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Framed symplectic sheaves on surfaces and their ADHM data

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Contents

Abstract

This dissertation is centered on the moduli space of what we call framed symplectic sheaves on a surface, compactifying the corresponding moduli space of framed principal $SP-$bundles.

It contains the construction of the moduli space, which is carried out for every smooth projective surface X with a big and nef framing divisor, and a study of its deformation theory.

We also develop an in-depth analysis of the examples $X = \mathbb{P}^2$ and $X = Bl_p(\mathbb{P}^2)$, showing that the corresponding moduli spaces enjoy an ADHM-type description. In the former case, we prove irreducibility of the space and exhibit a relation with the space of framed ideal instantons on S^4 in type C.

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1 Introduction

1.1 Historical notes

Consider the complex projective plane $\mathbb{P}^2_{\mathbb{C}} = \mathbb{P}^2$ with a fixed line $l_\infty \subseteq \mathbb{P}^2$. We call a *framed sheaf* on $(\mathbb{P}^2, l_\infty)$ a torsion free sheaf $E \in Coh(\mathbb{P}^2)$ which is "well-behaved at infinity", meaning that E is trivial on l_∞ with a fixed trivialization. In the last twenty years, these purely algebro-geometric gadgets have been attracting the interest of many mathematicians from different areas, becoming particularly popular among Gauge theorists.

This is motivated by the discovery of important relations between moduli spaces of framed sheaves and $4D$ *Gauge theories*. At the core of such relations sits the definition of *(framed) instanton.*

Definition 1.1.1. Let r be a positive integer. An $SU(r)$ instanton on the real four sphere S^4 is a principal $SU(r)-$bundle P endowed with an anti-selfdual connection. Let us fix a point $\infty \in S^4$. A framed instanton consists of an instanton together with the specification of an element p of the fiber P_∞.

We refer to the standard book [DK] and references therein for an in-depth analysis of this topic.

Instantons on the sphere are topologically classified by a nonnegative integer n, which we call *charge.* We call $\mathcal{M}^{reg}(r,n)$ the set of charge n framed instantons modulo Gauge equivalence, which is indeed a noncompact smooth manifold. As a framed instanton is essentially the solution of a differential equation, partial compactifications of $\mathcal{M}^{reg}(r,n)$ may be obtained by adding a boundary of "generalized solutions". In particular, a partial compactification $\mathcal{M}^{reg}(r,n) \subseteq \mathcal{M}_0(r,n)$ may be constructed allowing instantons to degenerate to a so-called *ideal instanton*; such boundary points are bundles with a connection whose square curvature density has distributional degenerations at a finite number of points. The space $\mathcal{M}_0(r,n)$ is often called *Uhlenbeck space* in the literature, as its construction is based on the famous paper [U].

Instantons can in fact be described by means of linear algebraic data. This was first pointed out in 1978 ([ADHM]) and this is where algebraic geometry actually enters the picture; a few years later, in his ground-breaking paper [Do], Donaldson noted that such linear algebraic data could be used to give an interpretation of the moduli space of instantons as a moduli space of holomorphic vector bundles on the projective plane with a framing on a line. To get a visual understanding of this claim, one should think of framed instantons as instantons on $\mathbb{R}^4$ with "finite energy", i.e. admitting an extension to the compact foufold S^4. If one fixes a complex structure on $\mathbb{R}^4 \cong \mathbb{C}^2$ and considers a holomorphic compactification $\mathbb{C}^2 \cup l_\infty = \mathbb{P}^2$, one can see that every framed instanton can be associated with an holomorphic bundle on $\mathbb{C}^2$ extending to $\mathbb{P}^2$.

Donaldson's construction is based on the fact that the linear data (from now on: *ADHM data*) may be thought of as parameter spaces for *monads,* which are special three term complexes of bundles with fixed entries; varying the differentials and taking cohomology, one can obtain every possible framed bundle on the plane. Similarly flavored technologies were already known to experts, see for example [OSS, BH].

Also the Uhlenbeck space $\mathcal{M}_0(r,n)$ is an algebro-geometric object: by means of the ADHM data, it can be realized as an affine $\mathbb{C}-$scheme hosting $\mathcal{M}^{reg}(r,n)$ as an open subset, which is nonempty and dense whenever $r > 1$ (see e.g. [Na]). The scheme $\mathcal{M}_0(r,n)$ has a stratification into locally closed subsets

$$\mathcal{M}_0(r,n) = \bigsqcup_{k=0}^{n} \mathcal{M}^{reg}(r,n-k) \times (\mathbb{C}^2)^{(k)}. \tag{1.1.1}$$

It is very singular (its singular locus coincides with the union of the $k > 0$ strata) and does not have very well defined modular properties (despite some "tricky" modular interpretations have been carried out in [Ba2, BFG]). However, there exists a resolution of singularities

$$\pi : \mathcal{M}(r,n) \to \mathcal{M}_0(r,n),$$

where $\mathcal{M}(r,n)$ is a *fine moduli scheme* for framed sheaves on $\mathbb{P}^2$.

1.2 Adding parameters to the theory

The moduli space $\mathcal{M}(r,n)$ enjoys remarkable geometric properties: it is a generalization of the Hilbert scheme of points on the affine plane $\mathbb{A}^2$, it is perhaps the most famous example of a *Nakajima quiver variety* (and thus admits a *Hyperkähler structure*) and it admits a toric action which turns out to be a fundamental tool define and perform *instanton counting* in this framed (i.e. noncompact) case. For this and other reasons, in the recent years, many people from algebraic geometry have been studying it and constructing generalizations.

For example, one may change the base surface: the moduli space of framed sheaves has been constructed by means of monadic descriptions for multi blow-ups of $\mathbb{P}^2$ [Bu, He] and Hirzebruch surfaces [BBR]. Furthermore, the moduli space has been shown to exist on an arbitrary smooth projective surface X and a "reasonable" framing divisor $D \subseteq X$ in [BM]; the authors recognize the sought-for space as an open subscheme of the fine moduli space of *framed pairs* on X, defined and studied in [HL1, HL2].

A somehow more delicate change of parameter one can study is the *change of structure group.* Indeed, on $\mathbb{P}^2$, the category of rank r framed vector bundles is equivalent to the category of *framed principal SL_r-bundles.* The latter simply are pairs (P, a) where P is a SL_r-bundle and

$$a : P\mid_{l_\infty} \to l_\infty \times SL_r$$

is an isomorphism. The definition of (families of) framed principal $G-$bundles clearly makes sense for an arbitrary complex algebraic group G.

Such a generalization is relevant from a Gauge-theoretic viewpoint. Indeed, if we fix a simple compact Lie group K, the set of charge n framed $K-$instantons on the four sphere modulo Gauge equivalence is in bijection with its natural algebro-geometric counterpart: the set of framed principal $K^{\mathbb{C}}-$bundles on the plane modulo isomorphisms.

If $G \subseteq GL(W)$ is the group of transformations preserving a fixed nondegenerate quadratic form $Q : W \to W^\vee$ (i.e. if G is a classical group), a framed $G-$bundle is in fact the same thing as a rank r framed vector bundle E with an isomorphism $E \to E^\vee$ which behaves like Q once restricted to the framing divisor. ADHM-theoretic descriptions of the moduli space for these groups have been known for a long time (see for example [BS, JMW, NS]).

In [BFG], the authors present a differently flavored construction of the moduli space of framed $G-$bundles (from now on: $\mathcal{M}_G^{reg}$), holding for every simple, affine and simply connected G. One of the major achievements of their work is the construction of an affine partial compactification

$$\mathcal{M}_G^{reg} \subseteq \mathcal{M}_{G,0}$$

which behaves pretty much like the Uhlenbeck compactification; for instance, if K is the compact form of G, its set of $\mathbb{C}-$points may be identified with the set of framed ideal $K-$instantons on S^4. Furthermore, $\mathcal{M}_{G,0}$ has a stratification in locally closed subschemes analogue to 1.1.1. For $G = SP_r$ and O_r, Uhlenbeck spaces have been thoroughly studied in [Ch] by making use of ADHM-theoretic tools.

In the light of the previous discussion, it is rather natural to ask about the existence of a $G-$analogue of $\mathcal{M}(r, n)$. More explicitly:

Question. *Can we find a nonsingular modular compactification $\mathcal{M}_G \supseteq \mathcal{M}_G^{reg}$ with a proper surjective morphism $\mathcal{M}_G \to \mathcal{M}_{G,0}$?*

Up to date, the question is still open.

1.3 Weak $G-$bundles

Given a principal $G-$bundle P on a variety X, it is possible to functorially construct a rank r vector bundle $E_\rho(P)$ for any representation

$$\rho : G \to GL_r.$$

To compactify moduli spaces of $G-$bundles on X one has to produce a definition of "weak principal bundle." Aware of the vastity of the literature about compactifications of moduli of bundles, a reasonable path to follow is to fix a suitable (i.e. faithful) representation ρ and weaken the notion of associated vector bundle. Following this idea, the notions of *principal $G-$sheaf* and *honest singular principal $G-$bundle* were proposed by Gómez and Sols [GS1, GS2] and by Schmidt [Sch], respectively, under mild hypothesis on G and depending on a fixed representation ρ. In both cases, a suitable (semi)stability condition is introduced, producing a projective moduli space. Every object in both compactifications happens to give rise to a vector bundle on an open set $U \subseteq X$ satisfying $codim(X \backslash U) \geq 2$. If G is a classical group, there is of course a standard representation $G \subseteq GL_r$. In this case, Schmidt noted ([Sch, 5.2]) that the two definitions coincide "pointwise", meaning that the moduli spaces are isomorphic up to nilpotents.

Remark 1.3.1. Gómez and Sols' moduli have been mentioned in [Ba2, Bal] in relation to Uhlenbeck spaces in the unframed case on a surface; the authors suggest the latter spaces may be obtained as *generalized blow-downs* of the former, motivated by the analogous phenomenon in the classic case $G = GL_r$, studied in detail in the milestone papers [Li, Mo].

Let us consider the case $G = SP_r$ with the standard representation $SP_r \subseteq GL_r$ given by a fixed symplectic form $\Omega : \mathbb{C}^r \to \mathbb{C}^{r\vee}$. In this case, the above definitions of weak $G-$bundle lead to the following.

Definition 1.3.2. A *symplectic sheaf* on X is a pair (E, φ) where

- E is a torsion-free sheaf on X with generic rank r and $det(E) \cong \mathcal{O}_X$;
- $\varphi : E \to E^\vee$ is a morphism whose restriction to the locally free locus

$$U = \{x \in X \mid E_x \text{ is free}\} \subseteq X$$

is a symplectic form, i.e. a skew-symmetric isomorphism.

1.4 My work

The considerable amount of literature regarding the example $G = SP_r$ [BFG, BS, Ch, Do, JMW, NS] makes it a natural first candidate to consider in order to investigate the question 1.2. Motivated by the previous considerations, I have constructed a framed variant of the moduli space of symplectic sheaves on a surface X and studied some of its geometric properties. The case $X = \mathbb{P}^2$ is treated more in depth. A sizeable part of this thesis is based on my paper [Sca].

The moduli space $\mathcal{M}^D_{X,\Omega}$

Let us fix a smooth projective surface X, an effective divisor $D \subseteq X$ and a symplectic vector space (W, Ω). We define a *framed symplectic sheaf* over (X, D) to be a triple (E, a, φ), where (E, a) is a framed sheaf on (X, D) with $det(E) \cong \mathcal{O}_X$ and $\varphi : E^{\otimes 2} \to \mathcal{O}_X$ is a skew-symmetric morphism whose restriction to D coincides with the pullback of Ω via $a : E_D \to \mathcal{O}_D \otimes W$.

By means of a suitable variation of the construction of framed modules (reviewed in Chap. 2), it is possible to define a fine moduli scheme for framed symplectic sheaves, if D is *big and nef.* The construction is carried out in Chapter 3, where we prove the following theorem.

Theorem (Existence of $\mathcal{M}^D_{X,\Omega}$). *For every pair of integers (r, n) there exists a fine moduli scheme $\mathcal{M}^D_{X,\Omega}(r, n)$ representing the functor of families of framed symplectic sheaves on (X, D) with generic rank r and $c_2 = n$. There exists a closed embedding*

$$\iota : \mathcal{M}^D_{X,\Omega}(r, n) \to \mathcal{M}^D_X(r, n)$$

where $\mathcal{M}^D_X(r, n)$ is the moduli space of framed sheaves on (X, D) with $(rk, c_1, c_2) = (r, 0, n)$. $\mathcal{M}^D_{X,\Omega}(r, n)$ contains an open subscheme $\mathcal{M}^{D,reg}_{X,\Omega}(r, n)$ which is a fine moduli space for framed symplectic bundles on (X, D).

Deformations

The last part of Chapter 3 is devoted to compute the tangent space to $\mathcal{M}^D_{X,\Omega}$ at a point $[E, a, \varphi]$. We achieve this by computing deformations of the relevant parameter space and modding out symmetries, employing the techniques developed in [HL2].

Theorem (Description of $T_{[E,a,\varphi]}\mathcal{M}^D_{X,\Omega}$). *The tangent space $T_{[E,a,\varphi]}\mathcal{M}^D_{X,\Omega}(r, n)$*

is naturally isomorphic to the kernel of a canonically defined linear map

$$p_\varphi : Ext^1_{\mathcal{O}_X}(E, E(-D)) \to Ext^1_{\mathcal{O}_X}(\Lambda^2 E, \mathcal{O}_X(-D)),$$

and the induced inclusion

$$T_{[E,a,\varphi]}\mathcal{M}^D_{X,\Omega}(r,n) \to Ext^1_{\mathcal{O}_X}(E, E(-D)) \cong T_{[E,a]}\mathcal{M}^D_X(r,n)$$

identifies with the tangent map $T_{[E,a,\varphi]}\iota$. If p_φ is surjective and $[E,a]$ is a smooth point of $\mathcal{M}^D_X(r,n)$, then $[E,a,\varphi]$ is a smooth point of $\mathcal{M}^D_{X,\Omega}(r,n)$.

A more natural approach to deformations of symplectic sheaves is presented in Chapter 4. Here we construct what we call the *adjoint complex* associated to a sheaf with a quadratic form (E,φ), which is conjectured to rule the deformations of (E,φ) (i.e., to be an obstruction theory for the corresponding moduli stack).

The planar case

The second part of the thesis is devoted to a detailed analysis of two examples of the moduli space. The first example, which we study in more depth, is about the moduli space of framed symplectic sheaves over the complex projective plane with a fixed line l_∞ as framing divisor. In Chapter 6, we produce an alternative description of the space $\mathcal{M}^{l_\infty}_{\mathbb{P}^2,\Omega}(r,n) =: \mathcal{M}_\Omega(r,n)$ by means of linear data analogue to the classic ADHM description of framed sheaves on the plane (reviewed at the beginning of the chapter).

Theorem (Symplectic ADHM data on $\mathbb{P}^2$). *Let V be an $n-$dimensional vector space. Consider the affine variety*

$$\mathbb{M}_\Omega(r,n) \subseteq End(V)^{\oplus 2} \oplus Hom(W,V) \oplus S^2V^\vee \ni (A,B,I,G)$$

cut out by the equations

$$A^\vee G - GA = B^\vee G - GB = [A,B] - I\Omega^{-1}I^\vee G = 0.$$

The natural action of the group $GL(V)$ on $\mathbb{M}_\Omega(r,n)$ restricts to a free and locally proper action on a (suitably defined) invariant open subset $\mathbb{M}^s_\Omega(r,n) \subseteq \mathbb{M}_\Omega(r,n)$. The resulting geometric quotient $\mathbb{M}^s_\Omega(r,n)/GL(V)$ is isomorphic to $\mathcal{M}_\Omega(r,n)$.

Relations with Uhlenbeck spaces and singularities

Let us denote by $\mathcal{M}_{\Omega,0}(r,n)$ the symplectic variant of the Uhlenbeck space. $\mathcal{M}_{\Omega,0}(r,n)$ may be identified with a closed subscheme of $\mathcal{M}_0(r,n)$ and the birational morphism

$\pi : \mathcal{M}(r,n) \to \mathcal{M}_0(r,n)$ restricts in fact to a morphism $\pi_\Omega : \mathcal{M}_\Omega(r,n) \to \mathcal{M}_{\Omega,0}(r,n)$. In the second part of Chapter 6, we prove that the locally free locus $\mathcal{M}_\Omega^{reg}(r,n) \subseteq \mathcal{M}_\Omega(r,n)$ is dense, providing a description of $\mathcal{M}_{\Omega,0}(r,n)$ as a contraction of $\mathcal{M}_\Omega(r,n)$ along a divisor, in the spirit of Remk. 1.3.1. The density result is indeed equivalent to the following theorem.

Theorem (Irreducibility). *The moduli space $\mathcal{M}_\Omega(r,n)$ is irreducible.*

As $\mathcal{M}_\Omega(r,n)$ is not smooth in general, the proof of the theorem requires some nontrivial manipulations on the ADHM data. In Chapter 5 we recollect the necessary tools from matrix analysis and prove some related technical lemmas.

We also present a discussion on the singularities of $\mathcal{M}_\Omega(r,n)$, together with explicit descriptions for low n.

The blown-up case

The last chapter of this dissertation is devoted to the study of the case $(X,D) = (\hat{\mathbb{P}}^2, l_\infty)$, where $\hat{\mathbb{P}}^2 = Bl_p\mathbb{P}^2$ for $p \in \mathbb{P}^2$ and l_∞ is the pullback of a line disjoint from p. We prove the following ADHM description of the corresponding moduli space $\mathcal{M}^{l_\infty}_{\hat{\mathbb{P}}^2,\Omega}(r,n) =: \hat{\mathcal{M}}_\Omega(r,n)$.

Theorem (Symplectic ADHM data on $\hat{\mathbb{P}}^2$). *Let V_0, V_1 be $n-$dimensional vector spaces. Consider the affine variety*

$$\hat{\mathbb{M}}_\Omega(r,n) \subseteq Hom(V_1,V_0)^{\oplus 2} \oplus Hom(W,V_0) \oplus Hom(V_1,V_0^\vee) \oplus Hom(V_0,V_1) \ni (A,B,I,G,D)$$

cut out by the equations

$$A^\vee G - G^\vee A = B^\vee G - G^\vee B = GD - D^\vee G^\vee = 0,$$

$$ADB - BDA - I\Omega^{-1}I^\vee G = 0.$$

The natural action of the group $GL(V_0) \times GL(V_1)$ on $\hat{\mathbb{M}}_\Omega(r,n)$ restricts to a free and locally proper action on a (suitably defined) invariant open subset $\hat{\mathbb{M}}^s_\Omega(r,n) \subseteq \hat{\mathbb{M}}_\Omega(r,n)$. The resulting geometric quotient $\hat{\mathbb{M}}^s_\Omega(r,n)/GL(V_0)\times GL(V_1)$ is isomorphic to $\hat{\mathcal{M}}_\Omega(r,n)$.

We believe that also in this case the moduli space is irreducible. At the end of the chapter the ADHM description is employed to provide some evidence.

Conventions

Throughout the thesis, we shall fix an algebraically closed field $\mathbb{K}$ of characteristic 0. From Chapter 5 on, we will fix $\mathbb{K} = \mathbb{C}$. Unless otherwise stated, every scheme will be a $\mathbb{K}-$scheme of finite type.

For a scheme X and an $\mathcal{O}_X-$module F, we will denote by

$$\underline{Hom}_{\mathcal{O}_X}(F, \_) : Mod(\mathcal{O}_X) \to Mod(\mathcal{O}_X)$$

and

$$Hom_{\mathcal{O}_X}(F, \_) : Mod(\mathcal{O}_X) \to Vec_{\mathbb{K}}$$

the functor of local and global homomorphisms, respectively. The associated derived functors will be denoted $\underline{Ext}^i_{\mathcal{O}_X}(F, \_)$ and $Ext^i_{\mathcal{O}_X}(F, \_)$.

Given another module G, we shall denote by $\underline{Tor}_i^{\mathcal{O}_X}(F, G)$ the left derived functor of $F \underset{\mathcal{O}_X}{\otimes} \_$ applied to G, well defined up to isomorphism whenever F has a locally free resolution. In the dissertation, Tor sheaves will be exclusively applied to coherent sheaves on smooth varieties.

The dual sheaf $\underline{Hom}_{\mathcal{O}_X}(F, \mathcal{O}_X)$ will be often denoted $F^\vee$.

The $\mathcal{O}_X$ subscripts will be omitted whenever there is no ambiguity.

2 The setting

2.1 Preliminary results

We collect in this section some standard results and easy lemmas we will be using throughout this thesis.

A criterion for closed embeddings

The following almost tautological lemma gives a criterion for a morphism of schemes to be a closed embedding. We shall be frequently applying it in the sequel.

Lemma 2.1.1. *Let $f : X \to Y$ be a morphism of $\mathbb{K}-$schemes of finite type. Assume that the following conditions hold:*

1. *the map of sets $f(\mathbb{K}) : X(\mathbb{K}) \to Y(\mathbb{K})$ is injective;*
2. *for any closed point $x \in X(\mathbb{K})$ the linear map $Tf_x : T_xX \to T_{f(x)}Y$ is a monomorphism;*
3. *f is proper.*

Then f is a closed embedding.

Serre duality

We will sometimes use the following form of Serre-Grothendieck duality.

Theorem 2.1.2. *Let X be a smooth projective variety of dimension d. Then for any pair of coherent sheaves F, G on X and for every integer i, there is an isomorphism*

$$Ext^i(F, G) \cong Ext^{d-i}(G, F \otimes \omega_X)^\vee,$$

where ω_X is the canonical bundle.

A result by Gomez-Sols

This technical lemma will turn out to be very useful to construct parameter spaces form moduli spaces of sheaves "with decorations."

Lemma 2.1.3. *[GS2, Lemma 0.9] Let X be a projective variety and Y be a scheme. Let $h : \mathcal{F} \to \mathcal{G}$ be a morphism of coherent sheaves on $X \times Y$. Assume $\mathcal{G}$ is flat over Y. There exists a closed subscheme $Z \subseteq Y$ such that the following universal property is satisfied: for any $s : S \to Y$ with $(1_X \times s)^\star h = 0$, one has a unique factorization of s as $S \to Z \to Y$.*

Bilinear forms on sheaves

Let X be a $\mathbb{K}-$scheme and let E be an $\mathcal{O}_X-$module. The skew-symmetric square $\Lambda^2 E$ and the symmetric square $S^2 E$ of E are defined as the sheafifications of the presheaves

$$U \mapsto S^2 E(U),\ U \mapsto \Lambda^2 E(U).$$

In particular, they can be naturally written as quotients of $E^{\otimes 2}$. Let $i \in Aut(E^{\otimes 2})$ be the natural switch morphism (on stalks: $i(e \otimes f) = f \otimes e$). Let G be another $\mathcal{O}_X-$module and $\varphi : E^{\otimes 2} \to G$ a morphism. We call φ symmetric (resp. skew-symmetric) if $\varphi \circ i = \varphi$ (resp. $\varphi \circ i = -i$). $S^2 E$ and $\Lambda^2 E$ satisfy the obvious universal properties

$$\begin{array}{ccc} E^{\otimes 2} & \xrightarrow{\forall symm} & G \\ \downarrow & \nearrow_{\exists!} & \\ S^2E & & \end{array} \qquad \begin{array}{ccc} E^{\otimes 2} & \xrightarrow{\forall skew} & G \\ \downarrow & \nearrow_{\exists!} & \\ \Lambda^2E & & \end{array}$$

and in fact fit into a split-exact sequence

$$0 \to S^2 E \to E^{\otimes 2} \to \Lambda^2 E \to 0.$$

Remark 2.1.4. A bilinear form $\varphi \in Hom(E^{\otimes 2}, G)$ naturally corresponds to an element of

$$Hom(E, G \otimes E^\vee) \cong Hom(E^{\otimes 2}, G).$$

If φ is symmetric or skew, it corresponds to a unique form in $Hom(S^2 E, G)$ or $Hom(\Lambda^2 E, G)$. We shall make a systematic abuse of notation by calling φ all these morphisms.

The modules $S^2 E$ and $\Lambda^2 E$ are coherent or locally free if E is and are well-behaved with respect to pullbacks, meaning that for a given morphism of schemes $f : Y \to X$

one has natural isomorphisms

$$S^2 f^\star E \cong f^\star S^2 E,\ \Lambda^2 f^\star E \cong f^\star \Lambda^2 E.$$

The following straightforward lemma will be useful later.

Lemma 2.1.5. *Let $K \to H \to E \to 0$ be an exact sequence of coherent $\mathcal{O}_X-$modules. There is an exact sequence*

$$K \otimes H \to \Lambda^2 H \to \Lambda^2 E \to 0.$$

If furthermore H is locally free and $K \to H$ is injective, define $K \wedge H$ to be the image of the subsheaf $K \otimes H \subseteq H^{\otimes 2}$ under $H^{\otimes 2} \to \Lambda^2 H$. Then there is a natural isomorphism

$$K \wedge H \cong K \otimes H / (K \otimes H \cap S^2 H)$$

and thus an exact sequence

$$0 \to K \wedge H \to \Lambda^2 H \to \Lambda^2 E \to 0.$$

Monads

Some surfaces admit alternative definitions of the moduli space of framed (symplectic) sheaves by means of monads. We recall here the definition of monads and present two useful lemmas.

Definition 2.1.6. A *monad* on a scheme X is a three-term complex of coherent locally free sheaves on X (in degrees $-1, 0, 1$):

$$M: \ \mathcal{U} \xrightarrow{\alpha} \mathcal{W} \xrightarrow{\beta} \mathcal{T}$$

which is exact but in degree 0. In other words, we require α be injective and β be surjective. We denote by $\mathcal{E} = \mathcal{E}(M)$ the coherent sheaf $H^0(M)$.

Remark 2.1.7. Let X be a smooth projective surface and let M be a monad on X. Then $\mathcal{E}(M)$ is a torsion-free sheaf and it is locally free sheaf if and only if α is injective as a map of vector bundles.

Every monad comes endowed with a diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathcal{U} & = & \mathcal{U} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \to & ker(\beta) & \to & \mathcal{W} & \to & \mathcal{T} & \to & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \to & \mathcal{E} & \to & coker(\alpha) & \to & \mathcal{T} & \to & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

which is called the *display* of the monad.

We shall be dealing soon with sheaves $\mathcal{E}$ which may be realized as cohomologies of monads. The following proposition will be useful to deal with quadratic forms $\varphi : \mathcal{E} \to \mathcal{E}^\vee$ on such sheaves.

Proposition 2.1.8. *Let $\mathcal{E}$ be the cohomology of a monad M. Let $M^\vee$ be the dual complex (it may be no longer a monad). Then we have an isomorphism $\mathcal{E}^\vee \cong H^0(M^\vee)$.*

Proof. Dualizing the display of M we get

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{T}^\vee & \to & coker(\alpha)^\vee & \to & \mathcal{E}^\vee & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{T}^\vee & \to & \mathcal{W}^\vee & \to & ker(\beta)^\vee & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{U}^\vee & = & \mathcal{U}^\vee & &
\end{array}$$

as the sheaf $\underline{Ext}^i_{\mathcal{O}_X}(\mathcal{T}, \mathcal{O}_X)$ vanishes for $i \neq 0$. Since $coker(\alpha)^\vee = ker(\alpha^\vee)$ and $ker(\beta)^\vee = coker(\beta^\vee)$, we obtain the sought for isomorphism. □

The following proposition is a slight generalization of [Ki, Prop. 2.2.1].

Proposition 2.1.9. *Let*

$$M: \ \mathcal{U} \xrightarrow{\alpha} \mathcal{W} \xrightarrow{\beta} \mathcal{T} \ ;$$

$$M': \ \mathcal{U}' \xrightarrow{\alpha'} \mathcal{W}' \xrightarrow{\beta'} \mathcal{T}'$$

be two complexes on X such that M is a monad and α' is injective. Let $\mathcal{E} = H^0(M)$ and $\mathcal{E}' = H^0(M')$. Assume the following vanishings hold:

$$Ext^1_{\mathcal{O}_X}(\mathcal{T}, \mathcal{W}') = Ext^1_{\mathcal{O}_X}(\mathcal{W}, \mathcal{U}') = Ext^2_{\mathcal{O}_X}(\mathcal{T}, \mathcal{U}') = 0.$$

Then the natural map $H^0 : Hom(M, M') \to Hom_{\mathcal{O}_X}(\mathcal{E}, \mathcal{E}')$ is surjective. If in addition

$$Hom_{\mathcal{O}_X}(\mathcal{T}, \mathcal{W}') = Hom_{\mathcal{O}_X}(\mathcal{W}, \mathcal{U}') = 0,$$

the kernel of the map H^0 is naturally identified with the vector space $Ext^1_{\mathcal{O}_X}(\mathcal{T}, \mathcal{U}')$.

