aligned}
A^1_{11} &= -2\,F_1 + 2\,q\frac{\partial}{\partial q}F_1 + 4\,p\frac{\partial}{\partial p}F_1 \\
A^1_{12} &= -F_2 + q\frac{\partial}{\partial q}F_2 + q\frac{\partial}{\partial p}F_1 + 2\,p\frac{\partial}{\partial p}F_2 \\
A^1_{22} &= 2\,q\frac{\partial}{\partial p}F_2 \\
A^2_{11} &= 2\,p\frac{\partial}{\partial q}F_1 \\
A^2_{12} &= -F_1 + 2\,q\frac{\partial}{\partial q}F_1 + p\frac{\partial}{\partial q}F_2 + p\frac{\partial}{\partial p}F_1 \\
A^2_{22} &= -2\,F_2 + 4\,q\frac{\partial}{\partial q}F_2 + 2\,p\frac{\partial}{\partial p}F_2 \\
B^{11}_{(A)} &= \frac{1}{2}\left(-\frac{\partial}{\partial p}F_2 - q\frac{\partial^2}{\partial q\partial p}F_2 + q\frac{\partial^2}{\partial p^2}F1 - 2\,p\frac{\partial^2}{\partial p^2}F2\right) \\
B^{12}_{(A)} &= \frac{1}{2}\left(-q\frac{\partial^2}{\partial q^2}F_2 + \frac{\partial}{\partial p}F_1 + q\frac{\partial^2}{\partial q\partial p}F_1 - 2\,p\frac{\partial^2}{\partial q\partial p}F_2\right) \\
B^{21}_{(A)} &= \frac{1}{2}\left(-\frac{\partial}{\partial q}F_2 + 2\,q\frac{\partial^2}{\partial q\partial p}F_1 - p\frac{\partial^2}{\partial q\partial p}F_2 + p\frac{\partial^2}{\partial p^2}F_1\right) \\
B^{22}_{(A)} &= \frac{1}{2}\left(+\frac{\partial}{\partial q}F_1 + 2\,q\frac{\partial^2}{\partial q^2}F_1 - p\frac{\partial^2}{\partial q^2}F_2 + p\frac{\partial^2}{\partial q\partial p}F_1\right)
\end{aligned}
$$

B.2 First order deformations

We provide here the form of the coefficients of the compatible deformation (3.34), after imposing the skewsymmetry conditions (3.36), that can be algebraically solved for in the cocycle conditions.

B.2.1 Deformations of (3.1)

$$
\begin{aligned}
&B^{1,11}_{11} = 0 && B^{2,22}_{22} = 0 && B^{2,22}_{11} = 0 \\
&B^{1,11}_{22} = 0 && B^{2,21}_{11} = 0 && B^{1,12}_{22} = 0 \\
&\tilde{B}^{1,11}_{12} = 0 && \tilde{B}^{2,22}_{12} = 0 && B^{1,21}_{11} = -B^{2,11}_{11} \\
&B^{2,12}_{22} = -B^{1,22}_{22} && \tilde{B}^{1,12}_{12} = -B^{2,11}_{22} + \frac{\partial\tilde{A}^{11}}{\partial q} && \tilde{B}^{2,21}_{12} = -B^{1,22}_{11} - \frac{\partial\tilde{A}^{22}}{\partial p} \\
&\tilde{B}^{2,11}_{12} = -\frac{1}{2}B^{1,12}_{11} - \frac{\partial\tilde{A}^{12}}{\partial p} && \tilde{B}^{1,22}_{12} = -\frac{1}{2}B^{2,21}_{22} + \frac{\partial\tilde{A}^{12}}{\partial q}
\end{aligned}
$$

$$\tilde{B}_{12}^{1,21} = \frac{\partial \tilde{A}^{12}}{\partial p} \qquad \tilde{B}_{12}^{2,12} = -\frac{\partial \tilde{A}^{12}}{\partial q}$$

$$B_{11}^{2,12} = B_{11}^{1,22} + 2\frac{\partial \tilde{A}^{22}}{\partial q} \qquad B_{22}^{1,21} = B_{22}^{2,11} - 2\frac{\partial \tilde{A}^{11}}{\partial p}$$

$$\tilde{D}^{11,1} = 0 \qquad \tilde{D}^{22,2} = 0$$

$$\tilde{D}^{12,1} = -\frac{1}{8}B_{11}^{1,12} \qquad \tilde{D}^{12,2} = \frac{1}{8}B_{22}^{2,21}$$

$$\tilde{D}^{22,1} = \frac{1}{2}\left(B_{22}^{2,11} - \frac{\partial \tilde{A}^{11}}{\partial q}\right) \qquad \tilde{D}^{22,2} = -\frac{1}{2}\left(B_{11}^{1,22} + \frac{\partial \tilde{A}^{22}}{\partial q}\right)$$

$$\tilde{C}^{11,11} = 0 \qquad \tilde{C}^{22,22} = 0$$

$$\tilde{C}^{11,12} = \frac{1}{4}\left(\frac{\partial B_{22}^{2,11}}{\partial p} - \frac{\partial^2 \tilde{A}^{11}}{\partial p \partial q}\right) \qquad \tilde{C}^{21,22} = -\frac{1}{4}\left(\frac{\partial B_{11}^{1,22}}{\partial q} + \frac{\partial^2 \tilde{A}^{22}}{\partial p \partial q}\right)$$

$$\tilde{C}^{11,21} = -\frac{1}{8}\frac{\partial B_{11}^{1,12}}{\partial p} \qquad \tilde{C}^{12,22} = \frac{1}{8}\frac{\partial B_{22}^{2,21}}{\partial p}$$

$$\tilde{C}^{12,12} = \frac{1}{2}\left(\frac{\partial B_{22}^{2,11}}{\partial q} - \frac{\partial^2 \tilde{A}^{11}}{\partial q^2}\right) \qquad \tilde{C}^{21,21} = -\frac{1}{2}\left(\frac{\partial B_{11}^{1,22}}{\partial p} + \frac{\partial^2 \tilde{A}^{22}}{\partial p^2}\right)$$

$$\tilde{C}^{12,21} = 0 \qquad \tilde{C}^{11,22} = \frac{1}{8}\left(\frac{\partial B_{22}^{2,21}}{\partial p} - \frac{\partial B_{11}^{2,12}}{\partial q}\right)$$

B.2.2 Deformations of (3.2)

$$B_{22}^{1,11} = 0 \qquad B_{11}^{2,21} = 0$$

$$B_{11}^{1,12} = 0 \qquad B_{11}^{2,22} = 0$$

$$B_{22}^{2,22} = 0 \qquad B_{22}^{1,21} = -\left(B_{22}^{1,12} + B_{22}^{2,11}\right)$$

$$B_{11}^{1,22} = -\frac{\partial \tilde{A}^{22}}{\partial q} \qquad B_{11}^{2,11} = \frac{\partial \tilde{A}^{22}}{\partial p}$$

$$B_{11}^{2,12} = \frac{\partial \tilde{A}^{22}}{\partial q} \qquad B_{22}^{1,22} - \left(B_{22}^{2,12} + B_{22}^{2,21}\right)$$

$$\tilde{B}_{11}^{2,21} = 0 \qquad \tilde{B}_{12}^{2,22} = 0$$

$$\tilde{B}_{12}^{1,12} = -\frac{1}{2}B_{11}^{1,11} \qquad \tilde{B}_{12}^{1,11} = -\frac{1}{2}B_{22}^{1,12}$$

$$\tilde{B}_{12}^{1,22} = -B_{11}^{2,11} + \frac{\partial \tilde{A}^{12}}{\partial q} \qquad \tilde{B}_{12}^{2,21} = \frac{1}{2}\left(B_{11}^{1,21} + \frac{\partial \tilde{A}^{22}}{\partial p}\right)$$

$$\tilde{B}_{12}^{2,11} = B_{22}^{2,12} + \frac{1}{2}B_{22}^{2,21} - \frac{\partial \tilde{A}^{11}}{\partial q} + \frac{\partial \tilde{A}^{12}}{\partial p} \qquad \tilde{B}_{12}^{2,12} = B_{22}^{2,11} - \frac{1}{2}B_{11}^{1,21} - \frac{\partial \tilde{A}^{12}}{\partial q} - \frac{1}{2}\frac{\partial \tilde{A}^{22}}{\partial p}$$

