SISSA - SCUOLA
INTERNAZIONALE
SUPERIORE
DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

THERMODINAMIC LIMIT AND BOUNDARY EFFECTS IN LONG RANGE SPIN MODELS

A thesis submitted for the Degree of
Magister Philosophiae

Academic Year 1988-89

Candidate:
Fabio BAGARELLO

Supervisor:
Prof.F.STROCCHI

TRIESTE

THERMODYNAMICAL LIMIT AND BOUNDARY EFFECTS IN LONG RANGE SPIN MODELS

Thesis submitted for the
Master degree by
Fabio Bagarello

Advisor
Prof. F. Strocchi

Contents

INTRODUCTION

The main aim of this thesis is to introduce the reader into the world of long range interaction (LRI). We will discuss many of the effects deriving from such interactions and which have no counterparts for potentials with short range.
In particular we will show all the details for spin models which are by far the easiest non trivial physical models we can imagine.
Our work is divided in six chapters:

in Chapter I we discuss some physical motivation that suggests the relevance of (spin models with) LRI in physics. In particular we discuss Fractional Quantum Hall Effect (FQHE) and High Tc Superconductivity;

in Chapter II some more technical insight is given. We discuss symmetry breaking with Goldstone mechanism. Two models with LRI, namely Schwinger model and Stuckelberg-Kibble model, are sketched;

in Chapter III a (almost) self-contained mathematical formulation relevant for LRI is developed. Of course not all the details are showed;

in Chapter IV we start dealing with mean field spin models. In particular the removing of the infrared cutoff in these models is discussed and we show that the breaking of the global symmetries gives rise to massive particles, even if no Higgs mechanism can be advocated;

in Chapter V the same models are treated at a non zero temperature. The framework used for T=0 is extended to T>0 and a phase transition is shown to happen at a critical temperature;

in Chapter VI we end studing a long range Ising model. Again the removing of the infrared cutoff is studied and Goldstone mechanism, symmetry breaking and mass gap are investigated.

CHAPTER I : MOTIVATIONS

I.1 GENERAL PROBLEMS

In its original formulation, Quantum Mechanics (QM) was essentially dealing with systems with a finite number of degrees of freedom. Many of the original problems arising for such systems were in fact explained by QM which moreover, in the past years, has been taken to an high level of rigour. However there are physical systems whose theoretical description requires infinite degrees of freedom. The historical motivations of such an extension come from the problem of combining QM and (special) Relativity. The need of describing interactions (or forces) for relativistic systems no longer as "actions at a distance" but as contact actions, mediated by "fields", necessarily leads to considering system with infinite degrees of freedom. For such systems QM has to be replaced by QM_∞ and by its branches: Quantum Field Theory, Many Body Theory and so on.
Many questions of principle of QM_∞ are still unsolved, expecially from a non-perturbative point of view but the main features and structures of the theory are becoming more and more established. We know that important phenomena like collective phenomena and spontaneous symmetry breaking (SSB), which plays such a crucial role in modern Theoretical Physics, have no counterpart in ordinary QM.
Inside the structure itself of QM in these last years another trend has become more and more appealing: it has become clear since some years that LRI plays a very crucial role in QM , quite different from the one played by short range interactions (SRI); infact many new phenomena appear which are not observed in the SRI case.
A tipical example is given by the SSB mechanism and the related mass gap. It is well known that if the Standard Goldstone Theorem (SGT) holds a broken symmetry, that is the existence of an order parameter, implies the existence of massless particles. This result is however not verified for some relevant model such as BCS, spin models with LRI, and it seems that also the U(1) problem for QCD should be considered in this perspective (U_A (1) has to be broken since no parity doubling is observed and, being our symmetry global, we expect a massless particle which, however, is not found).
The Higgs mechanism is well known as a possible way out to the existence of Goldstone particles at least at a perturbative level.
Our symmetry is now local and the gauge fields introduced in the lagrangian can explain the disappearance of the massless

particles.
However, of course, Higgs mechanism cannot be advocated for global symmetries (BCS for instance) and therefore we are led to think that the main reason for the absence of massless bosons is more substantial than the locality of the symmetry. Infact it lies in the impossibility of applying the SGT since some of its assumptions is not verified anymore. In fact Swieca, [1], showed that one of the main hypothesis of this theorem, namely the possibility of using local charges for describing the action of the symmetry at each time

$$\delta A_t = i \lim_{R,\infty} [Q_R, A_t],$$

is strongly related to the range of the interaction. In general in fact we cannot apply the SGT if we are dealing with LRI. The reason is that, as it will be discussed much more in details in the following, LRI implies the existence of unavoidable delocalization effects in the equations of motion, where therefore also variables localized outside any finite region appear. Of course, such variables commute with any localized operator, in particular with Q_R, so that the equation for δA_t is not well posed anymore.
This is not the only problem due to LRI. Another tipical example is given by the boundary effects in functional integral. This is computed, for instance by means of computers, fixing a volume V and a lattice spacing a. Once the result is obtained one lets the volume go to infinity (thermodynamical limit) and a go to zero. The thermodynamical limit is quite subdle here. If we are dealing with SRI the boundary effects are smaller and smaller when $V\to\infty$.

This is not the case if our interaction has long range: the boundary can affect strongly the final result and in general we cannot neglect such boundary terms.
The thermodynamical limit has to be considered carefully even in the definition of the "dynamics of the system"(see note (1) at the end of the chapter).
The standard method used for finding the dynamics of the system in dealing with QM_∞ is fixing ourself in a finite volume V and to use the cutoffed Hamiltonian H_V instead of H in (1.1) to find α_V^t.
It will be discussed in Chapter II that when we take the thermodynamical limit no problem arises in the SRI case while many points have to be treated more carefully if the interaction is a long range one. In particular we will show that we have to "weaken" in some sense the definition itself of α^t, in such a way that, however, the group nature of α^t, $\alpha^t\alpha^\tau = \alpha^{t+\tau}$, is preserved.
This is not the end of the story. Another critical difference

between SRI and LRI ,also related to the above discussion, is the relevance that the states assume for LRI. This is due to the fact that if we make a physical measure of some localized variable no effect comes out from the boundary if the interaction has short range while for LRI every localized measure is in general affected by boundary effects just as a consequence of the range of the interaction.
The last point I want to mention here is the role that for LRI may take terms related to space delocalization and usually descharged for SRI.
In particular the topological term introduced to explain the U(1) problem in QCD, $\Theta\,\mathrm{Tr}(F\tilde{F})$, and the Chern-Simon action used to obtain charged vortices in 2+1 dimensions, are related to effects of strongly delocalization. The first term can be written, when integrated over d^4x, as an integral on a surface at infinity of a vector which is different from zero since instanton solutions are considered; the second term gives rise to particles with "any" spin, anyons, which therefore cannot be generated by local fields,as showed in [2].
The above discussion shows the relevance of delocalized objects in Physics and therefore an algebraic mathematical framework in which such objects could be treated appears to be more and more useful.

I.2 HALL EFFECT

The main effort of this thesis is to study long range spin models and their peculiarities. The relevance of these models relies on the fact that often they are the discretized form of much more complicated models and allow a relative easy discussion of many crucial points that in the original model can be quite difficult to analyse.
A typical example of such a "trick" is given by the Anderson's approach to BCS model, [3], where some consideration on the wave function and on the spectrum of the original model gives the right hint for translating the problem in terms of spin variables.
In this paragraph we will show, using heuristic arguments, that it seems possible to rewrite the hamiltonian used in [4] for describing the FQHE introducing spin variables. The Hamiltonian that one gets looks very much the same as the one that will be studied in Chapter VI.
Of course since a lot of conditions are imposed during the "spin

translation" there are some points that need to be clarified. However in our opinion the strong analogy between the two hamiltonians urges to better study Hamiltonian (6.1)

Let us suppose to have some semiconductor devise. If we apply a magnetic field B in the z direction an Hall voltage E_y can be measured along y (open circuit direction), when a current j_x is forced along the orthogonal direction x (closed circuit direction).
The Hall resistivity

$$\rho_{xy}=E/j$$

is expected to increase linearly with B while we get σ:

$$\sigma_{xy}=\nu e^2/h , \qquad \sigma_{xx}=\sigma_{yy}=0$$

where $\nu=N/N_0$ is the "filling factor", that is the number of filled Landau levels.
Experimentally it has been seen that ν can take integer values (Integer Quantum Hall Effect) or fractional values : $\nu=1/3,2/3,2/5,3/5,2/7,3/7,\ldots$ (see ref.[5]). The current belief is that this fractional effect is due to electron-electron interaction and that the disorder does not play any crucial role, countrarily to what happens for IQHE in which disorder is at the basis of the process. This belief is mainly due to the finding that increasing disorder tends to wipe out the fractional effect, making integer effect wider and wider.
In the model of FQHE one deals with N electrons confined along z by a strong electrostatic potential, mobile along x,y in a neutralizing background potential $V_B(x,y)$ (We will ignore possible impurity effects). The electrons interact each other via a Coulomb interaction e^2/r (no screaning effect will be considered here) , $r^2=x^2+y^2$, and are immersed in a strong magnetic field parallel to z, plus an electric field E_y. If $\underline{A}$ is the vector potential describing the magnetic field, $\underline{B}=\nabla\wedge\underline{A}$, then we have

$$H=\sum_{i=1}^{N}(1/2m)(\underline{p}_i+e/c\,\underline{A})^2+1/2\sum_{ij}\frac{e^2}{|r_i-r_j|}+\sum_i V_B(\underline{r}_i)-eE_y\sum_i \underline{y}_i \qquad (1.2)$$

Many approximations are done in order to make easier this formidable problem:

- we suppose that V_B arises from an uniform density N/L^2 ;
- B is taken so strong that only the first Landau level can be filled; all higher Landau levels can be dropped from the problem;
- E_y is taken equal to zero.

As it is discussed in ref.[4] the Hamiltonian can be written as

$$H = \frac{\hbar\omega_c}{2}\sum_i c_i^* c_i + 1/2 \sum_{ijkl} \langle ij|\ \frac{e^2}{|x-y|}\ |kl\rangle\ c_i^* c_j^* c_k c_l \qquad (1.3)$$

where second-quantized notation is adopted. The creation operator c_i^* spans the lowest Landau level ($i=1,\ldots,N$). We have $N=\sum_i c_i^* c_i < N_o$. Moreover

$$\{c_i, c_j\} = \{c_i^*, c_j^*\} = 0 \qquad \{c_i, c_j^*\} = \delta_{ij}$$

The first contribution in H has no particular interest for our purposes since it only counts the electrons in the first Landau level. The second term is instead the one that contains all the relevant informations.

We recall that the wavefunction has the form

$$\phi(n_1, n_2, \ldots\ldots, n_{N_o})$$

where n_i is equal to one or to zero according to have the i-th position occupied or not. We have to make some approximation in order to transform Hamiltonian (1.3) in a more suitable form. We will show that the relevant ones are:

i) restrict in some way the sum over the four indices
ii) suppose that the matrix elements satisfy some simple rules.

Let us define

$$V_{ijkl} = \langle ij|\ \frac{e^2}{|x-y|}\ |kl\rangle = \int \frac{\Psi_i^*(x)\,\Psi_j^*(y)\,\Psi_k(x)\,\Psi_l(y)}{|x-y|}\, d^2x\, d^2y \qquad (1.4)$$

The main steps now are:

$$H = 1/2\sum_{ijkl} V_{ijkl}\, c_i^* c_j^* c_k c_l = 1/2\sum_{ijkl} V_{ijkl}\, c_i^*(\delta_{jk} - c_k c_j^*) c_l =$$

$$= 1/2\sum_{ijl} V_{ijjl}\, c_i^* c_l - 1/2\sum_{ijkl} V_{ijkl}\, c_i^* c_k c_j^* c_l$$

Using restriction i) we get:

$$H = 1/2\sum_{ijl} V_{ijjl}\, c_i^* c_l \underset{i=l}{\overset{\uparrow}{=}} 1/2\sum_{ij} V_{ijji}\, \hat{n}_i$$

$$H = -1/2\sum_{ijkl} V_{ijkl}\, c_i^* c_k c_j^* c_l \underset{i=k,\,j=l}{\overset{\uparrow}{=}} -1/2\sum_{ij} V_{ijij}\, \hat{n}_i \hat{n}_j$$

and therefore

$$H = 1/2\sum_{ij} V_{ijji}\, \hat{n}_i - 1/2\sum_{ij} V_{ijij}\, \hat{n}_i \hat{n}_j \qquad (1.5)$$

We further suppose that the relevant physics is obtained if

$$V_{ijij} \simeq V_{ijji} \equiv V_{ij} \tag{1.6}$$

and so we get

$$H=1/2\sum_{ij} V_{ij}\,(\hat{n}_i - \hat{n}_i\hat{n}_j)$$

that can be written, a part a c-number, in the form:

$$H=-1/8\sum_{ij} V_{ij}\,(-1+2\hat{n}_i)\,(-1+2\hat{n}_j) \tag{1.7}$$

For reasons that will be clear in the following we add to (1.7) a term which looks quite irrilevant since it is already contained in the same H . We will see that it has a strong role in the convergence discussion.

$$H=-1/8\sum_{ij} V_{ij}\,(-1+2\hat{n}_i)\,(-1+2\hat{n}_j)+\,4\alpha\sum_{ij} V_{ij}\,\hat{n}_i \tag{1.8}$$

We notice that

$$(-1+2\hat{n}_i)\ \ \phi(n_1,\ldots n_{\nu_0}) = \begin{cases} +\ \phi(n_1,\ldots\ldots n_{\nu_0}) & \text{if } n_i=1 \\ -\ \phi(n_1,\ldots\ldots n_{\nu_0}) & \text{if } n_i=0 \end{cases}$$

Using this result we can apply a trick quite similar to the Anderson's one. We introduce spin variables:

$$\phi(n_1,\ldots,1_i,\ldots n_{\nu_0})=\alpha_i \quad ; \quad \phi(n_1,\ldots,0_i,\ldots n_{\nu_0})=\beta_i \tag{1.9}$$

$$(-1+2\hat{n}_i)=\sigma_3^{\,i} \tag{1.10}$$

With these positions we get, a part a c-number ,

$$H=-1/8\sum_{ij} V_{ij}\,\sigma_3^{\,i}\sigma_3^{\,j}\ +2\alpha\sum_{ij} V_{ij}\,\sigma_3^{\,i} \tag{1.11}$$

which has just he same form of the Hamiltonian (6.1), with α playing the role of $\bar{\sigma}$.
As it will be discussed in Chapter VI $\bar{\sigma}$ is basic if the potential is not integrable since it allows the cancellation of the otherwise unavoidable divergences and therefore it needs to be introduced in our model.

I.3 HIGH TEMPERATURE SUPERCONDUCTIVITY

Spin models are believed to play a crucial role also in understanding High Tc Superconductivity.
In particular ancestor of all the actual models of High Tc 2-dim Superconductors is the Hubbard model, described by the following Hamiltonian:

$$H=-\sum_{\langle i,j\rangle,\sigma} t\, C^{*}_{i\sigma} C_{j\sigma} + U\sum_{i} n_{i\sigma} n_{i-\sigma} \tag{1.12}$$

The first contribution is an hopping term since it annihilates an electron with spin σ in the site j and creates another electron with the same spin at site i. The U term is a repulsive one: it gives a great positive contribution to the energy if, in the site i, we have an electron with spin up, $n_{i\sigma}=1$, and an electron with spin down, $n_{i-\sigma}=1$ (Of course only opposite spins are possible in a single site due to Pauli principle). In order to minimize the energy is therefore favourite a configuration of two electrons (far) away from each other (even if with the same spin), than one with two electrons with opposite spin in the same site.
It is possible to show that, in the limit of large U, the above Hamiltonian can be written in the form:

$$H \simeq H'=-t\sum_{\langle i,j\rangle} (1-n_{i\sigma}) C^{*}_{i\sigma} C_{j\sigma} (1-n_{j-\sigma}) + (t^2/U)\sum_{\langle i,j\rangle} \underline{S}_i \cdot \underline{S}_j \tag{1.13}$$

where

$$\underline{S}_i = C^{*}_i \underline{\sigma} C_i \tag{1.14}$$

If the number of the electrons is equal to the number of sites (half-filling) the U term in (1.12) distributes the electrons in such a way that each site in the lattice is occupied and therefore we have $n_{i\sigma}=1$ in (1.13). We get therefore the following result:

$$H' \simeq H_H = (t^2/U)\sum_{\langle i,j\rangle} \underline{S}_i \cdot \underline{S}_j \tag{1.15}$$

which is an Heisemberg antiferromagnetic Hamiltonian for spin 1/2 with interaction only between nearest sites.
Suggested continuum limits of this H' give rise to different models like CP^1 model with Chern Simon term

$$S_{eff} = \sum_{i=1}^{N} |(\partial_\mu - A_\mu) z_i|^2 + \frac{i\mu}{2\pi} \varepsilon_{\mu\nu\rho} A^\mu \partial^\nu A^\rho , \qquad \sum_i |z_i|^2 = 1$$

or SU(2) massless fermions

$$S_{eff}(\Psi, A) = \bar{\Psi}^a (\not\partial + \not{A}^{ab}) \Psi^b$$

or two families of U(1) massive fermions

$$S_{eff}(\Psi, A) = \sum_{i=1}^{2} \bar{\Psi}_i (\not{\partial} + \not{A} + m)\Psi_i$$

It is worthwhile to note that since all these S_{eff} involve non local term, while H' is "highly" local, the continuum limits proposed have in some way introduced LRI that does not appear in the original problem and therefore some care is required.
We conclude this paragraph noticing that even if BCS spin model has to be rejected for sperimental reasons for High Tc Superconductivity (the critical temperature given by BCS is much less than the one obtained using new materials; for these materials the isotope effect, $M^{1/2} T_c \simeq$ constant, is not observed;..) other spin models seem to be relevant to obtain some information and in particular antiferromagnetic Heisemberg model for spin 1/2 is believed to contain most of the physical features of High Tc.

