One-Loop D=4 Superstrings in the Case of Absence of Non–Renormalization Theorems

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CANDIDATE

Kurt Lechner

SUPERVISOR

Prof. Roberto Iengo

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Contents

1. Introduction

One of the open problems in Quantum Field Theory (Q.F.T.) regards the possibility of constructing a consistent quantum theory of gravitation. The perturbative computations of t'Hooft and Veltman (1975), showing that pure gravity is one–loop finite on shell, revealed the possible existence of a hidden symmetry in the pure gravity sector, ensuring the absence of divergences, and raised the hope that the theory is actually finite. But recently, the analysis of Goroff and Sagnotti (1986) [GS] proved that pure gravity is two–loop divergent even on shell: there is no hidden symmetry, neither in the simplest gravitational theory, which ensures its finiteness.

Although the introduction of supersymmetry improves the ultraviolet behavior of gravity ($N = 1$ supergravity is expected to be two–loop finite) nowadays also the $N = 8$ supergravity theory in four dimensions is believed to be not renormalizable: there are candidates for counterterms from seven loops onwards.

Various attempts of unification of the fundamental forces during the last years, revived the ideas of Kaluza and Klein of extra dimensions, enlarging in this way the scenario of the conventional Q.F.T.'s, but introducing, at the same time, also several drawbacks with respect to four dimensional models. Yang–Mills theories, at present the cornerstones in the description of the fundamental interactions, are not renormalizable for $D > 4$, there are Lorentz anomalies in $D = 4n - 2$, and in particular in ten dimensions, and, moreover, the introduction of new dimensions implies that one has to "invent" some compactification schemes which bring us down to our four observed dimensions.

The discovery of the Green–Schwarz anomaly cancellation mechanism [GSW] in the minimal Supergravity Super–Yang–Mills theory in ten dimensions, which can be thought of as an effective low energy theory of superstrings, and especially the proof of the finiteness of one–loop scattering amplitudes in string theories (1984), raised the hope that (at least at the perturbative level) a consistent quantum theory of gravitation can be found. One of the appealing features of string theories is that the presence of gravity is *required* by consistency, for example in the Green–Schwarz mechanism, where Yang–Mills anomalies cancel versus Lorentz–anomalies, while in conventional Q.F.T.'s the introduction of gravity usually spoils their consistency.

Although presently no deep understanding of the rôle of Superstring Theory in the outstanding problems of elementary particle physics, like the fermion family replication or the large splitting between the fermion mass scales, has been reached, String Theories offer for the first time the possibility of computing, in principle, quantum gravitational effects. One expects not only to reproduce classical General Relativity in some large–distance regime, but to get also a deeper insight into the nature of quantum corrections to General Relativity.

String Theory is an S–matrix theory, so only on–shell transition elements of scattering processes can be obtained. Although the perturbative approach for closed strings has been very well established, at least up to two loops, presently no non perturbative formulation of String Theories is available. Nevertheless in various asymptotic regions of the (s,t) space, s being the total energy squared in the center–of–mass frame and t the square of the momentum transfer, the evaluation of scattering amplitudes, especially of four–graviton amplitudes, has been possible also at higher genus: the employed techniques are mainly Regge–Gribov methods [ACV,SU], which are suitable for the study of large s and small t, and saddle point evaluations [GM], the latter being appropriate for large momenta and fixed angle configurations.

One of the peculiar features of the situations studied in [ACV,SU,GM] is the existence of non–renormalization theorems. These theorems, which are rigorously proved up to two loops and supposed to hold at all orders in perturbation theory, ensure the vanishing of one–, two– and three–point functions with external massless legs (even "off shell") in ten dimensional type II and heterotic Superstring Theories [MA]. The absence of those radiative corrections implies, in particular, that in the limit of $s \to \infty$ and $t \to 0$, in the four–graviton amplitude at one–loop only rescattering diagrams are present, and that, at all orders, no *one–particle–reducible* (1PR) contribution appears.

In this thesis we are concerned, at one loop, with situations in which the non–renormalization theorem does no longer hold, devoting special attention to the 1PR contributions to the amplitudes.

Two are the cases that we have analysed. The first case treats with a scattering of a scalar *massive* particle, of the first excited level of a type II superstring model in a four dimensional target space, with a massless graviton. This can be seen as a kind of gravitational version of the Compton scattering in QED. Our analysis of the 1PR contributions to the amplitude, for large s and small t, reveals, as leading contribution, a term proportional to $\frac{1}{\sqrt{t}}$ from which the α'–dependence drops out. It can be interpreted as a long range correction to Newton's potential like the first order expansion of the Schwarzschild metric. We show also that this leading term is produced, as expected by general arguments, by the "elastic" part of the 1PR contribution to the amplitude. For this leading term and for other (for $t \to 0$ subleading) contributions we found an interpretation in terms of ordinary Feynman diagrams, where also a state of the second excited level of the string spectrum appears.

The second topic, that we considered, is four–graviton scattering (to which in the literature great attention has been paid) at one–loop in a four dimensional model with *low* ($N < 8$) space–time supersymmetry. The break–down of the $N = 8$ supersymmetry, in the case considered by us down to $N = 4$, implies also the break–down of the non–renormalization theorem. The amplitude turns out to be a sum of two parts, $A^{N=8} + A^{N=4}$, where the first one is the amplitude of the model with unbroken $N = 8$ supersymmetry, while the second one, $A^{N=4}$, breaks this high supersymmetry and allows for non vanishing graviton two– and three–point functions. Performing also in this case an analysis at large s and small t we find 1PR configurations containing three–graviton vertex corrections, corresponding to one–loop three–point functions, and graviton self–energies, corresponding to one–loop two–point functions. These features resemble much what happens in ordinary Q.F.T's; we found, however, the interesting result, that, in spite of absence of non–renormalization theorems, neither the tree–level graviton propagator 1/t, nor the tree–level "three-graviton-vertex" $\sqrt{\mathrm{G}}$s, get renormalized. In this case "Schwarzschild-like" corrections, exhibiting a factor of $\frac{1}{\sqrt{t}}$, are seen to be absent. The leading behavior of the amplitude is in both cases rather be seen to be $G^2s^2(\ln t)^n$, n=2,1 respectively, which has no "classical" counterpart and which is subleading with respect to the leading rescattering contributions.

Furthermore we detected infrared divergences in the vertex correction, typical of four dimensions, whose presence can already be understood at the Q.F.T. level and which should be of the Bloch–Nordsieck type. Of course, the check of such a statement would involve the explicit implementation of the Bloch–Nordsieck cancellation mechanism, a problem which is outside the present investigation.

Throughout this thesis a constant attention has been paid to the comparison of the obtained results with ordinary Feynman diagrams in their parametric representation. In all cases under investigation String Theory is seen to reduce to Q.F.T. in the limit in which $s \to \infty$ and $t \to 0$. Moreover, the parametric representations of Feynman diagrams are seen to be reproduced by the string in certain "pinching limits", which correspond to degenerate configurations in the moduli space. This allows us to speculate about a more deep correspondence between strings and ordinary field theories, which, perhaps covers also certain subleading terms for $s \to \infty$ and $t \to 0$.

The thesis is organized as follows. In Chapter 2 we describe the compactification scheme of Antoniadis et al. and work out the particular four dimensional string models, on which our investigations in Chapters 4 and 5 are based. Particular attention is devoted to the sums over spin structures.

In Chapter 3 we present the hyperelliptic language, in which all our investigations are carried out, and establish also its relation with the θ–function formalism.

We derive several "pinching limit" identities that are then extensively used in the subsequent chapters.

In Chapter 4 we present our "Compton–like" scattering, perform the relevant pinching limit, we are interested in and derive our results, among which the Schwarzschild correction is probably the most significant one. In the last section of this chapter we make a comparison with Feynman diagrams.

In Chapter 5 we study four–graviton scattering at $N = 4$ supersymmetry. We revisit the leading rescattering terms in the $N = 8$ contribution to the amplitude, with the aim of depicting their relation with the parametric representations of the corresponding ordinary Feynman diagrams. In the last section we study the effects of non vanishing two– and three–point functions in the supersymmetry breaking contribution to the amplitude, proving "non–renormalization–theorems" when there are no a priori non–renormalization theorems. We depict the infrared divergence and make also here a comparison with Feynman diagrams. Chapter 6 is devoted to some concluding remarks.

2. Four dimensional strings

In a fermionic formulation of compactified strings the set of free Majorana-Weyl fermions which enter in D - dimensional type II string theories is given by the set [SC]

$$F = \{\psi^\mu, \bar{\psi}^\mu, \lambda^A, \bar{\lambda}^A\} \tag{2.1}$$

Here μ runs from one to $D-2$, the number of transverse space-time dimensions, and A goes from 1 to $3(10-D)$ and labels the internal compactified degrees of freedom. Bared and unbared fields refer to right- and left-moving modes respectively.

In addition to these fermions we have a space-time vector of uncompactified bosonic coordinates, $\partial_z X^\mu$ and $\partial_{\bar{z}} X^\mu$, where $\mu = 1, 2, ..., D$.

In a covariant BRS-quantization of strings one has to add (for each chirality sector) also Fadeev-Popov ghosts; these are made out of an anticommuting b,c system of conformal weight 2 and -1 respectively and of their commuting super-partners β, γ with conformal weight 3/2 and -1/2 respectively.

For completeness we remark that for the heterotic string we have to replace the right sector with the following fields:

$$\begin{aligned} bosonic\ coordinates &\rightarrow \partial_{\bar{z}} X^\mu,\ \mu = 1, 2, \ldots, D \\ compensating\ fermions &\rightarrow \bar{\psi}^A,\ A = 1, 2, \ldots, 2(26 - D) \\ reparametrization\ ghosts &\rightarrow b, c \end{aligned}$$

This content of primary fields is basically fixed by the requirement of cancellation of the conformal anomaly. In fact, each Majorana-Weyl fermion contributes to the central charge with $c = 1/2$ and each bosonic coordinate with $c = 1$ while the conformal systems, (b,c) and (β, γ), give universal contributions of $c = -26$ and $c = +15$ respectively. Thus the conformal anomaly cancels in each sector separately.

