ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

Homoclinic solutions for a class of non autonomous second order Hamiltonian systems

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Piero Montecchiari

SUPERVISOR

Prof. Vittorio Coti Zelati

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§0. Introduction

This thesis concerns the existence of non trivial homoclinic orbits for second order asymptotically periodic hamiltonian systems with superquadratic potential.

To be precise we study the problem:

$$\underline{\text{find}}\ q \in C^2(\mathrm{R}, \mathrm{R}^m) \setminus \{0\}\ \underline{\text{such that:}}$$

$$(HS) \quad \begin{cases} \ddot{q} = L(t)q + \nabla V(t,q) \\ q(t) \underset{|t|\to\infty}{\longrightarrow} 0 \\ \dot{q}(t) \underset{|t|\to\infty}{\longrightarrow} 0 \end{cases}$$

where $L \in C(\mathrm{R}, \mathrm{R}^{m^2})$ is a symmetric and positive definite matrix uniformly for $t \in \mathrm{R}$ and where $V \in C^1(\mathrm{R} \times \mathrm{R}^m, \mathrm{R})$ is a superquadratic potential, that is

$$\exists \mu > 2 \,/\, 0 < \mu V(t,x) \leq \nabla V(t,x)x \ , \ \ \forall\, t \in \mathrm{R}, \forall\, x \in \mathrm{R}^m \setminus \{0\},$$

which satisfies also $\nabla V(t,x) = o(x)$ for $|x| \to 0$ and $\forall\, R > 0\ \exists M_R > 0\ /\ |\nabla V(t,x) - \nabla V(t,y)| \leq M_R|x-y|$, $\forall\, x$, $y \in B(0,M)$. Moreover we ask that the functions V and L approximate at infinity some periodic functions. Precisely, for the function V we suppose that there exists V_- , $V_+ \in C^1(\mathrm{R}\times \mathrm{R}^m, \mathrm{R})$, which verify the same hypotheses as V with constants μ_i , $M_R^{(i)}$, $i = 1,2$ respectively, such that

$$|\nabla V(t,x) - \nabla V_+(t,x)| \underset{t\to+\infty}{\longrightarrow} 0 \quad ,$$
$$|\nabla V(t,x) - \nabla V_-(t,x)| \underset{t\to-\infty}{\longrightarrow} 0 \quad , \textit{ uniformly on the compacts of } \mathrm{R}^m$$

and

$$\exists T_- , T_+ \geq 0 \,/\, V_i(t+T_i, x) = V_i(t,x)\ \forall\, (t,x) \in \mathrm{R} \times \mathrm{R}^m , \, i = +, - ,$$

for the function L that there exists $L_-(t)$, $L_+(t)$ symmetric $m \times m$ matrices, positive definite uniformly for $t \in \mathrm{R}$, with continuous and periodic coefficients of minimal period T_- , $T_+ \geq 0$ respectively, such that

$$|L_{i,j}(t) - L_{-,i,j}(t)| \underset{t\to-\infty}{\longrightarrow} 0 \quad i\,,\, j = 1,\ldots,m$$
$$|L_{i,j}(t) - L_{+,i,j}(t)| \underset{t\to+\infty}{\longrightarrow} 0 \quad i\,,\, j = 1,\ldots,m\,.$$

We approach the problem using variational methods.

We define $X = W^{1,2}(\mathrm{R}, \mathrm{R}^m)$ equiped with the norm $\|q\|^2 = \int_{\mathrm{R}} |\dot{q}|^2 + L(t)q \cdot q\, dt$.
It's easy to show that $X \subset C_0(\mathrm{R}, \mathrm{R}^m)$ the space of continuous functions q on R such that $q(t) \underset{|t|\to\infty}{\longrightarrow} 0$.
We observe that if $q \in X \cap C^2(\mathrm{R}, \mathrm{R}^m)$ is a solution of the equation then also $\ddot{q} \in C_0(\mathrm{R}, \mathrm{R}^m)$ and in that case we have $\dot{q} \underset{|t|\to\infty}{\longrightarrow} 0$.

We seek solutions of (HS) as critical points of the functional

$$\varphi(u) = \frac{1}{2}\int_{\mathbb{R}} |\dot{q}|^2 + L(t)q \cdot q \, dt - \int_{\mathbb{R}} V(t,q)\, dt \, .$$

In fact, by standard regularization arguments, it is possible to show that every critical point of φ is a C^2 solution of the equation and so, by the above observation, it is also a classical solution of (HS).

To find a non zero critical point of φ we use a mini-max type method.

We easily see that all the hypotheses of the mountain pass theorem but the (PS) condition are verified. Therefore we can't say a priori that the mountain pass level, which we call c, is a critical level for φ.

By a careful analysis of the Palais Smale sequences, using concentration-compactness arguments, we succeed in overcoming the lack of compactness if we make some hypothesis *'at infinity'*.

Precisely, if we call φ_- , φ_+ the functionals associated to the pairs of functions (L_-, V_-) , (L_+, V_+), K_- , K_+ , K the sets of non zero critical points of φ_- , φ_+ , φ and finally if we set

$$\begin{aligned} K_-^{c^*} &= K_- \cap \{x \in X \,/\, \varphi_-(x) \le c^*\} \\ K_+^{c^*} &= K_+ \cap \{x \in X \,/\, \varphi_+(x) \le c^*\} \\ K^{c^*} &= K \cap \{x \in X \,/\, \varphi(x) \le c^*\} \, , \end{aligned}$$

then

Theorem 2.7 *If $\exists\, c^* > c$ such that $K_-^{c^*}$, $K_+^{c^*}$ are constituted of isolated points, then*

$$K^{c^*} \neq \emptyset \quad .$$

The ideas used in this thesis are mainly contained in [CZES], [S1], [CZR1], [CL]. These works deal with the existence and multiplicity of homoclinic orbits for periodic first and second order superquadratic hamiltonian systems. While there is by now a large literature on the periodic problem (see [S2] and [CZR] for the best results) the asymptotically periodic one remains yet an open problem. The best result I know is contained in a paper of Rabinowitz and Tanaka [RT] which, as we will show in §2, is a consequence of theorem 2.7.

The thesis is divided in two chapter. In the first one we caracterize the Palais Smale sequences and, using this caracterization, in the second one we construct a deformation lemma which is basic in the proof of theorem 2.7.

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§1. Analysis of the Palais Smale sequences.

Let $V \in C^1(\mathrm{R} \times \mathrm{R}^m, \mathrm{R})$ such that

$$o) \quad \sup_{t\in\mathrm{R}}(|V(t,x)| + |\nabla V(t,x)|) < +\infty \quad \forall x \in \mathrm{R}^m$$

$$i) \quad |\nabla V(t,x)| = o(|x|) \quad |x| \to 0 \qquad \textit{uniformly for } t \in \mathrm{R}$$

$$ii) \quad \exists \mu > 2 \,/\, 0 < \mu V(t,x) \leq \nabla V(t,x)x \quad \forall\, t \in \mathrm{R}\,, \forall\, x \in \mathrm{R}^m \setminus \{0\}$$

$$iii) \quad \forall\, R > 0\ \exists M_R > 0 \,/\, |\nabla V(t,x) - \nabla V(t,y)| \leq M_R|x-y| \quad \forall\, x\,,\, y \in B(0,R)\,, \forall\, t \in \mathrm{R}.$$

Let $L_{i,j} \in C(\mathrm{R},\mathrm{R})\,,\, i,j = 1,\ldots,m$ such that the matrix $L(t)$, who's components are $L_{i,j}$, is symmetric and positive definite uniformly for $t \in \mathrm{R}$. Let also $\sup_{t\in\mathrm{R}} \sup_{i,j} |L_{i,j}(t)| < +\infty$. Let $X = W^{1,2}(\mathrm{R},\mathrm{R}^m)$ equiped with the norm

$$\|u\| = (\int_{\mathrm{R}} |\dot{u}|^2 + L(t)u\cdot u\, dt)^{\frac{1}{2}} \quad u \in X$$

which is clearly equivalent with the usual one and derivable from the scalar product

$$< u, v >= \int_{\mathrm{R}} \dot{u}\dot{v} + L(t)u \cdot v\, dt \quad u\,,\, v \in X\,.$$

We define also, for $A \subset \mathrm{R}$, the functional

$$\|u\|_A = (\int_A |\dot{u}|^2 + L(t)u \cdot u\, dt)^{\frac{1}{2}} \quad u \in X$$

and the bilinear form

$$< u, v >_A= \int_A \dot{u}\dot{v} + L(t)u\cdot v\, dt \quad u\,,\, v \in X\,.$$

Let $\varphi(u) = \frac{1}{2}\|u\|^2 - \int_{\mathrm{R}} V(t,u)\, dt$ for $u \in X$.
For the hypothesis $i)$ we have that

$$|V(t,x)| = |\int_0^{|x|} \frac{d}{ds}V(t,s\frac{x}{|x|})\, ds| = |\int_0^{|x|} \nabla V(t,s\frac{x}{|x|})\frac{x}{|x|}\, ds| \leq$$
$$\leq \sup_{t\in\mathrm{R}}\ \sup_{\xi\in B(0,|x|)} |\nabla V(t,\xi)||x| = o(|x|^2) \qquad |x| \to 0$$

thus φ is well defined on X because if $u \in X$ then $u(t) \to_{|t|\to\infty} 0$.

We prove moreover

Theorem 1.1. $\varphi \in C^1(X,\mathrm{R})$

Proof. Let's prove first that φ is Gateaux differentiable.

Let $u \in X$ and $h \in X \,/\, \|h\| = 1$. Then

$$\lim_{s\to 0} \frac{1}{s}(\varphi(u+sh) - \varphi(u)) = < u, h > - \lim_{s\to 0} \frac{1}{s} \int_{\mathbb{R}} V(t, u+sh) - V(t,u)\, dt \,.$$

The hypothesis $i)$ implies that $\forall\, \epsilon > 0$ there exists $R_\epsilon > 0$ such that

$$|t| > R_\epsilon \;\Rightarrow\; |\nabla V(t, u+sh)| \le |u(t)| + |h(t)| \quad \forall\, s \in (0,1)$$

therefore if we define

$$\psi_\epsilon(t) = \begin{cases} |h(t)| \cdot \sup\{|\nabla V(t,\xi)| \,/\, |t| \le R_\epsilon \,,\, |\xi| \le |u|_\infty + |h|_\infty\} & \text{if } |t| \le R_\epsilon, \\ |h(t)|(|u(t)| + |h(t)|) & \text{if } |t| > R_\epsilon \end{cases}$$

we have $\psi_\epsilon \in L^1(\mathbb{R})$ and

$$\frac{1}{s}|V(t,u+sh) - V(t,u)| = \frac{1}{s}\int_0^s |\nabla V(t, u+\tau h)||h|\, d\tau \le \psi_\epsilon(t) \quad \forall\, t \in (0,1).$$

Using the dominated convergence theorem we obtain that φ is Gateaux differentiable and

$$\varphi'_G(u)h = < u, h > - \int_{\mathbb{R}} \nabla V(t,u) h \, dt \,,\, \forall\, u \,,\, h \in X \,.$$

Let's prove now that φ'_G is continuous.

We must prove that $u_n \overset{X}{\to} u \;\Rightarrow\; \varphi'_G(u_n) \overset{X^*}{\to} \varphi'_G(u)$.

Let $\epsilon > 0$ and $R_\epsilon > 0$ such that

$$|\nabla V(t,u_n)| \le \|u_n\|_{\{|t|>R_\epsilon\}} < \epsilon \;,\; |\nabla V(t,u)| \le \|u\|_{\{|t|>R_\epsilon\}} < \epsilon \quad if\ |t| \ge R_\epsilon$$

and this is possible because of hypothesis $i)$ and the continuity of the immersion $X \to L^\infty(\mathbb{R}, \mathbb{R}^m)$.
In that case

$$\begin{aligned} |(\varphi'_G(u_n) - \varphi'_G(u))h| &\le \|u_n - u\|\|h\| + \Big(\int_{|t|<R_\epsilon} |\nabla V(t,u_n) - \nabla V(t,u)|^2\, dt\Big)^{\frac{1}{2}} \|h\| + \\ &\quad + c_1\epsilon\|h\| = \\ &= (o(1) + c_1\epsilon)\|h\| \quad n \to +\infty \end{aligned}$$

which, since ϵ is arbitrary, implies the continuity of φ'_G.

q.e.d.

Lemma 1.2. $\{u_n\}_{n\in\mathbb{N}} \in X \;,\; \varphi(u_n) \underset{n\to\infty}{\longrightarrow} c \;,\; \varphi'(u_n) \underset{n\to\infty}{\longrightarrow} 0 \;\Rightarrow\; \{u_n\}_{n\in\mathbb{N}}$ <u>is bounded in</u> X.