NOTES

(1)
Let us suppose to have a well defined Hamiltonian H.
The equation of motion for a localized variable A is given by

$$dA/dt = i\,[H, A] \tag{1.1}$$

which, once solved, defines the time evolution A(t) of A. We put $\alpha^t(A) \equiv A(t)$ and we call α^t the dynamics of the system.

CHAPTER II: PROBLEMS AND GENERAL STRUCTURES OF LRI

This chapter is devoted in looking with a bit more critical insight to the main features outlined in Chapter I, without going too much in the mathematical detais.

II.1 THERMODYNAMICAL LIMIT AND LONG DISTANCE BEHAVIOUR OF THE STATES

The only possible measures that can be done on a physical system, from a practical point of view, are localized in a certain region of space and time. This explain way the local observables, that is observables localized in a finite region, are so relevant.
As we have already sketched in the previous Chapter given a cutoffed Hamiltonian H_V we can find the equation of motion for every local variable using the Heisemberg equation

$$dA/dt = i\ [\ H_V,\ A] \tag{2.1}$$

This equation, when solved, gives the time dependence of A, $A^V(t)$ (the suffix V shows that we are at fixed volume). We put $\alpha_V^t(A) \equiv A^V(t)$. α_V^t is an automorphism of the local algebra (see note (1) at the end of this Chapter), that is, it maps localized variables into themselves as long as we are at finite volume. If we take the thermodynamical limit two kinds of problems arise:

1) we have to take the limit $V \to \infty$ in the correlation functions for any fixed t;
2) we have to define the "algebraic dynamics", α^t, using some limiting procedure starting from α_V^t.

We have already discussed the relevance of LRI for both problems. Here we want to give some more informations about the last point. We can fall into three different classes of physical situations:

i) let us suppose that our system is described by causal field so that the signals propagate at a finite speed (relativistic system). Therefore an observable which at t=0 is localized in a finite volume at any t>0 is still localized;
ii) if we have SRI the situation is similar. For concretness sake let us suppose that

$$H_V \sim \int_V \Psi^+(\underline{x})\ \Psi^+(\underline{y})\ U(\underline{x}-\underline{y})\ \Psi(\underline{y})\ \Psi(\underline{x})\ d^3\underline{x}\ d^3\underline{y}$$

which is a typical many-body interaction Hamiltonian, with potential U(x-y). If we calculate (2.1) for A= $\Psi(\underline{z})$ we get

$$(d/dt)\alpha_V^t(\Psi(z)) = \Psi(z)\int_V U(x-y)\,\rho(y)\,dy, \qquad \rho(y)=\Psi^*(y)\,\Psi(y) \qquad (2.2)$$

If the potential has short range, that is if $U=0$ outside a finite volume W, the only contributions to $\Psi(z,t)\equiv\alpha_V^t(\Psi(z))$ come from the values of U inside W.Therefore also if $V\to\infty$ $\Psi(z,t)$ is again localized inside W.

iii) The situation is quite different for LRI since, as it is clear from (2.2), contributions to $\Psi(z,t)$ come from each point of the space when $V\to\infty$. Delocalization effects therefore are unavoidable for such interactions.

The differences just outlined are reflected in the definition of the dynamics α^t as already anticipated in the previous Chapter. In ref.[6] it is shown that for a spin system if the interaction goes to zero faster than $|i-j|^{-3}$ then α^t can be defined as a norm limit of α_V^t (see the appendix 1 for a summary of the relevant convergence methods used in this work). This explicitly means that

$$\forall A\in\mathcal{Q} \qquad \|\alpha_V^t(A) - \alpha^t(A)\| \longrightarrow 0 \qquad (2.3)$$

We know that definition (2.3) does not make reference to any particular state. This is in agreement with what already pointed out, that is the fact that for SRI no particular relevance has to be given to the states which in particular need not to be regular at infinity.

The result above can be expressed even for continuous systems, stating that every time we deal with LRI α_V^t cannot be norm converging. We have therefore to substitute norm convergence of α_V^t with a weaker requirement that however has to be strong enough to preserve the group nature of α^t :

$$\alpha^t\alpha^\tau = \alpha^{t+\tau} \qquad (2.4)$$

The reason of this weakening is, how will be clarified in the explicit models, that LRI implies the existence of infinitely delocalized variables, that is variables localized outside any finite region and that therefore commute with every localized variable. A tipical example of "variable at infinity" which appear in Chapter IV is

$$< \sigma_\alpha^\infty > = <\lim_{V,\infty} \sigma_\alpha^V > = < \lim_{V,\infty} 1/|V|\sum_{i\in V} \sigma_\alpha^i > \qquad (2.5)$$

where <.> is a state on the spin algebra (see note (2)).

The necessity of introducing a state in the definition (2.5) comes from the fact that the mean σ_α^V is not norm converging and

therefore this kind of convergence, which is the 'strongest' one, must be substituted using a weaker topology.
In particular if an operator B is such that its correlation functions

$$\langle A\, B^V C\rangle_0 = (\Psi_0, A\, B^V C\, \Psi_0) \qquad \forall A, C \in \mathcal{A}$$

converge when $V \to \infty$ we say that B^V is weakly convergent.
For instance one can verify that if in (2.5) we take an highly irregular state

$$\Psi \sim \uparrow_1 \downarrow_2 \downarrow_3 \uparrow_4 \ldots \uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow\downarrow \cdots\cdots$$

(where $\uparrow_i$ and $\downarrow_i$ are spin up and down in the α direction at the site i) the limit above oscillates and therefore we cannot define the ergodic mean σ_α^∞. If on the countrary

$$\Psi \sim \uparrow\uparrow\uparrow\uparrow\uparrow\uparrow \cdots\cdots$$

(or local modification of this vector) the series in (2.5) is converging and therefore to σ_α^∞ can be given a well defined meaning. The characteristic of the states of being quite regular at infinity is known as "good behaviour of the states at large distances". The set of states which share this feature form a set of "physically relevant states", F.

II.2 SYMMETRY BREAKING

SGT is one of the most powerful tool available to theoretical physicists in order to obtain non perturbative informations on the spectrum of a given theory. However some care is required in order to get the right results. In fact we know that in some situation the conclusions of the SGT are in contrast with the explicit solution of the model.
Let β be a symmetry of the algebra $\mathcal{A}$, that is an automorphism of $\mathcal{A}$ such that $\beta(A^*) = \beta(A)^*$.
We say that the symmetry is not broken in the representation π (see note (3)) iff there exists an unitary operator which describes such a symmetry in π. Equivalently β is spontaneously broken iff

$$\exists\, A \in \mathcal{A} \quad \text{such that} \quad (\Psi_0, \beta(A)\Psi_0) \neq (\Psi_0, A\Psi_0) \qquad (2.6)$$

where Ψ_0 is the ground state of the Hilbert space H_π of the π representation (see appendix 2).
For SRI symmetry breaking can arise only if the ground state is not symmetric. All the standard wisdom on SSB relies on the following characterization: symmetric equations of motion for local variables and non-symmetric correlation functions. The asymmetry of the ground state is in general understood as the dependence of the correlation functions from the boundary conditions, which however cannot affect the equation of motion of local variables, as a consequence of the short range of the interaction.
In this framework the conclusion of the SGT become very clear. In fact the spectrum of the excitations associated to the breaking of the one parameter group of symmetries β^λ, $\lambda \in R$, generated by local charges, is given by the (support) of the Fourier transform $J(\omega)$ of

$$J(t) = i \lim_{R,\infty} < \Psi_0, [Q_R, A_t] \Psi_0 > = (d/d\lambda) < \beta^\lambda \alpha^t (A) >_0 \Big|_{\lambda=0} \qquad (2.7)$$

Since the equations of motion are β^λ symmetric, $\beta^\lambda \alpha^t = \alpha^t \beta^\lambda$, if they are symmetric at finite volume and under some technical assumption, we have from (2.7)

$$J(t) = (d/d\lambda) < \beta^\lambda \alpha^t (A) >_0 \Big|_{\lambda=0} = (d/d\lambda) < \alpha^t \beta^\lambda (A) >_0 \Big|_{\lambda=0} = J(0) \qquad (2.8)$$

using the stationarity of Ψ_0. Hence $J(\omega) \sim \delta(\omega)$ and the energy spectrum of the Goldstone boson, in the limit $k \to 0$ (or $R \to \infty$), reduces to the single point $\omega = 0$.
The existence of a mass gap associated to a SSB implies that the basic ingredient of the SGT, namely β^λ being generated by a local charge on an algebra stable under a symmetric algebraic dynamics, must fail.
As we told in advance for LRI two kinds of problems arise:

i) the local algebra $\mathcal{Q}$ is no more stable under time evolution since if $A \in \mathcal{Q}$ in general $\alpha^t(A)$ depends also from the variables at infinity which necessarily appear (and of course such variables are not local so that $\alpha^t(A) \notin \mathcal{Q}$);
ii) Since Q_R commutes with the variables at infinity the conservation of the total charge does not imply that, when $R \to \infty$, the vev of the commutator $[Q_R, A_t]$ is time independent.

It is shown in refs.[7] that under this condition the correct Goldstone spectrum can be obtained by considering a particolar factorial representation (see note (4)) and using the "effectively localized dynamics", α^t_π, whose main

characteristics will be discussed in the next Chapter. At this level it is relevant to stress simply that it differs from α^t since the variables at infinity, A_∞ , entering in the definition of α^t, are now substituted by the c-numbers $\pi(A_\infty)$, that is by their expectation values in the representation π.
The physical meaning of this structure is that the essential nonlocal effects of the algebraic dynamics are due to the involvement of the variables at infinity; once these variables are frozen to c-numbers, as it happens with the choice of a factorial representation π, then one obtains a dynamics which maps a complete set $\mathcal{Q}_\ell$ of "essentially local variables" into themselves.
This structure offers a convenient mathematical framework for the generalization of the SGT:
the "effective localization algebra" $\mathcal{Q}_\ell$ is in fact a natural algebra on which symmetries may be locally approximated and furthermore it is stable under a reduced dynamics, α^t_π , which coincides with α^t in π.
We will consider automorphisms $\beta^\lambda, \lambda \in R$, which are generated on $\mathcal{Q}_\ell$ by local charges Q_R, that is

$$(d/d\lambda)\ \phi_0(\beta^\lambda(A))\Big|_{\lambda=0} = i \lim_{R,\infty} \phi_0([Q_R,A])$$

on a space-time translationally invariant primary state (see note (4)) ϕ_0 . Since α^t_π maps $\mathcal{Q}_\ell$ into $\mathcal{Q}_\ell$ it can be shown that:

$$J(t) = i \lim_{R,\infty} \phi_0([Q_R,\alpha^t(A)]) = (d/d\lambda)\ \phi_0(\beta^\lambda\alpha^t_\pi(A))\Big|_{\lambda=0}$$

which is the natural extention of (2.7). This equation can be even simplified: in refs.[7] is proven that we can write

$$J(t) = (d/d\lambda)\ \phi_0(\beta^\lambda\alpha^t_\pi(A_\infty))\Big|_{\lambda=0} , \quad A_\infty = \lim_{V,\infty} (1/V)\int_V d^3\underline{x}\ A_{\underline{x}} \qquad (2.9)$$

and therefore the spectrum is given essentially by the motion of the variables at infinity. Since they form an abelian algebra, stable under time evolution, the problem is reduced to the study of a "classical dynamical system".
It is not difficult to understand now the reasons that allow the existence of a non zero mass gap. It is possible to show that if β^λ is broken in the representation π , see eq.(2.6), even if $\beta^\lambda\alpha^t = \alpha^t\beta^\lambda$
we have

$$\beta^\lambda\alpha^t_\pi \neq \alpha^t_\pi\beta^\lambda \qquad (2.10)$$

and therefore a non trivial time dependence of J(t) is in general obtained and its Fourier transform gives a massive particle. The

non symmetricity of the reduced dynamics α^t_π depends on the c-number nature of the variables at infinity in π ,since they do not transform anymore under β^λ .
It is worthwhile to note that this is a new mechanism of symmetry breaking with respect to what it happens in the SRI case, since as we see the boundary conditions affect directly the equation of motion and not only the correlation functions.

II.3 EXAMPLES

We will show now some easy effect of what untill now have been discussed with two specific models, without going too much into detais.

Stuckelberg-Kibble Model (D=4)
This model is relevant since it is usually regarded as a prototype of gauge theories exhibiting Higgs phenomenon. The cutoffed Hamiltonian is (in Coulomb gauge)

$$H_L=1/2 \int [(\nabla\varphi)^2 + \pi^2]d^3x + e^2/2\int d^3\underline{x}d^3\underline{y}\,\pi(\underline{x})V_L(\underline{x}-\underline{y})\,\pi(\underline{y})$$

where

$$V_L(\underline{x})= (1/|\underline{x}|)\ f_L(x) \quad ; \quad f_L(x)=\begin{cases} 1 & |x|<L \\ 0 & |x|>L(1+\varepsilon) \end{cases} \tag{2.11}$$

The equations of motion defining $\alpha^t_L(\varphi) \equiv \varphi(t)$, give:

$$\ddot{\varphi} = \nabla^2\varphi + e^2\, V_L*\nabla^2\varphi \tag{2.12}$$

It can be shown that the second term in the rhs can be written in the form

$$V_L* \nabla^2\varphi = -4\pi\,\varphi + \int\sigma_L(\underline{x}-\underline{y})\,\varphi(\underline{y})\,d^3\underline{y} = -4\pi\,\varphi + \sigma_L*\varphi$$

where

$$\sigma_L(\underline{x}-\underline{y}) = 4\pi\,\delta(\underline{x}-\underline{y}) + \nabla^2_y\,V_L(\underline{x}-\underline{y})$$

From the above definition can be verified that

$$\mathrm{supp}\,\sigma_L \subset \{\underline{y} :\ L<|y|<L(1+\varepsilon)\ \} \tag{2.13}$$

Therefore $\sigma_L * \varphi$ is the mean of the field in the space region defined by (2.13). Of course when the cutoff is removed, $L \to \infty$, $\sigma_L * \varphi$ is localized outside any bounded region and, on states sufficently regular at infinity, we can put

$$< \varphi_\infty > = (1/4\pi) \lim_{L,\infty} < \sigma_L * \varphi >$$

We see therefore that the equation (2.12) can be set in the form

$$\Box (\varphi - \varphi_\infty) = 4\pi e^2 (\varphi - \varphi_\infty) \tag{2.14}$$

and we conclude that:

1) an accurate removing of the cutoff is crucial in finding the right equation of motion (in fact if we use V(x) without the cutoff we would get an equation of motion which has not the same symmetry of H ($\varphi \to \varphi + c$; $\pi \to \pi$),while (2.14) is symmetric);
2) the good behaviour of the states on which φ_∞ converges is fundamental. In particular such states must be regular at infinity, since $\sigma_L * \varphi$ is localized at large distances;
3) In representation π, in which φ_∞ is a c-number and does not transform anymore, (2.14) shows that the symmetry is broken and that a massive particle nevertheless appears. Therefore SGT is evaded.