The proof of the statement in [Ki] (in which M and M' were required to be monads with locally free cohomology) generalizes with no changes to the present situation.

A spectral sequence

The proof of the following lemma may be found in [Mc, Thm 12.1].

Lemma 2.1.10. *Let $R \to S$ be a morphism of unitary commutative rings, let N be an $S-$module and N' be an $R-$module. There exists a spectral sequence with $E^2_{p,q} = Ext^q_S(Tor^R_p(S, N'), N)$ converging to $Ext_R(N', N)$. In particular, if N' is $R-$flat, we get an isomorphism*

$$Ext^q_S(S \otimes N', N) \cong Ext^q_R(N', N).$$

2.2 Framed sheaves on surfaces

In this section we quickly go through the construction of the moduli space of framed sheaves on a surface and present some of its deformation-theoretic aspects. We need to recall some technicalities of the construction as they will be helpful to define the moduli space of framed symplectic sheaves. We refer to [HL1], [HL2] and [BM] for a detailed treatment of the subject.

Notation

Throughout this section we fix:

- $(X, \mathcal{O}_X(1))$ a polarized projective surface over $\mathbb{K}$.
- a *big and nef* divisor $D \subseteq X$, also called *framing divisor*;
- numerical invariants $(r, c_1, n) \in \mathbb{N}^+ \times H^2(X, \mathbb{Z}) \times \mathbb{Z}$;
- an $r-$dimensional $\mathbb{K}-$vector space W.

We also call $P = P_{r,c_1,n} \in \mathbb{Q}[t]$ the Hilbert polynomial of a coherent sheaf on X with generic rank, c_1 and $n = c_2$ given by the corresponding invariants.

Definition of framed sheaves

We introduce the definition of framed sheaves and their families.

Definition 2.2.1. A *framed sheaf* on X is a pair (E, a) where E is a torsion-free coherent sheaf on X and $a : E \mid_D \to \mathcal{O}_D \otimes W$ is an isomorphism. A morphism of framed sheaves $(E, a) \to (E', a')$ consists of a morphism $f : E \to E'$ such that

$$a' \circ f \mid_D = \lambda a$$

for some $\lambda \in \mathbb{K}$.

We can easily define families of framed sheaves. If S is a scheme, an $S-$family of framed sheaves consists of a pair $(\mathcal{E}_S, a_S)$ where $\mathcal{E}_S$ is an $S-$flat sheaf on $S \times X = X_S$ and $a_S : \mathcal{E}_S \mid_{D_S} \to \mathcal{O}_{D_S} \otimes W$ is an isomorphisms, $D_S = S \times D$.

Let us pass to the definition of the moduli functor.

Definition 2.2.2. Let us fix a triple (r, c_1, n). We define the functor

$$\mathfrak{M}_X^D(r, c_1, n) : Sch_{\mathbb{K}}^{op} \to Set$$

assigning to a scheme S the set of isomorphism classes of $S-$families of framed sheaves $(\mathcal{E}_S, a_S)$ on X satisfying the additional conditions:

- the (open) subscheme of S whose closed points satisfy

  $$\mathcal{E}_S \mid_s = E_s \text{ is torsion-free}$$

  is the entire S;
- $(r(E_s), c_1(E_s), n(E_s)) = (r, c_1, n)\ \forall s \in S(\mathbb{K})$.

Framed sheaves as Huybrechts-Lehn framed pairs

It is possible to construct a fine moduli space for the functor $\mathfrak{M}^D_X(r,n)$. Let F be a fixed sheaf on X. A framed module is a pair (E,a) where E is a coherent sheaf on X, and $a: E \to F$ is a morphism. A framed sheaf (E,a) is a special example of framed module, with $F = \mathcal{O}_D \otimes W$, E torsion-free and a inducing an isomorphism once restricted to D. In [HL2, Def. 1.1 and Thm 2.1], a (semi)stability condition depending on a numerical polynomial δ and on a fixed polarization H is defined, and a boundedness result is provided for framed modules.

Let $c \in H^\star(X,\mathbb{Q})$. In [BM, Thm 3.1], it is shown that there exist a polarization H and a numerical polynomial δ such that any framed sheaf (E,a) with Chern character $c(E) = c$ is δ-stable as a framed module. This is a crucial step in order to realize the moduli space of framed sheaves $\mathcal{M}^D_X(r,n)$ as an open subscheme of the moduli space of δ-semistable framed modules as defined in [HL2].

Let $c \in H^\star(X,\mathbb{Q})$ be the Chern character of a sheaf $E \in Coh(X)$ with invariants (r, c_1, n) and, if necessary, change $\mathcal{O}_X(1)$ for a polarization as in [BM, Thm 3.1]. Let $P = P_{r,c_1,n} \in \mathbb{Q}[t]$ be the corresponding Hilbert polynomial. Keeping now in mind that in this setting framed sheaves are a particular type of *stable* framed modules, we obtain the following proposition as an immediate consequence of [HL2, Thm. 2.1] and [HL1, Lemma 1.6].

Proposition 2.2.3. *The following statements hold.*

1. *There exists a positive integer m_0 such that for any $m \geq m_0$ and for any framed sheaf (E,a) on X with Hilbert polynomial P, E is m−regular, $H^i(X,\mathcal{O}_X(m)) = 0\ \forall i > 0$, $H^1(X,\mathcal{O}_D(m)) = 0$ and $P(m) = h^0(E(m))$.*

2. *Any nonzero morphism of framed sheaves (E,a), (F,b) with the same Hilbert polynomial is an isomorphism. Furthermore, there exists a unique morphism $f : (E,a) \to (F,b)$ satisfying $f_D \circ b = a$, and any other morphism between them is a nonzero multiple of f.*

Construction of $\mathcal{M}^D_X(r,n)$

We need to recall how to endow the set of isomorphism classes of framed sheaves $\mathcal{M}^D_X(r,n)$ with a scheme structure, which makes it a fine moduli space for the functor $\mathfrak{M}^D_X(r,n)$. The moduli space is obtained as a geometric quotient of a suitable parameter space, defined using *Quot* schemes. This type of construction is somehow standard, as it is essentially the same one uses to construct moduli of (unframed) sheaves and bundles on smooth schemes (the reader may check the classic book [HL3] for an exhaustive presentation of these methods). The relevant parameter space for our problem will be defined in the present subsection.

Fix a polynomial $P = P_{r,c_1,n}$ and a positive integer $m \gg 0$ as in Prop. 2.2.3, and let V be a vector space of dimension $P(m)$. Let $H = V \otimes \mathcal{O}_X(-m)$. Consider the projective scheme

$$Hilb(H, P) \times \mathbb{P}(Hom(V, H^0(\mathcal{O}_D(m)) \otimes W)^\vee) =: Hilb \times P_{fr},$$

where $Hilb(H, P)$ is the Grothendieck *Quot* scheme parametrizing equivalence classes of quotients

$$q : H \to E,\ P_E = P$$

on X. Define Z as the subset of pairs $([q : H \to E], A)$ such that the map $H \to \mathcal{O}_D \otimes W$ induced by A factors through E:

$$\begin{array}{ccc} H & \xrightarrow{A} & \mathcal{O}_D \otimes W \\ {\scriptstyle q}\downarrow & \nearrow{\scriptstyle a} & \\ E & & \end{array}$$

Z can be in fact interpreted as a closed subscheme as follows. Let us fix a sheaf $\mathcal{E} \in Coh(X \times Hilb)$ which is a universal quotient

$$q_{Hilb} : V \otimes \mathcal{O}_{X \times Hilb} \otimes p_X^\star \mathcal{O}_X(-m) \twoheadrightarrow \mathcal{E}.$$

The pullback of the universal map

$$V \otimes \mathcal{O}_{P_{fr}} \to H^0(\mathcal{O}_D(m) \otimes W) \otimes \mathcal{O}_{P_{fr}}$$

to $X \times Hilb \times P_{fr}$ yields a morphism

$$\mathcal{A}_{univ} : V \otimes \mathcal{O}_{X \times Hilb \times P_{fr}} \otimes p_X^\star \mathcal{O}_X(-m) \to \mathcal{O}_{D \times Hilb \times P_{fr}} \otimes W.$$

The pullback of q_{Hilb}, induces a diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathcal{K} & \to & V \otimes \mathcal{O}_{X \times Hilb \times P_{fr}} \otimes p_X^\star \mathcal{O}_X(-m) & \to & p_{Hilb \times X}^\star(\mathcal{E}) & \to & 0 \\ & & & {\scriptstyle \mathcal{A}_{univ}^{\mathcal{K}}}\searrow & \downarrow {\scriptstyle \mathcal{A}_{univ}} & \swarrow & & & \\ & & & & \mathcal{O}_{D \times Hilb \times P_{fr}} \otimes W & & & & \end{array}$$

The closed subscheme $Z \subseteq Hilb \times P_{fr}$ we are looking for is the one defined by Lemma 2.1.3, where $Y = Hilb \times P_{fr}$ and $h = \mathcal{A}_{univ}^{\mathcal{K}}$. We denote by $\mathring{Z} \subseteq Z$ the open subscheme defined by requiring that the pullback $a : E_D \to \mathcal{O}_D \otimes W$ is a framing.

The natural action of the group $SL(V)$ on the bundle $H = V \otimes \mathcal{O}_X(-m)$ induces

$SL(V)$ actions on Z and the open subscheme $\overset{\circ}{Z}$ is invariant. As already announced, the following theorem holds:

Theorem 2.2.4. *The $SL(V)-$scheme $\overset{\circ}{Z}$ admits a geometric quotient $\overset{\circ}{Z}/SL(V)$; this scheme is a fine moduli space for the functor $\mathfrak{M}_X^D(r,n)$, and will be denoted $\mathcal{M}_X^D(r,n)$.*

Infinitesimal study of $\mathcal{M}_X^D(r,n)$

The aim of this subsection is to give a brief account on the deformation theory of $\mathcal{M}_X^D(r,n)$. We start from the deformation theory of Z.

Let A be an Artin local $\mathbb{K}-$algebra. We define the sheaves on $X_A = X \times Spec(A)$

$$\mathcal{H} := H \otimes \mathcal{O}_A = V \otimes O_{X_A}(-m);$$

$$\mathcal{D} = \mathcal{O}_D \otimes \mathcal{O}_A \otimes W.$$

Consider the scheme

$$Z^A \subseteq Quot_{X_A}(\mathcal{H}, P) \times \mathbb{P}(Hom(\mathcal{H}, \mathcal{D})^\vee)$$

defined as the closed subscheme representing the functor assigning to an $A-$scheme T the set

$$\{q_T : V \otimes \mathcal{O}_{X_T}(-m) \twoheadrightarrow \tilde{\mathcal{E}}, a_T : \tilde{\mathcal{E}} \to \mathcal{D}_T \mid \tilde{\mathcal{E}}\ T-\text{flat},\ P_{\tilde{\mathcal{E}}} = P\}.$$

The computation of the tangent spaces to Z^A was already performed in [HL2], and goes as follows.

Let $q : \mathcal{H} \to \mathcal{E}$ be a quotient with $P_{\mathcal{E}} = P$. Let $ker(q) := \mathcal{K} \overset{\iota}{\hookrightarrow} \mathcal{H}$. The tangent space to $Quot_{X_A}(\mathcal{H}, P)$ at the point $[q : \mathcal{H} \to \mathcal{E}]$ is naturally isomorphic to the vector space $Hom_{X_A}(\mathcal{K}, \mathcal{E})$. Indeed, writing $S = Spec(A[\varepsilon])$, and using the universal property of *Quot*, the tangent space may be indeed identified with the set of equivalence classes of quotients $\tilde{q} : \mathcal{H}_S \to \tilde{\mathcal{E}}$ on X_S (where

$$\mathcal{H}_S = \mathcal{H} \otimes \mathbb{C}[\varepsilon] = H \otimes A[\varepsilon]$$

and $\tilde{\mathcal{E}}$ is an $S-$flat sheaf on X_S) reducing to q mod ε. Write $q_S : \mathcal{H}_S \to \mathcal{E}_S$ for the pullback of q to X_S. For a given $\tilde{q}$ with $ker(\tilde{q}) = \tilde{\mathcal{K}}$, the map $q_S \mid_{\tilde{\mathcal{K}}}$ takes values in $\varepsilon \cdot \mathcal{E}_S$ and factors through $\mathcal{K}$, defining a morphism $\mathcal{K} \to \varepsilon \cdot \mathcal{E}_S \cong \mathcal{E}$. *Vice versa*, a given map $\gamma : \mathcal{K} \to \mathcal{E}$ defines

$$\tilde{\mathcal{K}} \subseteq \mathcal{H}_S,\ \tilde{\mathcal{K}} = \rho^{-1}(\mathcal{K}) \cap ker(q_S + \gamma \circ \rho),$$

where $\rho : \mathcal{H}_S \twoheadrightarrow \mathcal{H}$ is the natural projection induced by $A[\varepsilon] \twoheadrightarrow A$. The tangent space to $\mathbb{P}(Hom(\mathcal{H}, \mathcal{D})^\vee)$ at a point $[\mathcal{A}]$ is naturally identified with the quotient $Hom(\mathcal{H}, \mathcal{D})/A \cdot \mathcal{A}$. To see why this is the case, we apply again a universal property. An element of

$$\mathbb{P}(Hom(\mathcal{H}, \mathcal{D})^\vee)(S \to Spec(A)),\ Spec(\mathbb{K}) \mapsto [\mathcal{A}] \in \mathbb{P}(Hom(\mathcal{H}, \mathcal{D})^\vee)$$

is the same thing as a morphism $\tilde{\mathcal{A}} : \mathcal{H}_S \to \mathcal{D}_S$ (up to units in $A[\varepsilon]$), reducing to $\mathcal{A} : \mathcal{H} \to \mathcal{D}$ mod ε (up to units in A). Such a morphism may be represented as

$$h + \varepsilon h' \mapsto \mathcal{A}(h) + \varepsilon(\mathcal{A}(h') + \mathcal{B}(h))$$

with $\mathcal{B} : \mathcal{H} \to \mathcal{D}$ an $\mathcal{O}_{X_A}-$linear map. The units in $A[\varepsilon]$ preserving this map modulo ε are exactly those of the form $1 + \lambda\varepsilon$ with $\lambda \in A$, and they send $\mathcal{B}$ to $\mathcal{B} + (\lambda - 1)\mathcal{A}$. This proves the claim.

Let $(q, \alpha) \in Z^A$ with $\alpha \circ q = \mathcal{A}$. One can prove that the pairs

$$(\gamma, \mathcal{B}) \in Hom(\mathcal{K}, \mathcal{E}) \oplus Hom(\mathcal{H}, \mathcal{D})/A \cdot \mathcal{A} = T_{(q,\mathcal{A})}(Quot(\mathcal{H}, P) \times \mathbb{P}(Hom(\mathcal{H}, \mathcal{D})^\vee))$$

belonging to $T_{(q,\alpha)} Z^A$ are characterized by the equation

$$\mathcal{B} \circ \iota = \alpha \circ \gamma.$$

The previous considerations provide information on the infinitesimal behavior of $\mathcal{M}_X^D(r, n)$ as follows. Consider the natural action of $Aut(\mathcal{H})$ on $\overset{\circ}{Z}{}^A$. To obtain the tangent space to $\overset{\circ}{Z}{}^A/Aut(\mathcal{H})$ at a point $[(q, \mathcal{A})]$ one has to mod out the image of the induced tangent orbit map

$$End(\mathcal{H}) \to T_{(q,\mathcal{A})} Z^A.$$

Such map factors through

$$End(\mathcal{H}) \to Hom(\mathcal{H}, \mathcal{E}),\ x \mapsto q \circ x$$

and is described by

$$Hom(\mathcal{H}, \mathcal{E}) \ni \lambda \mapsto (\lambda \circ \iota, \alpha \circ \lambda) \in T_{(q,\mathcal{A})} Z^A;$$

The quotient can be shown to be isomorphic to the hyperext group

$$\mathbb{E}xt^1(\mathcal{E}, \mathcal{E} \to \mathcal{D}),$$

because we can interpret a pair $(\gamma, \mathcal{B}) \in T_{(q,\mathcal{A})}Z^A$ as a morphism of complexes

$$\begin{array}{ccc} \mathcal{K} & \xrightarrow{\iota} & \mathcal{H} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\mathcal{B}} \\ \mathcal{E} & \xrightarrow[\mathcal{A}]{} & \mathcal{D} \end{array}$$

and it is immediate to see that the subspace of nullhomotopic morphisms coincides with the image of $Hom(\mathcal{H}, \mathcal{E}) \to T_{(q,\mathcal{A})}Z^A$. We obtain a chain of natural isomorphisms

$$T_{(q,\mathcal{A})}Z^A / Hom(\mathcal{H}, \mathcal{E}) \cong Hom_K(\mathcal{K} \to \mathcal{H}, \mathcal{E} \to \mathcal{D}) \cong \mathbb{E}xt^1(\mathcal{E}, \mathcal{E} \to \mathcal{D}).$$

Remark 2.2.5. $\mathbb{E}xt^1(\mathcal{E}, \mathcal{E} \to \mathcal{D})$ is in fact isomorphic to $Ext^1(\mathcal{E}, \mathcal{E}(-D_A))$ as $\mathcal{E}$ is locally free on D_A by hypothesis.

We collect the previous remarks in a theorem and add a smoothness criterion. We refer as usual to [HL2] for the full proof.

Proposition 2.2.6. *Let* $\xi = [E, a] \in \mathcal{M}_X^D(r, n)$.

1. *The tangent space* $T_\xi \mathcal{M}_X^D(r, n)$ *is naturally isomorphic to the group* $Ext^1(E, E(-D))$.
2. *If the group* $Ext^2(E, E(-D))$ *vanishes,* ξ *is a nonsingular point of* $\mathcal{M}_X^D(r, n)$.

3 Framed symplectic sheaves

Throughout the chapter, we fix the set of data $(X, \mathcal{O}_X(1))$, D, (r, c_1, n) and W as in 2.2, modified as follows:

- we add the datum of a symplectic form $\Omega : W \to W^\vee$;
- we impose the constraint $c_1 = 0$.

The symplectic form Ω induces a symplectic form

$$1_{\mathcal{O}_D} \otimes \Omega : \mathcal{O}_D \otimes W \to \mathcal{O}_D \otimes W^\vee;$$

by abuse of notation, we shall denote also the latter by Ω.

The definition of framed symplectic sheaf we are going to deal with in this chapter is inspired from [GS1], where the authors present the construction of a coarse moduli space for what they call *semistable symplectic sheaves*. A symplectic sheaf is a pair (E, φ) where E is a coherent torsion-free sheaf on X and $\varphi : \Lambda^2 E \to \mathcal{O}_X$ is a morphism inducing a symplectic form on the maximal open subset of X over which E is locally free. Our construction will simply be a “framed version” of this.

3.1 The definition

Definition 3.1.1. A framed symplectic sheaf on X is a triple (E, a, φ) where (E, a) is a framed sheaf with $det(E) \cong \mathcal{O}_X$ and $\varphi : \Lambda^2 E \to \mathcal{O}_X$ is a morphism satisfying

$$\varphi_D = \Omega \circ \Lambda^2 a. \tag{3.1.1}$$

Remark 3.1.2. The morphism $\varphi : E \to E^\vee$ is an isomorphism once restricted to D; consequently, the same holds on an open neighborhood of D. Since

$$det(E) \cong det(E^\vee) \cong \mathcal{O}_X,$$

we have $c_1(E) = 0$. Furthermore, we deduce that $det(\varphi)$ must be a nonzero constant; hence, φ induces in fact an isomorphism $E_U \to E_U^\vee$ on the entire 2−codimensional subset $X - Sing(E) = U \supseteq D$. If (E, a) is a framed sheaf whose first Chern class vanishes and we are given a morphism $\varphi : \Lambda^2 E \to \mathcal{O}_X$ satisfying the compatibility condition 3.1.1, we get in fact $det(E) \cong \mathcal{O}_X$. Indeed, since φ is nondegenerate in a neighborhood of D, we get an exact sequence

$$0 \longrightarrow E \xrightarrow{\varphi} E^\vee \longrightarrow coker(\varphi) \longrightarrow 0.$$

The sheaf $coker(\varphi)$ must be supported on a subscheme whose dimension is at most 1, and $c_1(coker(\varphi)) = 0$ forces such dimension to be 0. In particular, φ must be an isomorphism outside a 2−codimensional subscheme of X, from which

$$det(E) \cong det(E^\vee)$$

follows. Finally, consider the dual map

$$\varphi^\vee : E^{\vee\vee} \to E^\vee.$$

It is a skew-symmetric isomorphism of vector bundles (i.e., a classical symplectic form): we conclude that $det(E^\vee) \cong \mathcal{O}_X$. In addition, we observe that if (E, a) is a framed sheaf such that $(E^{\vee\vee}, a^{\vee\vee})$ admits a structure of framed symplectic bundle $\bar{\varphi} : E^{\vee\vee} \to E^\vee$, the composition $\bar{\varphi} \circ \iota = \varphi$, where $\iota : E \to E^{\vee\vee}$ is the natural inclusion, makes (E, a, φ) a framed symplectic sheaf.

From now on, we shall always omit the hypothesis on the determinant while working with framed symplectic sheaves, as we already fixed the hereby equivalent constraint $c_1 = 0$.

Definition 3.1.3. We define a morphism of framed symplectic sheaves $(E, a, \varphi) \to (E', a', \varphi')$ to be a morphism f of framed sheaves where $\varphi' \circ \Lambda^2 f = \lambda\varphi$ for a nonzero $\lambda \in \mathbb{K}$.

Remark 3.1.4. Prop. 2.2.3 shows that for a given pair of framed sheaves (E, a) and (E', a') with the same invariants, there is at most one isomorphism $f : E \to E'$ satisfying $a = a' \circ f_D$. This implies that (E, a) can support *at most one* structure of framed symplectic sheaf, in the following sense. If $\varphi : \Lambda^2 E \to \mathcal{O}_X$ is a symplectic form, it induces a morphism $\varphi : E \to E^\vee$, whose dual $\varphi^\vee : E^{\vee\vee} \to E^\vee$ is an isomorphism. Furthermore, $E^{\vee\vee}$ and $E^\vee$ inherit framings from (E, a), and $\varphi^\vee$ preserves these framings, as φ is Ω−compatible with a. As a consequence, any other symplectic form φ' on E satisfies $(\varphi')^\vee = \varphi^\vee$, from which $\varphi = \varphi'$ follows. In particular:

Lemma 3.1.5. *If (E, α, φ) is a framed symplectic sheaf, $Hom(\Lambda^2 E, \mathcal{O}_X(-D)) = 0$ (i.e., the symplectic form has no nontrivial infinitesimal automorphisms).*

Proof. Let $\psi : \Lambda^2 E \to \mathcal{O}_X(-D)$ be a morphism. By means of the exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0$$

we can think of ψ as a morphism $\Lambda^2 E \to \mathcal{O}_X$ which vanishes on D. Now, for a nonzero scalar λ consider $\psi_\lambda = \psi + \lambda\varphi$. If we choose a square root of $\lambda^{1/2}$, we obtain that $(E, \lambda^{1/2}a, \psi_\lambda)$ is a framed symplectic sheaf, but also $(E, \lambda^{1/2}a, \lambda\varphi)$ is. This forces $\psi_\lambda = \lambda\varphi$, i.e. $\psi = 0$. □

We can define framed symplectic sheaves in families again; an $S-$family of framed symplectic sheaves will be a triple $(\mathcal{E}_S, a_S, \varphi_S)$ with $(\mathcal{E}_S, a_S)$ an $S-$family of framed sheaves, and $\varphi_S : \Lambda^2\mathcal{E}_S \to \mathcal{O}_{X_S}$ a morphism such that $\varphi_S\ |_{D_S} = \Omega \circ \Lambda^2 a_S$. The corresponding functor will be denoted $\mathfrak{M}^D_{X,\Omega}(r,n)$.

Definition 3.1.6. We denote by $\mathfrak{f}$ the forgetful natural transformation of functors

$$\mathfrak{M}^D_{X,\Omega}(r,n) \to \mathfrak{M}^D_X(r,n) := \mathfrak{M}^D_X(r,0,n).$$

3.2 The moduli space

In this section we will construct a fine moduli space for symplectic sheaves. There are two possible approaches to the problem, both exploiting the existence of a fine moduli space for framed sheaves:

1. one may prove that the forgetful functor $\mathfrak{f}$ is representable and deduce the representability of $\mathfrak{M}^D_{X,\Omega}(r,n)$, or
2. one may directly construct the moduli space as a geometric quotient of a suitable parameter space as in [HL2].

Despite the first path is somehow more natural, we will follow the second, as it will allow us to quickly obtain some infinitesimal information on the moduli space, making once again use of the deformation theory of *Quot* schemes. This method was already used in [Sca]. Anyway, in the following chapter we shall give an account on the first approach, together with a general discussion on the deformation theory of $2-$tensors.

The parameter space and its deformations

We go back to the setting and notation of subsection 2.2, where we defined the subscheme

$$Z \subseteq Hilb \times P_{fr}$$

parametrizing quotients $H \twoheadrightarrow E$ of a fixed bundle H on X together with a morphism $E \to \mathcal{O}_D \otimes W$, and its open subscheme

$$\mathring{Z} \subseteq Z$$

over which the morphism induces a framing. Take now the product

$$Hilb(H,P) \times \mathbb{P}(Hom(\Lambda^2 V \to H^0(\mathcal{O}_X(2m)))^\vee) =: Hilb \times P_{symp}$$

and consider the closed subscheme $Z' \subseteq Hilb \times P_{symp}$ given by pairs $([q], \phi)$ such that the map $\Lambda^2 H \to \mathcal{O}_X$ induced by ϕ descends to some $\varphi : \Lambda^2 E \to \mathcal{O}_X$:

$$\begin{array}{ccc} \Lambda^2 H & \xrightarrow{\phi} & \mathcal{O}_X \\ {\scriptstyle \Lambda^2 q}\downarrow & \nearrow{\scriptstyle \varphi} & \\ \Lambda^2 E & & \end{array}$$

We define the closed subscheme $Z_\Omega \subseteq Hilb \times P_{fr} \times P_{symp}$ as follows. First, consider the scheme-theoretic intersection

$$p^{-1}_{Hilb \times P_{fr}}(Z) \cap p^{-1}_{Hilb \times P_{symp}}(Z'),$$

whose closed points are triples $([q], A, \phi)$ satisfying: A descends to $a : E \to \mathcal{O}_D \otimes W$, ϕ descends to $\varphi : \Lambda^2 E \to \mathcal{O}_X$. This scheme has again a closed subscheme defined by triples satisfying the following compatibility on D:

$$\varphi_D = \Omega \circ \Lambda^2 a_D : \Lambda^2 E_D \to \mathcal{O}_D.$$

Call this subscheme Z_Ω. We denote by $\mathring{Z}_\Omega$ the preimage of $\mathring{Z} \subseteq Z$ under the natural projection $\pi : Z_\Omega \to Z$.

Theorem 3.2.1. *The restriction $\mathring{\pi} : \mathring{Z}_\Omega \to \mathring{Z}$ is a closed embedding.*

Remark 3.2.2. The schemes we are dealing with are in fact of finite type; this means that we can apply Lemma 2.1.1 to prove the theorem. The morphism $\mathring{\pi}$ is clearly proper, as a base change of a map between projective schemes. For injectivity, it is enough to note that there exists only one structure of symplectic sheaf on a given framed sheaf, in the sense of Remk. 3.1.4. It follows that we only need to prove the claim on tangent spaces: for any triple $([q], A, \phi)$ the tangent map $T_{([q],A,\phi)}\mathring{Z}_\Omega \to T_{([q],A)}\mathring{Z}$ is injective.

The previous remark motivates the following infinitesimal study for the parameter spaces. Let us remind the notation of subsection 2.2.