In the following we are interested in $D = 4$ and in the case of type II superstrings.

In the construction of string models in $D < 10$ dimensions one has to take into account the following consistency requirements [ABK,ABKW,FG,KLT,N]:

1) Local world-sheet supersymmetry should be realized (non linearly) in both chirality sectors. This property allows a consistent decoupling of negative norm states and requires, in turn, that the supercurrent has a definite spin structure.

2) Multiloop string amplitudes on higher genus Riemann surfaces must exhibit modular invariance. This implies, in particular, that we have to sum over spinstructers [SW] and that GSO-projections [GSO] are automatically performed.

3) We make the unitarity requirement of factorization of multiloop amplitudes.

These properties will (in particular models) imply the absence of tachyons, ensure the presence of space-time gravitinos and gravitons and of space-time supersymmetry; the cosmological constant in these models was shown to be zero up to two loops [ABRV,ISZ,MO] and some of the models have, from the phenomenological point of view, promising underlying gauge groups.

2.1 The compactification scheme of Antoniadis et al.

The models which satisfy the requirements (1–3) of the preceding paragraph can be classified (for example) by means of the set-notation, introduced by Antoniadis et. al. [ABK], which is appropriate for the description of spin structures for the fermionic variables. We will use this approach in the following.

On the torus, which constitutes, via factorization, the basic block for higher genus Riemann surfaces, the homology basis is given by the a- and b-cycle which are often also called the **1**- and τ-cycle respectively. The spin structure for one fermion is then specified by a couple of numbers, $\left[{a \atop b}\right]$, where a and b can assume the values 1 and 0, which refer to periodicity and antiperiodicity respectively along the a- and b-cycle.

The possible spin structures on the torus are therefore given by $\left[{0 \atop 0}\right]$, $\left[{1 \atop 0}\right]$ and $\left[{0 \atop 1}\right]$, which are the even ones, corresponding to an even number of zero modes of the Dirac operator (in fact, they have none), and by $\left[{1 \atop 1}\right]$ which is the odd one, corresponding to one Dirac zero mode.

A given spin structure of all the fermions can then be specified by an ordered pair of subsets of F, $(\alpha|\beta)$, where α and β are the sets of fermions that are periodic in the directions $\mathbf{1}$ and τ respectively. The one-loop vacuum to vacuum amplitude for a consistent string model can then be written as (see for example [ABK]):

$$Z = \sum_{\alpha,\beta \in \Sigma} C(\alpha|\beta) Z_{(\alpha|\beta)} \tag{2.2}$$

In (2.2) Σ is a subgroup of $Y \equiv 2^F$, the set of all subsets of F, where the composition law is given by the "symmetric difference"

$$\alpha\beta \equiv \alpha \cup \beta - \alpha \cap \beta \tag{2.3}$$

$Z_{(\alpha|\beta)}$, whose explicit form will be given later on in the hyperelliptic language, is the contribution to Z of a fixed spin structure and involves a factor $\theta^{1/2}\left[{a \atop b}\right] \equiv \theta^{1/2}\left[{a \atop b}\right](0|\tau)$ for each left-moving fermion with spin structure $\left[{a \atop b}\right]$; this factor has to be complex conjugated for each right-moving fermion. $C(\alpha|\beta)$ are phases which can assume the values +1 and -1 and are constrained by the requirements (1-3).

Defining $n(\alpha)$ as the number of left-moving fermions minus the number of right-moving fermions in the set α:

$$n(\alpha) = n_L(\alpha) - n_R(\alpha) \tag{2.4}$$

and setting:

$$\epsilon(\alpha) = e^{i\pi n(\alpha)/8} \tag{2.5}$$

the requirement 2) at one loop becomes [SC]:

$$\begin{aligned} C(\alpha|\beta) &= \epsilon(F)\epsilon(\alpha)C(\alpha|F\alpha\beta) \\ C(\alpha|\beta) &= \epsilon^2(\alpha \cap \beta)C(\beta|\alpha) \end{aligned} \tag{2.6}$$

where conventionally we set:

$$C(\emptyset|\emptyset) = 1 \tag{2.7}$$

We remark that for even spin structures $\epsilon(\alpha \cap \beta) = 1$ and that $\epsilon(F)$ equals -1 for for the heterotic string and +1 for type II strings.

Requirement 3), applied to the factorization of a two-loop amplitude, gives [ABK]:

$$C(\alpha|\beta)C(\alpha|\gamma) = \delta(\alpha)C(\alpha|\beta\gamma) \tag{2.8}$$

Here $\delta(\alpha)$ is defined as -1 if α contains ψ or $\bar{\psi}$ and as +1 if it contains both or none of them.

Equations (2.6) and (2.8) suffice to determine all coefficients C once a basic set of coefficients has been given. To be more explicit: when computing the cosmological constant or bosonic 4-point amplitudes the odd spin structures drop out; therefore we are interested only in the even ones, which means, only in the coefficients $C(\alpha|\beta)$ such that $\alpha \cap \beta = \emptyset$. For those coefficients (2.5)-(2.8) imply the following relations (for type II strings):

$$\begin{gathered} C(\alpha|\beta) = C(\beta|\alpha) \\ C(\alpha|\emptyset) = \delta(\alpha) \\ C(F\alpha|\emptyset) = \delta(\alpha) \\ C(\alpha|F\alpha) = \epsilon(\alpha)\delta(\alpha) \\ \epsilon(F) = \epsilon(\emptyset) = \delta(F) = \delta(\emptyset) = 1 \end{gathered} \tag{2.9}$$

At this point, in order to determine all the phases C, we have to make a set of assignments ± 1 for $C(b_i|b_j), i < j$, for those generators b_i, b_j of Σ such that

$$b_i \cap b_j = \emptyset \tag{2.10}$$

To fix the model completely we have to give the generators $G = \{b_i\}$ of Σ. This set $\{b_i\}$ has again be constrained by a set of rules in reference [ABK]. In particular the requirement 1) imposes on the 18 compactified fermions the $SU(2)^6$-structure. Consequently the supercurrent assumes the following form:

$$G(z) = \sum_{\mu=0}^{3} \psi^\mu \partial_z X_\mu + \sum_{I=1}^{6} x^I y^I z^I + 2c\partial\beta - \gamma b + 3\partial c\beta \tag{2.11}$$

where

$$\lambda^A = \{x^I, y^I, z^I\} \quad I = 1, \ldots, 6 \tag{2.12}$$

and analogous relations hold for the right-movers. Supersymmmetry requires then G(z), ψ and the world sheet gravitino to have the same spin structure.

2.2 Specific models

Without entering in the details of the rules mentioned above we will just give some representative examples (which fulfill those rules) and which will then be used in the following. For this purpose we define the generators:

$$\begin{aligned} s &= \{\psi^1, \psi^2, x^1, \ldots, x^6\} \\ \bar{s} &= \{\bar{\psi}^1, \bar{\psi}^2, \bar{x}^1, \ldots, \bar{x}^6\} \\ b &= \{\psi^1, \psi^2, x^1, x^2, y^3, \ldots, y^6, \bar{x}^3, \ldots, \bar{x}^6, \bar{y}^3, \ldots, \bar{y}^6\} \end{aligned} \tag{2.13}$$

and consider the following sets of generators each of which corresponds to a consistent 4-dimensional string model:

$$\begin{aligned} G_1 &= \{F, s\} \\ G_2 &= \{F, s, \bar{s}\} \\ G_3 &= \{F, s, \bar{s}, b\} \end{aligned} \tag{2.14}$$

The set Σ_1 is given for example by:

$$\Sigma_1 = \{F, s, Fs, \emptyset\} \tag{2.15}$$

while Σ_2 and Σ_3 involve correspondingly more elements. We note that the models corresponding to Σ_1, Σ_2 and Σ_3 have space-time supersymmetry $N = 4, 8$ and 4 respectively (see Chapter 5). Although this high supersymmetry seems extremely unreasonable from a "phenomenological" point of view there exists an entire class of more realistic models [DKV], with $N = 4, 2$ or 1, whose amplitudes contain as principal building block the amplitudes of the models described by Σ_1, Σ_2 and Σ_3.

To illustrate how to build a model from the above rules we construct now the sum over spin structures, and correspondingly the scattering amplitudes, for the simplest model, which is the one given by Σ_1, even though in the following we will consider mainly the models $\Sigma_{2,3}$.

We begin by writing (2.2)) for Σ_1 (writing only the spin structure dependent part):

$$\begin{aligned}
Z_1 = \; & C(\emptyset|\emptyset)\theta^4\begin{bmatrix}0\\0\end{bmatrix}\theta^6\begin{bmatrix}0\\0\end{bmatrix}\bar{\theta}^{10}\begin{bmatrix}0\\0\end{bmatrix} + C(\emptyset|s)\theta^4\begin{bmatrix}0\\1\end{bmatrix}\theta^6\begin{bmatrix}0\\0\end{bmatrix}\bar{\theta}^{10}\begin{bmatrix}0\\0\end{bmatrix} \\
+ \; & C(s|\emptyset)\theta^4\begin{bmatrix}1\\0\end{bmatrix}\theta^6\begin{bmatrix}0\\0\end{bmatrix}\bar{\theta}^{10}\begin{bmatrix}0\\0\end{bmatrix} + C(\emptyset|Fs)\theta^4\begin{bmatrix}0\\0\end{bmatrix}\theta^6\begin{bmatrix}0\\1\end{bmatrix}\bar{\theta}^{10}\begin{bmatrix}0\\1\end{bmatrix} \\
+ \; & C(\emptyset|F)\theta^4\begin{bmatrix}0\\1\end{bmatrix}\theta^6\begin{bmatrix}0\\1\end{bmatrix}\bar{\theta}^{10}\begin{bmatrix}0\\1\end{bmatrix} + C(s|Fs)\theta^4\begin{bmatrix}1\\0\end{bmatrix}\theta^6\begin{bmatrix}0\\1\end{bmatrix}\bar{\theta}^{10}\begin{bmatrix}0\\1\end{bmatrix} \\
+ \; & C(Fs|\emptyset)\theta^4\begin{bmatrix}0\\0\end{bmatrix}\theta^6\begin{bmatrix}1\\0\end{bmatrix}\bar{\theta}^{10}\begin{bmatrix}1\\0\end{bmatrix} + C(Fs|s)\theta^4\begin{bmatrix}0\\1\end{bmatrix}\theta^6\begin{bmatrix}1\\0\end{bmatrix}\bar{\theta}^{10}\begin{bmatrix}1\\0\end{bmatrix} \\
+ \; & C(F|\emptyset)\theta^4\begin{bmatrix}1\\0\end{bmatrix}\theta^6\begin{bmatrix}1\\0\end{bmatrix}\bar{\theta}^{10}\begin{bmatrix}1\\0\end{bmatrix}
\end{aligned} \tag{2.16}$$