<u>Schwinger model</u> (D=2)
This model is relevant in understanding the breaking of chiral U(1) symmetry in QCD.
Originally Scwinger model is simply QED in two dimensions:

$$\mathcal{L} = \bar{\Psi} (i\gamma\partial - \not{A}) \Psi - 1/4 \; F_{\mu\nu}^2 \tag{2.15}$$

We treat it in Coulomb gauge, $\partial_1 A_1 = 0$, and in the bosonized form. Without giving the details we claim that our problem above is equivalent to the one described by the following cutoffed Hamiltonian

$$H_L = 1/2 \int dx \; [\pi^2 + (\partial_1 \varphi)^2] - (e^2/2) \int dxdy \; \partial_1 \varphi(x) \; V_L(x-y) \; \partial_1 \varphi(y)$$

Where like in the S.K. model we have $V_L(x) = V(x) \; f_L(x)$ and V(x) is the Coulomb potential in two dimensions. The equations of motion are :

$$\dot{\varphi}(x) = \pi(x)$$

$$\dot{\pi}(x) = \partial_1^2 \varphi(x) - e^2 (\partial_1^2 V_L) * \varphi(x)$$

But:

$$(\partial_1^2 V_L) * \varphi = V_L * (\partial_1^2 \varphi) = -4\pi\, \varphi + \sigma_L * \varphi$$

$$\sigma_L(x) = \partial_1^2 V_L(x) + 4\pi\, \delta(x)$$

Condition (2.13) is verified even in this model and again we put

$$\langle \varphi_\infty \rangle = \lim\, (1/4\pi)\, \langle \sigma_L * \varphi \rangle$$

The equation of motion is

$$\Box(\varphi - \varphi_\infty) = 4\pi e^2 (\varphi - \varphi_\infty) \qquad (2.16)$$

We can now discuss the symmetry of the model. The original lagrangian (2.15) is symmetric under the chiral transformation $\psi \longrightarrow \exp(i\lambda\gamma_5)\,\psi$ .In the bosonized form this becomes $\varphi \longrightarrow \varphi + \lambda/\sqrt{\pi}$ and we see that (2.16) is symmetric under this transformation. Like for the S.K. model we can show that if the infrared cutoff is not introduced a non-symmetric equation of motion is obtained starting by the symmetric Hamiltonian !!
However in each representation π φ_∞ is a c-number and does not transform under the symmetry, which is said to be broken in π . Nevertheless we get a massive particle which shows that no SGT can be applied. So even if the chiral symmetry is broken no massless particle is obtained when the model is solved correctly.

From both models it appears evident that the mechanism by which the symmetry is broken is related to the fact that φ_∞ are c-numbers in each (factorial) representation, and therefore do not transform anymore under the symmetry. Since the appearance of these variables is due essentially to the long range of the interaction we conclude that LRI gives a possible way out to the SGT. We remark that this is the same conclusion of the Swieca's analysis, [1], which was obtained through heuristic motivations. Other models showing in detail all these points are tipically long range spin models, which we are going to treat in the following Chapters.

NOTES

(1)
In general given an algebra $\mathcal{A}$ we say that β is an automorphism of $\mathcal{A}$ if:

a) $\forall A \in \mathcal{A} \quad \beta(A) \in \mathcal{A}$
b) $\beta(AB) = \beta(A)\,\beta(B)$
c) β is linear

(2)
For instance we can think $< * > = (\Psi, \ * \Psi)$, where Ψ is a vector describing the spin in each lattice site in the up position.

(3)
A representation π of an abstract (C^*)-algebra is a map $\pi(\mathcal{A})$ such that $\pi : \mathcal{A} \longrightarrow B(H)$ where $B(H)$ is the set of bounded operators on a given Hilbert space H.
π is a continuous map preserving the algebraic properties included the $*$-operation.

(4)
A representation π is said factorial if its center, that is $\pi(\mathcal{A}) \cap \pi(\mathcal{A}')$ ($\pi(\mathcal{A}')$ is the set of all operators commuting with all the operators of $\pi(\mathcal{A})$), consists only in multiples of the unit operator.
Moreover a state is called primary if its GNS-representation (see app.2) is factorial.

CHAPTER III : MATHEMATICAL FORMULATION AND PROBLEMS

The general motivations and the models discussed in the previous Chapters clearly show how the role played by LRI is crucial for Physics and how new phenomena in general arise.
All this urges a mathematical formalization and the developing of some techniques that seem to be essential for dealing with LRI. In this Chapter some of these techniques will be explained. They form the relevant framework by which LRI has to be treated and in which this work is inserted.
For SRI the natural algebraic framework is given by a (quasi) local algebra: we call $\mathcal{Q}_V$ the algebra generated by the set of all the variables localized in V, satisfying the natural isotony property $\mathcal{Q}_{V_2} \supset \mathcal{Q}_{V_1}$ if $V_2 \supset V_1$. We further consider the union of all such algebras

$$\mathcal{Q}_0 = \bigcup_V \mathcal{Q}_V \qquad (3.1)$$

For technical reasons is actually convenient to consider also the norm limit of the element of $\mathcal{Q}_0$, namely the norm closure $\mathcal{Q}$ of $\mathcal{Q}_0$. The space-time translations are in general assumed to be automorphisms of $\mathcal{Q}$. The symmetries are generated by local charges, Q_R, on $\mathcal{Q}_0$. For instance for a one parameter group β^λ, $\lambda \in R$, we have

$$\forall A \in \mathcal{Q}_0 \quad \beta^\lambda(A) = \text{norm-}\lim_{R,\infty} \beta^\lambda_R(A) = \text{norm} \lim_{R,\infty} \exp(iQ_R\lambda)\ A \exp(-iQ_R\lambda)$$

The mechanism that allows breaking of the symmetry is the dependence of ground state by the boundary conditions, so that condition (2.6) is verified and, as showed in (2.8) a massless Goldstone particle appears.
For LRI we have to generalize the above structure. First of all the norm convergence of α^t_V has to be replaced with the convergence of the expectation values of α^t_V on a family F of "physically relevant" representations of $\mathcal{Q}_0$. More precisely, the algebraic dynamics is defined as a weak limit of α^t_V with respect to the weak topology τ_F of the states of F:

$$\alpha^t = \tau_F - \lim_{V,\infty} \alpha^t_V \qquad (3.2)$$

Consequently the algebraic dynamics is naturally defined on the τ_F closure M of $\mathcal{Q}_0$, (see app.1), rather than on $\mathcal{Q}$.
The set F is assumed to be:

1) closed under linear combinations;
2) norm closed and separating, i.e. $\phi(A)=0 \quad A \in \mathcal{Q}, \quad \forall \phi \in F$ implies $A=0$

3) stable under local operations, in the sense that if $\phi \in F$ also $\phi_{AB}(\cdot) \equiv \phi(A.B)$, $A,B \in \mathcal{A}$, belongs to F.

If the interaction is of sufficently long range, the commutator $[H_V, A]$, A a local variable, involves in a substantial way operators localized on (or near) the boundary of V, which, when $V \to \infty$, become variables localized outside any bounded region, the so called variables at infinity, which obviously commute with any local variable.
Typical example of variables at infinity are

$$\lim_{V,\infty} 1/V \int_V d^3\underline{x}\ A(\underline{x}) = \lim_{V,\infty} A_V = A_\infty$$

or

$$\lim_{R,\infty} \int d^3\underline{x}\ f_R(\underline{x})\ A(\underline{x}) = \lim_{R,\infty} A_{S_R}$$

where $f_R(\underline{x})$ is a regular function with support in $R<|\underline{x}|< R(1+\varepsilon)$ and normalized. As it was already discussed the above limits do not exist in the norm topology but only in the weak topology defined by the states sufficently regular at infinity.
Thus, quite generally, when the interaction is of sufficently long range so that the time evolution of local variables $\alpha_V^t(A)$ involves largely delocalized variables, the infinite volume limit requires to make reference to a family F of regular states in order to define α^t by (3.2).
It can be proven that α^t satisfies the group law (2.4) if the convergence of the cutoffed dynamics is (ultra)strong (see app. 1).
The presence of variables at infinity in $\alpha^t(A)$, A a local variable, implies moreover that $\mathcal{A}$ is no longer stable under α^t. What is now true is that α^t defines a one parameter group of automorphisms of M and actually α^t leaves stable the algebra generated by an essentially local algebra $\mathcal{A}$ with trivial center (weakly dense in M) (see note (1)), and by an algebra of variables at infinity.
It is worthwhile to stress that in the above generalization of dynamical system, the algebraic structure is fully determined by the algebra $\mathcal{A}$ and by the family F since M is obtained as weak closure of $\mathcal{A}$ with respect to the weak topology of F. With respect to the local formulation, the new ingredient is the non trivial role played by the states.
Within this framework a symmetry α is naturally defined as a *-automorphism of the $\mathcal{A}$ algebra and it is possible to prove that if the family of states F is stable under α^* and $(\alpha^*)^{-1}$, that is if for $\phi \in F$ then $\alpha^*\phi(\cdot) = \phi(\alpha(\cdot)) \in F$, then α can be extended to an automorphism of M.

Moreover one can prove that
- If α is a *-automorphism of $\mathcal{Q}$ and α_V^t is α symmetric, $\alpha_V^t \alpha = \alpha \alpha_V^t$ (e.g. if α_V^t is generated by a symmetric finite volume Hamiltonian H_V), then F being α^* stable implies $\alpha^t \alpha = \alpha \alpha^t$ on M.

The algebraic structure desribed above can be in some sense trivialized by making reference to a particular factorial representation π (of the algebra $\mathcal{Q}$) and by defining a 'new' dynamics replacing the variables at infinity with their c-number expectation values in π .
This explicitly means that if $\alpha^t(A) = F(\mathcal{Q}, \mathcal{Q}_\infty)$ then we define another dynamics $\alpha^t_\pi(A) = F(\mathcal{Q}, \pi(\mathcal{Q}_\infty))$.
The so defined α^t_π does no longer contain variables at infinity and in general it leaves stable some essentially local algebra with trivial center.
We say that α^t gets effectively localized, with respect to a factorial representation π of F, if there exists a subalgebra $\mathcal{Q}_\ell \subset$ M such that

1) $\mathcal{Q}_\ell$ has trivial center
2) $\mathcal{Q}_\ell$ is weakly dense in M
3) there exists a one parameter group of automorphisms α^t_π of $\mathcal{Q}_\ell$ such that
$$\phi(\alpha^t(A)) = \phi(\alpha^t_\pi(A)) \qquad \forall A \in \mathcal{Q}_\ell$$

The above property has the very simple physical interpretation that the algebra of essentially localized observables $\mathcal{Q}_\ell$ is stable under time evolution once the boundary conditions are fixed.
The property of effective localization may be regarded as the clear and rigorous version of the seizing of the vacuum advocated by Kogut & Susskind on the basis of the Schwinger model, [8]. In this paper it is stated that this effect, caused by long range forces, prevent distant parts of the vacuum from behaving independently.
Clearly the fact of making reference to a particular factorial representation, even if allows the recovering of the Haag & Kastler local formulation, makes one loses the algebraic characteristics of the problem that are quite important for their unifying nature. For instance different "phases" of the same physical system are described by the same algebraic dynamics α^t , but by different effective dynamics α^t_π , depending on the π representation and therefore by the expectation values in π of the variables at infinity.
Moreover one loses the general mechanism by which symmetries of the finite volume Hamiltonian get broken in each factorial

representation: if in fact the boundary conditions are already been fixed (and we are therefore dealing with α^t_π) our symmetries are clearly the ones which commute with α^t_π and they cannot be broken in the sense of equation (2.10). On the countrary we have already discussed that if we start with the algebraic dynamics α^t a symmetry β , $\beta\alpha^t = \alpha^t\beta$, can give, frezing the variables at infinity, $\beta\alpha^t{}_\pi \neq \alpha^t_\pi\beta$ and therefore another way to break the symmetry arises. This is the relevant one, as we will see in the next chapter, since provides a new mechanism by which J(t) can get a non trivial time dependence (and in this way one can get a massive Goldstone boson).
This discussion of course does not conclude the mathematical treatment of LRI, but can be sufficient for our aims. A much more complete technical treatment can be found in ref.[7].

NOTES

(1)
"Essentially" is due to possible small delocalization effects associated to the infinite propagation speed of non-relativistic dynamics that simply change the class of the test functions used to regularize the field operators. $\mathfrak{D}$ is now too restrictive and we use $\mathcal{S}$ or $\mathcal{L}^2$.
"With trivial center" means that the element of the center of the algebra are multiples of the unit operator.
"Weakly dense in M" means that the weak closure of $\mathfrak{A}$ is the same M.

CHAPTER IV : HEISEMBEG-WEISS AND BCS MODELS

IV.1 GENERAL PROBLEMS AND RESULTS

This Chapter will be devoted to the discussion of the well known BCS and Heisemberg-Weiss models. Both of them are mean field models, that is models with infinite range interactions; every spin interacts with every other spin with the same strenght. Clearly all the problems of LRI discussed untill now arise.
We will focus our attention to the basic problem of the definition of the dynamics in the thermodynamical limit, namely when the infrared cutoff is removed, in the dynamics α_V^t .
As briefly discussed in Chapter II, for such models the dynamics of a spin at the site i involves the operator σ^V , localized on the volume V. For instance for

$$H_V = J/|V| \sum_{ij\in V} \underline{\sigma}^i \cdot \underline{\sigma}^j$$

we get

$$d/dt \ \alpha_V^t (\sigma_\delta^i) = -2 \ J \ \varepsilon_{\alpha\delta\gamma} \ \{ \alpha_V^t (\sigma_\gamma^i), \alpha_V^t (\sigma_\alpha^V) \} \ , \qquad \sigma_\alpha^V = 1/|V| \sum_{i\in V} \sigma_\alpha^i$$

The problem is then to discuss what meaning, if any, can be given to σ^V as $V \to \infty$. It is easy to see that in general such limit does not exist, as already anticipated in paragraph II.1, where it is very well illustrated that the removal of the infrared cutoff ($V \to \infty$) cannot be done without making restrictions on the large distance behaviour of the states, namely without identifying the "physically relevant" states.
The main result of this Chapter is to prove that for such models one can determines the condition which identifies the family F, i.e. the states with respect to which α_V^t converges to an automorphism α^t which obeys the group law.
Such condition is that σ^V converges (ultra)strongly with respect to such states.
Thus the thermodynamical limit of the dynamics gives a meaningful result only if the states are sufficently regular at large distances (<u>infrared regular states</u>).
The idea of the proof is to study the closed system of equations of motion

$$d/dt \, \alpha_V^t (\underline{\sigma}) = \underline{f}(\alpha_V^t (\underline{\sigma}), \alpha_V^t (\underline{\sigma}^V)) \ ,$$

where $\underline{f}$ does not depend explicitly on V, and to deduce that the solution of such system is an analytical function of its variables:

$$\alpha_V^t(\underline{\sigma}) = \underline{X}(t \ , \underline{\sigma}, \underline{\sigma}^V)$$

Hence $\underline{X}$ can be expanded in a (norm) converging power series of σ^V, which is the relevant variable. We know that the only topology that preserve the product, apart the norm topology that cannot be used here as we have already discussed, is the (ultra) strong topology. Asking that σ_α^V converges (ultra)strongly to σ_α^∞ we have also $(\sigma_\alpha^V)^n \longrightarrow (\sigma_\alpha^\infty)^n$ for every n integer and therefore $\underline{X}(\sigma^V) \longrightarrow \underline{X}(\sigma^\infty)$.

Actually such arguments apply generally to all mean field models described by a finite volume Hamiltonian of the form

$$H_V = 1/|V| \sum_{ij\in V} \sum_{\alpha\beta} \sigma_\alpha^i \sigma_\beta^j A^{\alpha\beta} + \sum_{i\in V} \sum_{\alpha} C_\alpha \sigma_\alpha^i \qquad (4.1)$$

where i and j are lattice site indexes while α and β are spin components.

In (4.1) the particular choice

$$A^{\alpha\beta} = -(T_c/2) \begin{pmatrix} 1 & i & 0 \\ -i & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad C_\alpha = -\mathcal{E}\, \delta_{\alpha 3} \qquad (4.2)$$

gives the BCS model of Superconductivity, see [3],[9], while

$$A^{\alpha\beta} = J\, \delta^{\alpha\beta} \qquad C_\alpha = B\, \delta_{\alpha 3} \qquad (4.3)$$

gives the Heisemberg-Weiss model.