Let us consider the scheme

$$Z^A_\Omega \subseteq Z^A \times \mathbb{P}(Hom(\Lambda^2\mathcal{H}, \mathcal{O}_{X_A})^\vee)$$

representing the functor assigning to an $A-$scheme T the set

$$\{(q_T, \alpha_T) \in Z^A(T), \varphi_T : \Lambda^2\tilde{\mathcal{E}} \to \mathcal{O}_{X_T} \mid \varphi_T \mid_{D\times T} = \Omega \circ \Lambda^2\alpha_T \mid_{D\times T}\}.$$

We denote once again $S = Spec(A[\varepsilon])$ and compute $Z^A_\Omega(S)$. Let $[\Phi] \in \mathbb{P}(Hom(\Lambda^2\mathcal{H}, \mathcal{O}_A)^\vee$. An element of

$$\mathbb{P}(Hom(\Lambda^2\mathcal{H}, \mathcal{O}_A)^\vee)(S \to Spec(A)),\ Spec(\mathbb{K}) \mapsto [\Phi] \in \mathbb{P}(Hom(\Lambda^2\mathcal{H}, \mathcal{O}_A)^\vee)$$

is the same thing as a morphism $\tilde{\Phi} : \Lambda^2\mathcal{H}_S \to \mathcal{O}_S$ (up to units in $A[\varepsilon]$), reducing to Φ modulo ε (up to units in A). Indeed, one can choose a representative

$$(h + \varepsilon h') \otimes (g + \varepsilon g') = \Phi(h \otimes g) + \varepsilon(\Phi(h' \otimes g) + \Phi(h \otimes g') + \psi(h \otimes g))$$

with $\psi : \Lambda^2\mathcal{H} \to \mathcal{O}_{X_A}$, and two different maps ψ will induce the same morphism up to units if and only if they differ by an $A-$multiple of Φ.

Now, let $(\tilde{q}, \tilde{A}, \tilde{\Phi}) \in Z^A_H(S)$. This means that the morphism $\tilde{\Phi} : \Lambda^2\mathcal{H}_S \to \mathcal{O}_S$ descends to some $\tilde{\varphi} : \Lambda^2\mathcal{E}_S \to \mathcal{O}_S$ via $\tilde{q}$, i.e. its restriction to $\tilde{\mathcal{K}} \otimes \mathcal{H}_S$ vanishes. In terms of local sections, an element $h + \varepsilon h' \in \mathcal{H}_S$ belongs to $\tilde{\mathcal{K}}$ if and only if $q(h) = 0$ and $\gamma(h) = -q(h')$. We obtain:

$$\tilde{\Phi}((h + \varepsilon h') \otimes (g + \varepsilon g')) = \varepsilon(\Phi(h' \otimes g) + \psi(h \otimes g)) = \varepsilon(-\varphi(\gamma(h) \otimes q(g)) + \psi(h \otimes g)).$$

This quantity vanishes if and only if the equation

$$\psi(\iota \otimes 1) = \varphi(\gamma \otimes q)$$

holds.

The triple is required to satisfy another condition, namely the compatibility on the divisor D_S:

$$(\tilde{\varphi}) \mid_{D_S} = \Omega \circ (\tilde{\mathcal{A}}^{\otimes 2}) \mid_{D_S}.$$

We make an abuse of notation by writing $h + \varepsilon h'$ for sections of $\mathcal{H}_S \mid_{D_S} \cong V \otimes \mathcal{O}_{D_S}(-m)$; we get

$$\tilde{\Phi}((h + \varepsilon h') \otimes (g + \varepsilon g')) = \Omega((\mathcal{A}(h) + \varepsilon(\mathcal{A}(h') + \mathcal{B}(h))) \otimes (\mathcal{A}(g) + \varepsilon(\mathcal{A}(g') + \mathcal{B}(g))))$$

$$\Phi(h \otimes g) + \varepsilon(\Phi(h' \otimes g) + \Phi(h \otimes g') + \psi(h \otimes g)) = \Omega(\mathcal{A}(h) \otimes \mathcal{A}(g)) +$$

$$+\varepsilon(\Omega(\mathcal{A}(h)\otimes\mathcal{A}(g'))+\Omega(\mathcal{A}(h)\otimes\mathcal{B}(g))+\Omega(\mathcal{A}(h')\otimes\mathcal{A}(g))+\Omega(\mathcal{B}(h)\otimes\mathcal{A}(g)))$$

Since

$$(\varphi)\mid_{D_S}=\Omega\circ(\mathcal{A}^{\otimes 2})\mid_{D_S},\ \alpha\circ q=\mathcal{A}$$

holds by hypothesis, we can simplify:

$$\psi(h\otimes g)=\Omega(\alpha\circ q(h)\otimes\mathcal{B}(g))+\Omega(\mathcal{B}(h)\otimes\alpha\circ q(g))).$$

We have obtained the description of the tangent space we needed.

Proposition 3.2.3. *There is an isomorphism*

$$T_{(q,\mathcal{A},\Phi)}Z^A_\Omega\cong\{((\gamma,\mathcal{B}),\psi)\in T_{(q,\mathcal{A})}Z^A\oplus Hom(\Lambda^2\mathcal{H},\mathcal{O}_A)/A\cdot\Phi\mid\psi(\iota\otimes 1)=\varphi(\gamma\otimes q),$$

$$\psi\mid_{D_A}=\Omega(\mathcal{A}\otimes\mathcal{B}+\mathcal{B}\otimes\mathcal{A})\}.$$

We can now apply the proposition to prove Thm. 3.2.1.

Proof. For a quotient $q:H\to E$, we denote by $\iota:K\to H$ its kernel. We know that the tangent space $T_{([q],A)}Z$, which we think of as a subspace of

$$Hom(K,E)\oplus(Hom(H,D_W)/\mathbb{K}A),$$

can be described as the subset of pairs $(\gamma,[B])$ satisfying the equation

$$\bar{A}\circ\gamma=B\mid_K.$$

The space

$$T_{([q],A,\phi)}\subseteq T_{([q],A)}Z\oplus(Hom(\Lambda^2H,\mathcal{O}_X)/\mathbb{K}\phi$$

can be instead identified with the subspace of triples $(\gamma,[B],[\psi])$ defined by the equations

$$\psi(\iota\otimes 1_H)=\varphi(\gamma\otimes q);\ \psi_D=\Omega(A\otimes B+B\otimes A).$$

The differential map $T_{([q],A,\phi)}\mathring{Z}_\Omega\to T_{([q],A)}\mathring{Z}$ is just the projection

$$(\gamma,[B],[\psi])\mapsto(\gamma,[B]).$$

Now, suppose an element of type $(0,\lambda\mathcal{A},[\psi])$ belongs to $T_{([q],A,\phi)}\mathring{Z}_\Omega$. This means

$$\psi(\iota\otimes 1_\Omega)=0,$$

so that ψ descends to some $\bar{\psi} \in Hom(\Lambda^2 E, \mathcal{O}_X)$. Furthermore, we get

$$\psi_D = \lambda^2\Omega(\mathcal{A}^{\otimes 2} + \mathcal{A}^{\otimes 2} \circ i) = 0 \implies \psi_D \in Hom(\Lambda^2 E, \mathcal{O}_X(-D)) = 0.$$

We have proved injectivity for the tangent map, and this finishes the proof thanks to Remk. 3.2.2. □

3.2.1 Construction of the moduli space.

The natural action of the group $SL(V)$ on the bundle $H = V \otimes \mathcal{O}_X(-m)$ induces $SL(V)$ actions on the schemes Z and Z_Ω, and the map $Z_\Omega \to Z$ is equivariant. In addition, the open subschemes $\mathring{Z}$ and $\mathring{Z}_\Omega$ are invariant.

Definition 3.2.4. We define the scheme $\mathcal{M}^D_{X,\Omega}(r,n)$ to be the closed subscheme

$$\mathring{Z}_\Omega / SL(V) \subseteq \mathcal{M}^D_X(r,n).$$

Theorem 3.2.5. The scheme $\mathcal{M}^D_{X,\Omega}(r,n)$ is a fine moduli space for the functor $\mathfrak{M}^D_X(r,n)$.

Proof. Let S be any scheme of finite type and let $(\mathcal{E}_S, a_S, \varphi_S)$ be an $S-$family of framed symplectic sheaves. Consider the sheaf $\mathcal{V}_S := p_{S\star}(\mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m))$. Since for any $s \in S(\mathbb{K})$ we have

$$H^i(\mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m) \mid_{\{s\}\times X}) = 0 \ \forall i > 0$$

by Lemma 2.2.3, $\mathcal{V}_S$ is locally of rank $P(m)$. Furthermore, the natural map

$$q_S : p_S^\star \mathcal{V}_S \to \mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m)$$

is surjective, as it restricts to

$$H^0(\mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m) \mid_{\{s\}\times X}) \otimes \mathcal{O}_X \to \mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m) \mid_{\{s\}\times X}$$

on fibers, and we may apply again Lemma 2.2.3. From φ_S we obtain a map

$$\mathcal{V}_S^{\otimes 2} \to p_{S\star}(\mathcal{O}_{X_S} \otimes p_X^\star \mathcal{O}_X(2m)) = \mathcal{O}_S \otimes H^0(X, \mathcal{O}_X(2m)).$$

Indeed, since the higher p_S-pushforwards of $\mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m)$ vanish, the formula

$$Rp_{T\star}(\mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m) \overset{L}{\otimes} \mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m)) \cong$$

$$\cong Rp_{S\star}(\mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m)) \overset{L}{\otimes} Rp_{S\star}(\mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m)) \cong \mathcal{V}_S^{\otimes 2}$$

holds, and induces the desired map. It still is skew-symmetric, and thus yields

$$p_{S\star}(\varphi_S \otimes p_X^\star \mathcal{O}_X(2m)) : \Lambda^2 \mathcal{V}_S \to \mathcal{O}_S \otimes H^0(X, \mathcal{O}_X(2m)).$$

We define the morphism ϕ_S to be the composition

$$\phi_S : \Lambda^2 p_X^\star \mathcal{V}_S \to \mathcal{O}_{X_S} \otimes H^0(X, \mathcal{O}_X(2m)) \to \mathcal{O}_{X_S} \otimes p_X^\star \mathcal{O}_X(2m).$$

Also a_S similarly induces a map

$$\mathcal{V}_S \to \mathcal{O}_S \otimes H^0(X, \mathcal{O}_D(m) \otimes W)$$

and we define as above

$$A_S : p_X^\star \mathcal{V}_S \to \mathcal{O}_{D_S} \otimes p_X^\star \mathcal{O}_X(m) \otimes W.$$

By construction, the diagrams

$$\begin{array}{ccc} \Lambda^2 p_X^\star \mathcal{V}_S & \xrightarrow{\phi_S} & \mathcal{O}_{X_S} \otimes p_X^\star \mathcal{O}_X(m) \\ {\scriptstyle \Lambda^2 q_s}\downarrow & \nearrow{\scriptstyle \varphi_S(2m)} & \\ \Lambda^2 \mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(2m) & & \end{array}$$

$$\begin{array}{ccc} p_X^\star \mathcal{V}_S & \xrightarrow{A_S} & \mathcal{O}_{D_S} \otimes p_X^\star \mathcal{O}_X(m) \otimes W \\ {\scriptstyle q_s}\downarrow & \nearrow{\scriptstyle a_S(m)} & \\ \mathcal{E}_S \otimes p_X^\star \mathcal{O}_X(m) & & \end{array}$$

are commutative.

Define an open covering $S = \bigcup S_i$ such that $\mathcal{V}_S$ trivializes over each S_i, and fix isomorphisms $\mathcal{O}_{S_i} \otimes V \cong \mathcal{V}_{S_i}$, where V is a vector space of a dimension $P(m)$; the trivializations differ on the overlaps S_{ij} by a map $S_{ij} \to GL(V)$. Restricting the maps q_S, ϕ_S and A_S to $S_i \times X$ and twisting by $p_X^\star \mathcal{O}_X(-m)$ we obtain maps $S_i \to \mathring{Z}_\Omega$, which glue to a map $S \to \mathring{Z}_\Omega / SL(V) = \mathcal{M}^D_{X,\Omega}(r, n)$. We note that acting via $SL(V)$ or $GL(V)$ does not make a real difference as the natural action by $\mathbb{G}_m$ on the parameter spaces is trivial.

The resulting natural transformation

$$\mathfrak{M}^D_{X,\Omega}(r, n) \to \mathcal{M}^D_{X,\Omega}(r, n)$$

makes $\mathcal{M}^D_{X,\Omega}(r, n)$ into a coarse moduli space for framed symplectic sheaves; indeed, since $\overset{\circ}{Z}_\Omega$ parametrizes a tautological family of framed symplectic sheaves, for any scheme N and for any natural transformation $\mathfrak{M}^D_{X,\Omega}(r,n) \to Hom(\_, N)$ we obtain a map $\overset{\circ}{Z}_\Omega \to N$. This map has to be $SL(V)$ invariant as two points of $\overset{\circ}{Z}_\Omega$ that lie in the same orbit define isomorphic framed sheaves. The fact that the moduli space is indeed fine can be proved by noting that framed symplectic sheaves are rigid, i.e., by applying Remk. 3.1.4 and proceeding as in [HL2, proof of Main Theorem]. □

Remark 3.2.6. The moduli spaces $\mathcal{M}^D_X(r,n)$ and $\mathcal{M}^D_{X,\Omega}(r,n)$ both contain open subschemes of isomorphism classes of locally free sheaves. These are fine moduli spaces for framed SL_r and SP_r principal bundles, and will be respectively denoted $\mathcal{M}^{D,reg}_X(r,n)$ and $\mathcal{M}^{D,reg}_{X,\Omega}(r,n)$ in the sequel.

3.3 The tangent space

We can apply the results of the previous section to describe the tangent spaces to $\mathcal{M}^D_{X,\Omega}(r,n)$. Specifically, we aim to prove:

Theorem 3.3.1. *Let $\xi = [E, a, \varphi] \in \mathcal{M}^D_{X,\Omega}(r,n)$. The tangent space $T_\xi \mathcal{M}^D_{X,\Omega}(r,n)$ is naturally isomorphic to the kernel of a canonically defined linear map*

$$p_\varphi : Ext^1_{\mathcal{O}_X}(E, E(-D)) \to Ext^1_{\mathcal{O}_X}(\Lambda^2 E, \mathcal{O}_X(-D)).$$

For a given Artin $\mathbb{K}-$algebra A, consider the natural action of $Aut(\mathcal{H})$ on

$$\overset{\circ}{Z}{}^A_\Omega \subseteq Z^A \times \mathbb{P}(Hom(\Lambda^2\mathcal{H}, \mathcal{O}_{X_A})^\vee)$$

Analogously to the non symplectic case, in order to obtain the tangent space to $\overset{\circ}{Z}{}^A_\Omega / Aut(\mathcal{H})$ at a point $[(q, \mathcal{A}, \Phi)]$ one has to mod out the image of the induced tangent orbit map

$$End(\mathcal{H}) \to T_{(q,\mathcal{A},\Phi)} Z^A_\Omega.$$

Such a map factors through

$$End(\mathcal{H}) \to Hom(\mathcal{H}, \mathcal{E}),\ x \mapsto q \circ x$$

and reads

$$Hom(\mathcal{H}, \mathcal{E}) \ni \lambda \mapsto (\lambda \circ \iota, \alpha \circ \lambda, \varphi(\lambda \otimes q + q \otimes \lambda)) \in T_{(q,\mathcal{A})} Z^A_\Omega.$$

Using $\varphi : \Lambda^2\mathcal{E} \to \mathcal{O}_{X_A}$, we define a map

$$p_\varphi : Hom_K(\mathcal{K} \to \mathcal{H}, \mathcal{E} \to \mathcal{D}) \to Hom_K(\mathcal{K} \wedge \mathcal{H} \to \Lambda^2\mathcal{H}, \mathcal{O}_{X_A} \to \mathcal{O}_{D_A})$$

by assigning to a morphism of complexes $(\gamma, \mathcal{B})$ the morphism

$$\varphi(\gamma \otimes q) : \mathcal{K} \wedge \mathcal{H} \to \mathcal{O}_{X_A},\ \Omega(\alpha q \otimes \mathcal{B} + \mathcal{B} \otimes \alpha q) : \Lambda^2\mathcal{H} \to \mathcal{O}_{D_A}.$$

The map $\varphi(\gamma \otimes q)$ is naturally defined on $\mathcal{K} \otimes \mathcal{H}$ but, due to the skew-symmetry of φ, it vanishes on the subsheaf $S^2\mathcal{H} \cap (\mathcal{K} \otimes \mathcal{H})$; we use the same notation for the naturally induced map on the quotient $\mathcal{K} \wedge \mathcal{H}$, see Lemma 2.1.5. p_φ is well defined since if $(\gamma, \mathcal{B})$ is homotopic to 0 and $\lambda : \mathcal{H} \to \mathcal{E}$ is an homotopy, then $\varphi(\lambda \otimes q + q \otimes \lambda)$ will be an homotopy for $p_\varphi(\gamma, \mathcal{B})$. By definition, $p_\varphi(\gamma, \mathcal{B}) = 0$ if and only if there exists a morphism $\psi : \Lambda^2\mathcal{H} \to \mathcal{O}_{X_A}$ such that $\psi(\iota \otimes 1) = \varphi(\gamma \otimes q)$ and $\psi\ |_{D_A} = \Omega(\alpha q \otimes \mathcal{B} + \mathcal{B} \otimes \alpha q)$.

Remark 3.3.2. The natural map

$$Hom_K(\mathcal{K} \wedge \mathcal{H} \to \Lambda^2\mathcal{H}, \mathcal{O}_{X_A} \to \mathcal{O}_{D_A}) \to \mathbb{H}om_{D^b}(\mathcal{K} \wedge \mathcal{H} \to \Lambda^2\mathcal{H}, \mathcal{O}_{X_A} \to \mathcal{O}_{D_A})$$

is an isomorphism.

Proof. We prove surjectivity first. We represent elements of the target group as roofs in the derived category, i.e. as pairs given by a quasi-isomorphism $M^\bullet \to (\mathcal{K} \wedge \mathcal{H} \to \Lambda^2\mathcal{H})$ and a morphism $M^\bullet \to (\mathcal{O}_{X_A} \to \mathcal{O}_{D_A})$, where $M^\bullet$ is a complex concentrated in degrees zero and one. We get a morphism of short exact sequences

$$\begin{array}{ccccc}
M^0 & \longrightarrow & M^1 & \longrightarrow & \Lambda^2\mathcal{E} \\
\downarrow & & \downarrow & & \| \\
\mathcal{K} \wedge \mathcal{H} & \longrightarrow & \Lambda^2\mathcal{H} & \longrightarrow & \Lambda^2\mathcal{E}
\end{array}$$

Apply $Hom(\_, \mathcal{O}_{X_A})$ and use

$$Ext^1(\Lambda^2\mathcal{H}, \mathcal{O}_{X_A}) \cong \Lambda^2 V \otimes H^1(X_A, \mathcal{O}_{X_A}(2m)) = 0;$$

get

$$\begin{array}{ccccccc}
Hom(\Lambda^2\mathcal{E}, \mathcal{O}_{X_A}) & \longrightarrow & Hom(\Lambda^2\mathcal{H}, \mathcal{O}_{X_A}) & \longrightarrow & Hom(\mathcal{K} \wedge \mathcal{H}, \mathcal{O}_{X_A}) & \twoheadrightarrow & Ext^1(\Lambda^2\mathcal{E}, \mathcal{O}_{X_A}) \\
\| & & \downarrow & & \downarrow & & \| \\
Hom(\Lambda^2\mathcal{E}, \mathcal{O}_{X_A}) & \longrightarrow & Hom(M^1, \mathcal{O}_{X_A}) & \longrightarrow & Hom(M^0, \mathcal{O}_{X_A}) & \longrightarrow & Ext^1(\Lambda^2\mathcal{E}, \mathcal{O}_{X_A})
\end{array}$$

We deduce that the natural map

$$Hom(\mathcal{K} \wedge \mathcal{H}, \mathcal{O}_{X_A}) \to Hom(M^0, \mathcal{O}_{X_A})/Hom(M^1, \mathcal{O}_{X_A})$$

is surjective; in other words we can suppose that the morphism $M^0 \to \mathcal{O}_{X_A}$ comes from $Hom(\mathcal{K} \wedge \mathcal{H}, \mathcal{O}_{X_A})$ up to homotopy. To show that also $M^1 \to \mathcal{O}_{D_A}$ comes from a map $\Lambda^2\mathcal{H} \to \mathcal{O}_{D_A}$, we may proceed as above: apply $Hom(\_, \mathcal{O}_{D_A})$ and use

$$Ext^1(\Lambda^2\mathcal{H}, \mathcal{O}_{D_A}) \cong \Lambda^2 V \otimes H^1(X_A, \mathcal{O}_{D_A}(2m)) = 0.$$

This proves surjectivity.

To prove injectivity, let $f \in Hom_K(\mathcal{K} \wedge \mathcal{H} \to \Lambda^2\mathcal{H}, \mathcal{O}_{X_A} \to \mathcal{O}_{D_A})$ be such that there exists a quasi isomorphism $M^\bullet \to (\mathcal{K} \wedge \mathcal{H} \to \Lambda^2\mathcal{H})$ whose composition with f admits a homotopy $h : M^1 \to \mathcal{O}_{X_A}$. We only need to prove that h factors through another homotopy $\Lambda^2\mathcal{H} \to \mathcal{O}_{X_A}$; this is again achieved by chasing the diagram above. □

Since there is an isomorphism

$$\mathbb{H}om_{D^b}(\mathcal{K} \wedge \mathcal{H} \to \Lambda^2\mathcal{H}, \mathcal{O}_{X_A} \to \mathcal{O}_{D_A}) \cong Ext^1(\Lambda^2\mathcal{E}, \mathcal{O}_{X_A}(-D_A)),$$

the previous discussion leads us to conclude that the tangent space $T^{1,A}_\Omega$ to $\overset{\circ}{Z}{}^A_\Omega/Aut(\mathcal{H})$ at $[(q, \mathcal{A}, \Phi)]$ fits into an exact sequence of vector spaces

$$0 \to T^{1,A}_\Omega \to Ext^1(\mathcal{E}, \mathcal{E}(-D_A)) \to Ext^1(\Lambda^2\mathcal{E}, \mathcal{O}_{X_A}(-D_A)).$$

Remark 3.3.3. Let $A = \mathbb{K}$. The map p_φ is in fact canonical (i.e., it only depends on the triple (E, φ, a)) since it admits the following Yoneda-type description. Let $\xi \in Ext^1(E, E(-D))$ be represented by an extension

$$0 \longrightarrow E(-D) \xrightarrow{\iota} F \xrightarrow{\pi} E \longrightarrow 0\,.$$

Apply $E \otimes \_$ and pushout via $\varphi(-D)$:

$$\begin{array}{ccccccccc}
0 \to & ker(1_E \otimes \iota) & \to & E^{\otimes 2}(-D) & \xrightarrow{1_E \otimes \iota} & E \otimes F & \to & E^{\otimes 2} & \to 0 \\
& \downarrow & & \downarrow \varphi(-D) & & \downarrow & & \| & \\
0 \to & ker(\chi) & \to & \mathcal{O}_X(-D) & \xrightarrow{\chi} & M & \xrightarrow{p} & E^{\otimes 2} & \to 0
\end{array}$$

Now, the module $ker(1_E \otimes \iota)$ is a torsion sheaf; it is indeed an epimorphic image of the 0−dimensional sheaf $\mathcal{T}or_1(E, E)$. It follows that $ker(\chi)$ is torsion as well: it is then forced to vanish. The above construction defines a linear map

$$\varphi(1_E \otimes \_) : Ext^1(E, E(-D)) \to Ext^1(E^{\otimes 2}, \mathcal{O}_X(-D)).$$

Similarly, define the “adjoint” map $\varphi(\_ \otimes 1_E)$. It is immediate to verify that the map

$$\varphi(\_ \otimes 1_E) + \varphi(1_E \otimes \_) : Ext^1(E, E(-D)) \to Ext^1(E^{\otimes 2}, \mathcal{O}_X(-D))$$

takes values in the subspace of skew extensions

$$Ext^1(\Lambda^2 E, \mathcal{O}_X(-D)) \subseteq Ext^1(E^{\otimes 2}, \mathcal{O}_X(-D)),$$

and a direct check shows that the equation

$$\varphi(\_ \otimes 1_E) + \varphi(1_E \otimes \_) = p_\varphi$$

holds.

Proposition 3.3.4. *Let (E, α, φ) be a framed symplectic sheaf whose underlying framed sheaf (E, α) corresponds to a smooth point $[(E, \alpha)] \in \mathcal{M}_X^D(r, n)$, and suppose that the map p_φ is an epimorphism. Then $[(E, \alpha, \varphi)]$ is a smooth point of $\mathcal{M}_{X,\Omega}^D(r, n)$.*

Proof. Let A be an Artin local $\mathbb{K}-$algebra and let $(\mathcal{E}, \mathcal{A}, \Phi) \in \mathfrak{M}_{X,\Omega}^D(A)$ be a framed symplectic sheaf over X_A. We proved that the space of its infinitesimal deformations can be written as

$$T^1(\mathcal{E}, \mathcal{A}, \Phi)_A = ker(Ext^1_{\mathcal{O}_{X_A}}(\mathcal{E}, \mathcal{E}(-D_A)) \to Ext^1_{\mathcal{O}_{X_A}}(\Lambda^2\mathcal{E}, \mathcal{O}_{X_A}(-D_A))).$$

We want to give a sufficient condition for the smoothness at a closed point (E, α, φ) by means of the T^1-lifting property. In our setting, this property may be expressed in the following way. Let $A_n \cong \mathbb{C}[t]/t^{n+1}$, $n \in \mathbb{N}$. Let $(\mathcal{E}_n, \mathcal{A}_n, \Phi_n) \in \mathfrak{M}_{X,\Omega}^D(A_n)$ and $(\mathcal{E}_{n-1}, \mathcal{A}_{n-1}, \Phi_{n-1}) \in \mathfrak{M}_{X,\Omega}^D(A_{n-1})$ be its pullback via the natural map $A_n \twoheadrightarrow A_{n-1}$. We get a map

$$T^1(\mathcal{E}_n, \mathcal{A}_n, \Phi_n)_{A_n} \to T^1(\mathcal{E}_{n-1}, \mathcal{A}_{n-1}, \Phi_{n-1})_{A_{n-1}}$$

and the underlying closed point $[(E, \alpha, \varphi)]$, where $(E, \alpha, \varphi) = (\mathcal{E}_n, \mathcal{A}_n, \Phi_n)\, mod(t)$, turns out to be a smooth point of $\mathcal{M}_{X,\Omega}^D(r, n)$ if and only if the above map is surjective for any n. From the exact sequence of $\mathcal{O}_{X_n}(:= \mathcal{O}_{X_{A_n}})$ modules

$$0 \to \mathcal{O}_X \to \mathcal{O}_{X_n} \to \mathcal{O}_{X_{n-1}} \to 0$$

we get exact sequences

$$0 \to E \to \mathcal{E}_n \to \mathcal{E}_{n-1} \to 0$$

and

$$0 \to \Lambda^2 E \to \Lambda^2 \mathcal{E}_n \to \Lambda^2 \mathcal{E}_{n-1} \to 0.$$

Making extensive use of Lemma 2.1.10, we obtain a commutative diagram with exact rows and columns

$$\begin{array}{ccccccc}
0 \longrightarrow & T^1 & \longrightarrow & Ext_{\mathcal{O}_X}{}^1(E, E(-D)) & \xrightarrow{h} & Ext^1_{\mathcal{O}_X}(\Lambda^2 E, \mathcal{O}_X(-D)) \\
& \downarrow & & \downarrow & & \downarrow \\
0 \longrightarrow & T^{1,n} & \longrightarrow & Ext^1_{\mathcal{O}_{X_n}}(\mathcal{E}_n, \mathcal{E}_n(-D_n)) & \xrightarrow{p_{n,\varphi}} & Ext^1_{\mathcal{O}_{X_n}}(\Lambda^2 \mathcal{E}_n, \mathcal{O}_{X_n}(-D_n)) \\
& {\scriptstyle a}\downarrow & & {\scriptstyle b}\downarrow & & {\scriptstyle c}\downarrow \\
0 \longrightarrow & T^{1,n-1} & \longrightarrow & Ext^1_{\mathcal{O}_{X_{n-1}}}(\mathcal{E}_{n-1}, \mathcal{E}_{n-1}(-D_{n-1}) \otimes W) & \longrightarrow & Ext^1_{\mathcal{O}_{X_{n-1}}}(\Lambda^2 \mathcal{E}_{n-1}, \mathcal{O}_{X_{n-1}}(-D_{n-1}))
\end{array}$$

Call $\tilde{c}$ the restriction of c to the image of $p_{n,\varphi}$ in the diagram; by applying the snake lemma to the second and third row, we get an exact sequence of vector spaces

$$ker(a) \to ker(b) \to ker(\tilde{c}) \to coker(a) \to coker(b) \to coker(\tilde{c}).$$

We know by hypothesis that $coker(b) = 0$. A sufficient condition to get or claim $coker(a) = 0$ is $ker(b) \to ker(c)$ to be surjective ($\implies$ $ker(c) = ker(\tilde{c})$). This condition is clearly satisfied if h is surjective. □

We conclude this section with a direct application to the case of bundles.