$$\tilde{D}^{11,1} = \frac{1}{4}B_{22}^{1,12} \qquad \tilde{D}^{11,2} = -\frac{1}{4}B_{11}^{1,11}$$

$$\tilde{D}^{12,1} = \frac{1}{8}\left(B_{11}^{1,11} - B_{22}^{2,21}\right) \qquad \tilde{D}^{12,2} = -\frac{1}{8}\left(B_{11}^{1,21} + \frac{\partial \tilde{A}^{22}}{\partial p}\right)$$

$$\tilde{D}^{22,1} = -\frac{1}{2}\left(B_{11}^{1,21} + \frac{\partial \tilde{A}^{22}}{\partial p}\right) \qquad \tilde{D}^{22,2} = 0$$

$$\tilde{C}^{11,11} = \frac{1}{4}\frac{\partial B_{22}^{1,12}}{\partial p} \qquad \tilde{C}^{12,22} = -\frac{1}{8}\left(\frac{\partial B_{11}^{1,21}}{\partial q} + \frac{\partial^2 \tilde{A}^{22}}{\partial p \partial q}\right)$$

$$\tilde{C}^{21,22} = \frac{1}{8}\left(\frac{\partial B_{11}^{1,21}}{\partial q} + \frac{\partial^2 \tilde{A}^{22}}{\partial p \partial q}\right) \qquad \tilde{C}^{22,22} = 0$$

$$\tilde{C}^{11,21} = \frac{1}{8}\left(\frac{\partial B_{11}^{1,11}}{\partial p} - \frac{\partial B_{22}^{2,21}}{\partial p}\right) \qquad \tilde{C}^{11,12} = \frac{1}{8}\left(\frac{\partial B_{22}^{1,12}}{\partial q} - \frac{\partial B_{11}^{1,11}}{\partial p}\right)$$

$$\tilde{C}^{12,12} = -\frac{1}{4}\frac{\partial B_{11}^{1,11}}{\partial q} \qquad \tilde{C}^{21,21} = \frac{1}{4}\left(\frac{\partial B_{11}^{1,21}}{\partial p} + \frac{\partial^2 \tilde{A}^{22}}{\partial p^2}\right)$$

$$\tilde{C}^{11,22} = \frac{1}{8}\left(2\frac{\partial B_{11}^{1,11}}{\partial q} - 2\frac{\partial B_{22}^{2,12}}{\partial q} - \frac{\partial B_{22}^{2,21}}{\partial q} - 2\frac{\partial B_{11}^{1,21}}{\partial p} + 2\frac{\partial^2 \tilde{A}^{11}}{\partial q^2} - 4\frac{\partial^2 \tilde{A}^{12}}{\partial p \partial q}\right)$$

$$\tilde{C}^{12,21} = -\frac{1}{8}\left(2\frac{\partial B_{22}^{2,12}}{\partial q} + \frac{\partial B_{11}^{1,11}}{\partial q} - \frac{\partial B_{11}^{1,21}}{\partial p} + 2\frac{\partial^2 \tilde{A}^{22}}{\partial p^2} + 2\frac{\partial^2 \tilde{A}^{11}}{\partial q^2} - 4\frac{\partial^2 \tilde{A}^{12}}{\partial p \partial q}\right)$$

B.2.3 Deformation of (3.3)

$$B_{11}^{1,11} = \frac{q^2}{p^2}\left(\frac{\partial \tilde{A}^{22}}{\partial q} + 2B_{11}^{1,22}\right) + \frac{q}{p}\left(-2\frac{\partial \tilde{A}^{12}}{\partial q} + \frac{5}{2}\frac{\partial \tilde{A}^{22}}{\partial p} + 2B_{11}^{1,21} + B_{11}^{2,11}\right) +$$
$$- \left(-\frac{\partial \tilde{A}^{11}}{\partial q} + \frac{\partial \tilde{A}^{12}}{\partial p} + B_{22}^{1,22} + B_{22}^{2,12}\right) + \frac{p\left(\frac{\partial \tilde{A}^{11}}{\partial p} - 2B_{22}^{2,11}\right)}{2q} +$$
$$+ \frac{\tilde{A}^{12}}{p} - \frac{2q\tilde{A}^{22}}{p^2} - \frac{\tilde{A}^{11}}{q}$$

$$B_{22}^{2,22} = \frac{p^2}{q^2}\left(-\frac{\partial \tilde{A}^{11}}{\partial p} + 2B_{22}^{2,11}\right) + \frac{p}{q}\left(-\frac{5}{2}\frac{\partial \tilde{A}^{11}}{\partial q} + 2\frac{\partial \tilde{A}^{12}}{\partial p} + B_{22}^{1,22} + 2B_{22}^{2,12}\right) +$$
$$- \left(\frac{\partial \tilde{A}^{22}}{\partial p} - \frac{\partial \tilde{A}^{12}}{\partial q} + B_{11}^{1,21} + B_{11}^{2,11}\right) + \frac{q\left(-\frac{\partial \tilde{A}^{22}}{\partial q} - 2B_{11}^{1,22}\right)}{2p} +$$
$$+ \frac{2p\tilde{A}^{11}}{q^2} - \frac{\tilde{A}^{12}}{q} + \frac{\tilde{A}^{22}}{p}$$

$$B^{1,11}_{22} = 0$$
$$B^{2,22}_{11} = 0$$
$$B^{1,12}_{11} = -\frac{2p}{q}B^{1,11}_{11}$$
$$B^{2,21}_{22} = -\frac{2q}{p}B^{2,22}_{22}$$
$$B^{2,12}_{11} = 2\frac{p^2}{q^2}B^{1,11}_{11} - \frac{p}{q}\left(2B^{1,21}_{11} + 2B^{2,11}_{11}\right) - B^{1,22}_{11}$$
$$B^{1,21}_{22} = 2\frac{q^2}{p^2}B^{2,22}_{22} - \frac{q}{p}\left(2B^{1,22}_{22} + 2B^{2,12}_{22}\right) - B^{2,11}_{22}$$
$$B^{2,21}_{11} = \frac{p}{q}\left(B^{1,21}_{11} + B^{2,11}_{11}\right) - \frac{p^2}{q^2}B^{1,11}_{11}$$
$$B^{1,12}_{22} = \frac{q}{p}\left(B^{1,22}_{22} + B^{2,12}_{22}\right) - \frac{q^2}{p^2}B^{2,22}_{22}$$
$$\tilde{B}^{1,11}_{12} = -\frac{1}{2}\left(\frac{q}{p}\left(B^{1,22}_{22} + B^{2,12}_{22}\right) - \frac{q^2}{p^2}B^{2,22}_{22}\right)$$
$$\tilde{B}^{2,22}_{12} = -\frac{1}{2}\left(\frac{p}{q}(B^{1,21}_{11} + B^{2,11}_{11}) - \frac{p^2}{q^2}B^{1,11}_{11}\right)$$
$$\tilde{B}^{1,12}_{12} = \frac{1}{2}\left(-\frac{2q}{p}B^{2,22}_{22} - B^{1,11}_{11} + 2B^{1,22}_{22} + 2B^{2,12}_{22}\right)$$
$$\tilde{B}^{2,21}_{12} = \frac{1}{2}\left(-\frac{2p}{q}B^{1,11}_{11} + 2B^{1,21}_{11} + 2B^{2,11}_{11} - B^{2,22}_{22}\right)$$
$$\tilde{B}^{1,22}_{12} = -\frac{\partial\tilde{A}^{12}}{\partial q} + 2\frac{\partial\tilde{A}^{22}}{\partial p} + \frac{q}{p}\frac{\partial\tilde{A}^{22}}{\partial q} - \frac{2}{p}\tilde{A}^{22} + (2B^{1,21}_{11} + B^{2,11}_{11}) - \frac{3p}{2q}B^{1,11}_{11} + \frac{2q}{p}B^{1,22}_{11}$$
$$\tilde{B}^{2,11}_{12} = -2\frac{\partial\tilde{A}^{11}}{\partial q} + \frac{\partial\tilde{A}^{12}}{\partial p} - \frac{p}{q}\frac{\partial\tilde{A}^{11}}{\partial p} + \frac{2}{q}\tilde{A}^{11} + (B^{1,22}_{22} + 2B^{2,12}_{22}) + \frac{2p}{q}B^{2,11}_{22} - \frac{3q}{2p}B^{2,22}_{22}$$
$$\tilde{B}^{1,21}_{12} = \frac{1}{2}\left(-\frac{4\tilde{A}^{11}}{q} + 4\frac{\partial\tilde{A}^{11}}{\partial q} - 2\frac{\partial\tilde{A}^{12}}{\partial p} + \frac{2p\frac{\partial\tilde{A}^{11}}{\partial p}}{q} - \frac{4pB^{2,11}_{22}}{q} + \frac{3qB^{2,22}_{22}}{p} + \right.$$
$$\left. + 2B^{1,11}_{11} - 3B^{1,22}_{22} - 5B^{2,12}_{22}\right)$$
$$\tilde{B}^{2,12}_{12} = \frac{1}{2}\left(\frac{4\tilde{A}^{22}}{p} + 2\frac{\partial\tilde{A}^{12}}{\partial q} - 4\frac{\partial\tilde{A}^{22}}{\partial p} - \frac{2q\frac{\partial\tilde{A}^{22}}{\partial q}}{p} + \frac{3pB^{1,11}_{11}}{q} - \frac{4qB^{1,22}_{11}}{p} + \right.$$
$$\left. - 5B^{1,21}_{11} - 3B^{2,11}_{11} + 2B^{2,22}_{22}\right)$$