The equation of motion obtained from (4.1) are

$$d/dt\, \alpha_V^t(\sigma_\delta^i) = -2\, \mathcal{E}_{\alpha\delta\gamma} (A^S_{\alpha\beta} \{ \alpha_V^t(\sigma_\gamma^i), \alpha_V^t(\sigma_\beta^V) \} +$$
$$+ A^A_{\alpha\beta} [\alpha_V^t(\sigma_\gamma^i), \alpha_V^t(\sigma_\beta^V)]) - 2\, C_\alpha \mathcal{E}_{\alpha\delta\gamma}\, \alpha_V^t(\sigma_\gamma^i) \qquad (4.4)$$

This equation defines α_V^t which, as long as V is finite, is a one parameter group of automorphisms of $\mathcal{Q}$. The following quantities are defined:

$$A^S_{\alpha\beta} = (A_{\alpha\beta} + A_{\beta\alpha})/2; \quad A^A_{\alpha\beta} = (A_{\alpha\beta} - A_{\beta\alpha})/2; \quad \sigma_\beta^V = (1/|V|) \sum_{i\in V} \sigma_\beta^i \qquad (4.5)$$

Equations (4.4) and (4.5) give also the equation of motion for the mean $\alpha_V^t(\sigma_\delta^V)$:

$$d/dt\, \alpha_V^t(\sigma_\delta^V) = -2\, \mathcal{E}_{\alpha\delta\gamma} (A^S_{\alpha\beta} \{ \alpha_V^t(\sigma_\gamma^V), \alpha_V^t(\sigma_\beta^V) \} +$$
$$+ A^A_{\alpha\beta} [\alpha_V^t(\sigma_\gamma^V), \alpha_V^t(\sigma_\beta^V)]) - 2\, C_\alpha \mathcal{E}_{\alpha\delta\gamma}\, \alpha_V^t(\sigma_\gamma^V) \qquad (4.6)$$

We rewrite equations (4.4) and (4.6), which form a closed set, as:

$$dx_t^V/dt = f(x_t^V, y_t^V) \tag{4.7}$$

$$dy_t^V/dt = f(y_t^V, y_t^V) \tag{4.8}$$

where

$$x_t^V = \alpha_V^t(\sigma) \qquad y_t^V = \alpha_V^t(\sigma^V)$$

$$f(x,y) = -2\,\varepsilon_{\alpha\delta\gamma}\,(A^s_{\alpha\beta}\{x,y\} + A^a_{\alpha\beta}[x,y]) - 2\,C_\alpha\,\varepsilon_{\alpha\delta\gamma}\,x \tag{4.9}$$

(We note that no explicit dependence on V is present in $f(x,y)$) Following ref.[10] we claim that equation (4.8) admits, once the initial conditions are fixed, one and only one solution $y_t^V = Y(t,y^V)$ which is analytical entire and contains, obviously, no explicit dependence on V. When this solution is substituted in equation (4.7) we get

$$dx_t^V/dt = F(t, x_t^V, y^V) \tag{4.10}$$

with F again analytical entire and therefore, again ref.[10], it exists an unique analytical entire solution

$$x_t^V = X(t,x,y^V) \tag{4.11}$$

Returning to the original spin language this means that equation (4.4) admits one and only one solution

$$\alpha_V^t(\sigma_\delta^i) = X(t,\sigma^i,\sigma^V) \tag{4.12}$$

which is analytical entire in its variables. The following theorem can now be stated:

Theorem 1

Given the equation (4.4) this admits an unique solution and if σ^V converges (ultra)strongly to σ^∞ this solution converges in the (ultra)strong topology to the solution of the (ultra)strong limit of equation (4.4)

The main ideas of the proof of the theorem have been already discussed previously and we will prove it in details in a bit generalized form in the mathematical section of this Chapter.
This framework developed for solving, at least formally, the equations of motion for a given system can be used everytime the system of differential equations we deal with is closed. This is certainly what it happens for mean field models and even for the long range Ising model described in Chapter VI. Of course the 'good analytical properties' of the functions in the rhs of the

cutoffed differential equations are also crucial in deriving the existence of the solution at V finite and then to verify the existence of its thermodynamical limit.
As a family F of physically relevant states one can take the largest set of states on which the average in (4.5) converges (ultra)strongly to the ergodic mean

$$\sigma_\alpha^\infty = (u.)s. \lim_{V\uparrow\infty} \sigma_\alpha^V \tag{4.13}$$

This is of course the condition of sufficient regularity (see Chapter II) of the states at large distance needed to guarantee the (ultra)strong convergence of α_V^t to a one parameter group of automorphisms α^t of the (Von Neumann) algebra M obtained by the weak closure of $\mathcal{Q}$ with respect to F.
We further specify F as the set of product states ϕ with the property

$$\phi\,(\alpha^t(\sigma_\alpha^\infty)) = \phi\,(\sigma_\alpha^\infty) \tag{4.14}$$

On the states of F theorem 1 says that α_V^t converges in the (ultra)strong topology to α^t:

$$\alpha^t(\sigma_\delta^i) = X(t,\sigma^i,\sigma^\infty) \tag{4.15}$$

which is solution of the equation

$$d/dt\ \alpha^t(\sigma_\delta^i) = -2\,\mathcal{E}_{\alpha\delta\gamma}\, A^s_{\alpha\beta}\,\{\alpha^t(\sigma_\gamma^i),\alpha^t(\sigma_\beta^\infty)\} + \\ -\,2\,C_\alpha\,\mathcal{E}_{\alpha\delta\gamma}\,\alpha^t(\sigma_\gamma^i) \tag{4.16}$$

This equation shows that the algebraic dynamics α^t involve in its definition itself the presence of the variable at infinity $\underline{\sigma}^\infty$. So in these models we verify that α^t leaves stable the algebra generated by the local algebra $\mathcal{Q}$ and by the algebra at infinity $\mathcal{Q}_\infty$ generated by $\underline{\sigma}^\infty$.
Equation (4.16) moreover shows that α^t is nothing but a rotation around the vector

$$\underline{\varrho} = 4\,A^s\,\underline{\sigma}^\infty + 2\,\underline{C} \tag{4.17}$$

and so we write

$$\alpha^t(\sigma_\alpha^i) = (R_\varrho(t)\sigma^i)_\alpha \tag{4.18}$$

where $R_\varrho(t)$ is a rotation around the vector ϱ.
If we consider a factorial representation π of F the variables at infinity are "frozen" to their expectation values $\pi(\underline{\sigma}^\infty)$

which are now c-number becouse of the factoriality of π .
Different representations are defined by the different values that these variables at infinity assume.
The essentially localized dynamics α^t_π is simply given by the rotations around the vector

$$\varrho^\pi \equiv \pi(4\ A^s\ \underline{\sigma}^\infty + 2\ \underline{C}\ ) = 4\ A^s\ \pi\ (\underline{\sigma}^\infty) + 2\ \underline{C} \tag{4.19}$$

$$\alpha^t_\pi(\sigma^i_\alpha) = (R_{\varrho^\pi}(t)\sigma^i)_\alpha \tag{4.20}$$

and of course no variable at infinity is anylonger present so that we can say that α^t_π maps $\mathcal{Q}_e$ in $\mathcal{Q}_e$ (Due to lattice formulation the algebra of essential localization $\mathcal{Q}_e$ coincide with $\mathcal{Q}$). We conclude that stability under time evolution is regained once the boundary conditions are fixed and that the effective dynamics α^t_π depends on the boundary conditions themselves. Thus different phases are described by different effective dynamics.
The π representations used above are the ones that are derived (see app.2) by the pure product states invariant under space translations, $\phi^{\vec{n}}_0$:

$$\phi^{\vec{n}}_0(\sigma^i_\alpha) = n_\alpha \quad , \qquad |\vec{n}| = 1 \tag{4.21}$$

Moreover we require also invariance under time-translations, equation (4.14), which gives some restriction on the vector $\vec{n}$:

$$\text{BCS} \qquad n_\alpha = (0,0,\pm 1) \qquad \text{or} \qquad n_\alpha = (n_1, n_2, \mathcal{E}/T_c) \tag{4.22}$$

$$\text{HW} \qquad n_\alpha = (0,0,-B/2J) \qquad \text{or} \qquad n_\alpha = (n_1, n_2, n_3) \quad B=0 \tag{4.23}$$

(The choice of one or the other solution is crucial for the breaking of the symmetries)

Let us discuss now the symmetries for the HW model, with B=0.
Here the symmetries are the three dimensional spin rotations. They define automorphisms of $\mathcal{Q}$ which commute with α^t_V , and therefore they can be extended to automorphisms of M which commute with α^t (This follows from a theorem given in Chapter III, thanks to the stability of F under β^*).
Of course if $\pi_{\vec{n}}$ is the representation defined by the ground state $\phi^{\vec{n}}_0$ only the rotations around $\vec{n}$ are unbroken in $\pi_{\vec{n}}$.
The classical motion at infinity defined by α^t_π is the group of the rotations of $\underline{\sigma}^\infty$ around $\vec{n}'=\pi\ (\underline{\sigma}^\infty)$ with frequency $\omega=4J$, as it can be obtained from equation (4.16) recalling that $C_\alpha = 0$ and that $A^s_{\alpha\beta} = A_{\alpha\beta} = J\,\delta_{\alpha\beta}$. This is the energy gap at $k\to 0$ of the

generalized Goldstone boson associated to the spontaneous breaking of rotations in each representation definited by the states $\phi_0^{\vec{n}}$ $(\underline{n} \neq \pi(\underline{\sigma}^\infty))$.

The discussion is quite similar for BCS model. Here the symmetries are the rotations around the z axis. These are automorphisms $\beta^\lambda, \lambda \in R$, of the algebra $\mathfrak{A}$ which commute with α_V^t. Again we can extend β^λ to automorphisms of M thanks to the stability of F under $(\beta^\lambda)^*$.
In the representation $\pi_{\vec{n}}$ with $\underline{n} = (n_1, n_2, \mathcal{E}/T_c)$, β^λ is broken since $\underline{n}$ is not alligned with the z axis. $\alpha^t_{\vec{n}}$ defines rotations around the vector $\pi(\varrho)$ with a frequency $\omega = 2\,T_c$ when $k \to 0$. This mass gap is again due to the long range of the interaction and it can be foreseen using the Generalized Goldstone Theorem (see for instance ref.[7a]).
We see from this analysis that all the tipical features discussed in [7] naturally appear in these models: the need for enlarging the algebra by introducing variables at infinity, the relevance of the family F in order to define an algebraic dynamics, the essential localization of the dynamics and finally the Generalized Goldstone Theorem with its mass generation (we note that no Higgs mechanism can be advocated here !!).
Moreover it will appear clear in Chapter VI that the appearance of the variables at infinity is not a tipical feature of the mean field approximation but naturally follows from the long range of the interaction.

IV.2 MATHEMATICAL DETAILS AND PROOFS

Here all the relevant mathematical results claimed in the previous discussion will be proven. Since no physical information is contained in this paragraph it can be omitted at a first reading.
In particular

1) the existence and unicity of the solution of the equation of motion will be recaved;
2) a generalization of theorem 1 will be stated and proven;
3) the group nature of the algebraic dynamics α^t in (4.15) will be verified;
4) conditions (4.22) and (4.23) will be obtained.

(1)
We start proving that equation (4.4) can be solved and that the solution is analytical entire in its variables.
Equation (4.6) is really a system of three equations in three variables and can be written in the following form:

$$\begin{cases} (d/dt)y_1^v(t) = f_1(y_1^v(t), y_2^v(t), y_3^v(t)) \\ (d/dt)y_2^v(t) = f_2(y_1^v(t), y_2^v(t), y_3^v(t)) \\ (d/dt)y_3^v(t) = f_3(y_1^v(t), y_2^v(t), y_3^v(t)) \end{cases} \qquad (4.24)$$

where $y_\alpha^v(t) = \alpha_v^t(\sigma_\alpha^m)$ and where the functions $f_\alpha(x,y,z)$ are polynomials, whose explicit form can be deduced directly from (4.6), and therefore they are analytical entire functions.
We have to consider together with (4.24) the intial conditions:

$$y_i^v(t_o) = y_{i_o} \quad , \quad \underline{y}_o = (y_{1_o}, y_{2_o}, y_{3_o}) \qquad (4.25)$$

This is a typical example of a system of differential equations in normal form on a Banach algebra. We can therefore use theorem 2.2.2 , [10], that guarantees the existence and the unicity of the vectorial solution $y(t;t_o,\underline{y}_o)$ of the above system.
This solution is defined and holomorphic in the disk : $|t-t_o| \leq r$ where

$$r < \min\{ a, b_1/M, b_2/M, b_3/M, 1/k\} \qquad (4.26)$$

The above coefficents are determined by the following conditions:

i) Let us call $\underline{F} = (f_1, f_2, f_3)$. $\underline{F}$ is defined in a cylinder 3+1 dymensional:

$$D: |t - t_o| \leq a \qquad |y_i - y_{i_o}| \leq b_i \quad i=1,2,3$$

ii) $M=\max\{M_i\}$, where the M_i are the sups $|f_i(t,y)|$, with t and y running in D

iii) k is obtained from Lipschitz conditions

$$|f_i(t,\underline{y}) - f_i(t,\bar{\underline{y}})| < k\,|\underline{y} - \bar{\underline{y}}|$$

What we get for our present condition is that:

$a = b = +\infty$ becouse of the polynomial nature of our f ;
M is a finite quantity being $f_i(t,y)$ continuous in D and therefore finite in every compact region of D;

k can be small as we like becouse of the continuity of the f.

The result is therefore that the solution $y(t;t_0,y_0)$ is holomorphic in a dysc D : $|t - t_0| < +\infty$, so y is an entire function.
If we now substitute $y(t;t_0,y_0)$ in equation (4.7) we get a differential equation whose rhs is again an entire function depending on an external parameter, y_0.
We use again ref.[10], theorem 2.8.5, that ensures us the existence and the unicity of the solution, which is again analytical entire.

(2)
The more general form of theorem 1 we will now prove is:

-If $x_t^\nu = X(t,x,y^\nu)$ is a formal norm-converging power series and if y^ν converges to y in the ultrastrong topology, then x_t^ν converges in the ultrastrong topology to $x_t = X(t,x,y)$. Moreover if x_t^ν is the solution of equation (4.7) then x_t is the solution of

$$\text{u.s.} \lim_{\nu \to \infty} (dx_t^\nu/dt) = \text{u.s} \lim_{\nu \to \infty} f(x_t^\nu, y_t^\nu) \quad \text{,that is}$$

$$dx_t/dt = f(x_t, y_t) \tag{4.27}$$

We first recall that if y^ν converges ultrastrongly to y then uniformly in V we have $\| y^\nu \| \leqslant C$ (C is a constant).
Moreover also the power n of y^ν converges ultrastrongly to $(y)^n$ for every n.
So if $p_j(y^\nu)$ is a polynomial in y^ν of the j-th degree we have

$$p_j(y^\nu) \xrightarrow{u.s.} p_j(y) \tag{4.28}$$

Since $X(t,x,y^\nu)$ is analitycal in norm it can be approximated as well as we want by polynomials. That is, expliciting only the relevant variable y^ν,

$$X(y^\nu) = \text{norm} \lim_{j \to \infty} p_j(y^\nu) \qquad p_j(y^\nu) = \sum_{k=0}^{j} a_k (y^\nu)^k$$

Let us indicate with $\| \cdot \|_{(\varphi_i)}$ a seminorm of the ultrastrong topology. Obviously becouse of the norm convergence of p_j to X we have also its ultrastrong convergence.
In order to prove that $X(y^\nu)$ converges to X(y) we study the following quantity:

$$\| X(y^\nu) - X(y) \|_{(\varphi_i)} = \| X(y^\nu) - p_j(y^\nu) + p_j(y^\nu) - p_j(y) + p_j(y) - X(y) \|_{(\varphi_i)} \leq$$

$$\| X(y^{\vee})-p_j(y^{\vee})\|_{(\varphi_i)} + \| p_j(y^{\vee})-p_j(y)\|_{(\varphi_i)} + \| p_j(y)-X(y)\|_{(\varphi_i)}$$

Now we note that:
-the first term in the last line goes to zero becouse of the norm convergence of $p_j(y^{\vee})$ to $X(y^{\vee})$ (which implies also its ultrastrong convergence);
-the second term goes to zero becouse of the (4.28);
-the third term is zero for the same reasons discussed for first one.