Corollary 3.3.5. *If (E, α, φ) is a framed symplectic bundle, the corresponding point $[(E, \alpha, \varphi)] \in \mathcal{M}^{D,reg}_{X,\Omega}(r, n)$ is smooth if $[(E, \alpha)] \in \mathcal{M}^{D,reg}_X(r, n)$ is.*

Proof. Let (E, α, φ) be a symplectic bundle. Consider the following map:

$$\underline{Hom}(E, E(-D)) \to \underline{Hom}(\Lambda^2 E, \mathcal{O}(-D)) \subseteq \underline{Hom}(E, E^\vee(-D))$$

defined on sections by

$$f \mapsto \varphi(-D) \circ f + (f(-D))^\vee \varphi,$$

where φ is interpreted as an isomorphism $E \to E^\vee$. The kernel of this map is identified with the bundle $Ad^\varphi(E)(-D)$, i.e. the twisted adjoint bundle associated with the principal $SP-$bundle defined by (E, φ). The map defined above is easily proved to be surjective since φ is an isomorphism. We obtain a short exact sequence of bundles which is in fact split-exact, as the map

$$\underline{Hom}(\Lambda^2 E, \mathcal{O}(-D)) \to \underline{Hom}(E, E(-D)),\ \psi \mapsto \frac{1}{2}\psi(\varphi(-D))^{-1}$$

gives a splitting. In particular, the H^1-factors of the corresponding long exact sequence in cohomology define an exact sequence

$$0 \to H^1(X, Ad^{\varphi}(E)(-D)) \to Ext^1(E, E(-D)) \to Ext^1(\Lambda^2 E, \mathcal{O}(-D))) \to 0$$

whose second map is just the map p_{φ}. The surjectivity of the latter provides the result. We remark that the result does not require the natural obstruction space $H^2(X, Ad^{\varphi}(-D))$, coming from the theory of principal bundles, to vanish. □

4 Deformations of quadratic sheaves

The purpose of the present chapter is to discuss some results and speculations on the deformation theory of a generalization of symplectic sheaves, namely *quadratic sheaves.* Hopefully, the following discussion will shed some more light on the constructions we introduced in Sect. 3.3, making them less *ad hoc* than they are right now. The interested reader may consult for example [FM, Ha, HL3, Ol] for a more general treatment of general deformation theory, T^i functors and deformations of coherent sheaves.

The conjectures proposed in the last part of this chapter are the main goals of a work in progress, joint with G. Scattareggia[1].

4.1 T^i spaces

We denote by *Art* the category of local Artin $\mathbb{K}-$algebras. A *functor of Artin rings* is any covariant functor $Art \to Set$ satisfying

$$F(\mathbb{K}) = \{\star\}.$$

We recall the definition of semi-small extensions.

Definition 4.1.1. A surjection of local Artin $\mathbb{K}-$algebras $A \twoheadrightarrow B$ is said to be a *semi-small extension* if the kernel I satisfies $m_A \cdot I = 0$, where m_A is the maximal ideal in A. If $I \cong \mathbb{K}$, we call it a *small extension.*

We also recall the definition of tangent and obstruction spaces for functors of Artin rings and for their natural transformations.

Definition 4.1.2. Let F be a functor of Artin rings. Two vector spaces T^1_F and T^2_F are said to be the *tangent space* and an *obstruction space* for F if for every semismall extension of Artin rings

$$0 \longrightarrow I \longrightarrow A \longrightarrow B \longrightarrow 0$$

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there exists a functorial exact sequence of groups and sets

$$0 \longrightarrow T^1_F \otimes I \longrightarrow F(A) \longrightarrow F(B) \longrightarrow T^2_F \otimes I \ .$$

Let $\phi : F \to G$ be a natural transformation of functors of Artin rings. Two vector spaces T^1_ϕ and T^2_ϕ are said to be the *tangent space* and an *obstruction space* for ϕ if for every semismall extension of Artin rings as above, there exists a functorial exact sequence of groups and sets

$$0 \longrightarrow T^1_\phi \otimes I \longrightarrow F(A) \longrightarrow F(B) \times_{G(B)} G(A) \longrightarrow T^2_\phi \otimes I \ .$$

Remark 4.1.3. The definition of tangent and obstruction spaces may be tested on small extensions only, due to the fact that any semi-small extension factorizes as a sequence of small ones.

4.2 Quadratic sheaves

Let X be a smooth projective variety and suppose (E, φ) is a quadratic sheaf on X, meaning E is a coherent torsion-free sheaf on X and $\varphi : E^{\otimes 2} \to \mathcal{O}_X$ is a morphism. We consider the following two functors $G, F : Art \to Set$.

For $A \in Ob(Art)$, the $A-$points of G are equivalence classes of $A-$flat coherent sheaves $\mathcal{E}_A$ on $X_A = Spec(A) \times X$ with an isomorphism $f : E \to \mathcal{E}_A \mid_X$. The $A-$points of F are equivalence classes of triples $(\mathcal{E}_A, f, \varphi_A)$ such that $[(\mathcal{E}_A, f)] \in G(A)$, $\varphi_A : \mathcal{E}_A^{\otimes 2} \to \mathcal{O}_{X_A}$ is a morphism and

$$\varphi = f^\star(\varphi_A \mid_X) \in Hom_{\mathcal{O}_X}(E^{\otimes 2}, \mathcal{O}_X).$$

If we assume E to be *simple,* we see both functors have a single $\mathbb{K}-$point. There is an obvious forgetful natural transformation

$$\mathfrak{f} : F \to G.$$

Remark 4.2.1. The natural transformation $\mathfrak{f}$ is representable and affine. In order to prove this, we have to verify that for any Artin algebra A and for any $\mathcal{E}_A \in G(A)$, the functor assigning to an Artin algebra B the set

$$\{((\mathcal{E}_B, \varphi_B), p) \mid p^\star \mathcal{E}_A \cong \mathcal{E}_B\} \subseteq F(B) \times Mor(Spec(B), Spec(A))$$

is representable by an affine $\mathbb{K}-$scheme. This is equivalent to proving that the underlying functor

$$Art(A) \to Set;\ (p : Spec(B) \to Spec(A)) \mapsto Hom_{\mathcal{O}_{X_B}}((p^\star \mathcal{E}_A)^{\otimes 2}, \mathcal{O}_{X_B})$$

is representable by an affine $A-$scheme; as

$$Hom_{\mathcal{O}_{X_B}}((p^\star \mathcal{E}_A)^{\otimes 2}, \mathcal{O}_{X_B}) \cong Hom_{\mathcal{O}_{X_A}}({\mathcal{E}_A}^{\otimes 2}, p_\star(\mathcal{O}_{X_B})) \cong$$

$$\cong Hom_{\mathcal{O}_{X_A}}({\mathcal{E}_A}^{\otimes 2}, \mathcal{O}_{X_A})) \otimes_{\mathbb{K}} B,$$

clearly the vector space $Hom_{\mathcal{O}_{X_A}}({\mathcal{E}_A}^{\otimes 2}, \mathcal{O}_{X_A}))$ does the job.

Lemma 4.2.2. *The vector spaces $Hom(E^{\otimes 2}, \mathcal{O}_X)$ and $Ext^1(E^{\otimes 2}, \mathcal{O}_X)$ are respectively the tangent space and an obstruction space for $\mathfrak{f}$.*

Proof. By Remk. 4.2.1, we deduce that $Hom(E^{\otimes 2}, \mathcal{O}_X) \cong T^1_\phi$. Let $p : A \twoheadrightarrow B$ be a small extension. The set $F(B) \underset{G(B)}{\times} G(A)$ is in bijection with the set of equivalence classes of pairs $(\mathcal{E}^A, \varphi_B)$ such that $\mathcal{E}^A \in G(A)$ and $\varphi^B \in Hom_{\mathcal{O}_{X_B}}(p^\star(\mathcal{E}^A)^{\otimes 2}, \mathcal{O}_{X_B})$. Let us tensor the extension

$$0 \longrightarrow \mathbb{K} \longrightarrow A \longrightarrow B \longrightarrow 0$$

by $\mathcal{O}_{X_A}$, getting an exact sequence of $\mathcal{O}_{X_A}-$modules

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_{X_A} \longrightarrow \mathcal{O}_{X_B} \longrightarrow 0\,. \tag{4.2.1}$$

Here, we are identifying $\mathcal{O}_{X_B}$ with the quotient $p_\star \mathcal{O}_{X_B}$ of $\mathcal{O}_{X_A}$, so that we have a natural isomorphism

$$Hom_{\mathcal{O}_{X_B}}(p^\star(\mathcal{E}^A)^{\otimes 2}, \mathcal{O}_{X_B}) \cong Hom_{\mathcal{O}_{X_A}}(\mathcal{E}^{A \otimes 2}, \mathcal{O}_{X_B}).$$

We obtain a canonical map

$$ob : F(B) \underset{G(B)}{\times} G(A) \to Ext^1(E^{\otimes 2}, \mathcal{O}_X)$$

by means of a coboundary map in the following way: consider the long exact sequence induced applying the functor $Hom_{\mathcal{O}_{X_A}}(\mathcal{E}^{A\otimes 2}, \_)$ to the sequence 4.2.1. In the light of Lemma 2.1.10 and of the flatness hypothesis on the families of sheaves under consideration, we obtain for every nonnegative integer i natural isomorphisms

$$Ext^i_{\mathcal{O}_{X_A}}(\mathcal{E}^{A \otimes 2}, \mathcal{O}_X) \cong Ext^i_{\mathcal{O}_X}(E^{\otimes 2}, \mathcal{O}_X)$$

and

$$Ext^i_{\mathcal{O}_{X_A}}(\mathcal{E}^{A\otimes 2}, \mathcal{O}_{X_B}) \cong Ext^i_{\mathcal{O}_{X_B}}(\mathcal{E}^{B\otimes 2}, \mathcal{O}_{X_B}).$$

It turns out then that the coboundary morphism

$$Hom_{\mathcal{O}_{X_A}}(\mathcal{E}^{A\otimes 2}, \mathcal{O}_{X_B}) \to Ext^i_{\mathcal{O}_{X_A}}(\mathcal{E}^{A\otimes 2}, \mathcal{O}_X)$$

yields a map

$$Hom_{\mathcal{O}_{X_A}}(\mathcal{E}^{A\otimes 2}, \mathcal{O}_{X_B}) \to Ext^i_{\mathcal{O}_X}(E^{\otimes 2}, \mathcal{O}_X),$$

providing the sought for definition for *ob*. Now, $ob(\mathcal{E}^A, \varphi_B)$ vanishes if and only if φ_B lifts to $\varphi_A \in Hom_{\mathcal{O}_{X_A}}(\mathcal{E}^{A\otimes 2}, \mathcal{O}_{X_A})$ by exactness, but this is indeed equivalent to ask $(\mathcal{E}^A, \varphi_B)$ to come from $F(A)$. This concludes the proof. □

4.3 The adjoint complex

As in the previous section, let X be $d-$dimensional smooth projective variety. Given an object P in the bounded derived category $D^b(X)$, we denote by $P^{R\vee}$ the object

$$R\underline{Hom}_{\mathcal{O}_X}(P, \mathcal{O}_X).$$

We recall that if $Q \in D^b(X)$ and both P and Q are perfect, we have natural isomorphisms

$$P^{R\vee} \overset{L}{\otimes} Q \cong R\underline{Hom}_{\mathcal{O}_X}(P, Q);$$
$$i : P \overset{L}{\otimes} Q \to Q \overset{L}{\otimes} P. \tag{4.3.1}$$

Definition 4.3.1. Let (E, φ) be a quadratic sheaf on X. We define the *adjoint morphism*

$$ad^{\varphi} \in Hom_{D^b(X)}(E^{R\vee} \overset{L}{\otimes} E, E^{R\vee} \overset{L}{\otimes} E^{R\vee})$$

as follows. Making an abuse of notation, we call φ the composition of $\varphi : E \to E^{\vee}$ with the natural map $E^{\vee} \to E^{R\vee}$ and consider the switch morphism

$$i : E^{R\vee} \overset{L}{\otimes} E^{R\vee} \to E^{R\vee} \overset{L}{\otimes} E^{R\vee};$$

let us define

$$ad^{\varphi} = 1_{E^{R\vee}} \overset{L}{\otimes} \varphi + i \circ (1_{E^{R\vee}} \overset{L}{\otimes} \varphi).$$

We call $Ad^{\varphi}(E) \in D^b(X)$ the object (unique up to quasi-isomorphism) fitting in an exact triangle

$$Ad^{\varphi} \longrightarrow E^{R\vee} \overset{L}{\otimes} E \xrightarrow{ad^{\varphi}} E^{R\vee} \overset{L}{\otimes} E^{R\vee} \longrightarrow Ad^{\varphi}[1]$$

Remark 4.3.2. It is possible to express the morphism ad^{φ} in a more explicit fashion once we are given a free resolution of E. To simplify the exposition we assume for a moment X is a projective surface, although the general case can be treated analogously without any conceptual effort. Choose a locally free resolution of E, i.e a complex

$$0 \longrightarrow F_{-1} \xrightarrow{d} F_0 \xrightarrow{q} E \ ;$$

this allows to represent $E^{R\vee}$ by the complex

$$F_0^{\vee} \xrightarrow{d^{\vee}} F_{-1}^{\vee}$$

in degrees $0, 1$. Consequently we obtain:

- the complex

$$F_0^\vee \otimes F_{-1} \longrightarrow (F_0^\vee \otimes F_0) \oplus (F_{-1}^\vee \otimes F_{-1}) \longrightarrow F_{-1}^\vee \otimes F_0 \ ,$$

  in degrees $-1, 0, 1$, representing $E^{R\vee} \overset{L}{\otimes} E$, with differentials

$$\delta_{-1} = \begin{pmatrix} 1 \otimes d \\ d^\vee \otimes 1 \end{pmatrix}, \ \delta_0 = \begin{pmatrix} d^\vee \otimes 1 & -1 \otimes d \end{pmatrix};$$

- the complex

$$F_0^\vee \otimes F_0^\vee \longrightarrow (F_0^\vee \otimes F_{-1}^\vee) \oplus (F_{-1}^\vee \otimes F_0^\vee) \longrightarrow F_{-1}^\vee \otimes F_{-1}^\vee \ ,$$

  in degrees $0, 1, 2$, representing $E^{R\vee} \overset{L}{\otimes} E^{R\vee}$, with differentials

$$\delta_0' = \begin{pmatrix} 1 \otimes d^\vee \\ d^\vee \otimes 1 \end{pmatrix}, \ \delta_0 = \begin{pmatrix} d^\vee \otimes 1 & -1 \otimes d^\vee \end{pmatrix}.$$

If we denote $\varphi_0 : F_0 \to F_0^\vee$ the composition

$$F_0 \to E \overset{\varphi}{\to} E^\vee \to F_0^\vee$$

we can define a morphism of complexes

$$\begin{array}{ccccccc}
F_0^\vee \otimes F_{-1} & \longrightarrow & (F_0^\vee \otimes F_0) \oplus (F_{-1}^\vee \otimes F_{-1}) & \longrightarrow & F_{-1}^\vee \otimes F_0 & \longrightarrow & 0 \\
\downarrow & & \downarrow \chi & & \downarrow \psi & & \downarrow \\
0 & \longrightarrow & F_0^\vee \otimes F_0^\vee & \longrightarrow & (F_0^\vee \otimes F_{-1}^\vee) \oplus (F_{-1}^\vee \otimes F_0^\vee) & \longrightarrow & F_{-1}^\vee \otimes F_{-1}^\vee
\end{array}$$

where

$$\chi = \begin{pmatrix} 1 \otimes \varphi_0 + i_{00} \circ (1 \otimes \varphi_0) & 0 \end{pmatrix}$$

and

$$\psi = \begin{pmatrix} i_{0,-1} \circ (1 \otimes \varphi_0) \\ 1 \otimes \varphi_0 \end{pmatrix},$$

which can be verified to induce ad^φ.

Let us consider the natural morphism $E \overset{L}{\otimes} E \to E \otimes E$ in the derived category $D^b(X)$.

Remark 4.3.3. As E is torsion-free, we know that the Tor sheaves

$$\underline{Tor}_i(E, E) = H^{-i}(E \overset{L}{\otimes} E)$$

are supported in codimension at least 2 for any $i > 0$; indeed, or every point $x \in X$, there is a natural isomorphism

$$\underline{Tor}_i^{\mathcal{O}_X}(E,E)_x \cong Tor_i^{\mathcal{O}_{X,x}}(E_x,E_x)$$

and the latter group vanishes whenever E_x is free.

Lemma 4.3.4. *The natural map*

$$Ext^1(E \otimes E, \mathcal{O}_X) \to Ext^1(E \overset{L}{\otimes} E, \mathcal{O}_X)$$

is an isomorphism.

Proof. Let us consider the induced exact triangle

$$K \longrightarrow E \overset{L}{\otimes} E \longrightarrow E \otimes E \longrightarrow K[1]\ .$$

K can be realized by means of the truncation $\tau_{\leq 0}(E \overset{L}{\otimes} E)$. For this reason, we can apply Remk. 4.3.3 to show that the sheaves $H^i(K)$ are supported in codimension at least 2 for any i. Applying $RHom(\_, \mathcal{O}_X)$ to the triangle and considering the corresponding long exact sequence, we see that to prove the lemma it is enough to prove the vanishings

$$\mathbb{H}om(K, \mathcal{O}_X) = 0;$$

$$\mathbb{E}xt^1(K, \mathcal{O}_X) = \mathbb{H}om(K, \mathcal{O}_X[1]) = 0.$$

Let us start from the first vanishing. Any element of the group $\mathbb{H}om(K, \mathcal{O}_X) = Hom_{D^b(X)}(K, \mathcal{O}_X)$ may be represented by a complex $M^\bullet$ concentrated in nonpositive degrees together with a quasi-isomorphism $M^\bullet \to K$ and a morphism $M^\bullet \to \mathcal{O}_X$. Let us write:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M^{-2} & \xrightarrow{d^{-2}} & M^{-1} & \xrightarrow{d^{-1}} & M^0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \chi & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0
\end{array}$$

We see that χ has to factor through a morphism $M^0/im(d^{-1}) \to \mathcal{O}_X$, but as $M^0/im(d^{-1})$ is a torsion sheaf (isomorphic to $H^0(K)$); we conclude that $\chi = 0$. For the other vanishing, we similarly obtain a diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M^{-2} & \xrightarrow{d^{-2}} & M^{-1} & \xrightarrow{d^{-1}} & M^0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \chi & & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

Its commutativity tells us χ factors through $M^{-1}/im(d^{-2})$. As the restriction to the torsion sheaf $ker(d^{-1})/im(d^{-2}) \cong H^{-1}(K)$ obviously vanishes, we see that χ induces a map $im(d^{-1}) \to \mathcal{O}_X$. In addition, looking at the $Hom(_, \mathcal{O}_X)-$long exact sequence from

$$0 \to im(d^{-1}) \to M^0 \to H^0(K) \to 0$$

and using

$$Ext^1(H^1(K), \mathcal{O}_X) \cong H^{n-1}(X, H^1(K))^\vee = 0,$$

we discover that χ actually lifts to a morphism $\bar{\chi} : M^0 \to \mathcal{O}_X$. But then the diagram

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & M^{-2} & \xrightarrow{d^{-2}} & M^{-1} & \xrightarrow{d^{-1}} & M^0 & \longrightarrow & 0 \\
\downarrow & & \downarrow & \overset{0}{\swarrow} & \downarrow \chi & \overset{\bar{\chi}}{\swarrow} & \downarrow & & \downarrow \\
\cdots & \longrightarrow & 0 & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

shows that our morphism $M^\bullet \to \mathcal{O}_X$ is in fact nullhomotopic, providing the required vanishing. □

The reason why we introduced the adjoint complex lies in the following conjecture.

Conjecture 4.3.5. *The exact triangle*

$$Ad^\varphi \longrightarrow E^{R\vee} \overset{L}{\otimes} E \xrightarrow{ad^\varphi} E^{R\vee} \overset{L}{\otimes} E^{R\vee} \longrightarrow Ad^\varphi[1]$$

rules the deformation theory of $\mathfrak{f} : F \to G$, *i.e. for* $i = 1, 2$ *there are isomorphisms*

1. $\mathbb{H}^i(X, Ad^\varphi) \cong T^i_F$;
2. $\mathbb{H}^i(X, E^{R\vee} \overset{L}{\otimes} E) \cong T^i_G$;
3. $\mathbb{H}^{i-1}(X, E^{R\vee} \overset{L}{\otimes} E^{R\vee}) \cong T^i_{\mathfrak{f}}$.

As obstruction spaces are not uniquely defined, the claimed existence of isomorphisms with the T^2s just means the corresponding hypercohomology groups are in fact obstruction spaces.

Items 2 and 3 in the conjecture are known. For item 2, we note

$$\mathbb{H}^i(E^{R\vee} \overset{L}{\otimes} E) \cong Ext^i(E, E),$$

and the claim reduces to the existence of equalities

$$Ext^i(E, E) \cong T^i_G$$

which are well known, see for example [HL3]. We already discussed item 3, which is obtained by putting together Lemmas 4.2.2 and 4.3.4.

4.4 The framed symplectic adjoint complex

Let us go back to framed symplectic sheaves on (X, D), $dim(X) = 2$. Suppose (E, a, φ) is a framed symplectic bundle. We mentioned in the proof of Cor. 3.3.5 that the exact sequence

$$0 \longrightarrow Ad^{\varphi}(E)(-D) \longrightarrow \underline{Hom}(E, E)(-D) \longrightarrow \underline{Hom}(\Lambda^2 E, \mathcal{O}_X(-D)) \longrightarrow 0$$

rules the deformation theory of (E, a, φ) in the sense of Conj. 4.3.5. This actually implements for symplectic bundles a general fact in the theory of principal bundles (explained, for instance, in [So]): the T^i functors for moduli of principal bundles are i^{th} cohomologies of the associated vector bundles via the adjoint representations. The definition of adjoint complex we want to give is aimed to generalize this concept to the the context of quadratic torsion free sheaves.

Definition 4.4.1. Given an object $C \in D^b(X)$, we define the *derived square wedge* $C \overset{L}{\wedge} C$ to be the unique (up to quasi isomorphism) object in $D^b(X)$ fitting in the exact triangle

$$C \overset{L}{\otimes} C \xrightarrow{i+1} C \overset{L}{\otimes} C \longrightarrow C \overset{L}{\wedge} C \longrightarrow C \overset{L}{\otimes} C[1] \ ,$$

where i was defined in 4.3.1 (and $1 = 1_{C \overset{L}{\otimes} C}$).

Remark 4.4.2. Let $E \in Coh(X)$ and let

$$\cdots \longrightarrow F_{-1} \xrightarrow{d} F_0 \xrightarrow{q} E$$

be a finite free resolution of E. Each entry T_k of the total complex associated to the double complex $F_\bullet \otimes F_\bullet$ has a natural involution i_k. With some patience, one can show that the term-by-term quotient $T_k/im(i_k + 1_{T_k})$ inherits differentials from T_k and that the induced complex is a representative for $E \overset{L}{\wedge} E \in D^b(X)$. Furthermore, if $\varphi : \Lambda^2 E \to \mathcal{O}_X$ is a skew form on E, we can employ Remk. 4.3.2 to show that the adjoint map

$$ad^{\varphi} : R\underline{Hom}(E, E) \to R\underline{Hom}(E \overset{L}{\otimes} E, \mathcal{O}_X)$$

from Defn. 4.3.1 lifts to a map

$$R\underline{Hom}(E, E) \to R\underline{Hom}(E \overset{L}{\wedge} E, \mathcal{O}_X)$$

which we keep calling ad^φ by abuse of notation. We may similarly define $Ad^\varphi(E) \in D^b(X)$ to be the "kernel" ad^φ.

The notation we just introduced allows to restate Conj. 4.3.5 in a framed symplectic flavor.

Conjecture 4.4.3. *Let (E, a, φ) be a framed symplectic sheaf. The exact triangle*

$$Ad^\varphi(-D) \longrightarrow R\underline{Hom}(E,E)(-D) \xrightarrow{ad^\varphi} R\underline{Hom}(E \overset{L}{\wedge} E, \mathcal{O}_X)(-D) \longrightarrow Ad^\varphi(-D)[1]$$

rules the deformation theory of (the localization at $[E, a, \varphi] \mapsto [E, a]$ of)

$$\iota : \mathcal{M}^D_{X,\Omega} \to \mathcal{M}^D_X,$$

i.e.:

1. *there is an isomorphism*

$$\mathbb{H}^1(X, Ad^\varphi(-D)) \cong T_{[E,a,\varphi]}\mathcal{M}^D_{X,\Omega}$$

   and $\mathbb{H}^2(X, Ad^\varphi(-D))$ is an obstruction space for $\mathcal{M}^D_{X,\Omega}$ at $[E, a, \varphi]$;

2. *there is an isomorphism*

$$\mathbb{H}^1(X, R\underline{Hom}(E,E)(-D)) \cong T_{[E,a]}\mathcal{M}^D_X$$

   and $\mathbb{H}^2(X, R\underline{Hom}(E,E)(-D))$ is an obstruction space for $\mathcal{M}^D_X$ at $[E, a]$;

3. $\mathbb{H}^{i-1}(X, R\underline{Hom}(E \overset{L}{\wedge} E, \mathcal{O}_X)(-D)) \cong T^i_\iota$.

Remark 4.4.4. We conclude this chapter with some quick comments on the conjecture.

- As the conjecture is just a twisted skew-symmetric variant of Conj. 4.3.5, it is possible to verify that it holds if the latter does.
- The only new piece of information is the statement about the obstruction to $\mathcal{M}^D_{X,\Omega}$. Indeed, the tangent and obstruction spaces for $\mathcal{M}^D_X$ and the tangent space to $\mathcal{M}^D_{X,\Omega}$ were already discussed in Chapter 2 and 3, respectively.
- The opening remarks of the present section show that the conjecture holds if we suppose E to be locally free.
- The hypothesis of Prop. 3.3.4, together with the vanishing $Ext^2(E, E(-D)) = 0$, imply $\mathbb{H}^2(X, Ad^\varphi(-D)) = 0$.

5 Some matrix analysis

From now on, we only consider $\mathbb{K} = \mathbb{C}$.

This short chapter is meant to introduce some definitions and results in matrix theory for later use in Chapters 6 and 7. The interested reader may check the standard referece [HJ] for a general discussion on the subject and in particular for the proof of the following basic lemma.

Lemma 5.0.5. *Let $A \in Mat_{\mathbb{C}}(n)$ be a matrix. The following statements are equivalent:*

1. *the minimal polynomial and the characteristic polynomial of A coincide;*
2. *if $[A, B] = 0$ for a matrix $B \in Mat_{\mathbb{C}}(n)$, there exists a polynomial $p(t) \in \mathbb{C}[t]$, $deg(p) \leq n - 1$, such that $B = p(A)$;*
3. *the geometric multiplicity of any eigenvalue of A is 1;*
4. *there exists a vector $v \in \mathbb{C}^n$ such that*

$$< v, Av, \ldots, A^{n-1}v >= \mathbb{C}^n.$$

In case one of the above conditions holds for A, we call A a *nonderogatory (or cyclic)* matrix.