In this case (2.9)) is sufficient to fix all the phases C because $b_i \cap b_j \neq \emptyset$ for all $b_i, b_j \in G_1$:

$$\begin{aligned}
& C(\emptyset|\emptyset) = C(\emptyset|F) = C(s|Fs) = C(Fs|s) = C(F|\emptyset) = +1 \\
& C(\emptyset|s) = C(s|\emptyset) = C(\emptyset|Fs) = C(Fs|\emptyset) = -1
\end{aligned} \tag{2.17}$$

Defining:

$$\begin{aligned} \theta_1 &= \theta\begin{bmatrix}1\\1\end{bmatrix} = 0 \\ \theta_2 &= \theta\begin{bmatrix}1\\0\end{bmatrix} \\ \theta_3 &= \theta\begin{bmatrix}0\\0\end{bmatrix} \\ \theta_4 &= \theta\begin{bmatrix}0\\1\end{bmatrix} \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \eta_2 &= -1 = \eta_4 \\ \eta_3 &= +1 \end{aligned} \tag{2.19}$$

we obtain:

$$Z_1 = \left(\sum_{i=2}^{4} \eta_i |\theta_i|^{12} \bar{\theta}_i^4\right) \left(\sum_{j=2}^{4} \eta_j \theta_j^4\right) \tag{2.20}$$

The cosmological constant, eq.(2.20), is manifestly modular invariant due to the well known transformation properties of the Jacobi theta-functions; in fact it vanishes due to the Jacobi identity as it should for "realistic" models.

When we have also insertion of vertex operators, or, more specifically, of an operator O_{ij} which depends on the spin structures only through ψ_i, with spin structure i, and $\bar{\psi}_j$, with spin structure j, we get:

$$< O >_1 = \sum_{i=2}^{4} \sum_{j=2}^{4} \eta_i \eta_j \langle O_{ij} \rangle |\theta_i|^{12} \theta_i^4 \bar{\theta}_j^4 \tag{2.21}$$

Actually (2.21) holds if we suppose that 1/2–fields with odd spin structure do not contribute. This depends on how many, and which, 1/2-fields are present in the operator O_{ij}. As we shall explain in section 4.2, in our applications the odd spin structures will never contribute to the amplitudes and so we can apply (2.21).

For what concerns the models $\Sigma_{2,3}$ we note that their partition functions contain always Z_1 and in addition pieces corresponding to the elements in $\Sigma_{2,3}$ which are not present in Σ_1. In these cases, due to the fact that $s \cap \bar{s} = \emptyset$, the unique phase which is not fixed by (2.5)-(2.9) is $C(s|\bar{s})$ while all other generators have intersections among them self which are different from $\emptyset$. Remarkably enough both choices, $C(s|\bar{s}) = 1$ and $C(s|\bar{s}) = -1$, lead (as must be) to modular invariant theories while only $C(s|\bar{s}) = 1$ leads to a vanishing cosmological constant. We will therefore choose:

$$C(s|\bar{s}) = +1 \tag{2.22}$$

Repeating then the above procedure, which is now slightly more involved, (2.21) is completed to the following left-right symmetric form for $< O >_2$:

$$< O >_2 = \left(\sum_{i=2}^{4} \sum_{j=2}^{4} \eta_i \eta_j \langle O_{ij} \rangle \, \theta_i^4 \bar{\theta}_j^4 \right) \left(\sum_{k=2}^{4} |\theta_k|^{12} \right) \tag{2.23}$$

For $< O >_3$ we get instead:

$$< O >_3 = < O >_2 + \sum_{i<j=2}^{4} |\theta_i|^8 |\theta_j|^8 \left(|\theta_i|^4 + |\theta_j|^4 \right) \langle O_{ii} - O_{ij} - O_{ji} + O_{jj} \rangle \tag{2.24}$$

Usually in string theory the spin structure dependent part of O_{ij} factorizes in a left and a right part:

$$O_{ij} = O_i \bar{O}_j \tag{2.25}$$

In this case we get:

$$< O >_3 = < O >_2 + \sum_{i<j=2}^{4} |\theta_i|^8 |\theta_j|^8 \left(|\theta_i|^4 + |\theta_j|^4 \right) \left\langle \left(O_i - O_j \right) \left(\bar{O}_i - \bar{O}_j \right) \right\rangle \tag{2.26}$$

Also in (2.26) the odd spin structures are supposed to not contribute to the right hand side (see chapter 5). The cosmological constants are recovered from (2.23) and (2.24) setting $O_{ij} = 1$ and are easily seen to vanish.

The model described by Σ_2 (which contains in some sense the model Σ_1) is the simplest one and will be used to study the scattering of a massive particle with a graviton; in fact, in this case the non-renormalization theorem does not hold (even in 10 dimensions) and we expect genuine field theoretic one-loop corrections to gravity.

The model Σ_3 will be used to investigate 4-graviton scattering amplitudes; it is this model, with low space-time supersymmetry, which allows non vanishing two-and three-graviton amplitudes. Thus this is the appropriate situation where the new effects we are searching for should appear.

Before entering in the details of the computation of these amplitudes we will in the next section present the hyperelliptic language, which is appropriate, as we will see, for the extraction of field-theoretic corrections from superstrings and give some useful formulae.

3. The hyperelliptic language

3.1 The hyperelliptic torus and the main correlators

Every genus one (and two) Riemann surface can be realized as a hyperelliptic surface in CP^2. We will give here only the explicit formulas for the genus one case, which is the one needed in the following, generalizations being almost straightforward [BR,GIS,KN,MO].

The genus one surface is described by:

$$y = \pm \sqrt{\prod_{i=1}^{4} (z - a_i)} \tag{3.1}$$

where the complex numbers $\{a_i\}$ are the four branch points three of which are fixed by Möbius invariance while the remaining one is the modulus of the torus. However, sometimes other choices for the gauge fixing will be more convenient. Modular invariance, in this language, corresponds to complete symmetry in the branch points. As a convention the + sign in (3.1) refers to the upper sheet while the - sign refers to the lower sheet; we denote the corresponding points on the two sheets as $z\pm$. The torus is thus seen as a double covering of the Riemann sphere, S^2, introducing, for example, analyticity cuts from a_1 to a_2 and from a_3 to a_4, see Fig.2.

The unique abelian holomorphic differential is given by:

$$\Omega(z) = \frac{dz}{y(z)} \tag{3.2}$$

where as uniformizer coordinate near the branch points on should use u:

$$u^2 = z - a_i \tag{3.3}$$

Choosing the canonical homology basis as shown in Fig.2 we define:

$$K = \oint_a \Omega(z) dz \tag{3.4}$$

and

$$\omega(z) = \frac{\Omega(z)}{K} \tag{3.5}$$

$\omega(z)$ is then the usual abelian differential with the canonical normalization:

$$\oint_a \omega(z) dz = 1 \tag{3.6}$$

The period "matrix" is defined as:

$$\tau = \oint_b \omega(z) dz \tag{3.7}$$

A spin structure on the hyperelliptic torus is defined by splitting the branch points into two non intersecting sets. The three even ones are given by the pairs:

$$(A_1, A_2 | B_1, B_2) \tag{3.8}$$

while the odd one is given by:

$$(a_1, a_2, a_3, a_4 | \emptyset) \tag{3.9}$$

with the identifications:

$$\begin{aligned}
(12\ 34) &\longleftrightarrow \begin{bmatrix}1\\1\end{bmatrix} \longleftrightarrow \mathbf{1}\\
(12|34) &\longleftrightarrow \begin{bmatrix}1\\0\end{bmatrix} \longleftrightarrow \mathbf{2}\\
(13|24) &\longleftrightarrow \begin{bmatrix}0\\0\end{bmatrix} \longleftrightarrow \mathbf{3}\\
(14|23) &\longleftrightarrow \begin{bmatrix}0\\1\end{bmatrix} \longleftrightarrow \mathbf{4}
\end{aligned} \tag{3.10}$$

The connection with the θ-function formalism is established by the Thomae [F] formula (for i=2,3,4):

$$\theta_i^4(0|\tau) = \pm K^2(a) Q_i(a) \tag{3.11}$$

where Q_i is defined by:

$$Q_i = \prod_{k<l=1}^{2} A_{kl}B_{kl} \tag{3.12}$$

and $A_{kl} \equiv A_k - A_l$, $B_{kl} \equiv B_k - B_l$.