This proves the first part of the theorem showing that in fact $X(y^{\vee})$ converges ultrastrongly to $X(y)$.
For what concerns the second part we compute the limit below:

$$\text{u.s.} \lim_{V,\infty} dx_t^{\vee}/dt = \text{u.s.} \lim_{V,\infty} f(x_t^{\vee}, y_t^{\vee}) \tag{4.29}$$

We have just proven that if $y^{\vee} \longrightarrow y$ then $x_t^{\vee} \longrightarrow x_t$. An immediate consequence is that in this hypothesis , $y^{\vee} \longrightarrow y$, we have

$$\text{u.s} \lim_{V,\infty} f(x_t^{\vee}, y_t^{\vee}) = f(x_t, y_t)$$

The lhs of (4.29) it is showed to give

$$\text{u.s.} \lim_{V,\infty} dx_t^{\vee}/dt = d/dt \ \text{u.s.} \lim_{V,\infty} x_t^{\vee} = dx_t/dt$$

This is due to the analitycity of $X(t,x,y^{\vee})$ and to the ultrastrong convergence of $y^{\vee}$. In fact we can easily prove that $dx_t^{\vee}/dt$ is uniform in V:

$$\text{norm} \lim_{\varepsilon,0} (x_{t+\varepsilon}^{\vee} - x_t^{\vee})/\varepsilon = \lim_{\varepsilon,0} 1/\varepsilon \ \| X(t+\varepsilon, x, y^{\vee}) - X(t,x,y^{\vee}) \| =$$
$$= \lim 1/\varepsilon \ \| \sum_{n=0}^{\infty} (C_n(t+\varepsilon) - C_n(t))(y^{\vee})^n \| \leq \sum_n \| \dot{C}_n(t) \| \ \| y^{\vee} \|^n \leq$$
$$\leq \sum_n \| \dot{C}_n(t) \| \ C$$

which is independent on V , as we needed to prove. This uniformity lets one changes the limit in V with the time derivative giving the desidered result.

(3)
We have just proven that (4.29) is nothing but the differential equation for the algebraic dynamics α^t. Shifting back to spin language we get:

$$d/dt \ \alpha^t(\sigma) = f(\alpha^t(\sigma), \alpha^t(\sigma^{\infty})) \tag{4.30}$$

This equation allows us to prove that

$$\alpha^t \alpha^\tau = \alpha^{t+\tau} \tag{4.31}$$

and therefore that α^t define a one parameter group of automorphisms of M. In fact we have

$$d/dt\ \alpha^{t+\tau}(\sigma) = f\,(\alpha^{t+\tau}(\sigma),\alpha^{t+\tau}(\sigma^\infty))$$

$$d/dt\ \alpha^t\alpha^\tau(\sigma) = d/dt\ \alpha^t[\alpha^\tau(\sigma)] =$$

$$= d/dt\ f(\alpha^t[\alpha^\tau(\sigma)],\alpha^t[\alpha^\tau(\sigma^\infty)]) = d/dt\ f(\alpha^t\alpha^\tau(\sigma),\alpha^t\alpha^\tau(\sigma^\infty))$$

This means that $\alpha^{t+\tau}(\sigma)$ and $\alpha^t\alpha^\tau(\sigma)$ satisfy the same equation of motion. But we know, for what already has been discussed in point (1), that when we fix the initial conditions the solution of the equation is unique, due to the analitycity of f.
Since in t=0

$$\alpha^{t+\tau}(\sigma)\Big|_{t=0} = \alpha^t\alpha^\tau(\sigma)\Big|_{t=0} = \alpha^\tau(\sigma)$$

then $\alpha^{t+\tau}(\sigma)$ and $\alpha^t\alpha^\tau(\sigma)$ must be the same function. So equation (4.31) is verified.

(4)
Equations (4.22) and (4.23) can be easily obtained simply by imposing condition (4.14), or the equivalent one

$$\phi(d/dt\,\alpha^t(\sigma^\infty)) = 0$$

We use (4.16) and therefore we get

$$d/dt\,\alpha^t(\sigma^\infty_\delta) = -4\,\varepsilon_{\alpha\delta\gamma}\,A^s_{\alpha\beta}\,\alpha^t(\sigma^\infty_\gamma)\,\alpha^t(\sigma^\infty_\beta) - 2\,C_\alpha\,\varepsilon_{\alpha\delta\gamma}\,\alpha^t(\sigma^\infty_\gamma)$$

Since $\phi(\sigma^\infty_\gamma) = n_\gamma$ and $\phi(\alpha^t(\sigma^\infty_\alpha)) = \phi(\alpha^t_\pi(\sigma^\infty_\alpha))$ we obtain

$$(2\,A^s\,\underline{n} + \underline{C}) \wedge \underline{n} = 0 \tag{4.32}$$

The solutions are of two kinds:

$$\left.\begin{cases} 2\,A^s\,\underline{n} + \underline{C} = 0 \\ 2\,A^s\,\underline{n} + \underline{C} = \lambda\,\underline{n} \end{cases}\right. \tag{4.33}$$

And from these equation one easily gets the conditions (4.22) and (4.23).

CHAPTER V : KMS CONDITION AND FINITE TEMPERATURE STATES

This Chapter is devoted to discuss the peculiarities of the spin models introduced in the previous Chapter at a non-zero temperature. We will show what of the framework previously discussed can be extended to the T>0 situation. Moreover the existence of a phase transition will be proven in all details.
We will make extensive use of the powerful algebraic tool given by the KMS states which we will briefly discuss in the first paragraph.

V.1 KMS CONDITION

We discuss here some characteristic of KMS states. For more details we refer to [6] and [11] as a first reading and to [12] for a more sofisticate review of the subject.
We start defining an equilibrium state for a finite volume V lattice by:

$$\omega_V(A) = \mathrm{tr}\,(\rho_V A)\ , \qquad \rho_V = \exp(-\beta H_V)/\ \mathrm{tr}(\exp(-\beta H_V))$$

where $\beta = (KT)^{-1}$.
This is the state of our lattice corresponding to the Canonical Ensemble at temperature T.
Some problem arises when we try to take the thermodynamical limit since as it is clear from the previous Chapters the Hamiltonian H_V may not be well defined in this limit, being representation dependent since variables at infinity in general appear.
This problem is usually overcome introducing, instead of H_V, the relative cutoffed dynamics α_V^t, $\alpha_V^t(A) = \exp(iH_V t)\ A\ \exp(-iH_V t)$.
Using the definition of ω_V and the fact that the trace is invariant uncer cyclic permutations we can prove that

-If $A,B \in \mathcal{Q}_V$ then $\omega_V(\alpha_V^t(A)B)$ can be extended to an entire function of a complex variable t which is uniformly bounded in $-\beta \leq \mathrm{Im}\ t \leq 0$; similarly $\omega_V(B\,\alpha_V^t(A))$ can be extended to an entire function of t which is uniformly bounded in $0 \leq \mathrm{Im}\ t \leq -\beta$.
Moreover we have

$$\omega_V(\alpha_V^t(A)B) = \omega_V(B\alpha_V^{t+i\beta}(A)) \qquad -\beta \leq \mathrm{Im}\ t \leq 0 \qquad (5.1)$$

In ref.[6] is showed that if α_V^t converges to α^t in the norm topology and if $\omega_V(A) \to \omega(A)$, $\forall\ A \in \mathcal{Q}_V$, we can derive from (5.1), without making reference to the Gibbs states, the KMS condition:

$$\omega(\alpha^t(A)B) = \omega(B\alpha^{t+i\beta}(A)) \qquad -\beta \le \mathrm{Im}\, t \le 0 \qquad (5.2)$$

States which satisfy equation (5.2) are equilibrium states for the infinite lattice at the temperature $T = (\beta K)^{-1}$. They look very much the same as the ground state for zero temperature. This analogy is showed, for instance, noticing that, just like the ground state, also KMS states are stationary, i.e. invariant under α^t.

KMS states have also a clear physical interpretation: let us consider a finite system S coupled with an infinite system W at t=0, when W is in the state f. One can prove that, under some technical requirement, S is taken to the Gibbs state at β^{-1} temperature iff f satisfies equation (5.2). This means that W behaves like a thermostat a β^{-1} temperature.

Even if in our models norm convergence of α_V^t is not verified and therefore (5.2) cannot be deduced from (5.1), we will take (5.2) as the definition of equilibrium state for infinite system as its physical interpretation suggests to do.

Under some technical assumptions that need no to be emphasized here, see [11], one can deduce that the space average of an observable (ergodic mean) exist and is a c-number on the KMS state ω.

Therefore the mechanism of essential localization of the dynamics can be applied also for representations π constructed starting by KMS states via GNS construction (see app.2). The reduced dynamics obtained maps $\mathcal{Q}_\ell$ in $\mathcal{Q}_\ell$.

Since we know from the discussion in Chapter III on effective localization that if $\phi \in \pi$ then $\phi(\alpha^t(A)) = \phi(\alpha^t_\pi(A))$ we can implement (5.2) using α^t_π instead of α^t:

$$\omega(\alpha^t_\pi(A)B) = \omega(B\alpha^{t+i\beta}_\pi(A)) \qquad (5.3)$$

We note that even if the above equation widely simplifies the explicit calculation it also present a drawback: it is valid only if the π representation is the one GNS related to the state ω. If we use in (5.3) another reduced dynamics we obviousy get wrong results and therefore some care is required.

V.II SYMMETRY BREAKING AND PHASE TRANSITION

In order to implement equation (5.3) we have to find the explicit time dependence of σ_α^i. We have already discussed in the previous Chapter the main problems related to the thermodynamical

limit for our mean field spin models and we have showed that an algebraic dynamics can be defined even if we are dealing with a LRI since α_V^t converges (ultra) strongly to α^t.
The dynamics so obtained satisfies the same equation of motion that one gets using the following effective Hamiltonian:

$$\begin{cases} H = \underline{F} \cdot \sum_i \underline{\sigma}^i & (5.4) \\ F_\alpha = 2\sigma_\beta^\infty \, A_S^{\alpha\beta} + C_\alpha & (5.5) \end{cases}$$

HW $\qquad A_S^{\alpha\beta} = J\,\delta^{\alpha\beta} \qquad C_\alpha = B\,\delta_{\alpha 3} \qquad (5.6)$

BCS $\qquad A_S^{\alpha\beta} = -\,T_c/2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad C_\alpha = -\varepsilon\delta_{\alpha 3} \qquad (5.7)$

We can use Hamiltonian (5.4) to deduce the time evolution of the spin variables. It turns out to be convenient to use two different explicit form of the dynamics for the two models since otherwise some problem in understanding the final results may arise for HW model.
for BCS we obtain

$$\alpha^t(\sigma_\alpha^i) = \exp(iHt)\,\sigma_\alpha^i\,\exp(-iHt) = \cos^2(Ft)\,\sigma_\alpha^i + i\,\sin(Ft)\,\cos(Ft)\cdot$$
$$\cdot\,\underline{F}/F\,[\underline{\sigma}^i, \sigma_\alpha^i] + 1/F^2\,\sin^2(Ft)\,(-F^2\sigma_\alpha^i + 2F_\alpha(\underline{F}\cdot\underline{\sigma}^i)) \qquad (5.8)$$

while for HW model with B=0, which is the one that will be considered in the following

$$\alpha^t(\sigma_\alpha^i) = \exp(iJ\,\underline{\sigma}^\infty \cdot \sum_i \underline{\sigma}^i t)\;\sigma_\alpha^i\;\exp(-iJ\,\underline{\sigma}^\infty \sum_i \underline{\sigma}^i t) =$$
$$= \cos^2(J|\underline{\sigma}^\infty|\,t)\,\sigma_\alpha^i + i\sin(J|\underline{\sigma}^\infty|\,t)\,\cos(J|\underline{\sigma}^\infty|\,t)\frac{\underline{\sigma}^\infty}{|\underline{\sigma}^\infty|}[\underline{\sigma}^i, \sigma_\alpha^i] +$$
$$+ \frac{1}{|\underline{\sigma}^\infty|^2}\sin^2(J|\underline{\sigma}^\infty|\,t)\;(-|\underline{\sigma}^\infty|^2\sigma_\alpha^i + 2\sigma_\alpha^\infty(\underline{\sigma}^\infty \cdot \underline{\sigma}^i)) \qquad (5.9)$$

We have used the algebra of the Pauli matrices and the shorthand notation $\sigma_\alpha^i = \sigma_\alpha^i\,(t=0)$, $F = |\underline{F}|$.
The effective dynamics α_π^t related to the representation π generated by the KMS state ω via GNS construction is obtained substituting in (5.8) and (5.9)

$$\begin{aligned} \sigma_\alpha^\infty &\longrightarrow \qquad m_\alpha = \pi\,(\sigma_\alpha^\infty) \\ \underline{F} &\longrightarrow \qquad \underline{f} = \pi\,(\underline{F}) = 2A^S\,\underline{m} + \underline{C} \end{aligned} \qquad (5.10)$$

Therefore we get for BCS

$$\alpha^t_\pi(\sigma^i_\alpha) = \cos^2(ft)\,\sigma^i_\alpha + i\,\sin(ft)\,\cos(ft)\,\underline{f}/f\,[\underline{\sigma}^i,\sigma^i_\alpha] + $$
$$+\,1/f^2\,\sin^2(ft)\,(-f^2\sigma^i_\alpha+2f_\alpha(\underline{f}\cdot\underline{\sigma}^i)) \tag{5.11}$$

and for HW

$$\alpha^t_\pi(\sigma^i_\alpha) = \cos^2(Jmt)\,\sigma^i_\alpha + i\,\sin(Jmt)\,\cos(Jmt)\,\underline{m}/m\,[\underline{\sigma}^i,\sigma^i_\alpha] + $$
$$+\,1/m^2\,\sin^2(Jmt)\,(-m^2\sigma^i_\alpha+2m_\alpha(\underline{m}\cdot\underline{\sigma}^i)) \tag{5.12}$$

We use the above results in equation (5.3), taking for simplicity $t=0$ and $A=B=\sigma^i_\alpha$ (we will omit the unessential index i in the following computation).
Using the relations:

$$\omega(\sigma_\alpha\sigma_\alpha) = 1 \qquad \sin iz = i\sinh z$$

$$\cos iz = \cosh z \qquad \cosh^2 z - \sinh^2 z = 1$$

condition (5.3), which reads

$$\omega(\sigma_\alpha\sigma_\alpha) = \omega(\sigma_\alpha\,\alpha^{i\beta}_\pi(\sigma_\alpha)) \tag{5.13}$$

gives hyperbolic equations which depend on the model.

HW MODEL
Equation (5.13) after some minor manipulation gives the following condition

$$\mathrm{tgh}(Jm\beta) = -m \tag{5.14}$$

for each $\alpha=1,2,3$. This is solved by $m = 0$. However also a not trivial solution, $m = 0$, is allowed iff $J < 0$ and $-J\beta > 1$.
In these conditions we can define a critical temperature T_K, in unity $K = 1$, by the

$$T_K = -J \tag{5.15}$$

We conclude that for $T < T_K$ KMS condition gives a magnetization different fom zero while $m = 0$ is the only allowed solution for $T > T_K$. A phase transition therefore arises at the critical temperature.

BCS MODEL

The situation is a little bit more involved here essentially becouse we are dealing with an anisotropic situation.

The KMS condition can be simplified noticing that Hamiltonian (5.4) describes rotations of the spin $\underline{\sigma}^{\iota}$ around the vector $\underline{F}$ and therefore (Stationarity Condition) its mean value, $\underline{\sigma}^{\infty}$, has to be alligned with F itself:

$$\underline{F} = \mu\, \underline{\sigma}^{\infty} \qquad \text{and then} \qquad \underline{f} = \mu\, \underline{m} \tag{5.16}$$

Equation (5.13) with the above condition gives

$$(1 - f_{\alpha}^{2}/f^{2})\ \mathrm{tgh}(f\beta) = -1/f\ [f_1 m_1 (1-\delta_{\alpha 1}) + f_2 m_2 (1-\delta_{\alpha 2}) + f_3 m_3 (1-\delta_{\alpha 3})] \tag{5.17}$$

We will show in a while that even now a critical temperature, T_K, can be defined and that the conclusions are very similar to those of the previous model:

a spontaneous breaking of the symmetry is allowed for $T < T_K$ and a mass gap appears while for $T > T_K$ the only solution admitted is $m_1 = m_2 = 0$, m_3 arbitrary, and therefore no order parameter arises and no Goldstone Theorem can be used.