Remark 5.0.6. The subset of $Mat_{\mathbb{C}}(n)$ consisting of nonderogatory matrices is open and invariant under conjugation, due to item 4 in Lemma 5.0.5. Indeed, it is the complement of the closed subset cut out by the ideal generated by the coefficients of the polynomial

$$det \begin{pmatrix} v & Av & \cdots & A^{n-1}v \end{pmatrix} \in \mathbb{C}[v_1, \ldots, v_n].$$

Examples of nonderogatory matrices are diagonal matrices with distinct eigenvalues and Jordan blocks. Item 3 in Lemma 5.0.5 gives a criterion to distinguish a nonderogatory matrix from its Jordan form: distinct Jordan blocks have to correspond to distinct eigenvalues.

Remark 5.0.7. Let $B \in Mat_{\mathbb{C}}(n)$. Then there exists a nonderogatory matrix N such that $[N, B] = 0$. To see why, just put $B = J_1(\lambda_1) \oplus \cdots \oplus J_k(\lambda_k)$ in Jordan form, and note that any matrix $N = J_1(\lambda'_1) \oplus \cdots \oplus J_k(\lambda'_k)$ whose ordered Jordan blocks have the same sizes commutes with B. If we choose pairwise distinct λ'_is, we are done.

We quote the main theorem of [TZ]:

Theorem 5.0.8. *For any fixed matrix A there exists a nonsingular symmetric matrix g such that $gAg^{-1} = A^\top$. Any matrix g transforming A into its transpose is symmetric if and only if A is nonderogatory.*

A consequence of this is that any complex matrix A is similar to a symmetric one. Let $gAg^{-1} = A^\top$ with g symmetric. Write $g = s \cdot s^\top$ with s nonsingular. This gives

$$s^\top A s^{-\top} = s^{-1} A^\top s = (s^\top A s^{-\top})^\top.$$

As an immediate corollary we get:

Corollary 5.0.9. *Suppose (A, B) is a pair of commuting matrices, and suppose A is nonderogatory. There exists a nonsingular matrix g such that $gAg^{-1}$and gBg^{-1} are symmetric.*

Remark 5.0.10. Let us consider $S^2(\mathbb{C}^{n\star})$ and $\Lambda^2(\mathbb{C}^{n\star})$ the linear subspaces in $Mat_{\mathbb{C}}(n)$ consisting of symmetric ans skew-symmetric matrices. Let $A \in S^2(\mathbb{C}^{n\star})$. If A is nonderogatory, the linear map

$$[A, \_] : S^2(\mathbb{C}^{n\star}) \to \Lambda^2(\mathbb{C}^{n\star})$$

is onto. Indeed, the kernel coincides with the space of polynomials in A, which has dimension exactly $n = dim(S^2(\mathbb{C}^{n\star})) - \Lambda^2(\mathbb{C}^{n\star})$.

We state and prove here two technical lemmas.

Lemma 5.0.11. *Let $S \in Mat(k \times k)$ be symmetric and $\sigma \in Mat((n-k) \times (n-k))$. Suppose that S and σ share no eigenvalues. Then the linear map*

$$T \in End(Mat((n-k) \times k)),\ T(v) = vS - \sigma v$$

is invertible.

Proof. Choose coordinates on $\mathbb{C}^{n-k}$ so that σ is lower triangular:

$$\sigma = \begin{pmatrix} s_1 & 0 & \cdots \\ \star & s_2 & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

Suppose there exists a matrix $v = \begin{pmatrix} v_1 \\ \vdots \\ v_{n-k} \end{pmatrix}$, $v_i \in \mathbb{C}^k$ satisfying $vS = \sigma v$. We have $\sigma v = \begin{pmatrix} s_1 v_1 \\ \star \\ \star \end{pmatrix} = vS = \begin{pmatrix} v_1 S \\ \vdots \\ v_{n-k} S \end{pmatrix}$. We obtain $S(v_1^\top) = s_1 v_1^\top$. So if S and σ have disjoint spectra, T must be injective, i.e. an isomorphism. □

Lemma 5.0.12. *Let R be a $n-$dimensional vector space, let $T \in End_{\mathbb{C}}(R)$ and $L \subseteq R$ be a subspace. Let $v \in R$ be $T-$reachable from L, meaning*

$$v \in L + TL + T^2 L + \cdots + T^{n-1} L.$$

There exist parametrized curves $r : \mathbb{C} \to R$ and $l : \mathbb{C} \to L$ with $r(0) = 0$, $l(0) = 0$ satisfying

$$(T - t \cdot Id_R) r(t) = t \cdot v + l(t).$$

Proof. Write $v = \sum_{i=0}^{n-1} T^i l_i$, $l_i \in L$. Define

$$r(t) = \sum_{i=1}^{n-1} t^i \Big(\sum_{j=i}^{n-1} T^{j-i} l_j\Big).$$

We obtain

$$(T - tId_R) r(t) = \sum_{i=1}^{n-1} t^i \Big(\sum_{j=i}^{n-1} T^{j-i+1} l_j\Big) - \sum_{i=1}^{n-1} t^{i+1} \Big(\sum_{j=i}^{n-1} T^{j-i} l_j\Big) =$$

$$= t \cdot v - t l_0 + \sum_{i=2}^{n-1} t^i \Big(\sum_{j=i}^{n-1} T^{j-i+1} l_j\Big) - \sum_{i=1}^{n-2} t^{i+1} \Big(\sum_{j=i+1}^{n-1} T^{j-i} l_j\Big) - \sum_{i=1}^{n-2} t^{i+1} l_i =$$

$$= t \cdot v - \sum_{i=0}^{n-1} t^{i+1} l_i + \sum_{i=2}^{n-1} t^i \Big(\sum_{j=i}^{n-1} T^{j-i+1} l_j\Big) - \sum_{k=2}^{n-1} t^k \Big(\sum_{j=k}^{n-1} T^{j-k+1} l_k\Big) =$$

$$= t \cdot v - \sum_{i=0}^{n-1} t^{i+1} l_i.$$

So, it is enough to set

$$l(t) = -\sum_{i=0}^{n-1} t^{i+1} l_i.$$

□

6 The moduli space on the plane

We concentrate now on the case of $\mathbb{P}^2$; we fix homogeneous coordinates $[z_0, z_1, z_2]$ and take the line $l_\infty = \{z_0 = 0\}$ as framing divisor.

6.1 Framed sheaves on $\mathbb{P}^2$

The main reference for this section is [Na, Chap 2]. Let n and r be nonnegative integers. Following the notation in *loc. cit.,* we shall denote the moduli space $\mathcal{M}_{\mathbb{P}^2,l_\infty}(n,r)$ and its locally free locus $\mathcal{M}^{reg}_{\mathbb{P}^2,l_\infty}(n,r)$ by $\mathcal{M}(n,r)$ and $\mathcal{M}^{reg}(n,r)$, respectively. The choice $n \geq 0$ guarantees non-emptiness of these spaces.

Let $V \cong \mathbb{C}^n$, $W \cong \mathbb{C}^r$ be vector spaces.

Definition 6.1.1. We define the *variety of ADHM configurations* of type (n,r) to be the subvariety $\mathbb{M}(n,r)$ of the affine space

$$End(V)^{\oplus 2} \oplus Hom(W,V) \oplus Hom(V,W) = \{(A,B,I,J)\}$$

cut out by the equation

$$[A,B] + IJ = 0.$$

We say that a configuration (A,B,I,J) is

- *stable* if there exists no subspace $0 \subseteq S \subsetneq V$ such that $A(S) \subseteq S$, $B(S) \subseteq S$ and $im(I) \subseteq S$;
- *co-stable* if there exists no subspace $0 \subsetneq S \subseteq V$ such that $A(S) \subseteq S$, $B(S) \subseteq S$ and $ker(J) \supseteq S$;
- *regular* if it is stable and co-stable.

The variety $\mathbb{M}(n,r)$ is naturally acted on by $GL(V)$:

$$g \cdot (A,B,I,J) = (gAg^{-1}, gBg^{-1}, gI, Jg^{-1}).$$

The open subvarieties $\mathbb{M}^s(n,r)$, $\mathbb{M}^c(n,r)$ and $\mathbb{M}^{reg}(n,r)$ of stable, co-stable and regular data are invariant with respect to this action.

Consider the vector bundles $\mathcal{U} = \mathcal{O}(-1) \otimes V$, $\mathcal{W} = \mathcal{O} \otimes (V^{\oplus 2} \oplus W)$ and $\mathcal{T} = \mathcal{O}(1) \otimes V$. We have a map

$$(\alpha, \beta) : End(V)^{\oplus 2} \oplus Hom(W, V) \oplus Hom(V, W) \to Hom_{\mathcal{O}_X}(\mathcal{U}, \mathcal{W}) \times Hom_{\mathcal{O}_X}(\mathcal{W}, \mathcal{T})$$

which assigns to a quadruple (A, B, I, J) the pair of morphisms

$$\alpha(A, B, I, J) = \begin{pmatrix} z_0 A + z_1 \\ z_0 B + z_2 \\ z_0 J \end{pmatrix}, \ \beta(A, B, I, J) = (-z_0 B - z_2, z_0 A + z_1, z_0 I).$$

One can check that a configuration $\xi = (A, B, I, J)$ sits in $\mathbb{M}(n, r)$ if and only if $\beta(\xi) \circ \alpha(\xi) = 0$. Furthermore, $\beta(\xi)$ is surjective if and only if ξ is stable and $\alpha(\xi)$ is injective *as a map of bundles* if and only if ξ is co-stable. We note that $\alpha(\xi)$ is always a monomorphism of sheaves. We thus assigned to every $\xi \in \mathbb{M}(n, r)^s$ a monad on $\mathbb{P}^2$, whose cohomology $E(\xi)$ is a torsion free sheaf. A simple computation with Chern characters shows $rank(E(\xi)) = r$, $c_1(E(\xi)) = 0$ and $c_2(E(\xi)) = n$. If we pull back the monad to l_∞, we obtain an isomorphism $a : E(\xi) \mid_{l_\infty} \to \mathcal{O}_{l_\infty} \otimes W$. The fundamental result in [Na, Chap 2] states that every framed sheaf on $\mathbb{P}^2$ is obtained by means of this procedure. In fact, this really leads to an alternative description of the moduli space $\mathcal{M}(n, r)$.

Theorem 6.1.2. *The $GL(V)$ action on $\mathbb{M}^s(n, r)$ is free and locally proper. The quotient $\mathbb{M}^s(n, r)/GL(V)$ exists as an algebraic variety, which is smooth of dimension $2rn$ and connected. There exists an isomorphism*

$$\mathbb{M}^s(n, r)/GL(V) \cong \mathcal{M}(n, r)$$

which maps the open subscheme $\mathbb{M}^{reg}(n, r)/GL(V)$ onto $\mathcal{M}^{reg}(n, r)$.

Let us fix $(E_{\mathcal{M}}, a_{\mathcal{M}})$ a universal framed sheaf on $\mathcal{M}(n, r) \times X$. As we just saw that any framed sheaf on $\mathbb{P}^2$ can be obtained as the cohomology of a monad, it makes sense to ask whether this procedure can be made universal, in virtue of the just stated isomorphism. The answer is affirmative:

Proposition 6.1.3. *There exists a monad on $\mathcal{M}(n, r) \times \mathbb{P}^2$*

$$M : \ \mathcal{U}_{\mathcal{M}} \xrightarrow{\alpha_{\mathcal{M}}} \mathcal{W}_{\mathcal{M}} \xrightarrow{\beta_{\mathcal{M}}} \mathcal{T}_{\mathcal{M}}$$

whose cohomology is isomorphic to the universal sheaf $E_{\mathcal{M}}$ and whose pullback to $\mathcal{M}(n, r) \times l$ induces the universal trivialization $a_{\mathcal{M}}$. In addition, there exists an open cover $\mathcal{M}(n, r) = \bigcup_i U_i$ by open affine subschemes such that the following conditions are satisfied:

1. *the principal $GL(n)-$bundle $\mathbb{M}^s(n,r) \to \mathcal{M}(n,r)$ is trivial on U_i;*
2. *the pullback of the universal monad to $U_i \times \mathbb{P}^2$ is isomorphic to the monad*

$$M_i : \quad p_i^\star \mathcal{U} \xrightarrow{\alpha_i} p_i^\star \mathcal{W} \xrightarrow{\beta_i} p_i^\star \mathcal{T}$$

where p_i is the projection $U_i \times \mathbb{P}^2 \to \mathbb{P}^2$ and

$$(\alpha_i, \beta_i) \in Hom(U_i, Hom(\mathcal{U}, \mathcal{W}) \oplus Hom(\mathcal{W}, \mathcal{T}))$$

is defined as the composition $(\alpha, \beta) \circ \sigma_i$, where $\sigma_i : U_i \to \mathbb{M}(n,r)^s$ is any section.

Remark 6.1.4. The proof of this proposition can be found in [He, Chap 4-7], where the result is estabilished for a bigger class of surfaces, namely multiple blow-ups of the projective plane at distinct points. A similar result holds also for Hirzebruch surfaces, despite in this case a description by simple linear data as above is not available, as the entries of the monads are no longer trivial bundles. See [BBR] for a detailed discussion of the topic.

We conclude this section illustrating a recipe which will be fundamental later on.

Remark 6.1.5. Let (E, a) be a framed sheaf on $\mathbb{P}^2$. The double dual $E^{\vee\vee}$ is a locally free sheaf on $\mathbb{P}^2$ fitting into an exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow E^{\vee\vee}/E \longrightarrow 0\,.$$

Since E is trivial on a neighborhood of l, the double dual inherits a framing $a^{\vee\vee}$, hence we think of $E^{\vee\vee}$ as a framed bundle on $\mathbb{P}^2$. A simple computation with Chern characters shows $c_2(E^{\vee\vee}) = c_2(E) - length(E^{\vee\vee}/E)$. Let (E, a) be represented by an ADHM quadruple $\xi = (A, B, I, J)$. We present a canonical procedure to extract an ADHM quadruple for the double dual.

Let $S \subseteq V$ be the maximal $A, B-$stable subspace such that $S \subseteq ker(J)$. One has $S = 0 \iff E$ is locally free. Call $\bar{V} = V/S$. As S is A, B stable we get induced endomnorphisms $\bar{A}, \bar{B} \in End_{\mathbb{C}}(\bar{V})$. Call $\bar{I}$ and $\bar{J}$ the composition $W \xrightarrow{I} V \to \bar{V}$ and the map $\bar{V} \to W$ induced by J, respectively. As any $\bar{A}, \bar{B}-$stable subspace of $\bar{V}$ gives rise to an A, B-stable subspace of V, we can infer regularity of the resulting datum $\bar{\xi} = (\bar{A}, \bar{B}, \bar{I}, \bar{J})$. Arguing on the morphism of monads on $\mathbb{P}^2$

$$\begin{array}{ccccc} \mathcal{O}(-1) \otimes V & \longrightarrow & \mathcal{O} \otimes (V^{\oplus 2} \oplus W) & \longrightarrow & \mathcal{O}(1) \otimes V \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{O}(-1) \otimes \bar{V} & \longrightarrow & \mathcal{O} \otimes (\bar{V}^{\oplus 2} \oplus W) & \longrightarrow & \mathcal{O}(1) \otimes \bar{V} \end{array}$$

we just defined, it is also possible to show that $\bar{\xi}$ indeed represents $E^{\vee\vee}$.

6.2 Framed symplectic sheaves on $\mathbb{P}^2$

In this section we want to present an ADHM description for framed symplectic sheaves, together with a symplectic analogue of Thm 6.1.2. We recall that (E, a, φ) is a framed symplectic sheaf on $\mathbb{P}^2$ if and only if the double dual $(E^{\vee\vee}, a^{\vee\vee}, -\varphi^{\vee})$ is a framed symplectic bundle, see Remk. 3.1.2. Our description is a slight generalization of the ADHM construction in the locally free case, mentioned for example in [Do] (see also [JMW] for a modern exposition). We start revising this construction.

Let us fix a symplectic vector space (W, Ω). Let (E, a) be a framed bundle on $\mathbb{P}^2$, identified with the cohomology of a monad

$$M: \ \mathcal{U} \xrightarrow{\alpha} \mathcal{W} \xrightarrow{\beta} \mathcal{T}\ .$$

Let $\varphi : E \to E^{\vee}$ be a symplectic form compatible with the framing. The local freeness of E guarantees the dual complex $M^{\vee}$ is still a monad with cohomology $E^{\vee}$, and φ lifts uniquely to a morphism $\Phi : M \to M^{\vee}$ by Prop. 2.1.9. We write:

$$\begin{array}{ccccc}
\mathcal{O}(-1)\otimes V & \xrightarrow{\alpha} & \mathcal{O}\otimes(V^{\oplus 2}\oplus W) & \xrightarrow{\beta} & \mathcal{O}(1)\otimes V \\
\downarrow{\scriptstyle G_1} & & \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle G_2} \\
\mathcal{O}(-1)\otimes V^{\vee} & \xrightarrow{\beta^{\vee}} & \mathcal{O}\otimes((V^{\vee})^{\oplus 2}\oplus W^{\vee}) & \xrightarrow{\alpha^{\vee}} & \mathcal{O}(1)\otimes V^{\vee}
\end{array}$$

The requirement $\varphi^{\vee} = -\varphi$ gives $\Phi^{\vee} = -\Phi$, which means $F^{\vee} = -F$ and $G_2^{\vee} = -G_1 = G$. In [JMW] it is proved that the commutativity of the diagram together with the compatibility on the framing forces F to have the form

$$\begin{pmatrix} 0 & G & 0 \\ -G & 0 & 0 \\ 0 & 0 & \Omega \end{pmatrix},$$

which in turn tells $G^{\vee} = G$. Furthermore, if $(\alpha, \beta) = (\alpha, \beta)(A, B, I, J)$, we obtain the equations

$$\begin{cases} GA = A^{\vee}G \\ GB = B^{\vee}G \\ J = -\Omega^{-1}I^{\vee}G \end{cases} \tag{6.2.1}$$

Finally, as φ is an isomorphism, we have to ask $det(G) \neq 0$.

Suppose now (E, a, φ) is a symplectic sheaf, and fix an ADHM datum (A, B, I, J)

representing the underlying framed sheaf in $\mathcal{M}(n,r)$, together with the correspondign monad M. Thanks to Prop. 2.1.9, we know that also in this case the symplectic form lifts to a unique morphism $\Phi : M \to M^\vee$. Let $(\bar{A}, \bar{B}, \bar{I}, \bar{J})$ be the induced quadruple for $E^{\vee\vee}$ (see Remk 6.1.5), and let $\bar{G} : \bar{V} \to \bar{V}^\vee$ be the symmetric form induced from the symplectic form $-\varphi^\vee$. We can translate the commutative diagram

$$\begin{array}{ccc} E & \longrightarrow & E^{\vee\vee} \\ {\scriptstyle\varphi}\big\downarrow & \swarrow{\scriptstyle -\varphi^\vee} & \\ E^\vee & & \end{array}$$

into a chain of monad homomorphisms

$$\begin{array}{ccccc}
\mathcal{O}(-1)\otimes V & \xrightarrow{\alpha} & \mathcal{O}\otimes(V^{\oplus 2}\oplus W) & \xrightarrow{\beta} & \mathcal{O}(1)\otimes V \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{O}(-1)\otimes \bar{V} & \xrightarrow{\bar{\alpha}} & \mathcal{O}\otimes(\bar{V}^{\oplus 2}\oplus W) & \xrightarrow{\bar{\beta}} & \mathcal{O}(1)\otimes \bar{V} \\
{\scriptstyle -\bar{G}}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle \bar{G}} \\
\mathcal{O}(-1)\otimes \bar{V}^\vee & \xrightarrow{\bar{\beta}^\vee} & \mathcal{O}\otimes((\bar{V}^\vee)^{\oplus 2}\oplus W^\vee) & \xrightarrow{\bar{\alpha}^\vee} & \mathcal{O}(1)\otimes \bar{V}^\vee \\
\big\downarrow & & \big\downarrow & & \big\downarrow \\
\mathcal{O}(-1)\otimes V^\vee & \xrightarrow{\beta^\vee} & \mathcal{O}\otimes((V^\vee)^{\oplus 2}\oplus W^\vee) & \xrightarrow{\alpha^\vee} & \mathcal{O}(1)\otimes V^\vee
\end{array}$$

so that the resulting morphism $M \to M^\vee$ is exactly Φ. The diagram shows that if we call G the composition $V \to \bar{V} \xrightarrow{\bar{G}} \bar{V}^\vee \to V^\vee$, the system 6.2.1 holds unchanged. The only difference with the locally free case is that this time we *do not require* G *to be invertible.* Motivated by this discussion, we give the following definition.

Definition 6.2.1. We define the *variety of symplectic ADHM configurations* of type (n,r) to be the subvariety $\mathbb{M}_\Omega(n,r)$ of the affine space

$$End(V)^{\oplus 2} \oplus Hom(W,V) \oplus Hom(S^2V, \mathbb{C}) = \{(A,B,I,G)\}$$

cut out by the equations:

- $GA = A^\vee G$ ($GA-$*symmetry*);
- $GB = B^\vee G$ ($GB-$*symmetry*);
- $[A,B] - I\Omega^{-1}I^\vee G = 0$ (*ADHM eqt*).

We say that a configuration (A,B,I,G) is

- *stable* if there exists no subspace $0 \subseteq S \subsetneq V$ such that $A(S) \subseteq S$, $B(S) \subseteq S$ and $im(I) \subseteq S$;
- *regular* if it is stable and G is invertible.

The variety $\mathbb{M}_\Omega(n,r)$ is naturally acted on by $GL(V)$:

$$g \cdot (A, B, I, G) = (gAg^{-1}, gBg^{-1}, gI, g^{-\vee}Gg^{-1}).$$

The open subvarieties $\mathbb{M}^s_\Omega(n,r)$ and $\mathbb{M}^{reg}_\Omega(n,r)$ of stable and regular data are invariant with respect to this action.

Definition 6.2.2. We define the $GL(V)-$equivariant map

$$\iota : \mathbb{M}_\Omega(n,r) \to \mathbb{M}(n,r),\ \iota(A,B,I,G) = (A,B,I,-\Omega I^\vee G).$$

By definition the equality $\iota^{-1}(\mathbb{M}^s(n,r) = \mathbb{M}^s_\Omega(n,r)$ holds. What is also true is that a symplectic configuration is regular if and only if its associated classic configuration is regular, but this requires some explanations.

Lemma 6.2.3. $\iota^{-1}(\mathbb{M}^{reg}(n,r)) = \mathbb{M}^{reg}_\Omega(n,r)$.

Proof. Let $(A,B,I,G) \in \mathbb{M}^s_\Omega(n,r)$ and let $S \subseteq ker(-\Omega^{-1}I^\vee G) \subseteq V$ be an $A,B-$stable subspace. Let $s \in S$, $G(s) \in V^\vee$. Then $G(s)^\perp \supseteq im(I)$. Let

$$T = \bigcap_{s\in S} G(s)^\perp \subseteq V.$$

Using the $GA-$symmetry, we prove T is $A-$stable:

$$t \in T \implies \langle G(s), A(t)\rangle = \langle A^\vee G(s), t\rangle =$$

$$= \langle GA(s), t\rangle = 0,$$

since $A(s) \in S$. The same holds for B. The stability of the datum forces $T = V$; but this means $G(S) = 0$, i.e. $S \subseteq ker(G)$. *Vice versa*, any subspace $S \subseteq ker(G) \subseteq ker(-\Omega^{-1}I^\vee G)$ is $A,B-$stable thanks to the symmetries. In particular, $\iota(A,B,I,G)$ is co-stable if and only if $ker(G) = 0$. □

The remainder of this section will be devoted to prove the following theorem.

Theorem 6.2.4. *The $GL(V)$ action on $\mathbb{M}^s_\Omega(n,r)$ is free and locally proper. There exists an isomorphism*

$$\mathbb{M}^s_\Omega(n,r)/GL(V) \cong \mathcal{M}_\Omega(n,r)$$

which maps the open subscheme $\mathbb{M}^{sc}_{\Omega}(n,r)/GL(V)$ *onto* $\mathcal{M}^{reg}_{\Omega}(n,r)$. *The induced closed embedding*

$$\iota : \mathcal{M}_{\Omega}(n,r) \to \mathcal{M}(n,r)$$

coincides with the one defined in Chapter 3.

We will need the following preliminary result.

Proposition 6.2.5. *The restriction of* ι *to the stable loci*

$$\iota : \mathbb{M}^{s}_{\Omega}(n,r) \to \mathbb{M}^{s}(n,r)$$

is a closed embedding.

Proof. We are going to apply Lemma 2.1.1. We start with injectivity a closed points. Let $\xi_1, \xi_2 \in \mathbb{M}^{s}_{\Omega}(n,r)$ be such that $\iota(\xi_1) = \iota(\xi_2)$. Then $\xi_i = (A, B, I, G_i)$ and

$$I^{\vee}(G_1 - G_2) = 0 \iff (G_1 - G_2)^{\vee} I = 0 \iff im(I) \subseteq ker(G_1 - G_2).$$

As $G_i A = A^{\vee} G_i$ and $G_i B = B^{\vee} G_i$, we see $ker(G_1 - G_2)$ is $A, B-$stable. The stability conditions force $ker(G_1 - G_2) = V$, meaning $G_1 = G_2 \implies \xi_1 = \xi_2$.

We pass to tangent spaces. Fix a point $\xi = (A, B, I, G) \in \mathbb{M}^{s}_{\Omega}(n,r)$. Then we can describe $T_{\xi}\mathbb{M}^{s}_{\Omega}(n,r)$ as the subset of the vector space

$$End(V) \oplus Hom(W,V) \oplus Hom(S^2V, \mathbb{C}) \ni (X_A, X_B, X_I, X_G)$$

cut out by

$$\begin{cases} GX_A - X_A^{\vee} G + X_G A - A^{\vee} X_G = 0 \\ GX_B - X_B^{\vee} G + X_G B - B^{\vee} X_G = 0 \\ [A, X_B] + [X_A, B] - (X_I \Omega^{-1} I^{\vee} + I\Omega^{-1} X_I^{\vee})G - I\Omega^{-1} I^{\vee} X_G = 0 \end{cases}$$

On the other hand the tangent space to $\mathbb{M}^{s}(n,r)$ at a point(A, B, I, J) is identified with the subspace of

$$End(V) \oplus Hom(W,V) \oplus Hom(V,W) \ni (X_A, X_B, X_I, X_J)$$

defined by

$$[A, X_B] + [X_A, B] + X_I J + I X_J = 0.$$

The tangent map $T_{\xi}\iota : T_{\xi}\mathbb{M}^{s}_{\Omega}(n,r) \to T_{\iota(\xi)}\mathbb{M}^{s}(n,r)$ writes

$$(X_A, X_B, X_I, X_G) \mapsto (X_A, X_B, X_I, -\Omega^{-1}(I^{\vee} X_G + X_I^{\vee} G)).$$

Suppose $(X_A, X_B, X_I, X_G) \mapsto 0$. This forces $(X_A, X_B, X_I) = 0$ and $im(I) \subseteq ker(X_G)$. Furthermore, $X_G A = A^\vee X_G$ and $X_G B - B^\vee X_G$ tells $ker(X_G)$ is $A, B-$stable, thus $X_G = 0$. Injectivity is proved.