Proof. In fact

$$\begin{aligned} \|u_n\|^2 &= 2\varphi(u_n) + 2\int_{\mathbb{R}} V(t,u_n) \le 2\varphi(u_n) + \frac{2}{\mu}\int_{\mathbb{R}} \nabla V(t,u_n)u_n \le \\ &\le 2\varphi(u_n) - \frac{2}{\mu}\varphi'(u_n)u_n + \frac{2}{\mu}\|u_n\|^2 \end{aligned}$$

then

$$\exists c_2\,,\, c_3 > 0\,/\,(\frac{1}{2}-\frac{1}{\mu})\|u_n\|^2 \leq c_2 + c_3\|u_n\|$$

which implies that $\{u_n\}_{n\in\mathrm{N}}$ is bounded in X.

Lemma 1.3. $\{u_n\}_{n\in\mathrm{N}} \subset X\,,\, u_n \overset{w-X}{\longrightarrow} u\,,\, \varphi(u_n) \underset{n\to\infty}{\longrightarrow} c \in \mathrm{R} \;\Rightarrow\; \varphi(u_n - u) \underset{n\to\infty}{\longrightarrow} c - \varphi(u)$

Proof. Set $u_n' = u_n - u\,,\, \forall\, n \in \mathrm{N}$ and let's we show $\varphi(u_n') \underset{n\to\infty}{\longrightarrow} c - \varphi(u)$.

For this, because of

$$\varphi(u_n) = \varphi(u_n') + \varphi(u) + < u_n', u > - \int_{\mathrm{R}} V(t, u_n' + u) - V(t, u_n') - V(t, u)\,dt\,,$$

it is enough to prove that

$$\int_{\mathrm{R}} V(t, u_n' + u) - V(t, u_n') - V(t, u)\,dt \underset{n\to\infty}{\longrightarrow} 0\,.$$

Let $\epsilon > 0$ and $R_\epsilon > 0$ such that $\|u\|_{\{|t|>R_\epsilon\}} < \epsilon$ and $\int_{\{|t|>R_\epsilon\}} V(t,u) < \epsilon$.
By the mean value theorem, $\forall\,|t| > R_\epsilon$ there exists $\theta(t) \in (0,1)$ such that

$$V(t, u_n' + u) - V(t, u_n') = \nabla V(t, u_n' + \theta u)u \quad .$$

The hypothesis $i)$ implies that $\forall\,\epsilon > 0$ there exists $\delta > 0$ such that $|\nabla V(t,z)| \leq \epsilon|z|\,,\, \forall\,|z| < \delta\,,\, \forall\, t \in \mathrm{R}$. On the other hand if $M > \delta$ and $\delta \leq |z| \leq M$ then

$$|\nabla V(t,z)| \leq \sup_{t\in\mathrm{R}\,,\,|\xi|\leq M} |\nabla V(t,\xi)|\frac{|z|}{\delta} = \frac{C_M}{\delta}|z|\,.$$

In general we can say that $\forall\, M > 0\,,\, \exists C(M) > 0$ such that

$$|\nabla V(t,z) \leq C(M)|z| \quad \forall\,|z| < M\,,$$

in fact we can choose $C(M) = \frac{C_M}{\delta} + \epsilon$.

It is clear that this is a direct consequence of the hypothesis $iii)$ but we want to show the lemma without use it.

Let's choose M such that $|u_n'|_\infty + |u|_\infty \leq M$ then

$$\begin{aligned}\int_{|t|>R_\epsilon} |V(t, u_n' + u) - V(t, u_n') - V(t,u)|\,dt &\leq \int_{|t|>R_\epsilon} |\nabla V(t, u_n' + \theta u)u|\,dt + \epsilon \\ &\leq (\int_{|t|>R_\epsilon} |\nabla V(t, u_n' + \theta u)|^2\,dt)^{\frac{1}{2}}\epsilon + \epsilon \leq \\ &\leq C(M)(\|u_n'\|^2 + \|u\|^2)^{\frac{1}{2}}\epsilon + \epsilon \leq c_4\epsilon\end{aligned}$$

Since ϵ is arbitrary we can conclude

$$\int_{|t|>R_\epsilon} |V(t,u_n'+u) - V(t,u_n') - V(t,u)|\, dt \underset{n\to\infty}{\longrightarrow} 0$$

as we claimed.

q.e.d.

Lemma 1.4. $\{u_n\}_{n\in\mathrm{N}} \subset X$, $u_n \overset{w-X}{\longrightarrow} u$, $\varphi'(u_n) \underset{n\to\infty}{\longrightarrow} 0 \Rightarrow \varphi'(u_n - u) \underset{n\to\infty}{\longrightarrow} 0$

Proof. For the fact that for all $h \in X$ we have

$$\varphi'(u_n - u)h = \varphi'(u_n)h - \int_{\mathrm{R}} \nabla(V(t,u_n-u) - V(t,u_n) - V(t,u))h\, dt\,,$$

to prove the lemma it is sufficient to show that

$$\sup_{\|h\|=1} |\int_{\mathrm{R}} \nabla(V(t,u_n-u) - V(t,u_n) - V(t,u))h\, dt| \underset{n\to\infty}{\longrightarrow} 0\,.$$

Given $\epsilon > 0$ we fix $R_\epsilon > 0$ such that $\|u\|_{\{|t|>R_\epsilon\}} < \epsilon$ and $\int_{|t|>R_\epsilon} |\nabla V(t,u)|^2\, dt < \epsilon^2$. Using the dominated convergence theorem we easily get

$$|\int_{\mathrm{R}} \nabla(V(t,u_n-u) - V(t,u_n) - V(t,u))h\, dt| \le o(1)\|h\| +$$
$$+ |\int_{|t|>R_\epsilon} \nabla(V(t,u_n-u) - V(t,u_n) - V(t,u))h\, dt|\,.$$

On the other hand, because of the choice of R_ϵ, we have

$$\int_{|t|>R_\epsilon} |\nabla V(t,u)h|\, dt \le \epsilon\|h\|$$

and because of *iii)*, if $C > |u_n|_\infty + |u|_\infty$, $\forall\, n \in \mathrm{N}$, we get

$$|\int_{|t|>R_\epsilon} \nabla(V(t,u_n-u) - V(t,u_n))h\, dt| \le M_C \int_{|t|>R_\epsilon} |u||h|\, dt \le M_C\epsilon\|h\|\,.$$

Since ϵ is arbitrary the lemma is proved.

q.e.d.

Lemma 1.5. <u>If</u> $\{u_n\}_{n\in\mathrm{N}} \subset X$, $u_n \overset{w-X}{\longrightarrow} 0$, $\varphi'(u_n) \underset{n\to\infty}{\longrightarrow} 0$ <u>then</u>,

$$\forall\, \{\underline{u}_n\}_{n\in\mathrm{N}} \subset \{u_n\}_{n\in\mathrm{N}} \,/\, \underline{u}_n \overset{L^\infty_{loc}(\mathrm{R},\mathrm{R}^m)}{\longrightarrow} 0$$

it verifies either

$$a)\ \underline{u}_n \underset{n\to\infty}{\longrightarrow} 0 \quad in\ X, \qquad \underline{or}$$
$$b)\ \|\underline{u}_n\|_{|t|<T} \underset{n\to\infty}{\longrightarrow} 0\ ,\ \forall T>0 \quad \underline{and} \quad \exists \eta>0\ |t_n| \underset{n\to\infty}{\longrightarrow} \infty \,/\, |\underline{u}_n(t_n)| \geq \eta\,,\ \forall n\in \mathrm{N}\,.$$

Proof. In fact it holds either that $\underline{u}_n \underset{n\to\infty}{\longrightarrow} 0$ in $L^\infty(\mathrm{R},\mathrm{R}^m)$ or $\exists \eta>0\,,\ \{t_n\}_{n\in\mathrm{N}} \subset \mathrm{R} \,/\, |\underline{u}_n(t_n)| \geq \eta$.

In the first case, because of the weakly convergence of u_n, we have that $\exists C>0 \,/\, \|\underline{u}_n\| \leq C\,,\ \forall n\in \mathrm{N}$. Moreover, because of the $L^\infty(\mathrm{R},\mathrm{R}^m)$ convergence and the hypothesis $i)$, we have also

$$\forall\,\epsilon>0\ \exists m_\epsilon \in \mathrm{N} \,/\, n>m_\epsilon \ \Rightarrow\ \int_{\mathrm{R}} \nabla V(t,\underline{u}_n)\underline{u}_n\,dt \leq \epsilon\|\underline{u}_n\|^2 \leq C^2\epsilon\,.$$

Finally, since $\underline{u}_n$ is bounded we get $\varphi'(\underline{u}_n)\underline{u}_n \underset{n\to\infty}{\longrightarrow} 0$ so

$$\|\underline{u}_n\|^2 = \varphi'(\underline{u}_n)\underline{u}_n + \int_{\mathrm{R}} \nabla V(t,\underline{u}_n)\underline{u}_n \leq o(1) + C^2\epsilon\,.$$

Since ϵ is arbitrary the first alternative is proved.

Let's consider the second case.

For $T>0$ let's define $g_T:\mathrm{R}\to\mathrm{R}^+$ in such a way that $g_T\in C^\infty(\mathrm{R},\mathrm{R})$ and

$$g_T(t) = \begin{cases} 1 & \text{if } |t|\leq T \\ 0 & \text{if } |t|>2T\,. \end{cases}$$

By the dominated convergence theorem we get

$$\int_{\mathrm{R}} \nabla V(t,\underline{u}_n) g_T \underline{u}_n\,dt = \int_{|t|\leq 2T} \nabla V(t,\underline{u}_n) g_T \underline{u}_n\,dt \underset{n\to\infty}{\longrightarrow} 0\,.$$

Moreover

$$|\varphi'(\underline{u}_n) g_T \underline{u}_n| \leq |\varphi'(\underline{u}_n)|\|g_T\underline{u}_n\| \leq c_5|\varphi'(\underline{u}_n)| \underset{n\to\infty}{\longrightarrow} 0$$

so we have

$$< \underline{u}_n, g_T\underline{u}_n >= \varphi'(\underline{u}_n) g_T \underline{u}_n + \int_{\mathrm{R}} \nabla V(t,\underline{u}_n) g_T \underline{u}_n\,dt \underset{n\to\infty}{\longrightarrow} 0\,.$$

Since

$$\begin{aligned} \|\underline{u}_n\|^2_{B(0,T)} &=< \underline{u}_n, g_T\underline{u}_n > - < \underline{u}_n, g_T\underline{u}_n >_{\{|t|>T\}} = \\ &= o(1) - \int_{|t|>T} \dot{g}_T \dot{\underline{u}}_n \underline{u}_n\,dt - \int_{|t|>T} g_T(\dot{\underline{u}}_n^{\,2} + \underline{u}_n^2)\,dt \end{aligned}$$

it is sufficient to show that

$$\int_{|t|>T} \dot{g}_T \dot{\underline{u}}_n \underline{u}_n\,dt = o(1)\,.$$

This is true because of

$$
\begin{aligned}
|\int_{|t|>T} \dot g_T \underline{\dot u}_n \underline{u}_n \,dt| &= |\int_T^{2T} \dot g_T \underline{\dot u}_n \underline{u}_n \,dt + \int_{-2T}^{-T} \dot g_T \underline{\dot u}_n \underline{u}_n \,dt| \leq \\
&\leq \frac{1}{2} \max_{\mathrm{R}} |\dot g_T(t)| |\int_T^{2T} \frac{d}{dt} \underline{u}_n^2 \,dt + \int_{-2T}^{-T} \frac{d}{dt} \underline{u}_n^2 \,dt| \leq \\
&\leq 2 \max_{\mathrm{R}} |\dot g_T(t)| |\underline{u}_n|_{L^\infty((-2T,2T),\mathrm{R}^m)} = o(1)\,.
\end{aligned}
$$

q.e.d.

Lemma 1.6. <u>let</u> $K = \{u \in X \setminus \{0\} \,/\, \varphi'(u) = 0\}$. <u>Then</u> $\Lambda = \inf_K \varphi(u) > 0$.

Proof. First we note that because of $i)$ we have $\underline{c} = \inf_K \|u\| > 0$.
Moreover it holds : $u \in K \;\Rightarrow\; \|u\| \leq (\frac{2\mu\varphi(u)}{\mu-2})^{\frac{1}{2}}$.
In fact

$$
\begin{aligned}
\varphi(u) = \varphi(u) - \frac{1}{2}\varphi'(u)u &= \int_{\mathrm{R}} \frac{1}{2}\nabla V(t,u)u - V(t,u)\,dt \geq \\
\geq (\frac{1}{2} - \frac{1}{\mu}) \int_{\mathrm{R}} \nabla V(t,u)u &= (\frac{1}{2} - \frac{1}{\mu})\|u\|^2 \quad \forall\, u \in K\,.
\end{aligned}
$$

The two observation are in evident contradiction if $\inf_K \varphi(u) \leq 0$.

q.e.d.