Equation (5.17) gives no information if α is taken in the same direction of the $\underline{f}$ vector. If we take α in the plane perpendicular to $\underline{f}$ we obtain

$$\mathrm{tgh}(f\beta) = -(\underline{f}\cdot\underline{m})/f \tag{5.18}$$

which has to be solved consistently with condition (5.16). This one together with definition (5.5) gives two possible conditions:

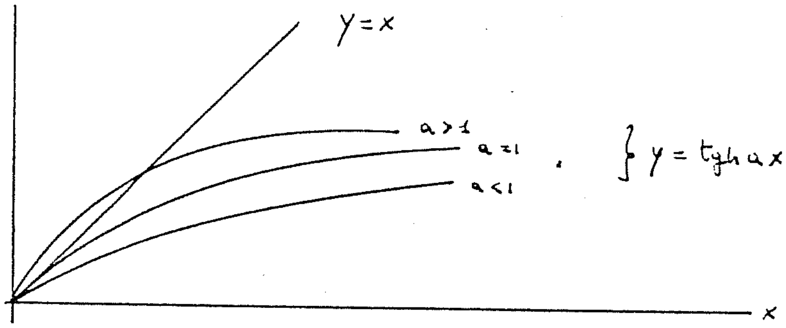
1. if $m_1 \neq 0$ ($m_2 \neq 0$ automatically) then $\mu = -T_c$ and $m_3 = \varepsilon/T_c$
2. if $m_1 = m_2 = 0$ then μ is not fixed and $m_3 = -\varepsilon/\mu$.

If we implement condition 1. equation (5.18) becames

$$\mathrm{tgh}(f\beta) = f/T_c \tag{5.19}$$

which admits a non zero solution, the relevant one, only if $\beta > 1/T_c$, that is for $T < T_c$.

Self-consistence conditions, however, cause some minor change in the critical temperature. In fact in order to ensure that $m_{1,2} \neq 0$ $T < T_c$ is not a sufficient condition. We see it if we plot the functions y=tgh (ax) and y=x, $a = \beta/\beta_c$, $x = f\beta_c$, $\beta_c = 1/T_c$.



We see that for a $\leq$ 1 the only solution of the equation tgh(ax) = x is x=0 and for a a bit greater than 1 the solution x is almost zero.
Recalling the definition of f and calling $\bar{f}$ the solution of the KMS condition (which surely exist for T < T_c) we see that

$$\bar{m}_1^2 + \bar{m}_2^2 = (\bar{f}^2 - \varepsilon^2)/\ T_c^2$$

and therefore $m_{1,2} \neq 0$ only if $\bar{f}^2 > \varepsilon^2$. It follows that for a certain range 1 < a < $\bar{a}$, $\bar{a}$ to be determined, this self-consistence check is not verified and therefore we must have $m_1 = m_2 = 0$, that is we go back to condition 2 above. We must therefore have a > $\bar{a}$ and so the critical temperature T_K is not equal to T_c but is obtained by $\bar{a}$ which is defined by asking that $\bar{f}^2 = \varepsilon^2$. This gives

$$T_K = T_c/\ \bar{a} \tag{5.19}$$

which shows that T_K is less than T_c, being $\bar{a}$ > 1.
For a < $\bar{a}$, which includes also T > T_c, the only solution is $m_1 = m_2 = 0$ and therefore no symmetry breaking is allowed. Only a magnetization in the direction of the magnetic field appears in this range of temperature (which moreover turns out to be continuous at the critical temperature).

We still have not calculated all correlation functions, which are of the form $\omega\,(\sigma_\alpha^i\,\sigma_\beta^j \ldots)$. Following ref.[11] we will consider the following expression:

$$\omega(AO)/\omega(O) \qquad A \in \mathfrak{A}_i\ ,\ O \in \mathfrak{A}_\Lambda \qquad \text{with } i \notin \Lambda$$

(i is a lattice site which does not belong to the lattice volume Λ).
If O is positive (that is it can be written as $O = C^* C$, for some $C \in \mathfrak{A}$) the state Ω definite by

$$\Omega(A) = \omega(AO)/\omega(O)$$

is a state on $\mathfrak{A}$ which satisfies KMS condition. Since this condition applied to a finite system determines the state univocally we deduce that $\Omega \doteq \omega$. Therefore

$$\omega(AO)/\omega(O) = \omega(A) \quad \text{and then} \quad \omega(AO) = \omega(A)\ \omega(O)$$

We conclude that for different lattice sites we can write

$$\omega\ (\sigma_\alpha^{i_1}\ \sigma_\beta^{i_2}\ \ldots\ldots) = m_\alpha m_\beta \ldots\ldots \tag{5.20}$$

that determines ω completely.

The last point of this Chapter consists in applying the Generalized Goldstone Theorem for this T > 0 situation. We simply substitute the ground state Φ_0 in (2.9) with the KMS state ω :

$$J(t) = d/d\lambda\ \omega(\ \beta^\lambda \alpha_\pi^t (A_\infty))\Big|_{\lambda=0} \tag{5.21}$$

The motion of the variables at infinity is derived by (5.11) (or by (5.12) for HW model) and we can put it in the form:

$$\alpha_\pi^t(\sigma_\alpha^\infty) = \cos(2ft)\ [\ \sigma_\alpha^\infty - (f_\alpha/f^2)\,\underline{f}\cdot\underline{\sigma}^\infty] + (f_\alpha/f^2)\,\underline{f}\cdot\underline{\sigma}^\infty +$$

$$-1/f\ \sin(2ft)\ \ (f_1 \varepsilon_{1\alpha\beta} + f_2 \varepsilon_{2\alpha\beta} + f_3\ \varepsilon_{3\alpha\beta}\ ) \tag{5.22}$$

that shows the existence of a frequence

$$\omega = 2f \tag{5.23}$$

which is different from zero for both models for $T < T_K$. For $T > T_K$ the HW model gives f=0 and therefore $\omega = 0$. For BCS we know that is not possible to break the symmetry (for $T > T_K$) in the allowed representations. Therefore no Goldstone mechanism can be advocated.
This result is in agreement with the result of the Landau analysis which shows that superconducting phoenomenon is allowed only if an energy gap appears in the energy spectrum.

CHAPTER VI : ISING MODEL WITH LRI

In this last Chapter we will discuss a spin model with LRI J_{ij}, assumed not to be integrable so that condition of ref.[6] is not verified. The Hamiltonian is

$$H_V = -\sum_{ij\in V} J_{ij} \, (\sigma_3^i - \bar{\sigma})(\sigma_3^j - \bar{\sigma}) \tag{6.1}$$

We have already discussed in paragraph I.2 the likeness between this Hamiltonian and the one that is actually believed to describe FQHE. σ_3^i is related to the occupation number of the original j-th electron while $\bar{\sigma}$ is an uniform neutralizing background. Hence we can think that a good control of this model is a first step in understanding FQHE.

VI.1 GENERAL PROBLEMS AND RESULTS

Because of the long range of the interaction we cannot define the algebraic dynamics α^t as a norm limit of α_V^t ,[6]. This is clearly due to the presence of a variable at infinity, $\bar{\sigma}$, in the equation of motion itself and we know that such variable cannot be norm converging. We have to use a weaker topology defined by a family F of relevant states which must be regular at large distances in order to get a well defined thermodynamical limit of the cutoffed dynamics.
The allowed form of J_{ij} will appear to be strictly connected with the definition of the family F. It is discussed in the following that if we take F as the set of states pointing (at large distances) in the z direction then no limitation needs to be introduced on J_{ij} . On the countrary if we assume that the states of F point in the generic $\underline{n}$ direction then α^t can be defined only if the potential is square-integrable.
These two different situations will be analyzed with different techniques which, however, turn out to be equivalent for what concerns the convergence discussion in the sense that both are sufficient to guarantee the group nature of α^t .
It is worthwhile to stress here that $\bar{\sigma}$ in H is really a variable at infinity, σ_3^∞ , as it will follow from the definition of the F family. We write it in this form only for simplicity of notation but we must have some care since $\bar{\sigma}$ is actually an operator. Nevertheless in each factorial representation this variable will take a definite expectation value and therefore it will become a c-number.

We also want to notice that this model is, as long as we know, the first spin model with LRI (not mean field) which has been exactly solved.

We suppose that in (6.1) the following conditions hold:

1) J_{ij} is not integrable
2) $|\bar{\sigma}| \leq 1$
3) $J_{ij} = J_{ji}$ and $J_{ii} = 0$ (6.2)

The equation of motion for α_V^t can be found using the canonical commutation relations for spin matrices and we get

$$d/dt\ \alpha_V^t(\sigma_\alpha^k) = T_k^V\ \varepsilon_{3\alpha\beta}\ \alpha_V^t(\sigma_\beta^k) \tag{6.3}$$

where we have defined

$$T_k^V = 4\sum_{i\in V} J_{ik}\,(\sigma_3^i - \bar{\sigma}) \tag{6.4}$$

System (6.3) can be easily solved noticing that T_k^V does not depend on time. This is due to the fact that σ_3^i is time independent as it is showed by (6.3) itself, so that we can integrate the equations of motion getting

$$\begin{cases} \alpha_V^t(\sigma_1^k) = \cos(T_k^V t)\ \sigma_1^k(0) + \sin(T_k^V t)\ \sigma_2^k(0) \\ \alpha_V^t(\sigma_2^k) = \cos(T_k^V t)\ \sigma_2^k(0) - \sin(T_k^V t)\ \sigma_1^k(0) \\ \alpha_V^t(\sigma_3^k) = \sigma_3^k(0) \end{cases} \tag{6.5}$$

It is however convenient to express these solutions in terms of σ_3^k, $\sigma_+^k = \sigma_1^k + i\,\sigma_2^k$, $\sigma_-^k = \sigma_1^k - i\,\sigma_2^k$. In fact we will use these variables to show the existence of the dynamics when the thermodynamical limit is taken. The equations of motion and their solution look now:

$$d/dt\ \alpha_V^t(\sigma_\pm^k) = \mp\, i\, T_k^V\, \alpha_V^t(\sigma_\pm^k)\ ; \qquad d/dt\ \alpha_V^t(\sigma_3^k) = 0 \tag{6.6}$$

and

$$\begin{aligned} \alpha_V^t(\sigma_+^k) &= \exp(-iT_k^V t)\ \sigma_+^k(0) \\ \alpha_V^t(\sigma_-^k) &= \exp(+iT_k^V t)\ \sigma_-^k(0) \\ \alpha_V^t(\sigma_3^k) &= \sigma_3^k(0) \end{aligned} \tag{6.7}$$

As we have already discussed we are interested in the case in which J_{ij} is not integrable since otherwise ref.[6] ensures the existence of α^t as a norm limit of α_V^t .

In order to define a one parameter group of automorphisms α^t of $M = \mathcal{Q}^{c_F}$ it is necessary to require strong convergence of T_k^V on a dense set of analytical vectors (see eq.(6.17) below for the definition of such vectors) or ultrastrong convergence of $\exp(\pm iT_k^V t)$.
This difference is due to the fact that T_k^V is not a bounded operator (since J_{ij} is not integrable) and therefore ultrastrong convergence cannot be asked for (see app.1) while $\exp(\pm iT_k^V t)$ is bounded and hence can be ultrastrongly convergent.

States of F are defined by the condition:

$$\forall \phi \in F \qquad \lim_{V \uparrow \infty} \phi(T_k^V) = \phi(T) < +\infty \tag{6.8}$$

which is satisfied requiring that

$$J_{ik}\, \phi(\sigma_3^i - \bar{\sigma}) \quad \text{is integrable.} \tag{6.9}$$

In particular if $J_{ik} \sim |i-j|^{-\alpha}$ a sufficient condition for (6.8) is

$$\phi(\sigma_3^i - \bar{\sigma}) \sim 1/|i|^{3-\alpha+\varepsilon} \qquad \alpha \leqslant 3, \quad \varepsilon > 0, \quad i > i_0 \tag{6.10}$$

As it is obvious from (6.10) condition (6.8) is a demand of sufficently regularity at infinity of the F states since for i going to infinity we get $\phi(\sigma_3^i) \sim \bar{\sigma}$.
We would like to recall that states $\phi \in F$ can be considered as (quasi) local perturbations of states $\phi^{\vec{n}}$ which are invariant under space translations. So rewriting (6.10) in terms of these states we obtain:

$$\phi^{\vec{n}}(A(\sigma_3^i - \bar{\sigma})B) \sim 1/|i|^{3-\alpha+\varepsilon} \qquad A,B \in \mathcal{Q} \tag{6.11}$$

We will show in the next paragraph that on these states it results

$$\lim_{V \uparrow \infty} \phi(1/|V| \sum_{i \in V} \sigma_3^i) = \phi(\sigma_3^\infty) = \bar{\sigma} \tag{6.12}$$

so that ,as we already pointed out before, the equations of motion are really affected by one variable at infinity.

It is worthwhile to note that even for this model the strategy developped in Chapter IV for proving the existence of the solution of the equation of motion could be used.

For simplicity of notation we write

$$d/dt\, \alpha_V^t(\sigma_\alpha^k) = X_\alpha(\alpha_V^t(\sigma_\beta^k), T_k^V) \tag{6.13}$$

$$\alpha_V^t(\sigma_\alpha^k) = F_\alpha(t, T_k^V, \sigma_\alpha^k(0)) \tag{6.14}$$

with both F_α and X_α analytical entire.
We deal now separately with two different situations that naturally appear, as we advanced before:
the one in which we study the behaviour of the limit for states in the z direction and the one in which the states point in a generic n direction.
As in Chapter IV the mathematical details will be given in a separate section.

<u>z-pointing states</u>
We can show that a sufficient condition that ensures strong convergence of T_k^V to T_k is

$$\| (\sigma_3^i - \bar{\sigma})\Psi \| \leq 1/|i|^{3-\alpha+\varepsilon} \quad i > i_o \tag{6.15}$$

which implies

$$\bar{\sigma} = \pm 1 \tag{6.16}$$

This means that if $\bar{\sigma}$ =+1 then the vector Ψ has to be a (quasi) local perturbation of the vector

$$\Psi_o^{up} = \uparrow \otimes \uparrow \otimes \uparrow \otimes \uparrow \; \ldots\ldots\ldots = \alpha_1 \alpha_2 \alpha_3 \; \ldots\ldots$$

where α_i are the eigenstate of σ_3^i with eigenvalue +1.
On the countrary if $\bar{\sigma}$ =-1 then the vectors which satisfy (6.15) are perturbations of

$$\Psi_o^{down} = \downarrow \otimes \downarrow \otimes \downarrow \otimes \downarrow \; \ldots\ldots\ldots = \beta_1 \beta_2 \beta_3 \; \ldots\ldots$$

Since T_k^V is not bounded we cannote require its ultrastrong convergence. However we can prove, and we will prove it in the next paragraph, that T_k^V converges strongly on a dense set D of analytical vectors which is defined by:

$$\begin{cases} D = \{ \Psi : \; \| (T_k^V)^n \Psi \| \leq C^n \} , & C \text{ independent of } V \\ \text{We write} \quad D\text{-}s \lim\limits_{V \uparrow \infty} T_k^V = T_k \end{cases} \tag{6.17}$$

It will be showed in the next paragraph that this is a sufficient condition for the existence of the D-strong limit of α_V^t defined in equation (6.14):

$$\alpha^t(\sigma_\alpha^k) = F_\alpha(t, T_k, \sigma_\alpha^k(0)) \tag{6.18}$$

This function is the solution of the infinite volume limit of the equation of motion (6.13), that is

$$d/dt\, \alpha^t(\sigma_\alpha^k) = X_\alpha(\alpha^t(\sigma_\beta^k), T_k) \tag{6.19}$$

The algebraic dynamics α^t defined by (6.18) satisfies the group property

$$\alpha^t \alpha^\tau = \alpha^{t+\tau} \tag{6.20}$$

We stress here that:

-<u>No condition needs to be imposed on J</u>
The strong limit of equation (6.5) gives:

$$\begin{cases} \alpha^t(\sigma_1^k) = \cos(Tt)\, \sigma_1^k(0) + \sin(Tt)\, \sigma_2^k(0) \\ \alpha^t(\sigma_2^k) = \cos(Tt)\, \sigma_2^k(0) - \sin(Tt)\, \sigma_1^k(0) \\ \alpha^t(\sigma_3^k) = \sigma_3^k(0) \end{cases} \tag{6.21}$$

where

$$T = \text{D-s} \lim_{V,\infty} T_k^V \; (= T_k) \tag{6.22}$$

(Of course T in the thermodynamical limit is independent of the k index for symmetry reason)
We can write system (6.21) in the form

$$\alpha^t(\sigma_\alpha^k) = (\, R_z^T(t)\, \sigma^k(0))_\alpha \tag{6.23}$$

where the matrix R describes rotations around the z axis with frequency T.
Each factorial representation π_z is defined here by a state ϕ^z in which $\bar{\sigma}$ can be frozen only to the values $\pm$ 1. The only continue symmetries of our Hamiltonian are given by the rotations around the z axis and therefore they cannot be broken in any representation π_z. In fact we have always

$$\alpha_{\pi_z}^t \beta^\lambda = \beta^\lambda \alpha_{\pi_z}^t \tag{6.24}$$

Thus no Goldstone mechanism can be applied and no Goldstone boson, massless or massive can be predicted on general ground.