$$\tilde{C}_{12}^{11,11} = \frac{1}{4}\frac{\partial}{\partial p}\left(\frac{q(B_{22}^{1,22}+B_{22}^{2,12})}{p} - \left(\frac{q}{p}\right)^2 B_{22}^{2,22}\right)$$

$$\tilde{C}_{12}^{22,22} = -\frac{1}{4}\frac{\partial}{\partial q}\left(\frac{p(B_{11}^{1,21}+B_{11}^{2,11})}{q} - \left(\frac{p}{q}\right)^2 B_{11}^{1,11}\right)$$

$$\begin{aligned}\tilde{C}_{12}^{11,12} = -\frac{1}{8}\Bigg(&\frac{\partial}{\partial q}\left(\left(\frac{q}{p}\right)^2 B_{22}^{2,22} - \frac{q(B_{22}^{1,22}+B_{22}^{2,12})}{p}\right)\\ &+\frac{\partial}{\partial p}\left(-\frac{2qB_{22}^{2,22}}{p} + B_{11}^{1,11} + 2B_{22}^{1,22} + 2B_{22}^{2,12}\right)\Bigg)\end{aligned}$$

$$\begin{aligned}\tilde{C}_{12}^{21,22} = \frac{1}{8}\Bigg(&\frac{\partial}{\partial q}\left(-\frac{2pB_{11}^{1,11}}{q} + 2B_{11}^{1,21} + 2B_{11}^{2,11} + B_{22}^{2,22}\right)+\\ &+\frac{\partial}{\partial p}\left(\left(\frac{p}{q}\right)^2 B_{11}^{1,11} - \frac{p(B_{11}^{1,21}+B_{11}^{2,11})}{q}\right)\Bigg)\end{aligned}$$

$$\tilde{C}_{12}^{12,12} = -\frac{1}{4}\frac{\partial}{\partial q}\left(-\frac{2qB_{22}^{2,22}}{p} + B_{11}^{1,11} + 2B_{22}^{1,22} + 2B_{22}^{2,12}\right)$$

$$\tilde{C}_{12}^{21,21} = \frac{1}{4}\frac{\partial}{\partial p}\left(-\frac{2pB_{11}^{1,11}}{q} + 2B_{11}^{1,21} + 2B_{11}^{2,11} + B_{22}^{2,22}\right)$$

$$\tilde{C}_{12}^{12,22} = -\frac{1}{8}\frac{\partial(B_{11}^{1,21}+B_{11}^{2,11}+2B_{22}^{2,22})}{\partial q}$$

$$\tilde{C}_{12}^{11,21} = \frac{1}{8}\frac{\partial(2B_{11}^{1,11}+B_{22}^{1,22}+B_{22}^{2,12})}{\partial p}$$

$$\begin{aligned}\tilde{C}_{12}^{11,22} = \frac{1}{4}\Bigg(&2\left(\frac{\partial^2\tilde{A}^{11}}{\partial q^2} - \frac{\partial^2\tilde{A}^{22}}{\partial p^2}\right) + \frac{p}{q}\frac{\partial B_{11}^{1,11}}{\partial p}+\\ &-\frac{\partial}{\partial p}\left(\frac{q\left(\frac{\partial\tilde{A}^{22}}{\partial q}+2B_{11}^{1,22}\right)}{p} - \frac{2\tilde{A}^{22}}{p} - \frac{3pB_{11}^{1,11}}{2q} + 3B_{11}^{1,21} + 2B_{11}^{2,11}\right)\\ &+\frac{\partial}{\partial q}\left(\frac{p\left(\frac{\partial\tilde{A}^{11}}{\partial p}-2B_{22}^{2,11}\right)}{q} - \frac{2\tilde{A}^{11}}{q} + \frac{3qB_{22}^{2,22}}{2p} + 2B_{11}^{1,11} - B_{22}^{1,22} - 2B_{22}^{2,12}\right)\Bigg)\end{aligned}$$

$$\begin{aligned}\tilde{C}_{12}^{12,21} = \frac{1}{8}\Bigg(&4\frac{\partial^2\tilde{A}^{11}}{\partial q^2} - 4\frac{\partial^2\tilde{A}^{22}}{\partial p^2} + \frac{2q\frac{\partial B_{22}^{2,22}}{\partial q}}{p}\\ &+\frac{\partial}{\partial p}\left(-\frac{2q\left(\frac{\partial\tilde{A}^{22}}{\partial q}+2B_{11}^{1,22}\right)}{p} + \frac{4\tilde{A}^{22}}{p} + \frac{3pB_{11}^{1,11}}{q} - 5B_{11}^{1,21} - 3B_{11}^{2,11} + 2B_{22}^{2,22}\right)\\ &+\frac{\partial}{\partial q}\left(\frac{2p\left(\frac{\partial\tilde{A}^{11}}{\partial p}-2B_{22}^{2,11}\right)}{q} - \frac{4\tilde{A}^{11}}{q} + \frac{3qB_{22}^{2,22}}{p} + 2B_{11}^{1,11} - 3B_{22}^{1,22} - 5B_{22}^{2,12}\right)\Bigg)\end{aligned}$$

$$\tilde{D}^{11,1}_{12} = \frac{1}{4}\left(\frac{q(B^{1,22}_{22} + B^{2,12}_{22})}{q} - \frac{q^2 B^{2,22}_{22}}{q^2}\right)$$
$$\tilde{D}^{22,2}_{12} = -\frac{1}{4}\left(\frac{q(B^{1,21}_{11} + B^{2,11}_{11})}{q} - \frac{q^2 B^{1,11}_{11}}{q^2}\right)$$
$$\tilde{D}^{11,2}_{12} = -\frac{1}{4}\left(-\frac{2qB^{2,22}_{22}}{q} + B^{1,11}_{11} + 2B^{1,22}_{22} + 2B^{2,12}_{22}\right)$$
$$\tilde{D}^{22,1}_{12} = \frac{1}{4}\left(-\frac{2qB^{1,11}_{11}}{q} + 2B^{1,21}_{11} + 2B^{2,11}_{11} + B^{2,22}_{22}\right)$$
$$\tilde{D}^{12,1}_{12} = \frac{1}{8}(2B^{1,11}_{11} + B^{1,22}_{22} + B^{2,12}_{22})$$
$$\tilde{D}^{12,2}_{12} = -\frac{1}{8}(B^{1,21}_{11} + B^{2,11}_{11} + 2B^{2,22}_{22})$$

B.3 Second order deformations

In this section we report the equations that constitute the Janet basis, as output by Maple; because of this, the naming convention is different that the one presented in Paragraph 3.3.3. Let us explain the naming convention – without providing the explicit form for all the 172 variables – by two examples.

$$\mathit{b21112p1y} \rightsquigarrow B^{12,21}_{12}$$
$$\mathit{f012p1xp2xy} \rightsquigarrow F^{122,11}_{12}$$

The first two characters denote the name of the coefficient in lower case (A, B, ...) and the related powers of λ in the definition (3.46). If the number of λ is greater than 0, the following two digits count the powers of λ_1 and λ_2; then there follow the two digits corresponding to the i-th and j-th component of the Poisson bivector. Finally, the expressions like *p1y* or *p2xy* denote the jets variables for which the coefficients are multiplied in the definition, *p1* corresponding to p and *p2* corresponding to q. This means that $\mathit{p1x} \mapsto \partial_1 p_1$, hence in the indices we will write 11, and so on.