We now give some hypotheses *'at infinity'*.

Let $V_-\,,\, V_+ \in C^1(\mathrm{R}\times\mathrm{R}^m,\mathrm{R})$ and let them verify the hypotheses $i)\,,\, ii)\,,\, iii)$ with constants $\mu_i\,,\, M_R^{(i)}\,,\; i = 1,2$ respectively.

Let moreover

$$
\begin{aligned}
iv)\quad & |\nabla V(t,x) - \nabla V_+(t,x)| \underset{t\to+\infty}{\longrightarrow} 0 \quad , \\
& |\nabla V(t,x) - \nabla V_-(t,x)| \underset{t\to-\infty}{\longrightarrow} 0 \quad , \textit{ uniformly on the compacts of } \mathrm{R}^m
\end{aligned}
$$

and finally

$$
v)\quad \exists T_-\,,\, T_+ \geq 0 \,/\, V_i(t+T_i,x) = V_i(t,x)\ \forall\,(t,x) \in \mathrm{R}\times\mathrm{R}^m\,,\ i = +\,,\,-\,.
$$

Obviously $iv),\, v)$ imply $o)$ and so from now on this hypothesis is superfluous.

Let also $L_-(t)\,,\, L_+(t)$ symmetric $m \times m$ matrices, positive definite uniformly for $t \in \mathrm{R}$, with continuous and periodic coefficients of minimal period $T_-\,,\, T_+ \geq 0$ respectively. Moreover we impose that

$$
\begin{aligned}
|L_{i,j}(t) - L_{-,i,j}(t)| \underset{t\to-\infty}{\longrightarrow} 0 \quad i\,,\, j = 1,\ldots,m \\
|L_{i,j}(t) - L_{+,i,j}(t)| \underset{t\to+\infty}{\longrightarrow} 0 \quad i\,,\, j = 1,\ldots,m\,.
\end{aligned}
$$

Let's define

$$\|u\|_+ = \Big(\int_{\mathbb{R}} |\dot u|^2 + L_+(t)u\cdot u\,dt\Big)^{\frac12}$$
$$\|u\|_- = \Big(\int_{\mathbb{R}} |\dot u|^2 + L_-(t)u\cdot u\,dt\Big)^{\frac12}$$
$$\varphi_+(u) = \frac12\|u\|_+^2 - \int_{\mathbb{R}} V_+(t,u)\,dt$$
$$\varphi_-(u) = \frac12\|u\|_-^2 - \int_{\mathbb{R}} V_-(t,u)\,dt \quad u\in X\,.$$

It's clear that $\|\cdot\|_-$, $\|\cdot\|_+$ are norms on X equivalent to $\|\cdot\|$ and also that φ_+ , $\varphi_- \in C^1(X,\mathbb{R})$.

Finally if we set K_+, K_- equal to the sets of non zero critical points of φ_+ , φ_- respectively, we have that $\Lambda_+ = \inf_{K_+}\varphi_+(u) > 0$ and $\Lambda_- = \inf_{K_-}\varphi_-(u) > 0$.

Let's fix $R>0$ and define the function η_R^+ , $\eta_R^- \in C^\infty(\mathbb{R},\mathbb{R})$ such that

$$\eta_R^+(t) = \begin{cases} 1 & t>2R \\ 0 & t<R \end{cases} \ ,\ \eta_R^+(s)\le\eta_R^+(t) \quad if\ s\le t\ ,\ \eta_R^-(t) = \eta_R^+(-t) \quad \forall\, t\in\mathbb{R}\,.$$

Lemma 1.7. *Let* $\{u_n\}_{n\in\mathbb{N}}\subset X$ *such that* $u_n \overset{w-X}{\longrightarrow} u$, $\varphi(u_n)\underset{n\to\infty}{\longrightarrow} c$, $\varphi'(u_n)\underset{n\to\infty}{\longrightarrow} 0$.
Then

$$\varphi(u_n-u) - \varphi_+(\eta_R^+(u_n-u)) - \varphi_-(\eta_R^-(u_n-u)) \underset{n\to\infty}{\longrightarrow} 0$$
$$\varphi_+'(\eta_R^+(u_n-u))\,,\ \varphi_-'(\eta_R^-(u_n-u)) \underset{n\to\infty}{\longrightarrow} 0$$

along a subsequence

Proof. First we observe

$$\begin{aligned}\varphi(u_n-u) &- \varphi_+(\eta_R^+(u_n-u)) - \varphi_-(\eta_R^-(u_n-u)) = o(1) + \\ &+ \frac12\|u_n-u\|^2_{\{|t|>2R\}} - \frac12\|\eta_R^+(u_n-u)\|^2_{+,\{|t|>2R\}} - \frac12\|\eta_R^-(u_n-u)\|^2_{-,\{|t|>2R\}} + \\ &- \int_{t>2R} V(t,u_n-u) - V_+(t,u_n-u)\,dt - \int_{t<-2R} V(t,u_n-u) - V_-(t,u_n-u)\,dt\end{aligned}$$

where

$$\begin{aligned}o(1) &= \frac12\|u_n-u\|^2_{\{|t|\le2R\}} - \frac12\|\eta_R^+(u_n-u)\|^2_{+,\{|t|\le2R\}} - \frac12\|\eta_R^-(u_n-u)\|^2_{-,\{|t|\le2R\}} + \\ &- \int_{|t|<2R} V(t,u_n-u) - V_+(t,\eta_R^+(u_n-u)) - V_-(t,\eta_R^-(u_n-u))\,dt\,.\end{aligned}$$

To obtain the first part of the lemma we prove that along a subsequence

$$\|u_n-u\|^2_{\{|t|>2R\}} - \|u_n-u\|^2_{+,\{t>2R\}} - \|u_n-u\|^2_{-,\{t<-2R\}} \underset{n\to\infty}{\longrightarrow} 0\,,$$
$$\int_{t>2R} V(t,u_n-u) - V_+(t,u_n-u)\,dt + \int_{t<-2R} V(t,u_n-u) - V_-(t,u_n-u)\,dt \underset{n\to\infty}{\longrightarrow} 0\,.$$

By lemma 1.5 we know that given a real sequence $t_j \underset{j\to\infty}{\longrightarrow} +\infty$ there exists $\{u_{n_j}\}_{j\in\mathbb{N}} \subset \{u_n\}_{n\in\mathbb{N}}$ such that

$$\|u_{n_j} - u\|_{\{|t|\le t_j\}} \underset{j\to\infty}{\longrightarrow} 0 .$$

In fact $\forall\, j > 0$, $\|u_n - u\|_{\{|t|\le t_j\}} \underset{n\to\infty}{\longrightarrow} 0$ so we can choose u_{n_j} such that for example $\|u_{n_j} - u\|_{\{|t|\le t_j\}} < \frac{1}{j}$.

We call again such subsequence $\{u_n\}_{n\in\mathbb{N}}$ and now it has the property that

$$\|u_n - u\|_{\{|t|\le t_n\}} \underset{n\to\infty}{\longrightarrow} 0 \; with \; t_n \underset{n\to\infty}{\longrightarrow} +\infty .$$

For the equivalence of the defined norms we have also $\|u_n - u\|_{i,\{|t|\le t_n\}} \underset{n\to\infty}{\longrightarrow} 0$, $i = +, -$.

We prove first that

$$\|u_n - u\|^2_{\{\, t>2R\}} - \|u_n - u\|^2_{+,\{t>2R\}} \underset{n\to\infty}{\longrightarrow} 0 .$$

In fact

$$\begin{aligned} \big|\|u_n - u\|^2_{\{t>2R\}} - &\|u_n - u\|^2_{+,\{t>2R\}}\big| \le \\ &\le o(1) + \int_{t|>t_n} |L(t) - L_+(t)||u_n - u|^2 \, dt \le \\ &\le o(1) + \sup_{t>t_n} |L(t) - L_+(t)| \int_{t|>t_n} |u_n - u|^2 \, dt \underset{n\to\infty}{\longrightarrow} 0 . \end{aligned}$$

In the same way we prove also

$$\|u_n - u\|^2_{\{t<-2R\}} - \|u_n - u\|^2_{-,\{t<-2R\}} \underset{n\to\infty}{\longrightarrow} 0 .$$

We prove now that

$$\int_{t>2R} V(t, u_n - u) - V_+(t, u_n - u) \, dt \underset{n\to\infty}{\longrightarrow} 0 .$$

Because of the mean value theorem $\forall\, t \in \mathbb{R}\ \exists \theta(t) \in (0, 1)$ such that

$$V(t, u_n - u) - V_+(t, u_n - u) = \nabla(V(t, \theta(u_n - u)) - V_+(t, \theta(u_n - u)))(u_n - u) .$$

If $M \ge |u_n|_\infty + |u|_\infty$, $\forall\, n \in \mathbb{N}$, we have that for all $\epsilon > 0$

$$\begin{aligned} \int_{t>2R} &V(t, u_n - u) - V_+(t, u_n - u) = \\ &= o(1) + \int_{t>t_n} \nabla(V(t, \theta(u_n - u)) - V_+(t, \theta(u_n - u)))(u_n - u) \, dt \le \\ &\le o(1) + (\sup_{t>t_n} \sup_{|\xi|\le M} |\nabla V(t, \xi) - \nabla V_+(t, \xi)| \frac{1}{\delta} + \epsilon)\|u_n - u\|^2 \le \\ &\le o(1) + \epsilon \sup_{n\in\mathbb{N}} \|u_n - u\|^2 . \end{aligned}$$

Since ϵ is arbitrary we obtain as we claimed $\int_{t>2R} V(t, u_n - u) - V_+(t, u_n - u) \, dt \underset{n\to\infty}{\longrightarrow} 0$.

In the same way we also obtain $\int_{t<-2R} V(t,u_n-u) - V_-(t,u_n-u)\,dt \underset{n\to\infty}{\longrightarrow} 0$.

For the second part we first claim that

$$\varphi'(\eta_R^+(u_n-u)),\ \varphi'(\eta_R^-(u_n-u)) \underset{n\to\infty}{\longrightarrow} 0 .$$

In fact

$$\begin{aligned}|\varphi'(\eta_R^+(u_n-u))| &= \sup_{\|h\|=1} |\varphi'(\eta_R^+(u_n-u))h| \leq \\ &\leq \sup_{\|h\|=1} |\varphi'(\eta_R^+(u_n-u))(\eta_R^+h)| + \sup_{\|h\|=1} \varphi'(\eta_R^+(u_n-u))(1\eta_R^+)h| ,\end{aligned}$$

but we have

$$\begin{aligned}|\varphi'(\eta_R^+(u_n-u))(1-\eta_R^+)h| &= | < \eta_R^+(u_n-u),(1-\eta_R^+)h > + \\ &- \int_R^{2R} \nabla V(t,\eta_R^+(u_n-u))(1-\eta_R^+)h\,dt| \leq \\ &\leq c_6\|u_n-u\|_{(R,2R)}\|h\| + (\int_R^{2R} |\nabla V(t,\eta_R^+(u_n-u))|^2)^{\frac{1}{2}}\|h\| = \\ &= o(1)\|h\|\end{aligned}$$

therefore

$$\begin{aligned}|\varphi'(\eta_R^+(u_n-u))| &\leq o(1) + \sup_{\|h\|=1} |\varphi'(\eta_R^+(u_n-u))(\eta_R^+h)| = \\ &= o(1) + \sup_{\|h\|=1} |\varphi'(u_n-u)(\eta_R^+h)| \leq \\ &\leq o(1) + c_7|\varphi'(u_n-u)| \underset{n\to\infty}{\longrightarrow} 0 .\end{aligned}$$

In the same way we get $|\varphi'(\eta_R^-(u_n-u))| \underset{n\to\infty}{\longrightarrow} 0$.

To finish the proof we utilize again lemma 1.5 in its well known consequence that if $t_j \underset{j\to\infty}{\longrightarrow} \infty$ then there exists a subsequence, which we call again $\{u_n\}_{n\in\mathbb{N}}$, such that $\|u_n-u\|_{(-t_n,t_n)} \underset{n\to\infty}{\longrightarrow} 0$ and we use again the fact that, because of the hypotesys $i)$, we have $\forall\,\epsilon>0\ \exists\delta>0$ such that $\forall\, z\in B(0,M)$

$$|\nabla V(t,z) - \nabla V_+(t,z)| \leq (\frac{1}{\delta}\sup_{|\xi|\leq M} |\nabla V(t,\xi) - \nabla V_+(t,\xi)| + \epsilon)|z|$$

and the analogous proposition with V_-.