The role of the Dedekind eta-function, $\eta(\tau)$, is played by the completely antisymmetric polynomial P:

$$P(a) = \prod_{i<j=1}^{4} a_{ij} = Q_2 Q_3 Q_4 \tag{3.13}$$

where $a_{ij} \equiv a_i - a_j$. In fact, translating Jacobi's triple product formula for $\eta(\tau)$ [HP] in the hyperelliptic language and using the Thomae formula (3.11) we obtain:

$$16\eta^{12}(\tau) = K^6(a)P(a) \tag{3.14}$$

Another useful formula is given by:

$$T(a) \equiv \int \frac{d^2z}{|y(z)|^2} = \frac{1}{2}|K(a)|^2 \mathrm{Im}\tau \tag{3.15}$$

We give now the correlators of primary fields which will be used in the computation of scattering amplitudes. These correlators are uniquely fixed imposing the right transformation properties under conformal transformations, requiring holomorphicity and by cancelling poles when points lie on opposite sheets. We introduce the convenient notation

$$\begin{aligned} z_{ab} &\equiv z_a - z_b \\ \nu_{ab} &\equiv \int_{z_a}^{z_b} \omega(z)dz \end{aligned} \tag{3.16}$$

First we give the correlator:

$$\begin{aligned} < \partial X(z_1)X(z_2) > &= \frac{1}{2}\frac{1}{z_{12}} + \frac{1}{2T}\int \frac{y(z_2)}{y(z_1)}\frac{1}{z_{12}}\frac{z_1 - z}{z_2 - z}\frac{d^2z}{|y(z)|^2} \\ &= \frac{1}{2}\frac{1}{z_{12}}\left(1 + \frac{y(z_2)}{y(z_1)}\right) + \frac{1}{2T}\frac{y(z_2)}{y(z_1)}\int \frac{1}{(z_2 - z)}\frac{d^2z}{|y(z)|^2} \end{aligned} \tag{3.17}$$

This correlator is well defined when it appears in a combination in which the total charge is 0:

$$\sum_{i=1}^{n} < \partial X(z_1)X(i) > c_i, \quad \sum_{i=1}^{n} c_i = 0 \tag{3.18}$$

Deriving (3.17) with respect to z_2 and symmetrizing in 1 and 2 one gets easily the one-loop version of the two-loop formula given by Knizhnik [KN2]:

$$< \partial X(z_1)\partial X(z_2) >= \left[\frac{1}{4z_{12}^2} + \frac{1}{4T}\frac{\partial}{\partial z_2}\int \frac{d^2z}{|y(z)|^2}\frac{y(z_2)}{y(z_1)}\frac{1}{z_{12}}\frac{z_1 - z}{z_2 - z}\right] + (1 \leftrightarrow 2) \tag{3.19}$$

Deriving (3.17) with respect to $\bar{z}_2$ we get:

$$< \partial X(z_1)\bar{\partial} X(z_2) >= \begin{cases} -\pi\delta^2(z_{12}) + \dfrac{\pi}{2Ty(z_1)\bar{y}(\bar{z}_2)} & z_1, z_2 \text{ on the same sheet} \\ \dfrac{\pi}{2Ty(z_1)\bar{y}(\bar{z}_2)} & z_1, z_2 \text{ on opposite sheets} \end{cases} \tag{3.20}$$

As a consistency check of equation (3.20) one verifies easily that

$$\int \omega(z_2) < \partial X(z_1)\bar{\partial} X(z_2) > d^2z_2 = 0 \tag{3.21}$$

where in (3.21) the left hand side has to be viewed as a double-sheeded integral.

Deriving, instead, (3.17) with respect to $\bar{z}_1$ and summing over a neutral system of charges we get:

$$\sum_{i=1}^{n} c_i < \partial\bar{\partial} X(z_1)X(z_i) >= \pi \sum_{i=1}^{n} c_i\delta^2(z_{1i}) \tag{3.22}$$

where again the delta function on the right hand side has to be dropped if z_1 and z_i lie on opposite sheets.

Equation (3.22) permits to derive the following formula whose importance will become clear in the next section:

$$\begin{gathered} < (X(1) - X(3))(X(2) - X(4)) >= \\ -\frac{1}{\pi}\int d^2z < \partial X(z)(X(1) - X(3)) >< \bar{\partial} X(z)(X(2) - X(4)) > \end{gathered} \tag{3.23}$$

Also the integral in (3.23) is an integral over a double sphere.

Sometimes it is more useful to work with a θ-function representation of the X-correlators. By general arguments [B] one can give a formula for the propagator $< X(1)X(2) >$ in terms of the odd θ-function $\theta\left[{1 \atop 1}\right](\nu|\tau) \equiv \Theta(\nu)$:

$$< X(1)X(2) >= \ln\left|\frac{\Theta(\nu_{12})}{h(1)h(2)}\right|^2 - 2\pi\frac{(\mathrm{Im}\nu_{12})^2}{\mathrm{Im}\tau} \tag{3.24}$$

In (3.24) we defined the 1/2 differential h(z) by:

$$h(z) \equiv \omega(z)\left.\frac{\partial\Theta(\nu)}{\partial\nu}\right|_{\nu=0} \tag{3.25}$$

while the odd Θ-function in turn is given by [GSW]:

$$\begin{aligned}
\Theta(\nu) &= 2f(q^2)q^{\frac{1}{4}}sen(\pi\nu)\prod_{n=1}^{\infty}(1-2q^{2n}\cos(2\pi\nu)+q^{4n}) \\
\left.\frac{\partial\Theta(\nu)}{\partial\nu}\right|_{\nu=0} &= 2\pi f^3(q^2)q^{\frac{1}{4}} \\
f(q^2) &= \prod_{n=1}^{\infty}(1-q^{2n}) = \eta(\tau)q^{-1/12} \\
q &= e^{i\pi\tau}
\end{aligned} \tag{3.26}$$

The last formula needed is the propagator for the 1/2 - differentials [MU]:

$$< \psi_\alpha(x)\psi_\beta(y) >_i= \frac{\delta_{\alpha\beta}}{x-y}\cdot\frac{u_i(x)+u_i(y)}{2\sqrt{u_i(x)u_i(y)}} \tag{3.27}$$

where:

$$u_i(x\pm) \equiv \pm\prod_{i=1}^{2}\left(\frac{x-A_i}{x-B_i}\right)^{1/2} \tag{3.28}$$

In the last equation the $\pm$ signs correspond to the two different sheets of the Riemann sphere and are needed in order to cancel the pole when x and y lie on opposite sheets.

3.2 A degenerate configuration in the moduli space

The formulae given in the preceeding section involve frequently integral representations which can not be simplified further. However, in the limit

$$
\begin{aligned}
a_2 &\to a_1 \qquad x \equiv \ln\left|\frac{a_{13}}{a_{12}}\right| \to +\infty \\
a_4 &\to a_3 \qquad y \equiv \ln\left|\frac{a_{13}}{a_{34}}\right| \to +\infty \ .
\end{aligned}
\tag{3.29}
$$

the integrals can be explicitly evaluated. The limit (3.29), to which in the following we will refer to as "pinching limit", is precisely the one we are interested in in our later calculations; in fact it corresponds, as we will see in Chapters 4 and 5, to the one–loop exchange of gravitons between energetic external particles [GIZ,IL]: the hyperelliptic surface splits into two spheres which are attached to each other through the punctures $\widehat{a_1, a_2}$ and $\widehat{a_3, a_4}$. We will now evaluate some of the previous formulae in this degenerate configuration of the moduli space. It is straightforward to obtain:

$$
\begin{aligned}
y(z) &\to \pm(z-a_1)(z-a_3) \\
P(a) &\to (a_{13})^4 a_{12} a_{34} \\
K &\to \frac{2\pi i}{a_{13}} \\
\omega(z) &\to \pm\frac{1}{2\pi i}\left(\frac{1}{z-a_1} - \frac{1}{z-a_3}\right) \\
T &\to \frac{2\pi}{|a_{13}|^2}(x+y) \\
\mathrm{Im}\tau &\to \frac{x+y}{\pi} \\
\mathrm{Im}\nu_{12} &\to \frac{1}{2\pi}\ln\left|\frac{(z_1-a_1)(z_2-a_3)}{(z_1-a_3)(z_2-a_1)}\right| \\
\mathrm{Im}\nu_{23} &\to \frac{x}{\pi} + \frac{1}{2\pi}\ln\left|\frac{(z_2-a_1)(z_3-a_1)}{(z_2-a_3)(z_3-a_3)}\right| \\
P(a) &\to a_{12} a_{34} (a_{13})^4 \\
\left(\sum_{k=2}^{4} |Q_k|^3\right) &\to 2|a_{13}|^6
\end{aligned}
\tag{3.30}
$$

The relations for $\mathrm{Im}\nu_{12}$ and $\mathrm{Im}\nu_{23}$ are valid if z_1 and z_2 lie on the (same) upper sheet and z_3 lies on the lower sheet. For ν_{23} we chose an integration path which crosses the cut between a_1 and a_2, giving an $\frac{\pi}{\pi}$. Analogous relations hold for other configurations, e.g. if z_1 and z_2 lie on the lower sheet the formula for $\mathrm{Im}\nu_{12}$ changes its sign and so on.

Equation (3.17) becomes, in first approximation:

$$< \partial X(z_1)X(z_2) > \longrightarrow$$

$$\begin{cases} \dfrac{1}{z_{12}} - \dfrac{1}{2}\dfrac{1}{\ln|a_{12}a_{34}|}\left[\dfrac{\ln|a_{12}|}{(z_1-a_3)} + \dfrac{\ln|a_{34}|}{(z_1-a_1)}\right] & z_1, z_2 \text{ on the same sheet} \\ +\dfrac{1}{2}\dfrac{1}{\ln|a_{12}a_{34}|}\left[\dfrac{\ln|a_{12}|}{(z_1-a_3)} + \dfrac{\ln|a_{34}|}{(z_1-a_1)}\right] & z_1, z_2 \text{ on different sheets} \end{cases} \tag{3.31}$$

Eq. (3.31), when inserted in (3.23), would produce a formula for $G(1,2,3,4) \equiv < (X(1)-X(3))(X(2)-X(4)) >$. We are interested in this formula when 1 and 2 lie on the upper sheet and 3 and 4 on the lower sheet (see Fig.4).