$\vec{n}$-pointing states

The situation is more complex for states $\phi^{\vec{n}}$ with $\vec{n}$ not aligned with the z axis. Moreover it turns out also to be more interesting since in the representations constructed with these states order parameters appear and therefore the symmetry is broken.

We have previously obtained that a sufficient condition for the existence of the dynamics is given by (6.15) which implies (6.16) and therefore that the only relevant states are the ones pointing in the z direction. However it is possible to search for some other characterization which permits the existence of α^t. This is in fact possible if we make an assumption on the nature of the potential which has to be square-integrable. In this situation $\bar{\sigma}$ needs not to be equal to $\pm$ 1 but is given by

$$\bar{\sigma} = \cos \theta \tag{6.25}$$

where θ is one of the angles defining the $\vec{n}$ vector.

We will prove in the next paragraph that the cutoffed dynamics defined by (6.7) is ultrastrongly convergent if we assume

$$\sum_i J_{ij}^2 < \infty \tag{6.26}$$

and of course the thermodynamical limit so obtained satisfies the group law.

The main steps of the derivation of this result are the followings:

1) first of all we prove that the exponentials appearing in (6.7), $\exp(\pm i T_k^V t)$, converge on the vector $\Psi_{\vec{n}}$ invariant under space translations if condition (6.26) is satisfied;
2) Once the above result is obtained we prove that $\exp(\pm i T_k^V t)$ converges also on $A_\alpha \Psi_{\vec{n}}$, $A_\alpha \in \mathfrak{A}$ and, also on its closure, $\overline{A_\alpha \Psi_{\vec{n}}}$;
3) Using the norm limitation of $\exp(\pm i T_k^V t)$ we conclude that these exponentials are ultrastrongly convergent.

(in the same fashion one can prove that also the polynomials in σ are ultrastrongly convergent)

Therefore we conclude that α_V^t converges ultrastrongly to α^t . Every $\Psi_{\vec{n}}$ defines a representation $\pi_{\vec{n}}$ on which (6.25) is satisfied. In general $\vec{n}$ is not aligned with the z axis and therefore an order parameter appear. Nevertheless α^t and α^t_π differ only for

having fixed the value of $\overline{\sigma} = \phi\,(\sigma_3^\infty) = \cos\theta$, which is in any way not affected by the continuous symmetry of our model which, we recall, are the rotations around the z axis.
We stress that the Generalized Goldstone Theorem cannot be applied here since the states we are now considering are not stationary and therefore no information about the spectrum can be deduced.
Since condition (6.26) should be verified a Coulomb potential can be treated only taking F as the set of z-pointing states where no condition has to be imposed on the potential.
The last point I want to discuss here is the reason why two different convergences have been used. The main reason is the fact that, instead of mean field models, in the equations of motion it appears here an unbounded operator which cannot be ultrastrongy convergent. We have to relax this condition or to use functions of $T_k^\vee$ which are bounded independtly of the nature of $T_k^\vee$. As we see from (6.17) the convergence on a set of analytical vectors is a good candidate since it gives a kind of uniformity in V which is often required in the proofs and which is clearly satisfied by the ultrastrong convergence. In this sense we consider these two kind of convergences 'equivalent'. Of course the use of the one or the other is only matter of convenience.
In particular for n-states convergence properties are easier discussed using ultrastrong convergence since many practical difficulties arise trying to prove the existence of the D set.
Moreover the form of the solution in (6.7) gives directly the norm limitation of the operators and therefore only strong convergence has to be proven. Nevertheless explicit calculations of both strategies look very much the same, showing once again the strng relation between the two possible methods.

VI.2 MATHEMATICAL DETAILS AND PROOFS

Some of the assertion of the previous paragraph will be now proven. In particular:

(1) We prove that on the states belonging to F $\overline{\sigma}$ coincides with σ_3^∞;
(2) Conditions (6.15) and (6.16) are obtained;
(3) D - strong convergence of $T_k^\vee$ is discussed and its consequences, (6.18) and (6.19), are deduced; in particular we prove the existence of the set D which is

by no means ensured by general properties;

(4) For n-pointing states we give the general steps of the convergence of $\exp(iT_k^V t)$ on the ground ciclyc state Ψ_α ;

(5) We use the result in (4) to prove the existence of the ultrastrong limit of the dynamics proving the various points outlined in the previous paragraph.

(1)

Our states satisfy condition (6.8) and therefore, for instance, (6.9). On such states we can write:

$$\left|\phi(1/V \sum_{i\in V} (\sigma_3^i - \bar{\sigma}))\right| \leq 1/V \sum_{i\in V} \left|\phi(\sigma_3^i - \bar{\sigma})\right| =$$

$$=1/V \sum_{i\in V} |i-k|^{\alpha-\varepsilon/2} / |i-k|^{\alpha-\varepsilon/2} |\phi(\sigma_3^i - \bar{\sigma})| \leq \qquad (\varepsilon > 0)$$

$$\leq (1/V \sup|i-k|^{\alpha-\varepsilon/2}) \; (\sum_{i\in V} 1/|i-k|^{\alpha-\varepsilon/2} \; 1/|i|^{3-\alpha+\varepsilon})$$

But

$$B(1)= 1/V \sup|i-k|^{\alpha-\varepsilon/2} = 1/V \sup|i|^{\alpha-\varepsilon/2} = 1/V \sup[(i_x^2+i_y^2+i_z^2)^{1/2}]^{\alpha-\varepsilon/2}$$

Considering a cubic lattice we have $i_x=i_y=i_z=i$ and sup $i= V^{1/3}$. It follows that

$$B(1)=1/V \sup (\sqrt{3}\, i)^{\alpha-\varepsilon/2} \simeq V^{\alpha/3-\varepsilon/6-1}$$

and since we are dealing with $\alpha \leq 3$ and $\varepsilon > 0$ we have

$$\lim_{V,\infty} B(1) = 0$$

The second term is approximately

$$B(2)= \sum_{i\in V} 1/ |i-k|^{\alpha-\varepsilon/2} \; 1/|i|^{3-\alpha+\varepsilon} \simeq \sum_{i\in V} 1/|i|^{3+\varepsilon/2}$$

which is convergent in the limit $V\to\infty$.
We conclude that

$$0 \leq \lim_{V,\infty} \left|\phi(1/V \sum_{i\in V} (\sigma_3^i - \bar{\sigma}))\right| \leq \lim_{V,\infty} (B(1)\cdot B(2)) = 0$$

and therefore:

$$\lim_{V,\infty} \phi(1/V \sum_{i\in V} \sigma_3^i) = \phi(\sigma_3^\infty) = \bar{\sigma}$$

(2)

Let us suppose that T_k^V converges strongly to T. This means that

$$\forall \varepsilon > 0 \quad \exists \Omega \subset \mathbb{Z}^3 : \forall V, V' \supset \Omega \quad \|(T_k^V - T_k^{V'})\Psi\| < \varepsilon \qquad D$$

We can take $V \supset V'$ and therefore

$$\|(T_k^V - T_k^{V'})\Psi\| = 4\|\sum_{i\in V/V'} J_{ik}(\sigma_3^i - \bar\sigma)\Psi\| < 4\sum_{i\in V/V'} |J_{ik}|\,\|(\sigma_3^i - \bar\sigma)\Psi\| < \varepsilon$$

which is surely satisfied if the series $a_i = |J_{ik}|\,\|(\sigma_3^i - \bar\sigma)\Psi\|$ is convergent due to the Cauchy theorem. So if we have in particular $J_{ij} \simeq |i-j|^{-\alpha}$, $\alpha \leq 3$, a sufficient condition for strong convergence of T_k^V is

$$\|(\sigma_3^i - \bar\sigma)\Psi\| \leq 1/|i|^{3-\alpha+\varepsilon} \qquad , \varepsilon > 0$$

It is now immediate to verify that the equation above implies (6.16). In fact since

$$\big|\|\sigma_3^i\Psi\| - \|\bar\sigma\Psi\|\big| \leq \|(\sigma_3^i - \bar\sigma)\Psi\| \leq 1/|i|^{3-\alpha+\varepsilon}$$ we have

$$\begin{cases} \|\sigma_3^i\Psi\| - \|\bar\sigma\Psi\| \leq 1/|i|^{3-\alpha+\varepsilon} \\ \|\bar\sigma\Psi\| - \|\sigma_3^i\Psi\| \leq 1/|i|^{3-\alpha+\varepsilon} \end{cases}$$

But $\|\sigma_3^i\Psi\| = \|\bar\sigma\Psi\| = 1$ and then, in the limit of large i, $1 \leq |\bar\sigma| \leq 1$ which implies (6.16).

(3)
We start proving that there is a set D, dense in H, for which equation (6.17) can be written.
To fix the ideas we suppose that $\bar\sigma = +1$ so that the ground state is Ψ_0^{up} and $H = \mathfrak{A}\Psi_0^{up}$. We begin proving (6.17) for Ψ_0^{up},

$$\|(T_k^V)^n \Psi_0^{up}\| \leq c^n \qquad (6.27)$$

For n=1 we have

$$T_k^V \Psi_0^{up} = 4\sum_{i\in V} J_{ik}(\sigma_3^i - 1)\Psi_0^{up} = 4\;[\,J_{1k}(\sigma_3^1 - 1)\Psi_0^{up} + J_{2k}(\sigma_3^2 - 1)\Psi_0^{up} + \ldots] = 0$$

and therefore

$$(T_k^V)^n \Psi_0^{up} = (T_k^V)^{n-1} T_k^V \Psi_0^{up} = 0 \quad \text{and then} \quad \|(T_k^V)^n \Psi_0^{up}\| = 0 \quad \forall n$$

so that (6.27) is trivially satisfied.
Let us take now $A \in \mathfrak{A}_{V'}$ so that the vector $\Psi = A\Psi_0^{up}$ coincides with Ψ_0^{up} outside the volume V' and it is different from Ψ_0^{up} inside such volume.

$$\|T_k^V\Psi\| = \|T_k^V A\Psi_0^{up}\| = \|(T_k^{V'} + T_k^{V/V'})A\Psi_0^{up}\| = \|T_k^{V'}A\Psi_0^{up}\| \leq \|T_k^{V'}\|\,\|A\Psi_0^{up}\| = K_{V'}d$$

$$K_{v'} = \| T_k^{v'} \| < \infty \qquad \text{since A is local,} \quad d = \| A \psi_0^{\text{up}} \|$$

$$\| (T_k^{v})^2 \psi \| = \| (T_k^{v'})^2 A \psi_0^{\text{up}} \| \leq \| T_k^{v'} \|^2 \| A \psi_0^{\text{up}} \| = (K_{v'})^2 d$$

and therefore

$$\| (T_k^{v})^n \psi \| \leq \| T_k^{v'} \|^n \| A \psi_0^{\text{up}} \| = K_{v'}^n d$$

so that (6.17) is verified.

The second step consists in deriving (6.18) from (6.14). We want to show that if T_k^v is D-stongly convergent then also $F(T_k^v)$ in (6.14) is D- strongly convergent.
Due to the analyticity of F , we can expand it in a norm converging power series

$$F_\alpha(T_k^v) = \sum_{n=0}^{\infty} c_n (T_k^v)^n$$

We want to study now the following quantity:

$$\| (F_\alpha(T_k^v) - F_\alpha(T_k^{v'}))\psi \| \qquad \forall V,V' \supset \mathcal{R} , \qquad \psi \in D$$

in order to verify the Cauchy nature of $F_\alpha(T_k^v)$. We have

$$\| (F_\alpha(T_k^v) - F_\alpha(T_k^{v'}))\psi \| \leq \sum_{n=0}^{\infty} \|c_n\| \, \| ((T_k^v)^n - (T_k^{v'})^n)\psi \| \tag{6.28}$$

Let us suppose that the rhs is uniform in V,V'. We can therefore take V and V' great as we want and since T is D-strongly convergent also any of its power $(T_k^v)^n$ converges strongly on D. This gives $\| ((T_k^v)^n - (T_k^{v'})^n)\psi \| < \varepsilon$, $\forall$ n. Moreover due to the entirety of $F_\alpha(T_k^v)$ we have $\sum_n \|c_n\| = N$ (becouse even in $\| T_k^v \| = 1$ our series converges). Therefore

$$\| (F_\alpha(T_k^v) - F_\alpha(T_k^{v'}))\psi \| < \varepsilon \cdot N$$

which proves the D-strong convergence of $F_\alpha(T_k^v)$.
Uniformity of the rhs of (6.28) directly follows from condition (6.17):

$$\sum_{n=0}^{\infty} \| c_n \| \, \| ((T_k^v)^n - (T_k^{v'})^n)\psi \| \leq \sum_{n=0}^{\infty} \| c_n \| \left(\| (T_k^v)^n \psi \| + \| (T_k^{v'})^n \psi \| \right) \leq 2 \sum_{n=0}^{\infty} \| c_n \| \, C^n$$

which is converging thanks again to the entireness of F , and it does not depend on V.

In a quite similar way we can derive that (6.19) from (6.13). We must consider

$$\text{D-s}\lim_{V\uparrow\infty} d/dt\, \alpha_V^t(\sigma_\alpha^k) = \text{D-s}\lim_{V\uparrow\infty} X_\alpha(\alpha_V^t(\sigma_\beta^k), T_k^V)$$

The rhs is obtained using the result just derived:

$$\text{D-s}\lim_{V\uparrow\infty} X_\alpha(\alpha_V^t(\sigma_\beta^k), T_k^V) = X_\alpha(\alpha^t(\sigma_\beta^k), T_k)$$

being X_α an analytical entire function. Of course if the lhs is uniform in V we can write

$$\text{D-s}\lim_{V\uparrow\infty} d/dt\, \alpha_V^t(\sigma_\alpha^k) = d/dt\ \text{D-s}\lim_{V\uparrow\infty} \alpha_V^t(\sigma_\alpha^k) = d/dt\, \alpha^t(\sigma_\alpha^k)$$

and uniformity is a consequence of (6.17), just as before.

(4)
This is the most involved point. We have to show that

$$\forall \varepsilon > 0 \quad \exists\, \Omega \subset \mathbb{Z}^3 : \forall\, V, V' \supset \Omega \qquad \| (\exp(iT_k^V t) - \exp(iT_k^{V'} t))\Psi_{\vec{n}} \| < \varepsilon$$

We have:

$$M(V/V') \equiv \| (\exp(iT_k^V t) - \exp(iT_k^{V'} t))\Psi_{\vec{n}} \| =$$
$$= \sqrt{(\Psi_{\vec{n}}, (\exp(-iT_k^V t) - \exp(-iT_k^{V'} t))(\exp(iT_k^V t) - \exp(iT_k^{V'} t))\Psi_{\vec{n}})} =$$
$$= \sqrt{2}\ \sqrt{(\Psi_{\vec{n}}, (A^2/2! - A^4/4! + A^6/6! - \ldots)\Psi_{\vec{n}})}$$

Using the following properties and definitions:

$$A \equiv (T_k^V - T_k^{V'})t = 4t \sum_{i \in V/V'} J_{ik}(\sigma_3^i - \bar{\sigma})$$

$$\exp(iA) + \exp(-iA) = 2\cos A$$

$$\cos A = 1 - A^2/2! + A^4/4! - \ldots$$

We see from the expression for M(V/V') that the relevant quantities that need to be studied in order to deduce convergence are the matrix elements $(\Psi_{\vec{n}}, A^n \Psi_{\vec{n}})$ $\forall n$.
In particular we will show that the sequence $(\Psi_{\vec{n}}, A^{2n}\Psi_{\vec{n}})/(2n)!$ is decrescent. Therefore M(V/V') is majorated by its first term which is or order ε. So the Cauchy nature of $\exp(\pm iT_k^V t)\Psi_{\vec{n}}$ is proven.
Let us calculate explicitly the matrix element for n=1:

$$(\Psi_{\vec{n}}, A^2\Psi_{\vec{n}}) = (4t)^2 \sum_{i,\ell \in V/V'} J_{ik} J_{\ell k} (\Psi_{\vec{n}}, (\sigma_3^i \sigma_3^\ell - \bar{\sigma}\sigma_3^i - \bar{\sigma}\sigma_3^\ell + \bar{\sigma}^2)\Psi_{\vec{n}})$$