In fact if $M \geq |u_n|_\infty + |u|_\infty$, $\forall\, n\in\mathbb{N}$ then

$$\begin{aligned}&|(\varphi'(\eta_R^+(u_n-u)) - \varphi'_+(\eta_R^+(u_n-u)))h| = \\ &= | \int_{t>R} (L(t)-L_+(t))\eta_R^+(u_n-u)h\,dt + \\ &\qquad - \int_{t>R} (\nabla V(t,\eta_R^+(u_n-u)) - \nabla V_+(t,\eta_R^+(u_n-u)))h\,dt| \leq \\ &\leq o(1)\|h\| + c_8 \sup_{t>t_n} |L(t)-L_+(t)|\|h\| + \\ &\qquad + (\frac{1}{\delta}\sup_{t>t_n}\sup_{\xi\in B(0,M)} |\nabla V(t,\xi) - \nabla V_+(t,\xi)| + \epsilon)\|h\| = \\ &= (o(1)+\epsilon)\|h\| .\end{aligned}$$

Since ϵ is arbitrary we get the claim. The proof for φ'_- is perfectly symmetric.

q.e.d.

We give now a proposition which is proved in [CZR]. It say that such *'tails at infinity'* generate critical points if the functional is invariant under translation.
We prove the proposition for φ_+ but the proof is identical in tha case of φ_-.

Proposition 1.8 *Let* $\{u_n\}_{n\in\mathbb{N}} \subset X$ *such that*

$$u_n \overset{w-X}{\longrightarrow} 0\,,\ \varphi_+(u_n) \underset{n\to\infty}{\longrightarrow} c > 0\,,\ \varphi'_+(u_n) \underset{n\to\infty}{\longrightarrow} 0$$

then

$$\exists k \in \mathbb{N}\cup\{0\}\,,\ \{\{t_n^i\}_{n\in\mathbb{N}} \subset \mathbb{R}\,/\, i = 1,\ldots,k\}\,,\ \{v_i \in K_+\,/\, i = 1,\ldots,k\}$$

such that

$$|t_n^j| \underset{n\to\infty}{\longrightarrow} +\infty\ ,\ j = 1\,,k\,,$$

$$|t_n^i - t_n^j| \underset{n\to\infty}{\longrightarrow} +\infty \quad i \neq j\,,$$

$$\|u_n - \sum_{i=1}^{k} v_i(\cdot - t_n^i)\| \underset{n\to\infty}{\longrightarrow} 0\,,$$

$$c = \sum_{i=1}^{k} \varphi_+(v_i)\,.$$

Proof. Since $\varphi_+(u_n) \underset{n\to\infty}{\longrightarrow} c > 0$ it can not be that $u_n \underset{n\to\infty}{\longrightarrow} 0$ so the alternative $b)$ of lemma 1.5 holds and there exists an $\alpha > 0$ and a sequence $\{s_n^1\}_{n\in\mathbb{N}}$ such that $|u_n(s_n^1)| \geq \alpha$. Without loss of generality we can suppose that $|u_n(s_n^1)| = \max_{\mathbb{R}}|u_n|$ and that $|u_n(t)| < |u_n(s_n^1)|\,, \forall t < s_n^1$.

If $T_+ = 0$ let's consider the sequence $t_n^1 = s_n^1$ and $u_n^1(\cdot) = u_n(\cdot + t_n^1)$.
If $T_+ > 0$ let's consider the sequence $t_n^1 = [\frac{s_n^1}{T_+}]T_+$ and $u_n^1(\cdot) = u_n(\cdot + t_n^1)$ where $[x]$ indicates the entire part of x for $x \in \mathbb{R}$.

It is clear that

$$|u_n^1|_{L^\infty((0,T_+),\mathbb{R}^m)} \geq \alpha\,,\ \varphi_+(u_n^1) \underset{n\to\infty}{\longrightarrow} c\,,\ \varphi'_+(u_n^1) \underset{n\to\infty}{\longrightarrow} 0\,.$$

From this we derive using the previous lemmata that $\exists v_1 \in X$ such that

$$u_n^1 \overset{w-X}{\longrightarrow} u\,,\ |v_1|_{L^\infty((0,T_+),\mathbb{R}^m)} \geq \alpha\,,\ \varphi_+(u_n^1 - v_1) \underset{n\to\infty}{\longrightarrow} c - \varphi_+(v_1)\,,\ \varphi'_+(v_1) = 0\,.$$

The possible cases are now: 1) $c = \varphi_+(v_1)$ *or* 2) $c > \varphi_+(v_1)$.

In the case 1) because of $\varphi'_+(u_n^1 - v_1) \underset{n\to\infty}{\longrightarrow} 0$, $\varphi_+(u_n^1 - v_1) \underset{n\to\infty}{\longrightarrow} 0$ it is necessarily that $u_n^1 \underset{n\to\infty}{\longrightarrow} 0$ in fact $(\frac{1}{2} - \frac{1}{\mu_1})\|u_n^1 - v_1\|^2 \le |\varphi_+(u_n^1 - u_1)| + \frac{1}{\mu_1}|\varphi'_+(u_n^1 - v_1)(u_n^1 - v_1)|$. Therefore, in the case 1), we can conclude that

$$|t_n^1| \underset{n\to\infty}{\longrightarrow} +\infty\,,$$
$$\|u_n - v_1(\cdot - t_n^1)\| \underset{n\to\infty}{\longrightarrow} 0\,,$$
$$c = \varphi_+(v_1)\,.$$

In the case 2) we have $\varphi'_+(u_n^1 - v_1) \underset{n\to\infty}{\longrightarrow} 0$, $\varphi_+(u_n^1 - v_1) \underset{n\to\infty}{\longrightarrow} c - \varphi_+(v_1) > 0$, so for the sequence $\{u_n^1 - v_1\}_{n\in\mathrm{N}}$ we are in the original situation.

We are again in the case $b)$ of lemma 1.5 so there exists a sequence $\{s_n^2\}_{n\in\mathrm{N}}$ such that $|s_n^2| \underset{n\to\infty}{\longrightarrow} +\infty$ and there exists $\alpha_1 > 0$ such that $|(u_n^1 - v_1)(s_n^2)| \ge \alpha_1$.

Again, without loss of generality, we can suppose that $|(u_n^1 - v_1)(s_n^2)| = \max_{\mathrm{R}} |u_n^1 - v^1|$ and that $|(u_n^1 - v_1)(t)| < |(u_n^1 - v_1)(s_n^2)|\,,\, \forall\, t < s_n^2$.

Let's consider the sequence $u_n^2(\cdot) = (u_n^1 - v_1)(\cdot + p_n^2)$ where $p_n^2 = [\frac{s_n^2}{T_+}]T_+$, if $T_+ > 0$, $p_n^2 = s_n^2$ if $T_+ = 0$. We have again

$$|u_n^2|_{L^\infty((0,T_+),\mathrm{R}^m)} \ge \alpha_1\,,\ \varphi_+(u_n^2) \underset{n\to\infty}{\longrightarrow} c - \varphi_+(v_1)\,,\ \varphi'_+(u_n^2) \underset{n\to\infty}{\longrightarrow} 0\,.$$

From this we derive that $\exists v_2 \in X$ such that

$$u_n^2 \overset{w-X}{\longrightarrow} v_2\,,\ |v_2|_{L^\infty((0,T_+),\mathrm{R}^m)} \ge \alpha\,,\ \varphi_+(u_n^2 - v_2) \underset{n\to\infty}{\longrightarrow} c - \varphi_+(v_1) - \varphi_+(v_2)\,,\ \varphi'_+(v_2) = 0\,.$$

The possible cases are now: 1) $c = \varphi_+(v_1) + \varphi_+(v_2)$ *or* 2) $c > \varphi_+(v_1) + \varphi_+(v_2)$.

In case 1) if we put $t_n^2 = t_n^1 + p_n^2$ we have that

$$|t_n^2| \underset{n\to\infty}{\longrightarrow} +\infty\,,$$
$$|t_n^2 - t_n^1| \underset{n\to\infty}{\longrightarrow} +\infty\,,$$
$$\|u_n - v_1(\cdot - t_n^1) - v_2(\cdot - t_n^2)\| \underset{n\to\infty}{\longrightarrow} 0\,,$$
$$c = \varphi_+(v_1) + \varphi_+(v_2)\,.$$

Note that $|p_n^2| \underset{n\to\infty}{\longrightarrow} +\infty$ because $u_n^2(\cdot - p_n^2) \overset{w-X}{\longrightarrow} 0$ while $u_n^2 \underset{n\to\infty}{\longrightarrow} v_2 \in K$.

In the second case continuing in this fashion in at most $[\frac{c}{\Lambda_+}]$ of steps the proposition follows.

q.e.d.

We can give now the main result of this chapter.

Proposition 1.9 *Let* $\{u_n\}_{n\in\mathrm{N}} \subset X$ *such that*

$$\varphi(u_n) \underset{n\to\infty}{\longrightarrow} c > 0 \quad , \quad \varphi'(u_n) \underset{n\to\infty}{\longrightarrow} 0$$

then

$$\exists k_- \,,\, k_+ \in \mathrm{N} \cup \{0\}\,,$$
$$\exists \{\{t^i_{n,+}\}_{n\in\mathrm{N}} \subset \mathrm{R} \,/\, i = 1, \ldots, k_+\}\,,\, \{\{t^i_{n,-}\}_{n\in\mathrm{N}} \subset \mathrm{R} \,/\, i, \ldots, k_-\}\,,$$
$$\exists u \in K \cup \{0\}\,,\, \{v^+_i \in K_+ \,/\, i = 1, \ldots, k_+\}\,,\, \{v^-_i \in K_- \,/\, i = 1, \ldots, k_-\}$$

such that

$$t^1_{n,+} \underset{n\to\infty}{\longrightarrow} +\infty \quad t^1_{n,-} \underset{n\to\infty}{\longrightarrow} -\infty\,,$$
$$t^j_{n,+} - t^{j-1}_{n,+} \underset{n\to\infty}{\longrightarrow} +\infty\,,\, j = 2, \ldots, k_+ \quad , \quad t^j_{n,-} - t^{j-1}_{n,-} \underset{n\to\infty}{\longrightarrow} -\infty\,,\, j = 2, \ldots, k_-\,,$$
$$u_n \overset{w-X}{\longrightarrow} u\,,$$
$$\|u_n - u - \sum_{i=1}^{k_+} v^+_i(\cdot - t^i_{n,+}) - \sum_{i=1}^{k_-} v^-_i(\cdot - t^i_{n,-})\| \underset{n\to\infty}{\longrightarrow} 0\,,$$
$$c = \varphi(u) + \sum_{i=1}^{k_+} \varphi_+(v^+_i) + \sum_{i=1}^{k_-} \varphi_-(v^-_i)$$

for a subsequence.

Proof. By lemma 1.2 we have that along a subsequence $u_n \overset{w-X}{\longrightarrow} u$.
By lemma 1.5 we have also that $\|u_n - u - \eta^+_R(u_n - u) - \eta^-_R(u_n - u)\| \underset{n\to\infty}{\longrightarrow} 0$.
From now on we do not distinguish subsequences.
By lemma 1.7 the sequences $\{\eta^+_R(u_n - u)\}_{n\in\mathrm{N}}$, $\{\eta^-_R(u_n - u)\}_{n\in\mathrm{N}}$ are Palais Smale sequences for φ_+ , φ_- respectively and also $\varphi_+(\eta^+_R(u_n - u)) + \varphi_-(\eta^-_R(u_n - u)) \underset{n\to\infty}{\longrightarrow} c - \varphi(u)$.
By proposition 1.8 there exist

$$k_- \,,\, k_+ \in \mathrm{N} \cup \{0\}\,,$$
$$\exists \{\{t^i_{n,+}\}_{n\in\mathrm{N}} \subset \mathrm{R} \,/\, i = 1, \ldots, k_+\}\,,\, \{\{t^i_{n,-}\}_{n\in\mathrm{N}} \subset \mathrm{R} \,/\, i, \ldots, k_-\}\,,$$
$$\exists \{v^+_i \in K_+ \,/\, i = 1, \ldots, k_+\}\,,\, \{v^-_i \in K_- \,/\, i = 1, \ldots, k_-\}$$

such that

$$|t^i_{n,+}| \underset{n\to\infty}{\longrightarrow} +\infty \quad i = 1, \ldots, k_+\,,$$
$$|t^i_{n,-}| \underset{n\to\infty}{\longrightarrow} -\infty \quad i = 1, \ldots, k_-\,,$$
$$|t^j_{n,+} - t^i_{n,+}| \underset{n\to\infty}{\longrightarrow} +\infty\,,\, i \neq j \quad , \quad |t^j_{n,-} - t^i_{n,-}| \underset{n\to\infty}{\longrightarrow} -\infty\,,\, i \neq j\,,$$
$$\|\eta^+_R(u_n - u) - \sum_{i=1}^{k_+} v^+_i(\cdot - t^i_{n,+})\| \underset{n\to\infty}{\longrightarrow} 0\,,$$
$$\|\eta^-_R(u_n - u) - \sum_{i=1}^{k_-} v^-_i(\cdot - t^i_{n,-})\| \underset{n\to\infty}{\longrightarrow} 0\,,$$
$$c - \varphi(u) = \sum_{i=1}^{k_+} \varphi_+(v^+_i) + \sum_{i=1}^{k_-} \varphi_-(v^-_i)\,.$$

If we look at the proof of proposition 1.8 we see that the sequences $\{t_n^j\}_{n\in\mathbb{N}}$ can be choosen in this case ordinately. Infact we know that $|\eta_R^+(u_n - u)|_{L^\infty((-\infty,2R),\mathbb{R}^m)} \underset{n\to\infty}{\longrightarrow} 0$ so we can take $\{s_n^1\}_{n\in\mathbb{N}}$ as the sequence of the smallest relative maximum point of $\eta_R^+(u_n - u)$ such that there exists an $\alpha > 0$ for which $\eta_R^+(u_n - u)(s_n^1) > \alpha\,,\,\forall\, n \in \mathbb{N}$. If we choose also the sequences $\{s_n^j\}_{n\in\mathbb{N}}$ in the same way we obtain that

$$t_{n,+}^1 \underset{n\to\infty}{\longrightarrow} +\infty\,,$$
$$t_{n,+}^j - t_{n,+}^{j-1} \underset{n\to\infty}{\longrightarrow} +\infty\,,\; j = 2,\ldots,k_+\,.$$

In a symmetric fashion we obtain also

$$t_{n,-}^1 \underset{n\to\infty}{\longrightarrow} -\infty\,,$$
$$t_{n,-}^j - t_{n,-}^{j-1} \underset{n\to\infty}{\longrightarrow} -\infty\,,\; j = 2,\ldots,k_-\,.$$

The proposition is completely proved.

q.e.d.