However, to illustrate the connection between the θ-function formalism and the hyperelliptic language we will derive now that formula from (3.24)–(3.26). We have in fact:

$$G(1,2,3,4) = \ln\left|\frac{\Theta(\nu_{12})\Theta(\nu_{34})}{\Theta(\nu_{23})\Theta(\nu_{14})}\right|^2 + 4\pi\frac{\mathrm{Im}\nu_{13}\cdot\mathrm{Im}\nu_{24}}{\mathrm{Im}\tau} \tag{3.32}$$

Under the pinching limit (3.29) $\mathrm{Im}\tau \to +\infty$ and $q \to 0$. Therefore:

$$\left|\frac{\Theta(\nu_{12})\Theta(\nu_{34})}{\Theta(\nu_{23})\Theta(\nu_{14})}\right|^2 \to \left|\frac{\sin(\pi\nu_{12})\sin(\pi\nu_{34})}{\sin(\pi\nu_{23})\sin(\pi\nu_{14})}\right|^2 \tag{3.33}$$

Under the same limit $\mathrm{Im}\nu_{23}$, $\mathrm{Im}\nu_{14} \to +\infty$, as can be seen from (3.30) and an analogous formula for $\mathrm{Im}\nu_{14}$, while, $\mathrm{Im}\nu_{12}$ and $\mathrm{Im}\nu_{34}$ acquire finite limits (see (3.30)). We have therefore for the formers:

$$|\sin(\pi\nu)|^2 = \frac{1}{2}[\cosh(2\pi\mathrm{Im}\nu) - \cos(2\pi Re\nu)] \to \frac{1}{4}e^{2\pi\mathrm{Im}\,\nu} \tag{3.34}$$

Using (3.34) for ν_{23} and ν_{14} we get:

$$\begin{aligned} G(1,2,3,4) \to \ln|4\sin(\pi\nu_{12})\sin(\pi\nu_{34})|^2 + 2\pi\left(\mathrm{Im}\nu_{12} - \mathrm{Im}\nu_{34}\right) \\ - 4\pi\frac{\mathrm{Im}\nu_{13}\cdot\mathrm{Im}\nu_{42}}{\mathrm{Im}\tau} \end{aligned} \tag{3.35}$$

Due to the well known transformation properties of the odd θ-function under the two independent modular transformations of the torus [GSW] the formula (3.24) is independent of the integration path we choose for ν_{12}. In contrast, in (3.35) the integration pathes of ν_{13} and ν_{42} are, by definition, the ones depicted in Fig.4 while ν_{12} and ν_{34} are obtained integrating on the upper and lower sheet respectively (without crossing any cut).

Substituting now the relations (3.30) in (3.35) we obtain finally:

$$\begin{aligned} G(1,2,3,4) \to & \ln|2\sin(\pi\nu_{12})|^2 + 2\pi\mathrm{Im}\nu_{12} \\ & + \ln|2\sin(\pi\nu_{34})|^2 - 2\pi\mathrm{Im}\nu_{34} \\ & - \frac{4xy}{x+y} - \frac{4\pi x}{x+y}(\mathrm{Im}\nu_{12} - \mathrm{Im}\nu_{34}) \\ & + \frac{2(x-y)}{x+y}(f(1)+f(3)) + \frac{(f(1)+f(3))(f(2)+f(4))}{x+y} \end{aligned} \tag{3.36}$$

where we defined:

$$f(z) = \ln\left|\frac{z-a_1}{z-a_3}\right| \tag{3.37}$$

The (only apparent) asymmetric form of (3.36) has been chosen for later convenience and adapted for the situation(s) we are going to study.

Finally we give a formula whose derivation requires a more accurate approximation then (3.31) in that it goes to zero under the pinching limit (3.29) (z_0 lies on the upper sheet and $z_{3,4}$ lie on the lower sheet):

$$< \partial X(0)X(3) > - < \partial X(0)X(4) > \to \pi\frac{a_{13}\mathrm{Im}\nu_{34}}{y(0)(x+y)} \tag{3.38}$$

(3.38) can be derived using the second equation of (3.17) and several times the pinching limit identity:

$$z_{ab}y(c) + z_{bc}y(a) + z_{ca}y(b) = z_{ab}z_{ac}z_{bc} \tag{3.39}$$

where here all y's are supposed (by definition) to have the plus sign in front of the square root.

A (more quick) alternative derivation of this formula runs through the application of (3.24) noting that in the limit (3.29) the θ-function contribution drops out of the difference at the r.h.s. of (3.38) while the "period matrix" part of (3.24) (using again (3.30)) gives directly (3.38). This illustrates, once more, the relation existing between the hyperelliptic language and the canonical θ-function formalism.

To conclude this chapter we note that in the hyperelliptic approach the projective invariance survives also at the loop level, due to the fact that we are working on a cut *sphere*. This means that, in order to be able to factorize out the (infinite) volume of the projective group, $G = PSL(2,C) = SL(2,C)/\{\pm 1\}$, of a scattering amplitude in a consistent manner, the amplitude itself must be invariant under G. To be more precise, we require invariance under the following Möbius transformations:

$$z \longrightarrow \frac{az+b}{cz+d} \tag{3.40}$$

with:

$$ad - bc = 1 \tag{3.41}$$

When checking that the formulae given in this chapter have the right "Möbius weight" the key transformation property is $z_{AB} \rightarrow z_{AB}/\gamma(A)\gamma(B)$ where $\gamma(z) \equiv cz + d$. When checking Möbius invariance in the approximated formulas (3.30)–(3.38) one has to take into account in a consistent manner the limit we have taken.

In the next chapter we will construct the 4-point amplitude corresponding to the scattering of a graviton with a massive scalar particle and use the relations of this section to extract the physical informations we are searching for.

4. A Compton-like scattering

4.1 Motivations

The present formulation of string perturbation theory deals with *on shell* scattering amplitudes and therefore questions about Quantum Gravity have to be put in terms of scattering experiments.

The first thing to study seems to be the scattering among the lowest states of the theory, i.e. massless particles, gravitons or photons etc. The scattering amplitude at tree level is dominated in the limit of $s \to \infty$ and $t \to 0$ (s being the Mandelstam variable squared energy in the CM frame and t the square of the momentum transfer) by the one graviton exchange, due to the fact that it is massless, giving a pole 1/t, and that it couples to the energy, giving a power of s. The tree level corresponds of course to (a relativistic version of) Newton's potential.

One can then compute the loop contributions in the same limit. Since $t \to 0$ corresponds to large distances, this will provide correction terms of the form of powers of {(energy) $\times$ (Newton's constant)/distance }. Indeed, this expansion is suitable to be studied by perturbation theory in a configuration where the distance is large as compared to the "Schwarzschild radius" (energy $\times$ Newton's constant) as it is likely to be typical of possible astronomical observations. A second order correction of this kind has been in fact evaluated in Superstring Theory at two loops in ref. [GIZ]. It happens that, due to non–renormalization theorems, the scattering amplitude for lowest (massless) string states receives interesting contributions coming from non linear gravity interactions beginning at two loops. This is the case for superstrings and heterotic strings in D=10 and also for compactified models with highest space-time supersymmetry (e.g. N=8 in D=4 for type II strings) which represent the simplest and more viable cases for doing explicit computations. In fact, in these models the one-loop order gives rise, in the above limit, to rescattering terms, which are dictated by unitarity [ACV,SU] where gravitons are exchanged between the external massless particles, but do not interact among themselves, whereas at two loops they begin to interact.

In this chapter we present an analysis of the one-loop contribution to the scattering amplitude of a massless particle (say, a graviton) against a massive one (a scalar of the first excited level) with mass $\alpha' M^2 = 4$ in a type II string model (the one characterized by G_2 in (2.14)). This can be seen as a gravitational version of a Compton scattering in Q.E.D. The just mentioned model lives in a four dimensional target space equipped with an N=8 space-time supersymmetry, but this time, due to the presence of massive vertices, the non renormalization theorem does no longer hold. Therefore the configuration of Fig.1, where the blob represents a string one-loop three-point function, gives a non vanishing contribution and can be interpreted as a first order correction to the gravitational interaction of the massless particle (wavy external lines) and the massive one (solid lines). The vertex correction of Fig.1 will provide a factor V(t) which multiplies the Gs^2/t term of the tree level single graviton exchange, G being Newton's constant. We are interested in the IR behavior for $t \to 0$, where V(t) is expected to contain a term of order $R_s = GM$, the Schwarzschild radius of the massive particle. This would give for dimensional reasons $V(t) \sim R_s\sqrt{t}$ corresponding in configuration space to a factor $\sim R_s/r$ multiplying the Newton potential. The singular behavior $\sim \sqrt{t}$ of the graph is related to the exchange of massless particles in the t-channel. In our high energy configuration the gravitons dominate and at the first order in G the effect is due to a three graviton interaction (section 4.4).

Our task is to look for a term of this kind at one-loop. By general reasons on infrared behavior it is expected to come from a particular degenerate configuration (a "pinching limit") in the moduli space. We found it convenient to carry out this analysis by using the hyperelliptic formalism which was also used in [GIZ] and has been proved to be very powerful in providing the first explicit description of the two-loop amplitude in Superstring Theory [IZ].

In section 4.2 we construct the amplitude and give the vertices which describe the specific scattering we are going to study. In section 4.3 we work out the relevant pinching limit in the moduli space and give some one–loop summation formula for the correlators which are needed when massive vertices are present. In section 4.4 we derive our main result, i.e. precisely a term of the above form, while in section 4.5 we make a comparison of our computation with would-be Quantum Gravity Feynman diagrams.

As a last remark we stress again that we are working (at one loop order) in a perturbative framework, corresponding to a study of large distances, i.e. of the infrared region $t \to 0$, the expansion parameter being the Schwarzschild radius over distance. This has not to be confused with different studies of asymptotic configurations in String Theory, like the $s, t \to \infty$ limit at fixed t/s of Ref. [GM] which requires an analysis of the amplitudes at all orders. In particular we do not attempt to resum a perturbative series of which we have only evaluated the first two terms (a task which seems beyond the available knowledges in String Theory) and therefore we cannot answer questions like the presence of a Schwarzschild singularity. Our computation should rather be seen as an evaluation of Post–Newtonian corrections in a String based Quantum Gravity.

4.2 Construction of the amplitude

The model that we will use in this chapter to search for Post–Newtonian corrections to Gravity is the one characterized by G_2 in (2.14) where the corresponding sum over spin structures is given in (2.23). We adopt the functional approach to string theory [P,VV] to write down the amplitude and will not enter into details because, at least in the case of the torus, the functional approach has been largely established [HP] and is well known in the literature. We observe only that on the torus we have one zero mode for each p-differential with p integer, in particular also for the b,c ghosts, and one zero mode for the 1/2–differentials with odd spin structure while we have none for the 1/2–differentials with even spin structure. The b,c-ghost zero modes are absorbed, as usual, by the insertion of b(x)c(y) in the path integral: the b zero mode represents the unique holomorphic two-differential on the torus, which in the hyperellptic language is the square of (3.2), while the c zero mode represents the unique conformal Killing vector, i.e. translational invariance. So, in addition to the volume of the group of "small" diffeomorphisms, we have to factor out also the volume of the group of translations, which in the θ–function approach is simply Imτ [HP], the area of the Fuchsian-torus.