B.3.1 Janet basis of the cocycle condition for a deformation of (3.1)

We provide here the basis as a list of expression that vanish for the solutions of the system and not as a set of equations. According to this rules, the Janet basis of the cocycle condition for brackets compatible with (3.1) is as follows.

$$\mathit{g012p1xp2xp1y} - 2\,\mathit{g012p1xp1xp2y}, \mathit{g012p2xp1yp1y} - 1/2\,\mathit{g012p1xp1yp2y}, \mathit{g012p1yp1yp1y},$$
$$\mathit{g012p1yp1yp2y}, \mathit{g012p1yp2yp2y}, \mathit{g012p2yp2yp2y}, \mathit{f012p1yp12x} - 1/3\,\mathit{f012p1xp1xy},$$
$$\mathit{f012p1yp22x} - 1/3\,\mathit{f012p2xp1xy}, \mathit{f012p1yp1xy} - \mathit{f012p1xp12y}, \mathit{f012p1yp2xy} - \mathit{f012p2xp12y},$$
$$\mathit{f012p1yp12y}, \mathit{f012p1yp22y}, \mathit{f012p2yp12x} - 1/3\,\mathit{f012p2xp1xy}, \mathit{f012p2yp1xy} - \mathit{f012p2xp12y},$$

$$f012p2yp12y, f012p2yp22y, e012p13y, e012p23y, d10111p22y, -2\, e012p13x + d11012p12x,$$
$$d11012p1xy - 3/2\, e012p12xy, -e012p1x2y + d11012p12y, -e012p2x2y + d11012p22y,$$
$$d11022p12x, d11022p1xy + d11012p22x - 3\, e012p23x, d11022p12y + d11012p2xy - 2\, e012p22xy,$$
$$c10111p1yp1y, c10111p1yp2y, c10111p2yp2y, c11012p1xp1x - 3/4\, f012p1xp12x,$$
$$c11012p1xp1y - 2/3\, f012p1xp1xy, c11012p2xp1y - 2/3\, f012p2xp1xy, c11012p1yp1y - 1/2\, f012p1xp12y,$$
$$c11012p1yp2y - f012p2xp12y, c11012p2yp2y - 1/2\, f012p2yp2xy,$$
$$c10112p1xp1y + c10111p1xp2x + c11011p2xp1y - 2\, c11011p1xp2y - f012p1xp12y,$$
$$c10112p1yp1y + c10111p2xp1y - 1/2\, c10111p1xp2y - 1/2\, c11011p1yp2y, c11022p1xp1x,$$
$$c11022p1xp2x, c11022p1xp1y + c11012p1xp2x - 2\, f012p1xp22x, c11022p2xp2x,$$
$$c11022p2xp1y - 2\, c11022p1xp2y - 2\, c11012p2xp2x + 2\, f012p2xp22x,$$
$$c11022p1yp1y + c11012p1xp2y - f012p1xp2xy,$$
$$c11022p1yp2y + c11012p2xp2y + 2\, f012p2yp22x - 2\, f012p2xp2xy,$$
$$c10122p2xp1y - c10122p1xp2y - c11012p2xp2y + 2\, f012p2yp22x,$$
$$b22012p2x - d10122p12x - d11012p22x + 2\, e012p23x, b22012p1y - 1/2\, e012p12xy,$$
$$b22012p2y - d11012p2xy + e012p22xy, b21112p1y - d10112p1xy + d10111p22x,$$
$$b20212p2x - b21112p2y - d10122p12y + 3\, e012p2x2y, b20212p1y - 2\, d10112p12y + d10111p2xy,$$
$$\frac{\partial}{\partial q} f012p1xp12x - 4/3\, g012p1xp1xp2x, \frac{\partial}{\partial q} f012p1xp1xy - 3\, g012p1xp1xp2y, \frac{\partial}{\partial q} f012p1xp12y - g012p1xp1yp2y,$$
$$\frac{\partial}{\partial q} f012p2xp1xy - 3\, g012p2xp2xp1y, \frac{\partial}{\partial q} f012p2xp12y - g012p2xp1yp2y, \frac{\partial}{\partial q} f012p2yp22x - g012p2xp2xp2y,$$
$$\frac{\partial}{\partial q} f012p2yp2xy - 2\, g012p2xp2yp2y, \frac{\partial}{\partial q} e012p13x - 1/2\, f012p2xp12x,$$
$$-2/3\, c11022p1xp2y - 2/3\, c11012p2xp2x + \frac{\partial}{\partial q} e012p23x + 1/3\, f012p2xp22x, \frac{\partial}{\partial q} e012p12xy - 2/3\, f012p2xp1xy,$$
$$\frac{\partial}{\partial q} e012p22xy + f012p2yp22x - f012p2xp2xy, \frac{\partial}{\partial q} e012p1x2y - f012p2xp12y, \frac{\partial}{\partial q} e012p2x2y - f012p2xp22y,$$
$$2/3\, c10112p1xp1x - 5/6\, c11011p1xp2x + \frac{\partial}{\partial q} d11011p12x - 2/9\, f012p1xp1xy,$$
$$-\, c10111p1xp2x - c11011p2xp1y + \frac{\partial}{\partial q} d11011p1xy,$$
$$-\, 1/2\, c10111p2xp1y - 1/2\, c10111p1xp2y - 1/2\, c11011p1yp2y + \frac{\partial}{\partial q} d11011p12y,$$
$$-\, 1/2\, c10111p1xp2x - 1/2\, c11011p2xp1y + 1/2\, c11011p1xp2y + \frac{\partial}{\partial q} d10111p12x,$$
$$-\, c10111p2xp2x + \frac{\partial}{\partial q} d10111p22x, -c10111p2xp1y + \frac{\partial}{\partial q} d10111p1xy,$$
$$-\, c10111p2xp2y + \frac{\partial}{\partial q} d10111p2xy, \frac{\partial}{\partial q} d10111p12y,$$
$$-\, c11022p1xp2y - 2\, c11012p2xp2x + \frac{\partial}{\partial q} d11012p22x + f012p2xp22x,$$
$$-\, c11012p2xp2y + \frac{\partial}{\partial q} d11012p2xy + 2\, f012p2yp22x - f012p2xp2xy,$$
$$-\, c10112p2xp1y - c10111p2xp2x + \frac{\partial}{\partial q} d10112p1xy,$$
$$-\, 1/2\, c10112p1yp2y - 1/2\, c10111p2xp2y + \frac{\partial}{\partial q} d10112p12y, \frac{\partial}{\partial q} d11022p22x,$$
$$-\, c11022p2xp2y + \frac{\partial}{\partial q} d11022p2xy, -c11022p2yp2y + \frac{\partial}{\partial q} d11022p22y,$$

$$-1/2\, c10122p1xp2x + 1/6\, c11022p1xp2y - 1/3\, c11012p2xp2x + \frac{\partial}{\partial q} d10122p12x - 1/3\, f012p2xp22x,$$

$$-c10122p2xp2x - c11022p2xp2y + \frac{\partial}{\partial q} d10122p22x, -c10122p1xp2y - c11012p2xp2y + \frac{\partial}{\partial q} d10122p1xy,$$

$$-c10122p2xp2y - 2\, c11022p2yp2y + \frac{\partial}{\partial q} d10122p2xy, -1/2\, c10122p1yp2y + \frac{\partial}{\partial q} d10122p12y - f012p2yp2xy$$

$$\frac{\partial}{\partial q} c11022p1xp2y + \frac{\partial}{\partial q} c11012p2xp2x - 2\, \frac{\partial}{\partial q} f012p2xp22x + 3\, g012p2xp2xp2x,$$

$$\frac{\partial}{\partial q} b22012p1x - c10122p1xp1x - 3/2\, f012p2xp12x + f012p1xp22x,$$