§2. An existence result.

2.1 *Some notation.*

Let K , K_- , K_+ be the sets of the non zero critical points of φ, φ_-, φ_+ respectively and let $\Lambda = \inf_K \|u\|$, $\Lambda_- = \inf_{K_-} \|u\|$, $\Lambda_+ = \inf_{K_+} \|u\|$. Let $K^c = K \cap \varphi^c$, $K^c_- = K_- \cap \varphi^c_-$, $K^c_+ = K_+ \cap \varphi^c_+$.

Given $A \in X$, $l \in \mathbb{N} \cup \{0\}$ we set $A^l = \{ (x_1, \ldots, x_l) \,/\, x_j \in A\,,\, j = 1, \ldots, l\}$ for $l \geq 1$, $A^0 \equiv \emptyset$. We set also $\mu(A) = \inf\{ \|x - y\| \,/\, x \neq y \in A\}$.

We define moreover:

$$\mathcal{P}_l = \{ \{(p_n^1, \ldots, p_n^l)\}_{n\in\mathbb{N}} \subset \mathbb{R}^l \,/\, (\frac{p_n^j}{T_+} \in \mathbb{N}\,, \forall n \in \mathbb{N}\,, j = 1, \ldots, l)\,, \\ (p_n^j - p_n^{j-1} \underset{n\to\infty}{\longrightarrow} +\infty\,, j = 2, \ldots, l)\,, (p_n^1 \underset{n\to\infty}{\longrightarrow} +\infty)\}$$

$$\mathcal{M}_l = \{ \{(\frac{p_n^1}{T_+} T_-, \ldots, \frac{p_n^l}{T_+} T_-)\}_{n\in\mathbb{N}} \subset \mathbb{R}^l \,/\, \{(p_n^1, \ldots, p_n^l)\}_{n\in\mathbb{N}} \in \mathcal{P}_l \}$$

$$\mathcal{S}_l = \{ \{(s_n^1, \ldots, s_n^l)\}_{n\in\mathbb{N}} \subset \mathbb{R}^l \,/\, |s_n^j| \underset{n\to\infty}{\longrightarrow} \infty\,, j = 1, \ldots, l\} \quad .$$

If $S = (s_1, \ldots, s_l) \in \mathbb{R}^l$ and $x \in X^l$ we set $S * x = \sum_{j=1}^l x_i(\cdot - s_j)$.

If $s \in \mathbb{R}$ we define $(s)^{\otimes l} = (s, \ldots, s) \in \mathbb{R}^l$.

If $k \in \mathbb{N}$ and $S^j = (s_1^j, \ldots, s_{l_j}^j) \in \mathbb{R}^{l_j}$, $j = 1, \ldots, k$ then we put

$$(S^1, \ldots, S^k) = (s_1^1, \ldots, s_{l_1}^1, \ldots, s_1^k, \ldots, s_{l_k}^k) \,.$$

If $Y_j = (y_1^j, \ldots, y_{l_j}^j) \in X^{l_j}$, $j = 1, \ldots, k$ let

$$(Y_1, \ldots, Y_k) = (y_1^1, \ldots, y_{l_1}^1, \ldots, y_1^k, \ldots, y_{l_k}^k)$$

It is immediate to see that:

$$(\{S_n^j\}_{n\in\mathbb{N}} \in \mathcal{S}_{l_j}\,,\ j = 1, \ldots, k) \Rightarrow (\{(S_n^1, \ldots, S_n^k)\}_{n\in\mathbb{N}} \in \mathcal{S}_{\sum_{j=1}^k l_j})$$

$$(\{p_n\}_{n\in\mathbb{N}} \in \mathcal{P}_1\,,\ \{M_n\}_{n\in\mathbb{N}} \in \mathcal{M}_l) \Rightarrow (\{M_n - (p_n)^{\otimes l}\}_{n\in\mathbb{N}} \in \mathcal{M}_l)$$

$$(\{t_n\}_{n\in\mathbb{N}} \in \mathcal{M}_1\,,\ \{P_n\}_{n\in\mathbb{N}} \in \mathcal{P}_l) \Rightarrow (\{P_n - (t_n)^{\otimes l}\}_{n\in\mathbb{N}} \in \mathcal{P}_l)$$

Finally we define, for $c \in \mathbb{R}^+$ and $l\,, m \in \mathbb{N} \cup \{0\}$, the set:

$$PS^c(l, m) = \{K^c \cup \{0\}\} \times \{(K_+^c)^l\} \times \{(K_-^c)^m\} \times \mathcal{P}_l \times \mathcal{M}_m$$

2.2 *Tecnical results.*

We first give a simple but usefull lemma.

Lemma 2.1. *Given* $l \in \mathrm{N}$ *then*

$$\forall x \in X \ , \ \forall Y = (y_1, \ldots, y_l) \in X^l \ , \ \forall \{S_n\}_{n\in\mathrm{N}} \in \mathcal{S}_l$$

we have that:

$$\forall \epsilon > 0 \ , \ \exists \overline{n} \in \mathrm{N} \ / \ n > \overline{n} \ \Rightarrow \ \|x + S_n * Y\| \geq \|x\| + \|S_n * Y\| - \epsilon$$

Proof. Let $R > 0$ such that $\sup\{\|x\|_{|t|>R}\, , \, \sup_{j=1,\ldots,l} \|y_j\|_{|t|>R}\} \leq \frac{\epsilon}{c(l)}$ where $c(l)$ is a costant which we will fix in the following.
Clearly it holds that

$$\begin{aligned}\|x + S_n * Y\| &= \|x + S_n * Y\|_{|t|<R} + \|x + S_n * Y\|_{|t|>R} \geq \\ &\geq \|x\|_{|t|<R} - \|S_n * Y\|_{|t|<R} + \|S_n * Y\|_{|t|>R} - \|x\|_{|t|>R} = \\ &= \|x\| + \|S_n * Y\| - 2(\|x\|_{|t|>R} + \|S_n * Y\|_{|t|<R}) \quad .\end{aligned}$$

If $\{S_n\}_{n\in\mathrm{N}} = \{(s_n^1, \ldots, s_n^l)\}_{n\in\mathrm{N}}$, then

$$\|S_n * Y\|_{|t|<R} \leq \sum_{j=1}^{l} \|y_j(\cdot - s_n^j)\|_{|t|<R} = \sum_{j=1}^{l} \|y_j\|_{-R+s_n^j<t<R+s_n^j}$$

and if we choose $\overline{n}$ such that $n > \overline{n} \ \Rightarrow \ \inf_{j=1,\ldots,l} |s_n^j| \ > \ 2R$ we get:

$$\|x + S_n * Y\| \geq \|x\| + \|S_n * Y\| - \frac{2(l+1)}{c(l)}\epsilon \quad .$$

We fix $c(l) = 2(l+1)$ and the lemma is proved.

q.e.d.

Now we make two different hypotheses to get two deformation theorems. We will analize in the following their principal consequences which will be basic in the proof of the above mentionated theorems.

(*) : $\exists c^* > 0$ *such that* $\min\{\mu(K^{c^*})\, , \, \mu(K_-^{c^*})\, , \, \mu(K_+^{c^*})\} > 0 \quad .$

(**): $\exists c^* > 0$ *such that* K^{c^*} , $K_-^{c^*}$, $K_+^{c^*}$ *are constituted by isolated points.*

It is clear that $(*) \Rightarrow (**)$.

Let $l\, , m\, , \overline{l}\, , \overline{m} \in \mathrm{N} \cup \{0\}$; then:

Lemma 2.2. *Let us assume hypothesis (**).*

Given $(u,U,V,\{P_n\}_{n\in\mathrm{N}},\{M_n\}_{n\in\mathrm{N}})\in PS^{c^{**}}(l,m)$ *then* $\exists\tilde r=\tilde r(u,U,V)>0$ *such that if*

$$(\overline{u},\overline{U},\overline{V},\{\overline{P}_n\}_{n\in\mathrm{N}},\{\overline{M}_n\}_{n\in\mathrm{N}})\in PS^{c^{**}}(\bar l,\overline{m})$$

verifies

$$\|u+P_n*U+M_n*V-\overline{u}-\overline{P}_n*\overline{U}-\overline{M}_n\overline{V}\|\le\tilde r+\epsilon_n\quad\forall n\in\mathrm{N}$$

with $\epsilon_n\underset{n\to\infty}{\longrightarrow}0$, *then we have*

$$l=\bar l\quad,\quad m=\overline{m}\quad,\quad u=\overline{u}\quad,\quad U=\overline{U}\quad,\quad V=\overline{V}$$

and

$$\exists n_0\in\mathrm{N}\,/\,(\;n>n_0\quad\Rightarrow\quad P_n=\overline{P}_n\quad M_n=\overline{M}_n\;)\quad.$$

Proof. Let $U=(u_1,\ldots,u_l)$, $V=(v_1,\ldots,v_m)$ and let

$$\tilde r=\frac{1}{2}\min_{i=1,\ldots,l\,,\,j=1,\ldots,m}\{\Lambda\,,\,\Lambda_+\,,\,\Lambda_-\,,\,d(u,K^{c^{**}}\setminus\{u\})\,,\,d(u_i,K_+^{c^{**}}\setminus\{u_i\})\,,\,d(v_j,K_-^{c^{**}}\setminus\{v_j\})\}.$$

Clearly if (**) is true then we have $\tilde r>0$.

We set also $P_n=(p_n^1,\ldots,p_n^l)$, $\overline{P}_n=(\overline{p}_n^1,\ldots,\overline{p}_n^{\bar l})\quad\forall n\in\mathrm{N}$.

We prove first that $u=\overline{u}$.

Let $\{S_n\}_{n\in\mathrm{N}}=\{P_n,\overline{P}_n,M_n,\overline{M}_n\}_{n\in\mathrm{N}}$ and $Y=(U,\overline{U},V,\overline{V})$.
It is clear that $\{S_n\}_{n\in\mathrm{N}}\in\mathcal{S}_{l+\bar l+m+\overline{m}}$ and using lemma 2.1 we obtain that there exists a real sequence $\epsilon'_n\underset{n\to\infty}{\longrightarrow}0$ such that

$$\tilde r+\epsilon_n\ge\|u-\overline{u}+S_n*Y\|\ge\|u-\overline{u}\|+\|S_n*Y\|-\epsilon'_n\quad.$$

Then we can conclude that $\|u-\overline{u}\|<\tilde r$ which implies as we claimed $u=\overline{u}$.

If $l=\bar l=m=\overline{m}$ there is nothing to prove. Let's suppose $l>0$.
We prove now that $\quad(l>0\quad\Rightarrow\quad\bar l>0)$.