For what concerns the 1/2–differentials with *odd* spin structure we note that, due to the presence of one zero mode, in order to get a non vanishing contribution to the amplitude, the product of all the vertex operators should contain at least one fermion field of each type, to soake up the zero modes of the corresponding Dirac determinants. If the vertices contain, besides the X–fields, only the ψ^μ (as in the case we will consider), there are a priori, see below, enough ψ^μ–fields to soake up the four zero modes of the space–time fermions. However, as one can deduce easily from (2.13,14), in the G_2–model the 6 x^I–fields have always the same spin structure as the ψ^μ, in the present case the odd one. Due to the fact that none of the x^I fields will be present in our vertices, their six zero modes send the contributions with odd spin structures to zero.

Far what concerns the world–sheet gravitini zero modes, i.e. holomorphic 3/2–differentials, we note that on the torus there is no such zero mode for the even spin structures, while there is precisely one zero mode for the odd ones. As we have just seen the last ones do not contribute and so we have no supercurrent insertions. This has to be viewed as opposite to the case $g > 1$ where there are

2(g-1) gravitini zero modes, independently of the spin structure.

Considering thus, in four dimensions, only the even spin structures, we can write the one–loop contribution to the amplitude following (2.21) [GIS,GIZ]:

$$
\begin{aligned}
A_1 = G^2 \int \frac{dM(a)}{(\det\ \mathrm{Im}\tau)^{D/2}} \frac{1}{T} \left|\det\partial_1\right|^{-4}_{X^\mu} \left|\det\partial_2\right|^2_{b,c} \left|\frac{(\det\partial_{1/2})_h}{(\det\partial_{3/2})_h}\right|^2_{(\beta,\gamma),\psi^L} \\
\left(\sum_{k=2}^{4} \left|(\det\partial_{1/2})_k\right|^{12}\right) \sum_{i,j=2}^{4} \eta_i\eta_j (\det\partial_{1/2})_i^{\frac{4}{}} (\det\bar\partial_{1/2})_j^{\frac{4}{}} \left\langle \prod_{l=1}^{4} O_l \right\rangle_{ij}
\end{aligned}
\tag{4.1}
$$

Here G denotes Newton's constant in four dimensions while our choice for α', which is consistent with the normalizations of the propagators (3.17), (3.24) and (3.27), is $\alpha' = 2$. We obmitted an overall normalization constant which has to be fixed by unitarity through factorization. In (4.1) we indicated explicitly the origin of the various determinants, (X, b, c, β, γ), and we put in evidence the determinants corresponding to two longitudinal space–time fermions ψ^L. The origin of the remaining fermionic determinants should be clear in view of equation (2.23). The index "h" refers to the spin structures of the ψ and of the β,γ–ghosts (as well as of the world sheet gravitino) which have to be equal as noted several times; dM(a) stands for the moduli measure whose explicit form will be given below. The factor $\frac{1}{T}$ in (4.1), corresponding to the inverse of the "area" of the hyperelliptic surface, results from factoring out the finite volume of the group of translations as mentioned above.

On hyperelliptic surfaces the determinants admit explicit and rather simple expressions [G,KN1,MO]:

$$
\begin{aligned}
\det\bar\partial_1 &= K(a)P(a)^{1/4} \\
\det\bar\partial_2 &= P(a)^{5/4} \\
\left(\det\bar\partial_{3/2}\right)_i &= P(a)^{3/8} Q_i^{1/4} \\
\left(\det\bar\partial_{1/2}\right)_i &= P(a)^{-1/8} Q_i^{1/4}
\end{aligned}
\tag{4.2}
$$

The normalizations of the determinants in (4.2), dictated by the normalizations of the zero modes (if any), have been taken from ref. [GIS]. Correspondingly the moduli measure takes the form:

$$dM(a) = \frac{\prod_{i=1}^{4} d^2 a_i}{dV_{pr}|P(a)|^2} \tag{4.3}$$

where dV_{pr} is the projective volume associated to the Möbius invariance of the hyperelliptic torus. Putting everthing together we get:

$$A_1 = G^2 \int \frac{\prod_{i=1}^{4} d^2 a_i}{|P(a)|^4 T^3 dV_{pr}} \left(\sum_{i,j=2}^{4} \eta_i \eta_j Q_i \bar{Q}_j \left\langle \prod_{l=1}^{4} O_l \right\rangle_{ij} \right) \left(\sum_{k=2}^{4} |Q_k|^3 \right) \tag{4.4}$$

We have still to specify the vertex operators $O_l = \int O_l(z) d^2 z$. We choose for $O_{1,2} = V_{1,2}$ the usual superstring vertex for gravitons [GSW]

$$V^j(z_j, \bar{z}_j) = (\varepsilon^j \cdot \partial X(j) + i A^j_{\alpha\beta} \psi^\alpha(j) \psi^\beta(j))(\bar{\varepsilon}^j \bar{\partial} X(j) + i \bar{A}^j_{\alpha\beta} \bar{\psi}^\alpha(j) \bar{\psi}^\beta(j)) e^{ik^j \cdot X(j)} \tag{4.5}$$

with

$$A^j_{\alpha\beta} \equiv k^j_{[\alpha} \varepsilon^j_{\beta]} \tag{4.6}$$

where the antisymmetrization is understood with unit weight. For $O_{3,4} = W_{3,4}$ we choose a scalar massive particle of the first excited level of the type II superstring spectrum with mass $M^2 = 2$ [ABIN2]:

$$\begin{aligned} W^j(z_j, \bar{z}_j) =& (3B^j_{\alpha\beta\gamma} \partial X(j)^\alpha \psi^\beta(j) \psi^\gamma(j) + i k^j_{[\alpha} B^j_{\beta\gamma\delta]} \psi^\alpha(j) \psi^\beta(j) \psi^\gamma(j) \psi^\delta(j)) \cdot \\ & (3\bar{B}^j_{\alpha\beta\gamma} \bar{\partial} X(j)^\alpha \bar{\psi}^\beta(j) \bar{\psi}^\gamma(j) + i k^j_{[\alpha} \bar{B}^j_{\beta\gamma\delta]} \bar{\psi}^\alpha(j) \bar{\psi}^\beta(j) \bar{\psi}^\gamma(j) \bar{\psi}^\delta(j)) \cdot \\ & e^{ik^j \cdot X(j)} \end{aligned} \tag{4.7}$$

The mass shell (and irreducibility) conditions are:

$$\begin{cases} \varepsilon_j k_j = k_j^2 = \bar{\varepsilon}_j \cdot \varepsilon_j = 0 & \\ \varepsilon_j^\alpha \bar{\varepsilon}_j^\beta = \varepsilon_j^{(\alpha} \bar{\varepsilon}_j^{\beta)} & \text{for gravitons} \\ k_j^\alpha B^j_{\alpha\beta\gamma} = 0, \; k_j^2 = M^2 & \\ B^j_{\alpha\beta\gamma} = B^j_{[\alpha\beta\gamma]} & \text{for scalars} \end{cases} \tag{4.8}$$

These relations imply, in particular, that we have not to worry about selfcontractions [ABIN2] and that the polarization tensors of the scalar satisfy:

$$\begin{aligned} B^j_{\alpha\beta\gamma} &= \varepsilon_{\alpha\beta\gamma\delta} \frac{k_j^\delta}{M} \\ k^j_{[\alpha} B^j_{\beta\gamma\delta]} &= -\frac{1}{4} M \varepsilon_{\alpha\beta\gamma\delta} \end{aligned} \tag{4.9}$$

Our key observation is that contrary to the four graviton amplitude in ten dimensional strings, or in four dimensional models with highest space-time supersymmetry, for the amplitude (4.4) the non renormalization theorem [MA,NNS] does no longer hold. In fact, there are now diagrams like the one in Fig.1: the exchange of a graviton in the t-channel which couples to the massive scalar via a non vanishing one-loop three-point function. In this paper we will examine this new feature and evaluate the contribution represented by Fig.1 in the limit:

$$\begin{aligned} s &= (k_1 + k_3)^2 \rightarrow \infty \\ t &= (k_1 + k_2)^2 \rightarrow 0 \end{aligned} \tag{4.10}$$

We should note that, properly speaking, the expression (4.4) is divergent due to the fact that string theories provide us with on-shell Green's functions which are one particle *reducible* and contain therefore (physical) singularities corresponding to the on shell poles of the propagators of the massive external lines as indicated in Fig.3. The physical scattering amplitude should then be obtained after a suitable subtraction. However, as we will see, the contribution we are going to study is not affected by these singularities and will in fact be finite.