We are left with properness, which we settle by making use of the following version of the valuative criterion (see [GD, 7.3.9]). Let $Spec(\mathbb{C}((t))) \to Spec(\mathbb{C}[[t]])$ be the natural one dimensional embedding of the pointed formal disc into the formal disc. Then ι is proper if and only if any commutative diagram

$$\begin{array}{ccc} Spec(\mathbb{C}((t))) & \longrightarrow & \mathbb{M}^s_\Omega(n,r) \\ \downarrow & \nearrow & \downarrow \iota \\ Spec(\mathbb{C}[[t]]) & \longrightarrow & \mathbb{M}^s(n,r) \end{array}$$

admits a lifting as specified by the dashed diagonal arrow. This statement can be rephrased as follows. Suppose $(A_t, B_t, I_t, J_t) \in \mathbb{M}(\mathbb{C}[[t]]])$ is such that its underlying closed point (A_0, B_0, I_0, J_0) belongs to $\mathbb{M}^s$, and let $G_t \in (S^2V)^\vee((t))$ such that $J_t = -\Omega^{-1} I_t^\vee G_t$ and $(A_t, B_t, I_t, G_t) \in \mathbb{M}_\Omega(\mathbb{C}((t)))$. Then we only need to prove $G_t \in (S^2V)^\vee[[t]]$. Arguing by contradiction, suppose we can write

$$G_t = t^{-k} G_0 + t^{-k+1} G_1 + t^{-k+2} G_2 + \dots$$

with $G_0 \in (S^2V)^\vee \backslash \{0\}$ and $k > 0$. We can put coordinates on V such that G_0 has the form $\mathbb{1}_m \oplus 0_{n-m}$, $0 < m \leq n$. The term of order $-k$ of the equation $G_t A_t = A_t^\vee G_t$ reads $G_0 A_0 = A_0^\vee G_0$, so that A_0 reads

$$\begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & A_0^{22} \end{pmatrix},$$

and the same goes for B. We write

$$I_t = I_0 + tI_1 + t^2 I_2 + \dots$$

and decompose $I_0 = \begin{pmatrix} I_0^1 \\ I_0^2 \end{pmatrix}$ coherently with our coordinates. From $J_t = -\Omega^{-1} I_t^\vee G_t \in Hom(V, W)[[t]]$ we deduce that $I_0^\vee G_0 = 0$, thus $I_0^1 = 0$. This tells us that the subspace of V spanned by vectors of type $\begin{pmatrix} 0 \\ v \end{pmatrix}$ contains the image of I_0 and is mapped into itself by A_0 and B_0, contradicting the stability of the triple (A_0, B_0, I_0); the claim is thus proved. $\square$

The following corollary is immediate.

Corollary 6.2.6. *The $GL(V)$ action on $\mathbb{M}^s_\Omega(n,r)$ is free and locally proper. In par-*

ticular, the quotient set $\mathbb{M}^s_\Omega(n,r)/GL(V)$ *inherits a scheme structure. Furthermore, there is a closed embedding*

$$\iota : \mathbb{M}^s_\Omega(n,r)/GL(V) \to \mathbb{M}^s(n,r)/GL(V) \cong \mathcal{M}(n,r)$$

so that the principal $GL(V)-$*bundle* $\mathbb{M}^s_\Omega(n,r) \to \mathbb{M}^s_\Omega(n,r)/GL(V)$ *is isomorphic to the pullback of the bundle* $\mathbb{M}^s(n,r) \to \mathcal{M}(n,r)$ *via* ι.

We may proceed to prove the main theorem of this section.

Proof. (Thm 6.2.4) Fix a universal sheaf $(\mathcal{E}_\mathcal{M}, a_\mathcal{M})$ on $\mathcal{M}(n,r) \times \mathbb{P}^2$, which one realizes as the cohomology of a universal monad

$$M : \ \mathcal{U}_\mathcal{M} \xrightarrow{\alpha_\mathcal{M}} \mathcal{W}_\mathcal{M} \xrightarrow{\beta_\mathcal{M}} \mathcal{T}_\mathcal{M}$$

as in Prop 6.1.3. Let $(\mathcal{E}_S, a_S, \varphi_S)$ be an $S-$point of $\mathcal{M}_\Omega(n,r)$. The pair $(\mathcal{E}_S, a_S)$ induces a morphism $S \to \mathcal{M}(n,r)$, and by construction the pullback monad M_S on $S \times \mathbb{P}^2$ satisfies $H^1(M_S) \cong \mathcal{E}_S$. We select an affine open cover $\{S_i\}_i$ of S so that $S \to \mathcal{M}(n,r)$ lifts to sections $\sigma_i : S_i \to \mathbb{M}^s(n,r)$ and the pullback M_{S_i} is isomorphic to

$$M_i : \ \mathcal{O}_{S_i\times\mathbb{P}^2}(-1) \otimes V \xrightarrow{\alpha_i} \mathcal{O}_{S_i\times\mathbb{P}^2} \otimes (V^{\oplus 2} \oplus W) \xrightarrow{\beta_i} \mathcal{O}_{S_i\times\mathbb{P}^2}(1) \ \otimes V$$

with $\alpha_i = \alpha \circ \sigma_i$ and $\beta_i = \beta \circ \sigma_i$. This is verified for example on any affine refinement of the covering given by the preimages of the open set covering $\{U_i\}_i$ of $\mathcal{M}(n,r)$ as in Prop. 6.1.3. Consider

$$\varphi_{S_i} : H^1(M_i) \cong \mathcal{E}_{S_i} \to \mathcal{E}^\vee_{S_i} \cong H^1(M_i^\vee).$$

As the pair of complexes $M_i, M_i^\vee$ satisfies the hypothesis of Prop. 2.1.9, φ_{S_i} lifts uniquely to a morphism of complexes $\Phi_i : M_i \to M_i^\vee$. We want to show that this induces morphisms $f_i : S_i \to \mathbb{M}^s_\Omega(n,r)$, but this is easy, as we get maps

$$\Phi_{i,-1} \in Hom(S_i, Hom(V, V^\vee))$$

$$\Phi_{i,0} \in Hom(S_i, Hom(V^{\oplus 2} \oplus W, (V^\vee)^{\oplus 2} \oplus W^\vee))$$

$$\Phi_{i,1} \in Hom(S_i, Hom(V, V^\vee))$$

defining a framed symplectic sheaf once evaluated at any closed point $s \in S_i$; we may conclude in the light of the discussion at the beginning of the present section. Also, on the overlaps S_{ij} we have $f_i \sim f_j$ under the natural action of $GL(V)$, giving thus rise to a well defined morphism $f : S \to \mathbb{M}^s_\Omega(n,r)/GL(V)$. We constructed a natural transformation of functors inducing a morphism of schemes $\mathcal{M}_\Omega(n,r) \to \mathbb{M}^s_\Omega(n,r)/GL(V)$.

An inverse to this morphism may be constructed as follows. Call $\{U_{i,\Omega}\}_i$ the open cover of $\mathbb{M}^s_\Omega(n,r)/GL(V)$ induced by the cover $\{U_i\}_i$ of $\mathcal{M}(n,r)$ so that we can fix sections $\sigma_i : U_{i,\Omega} \to \mathbb{M}^s_\Omega(n,r)/GL(V)$. Write $\sigma_i = (A_i, B_i, I_i, G_i)$. The pullback of the universal monad M to $\mathbb{M}^s_\Omega(n,r)/GL(V)$ can be written as

$$M_{i,\Omega}: \quad \mathcal{O}_{U_{i,\Omega}\times\mathbb{P}^2}(-1)\otimes V \xrightarrow{\alpha(\sigma_i)} \mathcal{O}_{U_{i,\Omega}\times\mathbb{P}^2}\otimes(V^{\oplus 2}\oplus W) \xrightarrow{\beta(\sigma_i)} \mathcal{O}_{U_{i,\Omega}\times\mathbb{P}^2}(1)\otimes V$$

and we obtain a morphism

$$\begin{array}{ccccc}
\mathcal{O}_{U_{i,\Omega}\times\mathbb{P}^2}(-1)\otimes V & \xrightarrow{\alpha(\sigma_i)} & \mathcal{O}_{U_{i,\Omega}\times\mathbb{P}^2}\otimes(V^{\oplus 2}\oplus W) & \xrightarrow{\beta(\sigma_i)} & \mathcal{O}_{U_{i,\Omega}\times\mathbb{P}^2}(1)\otimes V \\
\downarrow G_i & & \downarrow \Omega_i & & \downarrow -G_i \\
\mathcal{O}_{U_{i,\Omega}\times\mathbb{P}^2}(-1)\otimes V^\vee & \xrightarrow{\beta(\sigma_i)^\vee} & \mathcal{O}_{U_{i,\Omega}\times\mathbb{P}^2}\otimes((V^\vee)^{\oplus 2}\oplus W^\vee) & \xrightarrow{\alpha(\sigma_i)^\vee} & \mathcal{O}_{U_{i,\Omega}\times\mathbb{P}^2}(1)\otimes V^\vee
\end{array}$$

where

$$\Omega_i = \begin{pmatrix} 0 & G_i & 0 \\ -G_i & 0 & 0 \\ 0 & 0 & \Omega \end{pmatrix},$$

Passing to cohomology, we get a collection of skew-symmetric morphism $\varphi_i : \mathcal{E}\mid_{U_{i,\Omega}} \to \mathcal{E}^\vee_{\mathcal{M}}\mid_{U_{i,\Omega}}$, which agree on the overlaps U_{ij}. The $\mathbb{M}^s_\Omega(n,r)/GL(V)-$family of symplectic sheaves we just defined gives the sought-for inverse. □

6.3 Irreducibility of $\mathcal{M}_\Omega(n,r)$

The aim of this section is to prove the irreducibility of the moduli space $\mathcal{M}_\Omega(n,r)$. The moduli space of framed symplectic bundles $\mathcal{M}^{reg}_\Omega(n,r)$ is irreducible, as it is a smooth connected algebraic variety of dimension $rn+2n$, see [BFG]. This suggests a strategy for our proof: it is enough to prove that the open subset $\mathcal{M}^{reg}_\Omega(n,r) \subseteq \mathcal{M}_\Omega(n,r)$ is dense. We will make use of its description as the orbit space of the action of $GL(V)$ on the space of stable symplectic ADHM configurations $\mathbb{M}^s_\Omega(r,n)$, as explained in the previous section. We fix Darboux coordinates on the symplectic vector space (W,Ω) so that $-\Omega^{-1} = \Omega$, $W \cong \mathbb{C}^r$. We recall that the space of quadruples (A,B,I,G) whose orbits correspond to symplectic bundles are the ones belonging to the open invariant subset $\{rk(G) = n\}$.

Remark 6.3.1. The double dual of a symplectic sheaf is a symplectic bundle with $c_2 = rk(G)$. Let $[E,a,\varphi] = [A,B,I,G]$. Fix coordinates so that $G = \begin{pmatrix} \mathbb{1}_{rk(G)} & 0 \\ 0 & 0_{n-rk(G)} \end{pmatrix}$.

The GA, GB symmetries imply that one has

$$A = \begin{pmatrix} A' & 0 \\ a & \alpha \end{pmatrix}, \; B = \begin{pmatrix} B' & 0 \\ b & \beta \end{pmatrix}$$

with A' and B'symmetric. Write $I = \begin{pmatrix} I' \\ X \end{pmatrix}$ according to the above decomposition. We obtain the symplectic ADHM quadruple $(A', B', I', \mathbb{1}_{rk(G)})$ which is a representative for the point $[E^{\vee\vee}, a^{\vee\vee}] \in \mathcal{M}_{\Omega}^{reg}(r, rk(G))$.

For any given (A, B, I, G), we shall provide a rather explicit construction of a rational curve in the moduli space passing through $[A, B, I, G]$ and whose general point lies in $\mathcal{M}_{\Omega}^{reg}$. For the sake of clarity, we shall start by studying the cases $G = 0$ and $rk(G) = n - 1$, as in the proof for the general case we will use a blend of the techniques for these two extremal cases. The results in matrix analysis we shall quote have been summarized in Chapter 5.

The case $G = 0$

The symplectic sheaves represented by quadruples of type $(A, B, I, 0)$ have trivial double dual. The $GA, GB-$symmetries are vacuous and the ADHM equation reduces to $[A, B] = 0$. Assume A is a nonderogatory matrix. By 5.0.9, we can change the coordinates so that both A and B are symmetric. By Remk. 5.0.10, we can find a symmetric matrix B' such that

$$[A, B'] = I\Omega I^{\top}.$$

For $t \in \mathbb{C}$, define the matrices $G_t = t \cdot \mathbb{1}_n$, $B_t = B + tB'$. Then $(A, B_t, I, G_t) \in \mathbb{M}_{\Omega}(n, r)$ by construction, and for small nonzero values of t the deformation sits in $\mathbb{M}_{\Omega}^{reg}(n, r)$, as stability is an open condition and G_t is invertible for $t \neq 0$.

Furthermore, we can actually drop the hypothesis on the cyclicity of A. Indeed, let A' be a nonderogatory matrix such that $[A', B] = 0$: it exists by Remk. 5.0.7. Then we have

$$[tA' + (1 - t)A, B] = 0 \, \forall t \in \mathbb{C}$$

and the generic point of the line $A_t = tA' + (1 - t)A$ is a nonderogatory matrix and gives rise to a stable triple (A_t, B, I) as both conditions are open. This means that up to a small deformation of the base quadruple we can assume A is nonderogatory.

The case $rk(G) = n - 1$

These ADHM data correspond to symplectic sheaves whose singular locus is concentrated in one point, with multiplicity 1. We can choose coordinates such that

$$G = \begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & 0 \end{pmatrix}.$$

The other matrices can be written as

$$\begin{pmatrix} A & 0 \\ a & \alpha \end{pmatrix}, \begin{pmatrix} B & 0 \\ b & \beta \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}.$$

with A and B symmetric. Without loss of generality we may assume $\alpha = \beta = 0$, since $\mathbb{A}^2$ acts on the space of ADHM configurations by adding multiples of the identity matrix to the endomorphisms. Of course, the choice of a and b is not unique: by changing the coordinates we can replace them respectively with $vA + a\lambda$, $vB + b\lambda$ for a nonzero $\lambda \in \mathbb{C}$, keeping A, B, I fixed (the transformation $g = \begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ v & \lambda \end{pmatrix}$ does the job).

Remark 6.3.2. The subspace of $\mathbb{C}^{n-1}$

$$S = im(A) + im(I) + im(BI) + im(B^2 I) + \cdots + im(B^{n-2} I)$$

coincides in fact with the whole $\mathbb{C}^{n-1}$. Indeed, the triple (A, B, I) is stable, and the subspace we are considering contains by definition the image of I, and it is A,$B-$invariant. $A-$invariance is obvious since S contains the image of A. To prove $B-$invariance, we first note that the subspace

$$im(I) + im(BI) + im(B^2 I) + \cdots + im(B^{n-2} I)$$

is $B-$invariant (as B^{n-1} can be written as a polynomial in B of degree $n - 2$ at most). Moreover,

$$B(Av) = A(Bv) + I(\Omega I^\top v) \in im(A) + im(I)$$

by $[A, B] - I\Omega I^\top = 0$.

Remark 6.3.3. If we can choose a (or b) to be 0, then we can do the following. Consider the family of configurations

$$(\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} B & tb \\ b & 0 \end{pmatrix}, I, \begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & t \end{pmatrix}).$$

(or the analogous one with $b = 0$). This gives a locally free deformation of the sheaf E. This condition is verified, for example, when A (or B) is invertible; in this case, we can write $vA = a$ (or $vB = b$) and change the coordinates accordingly.

We apply the lemma to the following situation. Given an ADHM quadruple

$$(\begin{pmatrix} A & 0 \\ a & 0 \end{pmatrix}, \begin{pmatrix} B & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & 0 \end{pmatrix}),$$

we set: $R = \mathbb{C}^{n-1}$, $L = im(I)$, $v = a^\top$. By Remk. 6.3.2 we can write

$$v = Av_A + Ix_0 + BIx_1 + \cdots + B^{n-1}Ix_{n-1}$$

for some vector $v_A \in R$ and $x_i \in W$, and we can find an equivalent triple with the same A, B and I so that $v = Ix_0 + BIx_1 + \cdots + B^{n-1}Ix_{n-1}$ (just remember we can move a by any vector in the image of A). Let $r(t)$ and $l(t) = I(Y(t)) \in im(I)$ satisfying the thesis, and write the deformation

$$(\begin{pmatrix} A & 0 \\ a + r(t)^\top & 0 \end{pmatrix}, \begin{pmatrix} B - tId & 0 \\ b & 0 \end{pmatrix}, \begin{pmatrix} I \\ X + Y^\top(t) \cdot \Omega^{-1} \end{pmatrix}, \begin{pmatrix} \mathbb{1}_{n-1} & 0 \\ 0 & 0 \end{pmatrix}).$$

Now, any point of this curve sits in the space of ADHM configurations, because $[A, B - tId] = [A, B]$, the GA, GB symmetries are obviously satisfied, and the (2,1) block of the commutator is written as

$$(a + r(t)^\top)(B - tId) - bA = aB - bA - bA + r(t)^\top(B - tId) - ta =$$

$$= X\Omega I^\top + ta + Y^\top(t) \cdot I^\top - ta = (X + Y^\top(t)\Omega^{-1})\Omega I^\top.$$

The previous calculation exhibits a small deformation of the given configuration which has an invertible matrix $(B - tId)$ in the (1,1) block, and $\beta = 0$ in the (2,2) entry: this must sit in $\overline{\mathcal{M}_\Omega^{reg}}$ by Remk 6.3.3.

The general case

We are ready to deal with the case of quadruples (A', B', I', G) with $k = rk(G) \in \{1, \ldots n-2\}$, where $n = dim(V)$ as usual. We can normalize G to

$$G = \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}$$

and thus write

$$A' = \begin{pmatrix} A & 0 \\ a & \alpha \end{pmatrix}, B' = \begin{pmatrix} B & 0 \\ b & \beta \end{pmatrix}, I' = \begin{pmatrix} I \\ X \end{pmatrix},$$

with A, B symmetric, $[A, B] = I\Omega I^\top$, $[\alpha, \beta] = 0$ and $(aB - \beta a) - (bA - \alpha b) = X\Omega I^\top$. We note that acting by the $G-$preserving transformation $g_v = \begin{pmatrix} 1_k & 0 \\ v & 1_{n-k} \end{pmatrix}$ we leave A, B, α, β and I unchanged and move a and b respectively to $a + vA - \alpha v$ and $b + vB - \beta v$. In order to deform our quadruple into a rank n one, we shall need once again to prove that we can slightly deform it and get a quadruple with vanishing a or b. In the $n-1$ case, this was guaranteed if α was not an eigenvalue for A (or similarly for β and B). In the general case, the equation

$$vA - \alpha v = a$$

has a solution if A and α do not share any eigenvalues, see Lemma 5.0.11 for a proof.

We need the following generalization of Remk. 6.3.2.

Lemma 6.3.4. *Let $R = Mat((n-k) \times k)$ and A, B, I as above (A and B symmetric $k \times k$ matrices, $I \in Hom(\mathbb{C}^r, \mathbb{C}^k)$ with $(\mathbb{C}^r, \Omega)$ a symplectic vector space, $[A, B] = I\Omega I^\top$, stability is satisfied).*

1. *Suppose that there exists a subspace $R' \subseteq R$ which is stable with respect to the maps $v \mapsto vA$, $v \mapsto vB$ and contains the image of the linear map*

$$\tilde{I} : Hom(\mathbb{C}^r, \mathbb{C}^{n-k}) \to R, \; X \mapsto XI^\top.$$

   Then $R' = R$.

2. *Let $\alpha, \beta \in Mat((n-k) \times (n-k))$, $[\alpha, \beta] = 0$ and let $T_{A,\alpha}, T_{B,\beta} \in End(R)$ defined by*

$$v \mapsto vA - \alpha v, \; v \mapsto vB - \beta v,$$

   respectively. Then the identity

$$im(T_{A,\alpha}) + im(\tilde{I}) + im(T_{B,\beta}\tilde{I}) + im(T^2_{B,\beta}\tilde{I}) + \cdots + im(T^{dim(R)-1}_{B,\beta}\tilde{I}) = R$$

   holds.

Proof. To show item 1, it is enough to prove that for a given R' as in the hypothesis, any matrix of the form

$$v = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ v_i \\ 0 \\ \vdots \\ 0 \end{pmatrix},\ v_i \in \mathbb{C}^k$$

belongs to R'. We will prove it only for $i = 1$, as the other cases are completely analogous. The linear subspace $S \subseteq \mathbb{C}^k$ given by vectors s such that $\begin{pmatrix} s \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R'$ must be necessarily the whole of $\mathbb{C}^k$. Indeed, since

$$\begin{pmatrix} s \\ 0 \\ \vdots \\ 0 \end{pmatrix} A = \begin{pmatrix} sA \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R',$$

we see that S is $A-$stable (and similarly, $B-$stable). Furthermore,

$$X_1 = \begin{pmatrix} x_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in Hom(\mathbb{C}^r, \mathbb{C}^{n-k}) \implies \tilde{I}(X_1) = X_1 I^\top = \begin{pmatrix} x_1 I^\top \\ 0 \\ \vdots \\ 0 \end{pmatrix} \in R',$$

so $S \supseteq im(I)$, and we are done thanks to the stability of the triple (A, B, I).

To prove the second part, we may apply item 1 to the linear subspace

$$R' = im(T_{A,\alpha}) + im(\tilde{I}) + im(T_{B,\beta}\tilde{I}) + im(T_{B,\beta}^2\tilde{I}) + \cdots + im(T_{B,\beta}^{dim(R)-1}\tilde{I}).$$

By definition R' contains $Im(\tilde{I})$, so we just need to prove it is A, $B-$ stable in the above sense. We note that any

$$v \in im(\tilde{I}) + im(T_{B,\beta}\tilde{I}) + im(T_{B,\beta}^2\tilde{I}) + \cdots + im(T_{B,\beta}^{dim(R)-1}\tilde{I})$$

can be rewritten as

$$v = X_0' I^\top + X_1' I^\top B + \cdots + X_{k-1}' I^\top B^{k-1},$$

because

$$T^m_{B,\beta}\tilde{I}X = XI^\top B^k - \binom{m}{2}\beta XI^\top B^{k-1} + \cdots - (1)^m\beta^m XI^\top.$$

So, we may write:

$$R' = im(T_{A,\alpha}) + im(\tilde{I}) + im(\tilde{B}\tilde{I}) + im(\tilde{B}^2\tilde{I}) + \cdots + im(\tilde{B}^{k-1}\tilde{I})$$

where $\tilde{B}v = vB$. Now we have:

$$\begin{gathered}
\tilde{A}(T_{A,\alpha}(v)) = T_{A,\alpha}(\tilde{A}v) \in R' \\
\tilde{A}(\tilde{B}^k\tilde{I}(X)) = XI^\top B^k A = XI^\top B^k A - \alpha XI^\top B^k + \alpha XI^\top B^k = \\
= T_{A,\alpha}(XI^\top B^k) + \tilde{B}^k\tilde{I}(\alpha X) \in R' \\
\tilde{B}(T_{A,\alpha}(v)) = vAB - \alpha vB = vBA + vI\Omega I^\top - \alpha vB = \\
= T_{A,\alpha}(vB) + \tilde{I}(vI\Omega) \in R' \\
\tilde{B}(\tilde{B}^k\tilde{I}(X)) \in Im(\tilde{B}^{k+1}\tilde{I}) \subseteq R'.
\end{gathered}$$

This guarantees A, $B-$stability of R', and concludes the proof. □

We are now able to apply Lemma 5.0.12 to our ADHM quadruple

$$(\begin{pmatrix} A & 0 \\ a & \alpha \end{pmatrix}, \begin{pmatrix} B & 0 \\ b & \beta \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix})$$

in the following way. Thanks to Lemma 6.3.4, we can write

$$a = vA - \alpha v + X_0 I^\top + T_{B,\beta}(X_1 I^\top) + \ldots T^{k-1}_{B,\beta}(X_{k-1}I^\top),$$

and we can change the coordinates to eliminate the addend $vA - \alpha v$. We apply Lemma 5.0.12 to find a small deformation of a, written as $a_t = a + v(t)$, satisfying

$$(T_{B,\beta} - tId)v(t) = ta + Y(t)\Omega I^\top \implies (T_{B,\beta} - tId)a_t + T_{B,\beta}(a) = Y(t)\Omega I^\top$$

with $Y(0) = 0$, and so we obtain

$$a_t(B - tId) - \beta a_t - bA + \alpha b =$$

$$= aB - \beta a - bA + \alpha b + Y(t)\Omega I^\top = (X + Y(t))\Omega I^\top.$$

In other words, the deformation

$$(\begin{pmatrix} A & 0 \\ a_t & \alpha \end{pmatrix}, \begin{pmatrix} B - tId & 0 \\ b & \beta \end{pmatrix}, \begin{pmatrix} I \\ X + Y(t) \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix})$$

still sits in the space of symplectic ADHM data, and remains stable for small t. Obviously, the spectra of β and $B - tId$ are disjoint for arbitrarily small nonzero values of t; therefore, up to small deformations, we can indeed assume $b = 0$, by means of Lemma 5.0.11 together with the usual change of coordinates $a \mapsto vA - \alpha v + a$, $b \mapsto vB - \beta v + b$.

So we can suppose without loss of generality that our quadruple is of the form

$$\left(\begin{pmatrix} A & 0 \\ a & \alpha \end{pmatrix}, \begin{pmatrix} B & 0 \\ 0 & \beta \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}\right).$$

Now we are only left to apply what we have learned from the rank 0 case, which is to employ nonderogatory matrices. First, we note that we can deform α as $\alpha_t = (1-t)\alpha + t\eta$ for any matrix η commuting with β, and this does not harm the ADHM equations or the symmetries (this is why we made all the work to get $b = 0$, to make sure that no term of type $b\alpha_t$ comes to ruin the party). This way, we can suppose that α is nonderogatory up to small deformations, and has no eigenvalues in common with A (it is enough to choose η nonderogatory and, if necessary, modify α once again by adding small multiples of Id_{n-k} to slide the eigenvalues). By changing the coordinates as usual, we obtain a quadruple of the type

$$\left(\begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} B & 0 \\ b & \beta \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix}\right),$$

which is exactly as before except for a crucial detail: α is nonderogatory. This means that by acting with a change of coordinates of type $\begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & g \end{pmatrix}$, we can suppose that the commuting matrices α and β are symmetric. Finally, let χ be a symmetric matrix satisfying $[\alpha, \chi] = X\Omega X^\top$: it must exist, due to the cyclicity of α.

Write down the final deformation

$$\left(\begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} B & tb^\top \\ b & \beta + t\chi \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix}\right). \tag{6.3.1}$$

Let us verify that this curve sits in the space of ADHM symplectic data:

$$\begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix} - \begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}^\top \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} =$$

$$\begin{pmatrix} A - A^\top & 0 \\ 0 & t\alpha - t\alpha^\top \end{pmatrix} = 0.$$

$$\begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} \begin{pmatrix} B & tb^\top \\ b & \beta + t\chi \end{pmatrix} - \begin{pmatrix} B & tb^\top \\ b & \beta + t\chi \end{pmatrix}^\top \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} =$$

$$= \begin{pmatrix} B - B^\top & tb^\top - tb^\top \\ tb - tb & t\beta - t\beta^\top + t^2\chi - t^2\chi^\top \end{pmatrix} = 0.$$

These give the G-symmetries. Recall that the ADHM equation for $t = 0$ is given by $\alpha b - bA = X\Omega I^\top$, and that $\Omega^\top = -\Omega$. We verify the ADHM equation along the curve:

$$[\begin{pmatrix} A & 0 \\ 0 & \alpha \end{pmatrix}, \begin{pmatrix} B & tb^\top \\ b & \beta + t\chi \end{pmatrix}] - \begin{pmatrix} I \\ X \end{pmatrix} \Omega \begin{pmatrix} I^\top & X^\top \end{pmatrix} \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix} =$$

$$= \begin{pmatrix} [A, B] - I\Omega I^\top & tAb^\top - tb^\top\alpha - tI\Omega X^\top \\ \alpha b - bA - X\Omega I^\top & t[\alpha, \chi] - tX\Omega X^\top \end{pmatrix} = 0$$

This finishes the proof of the irreducibility, since $G_t = \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & t\mathbb{1}_{n-k} \end{pmatrix}$ is invertible and so the general point of this curve lies in the locally free locus of the moduli space.

A note on the geometry behind the proof

We saw that we can assign to any rank k datum (A, B, I, G) two symmetric matrices $A', B' \in Mat(k)$ up orthogonal conjugation and two commuting matrices $\alpha, \beta \in Mat(n-k)$ up to conjugation. This means that, taking eigenvalues, we can produce invariants $\{x_1, \ldots, x_k, y_1, \ldots, y_k\}$ and $Z_{\alpha,\beta} = \{(\alpha_1, \beta_1), \ldots, (\alpha_{n-k}, \beta_{n-k})\}$. The first set of invariants defines a grid of lines inside $\mathbb{A}^2 = \{(x, y)\}$, given by the union

$$D' = \bigcup_{i=1,\ldots k} \{x = x_k\} \cup \{y = y_k\}.$$

The symplectic bundle associated with the triple (A', B', I') is trivial on the complement of (the closure of) D' inside $\mathbb{P}^2$(see [BFG, Sect. 4-5]). Instead $Z_{\alpha,\beta}$ is just the zero cycle assigned to the singular locus of $[A, B, I, G]$.