If $\bar l=0$ then we set $Y=(u_2,\ldots,u_l,V,-\overline{V})$ and

$$\{S_n\}_{n\in\mathrm{N}}=\{(p_n^2-p_n^1,\ldots,p_n^l-p_n^1,M_n-(p_n^1)^{\otimes m},\overline{M}_n-(p_n^1)^{\otimes\overline{m}}\}_{n\in\mathrm{N}}\;.$$

It is clear that $\{S_n\}_{n\in\mathrm{N}}\in\mathcal{S}_{l-1+m+\overline{m}}$, so, by lemma 2.1, there exists a real sequence $\epsilon'\underset{n\to\infty}{\longrightarrow}0$ such that

$$\tilde r+\epsilon_n\ge\|u_1+\sum_{j=2}^{l}u_j(\cdot-p_n^j+p_n^1)+(M_n-(p_n^1)^{\otimes m})*V-(\overline{M}_n-(p_n^1)^{\otimes\overline{m}})*\overline{V}\|=$$
$$=\|u_1+S_n*Y\|\ge\|u_1\|+\|S_n*Y\|-\epsilon'\ge\Lambda_+-\epsilon'\quad\forall n\in\mathrm{N}$$

which is in contrast with the hypothesis $\tilde{r} < \Lambda_+$.

We claim now that $\{p_n^1 - \overline{p}_n^1\}_{n\in\mathrm{N}}$ is bounded.

Suppose on the contrary that there exists a subsequence, which we call again $\{p_n^1 - \overline{p}_n^1\}_{n\in\mathrm{N}}$, such that for example $\overline{p}_n^1 - p_n^1 \underset{n\to\infty}{\longrightarrow} +\infty$.
In this case we define $Y = (u_2, \ldots, u_l, -\overline{U}, V, -\overline{V})$ and

$$\{S_n\}_{n\in\mathrm{N}} = \{p_n^2 - p_n^1, \ldots, p_n^l - p_n^1, \overline{P}_n - (p_n^1)^{\otimes\bar{l}}, M_n - (p_n^1)^{\otimes m}, \overline{M}_n - (p_n^1)^{\otimes\overline{m}}\}_{n\in\mathrm{N}} \ .$$

We check easily that $\{S_n\}_{n\in\mathrm{N}} \in \mathcal{S}_{l+\bar{l}+m+\overline{m}-1}$ and using again lemma 2.1 we obtain that there exists a real sequence $\epsilon' \underset{n\to\infty}{\longrightarrow} 0$ such that

$$\tilde{r} + \epsilon_n \geq \|u_1 + S_n * Y\| \geq \|u_1\| + \|S_n * Y\| - \epsilon' \geq \Lambda_+ - \epsilon' \quad \forall n \in \mathrm{N}$$

which is again an absurd.
With the same reasoning we obtain an absurd if we suppose that $\overline{p}_n^1 - p_n^1 \underset{n\to\infty}{\longrightarrow} -\infty$ and therefore our claim is proved.

Let's show now that $u_1 = \overline{u}_1$ and that if n is sufficiently large then $p_n^1 = \overline{p}_n^1$.

We set $Y = (u_2, \ldots, u_l, -\overline{u}_2, \ldots, -\overline{u}_{\bar{l}}, V, -\overline{V})$ and

$$\{S_n\}_{n\in\mathrm{N}} = \{p_n^2 - p_n^1, \ldots, p_n^l - p_n^1, \overline{p}_n^2 - p_n^1, \ldots, \overline{p}_n^{\bar{l}} - p_n^1, M_n - (p_n^1)^{\otimes m}, \overline{M}_n - (p_n^1)^{\otimes\overline{m}}\}_{n\in\mathrm{N}}.$$

We check again easily that $\{S_n\}_{n\in\mathrm{N}} \in \mathcal{S}_{l+\bar{l}+m+\overline{m}-2}$ and, for the boundness of the sequence $\{p_n^1 - \overline{p}_n^1\}_{n\in\mathrm{N}}$, we obtain as in the lemma 2.1 that there exists a real sequence $\epsilon_n' \underset{n\to\infty}{\longrightarrow} 0$ such that

$$\tilde{r} + \epsilon_n \geq \|u_1 - (p_n^1 - \overline{p}_n^1) * \overline{u}_1 + S_n * Y\| \geq \|u_1 - (p_n^1 - \overline{p}_n^1) * \overline{u}_1\| + \|S_n * Y\| - \epsilon_n' \quad .$$

This implies that if n is sufficiently large then $d(u_1, K_+^{c^*} \setminus \{u_1\}) > \|u_1 - (p_n^1 - \overline{p}_n^1) * \overline{u}_1\|$. This is possible only if $u_1 = (p_n^1 - \overline{p}_n^1) * \overline{u}_1$ for all such n and this permits us to conclude that $u_1 = \overline{u}$ and $p_n^1 = \overline{p}_n^1$ for all such n, in fact if it is not then it must be $u_1 = \overline{u}_1 = 0$ which is in contrast with the position $l > 0$.

Continuing in this way we obtain that $u_j = \overline{u}_j$ for $j = 1, \ldots, l$. Then we exclude the case $\bar{l} > l$ exactly in the same way we have proved that $l > 0 \to \bar{l} > 0$ so $l = \bar{l}$, $U = \overline{U}$ and there exists $n_+ \in \mathrm{N}$ such that $n > n_+$ implies $P_n = \overline{P}_n$.

It is obvious that the same algoritm can be used to prove now that $m = \overline{m}$, $V = \overline{V}$ and there exists $n_- \in \mathrm{N}$ such that $n > n_-$ implies $M_n = \overline{M}_n$.

We set finally $n_0 = \max n_+, n_-$ and the lemma is proved.

q.e.d.

We are going now to give two simple corollary of lemma 2.2 which we will use in the proofs of the deformation theorems.

Lemma 2.3. *Let's assume hypothesis (**). Then:*
given $(u, U, V, \{P_n\}_{n\in N}, \{M_n\}_{n\in N}) \in PS^{c^*}(l, m)$ *if we set* $y_n = u + P_n * U + M_n * V$ *we have that*

$$\forall r < \tilde{r}\ \exists \delta_r > 0\ \exists n_0 \in N \,/\, (n > n_0\,,\ q \in (B_r(y_n) \setminus B_{r/2}(y_n)) \cap \varphi^{c^*}) \ \Rightarrow\ |\varphi'(q)| \geq \delta_r$$

where $\tilde{r} = \tilde{r}(u, U, V)$ *is as in lemma 2.2.*

Proof. Suppose by absurd that

$$\exists \{\overline{y}_n\}_{n\in N} \subset \varphi^{c^*} \,/\, \frac{r}{2} \leq ||\overline{y}_n - y_n|| \leq r \text{ and } \varphi'(\overline{y}_n) \underset{n\to\infty}{\longrightarrow} 0$$

.

We claim first that $\{\overline{y}_n\}_{n\in N}$ is a Palais Smale sequence.
In fact we know that $\varphi(\overline{y}_n) \leq c^*$. Moreover $||y_n|| \leq ||u|| + \sum_{j=1}^{l} ||u_j|| + \sum_{j=1}^{m} ||v_j|| = c_1$ (where we have indicated with u_j, v_i the components of U, V respectively) therefore we have $||\overline{y}_n|| \leq c_1 + r$ and for this $\varphi'(\overline{y}_n)\overline{y}_n \underset{n\to\infty}{\longrightarrow} 0$. We have also

$$\varphi(\overline{y}_n) = \frac{1}{2}\varphi'(\overline{y}_n) + \frac{1}{2}\int_R \nabla V(t, \overline{y}_n)\overline{y}_n = o(1) + \frac{1}{2}\int_R \nabla V(t, \overline{y}_n)\overline{y}_n\ .$$

We can conclude that for all but a finite number of $n \in N$ we have $|\varphi(\overline{y}_n)| \leq c^*$ as we claimed.

For the characterization of the Palais Smale sequences given in the previous chapter, we have that $\exists \overline{l}\,, \overline{m} \in N \cup \{0\}$ and $(\overline{u}, \overline{U}, \overline{V}, \{\overline{P}_n\}_{n\in N}, \{\overline{M}_n\}_{n\in N}) \in PS^{c^*}(\overline{l}, \overline{m})$ such that

$$||\overline{y}_n - \overline{u} - \overline{P}_n * \overline{U} - \overline{M}_n * \overline{V}|| \underset{n\to\infty}{\longrightarrow} 0 \quad .$$

Therefore there exists a positive real sequence $\epsilon_n \underset{n\to\infty}{\longrightarrow} 0$ such that

$$\frac{r}{2} - \epsilon_n \leq ||u - \overline{u} + P_n * U - \overline{P}_n * \overline{U} + M_n * V - \overline{M}_n * \overline{V}|| \leq r + \epsilon_n \quad \forall n \in N\,.$$

But $r < \tilde{r}$, so by lemma 2.2 we have that $\exists n_0 \in N$ such that $(n > n_0 \Rightarrow ||u - \overline{u} + P_n * U - \overline{P}_n * \overline{U} + M_n * V - \overline{M}_n * \overline{V}|| = 0)$ which is contrary to $||u - \overline{u} + P_n * U - \overline{P}_n * \overline{U} + M_n * V - \overline{M}_n * \overline{V}|| \geq \frac{r}{2} - \epsilon_n$.
q.e.d.

If we assume hypothesis (*) then if we set

$$r_0 = \frac{1}{2}\min\{(K^{c^*})\,, \mu(K_-^{c^*})\,, \mu(K_+^{c^*})\,, \Lambda\,, \Lambda_-\,, \Lambda_+\}$$

then $r_0 > 0$ and $r_0 \leq \tilde{r}(u,U,V) \quad \forall\ (u,U,V) \in K^{c^*} \times K_+^{c^*} \times K_-^{c^*}$. This observation and lemma 2.2 give us the following lemma.

Lemma 2.4. *Let's assume hypothesis (*). Then:*

$$\forall r < r_0\ \exists \delta_r^K > 0 \,/\, q \in (N_r(K^{c^*}) \setminus N_{r/2}(K^{c^*})) \cap \varphi^{c^*} \ \Rightarrow\ |\varphi'(q)| \geq \delta_r^K \quad .$$

Proof. If it does not hold then $\exists \{q_n\}_{n\in\mathrm{N}} \subset \varphi^{c^*}$, $\exists \{y_n\}_{n\in\mathrm{N}} \subset \varphi^{c^*}$ such that

$$\frac{r}{2} \leq \|q_n - y_n\| \leq r \quad and \quad \varphi'(q_n) \underset{n\to\infty}{\longrightarrow} 0 \quad .$$

As above it is easy to see that $\{q_n\}_{n\in\mathrm{N}}$, $\{y_n\}_{n\in\mathrm{N}}$ are Palais Smale sequences therefore there exists a positive real sequence $\epsilon_n \underset{n\to\infty}{\longrightarrow} 0$ and

$$(u,U,V,\{P_n\}_{n\in\mathrm{N}},\{M_n\}_{n\in\mathrm{N}}) \in PS^{c^*}(l,m)\,,\,(\overline{u},\overline{U},\overline{V},\{\overline{P}_n\}_{n\in\mathrm{N}},\{\overline{M}_n\}_{n\in\mathrm{N}}) \in PS^{c^*}(\overline{l},\overline{m})$$

such that

$$\frac{r}{2} - \epsilon_n \leq \|u - \overline{u} + P_n * U - \overline{P}_n * \overline{U} + M_n * V - \overline{M}_n * \overline{V}\| \leq r + \epsilon_n \quad \forall n \in \mathrm{N}$$

which is, as we have seen above, an absurd.

q.e.d.

2.3 Deformation Theorems

Lemma 2.5. *Let's assume that (**) holds. If moreover*

$$\exists b \in (0,c^*) \quad / \quad K_b^{c^*} = \emptyset$$

then $\forall\, c \in (b,c)\,,\, \forall \epsilon \in (0,\min\{c^*-c, c-b\})$ *there exists* $\eta \in C([0,1]\times X, X)$ *such that*

a) $\eta(0,x) = x \quad \forall x \in X$
b) $\eta(s,x) = x \quad \forall s \in [0,1] \quad \underline{if} \quad x \notin \varphi_{c-\overline{\epsilon}}^{c+\overline{\epsilon}}$
c) $\varphi(\eta(s_1,x)) \geq \varphi(\eta(s_2,x)) \quad \forall x \in X \quad \underline{if} \quad 0 \leq s_1 \leq s_2 \leq 1$
d) $\eta(1,\varphi^{c+\epsilon}) \subset \varphi^{c-\epsilon}$

Proof. We fix $\tilde{\epsilon} > \epsilon$ such that $\tilde{\epsilon} \in (0,\min\{c^*-c, c-b\})$ and we define

$$f(x) = \frac{d(x,\varphi^{c-\tilde{\epsilon}} \cup \varphi_{c+\tilde{\epsilon}})}{d(x,\varphi^{c-\tilde{\epsilon}} \cup \varphi_{c+\tilde{\epsilon}}) + d(x,\varphi_{c-\epsilon}^{c+\epsilon})}.$$

Let $\mathcal{V}$ be a locally Lipschitz continuous function on $X \setminus K$ such that

$$\mathcal{V}1) \quad \|\mathcal{V}(x)\| \leq \frac{4\tilde{\epsilon}}{|\varphi'(x)|}$$
$$\mathcal{V}2) \quad \varphi'(x)\mathcal{V}(x) \geq 2\tilde{\epsilon} \qquad \forall x \in X \setminus K \quad .$$

The function η will be determined as the solution of an ordinary differential equation.