4.3 The relevant pinching limit

The emission of the graviton in the t-channel as in Fig.1 corresponds to the limit in which the Koba-Nielsen variables z_1 and z_2 collide; so we have to evaluate V(1)V(2) in this limit. Setting $z_1 = z_0 + \frac{\Delta}{2}$ and $z_2 = z_0 - \frac{\Delta}{2}$ we get:

$$
\begin{aligned}
V(1)V(2)_{\Delta\to 0} \longrightarrow & \, \mathrm{e}^{-\frac{t}{2}\langle X(1)X(2)\rangle_{\Delta\to 0}} \mathrm{e}^{ik_0 X(0)} \cdot \left\{ \frac{1}{|z_{12}|^4} (\varepsilon_1\cdot\varepsilon_2)(\bar\varepsilon_1\cdot\bar\varepsilon_2)\left(1+\frac{t}{2}\right)^2 \right. \\
+\frac{1}{|z_{12}|^2}\Big[& -(\varepsilon\cdot X(0) + iA_{\alpha\beta}\psi^\alpha(0)\psi^\beta(0))(right) + \frac{i}{4}(\varepsilon_1\cdot\varepsilon_2)(\bar\varepsilon_1\cdot\bar\varepsilon_2)k_0\cdot\partial\bar\partial X(0) \\
& + c(0)\big((\bar\varepsilon_1\cdot\bar\varepsilon_2)(k_1\cdot\varepsilon_2)(k_2\cdot\varepsilon_1) \\
& + (\varepsilon_1\cdot\varepsilon_2)(k_1\cdot\bar\varepsilon_2)(k_2\cdot\bar\varepsilon_1)\big) + \big((\bar\varepsilon_1\cdot\varepsilon_2)(k_1\cdot\bar\varepsilon_2)(k_2\cdot\varepsilon_1) \\
& \left. + (\varepsilon_1\cdot\bar\varepsilon_2)(k_1\cdot\varepsilon_2)(k_2\cdot\bar\varepsilon_1)\big)\left\langle \partial X(0)\bar\partial X(0)\right\rangle_{reg} \Big] \right\}
\end{aligned}
\tag{4.11}
$$

We defined:

$$
\begin{aligned}
k_0 &= k_1 + k_2 \\
\varepsilon^\alpha &= (\varepsilon_1\cdot k_2)\varepsilon_2^\alpha - (\varepsilon_2\cdot k_1)\varepsilon_1^\alpha + \frac{1}{2}(\varepsilon_1\cdot\varepsilon_2)(k_1-k_2)^\alpha \\
A^{\alpha\beta} &= k_0^{[\alpha}\left(\varepsilon_1\cdot k_2\varepsilon_2^{\beta]} - \varepsilon_2\cdot k_1\varepsilon_1^{\beta]}\right) - \frac{t}{2}\varepsilon_1^{[\alpha}\varepsilon_2^{\beta]} - (\varepsilon_1\cdot\varepsilon_2)k_1^{[\alpha}k_2^{\beta]} \\
< \partial X(1)X(2) >_{\Delta\to 0} &\rightarrow \frac{1}{z_{12}} + d(0) + \bar z_{12}c(0) + \cdots
\end{aligned}
\tag{4.12}
$$

Of the last correlator in (4.11) we have to take only the regular part and we neglected contributions to the poles in z_{12} coming from the regular parts in $\mathrm{e}^{-\frac{t}{2}\langle X(1)X(2)\rangle}$ these last ones being of order o(t).

In (4.11) we note the presence of a quadruple pole in $|z_{12}|$, corresponding to the propagation of a tachyon, which is however canceled by the factor $(1+t/2)^2$, and a double pole which corresponds to the propagation of the massless state we are looking for. We will therefore concentrate on this last contribution.

First we remark that $\partial\bar\partial X(0)$ does not contribute to the amplitude in the limit (4.10). In fact, after summing over spin structures (see later), due to the fact that $\partial\bar\partial X(0)$ is spin structure independent, the surviving X-dependence in W(3)W(4) sits in the exponentials and we remain with:

$$
\begin{aligned}
&\left\langle k_0\cdot\partial\bar\partial X(0) e^{ik_3X(3)} e^{ik_4X(4)} e^{ik_0X(0)}\right\rangle = \\
&-\frac{i}{2}t\left[\langle\partial\bar\partial X(0)X(3)\rangle + \langle\partial\bar\partial X(0)X(4)\rangle\right]\left\langle e^{ik_0X(0)} e^{ik_3X(3)} e^{ik_4X(4)}\right\rangle
\end{aligned}
\tag{4.13}
$$

In deriving (4.13) we used the kynematical relations $k_0 \cdot k_3 = k_0 \cdot k_4 = -t/2$. So we see that this term is suppressed by a factor of t, the correlators in the square bracket containing δ-functions, see (3.22), which go to zero under analytic continuation on the external momenta, and zero mode projectors, (3.24), which do not contain any singularity in the Koba-Nielsen variables, giving thus a contribution to the amplitude which is order o(t). Moreover (4.13) does not exhibit any power of s, while the leading contributions, as we will see in the following, carry an s^2 factor. On the same footing the c-number terms in (4.11), even if they are not suppressed by a power of t, will not give any power of s.

Thus we remain with the following effective vertex:

$$
\begin{aligned}
V(1)V(2) \to & -\frac{1}{|z_{12}|^2} \mathrm{e}^{-\frac{t}{2}\langle X(1)X(2)\rangle_{\Delta\to 0}} \mathrm{e}^{ik_0 X(0)} \\
& (\varepsilon \cdot \partial X(0) + iA_{\alpha\beta}\psi^\alpha(0)\psi^\beta(0))(\bar{\varepsilon} \cdot \bar{\partial} X(0) + i\bar{A}_{\alpha\beta}\bar{\psi}^\alpha(0)\bar{\psi}^\beta(0)) \\
& \equiv V(0)
\end{aligned}
\tag{4.14}
$$

Next we evaluate the sum in the first bracket of eq. (4.4):

$$
\sum_{i,j=2}^{4} \eta_i \eta_j Q_i \bar{Q}_j \langle V(0)W(3)W(4)\rangle_{ij} \tag{4.15}
$$

To this order we apply Wick's theorem (first to fermions) and sum over spin structures. To carry out this sum we need the following identities which can easily be derived by explicit computation:

$$
\begin{aligned}
&\sum_{i=2}^{4} \eta_i Q_i \left(\frac{u(1)}{u(2)} + \frac{u(2)}{u(1)}\right)_i = 0 \\
&\sum_{i=2}^{4} \eta_i Q_i \left(\frac{u(1)u(2)}{u(3)u(4)} + \frac{u(3)u(4)}{u(1)u(2)}\right)_i = -P(a)\frac{z_{13}z_{14}z_{23}z_{24}}{\prod_{i=1}^{4} y(i)}
\end{aligned}
\tag{4.16}
$$

Using (4.16) we can evaluate all the kinds of fermionic contractions which appear in (4.15):

$$
\begin{aligned}
&\sum_{i=2}^{4}\eta_i Q_i \left\langle B^{\alpha\beta}\psi_\alpha(1)\psi_\beta(1)C^{\gamma\delta}\psi_\gamma(2)\psi_\delta(2)D^{\mu\nu\rho\sigma}\psi_\mu(3)\psi_\nu(3)\psi_\rho(3)\psi_\sigma(3)\right\rangle_i \\
&= -\frac{4!}{2^4}B^{\alpha\beta}C^{\gamma\delta}D_{\alpha\beta\gamma\delta}\frac{P(a)}{y(1)y(2)y(3)^2} \\
&\sum_{i=2}^{4}\eta_i Q_i \left\langle F^{\alpha\beta\gamma\delta}\psi_\alpha(1)\psi_\beta(1)\psi_\gamma(1)\psi_\delta(1)G^{\mu\nu\rho\sigma}\psi_\mu(2)\psi_\nu(2)\psi_\rho(2)\psi_\sigma(2)\right\rangle_i \\
&= -\frac{3}{2}P(a)\frac{F^{\alpha\beta\gamma\delta}G_{\alpha\beta\gamma\delta}}{[y(1)y(2)]^2} \\
&\sum_{i=2}^{4}\eta_i Q_i \left\langle F^{\alpha\beta\gamma\delta}\psi_\alpha(1)\psi_\beta(1)\psi_\gamma(1)\psi_\delta(1)G^{\mu\nu\rho\sigma}\psi_\mu(2)\psi_\nu(2)\psi_\rho(2)\psi_\sigma(2)\right. \\
&\qquad\qquad \left. H^{\varepsilon\varphi}\psi_\varepsilon(3)\psi_\varphi(3)\right\rangle_i = 0
\end{aligned}
\tag{4.17}
$$

The F, G and D tensors in (4.17) are ε–tensors times a constant (in 4 dimensions). Due to the antisymmetry of $H^{\varepsilon\varphi}$ in the last line of (4.17) no Lorentz-scalar can be formed and so the last identity follows. Specializing the generic polarizations B,C, etc. of (4.17) to the particular ones appearing in (4.15) we obtain:

$$
\begin{aligned}
&\sum_{i,j=2}^{4}\eta_i\eta_j Q_i\bar{Q}_j \left\langle V(0)W(3)W(4)\right\rangle_{ij} = -\frac{1}{|z_{12}|^2}\mathrm{e}^{-\frac{t}{2}\langle X(1)X(2)\rangle_{\Delta\to 0}}\left|\frac{9}{4}P(a)\right|^2 \cdot \\
&\left\langle\left(\frac{M^2\varepsilon\cdot\partial X(0)}{y(3)^2y(4)^2} - \frac{2\partial X^\alpha(3)k_3^\beta A_{\alpha\beta}}{y(1)y(3)y(4)^2} - \frac{2\partial X^\alpha(4)k_4^\beta A_{\alpha\beta}}{y(1)y(4)y(3)^2}\right)\cdot\Big(right\Big)\cdot\right. \\
&\left.\mathrm{e}^{ik_0X(0)}\mathrm{e}^{ik_3X(3)}\mathrm{e}^{ik_4X(4)}\right\rangle
\end{aligned}
\tag{4.18}
$$

We are now left with X-correlators only which can be computed by standard techniques. In addition to the contractions which are non zero at tree level we have now at one–loop also non vanishing mixed contractions between the two chirality sectors. However, it can be seen that those contractions lead to subleading contributions for $t \to 0$ and $s \to \infty$. This is due to the fact that they are made out of a δ-function contribution which goes to zero (under analytic continuation on external momenta as before) and of a projector onto the zero modes, eq. (3.24), which contains $\mathrm{Im}\tau$ at the denominator. As we will see below the small t behavior comes from the region $\mathrm{Im}\tau \to \infty$.