$$\frac{\partial}{\partial q} b21112p1x - c10122p1xp1y - c10112p1xp2x - 4/3\, f012p2xp1xy + 2\, f012p1xp2xy,$$

$$\frac{\partial}{\partial q} b21112p2x - c10122p1xp2y - 2\, c10112p2xp2x - c11012p2xp2y - 2\, f012p2yp22x + 2\, f012p2xp2xy,$$

$$\frac{\partial}{\partial q} b21112p2y - c10112p2xp2y, \frac{\partial}{\partial q} b20212p1x - c10122p1yp1y - c10112p1xp2y - f012p2xp12y + 3\, f012p1xp22y,$$

$$-2\, c10112p2yp2y + \frac{\partial}{\partial q} b20212p2y, \frac{\partial}{\partial q} a33011 - 2\, d10112p12x + e012p12xy,$$

$$\frac{\partial}{\partial q} a32111 - 2\, d10112p1xy + 2\, e012p1x2y, -2\, d10112p12y + \frac{\partial}{\partial q} a31211,$$

$$\frac{\partial}{\partial q} a30311, \frac{\partial}{\partial q} a33012 - d10122p12x + e012p23x, \frac{\partial}{\partial q} a32112 - d10122p1xy - d10112p22x + 2\, e012p22xy,$$

$$\frac{\partial}{\partial q} a31212 - d10122p12y - d10112p2xy + 3\, e012p2x2y,$$

$$-d10112p22y + \frac{\partial}{\partial q} a30312, \frac{\partial}{\partial q} a33022 + 2/3\, d11022p22x, -2\, d10122p22x + 2\, d11022p2xy + \frac{\partial}{\partial q} a32122,$$

$$-2\, d10122p2xy + 6\, d11022p22y + \frac{\partial}{\partial q} a31222, \frac{\partial}{\partial p} g012p1xp1xp2x - 3\, \frac{\partial}{\partial q} g012p1xp1xp1x,$$

$$\frac{\partial}{\partial p} g012p1xp1xp2y - \frac{\partial}{\partial q} g012p1xp1xp1y, \frac{\partial}{\partial p} g012p1xp1yp2y - 2\, \frac{\partial}{\partial q} g012p1xp1yp1y,$$

$$\frac{\partial}{\partial p} g012p2xp2xp1y - \frac{\partial}{\partial q} g012p1xp1xp2y, \frac{\partial}{\partial p} g012p2xp2xp2y + \frac{\partial}{\partial q} g012p2xp2xp1y - \frac{\partial}{\partial q} g012p1xp2xp2y,$$

$$\frac{\partial}{\partial p} g012p2xp1yp2y - \frac{\partial}{\partial q} g012p1xp1yp2y, \frac{\partial}{\partial p} g012p2xp2yp2y - \frac{\partial}{\partial q} g012p1xp2yp2y,$$

$$\frac{\partial}{\partial p} f012p1xp12x - 4\, g012p1xp1xp1x, \frac{\partial}{\partial p} f012p1xp1xy - 3\, g012p1xp1xp1y,$$

$$\frac{\partial}{\partial p} f012p1xp12y - 2\, g012p1xp1yp1y, \frac{\partial}{\partial p} f012p2xp12x - 4/3\, g012p1xp1xp2x,$$

$$\frac{\partial}{\partial p} f012p2xp22x + 3/2\, \frac{\partial}{\partial q} f012p2xp12x - \frac{\partial}{\partial q} f012p1xp22x - 2\, g012p1xp2xp2x, \frac{\partial}{\partial p} f012p2xp1xy - 3\, g012p1xp1xp2y,$$

$$\frac{\partial}{\partial p} f012p2xp2xy - \frac{\partial}{\partial q} f012p1xp2xy + 2\, g012p2xp2xp1y - g012p1xp2xp2y, \frac{\partial}{\partial p} f012p2xp12y - g012p1xp1yp2y,$$

$$\frac{\partial}{\partial p} f012p2xp22y - \frac{\partial}{\partial q} f012p1xp22y, \frac{\partial}{\partial p} f012p2yp22x + g012p2xp2xp1y - g012p1xp2xp2y,$$

$$\frac{\partial}{\partial p} f012p2yp2xy - 2\, g012p1xp2yp2y, \frac{\partial}{\partial p} e012p13x - 1/2\, f012p1xp12x,$$

$$\frac{\partial}{\partial p} e012p23x + 1/2\, f012p2xp12x - f012p1xp22x, \frac{\partial}{\partial p} e012p12xy - 2/3\, f012p1xp1xy,$$

$$\frac{\partial}{\partial p} e012p22xy + 1/3\, f012p2xp1xy - f012p1xp2xy, \frac{\partial}{\partial p} e012p1x2y - f012p1xp12y, \frac{\partial}{\partial p} e012p2x2y - f012p1xp22y,$$

$$- 2\, c10111p1xp1x - c11011p1xp1y + \frac{\partial}{\partial p} d11011p1xy,$$
$$- c10111p1xp1y - c11011p1yp1y + \frac{\partial}{\partial p} d11011p12y, -c10111p1xp2y + \frac{\partial}{\partial p} d10111p2xy,$$
$$- c10111p1xp1x + \frac{\partial}{\partial p} d10111p12x, -c10111p1xp1y + \frac{\partial}{\partial p} d10111p1xy,$$
$$- 1/2\, c10111p1xp2x + 1/2\, c11011p2xp1y - 1/2\, c11011p1xp2y + \frac{\partial}{\partial p} d10111p22x,$$
$$\frac{\partial}{\partial p} d10111p12y, -c11012p1xp2x + \frac{\partial}{\partial p} d11012p22x + 3/2\, f012p2xp12x - f012p1xp22x,$$
$$- c11012p1xp2y + \frac{\partial}{\partial p} d11012p2xy + 2/3\, f012p2xp1xy - f012p1xp2xy,$$
$$- 2/3\, c10112p1xp1x - 1/6\, c11011p1xp2x + \frac{\partial}{\partial p} d10112p12x - 1/9\, f012p1xp1xy,$$
$$-c10112p1xp2x + \frac{\partial}{\partial p} d10112p22x + 3/2\, \frac{\partial}{\partial q} d10112p12x - 1/2\, \frac{\partial}{\partial q} d11011p22x - 1/6\, f012p2xp1xy,$$
$$1/2\, c10111p1xp2x + 1/2\, c11011p2xp1y - 3/2\, c11011p1xp2y + \frac{\partial}{\partial p} d10112p1xy - f012p1xp12y,$$
$$1/2\, c10112p2xp1y - c10112p1xp2y + c10111p2xp2x + \frac{\partial}{\partial p} d10112p2xy - 1/2\, \frac{\partial}{\partial q} d11011p2xy +$$
$$-1/2\, f012p2xp12y, 1/2\, c10111p2xp1y - 1/2\, c10111p1xp2y - 1/2\, c11011p1yp2y + \frac{\partial}{\partial p} d10112p12y,$$
$$- 1/4\, c10112p1yp2y + 1/4\, c10111p2xp2y + \frac{\partial}{\partial p} d10112p22y - 1/2\, \frac{\partial}{\partial q} d11011p22y,$$
$$\frac{\partial}{\partial p} d11022p22x, -2\, c11022p1xp2y - 2\, c11012p2xp2x + \frac{\partial}{\partial p} d11022p2xy + 2\, f012p2xp22x,$$
$$\frac{\partial}{\partial p} d11022p22y + 2\, f012p2yp22x - f012p2xp2xy, -c10122p1xp1x + \frac{\partial}{\partial p} d10122p12x - 1/2\, f012p2xp12x,$$
$$- 1/2\, c10122p1xp2x - 3/2\, c11022p1xp2y - c11012p2xp2x + \frac{\partial}{\partial p} d10122p22x + f012p2xp22x,$$
$$- c10122p1xp1y + \frac{\partial}{\partial p} d10122p1xy - 2/3\, f012p2xp1xy, -c10122p1xp2y + \frac{\partial}{\partial p} d10122p2xy +$$
$$+ 4\, f012p2yp22x - 2\, f012p2xp2xy, -c10122p1yp1y + \frac{\partial}{\partial p} d10122p12y - f012p2xp12y,$$
$$- 1/2\, c10122p1yp2y + \frac{\partial}{\partial p} d10122p22y + f012p2yp2xy - 2\, f012p2xp22y,$$
$$\frac{\partial}{\partial p} c11011p1xp2x - 2\, \frac{\partial}{\partial q} c11011p1xp1x, \frac{\partial}{\partial p} c11011p1xp2y - \frac{\partial}{\partial q} c11011p1xp1y,$$
$$\frac{\partial}{\partial p} c10111p1xp2x - 2\, \frac{\partial}{\partial q} c10111p1xp1x + \frac{\partial}{\partial p} c11011p2xp1y - \frac{\partial}{\partial q} c11011p1xp1y,$$
$$\frac{\partial}{\partial p} c10111p1xp2y - \frac{\partial}{\partial q} c10111p1xp1y + \frac{\partial}{\partial p} c11011p1yp2y - 2\, \frac{\partial}{\partial q} c11011p1yp1y,$$
$$\frac{\partial}{\partial p} c10111p2xp2x - 1/2\, \frac{\partial}{\partial q} c10111p1xp2x + 1/2\, \frac{\partial}{\partial q} c11011p2xp1y - 1/2\, \frac{\partial}{\partial q} c11011p1xp2y,$$
$$\frac{\partial}{\partial p} c10111p2xp1y - \frac{\partial}{\partial q} c10111p1xp1y, \frac{\partial}{\partial p} c10111p2xp2y - \frac{\partial}{\partial q} c10111p1xp2y,$$
$$\frac{\partial}{\partial p} c11012p2xp2x - \frac{\partial}{\partial q} c11012p1xp2x + 3/2\, \frac{\partial}{\partial q} f012p2xp12x - g012p1xp2xp2x,$$
$$\frac{\partial}{\partial p} c11012p2xp2y - \frac{\partial}{\partial q} c11012p1xp2y + 2\, g012p2xp2xp1y - g012p1xp2xp2y,$$