Let's set $W(x) = -f(x)\mathcal{V}(x)$ for $x \in X \setminus K$ and $W(x) = 0$ for $x \in K$. It is easy to see that $W : X \to X$ is a locally Lipschitz continuous function. Let $\eta(\cdot, x)$ be the local solution of the Cauchy problem

$$\frac{d\eta}{ds} = W(\eta), \qquad \eta(0, x) = x\,.$$

Clearly η satisfies *a), b)* and for the fact that

$$\frac{d}{ds}\varphi(\eta(s,x)) = -f(\eta(s,x))\varphi'(\eta(s,x))\mathcal{V}(\eta(s,x)) \leq -2\tilde{\epsilon}f(\eta(s,x)) \leq 0$$

we can conclude that also *c)* holds.

We claim now that the maximal existing time for $\eta(\cdot, x)$ is $+\infty$ for all $x \in X$.

This is obvious if $x \in \varphi^{c-\tilde{\epsilon}} \cap \varphi^{c+\tilde{\epsilon}}$ in fact in that case $W(x) = 0$ so $\eta(s,x) = x \ \forall\, s \geq 0$.
Let $x \in \varphi_{c+\tilde{\epsilon}}^{c+\tilde{\epsilon}}$ and suppose that the maximal existing time of $\eta(\cdot, x)$, which we will call s_x, is finite. In that case there exists a positive real sequence $s_n \nearrow s_x$ such that $\|W(\eta(s_n, x))\| \underset{n\to\infty}{\longrightarrow} +\infty$. We set $u_n = \eta(s_n, x)$ and we observe that $\varphi'(u_n) \underset{n\to\infty}{\longrightarrow} 0$ because of $\mathcal{V}1)$.
For the uniqueness of the solution of the Cauchy problem, we have also that $\varphi(u_n) \in \varphi_{c-\tilde{\epsilon}}^{c+\tilde{\epsilon}}$, $\forall\, n \in \mathrm{N}$.
Therefore we can suppose $\varphi(u_n) \underset{n\to\infty}{\longrightarrow} \overline{c} \in (c - \tilde{\epsilon}, c + \tilde{\epsilon})$ (in fact we can consider a subsequence).
For the caracterization of the Palais Smale sequences given in the previous chapter we have that there exists l, $m \in \mathrm{N} \cap \{0\}$ and $(u, U, V, \{P_n\}_{n\in\mathrm{N}}, \{M_n\}_{n\in\mathrm{N}}) \in PS^{c^*}(l,m)$ such that (along a subsequence)

$$\|u_n - u - P_n * U - M_n * V\| \underset{n\to\infty}{\longrightarrow} 0 \quad .$$

If $l = m = 0$ then $\|u_n - u\| \underset{n\to\infty}{\longrightarrow} 0$ but this is not possible because of $d(K, \varphi_{c-\tilde{\epsilon}}^{c+\tilde{\epsilon}}) > 0$. Therefore if we put $y_n = u + P_n * U + M_n * V$, we can assume that $y_n \neq y_{n-1}$, $\forall n \in \mathrm{N}$.
We prove now that $\liminf_{n\to\infty} \|y_n - y_{n-1}\| > 0$ which implies that $\inf_{n\in\mathrm{N}} \|y_n - y_{n-1}\| = \rho > 0$.
For this, suppose that along a subsequence we have $\lim_{n\to\infty} \|y_n - y_{n-1}\| = 0$. In that case we have that there exists a positive real sequence $\epsilon_n \underset{n\to\infty}{\longrightarrow} 0$ such that for any fixed $r < \tilde{r}(u, U, V)$ we have $r + \epsilon_n \geq \|y_n - y_{n-1}\|$, $\forall\, n \in \mathrm{N}$ and therefore using lemma 2.2 we have that there exists n_0 such that $n > n_0 \Rightarrow y_n = y_{n-1}$ which is contrary to our assumption $y_n \neq y_{n-1}$.
Let now $r < \min\{\tilde{r}(u, U, V), \frac{\rho}{3}\}$, $r > 0$. Lemma 2.3 gives us that

$$\exists \delta_r > 0\,,\ \exists n_0 \in \mathrm{N}\,/\,(\ n > n_0\,,\ \eta(s,x) \in B_r(y_n) \setminus B_{r/2}(y_n) \ \Rightarrow\ |\varphi'(\eta(s,x))| \geq \delta_r\ ).$$

We know also that $\|\eta(s_n, x) - y_n\| \underset{n\to\infty}{\longrightarrow} 0$ so there exists $\overline{n} \in \mathrm{N}$ such that if $n > \overline{n}$ then $\|\eta(s_n, x) - y_n\| < r/2$. Therefore, for the continuity of $\eta(\cdot, x)$ we must have that $\forall n > \overline{n}$, $\exists s_n^1, s_n^2 \in \mathrm{R}$

such that $s_n < s_n^1 < s_n^2 < s_{n+1}$, $\eta(s_n^1, x) \in \partial B_{r/2}(y_n)$, $\eta(s_n^2, x) \in \partial B_r(y_n)$ and finally $\eta(s,x) \in B_r(y_n) \setminus B_{r/2}(y_n)\ \forall\, s \in (s_n^1, s_n^2)$. Therefore if $n > \max\{n_0, \overline{n}\}$ we have

$$\frac{r}{2} \le \|\eta(s_n^1, x) - \eta(s_n^2, x)\| = \| \int_{s_n^1}^{s_n^2} \frac{d}{ds}\eta(s,x)\, ds\| =$$
$$= \| \int_{s_n^1}^{s_n^2} f(\eta(s,x))\mathcal{V}(\eta(s,x))\, ds\| \le \frac{4\tilde{\epsilon}}{\delta_r} \int_{s_n^1}^{s_n^2} f(\eta(s,x))\, ds \quad .$$

For all fixed $k \in \mathrm{N}$ let $(s, \tilde{s}) \,/\, (s_n^1, s_n^2) \in (s, \tilde{s})$, $j = n+1, \ldots, n+k$ where $n > \max\{n_0, \overline{n}\}$. Then

$$2\tilde{\epsilon} \ge |\varphi(\eta(s,x)) - \varphi(\eta(\tilde{s},x))| = | \int_s^{\tilde{s}} \frac{d}{ds}\varphi(\eta(s,x))\, ds| =$$
$$= | \int_s^{\tilde{s}} f(\eta(s,x))\varphi'(\eta(s,x))\mathcal{V}(\eta(s,x))\, ds| \ge$$
$$\ge | \sum_{j=1}^{k} \int_{s_{n+j}^1}^{s_{n+j}^2} f(\eta(s,x))\varphi'(\eta(s,x))\mathcal{V}(\eta(s,x))\, ds| \ge$$
$$\ge 2\tilde{\epsilon} \sum_{j=1}^{k} \int_{s_{n+j}^1}^{s_{n+j}^2} f(\eta(s,x))\, ds \quad .$$

This last inequality together with the previous one implies that $1 \ge k\frac{\delta_r r}{8\epsilon} \quad \forall\, k \in \mathrm{N}$ which is evidently an absurd.

We have just proved that $s_x = +\infty\ \forall\, x \in X$.

We claim finally that $\varphi(\eta(1,x)) \le c - \epsilon \quad \forall\, x \in \varphi^{c+\epsilon}$.

This is a simple consequence of the previous proved claim. In fact if there exists $x \in \varphi_{c-\epsilon}^{c+\epsilon}$ such that $\varphi(\eta(s,x)) > c - \epsilon$, $\forall s \in [0,1]$ then

$$2\epsilon \ge |\varphi(x) - \varphi(\eta(1,x))| = | \int_0^1 \frac{d}{ds}\varphi(\eta(s,x))\, ds| =$$
$$= | \int_0^1 \varphi'(\eta(s,x))\mathcal{V}(\eta(s,x))\, ds| \ge 2\tilde{\epsilon}$$

for the $\mathcal{V}2$).
This is obviously in contrast with the fact that $\epsilon < \tilde{\epsilon}$.

q.e.d.

Lemma 2.6. <u>*Let's assume that (*) holds. Then*</u>

$$\forall b \in (0, c^*)\, , \forall \overline{\epsilon} \in (0, c^* - b), \forall r \in (0, \frac{r_0}{3})$$

<u>*there exist*</u>

$$\epsilon \in (0, \overline{\epsilon})\, , \eta \in C([0,1] \times X, X)\, , \sigma \in C(\varphi^{b+\epsilon}, [0,1])$$

such that

$$\begin{array}{ll} a) & \eta(0,x) = x \quad \forall x \in X \\ b) & \eta(s,x) = x \quad \forall s \in [0,1] \ \underline{if} \quad x \notin \varphi_{b-\bar\epsilon}^{b+\bar\epsilon} \\ c) & \varphi(\eta(s_1,x)) \geq \varphi(\eta(s_2,x)) \quad \forall x \in X \ \ \underline{if} \quad 0 \leq s_1 \leq s_2 \leq 1 \\ d) & \eta(1, \varphi^{b+\epsilon} \setminus N_r(K_{b-\bar\epsilon}^{b+\bar\epsilon}) \subset \varphi^{b-\epsilon} \\ e) & \sigma(x) = 0 \ \ \underline{if} \quad x \in \varphi^{b-\epsilon} \setminus N_r(K_{b-\bar\epsilon}^{b+\bar\epsilon}) \quad \underline{and\ moreover} \\ & \varphi(\eta(\sigma(x),x) = b - \epsilon \quad \forall x \in \varphi_{b-\epsilon}^{b+\epsilon} N_r(K_{b-\bar\epsilon}^{b+\bar\epsilon}) \quad . \end{array}$$

Proof. Let's fix $\tilde\epsilon < \bar\epsilon$ and define

$$g(x) = \frac{d(x, N_{\frac{r}{2}}(K_{b-\tilde\epsilon}^{b+\tilde\epsilon}))}{d(x, N_{\frac{r}{2}}(K_{b-\tilde\epsilon}^{b+\tilde\epsilon})) + d(x, X \setminus N_{\frac{3r}{4}}(K_{b-\tilde\epsilon}^{b+\tilde\epsilon}))}$$

and

$$f(x) = \frac{d(x, \varphi^{c-\tilde\epsilon} \cup \varphi_{c+\tilde\epsilon})}{d(x, \varphi^{c-\tilde\epsilon} \cup \varphi_{c+\tilde\epsilon}) + d(x, \varphi_{c-\epsilon}^{c+\epsilon})},$$

where $\epsilon < \tilde\epsilon$ is free for now. As in the obove lemma let $\mathcal{V}$ be a locally Lipschitz continuous function on $X \setminus K$ such that

$$\begin{array}{ll} \mathcal{V}1) & \|\mathcal{V}(x)\| \leq \dfrac{4\tilde\epsilon}{|\varphi'(x)|} \\ \mathcal{V}2) & \varphi'(x)\mathcal{V}(x) \geq 2\tilde\epsilon \qquad \forall x \in X \setminus K \quad . \end{array}$$

The function η will be again determined as the solution of an ordinary differential equation.

Let's set here $W(x) = -f(x)g(x)\mathcal{V}(x)$ for $x \in X \setminus K$ and $W(x) = 0$ for $x \in K$. Again, it is easy to see that $W : X \to X$ is a locally Lipschitz continuous function. Let $\eta(\cdot, x)$ be the local solution of the Cauchy problem

$$\frac{d\eta}{ds} = W(\eta), \qquad \eta(0,x) = x \, .$$

Clearly η satisfies *a), b)* and for the fact that

$$\frac{d}{ds}\varphi(\eta(s,x)) = -f(\eta(s,x))g(\eta(s,x))\varphi'(\eta(s,x))\mathcal{V}(\eta(s,x)) \leq -2\tilde\epsilon f(\eta(s,x))g(\eta(s,x)) \leq 0$$

we can conclude that also *c)* holds.

As in the previous lemma we prove that the maximal existing time for $\eta(\cdot,x)$ is $+\infty$ for all $x \in X$. The proof is the same as the one we have given above and we don't repeat it here.

It remain to show *d), e)*.

Let's prove *d)*.