Being $\Psi_{\vec{n}}$ invariant under space translations we have:

$$(\Psi_{\vec{n}}, \sigma_3^i \Psi_{\vec{n}}) = (\Psi_{\vec{n}}, \sigma_3^\ell \Psi_{\vec{n}}) = \bar{\sigma}$$

$$(\Psi_{\vec{n}}, \sigma_3^i \sigma_3^\ell \Psi_{\vec{n}}) = \delta^{i\ell} + \bar{\sigma}^2(1-\delta^{i\ell})$$

and finally

$$(\Psi_{\vec{n}}, A^2 \Psi_{\vec{n}}) = (4t)^2 \quad (1-\bar{\sigma}^2) \sum_{i \in V/V'} J_{ik}^2 \tag{6.29}$$

This result gives an hint on the way to follow. We ask for a potential J_{ij} which is square-integrable so that, for Cauchy condition,

$$\sum_{i \in V/V'} J_{ik}^2 < \varepsilon \tag{6.30}$$

A direct consequence of the above condition (or equivalently of the condition (6.26)) is the fact that every other power of J_{ik}, except J_{ik} itself, gives a series which is again convergent. This easily follows from the next considerations:

$$J_{ik}^2 = |J_{ik}|^2 \quad ; \quad |J_{ik}|^2 \geq |J_{ik}|^n \qquad n \geq 2 \quad \text{definitively}$$

and therefore, by the confront theorem for series with positive elements, if (6.26) holds also

$$\sum_i |J_{ik}|^n < \infty \qquad \forall\, n \geq 2 \tag{6.31}$$

is satisfied. Moreover we known that absolute convergence implies convergence so that from (6.31) we deduce

$$\sum_i J_{ik}^n < \infty \qquad \forall\, n > 2 \tag{6.32}$$

This result will be used in the main demonstration since in all the other matrix elements, $(\Psi_{\vec{n}}, A^{2n} \Psi_{\vec{n}})$, $n > 1$, such terms appear. For example we have:

$$(\Psi_{\vec{n}}, A^4 \Psi_{\vec{n}}) = (4t)^4 \{B_1 \sum_{i \in V/V'} J_{ik}^4 + 6\, B_2 \sum_{i \in V/V'} J_{ik}^2 \sum_{\substack{\ell \in V/V' \\ |\ell| > |i|}} J_{\ell k}^2\} \tag{6.33}$$

$$B_1 = 1 + 2\bar{\sigma}^2 - 3\bar{\sigma}^4 \qquad B_2 = (1-\bar{\sigma}^2)^2$$

$$(\Psi_{\vec{n}}, A^6 \Psi_{\vec{n}}) = (4t)^6 \{C_1 \sum_{i \in V/V'} J_{ik}^6 + 15C_2 \sum_{i \in V/V'} J_{ik}^2 \sum_{\substack{\ell \in V/V' \\ |\ell| > |i|}} J_{\ell k}^4 + 15C_3 \sum_{i \in V/V'} J_{ik}^4 \sum_{\substack{\ell \in V/V' \\ |\ell| > |i|}} J_{\ell k}^2 +$$

$$+ 20C_4 \sum_{i \in V/V'} J_{ik}^3 \sum_{\substack{\ell \in V/V' \\ |\ell| > |i|}} J_{\ell k}^3 + 90C_5 \sum_{i \in V/V'} J_{ik}^2 \sum_{\substack{\ell \in V/V' \\ |\ell| > |i|}} J_{\ell k}^2 \sum_{\substack{j \in V/V' \\ |j| > |\ell|}} J_{jk}^2\} \tag{6.34}$$

(The C_i are functions in $\bar{\sigma}$ which can be easily found but have no particular interest in the convergence discussion since are all of order 1 and therefore will be neglected in the following). The numerical coefficents appearing in the equations above and in all the other matrix elements can be found, to bypass direct computation, by the

$$k = n!/(n_1!\, n_2! \ldots n_\ell!)\,; \quad n_1+n_2+n_3+\ldots = n; \quad n_i \neq 1 \quad \forall i \qquad (6.35)$$

Here n is the power of A in the relevant matrix element, n_i is the power at which is raised the relative J_{ik} and l is the number of ($\sum J_{ik}^{n_i}$) appearing in each term of the matrix element. For instance in (6.34) we have:

coefficient of $\sum J_{ik}^6 = 6!/6! = 1$

coefficient of $\sum_i J_{ik}^4 \sum_\ell J_{\ell k}^2 = 6!/(4!2!) = 15$

coefficient of $\sum_i J_{ik}^2 \sum_\ell J_{\ell k}^4 = 6!/(2!4!) = 15$

coefficient of $\sum_i J_{ik}^3 \sum_\ell J_{\ell k}^3 = 6!/(3!3!) = 20$

coefficient of $\sum_i J_{ik}^2 \sum_\ell J_{\ell k}^2 \sum_j J_{jk}^2 = 6!/(2!2!2!) = 90$

In particular in the last term we have n=6, $n_i=2$ $\forall$ i and l=3.
We write condition (6.30) in the more convinient form

$$\sum_i J_{ik}^2 \sim O(\varepsilon)$$

so that we can deduce that

$$\sum_{i \in V/V'} J_{ik}^2 \sum_{\ell \in V/V'} J_{\ell k}^2 \sim O(\varepsilon^2)\,; \qquad \sum_{i \in V/V'} J_{ik}^4 \sim O(\varepsilon^2)\,; \qquad \sum_{i \in V/V'} J_{ik}^3 \sim O(\varepsilon^{3/2})$$ and so on.

We write finally

$(\Psi_{\vec{n}}, A^4\, \Psi_{\vec{n}})$ $(4t)^4$ $[4!/4!+4!/(2!2!)]$ $= (4t)^4\, 4![1/4!+1/(2!2!)]$

$(\Psi_{\vec{n}}, A^6\, \Psi_{\vec{n}})$ $(4t)^6$ $6!$ $[1/6!+1(2!4!)+1/(3!3!)+1/(4!2!)+1/(2!2!2!)]$

$(\Psi_{\vec{n}}, A^8\, \Psi_{\vec{n}})$ $(4t)^8$ $8!$ $[1/8!+1/(2!6!)+1/(3!5!)+1/(4!4!)+1/(5!3!)+$

$+(1/(2!2!4!)+\text{perm.})+(1/(2!3!3!)+\text{perm.})+1/(2!2!2!2!)]$

and so on.
Let us call $a_i = [\ldots\ldots]_i$ above. We can calculate these coefficents:

$a_4 = 0.29$ $a_6 = 0.196$ $a_8 = 0.141$ $a_{10} = 0.105$ $a_{12} = 0.086$
$a_{14} = 0.064$ $a_{16} = 0.046$

This is a strong indication that the sequence $\{a_i\}$ is positive and decrescent. Therefore we can write

$$(\Psi_{\vec{n}}, A^n \Psi_{\vec{n}}) \approx (4t)^n \, n! \, a_n \, \varepsilon^{n/2} \tag{6.36}$$

and:

$$M(V,V') \simeq \sqrt{2((4t)^2 a_2 \varepsilon - (4t)^4 a_4 \varepsilon^2 + (4t)^6 a_6 \varepsilon^3 - \ldots\ldots)}$$

This is a series with alternate sign terms and it decreases certainly if

$$(4t)^2 \varepsilon < 1 \tag{6.37}$$

Therefore $M(V,V')$ is majorate by its first term and hence Cauchy condition is satisfied.

(5)
Once this heavy proof has been carried out the next points are easy to deal with. First of all we will prove that

<u>$\exp(iT_k^V t)$ is strongly convergent on all the vectors $A_\kappa \Psi_{\vec{n}}$, $A \in \mathcal{A}_0$</u>

We have to estimate the quantity

$$\|(1 - \exp i(T_k^V - T_k^{V'})t)A\Psi_{\vec{n}}\| = \|(1 - \exp(iT_k^{V/V'}t))A\Psi_{\vec{n}}\| =$$

$$= \|A\Psi_{\vec{n}} - [\exp(iT_k^{V/V'}t), A]\Psi_{\vec{n}} - A\exp(iT_k^{V/V'}t)\Psi_{\vec{n}}\| \leq$$

$$\leq \|A\Psi_{\vec{n}} - A\exp(iT_k^{V/V'}t)\Psi_{\vec{n}}\| + \|[\exp(iT_k^{V/V'}t), A]\Psi_{\vec{n}}\|$$

But

$$\|A(\Psi_{\vec{n}} - \exp(iT_k^{V/V'}t)\Psi_{\vec{n}})\| \leq \|A\| \, \|(1 - \exp(iT_k^{V/V'}t))\Psi_{\vec{n}}\| = \|A\| \cdot M(V,V') \leq$$

$$\leq \|A\| \varepsilon = \varepsilon'$$

$$\|[\exp(iT_k^{V/V'}t), A]\Psi_{\vec{n}}\| = \|\exp(-4i\bar{\sigma} t \sum_{i \in V/V'} J_{ik})[\exp(4it \sum_{i \in V/V'} J_{ik}\sigma_3^i), A]\Psi_{\vec{n}}\|$$

$$= \|[\exp(4it \sum_{i \in V/V'} J_{ik}\sigma_3^i), A]\Psi_{\vec{n}}\| < \delta.$$

as V and V' goes both to infinity since A is localized in a finite volume. Therefore

$$\|(1 - \exp(i(T_k^V - T_k^{V'})t))A\Psi_{\vec{n}}\| < \varepsilon + \delta$$

and this ends the prove.

Let us now prove that

<u>$\exp(iT_k^V t)$ is strongly convergent on $\overline{A\Psi_{\vec{\alpha}}}^{\,\kappa''}$.</u>

Let $\Psi \in H$. It satisfies the following relation:

$$\lim_{\alpha,\infty} \| \Psi - A_\alpha \Psi_{\vec{\alpha}} \| = 0 \qquad A_\alpha \in \mathcal{Q}_o$$

We have to verify now that

$$\| (\exp(iT_k^V t) - \exp(iT_k^{V'} t)) \Psi \| < \varepsilon \qquad \forall\, V, V' > \Omega$$

$$\| (\exp(iT_k^V t) - \exp(iT_k^{V'} t)) \Psi \| = \| (\exp(iT_k^V t) - \exp(iT_k^{V'} t)) (\Psi - A_\alpha \Psi_{\vec{\alpha}} + A_\alpha \Psi_{\vec{\alpha}}) \| \leq$$

$$\leq \| (\exp(iT_k^V t) - \exp(iT_k^{V'} t)) (\Psi - A_\alpha \Psi_{\vec{\alpha}}) \| + \| (\exp(iT_k^V t) - \exp(iT_k^{V'} t)) A_\alpha \Psi_{\vec{\alpha}} \|$$

Of course we have just proven that

$$\| (\exp(iT_k^V t) - \exp(iT_k^{V'} t)) A_\alpha \Psi_{\vec{\alpha}} \| < \varepsilon \qquad \forall\, V, V' > \Omega$$

For what concerns the first term we have:

$$\| (\exp(iT_k^V t) - \exp(iT_k^{V'} t)) (\Psi - A_\alpha \Psi_{\vec{\alpha}}) \| \leq \left(\| \exp(iT_k^V t) \| + \right.$$

$$\left. + \| \exp(iT_k^{V'} t) \| \right) \| \Psi - A_\alpha \Psi_{\vec{\alpha}} \| \leq 2\delta$$

and therefore we get

$$\| (\exp(iT_k^V t) - \exp(iT_k^{V'} t)) \Psi \| \leq 2\delta + \varepsilon \equiv \varepsilon'$$

The conclusion is that $\exp(iT_k^V t)$ converges strongly on H. Moreover since the exponential is unitary its norm is 1 $\forall\, V$, which means that the convergence is ultrastrong.
We conclude that if (6.26) is satisfied $\alpha_V^t \xrightarrow{u.s.} \alpha^t$ on this large family of states.

APPENDIX 1: Some Convergence Definition

Let us consider a sequence of operators $\{A_i\}$ and an operator A. We say that $\{A_i\}$ is converging to A:

in norm if

$$\| A_i - A \| \xrightarrow[i\to\infty]{} 0$$

$$\text{where } \| A \| = \sup_{\psi \in \mathcal{H}} \| A \psi \| / \| \psi \|$$

in the strong topology if

$$\forall \psi \in H \qquad \| (A_i - A) \psi \| \to 0$$

in the weak topology if

$$\forall \phi, \psi \in H \qquad | (\phi , (A_i - A) \psi) | \to 0$$

in the ultrastrong topology if

$$\forall \text{ sequence } \{\psi_k\} \text{ such that } \sum_k |\psi_k|^2 < \infty \quad \text{then}$$

$$\sum_k |(A_i - A) \psi_k|^2 \xrightarrow[i\to\infty]{} 0$$

The 'strongest' topology is of course the one given by the norm, in the sense that all the other kind of convergence follow from this one. Moreover ultrastrong convergence implies the strong and weak ones.
We note that the norm convergence does not depend on the states while all the other topologies do. This is related to the fact that SRI does not give delocalization effects and therefore no particular request needs to be done about the states. This is not the case for LRI as it is widely discussed in this work.
Moreover weak convergence of α_V^t to α^t is not enough to make α^t be a group; infact only (ultra)strong convergence "passes through the product", as it was discussed in IV.1(see ref.[12]).
For spin system strong convergence of bounded operators implies their ultrastrong convergence. While σ^v is a bounded operator T_k^v is not. Therefore many tecnical differences arise.

By norm closure $\mathcal{Q}$ of $\mathcal{Q}_o$ we mean that $\mathcal{Q}$ contains all the elements of $\mathcal{Q}_o$ and all the norm limits of such elements.
In the same way M, weak closure of $\mathcal{Q}_o$ with respect to F, differs from $\mathcal{Q}$ since it contains also all the weak limit of the elements of $\mathcal{Q}_o$ (with respect to the topology introduced by F).

APPENDIX 2: GNS Construction

We start noticing that, given a representation R and an Hilbert space H on which the operators $R(A)$, $A \in \mathcal{A}$, act, we can define many states f of the form

$$f(A)=(\Psi, R(A)\Psi) \qquad \Psi \in H$$

GNS construction shows that this process can be reversed: to every state f it is possible to associate a cyclic representation R with a cyclic vector Ω such that

$$f(A) = (\Omega, R(A)\Omega)$$

Given f let us consider the following set:

$$I = \{ A \in \mathcal{A} : f(A^* A) = 0 \}$$

We will not prove that this is a left-ideal:

i) $B \in I$ and $C \in \mathcal{A} \Rightarrow CB \in I$;
ii) I is a subalgebra of $\mathcal{A}$.

Let us now take the space $\mathcal{A}/I$ of the equivalence classes e(A) to which the element $A \in \mathcal{A}$ belongs:

$$\mathcal{A}/I = \{ e(A) : A \in e(A) \}$$

We introduce the equivalence relation

$$e(A) = e(B) \iff A - B \in I$$

and a scalar product

$$(e(A), e(B)) = f(A^* B)$$

Of course I is the zero of the factor space. If we close $\mathcal{A}/I$ we get an Hilbert space H. To construct a representation R of $\mathcal{A}$ in H we define (on a dense set)

$$R(A)\, e(B) = e(AB)$$

R(A) is bounded and is an homomorphism: therefore R is a representation of $\mathcal{A}$. Let us define $\Omega = e(1)$
This is a cyclic vector in fact $R(A)\Omega = R(A)\, e(1) = e(A)$ which is dense in H. Finally

$$(\Omega, R(A)\Omega) = (e(1), e(A)) = f(A)$$

Q.E.D.

REFERENCES

[1] Swieca; Commun.Math.Phys.4, 1, (1967)

[2] Doplicher, Haag, Roberts; Commun.Math.Phys.13, 1 (1969);15, 173 (1969); 23, 199 (1971); 35, 49 (1974)

[3] Kittel,Quantum theory of solids, Wiley and Sons

[4] Tosatti, Parrinello, Chen:" The Fractional quantized two-dimensional Hall effect", Varenna, 1983

[5] Tsui, Stormer,Gassard: Phys.Rev.Lett.,48,1562 (1982)

[6] Haag, Hugenholtz, Winnink; Commun.Math.Phys.5,215 (1967) D.W.Robinson;Commun.Math.Phys.7,337 (1968)

[7a] G.Morchio,F.Strocchi;Commun.Math.Phys.99,153 (1985)
[7b] G.Morchio,F.Strocchi;J.Math.Phys. 28 (3),1987
[7c] F.Strocchi:" Long range interaction and symmetry breaking in many-body and in gauge theories", lectures at Wien

[8] J.Kogut,L.Susskind,Phys.Rev.D11,3594 (1975)

[9] Anderson,Phys.Rev.112,1900 (1958)

[10] Hille,Differential Equation in the Complex Domain, WILEY (1976)

[11] Hugenholtz:"Equilibrium states in statistical mechanics. C - algebraic approach"

[12] O.Brattelli and D.Robinson,Operator Algebras and Quantum Statistical Mechanics, volII,Springer