$$\frac{\partial}{\partial p} c10112p2xp1y + \frac{\partial}{\partial q} c10111p1xp2x - \frac{\partial}{\partial q} c11011p1xp2y - g012p1xp1yp2y,$$

$$\frac{\partial}{\partial p} c10112p2xp2y + 1/2 \frac{\partial}{\partial q} c10112p2xp1y - \frac{\partial}{\partial q} c10112p1xp2y + \frac{\partial}{\partial q} c10111p2xp2x +$$

$$-1/2 \frac{\partial}{\partial q} c11011p2xp2y - 1/2\, g012p2xp1yp2y, \frac{\partial}{\partial p} c10112p1yp2y + \frac{\partial}{\partial q} c10111p2xp1y - \frac{\partial}{\partial q} c11011p1yp2y,$$

$$\frac{\partial}{\partial p} c10112p2yp2y - 1/4 \frac{\partial}{\partial q} c10112p1yp2y + 1/4 \frac{\partial}{\partial q} c10111p2xp2y - 1/2 \frac{\partial}{\partial q} c11011p2yp2y,$$

$$\frac{\partial}{\partial p} c11022p1xp2y + \frac{\partial}{\partial q} c11012p1xp2x - 2 \frac{\partial}{\partial q} f012p1xp22x, \frac{\partial}{\partial p} c11022p2xp2y - 2 \frac{\partial}{\partial q} f012p2xp22x + 6\, g012p2xp2xp2x,$$

$$\frac{\partial}{\partial p} c11022p2yp2y - \frac{\partial}{\partial q} f012p2xp2xy + 2\, g012p2xp2xp2y,$$

$$\frac{\partial}{\partial p} c10122p1xp2x - 2 \frac{\partial}{\partial q} c10122p1xp1x + \frac{\partial}{\partial q} c11012p1xp2x - 3 \frac{\partial}{\partial q} f012p2xp12x + 2\, g012p1xp2xp2x,$$

$$\frac{\partial}{\partial p} c10122p1xp2y - \frac{\partial}{\partial q} c10122p1xp1y + \frac{\partial}{\partial q} c11012p1xp2y - 4\, g012p2xp2xp1y + g012p1xp2xp2y,$$

$$\frac{\partial}{\partial p} c10122p2xp2x - 1/2 \frac{\partial}{\partial q} c10122p1xp2x + 1/2 \frac{\partial}{\partial q} c11012p2xp2x - 3/2\, g012p2xp2xp2x,$$

$$\frac{\partial}{\partial p} c10122p2xp2y - \frac{\partial}{\partial q} c10122p1xp2y,$$

$$\frac{\partial}{\partial p} c10122p1yp2y - 2 \frac{\partial}{\partial q} c10122p1yp1y - 2\, g012p2xp1yp2y + 4\, g012p1xp2yp2y,$$

$$\frac{\partial}{\partial p} c10122p2yp2y - 1/2 \frac{\partial}{\partial q} c10122p1yp2y,$$

$$\frac{\partial}{\partial p} b21112p2x - c10122p1xp1y - 2\, c10112p1xp2x - c11011p2xp2x + 3 \frac{\partial}{\partial q} d10112p12x$$

$$-5/3\, f012p2xp1xy + 2\, f012p1xp2xy, \frac{\partial}{\partial p} b21112p2y + 1/2\, c10112p2xp1y - c10112p1xp2y + c10111p2xp2x$$

$$- 1/2\, c11011p2xp2y - 1/2\, f012p2xp12y,$$

$$\frac{\partial}{\partial p} b20212p2y - 1/2\, c10112p1yp2y + 1/2\, c10111p2xp2y - c11011p2yp2y,$$

$$6\, d10111p12x - 2\, d11011p1xy + \frac{\partial}{\partial p} a32111, -2\, d11011p12y + \frac{\partial}{\partial p} a31211 + 2\, d10111p1xy,$$

$$\frac{\partial}{\partial p} a30311 + 2/3\, d10111p12y, \frac{\partial}{\partial p} a33012 - b22012p1x + e012p13x,$$

$$\frac{\partial}{\partial p} a32112 - b21112p1x + 3/2\, d10112p12x - 1/2\, d11011p22x - 1/4\, e012p12xy,$$

$$\frac{\partial}{\partial p} a31212 - b20212p1x + 1/2\, d10112p1xy + 1/2\, d10111p22x - 1/2\, d11011p2xy - 1/2\, e012p1x2y,$$

$$\frac{\partial}{\partial p} a30312 - 1/2\, d10112p12y + 1/2\, d10111p2xy - 1/2\, d11011p22y, \frac{\partial}{\partial p} a33022,$$

$$\frac{\partial}{\partial p} a32122 - 2\, d10122p12x + 2\, d11012p22x - 2\, e012p23x, \frac{\partial}{\partial p} a31222 - 2\, d10122p1xy + 2\, d11012p2xy,$$

$$-2\, d10122p12y + 4\, e012p2x2y + \frac{\partial}{\partial p} a30322, \frac{\partial^2}{\partial q^2} f012p2xp12x + 2 \frac{\partial}{\partial p} g012p2xp2xp2x - 2 \frac{\partial}{\partial q} g012p1xp2xp2x,$$

$$2/3 \frac{\partial}{\partial p} c10112p2xp2x - 2/3 \frac{\partial}{\partial q} c10112p1xp2x - 1/3 \frac{\partial}{\partial q} c11011p2xp2x + \frac{\partial^2}{\partial q^2} d10112p12x - 1/3\, g012p2xp2xp1y,$$

$$- 4\, c10122p2yp2y + 2 \frac{\partial}{\partial q} d10122p22y + \frac{\partial^2}{\partial q^2} a30322,$$

$$\frac{\partial^2}{\partial q\partial p}a33011 - 4/3\,c10112p1xp1x - 1/3\,c11011p1xp2x + 4/9\,f012p1xp1xy,$$
$$\frac{\partial^2}{\partial p^2}g012p2xp2xp2x - \frac{\partial^2}{\partial q\partial p}g012p1xp2xp2x + 2/3\,\frac{\partial^2}{\partial q^2}g012p1xp1xp2x,$$
$$\frac{\partial^2}{\partial p^2}c10112p2xp2x - \frac{\partial^2}{\partial q\partial p}c10112p1xp2x + \frac{\partial^2}{\partial q^2}c10112p1xp1x$$
$$-1/2\,\frac{\partial^2}{\partial q\partial p}c11011p2xp2x + 1/4\,\frac{\partial^2}{\partial q^2}c11011p1xp2x, -4\,c11011p1xp1x + 2\,\frac{\partial}{\partial p}d11011p12x + \frac{\partial^2}{\partial p^2}a33011.$$