Let $x \in \varphi^{b+\epsilon} \setminus N_r(K_{b-\bar\epsilon}^{b+\bar\epsilon})$ and let $\varphi(\eta(s,x)) > b-\epsilon$, $\forall\, s \geq 0$. Then $f(\eta(s,x)) = 1$, $\forall\, s \geq 0$ therefore

$$\begin{aligned} 2\epsilon &\geq \varphi(\eta(0,x)) - \varphi(\eta(s,x)) = \\ &= \int_s^0 \frac{d}{d\xi}\varphi(\eta(\xi,x))\,d\xi = \\ &= \int_0^s f(\eta(\xi,x))g(\eta(\xi,x))\varphi'(\eta(\xi,x))\mathcal{V}(\eta(\xi,x))\,d\xi \geq \\ &\geq 2\bar\epsilon \int_0^s g(\eta(\xi,x))\,d\xi \qquad \forall\, s \geq 0\,. \end{aligned}$$

From the above inequality we derive that

$$\int_0^s g(\eta(\xi,x))\,d\xi < 1\,,\ \forall\, s \geq 0$$

which implies $\liminf_{s\to\infty} g(\eta(s,x)) = 0$.

Therefore there exists $s_n \nearrow \infty$ such that $d(\eta(s_n,x), N_{r/2}(K_{b-\bar\epsilon}^{b+\bar\epsilon})) \underset{n\to\infty}{\longrightarrow} 0$. In other words if we set $u_n = \eta(s_n,x)$ then there exists $\{y_n\}_{n\in\mathbb{N}} \subset K_{b-\bar\epsilon}^{b+\bar\epsilon}$ such that $\|u_n - y_n\| \underset{n\to\infty}{\longrightarrow} r/2$.
Let's show that this is not possible.

Since $\|\eta(0,x) - y\| \geq r$, $\forall\, y \in K_{b-\bar\epsilon}^{b+\bar\epsilon}$ and there exists $n \in \mathbb{N}$ such that $\|\eta(s_n,x) - y_n\| < \frac{3}{4}r$ we must have, for the continuity of $\eta(\cdot,x)$, that there exists $s \in [0,s_n)$ and $\theta(x) \in (0, s_n - s)$ such that $\eta(t,x) \in B_r(y_n) \setminus B_{3r/4}(y_n)$, $\forall t \in (s, s+\theta(x))$ and moreover $\eta(s,x) \in \partial B_r(y_n)$, $\eta(s+\theta(x),x) \in \partial B_{3r/4}(y_n)$.
Then using lemma 2.4 and $\mathcal{V}1$) and for the fact that $f(\eta(t,x))g(\eta(t,x)) = 1$, $\forall\, t \in (s, s+\theta(x))$ we obtain

$$\begin{aligned} \frac{r}{4} &\leq \|\eta(s+\theta(x),x) - \eta(s,x)\| = \\ &= \|\int_s^{s+\theta(x)} \frac{d}{d\xi}\eta(\xi,x)\,d\xi\| \leq \\ &\leq \int_s^{s+\theta(s)} \|\mathcal{V}(\eta(\xi,x))\|\,d\xi \leq \\ &\leq \frac{4\bar\epsilon}{\delta_r^K}\theta(x) \end{aligned}$$

and

$$\begin{aligned} 2\epsilon &\geq \varphi(\eta(s,x)) - \varphi(\eta(s+\theta(x),x)) \geq \\ &\geq \int_{s+\theta(x)}^s \frac{d}{d\xi}\varphi(\eta(\xi,x))\,d\xi \geq 2\bar\epsilon \quad . \end{aligned}$$

Therefore we conclude that $\theta(x) \leq \frac{2\epsilon}{2\bar\epsilon} < 1$ and $\frac{\delta_r^K r}{16} \leq \epsilon$.
Choosing $\epsilon < \frac{\delta_r^K r}{16}$ we obtain an absurd as we claimed and we have just proved *d)*.

If $\omega(x)$ is the amount of time it takes η to reach $\partial\varphi^{b-\epsilon}$ the last inequality show that $\omega(x) < 1$. Set $\sigma(x) = \omega(x)$ for $x \in \varphi_{b-\epsilon}^{b+\epsilon} \setminus N_r(K_{b-\bar\epsilon}^{b+\bar\epsilon})$, $\sigma(x) = 0$ for $x \in \varphi^{b-\epsilon}$ and finally, for $x \in \varphi^{b+\epsilon} \cap N_r(K_{b-\bar\epsilon}^{b+\bar\epsilon})$, we take

$$\sigma(x) = \min\{1, \textit{time at which}\ \eta \in \varphi^{b-\epsilon}\}\,.$$

From the continuous dependence of data in O.D.E. we easily derive the continuity of σ and *e)* is proved.

q.e.d.

2.3 *An existence Theorem.*

We observe first that fro the hypothesis *i)* we derive that

$$|V(t,x)| = |\int_0^{|x|} \frac{d}{ds} V(t, s\frac{x}{\|x\|})\, ds = |\int_0^{|x|} \nabla V(t, s\frac{x}{\|x\|}) \frac{x}{\|x\|}\, ds \leq$$
$$\leq \sup_{t\in\mathbb{R}} \sup_{\xi\in B(0,|x|)} |\nabla V(t,\xi)||x| = o(|x|^2) \qquad |x| \to 0\,.$$

From this we derive

$$\varphi(q) = \frac{1}{2}\|q\|^2 + o(\|q\|^2) \qquad \|q\| \to 0\,.$$

Then there exists $\rho > 0$ and $\alpha > 0$ such that $\varphi|_{\partial B(0,\rho)} \geq \alpha$.
The hypothesis *ii)* says moreover that $V(t,x) \geq (\inf_{|\xi|=1} V(t,\xi))|x|^\mu$ and this easily implies that $\forall\, q \in X \setminus \{0\}$ we have that $\varphi(tq) \to_{t\to\infty} -\infty$.
Set $\Gamma = \{\gamma \in C([0,1],X)\,/\,\gamma(0) = 0\,,\, \gamma(1) < 0\}$ and define

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \varphi(\gamma(t)) \quad .$$

It is obvious that $c \geq \alpha$; also, if $\eta \in C(X,X)$ is such that $\eta|_{\varphi^{\alpha-\epsilon}} = id$, for an $\epsilon < \alpha$, then $\eta\circ\gamma \in \Gamma\,, \forall\,\gamma \in \Gamma$.

Then we have

Theorem 2.7 *If* $\exists\, c^* > c$ *such that* $K_-^{c^*}$, $K_+^{c^*}$ *are constituted of isolated points, then*

$$K^{c^*} \neq \emptyset \quad .$$

Proof. In fact if we suppose that $K^{c^*} = \emptyset$ then the hypothesis of lemma 2.5 are verified and therefore we can choose $\epsilon < \min\{\alpha, c^* - c\}$ for which there exists $\eta \in C([0,1]\times X, X)$ such that $\eta(1,x) = x\,, \forall x \in \varphi^{c-\epsilon}$ and $\eta(1,\varphi^{c+\epsilon}) \subset \varphi^{c-\epsilon})$. For the $\eta(1,\cdot)$ invariance of Γ we obtain the absurd $c \leq c - \epsilon$.

q.e.d.

We give as a corollary and as example of applicability of theorem 2.7 a generalization of a theorem given in [RT].

Theorem 2.8. *Let* $L\,, V\,, L_+\,, V_+\,, L_-\,, V_-\,,$ *as above.*

Suppose that

$\cdot)\quad L(t)x\cdot x - V(t,x) \le \min\{L_-(t)x\cdot x - V_-(t,x)\,,\, L_+(t)x\cdot x - V_+(t,x)\}\;,\;\forall\,(t,x)\in \mathbb{R}\times\mathbb{R}^m$

and

$$\begin{aligned} \cdot\cdot)\quad & \exists t_-, t_+ \in \mathbb{R} \quad \textit{such that} \\ & L(t_-)x\cdot x - 2V(t_-,x) < L_-(t_-)x\cdot x - 2V(t_-,x)\,, \\ & L(t_+)x\cdot x - 2V(t_+,x) < L_-(t_+)x\cdot x - 2V(t_+,x) \qquad \forall\, x\in\mathbb{R}^m\,. \end{aligned}$$

Suppose finally that

$$\begin{aligned} \cdots)\quad & s \longrightarrow \frac{1}{s}\nabla V_+(t,sx)x \\ & s \longrightarrow \frac{1}{s}\nabla V_-(t,sx)x \qquad s\in\mathbb{R} \end{aligned}$$

are strictly increasing functions of $s\in(0,+\infty)$ *for all* $t\in\mathbb{R}$ *and* $\forall\, x\in\mathbb{R}^m\setminus\{0\}$.
Then there exists a non trivial solution of (HS).

Proof. We prove infact that $c<\min\{\Lambda_+,\Lambda_-\}$.

Claim 1) $\forall\, u\in K_-$ *we have that* $\max_{s\in(0,+\infty)}\varphi_-(su)=\varphi_-(u)$ *and* $\varphi_-(su)<\varphi_-(u)\ \forall\, s\neq 1$.

In fact if $u\in K_-$ then $\varphi_-'(u)u=0$ therefore $\|u\|^2 \doteq \int_{\mathbb{R}}\nabla V_-(t,u)u\,dt$.
If we define $g(s)=\varphi_-(su)$ we have

$$\begin{aligned} \frac{d}{ds}g(s) &= s\|u\|^2 - \int_{\mathbb{R}}\nabla V_-(t,su)u\,dt = \\ &= s\int_{\mathbb{R}}\nabla V_-(t,u)u - \frac{1}{s}\nabla V_-(t,su)u\,dt\,. \end{aligned}$$

For the hypothesis $\cdots)$ the claim is proved.

Clearly the same proposition is true for φ_+ that is
$\forall\, u\in K_+$ we have that $\max_{s\in(0,+\infty)}\varphi_+(su)=\varphi_+(u)$ and $\varphi_+(su)<\varphi_+(u)\ \forall\, s\neq 1$.

We define c_- , c_+ as the mountain pass levels for φ_- , φ_+ respectively as we have done for c.
Then we have

Claim 2) $c_-=\Lambda_-$, $c_+=\Lambda_+$.

Infact we know from the previous analisys that if $\{u_n\}_{n\in\mathbb{N}}\subset X$ is such that $\varphi_-(u_n)\underset{n\to\infty}{\longrightarrow} c_-$ and $\varphi_-'(u_n)\underset{n\to\infty}{\longrightarrow} 0$ then exists $\{t_n\}_{n\in\mathbb{N}}\in\mathcal{S}_1$ and exists $u\in K_-$ such that $u_n(\cdot+t_n)\underset{n\to\infty}{\longrightarrow} u$ and $\varphi_-(u)\le c_-$. This implies that $c_-\ge\Lambda_-$. By claim 1) we know moreover that $c_-\le\varphi_-(\underline{u})\ \forall\,\underline{u}\in K_-$ therefore $c_-\le\Lambda_-$. We can conclude as we claimed that $c_-=\Lambda_-$.
The same proof for c_+.

Claim 3) $c<\min\{c_-,c_+\}$.

For the hypotesis $\cdot)$ if $u \in K$ and $\varphi_-(u) = c_-$ we have that

$$\varphi(su) \leq \varphi_-(su) \quad \forall s \in (0, +\infty).$$

Hence by claim 1)

$$c \leq \max_{(0,+\infty)} \varphi(su) \leq \varphi_-(u) = c_-.$$

If $c = c_-$ then there is an $\bar{s} \in (0, +\infty)$ such that $\varphi(\bar{s}u) = c_-$. In that case it must be $\bar{s} = 1$ because if it is not then $c_- = \varphi(\bar{s}u) \leq \varphi_-(\bar{s}u) < \varphi(u) = c_-$ which is impossible. Then $\varphi(u) = \varphi_-(u) = c = c_-$. From that we argue that $u(t) = 0$ in a neighborhood of $t = t_-$ for the hypothesys $\cdot\cdot)$ and this implies that also $\dot{u}(t) = 0$ in a neighborhood of $t = t_-$.
Observe now that u is a solution of the linear system of equations

$$\ddot{q}_j + \sum_{i=1}^{m} a_{i,j}(t) q_j = 0 \qquad 1 \leq j \leq m\ ,$$

where $a_{i,j}(t) = 0$ if $u(t) = 0$ and if $u(t) \neq 0$ then

$$a_{i,j}(t) = \frac{1}{|u(t)|^2}\Big(\sum_{l=1}^{m} L_{-,i,l}(t) u_l(t) - \frac{\partial}{\partial x_i} V_-(t,u)\Big) u_j \quad i\,,\, j = 1, \ldots, m\,.$$

The coefficients $a_{i,j}$ are continuous via the hypothesis $i)$. We can therefore conclude that, since $u(t), \dot{u}(t) = 0$ in a neighborhood of $t = t_-$ and u satisfies a linear system with continuous coefficients, $u(t) \equiv 0$ that is in contradiction with $u \in K_-$.
The proof is perfectly symmetric for c_+ and the theorem is completely proved.

q.e.d.

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