The "$G = 0$" part of the proof just tells us the singularities of framed sheaves with trivial double dual may be removed by families of symplectic bundles. What we see from the proof is that if the singularities of a general symplectic sheaf happen to be disjoint from the associated grid D', one can essentially resolve them and glue the family with the bundle $[A', B', I']$. The most difficult part of the work is to show that every symplectic sheaf can be deformed into a sheaf with $Z_{\alpha,\beta} \cap D' = \emptyset$.

6.4 The torus action on $\mathcal{M}_\Omega(n, r)$

Consider the algebraic torus $T_{\mathbb{P}} = (\mathbb{C}^\star)^2$ acting on $\mathbb{P}^2$ by

$$(t_1, t_2) \cdot [z_0, z_1, z_2] = [z_0, t_1 z_1, t_2 z_2]$$

and fix a maximal torus $T \subseteq GL(W)$. The moduli space $\mathcal{M}(r, n)$ is naturally acted on by $\tilde{T} = T_{\mathbb{P}} \times T$. Indeed, as $T_{\mathbb{P}}$ fixes l_∞, it acts by pullbacks on framed sheaves on $\mathbb{P}^2$, while T acts by rescalings on the framing. In [NY1, NY2] the authors classify the fixed points of this action; a framed sheaf $[E, a] \in \mathcal{M}(n, r)$ is a fixed point if and only if it can be written as a direct sum of r ideal sheaves of zero dimensional subschemes $Z_1, \ldots, Z_r$, all supported on $[1, 0, 0] \in \mathbb{P}^2$.

Let Ω be as usual a symplectic form on W, defining the subgroup $SP(W) \subseteq GL(W)$. Let $T_\Omega = SP(W) \cap T$ be the corresponding maximal torus. Of course one can also consider the induced restricted action of $\tilde{T}_\Omega = T_{\mathbb{P}} \times T_\Omega$ on $\mathcal{M}(n, r)$. Since this time we are dealing with symplectic rescalings of the framing, the closed subscheme $\mathcal{M}_\Omega(n, r)$ is invariant under the latter action. The aim of this section is to prove the following equality.

Theorem 6.4.1. $\mathcal{M}(n, r)^{\tilde{T}} = \mathcal{M}_\Omega(n, r)^{\tilde{T}_\Omega}$.

Proof. As any point in $\mathcal{M}(n, r)^{\tilde{T}}$ is represented by a sheaf whose double dual is trivial, it corresponds to a symplectic sheaf by Remk. 3.1.2. This means that the inclusion $\mathcal{M}(n, r)^{\tilde{T}} \subseteq \mathcal{M}_\Omega(n, r)^{\tilde{T}_\Omega}$ holds. In order to prove the reverse inclusion, it will be enough to prove that restricting the action on $\mathcal{M}(n, r)$ to $\tilde{T}_\Omega$ we do not produce more fixed points. This may be achieved manipulating the ADHM data. First, we use the fact that the $\tilde{T}_\Omega-$action on $\mathcal{M}(n, r)$ lifts to an action on $\mathbb{M}^s(n, r)$ in the following way:

$$(t_1, t_2, e)(A, B, I, J) = (t_1 A, t_1 B, I e^{-1}, e J t_1 t_2).$$

Let us consider the action of T_Ω only. Suppose $[A, B, I, J] \in \mathcal{M}(n, r)$ is fixed by T_Ω. This means that we can find a representation $\rho : T_\Omega \to GL(V)$ such that

$$(A, B, I e^{-1}, e J) = (A^{\rho(e)}, B^{\rho(e)}, \rho(e) I, J \rho(e)^{-1}).$$

Let us fix a Darboux basis $(x_1, \ldots, x_{r/2}, y_1, \ldots, y_{r/2})$ on W, meaning $\Omega(x_k, x_j) = \Omega(y_k, y_j) = 0$ and $\Omega(x_k, y_j) = \delta_{kj}$; get

$$T_\Omega = \{diag(e_1, \ldots e_{r/2}, e_1^{-1}, \ldots e_{r/2}^{-1}) \mid e_k \in \mathbb{C}^\star\}.$$

We get

$$\rho(e) I x_i = I e^{-1} x_i = e_i^{-1} I x_i; \ \rho(e) I y_i = I e^{-1} y_i = e_i I y_i.$$

Consider the eigenspaces of the representation ρ:

$$V(i_1,\dots i_{r/r}) = \{v \in V \mid \rho(e)v = e_1^{i_1}\cdots e_{r/2}^{i_{r/2}}v\}.$$

Since $im(I) \subseteq \bigoplus_i (V_i^- \oplus V_i^+)$ where $V_i^{\pm} = V(0,\dots,0,\overset{i}{\pm 1},0,\dots,0)$ and A and B send each $V_i^{\pm}$ to itself, we can conclude $V = \bigoplus_i (V_i^- \oplus V_i^+)$ by stability. Also, suppose $v \in V_i^+$; get

$$eJv = J\rho(e)^{-1}v = e_i^{-1}Jv \implies Jv \in \mathbb{C}y_1$$

and similarly we get $J(V_i^-) \subseteq \mathbb{C}x_i$. This implies one can decompose the ADHM datum as a direct sum of r rank 1 subdata

$$(A,B,I,J) = \bigoplus_i ((A_i^+,B_i^+,I_i^+,J_i^+) \oplus (A_i^-,B_i^-,I_i^-,J_i^-))$$

where $A_i^{\pm}, B_i^{\pm} \in End(V_i^{\pm})$, $I_i^- : \mathbb{C}x_i \to: V_i^-$, $I_i^- : \mathbb{C}x_i \to: V_i^-$, $J_i^- : V_i^- \to \mathbb{C}x_i$ and $J_i^+ : V_i^+ \to \mathbb{C}y_i$. Thus the sheaf represented by (A,B,I,J) is a direct sum of ideal sheaves of 0 dimensional subschemes Z_i. If we further require such framed sheaf to be fixed by T, we may conclude $supp(Z_i) = \{(0,0)\}$. □

Remark 6.4.2. We can write down explicitly the action of $\tilde{T}_\Omega$ on $\mathcal{M}_\Omega(n,r)$ as also in this case we can lift to ADHM configurations:

$$(t_1,t_2,e)(A,B,I,G) = ((t_1A, t_1B, Ie^{-1}, Gt_1t_2)).$$

Corollary 6.4.3. *The Euler characteristics of $\mathcal{M}_\Omega(n,r)$ and $\mathcal{M}(n,r)$ coincide.*

Proof. Let $\lambda : \mathbb{C}^\star \to \tilde{T}_\Omega$ be a generic $1-$parameter subgroup for the action on $\mathcal{M}_\Omega(n,r)$, meaning $\mathcal{M}_\Omega(n,r)^{\tilde{T}_\Omega} = \mathcal{M}_\Omega(n,r)^\lambda$ (for the existence of λ, see [Go, Sect. 2.2] and references therein). The proof follows then directly from [Bi, Cor. 2]. □

6.5 Uhlenbeck spaces

We want to construct a proper birational map from the moduli space of framed symplectic sheaves into the space of symplectic ideal instantons. The map will simply be the restriction of the so called *Gieseker-to-Uhlenbeck* map, defined on the moduli space of framed sheaves (we refer to [BMT] for the precise definition of this morphism). This provides a concrete example of the fact that Uhlenbeck spaces of type C can be obtained by generalized blow-downs of moduli spaces of symplectic sheaves, as suggested in [Bal, Ba2]. To this end, we will recall the definitions of Uhlenbeck spaces in a purely algebraic setting. In constrast with the classical case, the moduli space $\mathcal{M}_\Omega(r,n)$ is not smooth in general; we shall discuss its singularities in the next section.

Definition 6.5.1. Let (r,n) be positive integers. We call *Uhlenbeck space* or *space of framed ideal instantons* the affine scheme defined by the categorical quotient

$$\mathcal{M}_0(r,n) := \mathbb{M}(r,n)//GL(V).$$

The following theorem summarizes some of the properties of this scheme. For details, see [Na, BFG].

Theorem 6.5.2. *The following statements hold:*

1. $\mathcal{M}_0(r,n)$ *is reduced and irreducible;*
2. *the open embedding* $\mathbb{M}^{sc}(r,n) \hookrightarrow \mathbb{M}(r,n)$ *descends to an open embedding* $\mathcal{M}^{reg}(r,n) \to \mathcal{M}_0(r,n)$, *which is the smooth locus of* $\mathcal{M}_0(r,n)$*;*
3. $\mathcal{M}_0(r,n)$ *has a stratification into locally closed subsets of the form*

$$\mathcal{M}_0(r,n) = \bigsqcup_{k=0}^{n} \mathcal{M}^{reg}(r,n-k) \times (\mathbb{A}^2)^{(k)},$$

where $(\mathbb{A}^2)^{(k)}$ *is the* $k-th$ *symmetric power of the affine space* $\mathbb{P}^2 \backslash l_\infty$*;*

4. *the open embedding* $\mathbb{M}^s(r,n) \to \mathbb{M}(r,n)$ *descends to a projective morphism*

$$\pi : \mathcal{M}(r,n) \to \mathcal{M}_0(r,n)$$

which is a resolution on singularities.

Remark 6.5.3. As a set-theoretic map, π has a very simple description. Let (E,a) be a framed sheaf of charge n; the locally free sheaf $E^{\vee\vee}$ inherits a framing $a^{\vee\vee}$ from (E,a), and sits in an exact sequence

$$0 \longrightarrow E \longrightarrow E^{\vee\vee} \longrightarrow C \longrightarrow 0$$

where C is a $0-$dimensional sheaf supported away from l_∞. Let $Z(C)$ be the corresponding $0-$cycle on $\mathbb{A}^2$. As

$$length(C) + c_2(E^{\vee\vee}) = n,$$

we obtain a point

$$\pi([E,a]) = ([E^{\vee\vee}, a^{\vee\vee}], Z(C)) \in \mathcal{M}_0(r,n).$$

In particular, we see that the restriction of π to the locally free locus $\mathcal{M}^{reg}(r,n)$ induces an isomorphism onto the open subscheme $\mathcal{M}^{reg}(r,n) \subseteq \mathcal{M}_0(r,n)$.

It is possible to construct a symplectic variant of the Uhlenbeck space. Once again, let $V \cong \mathbb{C}^n$ and $W \cong \mathbb{C}^r$. Fix a symplectic structure Ω on W and a symmetric bilinear nondegenerate form 1_V on V, and call $End^+(V)$ the space of symmetric endomorphisms of V. Define the subspace of $End^+(V)^{\oplus 2} \oplus Hom(W,V)$:

$$\mathbb{X}(r,n) = \{(A,B,I) \mid [A,B] + I\Omega I^\top = 0\}$$

This space is naturally acted on by the orthogonal group $O(V)$.

Remark 6.5.4. Let $\tau : \mathbb{X}(r,n) \to \mathbb{M}_\Omega(r,n)$ be the embedding defined by

$$\tau(A,B,I) \to (A,B,I,1_V).$$

The morphism τ is equivariant with respect to the group homomorphism $O(V) \to GL(V)$ defined by 1_V. If we set $\mathbb{X}^s : \tau^{-1}(\mathbb{M}^s_\Omega) = \tau^{-1}(\mathbb{M}^{sc}_\Omega)$ (see Lemma 6.2.3), we obtain indeed an isomorphism of algebraic varieties

$$\mathbb{X}^s(r,n)/O(V) \to \mathbb{M}^{sc}_\Omega(r,n)/GL(V)(\cong \mathcal{M}^{reg}_\Omega(r,n)).$$

Definition 6.5.5. We define the symplectic Uhlenbeck space as the categorical quotient

$$\mathcal{M}_{0,\Omega}(r,n) = \mathbb{X}(r,n)//O(V).$$

We list some interesting properties of this affine scheme. For details, see [BFG, NS, Ch].

Theorem 6.5.6. *The following statements hold:*

1. *$\mathcal{M}_{0,\Omega}(r,n)$ is reduced and irreducible;*
2. *the open embedding $\mathbb{X}^s(r,n) \hookrightarrow \mathbb{X}(r,n)$ descends to an open embedding $\mathcal{M}^{reg}_\Omega(r,n) \to \mathcal{M}_{0,\Omega}(r,n)$, which is the smooth locus of $\mathcal{M}_{0,\Omega}(r,n)$;*
3. *$\mathcal{M}_{0,\Omega}(r,n)$ has a stratification into locally closed subsets of the form*

$$\mathcal{M}_{0,\Omega}(r,n) = \bigsqcup_{k=0}^{n} \mathcal{M}^{reg}_\Omega(r,n-k) \times (\mathbb{A}^2)^{(k)};$$

4. *the composition*

$$\mathbb{X}(r,n) \xrightarrow{\tau} \mathbb{M}_\Omega(r,n) \xrightarrow{\iota} \mathbb{M}(r,n)$$

is an equivariant closed embedding, inducing a closed embedding

$$\mathcal{M}_{0,\Omega}(r,n) \to \mathcal{M}_0(r,n)$$

which is compatible with the stratification, meaning that for any integer k

$$(\mathcal{M}^{reg}(r,n-k) \times (\mathbb{A}^2)^{(k)}) \cap \mathcal{M}_{0,\Omega}(r,n) = \mathcal{M}^{reg}_{\Omega}(r,n-k) \times (\mathbb{A}^2)^{(k)}$$

holds.

As a a direct consequence of the discussion of the previous two sections, we find a relation between symplectic Uhlenbeck spaces and moduli of framed symplectic sheaves.

Theorem 6.5.7. *The moduli space $\mathcal{M}_\Omega(r,n)$ is isomorphic to the strict transform of the closed subscheme $\mathcal{M}_{0,\Omega}(r,n) \subseteq \mathcal{M}_0(r,n)$ under the resolution π.*

Proof. The maximal open subset of $\mathcal{M}_0(r,n)$ over which π is an isomorphism is $\mathcal{M}^{reg}(r,n)$; it follows by definition that the strict transform in the statement is defined as

$$\overline{\pi^{-1}(\mathcal{M}_{0,\Omega}(r,n) \cap \mathcal{M}^{reg}(r,n))} = \overline{\pi^{-1}(\mathcal{M}^{reg}_{\Omega}(r,n))} = \mathcal{M}_\Omega(r,n),$$

since $\mathcal{M}_\Omega(r,n)$ is the smallest closed subscheme of $\mathcal{M}(r,n)$ containing the locus of symplectic bundles, see Sect. 6.3. □

Remark 6.5.8. We note that $\mathcal{M}_\Omega(r,n)$ also coincides with the total transform

$$\pi^{-1}(\mathcal{M}_{0,\Omega}(r,n)),$$

as its points are exactly the framed sheaves whose double dual is symplectic; indeed, if

$$\varphi' : E^{\vee\vee} \to E^{\vee\vee\vee} = E^{\vee}$$

is an "honest" symplectic form on the double dual, its composition with $E \to E^{\vee\vee}$ endows E with a structure of framed symplectic sheaf, as already noted in Remk. 3.1.2.

6.6 Singularities and examples with low c_2

Let us start from the following ADHM-theoretic smoothness criterion.

Proposition 6.6.1. *Let $\xi = [A,B,I,G] \in \mathcal{M}_\Omega(r,n)$. Suppose that A is nonderogatory (see 5.0.5). Then ξ is a smooth point.*

Proof. We recall the description of the tangent space we introduced in the proof of Lemma 6.2.5, i.e. we think of $T_{(A,B,I,G)}\mathbb{M}^s_\Omega(r,n)$ as the kernel of the the Jacobian matrix

$$\begin{pmatrix} X_A \\ X_B \\ X_I \\ X_G \end{pmatrix} \mapsto \begin{pmatrix} G\_ - \_^\top G & 0 & 0 & \_A - A^\top \_ \\ 0 & G\_ - \_^\top G & 0 & \_B - B^\top \_ \\ [\_, B] & [A, \_] & (I\Omega\_^\top + \_\Omega I^\top)G & I\Omega I^\top \_ \end{pmatrix} \begin{pmatrix} X_A \\ X_B \\ X_I \\ X_G \end{pmatrix} \tag{6.6.1}$$

$$\begin{pmatrix} X_A \\ X_B \\ X_I \\ X_G \end{pmatrix} \in End(V)^{\oplus 2} \oplus Hom(W,V) \oplus Hom(S^2V, \mathbb{C}).$$

Suppose A is nonderogatory. Applying 5.0.9, we change coordinates to make A symmetric, so that we can write $\_A - A^\top \_ = -[A, \_]$. The ciclicity of A guarantees the surjectivity of the map $[A, \_]$ as a morphism on the space of symmetric matrices with values in the space of skew-symmetric matrices. If we think $[A, \_]$ as a linear map on the space of general square matrices, its rank is $n^2 - n$. We deduce that the rank of the Jacobian is greater or equal to $\frac{3}{2}n(n-1)$, hence

$$dim(T_{(A,B,I,G)}\mathbb{M}^s_\Omega(r,n)) \leq 2n^2 + nr + \frac{1}{2}n(n+1) - \frac{3}{2}n(n-1) =$$

$$= n^2 + nr + 2n = dim(\mathcal{M}_\Omega(r,n)) + n^2 \leq dim(T_{(A,B,I,G)}\mathbb{M}^s_\Omega(r,n)).$$

This implies that the equation

$$dim T_\xi(\mathcal{M}_\Omega(r,n)) = dim(\mathcal{M}_\Omega(r,n))$$

holds, as required. □

We obtain the following straightforward corollary.

Corollary 6.6.2. *$\mathcal{M}_\Omega(r,n)$ is nonsingular in codimension one.*

Proof. Suppose that a quadruple $(A,B,I,G) \in \mathbb{M}^s_\Omega$ satisfies one of the following conditions:

1. G is invertible;
2. A is nonderogatory.

Then we know $\xi = [A,B,I,G] \in \mathcal{M}_\Omega$ is a smooth point. The first condition defines a $1-$codimensional opens subset of the moduli space. In fact, also the other conditions define open subschemes whose codimension is at least 1; indeed, call

$$\Delta_A : \mathbb{M}^s_\Omega \to \mathbb{C}$$

the function assigning to (A, B, I, G) the discriminant of the characteristic polynomial of A: it is a $GL(V)-$invariant function. The points where it does not vanish are exactly those quadruples whose A has n distinct eigenvalues, which in particular gives A nonderogatory. In particular, the singular locus of $\mathcal{M}_\Omega$ must be contained in the closed subscheme

$$(\{det(G) = 0\} \cap \{\Delta_A = 0\})/GL(V).$$

In order to prove the corollary, it is enough to prove that the divisors $\{det(G) = 0\}$ and $\{\Delta_A = 0\}$ don not share smaller divisors. To prove this, we will show that any quadruple in the intersection admits a deformation whose general point sits in one and only one of the divisors.

Suppose $(A, B, I, G) \in \{det(G) = 0\} \cap \{\Delta_A = 0\}$. Using the notation of Sect. 6.3, we normalize and rewrite the quadruple:

$$\begin{pmatrix} A & 0 \\ a & \alpha \end{pmatrix}, \begin{pmatrix} B & 0 \\ b & \beta \end{pmatrix}, \begin{pmatrix} I \\ X \end{pmatrix}, \begin{pmatrix} \mathbb{1}_k & 0 \\ 0 & 0_{n-k} \end{pmatrix},$$

with $k \neq 0$. In 6.3 we showed that we can slightly perturb the quadruple and get to a quadruple with $a = 0$ keeping G fixed. At this point there are two possibilities: either the deformed quadruple still sits in $\{\Delta_A = 0\}$, or it doesn't. In the first case we may proceed to deform to a quadruple for which G is invertible leaving the first endomorphism unchanged as in 6.3.1; in the latter case, we already moved the quadruple out of $\{\Delta_A = 0\}$ along a path contained in $\{det(G) = 0\}$, and the claim is proved. □

Remark 6.6.3. $\mathcal{M}_{\mathbb{P}^2,\Omega}(r, 1)$ is smooth for any even r.

Proof. We can apply Prop. 6.6.1 and conclude directly. We could also proceed in the following way. We know

$$\mathcal{M}_{\mathbb{P}^2,\Omega}(r, 1) \cong \mathbb{M}^s_\Omega(r, 1)/\mathbb{C}^\star$$

where $\mathbb{M}^s_\Omega(r, 1)$ is just the affine variety $\mathbb{C}^2 \times (\mathbb{C}^r \setminus \{0\}) \times \mathbb{C}$; indeed, the symmetries and the ADHM equation are vacuous in this case. Since the $\mathbb{C}^\star-$action is free on this space, no singularities can arise in the quotient. □

The case $n = 1$ is special, since the singular locus of the Uhlenbeck space $\mathcal{M}_{0,\Omega}(r, 1)$

is smooth (therefore, a single blow-up is enough to resolve the singularities). In fact, singularities appear in $\mathcal{M}_\Omega(r,n)$ as we take $n > 1$. In the next example, we draw our attention to the case $n = 2$.

Example 6.6.4. Let $I : W \to \mathbb{C}^2$ be a linear map, and consider the ADHM quadruple $\xi = (0,0,I,0) \in \mathbb{M}_\Omega(r,2)$. This configuration will be stable if we choose I to be surjective. Let us compute the dimension of $T_\xi\mathbb{M}^s_\Omega(r,1)$: using the description of the tangent space in the proof of Prop. 6.2.5, we have

$$T_\xi\mathbb{M}^s_\Omega(r,1) = \{(X_A, X_B, X_I, X_G) \mid I\Omega I^\top X_G = 0\}.$$

If we choose I so that $I\Omega I^\top$ is not invertible (it is enough to require the rows of the matrix I to span an isotropic subspace of W, and this can be done while keeping I surjective if $r \geq 4$), we obtain

$$dim(T_\xi\mathbb{M}^s_\Omega(r,1)) > 2 \cdot 2^2 + 2r = 8 + 2r.$$

The dimension of $\mathbb{M}^s_\Omega(r,1)$ is exactly $8 + 2r$, so we found a singular point.

Suppose (A,B,I,G) is a singular point of $\mathbb{M}^s_\Omega(r,2)$. By Prop. 6.6.1, we must have A and B derogatory, which for 2×2 matrices just means to be a multiple of the identity:

$$A = \lambda_A \cdot \mathbb{1},\ B = \lambda_B \cdot \mathbb{1}.$$

Also, we know $det(G) = 0$. Call $I^1, I^2 \in \mathbb{C}^r$ the rows of I; in order to satisfy stability of the quadruple, we must have that I^1 and I^2 span a $2-$dimensional $\Omega-$isotropic subspace of $\mathbb{C}^r$. Call J the Jacobian matrix 6.6.1; as explained in the proof of Prop. 6.6.1, (A,B,I,G) is a smooth point if and only if $rk(J) < \frac{3}{2}n(n-1) = 3$. If G has rank 1, it is immediate to check that the rank of J is at least 3, so that we get $G = 0$. We thus see that every singular point of $\mathcal{M}_\Omega(r,2)$ defines a unique isotropic $2-$subspace of (W,Ω) and a pair of scalars $(\lambda_A, \lambda_B) \in \mathbb{A}^2$. We conclude:

Lemma 6.6.5. *There is an isomorphism*

$$Sing(\mathcal{M}_\Omega(r,2)) \cong \mathbb{A}^2 \times \mathbb{G}^\Omega(2,r),$$

where $\mathbb{G}^\Omega(2,r)$ is the Grassmannian of $\Omega-$isotropic $2-$spaces in W. In particular, $Sing(\mathcal{M}_\Omega(r,2))$ is a smooth connected quasi-projective variety.

Remark 6.6.6. If we consider also $r = 2$, we obtain $\mathbb{G}^\Omega(2,2) = \emptyset$. This makes sense since $\mathcal{M}_\Omega(2,n) \cong \mathcal{M}(2,n)$, which is a direct consequence of the fact that every rank 2 vector bundle has a symplectic structure.

We conclude this section with the following remark.

Remark 6.6.7. Suppose that a point $x = [(E, \alpha, \varphi)] \in \mathcal{M}_\Omega(r, n)$ satisfies the hypothesis of Cor. 3.3.4. Let $\iota(x) \in \mathcal{M}(r, n)$ be the corresponding framed sheaf. We have an exact sequence of vector spaces

$$0 \longrightarrow T_x\mathcal{M}_\Omega(r, n) \longrightarrow T_{\iota(x)}\mathcal{M}(r, n) \longrightarrow Ext^1(\Lambda^2 E, \mathcal{O}_X(-1)) \longrightarrow 0$$

This forces

$$ext^1(\Lambda^2 E, \mathcal{O}_X(-1)) = 2nr - rn - 2n = rn - 2n.$$

By Serre duality, we have

$$Ext^1(\Lambda^2 E, \mathcal{O}_X(-1)) \cong H^1(\mathbb{P}^2, \Lambda^2 E(-2)).$$

Let us take $r = 2$. In this case, the map

$$T_x\mathcal{M}_\Omega(r, n) \to T_x\mathcal{M}(r, n)$$

is an isomorphism, see Remk. 6.6.6. Let us consider the $n = 2$ smooth point $E = \mathcal{I}_{(0,0)} \oplus \mathcal{O}_{\mathbb{P}^2}$, where $\mathcal{I}_{(0,0)}$ is the ideal sheaf of the origin. We have

$$\Lambda^2 E \cong \mathcal{I}_{(0,0)} \oplus \mathbb{C}_{(0,0)}$$

and thus

$$ext^1(\Lambda^2 E, \mathcal{O}_X(-1)) \cong h^1(\mathbb{P}^2, \mathcal{I}_{(0,0)}(-2)) = c_2(\mathcal{I}_{(0,0)}) = 1,$$

see [Na, Sect. 2.1]. This shows that 3.3.4 does not give necessary conditions for a point of the moduli space to be smooth.

7 The moduli space on the blown-up plane

Let $\hat{\mathbb{P}}^2$ be the blow-up of $\mathbb{P}^2$ at one point p. We fix as a framing divisor a line l_∞, $p \notin l_\infty$, and identify $\hat{\mathbb{P}}^2 \backslash l_\infty$ with $\mathbb{C}^2$ blown up at $(0,0)$. Call C the exceptional line and F the typical fiber of $\hat{\mathbb{P}}^2 \to C$. Keep in mind that $\hat{\mathbb{P}}^2$ is naturally identified with the projectivization of the bundle $\mathcal{O}_C(-1)$ on C. We have $l_\infty = F + C$, $l_\infty^2 = 1$, $C^2 = -1$, $F^2 = 0$. One has the usual line bundles on $\hat{\mathbb{P}}^2$

$$\mathcal{O}(p,q) = \mathcal{O}_{\hat{\mathbb{P}}^2}(p \cdot l_\infty) \otimes \mathcal{O}_{\hat{\mathbb{P}}^2}(q \cdot F).$$

We will make use of the following coordinate system on the surface. Take the flag manifold $\mathbf{F} = \mathbf{F}(\mathbb{C}^3)$ given by the hypersurface

$$\mathbf{F} \subseteq \mathbb{P}^2 \times \mathbb{P}^{2\star},\ \mathbf{F} = \{([x],[y]) \mid y(x) = \sum x_i y_i = 0\}.$$

We may identify $\hat{\mathbb{P}}^2$ with the hypersurface $y_3 = 0$ in $\mathbf{F}$. In these coordinates, we may recover $C = \{x_2 = 0\} = \{x_1 = 0\}$, $l_\infty = \{x_3 = 0\}$ and we get F as the typical fiber of $\pi : \mathbf{F} \to \mathbb{P}^{2\star}$. We have $\mathcal{O}(1,-1) = \mathcal{O}_{\hat{\mathbb{P}}^2}(C)$, and $H^0(\mathcal{O}(1,-1))$ is generated by the section $x_2/y_1 = -x_1/y_2$.

7.1 Framed sheaves on $\hat{\mathbb{P}}^2$

The main references here are [Ki] and [NY1]. We shall denote the moduli space $\mathcal{M}_{\hat{\mathbb{P}}^2, l_\infty}(r,n)$ and its locally free locus $\mathcal{M}^{reg}_{\hat{\mathbb{P}}^2, l_\infty}(r,n)$ by $\hat{\mathcal{M}}(r,n)$ and $\hat{\mathcal{M}}^{reg}(r,n)$, respectively. Of course framed sheaves on $\hat{\mathbb{P}}^2$ need not to satisfy $c_1 = 0$, but we are going to consider moduli spaces with $c_1 = 0$ only as we are interested in symplectic sheaves.