B.3.2 Coboundary conditions for (3.3)

$$p^3q^2a31222 + 2\,p^4q\,a32122 + 3\,p^5a33022 + p\,q^4a30312 - p^2q^3a31212 - 3\,p^3q^2a32112 -$$
$$-5\,p^4q\,a33012 - q^5a30311 + p^2q^3a32111 + 2\,p^3q^2a33011 = 0 \qquad \text{(B.1)}$$
$$p^2q^3a30322 - p^4q\,a32122 - 2\,p^5a33022 - 3\,p\,q^4a30312 - p^2q^3a31212 + p^3q^2a32112 +$$
$$+3\,p^4q\,a33012 + 2\,q^5a30311 + p\,q^4a31211 - p^3q^2a33011 = 0 \qquad \text{(B.2)}$$
$$-3/4\,p^5q\,\frac{\partial}{\partial p}a33012 - 1/4\,p\,q^4a30312 + 1/8\,p\,q^4a31211 - 5/2\,p^4q\,a33012 + \frac{7\,p^3q^2a33011}{16} +$$
$$+1/2\,p^3q^3\frac{\partial}{\partial p}a31212 - 1/32\,p^2q^3a32111 - 1/16\,p^4q^2\frac{\partial}{\partial p}a33011 - \frac{35\,p\,q^5\frac{\partial}{\partial p}a30311}{16} + \frac{7\,p^4q\,a32122}{8} +$$
$$+1/8\,p^2q^3a31212 - \frac{17\,p^3q^3\frac{\partial}{\partial q}a33011}{32} - 3/8\,p^3q^2a32112 + 1/16\,p^3q^3\frac{\partial}{\partial p}a32111 + 1/2\,p^2q^4\frac{\partial}{\partial p}a30312 -$$
$$-\frac{5\,p^2q^4\frac{\partial}{\partial p}a31211}{16} + 1/4\,p^4q^2\frac{\partial}{\partial p}a32112 - 7/4\,p\,q^5\frac{\partial}{\partial q}a30312 - 5/8\,p^2q^4\frac{\partial}{\partial q}a31212 - 1/32\,p\,q^5\frac{\partial}{\partial q}a31211 +$$
$$+1/32\,p^2q^4\frac{\partial}{\partial q}a32111 + 5/2\,p^5a33022 + \frac{49\,q^5a30311}{32} + 1/2\,p^3q^3\frac{\partial}{\partial q}a32112 - 5/8\,p^4q^2\frac{\partial}{\partial q}a32122 +$$
$$+\frac{13\,p^4q^2\frac{\partial}{\partial q}a33012}{8} - 3/8\,p^5q\,\frac{\partial}{\partial q}a33022 - 1/2\,p^5q\,d10122p12x + 1/4\,p^2q^4b20212p1y - 1/4\,p^4q^2b21112p1x +$$
$$+1/4\,p^5q\,b22012p1x + 1/4\,p^5q\,d11012p12x - p\,q^5d10112p22y + 1/2\,p^2q^4b20212p2x + 1/8\,p^2q^4b21112p2y +$$
$$+1/2\,p^4q^2d10112p12x - 1/4\,p^3q^3b21112p1y + 3/8\,p^3q^3b22012p2y - 1/2\,p^3q^3b20212p1x + \frac{15\,p\,q^5b20212p2y}{8} -$$
$$-1/2\,p^3q^3d10111p12x + 5/4\,p\,q^5d11011p22y + 1/4\,p^3q^3d11012p12y - 1/2\,p^3q^3d11022p22y - 1/2\,p^6d11022p12x +$$
$$+5/4\,q^6d10111p22y - \frac{71\,q^6\frac{\partial}{\partial q}a30311}{32} + p^6\frac{\partial}{\partial p}a33022 = 0 \qquad \text{(B.3)}$$
$$3/4\,p^5q\,\frac{\partial}{\partial p}a33012 + \frac{15\,p\,q^4a30312}{4} - \frac{15\,p\,q^4a31211}{8} + 15\,p^4q\,a33012 - \frac{39\,p^3q^2a33011}{16} -$$
$$-11/2\,p^3q^3\frac{\partial}{\partial p}a31212 - \frac{3\,p^2q^3a32111}{32} + \frac{17\,p^4q^2\frac{\partial}{\partial p}a33011}{16} + \frac{259\,p\,q^5\frac{\partial}{\partial p}a30311}{16} - \frac{43\,p^4q\,a32122}{8} -$$
$$-1/8\,p^2q^3a31212 + p^5q\,\frac{\partial}{\partial p}a32122 + \frac{93\,p^3q^3\frac{\partial}{\partial q}a33011}{32} + \frac{19\,p^3q^2a32112}{8} - 1/16\,p^3q^3\frac{\partial}{\partial p}a32111 -$$
$$-13/2\,p^2q^4\frac{\partial}{\partial p}a30312 + \frac{53\,p^2q^4\frac{\partial}{\partial p}a31211}{16} - \frac{13\,p^4q^2\frac{\partial}{\partial p}a32112}{4} + \frac{43\,p\,q^5\frac{\partial}{\partial q}a30312}{4} + \frac{33\,p^2q^4\frac{\partial}{\partial q}a31212}{8} +$$
$$+\frac{13\,p\,q^5\frac{\partial}{\partial q}a31211}{32} - \frac{13\,p^2q^4\frac{\partial}{\partial q}a32111}{32} - \frac{35\,p^5a33022}{2} - \frac{413\,q^5a30311}{32} - 5/2\,p^3q^3\frac{\partial}{\partial q}a32112 +$$

$$+\frac{41\,p^4q^2\,\frac{\partial}{\partial q}\,a32122}{8}-\frac{105\,p^4q^2\,\frac{\partial}{\partial q}\,a33012}{8}+\frac{63\,p^5q\,\frac{\partial}{\partial q}\,a33022}{8}+1/2\,p^5q\,d10122p12x-5/4\,p^2q^4\,b20212p1y+$$
$$+9/4\,p^4q^2\,b21112p1x-9/4\,p^5q\,b22012p1x+3/4\,p^5q\,d11012p12x+7\,p\,q^5\,d10112p22y-$$
$$-5/2\,p^2q^4\,b20212p2x-\frac{13\,p^2q^4\,b21112p2y}{8}-5/2\,p^4q^2\,d10112p12x+5/4\,p^3q^3\,b21112p1y-$$
$$-\frac{39\,p^3q^3\,b22012p2y}{8}+11/2\,p^3q^3\,b20212p1x-\frac{99\,p\,q^5\,b20212p2y}{8}+9/2\,p^3q^3\,d10111p12x-$$
$$-\frac{33\,p\,q^5\,d11011p22y}{4}-5/4\,p^3q^3\,d11012p12y+9/2\,p^3q^3\,d11022p22y+$$
$$+1/2\,p^6\,d11022p12x-\frac{33\,q^6\,d10111p22y}{4}+\frac{475\,q^6\,\frac{\partial}{\partial q}\,a30311}{32}=0 \qquad \text{(B.4)}$$

C

Computations for H^3

C.1 Janet basis for the cocycle condition of P_{LP}

$$p^2q\frac{\partial}{\partial q}A^{11}_{112} - p^3\frac{\partial}{\partial q}A^{11}_{212} + 2pq^2\frac{\partial}{\partial q}A^{12}_{112} - 2pqA^{12}_{112} - 2p^2q\frac{\partial}{\partial q}A^{12}_{212} + 2p^2A^{12}_{212} + q^3\frac{\partial}{\partial q}A^{22}_{112} - 2q^2A^{22}_{112} - pq^2\frac{\partial}{\partial q}A^{22}_{212} + 2pqA^{22}_{212} - 2\left(\frac{\partial}{\partial q}B^{1,2}_{222}\right)p^3 + 2\left(\frac{\partial}{\partial q}B^{1,2}_{111}\right)q^3 + 6\left(\frac{\partial}{\partial q}B^{1,2}_{122}\right)p^2q - 6\left(\frac{\partial}{\partial q}B^{1,2}_{112}\right)pq^2 = 0 \quad \text{(C.1)}$$

$$p^4\frac{\partial}{\partial p}A^{11}_{212} + p^3q\frac{\partial}{\partial q}A^{11}_{212} - 2p^3A^{11}_{212} + 4p^2q^2\frac{\partial}{\partial p}A^{12}_{112} + 4pq^3\frac{\partial}{\partial q}A^{12}_{112} - 8pq^2A^{12}_{112} - 2p^3q\frac{\partial}{\partial p}A^{12}_{212} - 2p^2q^2\frac{\partial}{\partial q}A^{12}_{212} + 4p^2qA^{12}_{212} + 4pq^3\frac{\partial}{\partial p}A^{22}_{112} + 4q^4\frac{\partial}{\partial q}A^{22}_{112} - 8q^3A^{22}_{112} - 3p^2q^2\frac{\partial}{\partial p}A^{22}_{212} - 3pq^3\frac{\partial}{\partial q}A^{22}_{212} + 6pq^2A^{22}_{212} - 2\left(\frac{\partial}{\partial q}B^{1,2}_{222}\right)p^3q + 2p^4\frac{\partial}{\partial p}B^{1,2}_{222} + 8\left(\frac{\partial}{\partial q}B^{1,2}_{111}\right)q^4 + 12\left(\frac{\partial}{\partial q}B^{1,2}_{122}\right)p^2q^2 - 6\left(\frac{\partial}{\partial p}B^{1,2}_{112}\right)p^2q^2 - 18\left(\frac{\partial}{\partial q}B^{1,2}_{112}\right)pq^3 + 4\left(\frac{\partial}{\partial p}B^{1,2}_{111}\right)pq^3 = 0 \quad \text{(C.2)}$$

$$p^3\frac{\partial}{\partial p}A^{11}_{112} - 2p^2A^{11}_{112} + p^3\frac{\partial}{\partial q}A^{11}_{212} + 6p^2q\frac{\partial}{\partial p}A^{12}_{112} + 4pq^2\frac{\partial}{\partial q}A^{12}_{112} - 10pqA^{12}_{112} - 4p^3\frac{\partial}{\partial p}A^{12}_{212} - 2p^2q\frac{\partial}{\partial q}A^{12}_{212} + 6p^2A^{12}_{212} + 5pq^2\frac{\partial}{\partial p}A^{22}_{112} + 4q^3\frac{\partial}{\partial q}A^{22}_{112} - 8q^2A^{22}_{112} - 4p^2q\frac{\partial}{\partial p}A^{22}_{212} - 3pq^2\frac{\partial}{\partial q}A^{22}_{212} + 6pqA^{22}_{212} + 8\left(\frac{\partial}{\partial q}B^{1,2}_{111}\right)q^3 - 18\left(\frac{\partial}{\partial q}B^{1,2}_{112}\right)pq^2 + 6\left(\frac{\partial}{\partial p}B^{1,2}_{111}\right)pq^2 + 12\left(\frac{\partial}{\partial q}B^{1,2}_{122}\right)p^2q - 12\left(\frac{\partial}{\partial p}B^{1,2}_{112}\right)p^2q + 6p^3\frac{\partial}{\partial p}B^{1,2}_{122} - 2\left(\frac{\partial}{\partial q}B^{1,2}_{222}\right)p^3 = 0 \quad \text{(C.3)}$$

$$
\begin{gathered}
3\left(\frac{\partial^2}{\partial q \partial p} B^{1,2}_{122}\right) p^3 q + \frac{39}{16}\left(\frac{\partial^2}{\partial q \partial p} B^{1,2}_{111}\right) pq^3 - \frac{81}{32}\left(\frac{\partial^2}{\partial q^2} B^{1,2}_{112}\right) pq^3 - \frac{11}{32}\left(\frac{\partial^2}{\partial q^2} B^{1,2}_{222}\right) p^3 q - \\
\frac{3}{4} p^3 q \frac{\partial^2}{\partial q \partial p} A^{12}_{212} - \frac{9}{32} p^2 q^2 \frac{\partial^2}{\partial q^2} A^{12}_{212} + p^2 q^2 \frac{\partial^2}{\partial p^2} A^{22}_{112} + \frac{3}{2} pq^3 \frac{\partial^2}{\partial q \partial p} A^{22}_{112} - \\
-\frac{9}{8} p^2 q^2 \frac{\partial^2}{\partial q \partial p} A^{22}_{212} - \frac{27}{64} pq^3 \frac{\partial^2}{\partial q^2} A^{22}_{212} - \frac{3}{4} p^3 q \frac{\partial^2}{\partial p^2} A^{22}_{212} - \frac{15}{8}\left(\frac{\partial^2}{\partial p^2} B^{1,2}_{112}\right) p^3 q + \frac{17}{16}\left(\frac{\partial^2}{\partial p^2} B^{1,2}_{111}\right) p^2 q^2 + \\
+ p^3 q \frac{\partial^2}{\partial p^2} A^{12}_{112} + \frac{9}{16} pq^3 \frac{\partial^2}{\partial q^2} A^{12}_{112} + \frac{3}{2} p^2 q^2 \frac{\partial^2}{\partial q \partial p} A^{12}_{112} + \frac{1}{64} p^3 q \frac{\partial^2}{\partial q^2} A^{11}_{212} + \frac{27}{16}\left(\frac{\partial^2}{\partial q^2} B^{1,2}_{122}\right) p^2 q^2 - \\
-\frac{81}{16}\left(\frac{\partial^2}{\partial q \partial p} B^{1,2}_{112}\right) p^2 q^2 - \frac{1}{2}\left(\frac{\partial^2}{\partial q \partial p} B^{1,2}_{222}\right) p^4 + \frac{9}{8}\left(\frac{\partial^2}{\partial q^2} B^{1,2}_{111}\right) q^4 + \frac{3}{4} p^4 \frac{\partial^2}{\partial p^2} B^{1,2}_{122} - \frac{1}{2} p^4 \frac{\partial^2}{\partial p^2} A^{12}_{212} + \\
+ \frac{9}{16} q^4 \frac{\partial^2}{\partial q^2} A^{22}_{112} + \frac{1}{8} p^3 q \frac{\partial}{\partial q} C^{1,11}_{112} + \frac{1}{8} p^4 \frac{\partial}{\partial p} C^{1,11}_{112} + \frac{1}{16} p^3 C^{1,11}_{112} + \frac{3}{2}\left(\frac{\partial}{\partial p} B^{1,2}_{111}\right) pq^2 - \\
-\frac{57}{16}\left(\frac{\partial}{\partial p} B^{1,2}_{112}\right) p^2 q - \frac{1}{2} pq^2 \frac{\partial}{\partial p} A^{22}_{112} + \frac{15}{8} p^3 \frac{\partial}{\partial p} B^{1,2}_{122} - \frac{3}{8} p^2 A^{11}_{112} - \frac{1}{4} p^3 \frac{\partial}{\partial p} A^{12}_{212} + \frac{7}{4}\left(\frac{\partial}{\partial q} B^{1,2}_{111}\right) q^3 + \\
+\frac{23}{16} p^2 A^{12}_{212} - \frac{1}{4} q^3 \frac{\partial}{\partial q} A^{22}_{112} - \frac{5}{8} q^2 A^{22}_{112} - -\frac{19}{16}\left(\frac{\partial}{\partial q} B^{1,2}_{222}\right) p^3 + \frac{1}{32} p^3 \frac{\partial}{\partial q} A^{11}_{212} - \frac{45}{8}\left(\frac{\partial}{\partial q} B^{1,2}_{112}\right) pq^2 - \\
- \frac{3}{32} pq^2 \frac{\partial}{\partial q} A^{22}_{212} + \frac{39}{8}\left(\frac{\partial}{\partial q} B^{1,2}_{122}\right) p^2 q + \frac{33}{32} pq A^{22}_{212} - \frac{1}{4} p^2 q \frac{\partial}{\partial q} A^{12}_{212} + \\
+ \frac{1}{8} pq^2 \frac{\partial}{\partial q} A^{12}_{112} - \frac{13}{8} pq A^{12}_{112} = 0 \qquad \text{(C.4)}
\end{gathered}
$$

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