Let n be a nonnegative integer and r a positive integer, and let $V_0 \cong V_1 \cong \mathbb{C}^k$, $W \cong \mathbb{C}^r$ be vector spaces.

Definition 7.1.1. We define the *variety of blown-up ADHM configurations* of type (r,n) to be the subvariety $\hat{\mathbb{M}}(r,n)$ of the affine space

$$Hom(V_1,V_0)^{\oplus 2} \oplus Hom(W,V_0) \oplus Hom(V_1,W) \oplus Hom(V_0,V_1) = \{(A,B,I,J,D)\}$$

cut out by the equation

$$ADB - BDA + IJ = 0.$$

We say that a configuration (A, B, I, J, D) is

- *stable* if there exists no subspaces $0 \subseteq S_0 \subsetneq V_0$, $0 \subseteq S_1 \subseteq V_1$ with $dim(S_0) = dim(S_1)$ such that $A(S_1) \subseteq S_0$, $B(S_1) \subseteq S_0$ and $im(I) \subseteq S_0$;
- *co-stable* if there exists no subspaces $0 \subseteq S_0 \subseteq V_0$, $0 \subsetneq S_1 \subseteq V_1$ with $dim(S_0) = dim(S_1)$ such that $A(S_1) \subseteq S_0$, $B(S_1) \subseteq S_0$ and $ker(J) \supseteq S_1$;
- *regular* if it is stable and co-stable.

The variety $\hat{\mathbb{M}}(r, n)$ is naturally acted on by $GL(V_0) \times GL(V_1)$:

$$(g_0, g_1) \cdot (A, B, I, J, D) = (g_0 A g_1^{-1}, g_0 B g_1^{-1}, g_0 I, J g_1^{-1}, g_1 D g_0^{-1}).$$

The open subvarieties $\hat{\mathbb{M}}^s(r, n)$, $\hat{\mathbb{M}}^c(r, n)$ and $\hat{\mathbb{M}}^{reg}(r, n)$ of stable, co-stable and regular data are invariant with respect to this action.

Exactly as in the planar case, this ADHM space can be viewed as a parameter space for monads. Consider the vector bundles

$$\mathcal{U} = (V_1 \otimes \mathcal{O}(-1, 0)) \oplus (V_0 \otimes \mathcal{O}(0, -1))$$

$$\mathcal{W} = (V_0 \oplus V_1 \oplus V_0 \oplus V_1 \oplus W) \otimes \mathcal{O}$$

$$\mathcal{T} = (V_0 \otimes \mathcal{O}(1, 0)) \oplus (V_1 \otimes \mathcal{O}(0, 1))$$

and define a morphism

$$(\alpha, \beta) : Hom(V_1, V_0)^{\oplus 2} \oplus Hom(W, V_0) \oplus Hom(V_1, W) \oplus Hom(V_0, V_1) \to$$

$$\to Hom_{\mathcal{O}_X}(\mathcal{U}, \mathcal{W}) \times Hom_{\mathcal{O}_X}(\mathcal{W}, \mathcal{T})$$

by

$$\alpha = \begin{pmatrix} Ax_3 & -y_2 \\ x_1 - DAx_3 & 0 \\ Bx_3 & y_1 \\ x_2 - DBx_3 & 0 \\ Jx_3 & 0 \end{pmatrix}, \ \beta = \begin{pmatrix} x_2 & Bx_3 & -x_1 & -Ax_3 & Ix_3 \\ Dy_1 & y_1 & Dy_2 & y_2 & 0 \end{pmatrix}.$$

Again, one verifies immediately that the locus $\beta \circ \alpha = 0$ coincides with $\hat{\mathbb{M}}(r, n)$ and that the stability and co-stability conditions correspond to the injectivity of α as a map of vector bundles and the surjectivity of β, respectively. We assigned to every $\xi \in \hat{\mathbb{M}}^s(r, n)$ a monad whose cohomology $E(\xi)$ defines a torsion free sheaf on $\hat{\mathbb{P}}^2$ with rank r, $c_1 = 0$ and $c_2 = n$. Imposing $x_3 = 0$ (i.e. pulling back to l_∞) we obtain a framing $E(\xi)\mid_{l_\infty} \to \mathcal{O}_{l_\infty} \otimes W$. Once again, every framed sheaf on $\hat{\mathbb{P}}^2$ can be obtained this way. Moreover:

Theorem 7.1.2. *[Ki, He]The $GL_{01} = GL(V_0) \times GL(V_1)-$action on $\hat{\mathbb{M}}^s(r,n)$ is free and locally proper. The quotient $\hat{\mathbb{M}}^s(r,n)/G$ exists as an algebraic variety, which is smooth of dimension $2rn$ and connected. There exists an isomorphism*

$$\hat{\mathbb{M}}^s(r,n)/GL_{01} \cong \hat{\mathcal{M}}(r,n)$$

which maps the open subscheme $\hat{\mathbb{M}}^{reg}(r,n)/G_{01}$ onto $\hat{\mathcal{M}}^{reg}(r,n)$.

In addition, we know that there exists a universal monad on $\hat{\mathbb{P}}^2$, see Remk. 6.1.4.

Let us denote by $\pi : \hat{\mathbb{P}}^2 \to \mathbb{P}^2$ the blow-up morphism. Then π induces a pullback morphism

$$\pi^\star : \mathcal{M}(r,n) \to \hat{\mathcal{M}}(r,n)$$

which is in fact an open embedding.

Remark 7.1.3. We can describe the morphism $\pi^\star$ by means of ADHM data in the following way. First, fix an identification $V_0 \cong V_1 \cong V$, and consider the morphism

$$\tilde{\pi}^\star : \mathbb{M}(r,n) \to \hat{\mathbb{M}}(r,n),\ (A,B,I,J) \mapsto (A,B,I,J,\mathbb{1}_V).$$

It is immediate to see that $\tilde{\pi}^\star$ preserves stability and co-stability. Furthermore, it is equivariant with respect to the diagonal morphism $GL(V) \to GL(V)^{\times 2}$, hence it descends to a morphism between the corresponding quotients. The induced map coincides with $\pi^\star$ because this is verified on the locally free locus $\mathcal{M}^{reg}(r,n)$, see [Ki, Sect. 3.6]. We see that $\pi^\star$ maps $\mathcal{M}(n,r)$ isomorphically onto the open subscheme $\{[A,B,I,J,D] \mid D \,\text{is invertible}\} \subseteq \hat{\mathcal{M}}(r,n)$.

7.2 Framed symplectic sheaves on $\hat{\mathbb{P}}^2$

Let (E,a) be a framed sheaf on $\hat{\mathbb{P}}^2$, defined by an ADHM configuration (A,B,I,J,D). Let M be the corresponding monad. Suppose now we have a morphism $\varphi : E \to E^\vee$ whose restriction to l_∞ is a matrix Ω. We claim that φ lifts to a morphism Φ between M and its dual:

$$\begin{array}{ccccc} \mathcal{U} & \longrightarrow & \mathcal{W} & \longrightarrow & \mathcal{T} \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle h} \\ \mathcal{T}^\vee & \longrightarrow & \mathcal{W}^\vee & \longrightarrow & \mathcal{U}^\vee \end{array}$$

This is proved once we verify the vanishings in the hypothesis of Prop. 2.1.9. We have $Ext^1(\mathcal{T}, \mathcal{W}^\vee) = 0$ as

$$H^1(\hat{\mathbb{P}}^2, \mathcal{O}(0,-1)) = H^1(\hat{\mathbb{P}}^2, \mathcal{O}(-1,0)) = 0,$$

$Ext^1(\mathcal{W}, \mathcal{T}^\vee) = 0$ as

$$H^1(\hat{\mathbb{P}}^2, \mathcal{O}(0,1)) = H^1(\hat{\mathbb{P}}^2, \mathcal{O}(1,0)) = 0,$$

and $Ext^2(\mathcal{T}, \mathcal{T}^\vee) = 0$ as

$$H^2(\hat{\mathbb{P}}^2, \mathcal{O}(-2,0)) = H^2(\hat{\mathbb{P}}^2, \mathcal{O}(0,-2)) = H^2(\hat{\mathbb{P}}^2, \mathcal{O}(-1,-1)) = 0.$$

This ensures the existence of the claimed lifting. Furthermore, the additional vanishings

$$H^0(\hat{\mathbb{P}}^2, \mathcal{O}(-1,0)) = H^0(\hat{\mathbb{P}}^2, \mathcal{O}(0,-1)) = 0$$

ensure $Hom(\mathcal{T}, \mathcal{W}^\vee) \cong Hom(\mathcal{W}, \mathcal{T}^\vee) = 0$, which allows us to identify the kernel of the map $Hom(M, M^\vee) \to Hom(\mathcal{E}, \mathcal{E}^\vee)$ with the vector space

$$Ext^1(\mathcal{T}, \mathcal{T}^\vee) \cong H^1(\mathcal{O}(0,-2)) \otimes Hom_{\mathbb{C}}(V_1, V_1^\vee) \cong Hom_{\mathbb{C}}(V_1, V_1^\vee).$$

Let us describe the set of all possible $\Phi = \Phi(g,p,h)$ lifting φ. We denote:

$$g : \mathcal{U} \to \mathcal{T}^\vee : \ \ g = \begin{pmatrix} G_{10} & 0 \\ G_{11}(x_2/y_1) & G_{01} \end{pmatrix}$$

with $G_{ij} \in Hom(V_i, V_j^\vee)$,

$$h : \mathcal{T} \to \mathcal{U}^\vee : \ \ h = \begin{pmatrix} H_{01} & H_{11}(x_2/y_1) \\ 0 & H_{10} \end{pmatrix}$$

with $H_{ij} \in Hom(V_i, V_j^\vee)$. By applying the constraints.

$$\beta^\vee g - p\alpha = 0, \ h\beta - \alpha^\vee p = 0,$$

we get a set of equations:

$$\begin{cases} G_{11} = -H_{11} \\ D^\vee G_{01} = -H_{10} D \\ G_{10} + D^\vee G_{11} + H_{10} = 0 & H_{01} + H_{11} D + G_{01} = 0 \\ I^\vee G_{10} = \Omega J & H_{01} I = J^\vee \Omega \\ H_{01} A - A^\vee H_{10} + A^\vee D^\vee H_{11} = 0 & H_{01} B - B^\vee H_{10} + B^\vee D^\vee H_{11} = 0 \\ B^\vee G_{10} - G_{01} B + G_{11} D B = 0 & A^\vee G_{10} - G_{01} A + G_{11} D A = 0 \end{cases} \tag{7.2.1}$$

and we may write

$$p = \begin{pmatrix} 0 & 0 & D^\vee G_{01} & -H_{10} & 0 \\ 0 & 0 & G_{01} & G_{11} & 0 \\ -D^\vee G_{01} & H_{10} & 0 & 0 & 0 \\ -G_{01} & -G_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \Omega \end{pmatrix}$$

Remark 7.2.1. *(Normalizing bilinear forms on monads)* Let $K \in Hom(V_1, V_1^\vee)$ be a morphism. We define morphisms

$$g(K) : \mathcal{U} \to \mathcal{T}^\vee,\ g(K) = \begin{pmatrix} 0 & 0 \\ K \cdot (x_2/y_1) & KD \end{pmatrix};$$

$$h(K) : \mathcal{T} \to \mathcal{U}^\vee,\ h(K) = \begin{pmatrix} 0 & -K \cdot (x_2/y_1) \\ 0 & -D^\vee K \end{pmatrix};$$

$$p(K) : \mathcal{W} \to \mathcal{W}^\vee,\ p(K) = \begin{pmatrix} 0 & 0 & D^\vee KD & D^\vee K & 0 \\ 0 & 0 & KD & K & 0 \\ -D^\vee KD & -D^\vee K & 0 & 0 & 0 \\ -KD & -K & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is immediate to verify, using 7.2.1, that g, p and h define a morphsim $M \to M^\vee$, providing a linear monomorphism

$$Hom_{\mathbb{C}}(V_1, V_1^\vee) \to Hom(M, M^\vee).$$

One can also check that that any such morphism induces the 0 map in cohomology; we described explicitly $Hom_{\mathbb{C}}(V_1, V_1^\vee)$ as the kernel of the map

$$Hom(M, M^\vee) \to Hom(\mathcal{E}, \mathcal{E}^\vee).$$

As a workaround to fix the uncertainty in the choice of a lifting for φ, we shall always consider in our situation a fixed complement of $Hom_{\mathbb{C}}(V_1, V_1^\vee)$ inside $Hom(M, M^\vee)$, identified in the above notation by the equation $G_{11} = 0$.

We impose now the skew-symmetry condition on φ, which amounts to ask

$$g^\vee = -h,$$

which in turn reads

$$G_{10} = -H_{01}^\top,\ G_{01} = -H_{10}^\top.$$

We may now perform some substitutions in the system 7.2.1, also keeping in account the normalization as in Remk. 7.2.1. We are left with

$$\begin{cases} G_{10} = G_{01}^\vee & A^\vee G_{10} - G_{01}A = 0 \\ D^\vee G_{01} = G_{10}D & B^\vee G_{10} - G_{01}B = 0 \\ J = \Omega^{-1} I^\vee G_{10} & \end{cases}$$

We call $-G_{10} = G$.

The previous discussion motivates the following definition of ADHM-type parameter space for symplectic sheaves on $\hat{\mathbb{P}}^2$.

Definition 7.2.2. We define the variety of *blown-up symplectic ADHM configurations* $\hat{\mathbb{M}}_\Omega(r, n)$ of type (r, n) to be the subvariety of the affine space

$$Hom(V_1, V_0)^{\oplus 2} \oplus Hom(W, V_0) \oplus Hom(V_1, V_0^\vee) \oplus Hom(V_0, V_1) = \{(A, B, I, G, D)\}$$

cut out by the equations

- $G^\vee A = A^\vee G$ (*GA-symmetry*);
- $G^\vee B = B^\vee G$ (*GB-symmetry*);
- $GD = D^\vee G^\vee$ (*GD-symmetry*);
- $ADB - BDA - I\Omega^{-1}I^\vee G = 0$ (*ADHM eqt*).

We say that a configuration (A, B, I, G) is

- *stable* if there exists no subspaces $0 \subseteq S_0 \subsetneq V_0$, $0 \subseteq S_1 \subseteq V_1$ with $dim(S_0) = dim(S_1)$ such that $A(S_1) \subseteq S_0$, $B(S_1) \subseteq S_0$ and $im(I) \subseteq S_0$;
- *regular* if it is stable and G is invertible.

The variety $\hat{\mathbb{M}}_\Omega(n,r)$ is naturally acted on by $GL(V_0)\times GL(V_1)$:

$$(g_0,g_1)\cdot(A,B,I,D,G)=(g_0Ag_1^{-1},g_0Bg_1^{-1},g_0I,g_0^{-\vee}Gg_1^{-1},g_1Dg_0^{-1}).$$

The open subvarieties $\mathbb{M}^s_\Omega(r,n)$ and $\mathbb{M}^{reg}_\Omega(r,n)$ of stable and regular data are invariant with respect to this action.

Definition 7.2.3. We define the $GL(V_0)\times GL(V_1)-$equivariant map

$$\hat{\iota}:\hat{\mathbb{M}}_\Omega(r,n)\to\hat{\mathbb{M}}(r,n),\ \iota(A,B,I,G,D)=(A,B,I,-\Omega I^\vee G,D).$$

By definition the equality $\hat{\iota}^{-1}(\hat{\mathbb{M}}^s(r,n)=\hat{\mathbb{M}}^s_\Omega(r,n)$ holds, and we have:

Lemma 7.2.4. $\hat{\iota}^{-1}(\hat{\mathbb{M}}^{reg}(r,n))=\hat{\mathbb{M}}^{reg}_\Omega(r,n)$.

Proof. Let $(A,B,I,G,D)\in\hat{\mathbb{M}}^s_\Omega(r,n)$, and suppose we have nonzero subspaces $S_1\subseteq ker(-\Omega I^\vee G)$, $S_0\subseteq V_0$ such that $dim(S_0)=dim(S_1)$, $A(S_1)\subseteq S_0$, $B(S_1)\subseteq S_0$ and $D(S_0)\subseteq S_1$. Suppose G is invertible; then the subspaces

$$S_0'=G(S_1)^\perp\subseteq V_0,\ S_1'=G^\vee(S_0)^\perp\subseteq V_1$$

are equidimensional and satisfy:

$$ker(-\Omega I^\vee G)\implies im(I)\subseteq S_0'.$$

Using the symmetries and the hypothesis it is immediate to verify

$$A(S_1')\subseteq S_0',\ B(S_1')\subseteq S_0',\ D(S_0')\subseteq S_1'.$$

Thus stability forces $S_1'=0$, which in turn implies $S_1,S_0=0$, i.e $\hat{\iota}(A,B,I,G,D)\in\hat{\mathbb{M}}^{reg}(n,r)$. *Vice versa*, suppose $ker(G)\neq 0$. Call S_1 and S_0 the nonzero equidimensional subspaces $ker(G)\subseteq V_1$ and $ker(G^\vee)\subseteq V_0$; one can readily check that S_0 and S_1 obstruct co-stability of $\hat{\iota}(A,B,I,G,D)$. □

Proposition 7.2.5. *The restriction of* ι *to the stable loci*

$$\hat{\iota}:\hat{\mathbb{M}}^s_\Omega(n,r)\to\hat{\mathbb{M}}^s(n,r)$$

is a closed embedding.

Proof. The proof follows the same strategy of the analogous proposition 6.2.5. For injectivity at closed points, suppose $(A,B,I,G_i,D)\in\hat{\mathbb{M}}^s_\Omega(r,n)$, $i=1,2$, have the same image under $\hat{\iota}$. Let $G=G_1-G_2$. We get $I^\vee G=0$, $A^\vee G=G^\vee A$, $B^\vee G=G^\vee B$, $GD=D^\vee G^\vee$. Let $S_0=ker(G^\vee)\subseteq V_0$ and $S_1=ker(G)\subseteq V_1$. We verify immediately $im(I) \subseteq S_0$, $A(S_1) \subseteq S_0$, $B(S_1) \subseteq S_0$ and $D(S_0) \subseteq S_1$. Then we must have $S_0 = V_0$ in order to get stability; we obtain $G_1 = G_2$.

Injectivity at tangent spaces is also very easy: if the tangent map

$$(X_A, X_B, X_I, X_G, X_D) \mapsto (X_A, X_B, X_I, \Omega^{-1}(I^\vee X_G + X_I^\vee G), X_D)$$

vanishes on a vector ξ, we must have $\xi = (0,0,0,X_G,0)$ with $I^\vee X_G = 0$ and

$$A^\vee X_G - X_G^\vee A = B^\vee X_G - X_G^\vee B = X_G D - D^\vee X_G^\vee = 0,$$

and we may use stability to conclude $X_G = 0$ as before.

To prove properness, we use the same valuative criterion. Let $(A_t, B_t, I_t, G_t, D_t)$ be a $\mathbb{C}((t))-$point of $\hat{\mathbb{M}}^s_\Omega(n,r)$ such that A_t, B_t, I_t, D_t and $I_t^\vee G_t$ have no poles. Suppose

$$G_t = t^{-k} G_0 + t^{-k+1} G_1 + t^{-k+2} G_2 + \dots$$

with $k > 0$ and $G_0 \neq 0$. Fix coordinates on $V_1 \cong V_0$ so that $G_0 = \mathbb{1}_m \oplus 0_{n-m}$ with $m \neq 0$. The symmetries force

$$A_0 = \begin{pmatrix} A_0^{11} & 0 \\ A_0^{21} & A_0^{22} \end{pmatrix},\ B_0 = \begin{pmatrix} B_0^{11} & 0 \\ B_0^{21} & B_0^{22} \end{pmatrix},\ D_0 = \begin{pmatrix} D_0^{11} & 0 \\ D_0^{21} & D_0^{22} \end{pmatrix}$$

and $I_0^\vee G_0 = 0$ implies

$$I = \begin{pmatrix} 0 \\ I_0^2 \end{pmatrix}.$$

We get subspaces $S_1 = S_0 = \{\begin{pmatrix} 0 \\ v \end{pmatrix} \mid v \in \mathbb{C}^{n-m}\}$ breaking the stability of the $\mathbb{C}-$point $(A_0, B_0, I_0, -\Omega^{-1} I_0^\vee, D_0)$, thus we may conclude G_t has no poles. □

Once again we may deduce that the GL_{01} action on $\hat{\mathbb{M}}^s(n,r)$ is free and locally proper, so that the geometric quotient $\hat{\mathbb{M}}(n,r)/GL_{01}$ is well defined. Furthermore, by means of the very same techniques we used in the proof of Thm. 6.2.4, we can prove:

Theorem 7.2.6. *There exists an isomorphism*

$$\hat{\mathbb{M}}^s_\Omega(n,r)/GL_{01} \cong \hat{\mathcal{M}}_\Omega(n,r)$$

which maps the open subscheme $\hat{\mathbb{M}}^{sc}_\Omega(n,r)/GL_{01}$ *onto* $\hat{\mathcal{M}}^{reg}_\Omega(n,r)$. *The induced closed embedding*

$$\iota : \hat{\mathcal{M}}_\Omega(n,r) \to \hat{\mathcal{M}}(n,r)$$

coincides with the one defined in Chapter 3.

Remark 7.2.7. The moduli space $\mathcal{M}_\Omega(n, r)$ may be identified with the open subscheme

$$\{det(D) \neq 0\}/GL_{01} \subseteq \hat{\mathcal{M}}_\Omega(n, r)$$

by restricting the pullback map $\pi^\star$ defined in 7.1,

7.3 About the irreducibility of $\hat{\mathcal{M}}_\Omega(n, r)$

We formulate the following conjecture generalizing the results of Sect. 6.3.

Conjecture 7.3.1. *The moduli space $\hat{\mathcal{M}}_\Omega(n, r)$ is irreducible.*

We devote the present section to discuss some pieces of evidence.

Firstly, we can observe that, due to the irreducibility of $\mathcal{M}_\Omega(r, n)$, the conjecture may be reformulated as follows: the open subscheme $\mathcal{M}_\Omega(r, n) \subseteq \hat{\mathcal{M}}_\Omega(r, n)$ is dense. In the following lemma, we exhibit a class of elements of $\hat{\mathcal{M}}_\Omega(r, n)$ sitting in $\overline{\mathcal{M}_\Omega(r, n)}$.

Lemma 7.3.2. *Let $\xi = [A, B, I, G, D] \in \hat{\mathcal{M}}_\Omega(r, n)$. Assume the existence of a pair $(\mu_A, \mu_B) \in \mathbb{C}^2\backslash\{(0, 0)\}$ such that $\mu_A A + \mu_B B$ is invertible. Then*

$$\xi \in \overline{\mathcal{M}_\Omega(r, n)} \subseteq \hat{\mathcal{M}}_\Omega(r, n).$$

Proof. Let $(\mu_A, \mu_B) \in \mathbb{C}^2\backslash\{(0, 0)\}$ be as in the hypothesis and suppose that $\mu_A \neq 0$. Then $A + tB$ is invertible for all except a finite number of values of $t \in \mathbb{C}$. As

$$(A + tB)DB - BD(A + tB) = ADB - BDA,$$

up to a small deformation we can suppose A is invertible. Let $D' = A^{-1}$; as

$$GA^{-1} = A^{-\vee}G^\vee$$

by the GA symmetry and of course

$$AD'B - BD'A = B - B = 0,$$

we see that

$$(A, B, I, G, D + tD')$$

sits in $\hat{\mathbb{M}}^s_\Omega(r, n)$ for small values of t, and $D + tD'$is invertible. □

Remark 7.3.3. The previous lemma has a geometric rephrasing: the condition in the hypothesis is satisfied if and only if the corresponding sheaf is trivial on the fiber

$F_{[\mu_A,\mu_B]}$ of the natural projection $\hat{\mathbb{P}}^2 \to \mathbb{P}^1$ or, equivalently, if it is trivial on the general fiber of the projection. Not all framed sheaves on $\hat{\mathbb{P}}^2$ need to satisfy this.

We are now able to prove a fundamental necessary condition.

Proposition 7.3.4. *The moduli space $\hat{\mathcal{M}}^{reg}_{\Omega}(n,r)$ is irreducible.*

Proof. We are going to prove that the open irreducible subset $\mathcal{M}^{reg}(n,r) \subseteq \hat{\mathcal{M}}^{reg}(n,r)$ is dense, see Remk. 7.2.7. Let us start with a symplectic bundle

$$[A,B,I,G,D] \in \hat{\mathcal{M}}^{reg}(n,r).$$

By Lemma 7.2.4, we know G is invertible. Choose an isomorphism $V_0 \cong V_1 \cong V$ and coordinates on V so that

$$D = \mathbb{1}_m \oplus 0_{n-m}.$$

Fix also Darboux coordinates on W. The symmetry $GD = D^\vee G^\vee$ yields

$$G = \begin{pmatrix} G' & g \\ 0 & \gamma \end{pmatrix}$$

with G' symmetric. The elements of $GL(V)^{\times 2}$ of the form

$$(g_0, g_1) = (\begin{pmatrix} X & 0 \\ s & \sigma \end{pmatrix}, \begin{pmatrix} X & t \\ 0 & \tau \end{pmatrix})$$

fix the present form of D and transform G as

$$g_0^{-\vee} G g_1^{-1} = \begin{pmatrix} X^{-\vee}G'X^{-1} & -X^{-\vee}G'X^{-1}t\tau^{-1} + X^{-\vee}g\tau^{-1} + X^{-\vee}s^\vee\sigma^{-1}\gamma\tau^{-1} \\ 0 & \sigma^{-1}\gamma\tau^{-1} \end{pmatrix}.$$

Consequently, we can choose X, t, σ and τ so that $G = \mathbb{1}_n$. Let us decompose also

$$A = \begin{pmatrix} A' & a_1 \\ a_2 & \alpha \end{pmatrix}, B = \begin{pmatrix} B' & b_1 \\ b_2 & \beta \end{pmatrix}, I = \begin{pmatrix} I' \\ X \end{pmatrix}.$$

The $A, B-$symmetries imply that A', B', α and β are symmetric and $a_1 = a_2^\top$, $b_1 = b_2^\top$. In particular, the matrices AD and DA are one the transpose of the other. Applying Thm. 5.0.8, we are able to find a nonsingular symmetric matrix C such that $ADC = CDA$. If we set $B_t = B + tC$, the deformation

$$(A, B_t, D, I, G)$$

sits in $\hat{\mathbb{M}}^s_{\Omega}(r,n)$ for small values of $t \in \mathbb{C}$. As the general point of the curve B_t is an invertible matrix, we obtain from Lemma 7.3.2

$$[A, B, D, I, G] \in \overline{\mathcal{M}^{reg}_{\Omega}(n, r)} \subseteq \hat{\mathcal{M}}^{reg}_{\Omega}(n, r);$$

hence, the irreducible open subset $\mathcal{M}^{reg}_{\Omega}(n, r)$ is dense. This provides the thesis. □

Remark 7.3.5. From the proof of the proposition one can extract the following description of $\hat{\mathcal{M}}^{reg}(r, n)$. Consider the space

$$\hat{\mathbb{X}}(r, n) = \{(A, B, I, D) \in S^2(V)^{\oplus 2} \oplus Hom(W, V) \oplus S^2(V^\vee) \mid ADB - BDA - I\Omega^{-1}I^\vee = 0\}$$

and let $\hat{\mathbb{X}}^s(r, n)$ be the open subspace of stable configurations, where stability is defined exactly as in Defn. 7.2.2. Then we have a natural free action of $GL(V)$ on $\hat{\mathbb{X}}^s(r, n)$ and an isomorphism

$$\hat{\mathbb{X}}^s(r, n)/GL(V) \cong \hat{\mathcal{M}}^{reg}(r, n),$$

see for example [BS].

Remark 7.3.6. We can prove that the moduli space $\hat{\mathcal{M}}^{reg}_{\Omega}(r, n)$ is smooth verifying that $H^2(\hat{\mathbb{P}}^2, Ad^{\varphi}(E)(-l_\infty)) = 0$ for a symplectic bundle (E, a, φ), see Sect. 4.4. This could in principle give a shortcut for the proof of Prop. 7.3.4, but I was not able to find a more concise proof for the connectedness.

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