Neglecting, therefore, the mixed contractions and computing the kynematical factors in the leading orders of s and t we get:

$$
\begin{aligned}
&\sum_{i,j=2}^{4} \eta_i \eta_j Q_i \bar{Q}_j \langle V(0)W(3)W(4)\rangle_{ij} \to \frac{s^2(\varepsilon_1\cdot\varepsilon_2)(\bar{\varepsilon}_1\cdot\bar{\varepsilon}_2)}{|z_{12}|^2}\left|\frac{9}{8}P(a)\right|^2 \cdot \\
&\left\{\frac{M^2}{y(3)^2y(4)^2}\left(\langle\partial X(0)X(3)\rangle - \langle\partial X(0)X(4)\rangle\right)\right. \\
&-\frac{t}{y(0)y(4)^2y(3)}\left(\langle\partial X(3)X(0)\rangle - \langle\partial X(3)X(4)\rangle\right) \\
&\left.+\frac{t}{y(0)y(3)^2y(4)}\left(\langle\partial X(4)X(0)\rangle - \langle\partial X(4)X(3)\rangle\right)\right\}\cdot\left\{right\right\}\cdot \\
&\quad e^{-\frac{t}{2}<(X(1)-X(3))(X(2)-X(4))>_{\Delta\to 0}+M^2<X(3)X(4)>}
\end{aligned}
\tag{4.19}
$$

In the last exponent we neglected a term proportional to s which vanishes as $z_{12} \to 0$. This amounts to take $t\ln s \to 0$. From (4.19) we read off a leading overall s^2 behavior of the amplitude while the t behavior is still encoded in the exponential. In our limit, eq. (4.10), we can neglect the terms which are multiplied by t in (4.19). (These terms contain also poles in the z–plane representing singularities of the massive external propagators. As we said at the beginning we do not dwell upon them.) We remain therefore with $(M^2 = 2)$:

$$
\begin{aligned}
A_1 \to & N\int \frac{\prod_{i=1}^4(d^2a_i)\left(\sum_{k=2}^4|Q_k|^3\right)}{|P(a)|^2T^3dV_{pr}}\int\frac{\prod_{j=1}^4 d^2z_j}{|z_{12}|^2|y(3)y(4)|^4}\cdot \\
& e^{-\frac{t}{2}\langle(X(1)-X(3))(X(2)-X(4))\rangle_{\Delta\to 0}+2\langle X(3)X(4)\rangle} \\
& \left(\langle\partial X(0)X(3)\rangle - \langle\partial X(0)X(4)\rangle\right)\cdot \\
& \left(\langle\bar{\partial} X(0)X(3)\rangle - \langle\bar{\partial} X(0)X(4)\rangle\right)
\end{aligned}
\tag{4.20}
$$

where we defined $N = c\ G^2s^2(\varepsilon_1\cdot\varepsilon_2)(\bar{\varepsilon}_1\cdot\bar{\varepsilon}_2)$ and c is a numerical constant in which in the following we will absorb all overall constants.

For our purposes it is convenient to fix the Möbius invariance by fixing a_1, a_3 and z_3. Then the projective volume becomes:

$$dV_{pr} = \frac{d^2a_1 d^2a_3 d^2z_3}{|a_{13}(z_3 - a_1)(z_3 - a_3)|^2} \tag{4.21}$$

The integration region in the remaining moduli space (a_2, a_4) which is responsible for the small t behavior is given by the following pinching limit:

$$\begin{aligned} a_2 &\longrightarrow a_1 \qquad & x \equiv \ln\left|\frac{a_{13}}{a_{12}}\right| \longrightarrow +\infty \\ a_4 &\longrightarrow a_3 \qquad & y \equiv \ln\left|\frac{a_{13}}{a_{34}}\right| \longrightarrow +\infty \end{aligned} \tag{4.22}$$

In taking this limit we keep z_1 and z_2 on the upper sheet and z_3 and z_4 on the lower sheet (Fig.4). In the limit (4.22) the upper and the lower sheet become an upper sphere attached with two punctures to a lower sphere: intermediate massless particles (i.e. gravitons in our limit) are exchanged through the punctures. We evaluated the relevant quantities, appearing in (4.20), in this limit already in section 3.2. The corresponding formulae are given in (3.30). Inserting (3.30) and (4.21) in (4.20) we get:

$$\begin{aligned} A_1 \cong & N \int \frac{d^2a_2 d^2a_4 |a_{13}|^4}{|a_{12}a_{34}|^2 T(x+y)^2} \int \frac{d^2z_1 d^2z_2 d^2z_4}{|z_{12}|^2 |y(3)y(4)^2|^2} \cdot \\ & (\langle \partial X(0) X(3) \rangle - \langle \partial X(0) X(4) \rangle) \cdot \quad (right) \cdot \\ & e^{-\frac{t}{2}\langle (X(1)-X(3))(X(2)-X(4)) \rangle_{\Delta\to 0} + 2\langle X(3)X(4)\rangle} \end{aligned} \tag{4.23}$$

As a check of the consistency of our limit one can readily verify the Möbius invariance of (4.23) (see section 3.2). So, for example, $x+y$ and the ν–variables (3.16) are Möbius invariant as well as the combination which multiplies t in the exponential of (4.23) which depends, in fact, only on ν_{ab} and τ. Using (3.17) and (3.24), and taking into account that also the moduli have to undergo a Möbius transformation, also the remaining part of (4.23) can easily be seen to be invariant.

In the limit (4.22) we can also evaluate the X-correlators, keeping $z_{1,2}$ on the upper sheet and $z_{3,4}$ on the lower one. Again, this has already been done in section

3.2. We observe that the combination $\langle\partial X(0)X(3)\rangle - \langle\partial X(0)X(4)\rangle$ is precisely the one which has been evaluated in (3.38) and can thus be directly substituted in (4.23). The combination of propagators

$$G(1,2,3,4) = \langle (X(1) - X(3))(X(2) - X(4))\rangle \tag{4.24}$$

has been evaluated in (3.36) for generic z_i–variables and is correct up to orders $o(1/x, 1/y, 1/x + y)$. A convenient Möbius gauge fixing is given by choosing a_1, a_3 and z_3 such that $|z_3 - a_1| = |z_3 - a_3|$ which is always possible; then we have $f(3) = 0$, see eq. (3.37). Moreover, in (3.36) we can neglect the last term, which goes to zero as $x, y \to \infty$, and, taking the limit $\Delta \to 0$, of the $z_{1,2}$ dependence we keep only the singularity arising from:

$$\ln|2\sin(\pi\nu_{12})|^2 \to \ln|\nu_{12}|^2$$

We obtain then:

$$G(1,2,3,4) \to \frac{-4xy}{x+y} + 2\ln|2\sin(\pi\nu_{34})| + 2\pi\frac{x-y}{x+y}\mathrm{Im}\nu_{34} \tag{4.25}$$

Eq. (4.25) can also be justified observing that we can still fix one of the integration variables, say z_1, by the translation invariance of the torus (see Chapter 5); so we could put also $f(1) = 0$ and, for $\Delta \to 0$, (3.36) would reduce directly to (4.25). (Clearly in this case we should factor out also an appropriate translational invariant measure, $\int d^2z_{12}/|y(1)|^2 = T$, see again chapter 5, and of course the result would be the same).

It remains to evaluate $\langle X(3)X(4)\rangle$ in the limit (4.22). Using again (3.24) we obtain:

$$\langle X(3)X(4)\rangle \to \ln\left|K^2 y(3)y(4)\left(\frac{\sin(\pi\nu_{34})}{\pi}\right)^2\right| - 2\frac{(\mathrm{Im}\nu_{34})^2}{x+y} \tag{4.26}$$

So, taking into account also the $\frac{1}{|y(0)|^2}$ factor arising from (3.38) we can integrate over $z_{1,2}$ observing that, in this limit:

$$\int \frac{d^2z_1 d^2z_2}{|z_{12}|^2|y(0)|^2} e^{-\frac{t}{2}\ln|\nu_{12}|^2} = \int \frac{d^2z_0}{|y(0)|^2}\int\frac{d^2\nu_{12}}{|\nu_{12}|^{2+t}} \to \frac{T}{t}$$

The $1/t$ pole represents the massless state emitted from the two external gravitons with momenta k_1 and k_2 in Fig.1.

Inserting the above formulae we have then:

$$
\begin{aligned}
A_1 \cong \frac{N}{t} \int \frac{d^2 a_2 d^2 a_4 |a_{13}|^2}{|a_{12} a_{34}|^2 (x+y)^4} \int \frac{d^2 z_4}{|y(4)|^2} (\mathrm{Im}\nu_{34})^2 |\sin(\pi \nu_{34})|^4 \cdot \\
e^{t\left(\frac{2xy}{x+y} - \ln|2\sin(\pi\nu_{34})| - \pi \frac{x-y}{x+y}\mathrm{Im}\nu_{34}\right)} e^{-4\pi^2 \frac{(\mathrm{Im}\nu_{34})^2}{x+y}}
\end{aligned}
\tag{4.27}
$$

In the next section we will evaluate this formula, investigate its physical content and state our results. To this order we will specify the integration limits of the various integration variables and, in particular, determine the region of $\mathrm{Im}\nu_{34}$ which contributes mainly to the amplitude.

4.4 Evaluation of the amplitude and results

To evaluate the amplitude it is convenient to switch from the variables a_2, a_4 and z_4 to x,y and ν_{34}:

$$
\nu_{34} = \int_{z_3}^{z_4} \omega(z) dz, \qquad \left|\frac{a_{13}}{y(4)}\right|^2 d^2 z_4 = (2\pi)^2 d^2 \nu_{34}
$$

We recall that one usually takes $-1/2 \leq \mathrm{Re}\tau \leq 1/2, (1 - (\mathrm{Re}\tau)^2)^{1/2} \leq \mathrm{Im}\tau < \infty$ and $0 \leq \mathrm{Im}\nu \leq \mathrm{Im}\tau, -1/2 \leq \mathrm{Re}\nu \leq 1/2$. However, our computation is sensitive to large values of $\mathrm{Im}\tau$, so we can safely restrict $\mathrm{Im}\tau \geq 1$. Furthermore, we are integrating z_4 on the lower sheet of the hyperelliptic torus, see Fig.4, and in our variables $\mathrm{Im}\nu_{34}$ can become negative, see (3.30):

$$
\begin{aligned}
-\frac{x}{\pi} \leq \mathrm{Im}\nu_{34} \leq \frac{y}{\pi} \\
-\frac{1}{2} \leq \mathrm{Re}\nu_{34} \leq \frac{1}{2} \\
\pi \leq x + y < \infty
\end{aligned}
\tag{4.28}
$$

Let us comment briefly on the integration region of Imν_{34}. The $z_{3,4}$ variables live, by our definition, on the lower sheet of the hyperelliptic torus which covers only one halph of Imτ. Therefore, the integration interval of Imν_{34} should have a length of $\frac{x+y}{2\pi}$. However, by unitarity, we should allow Imν_{34}, which is the last Koba Nielsen variable that survived, and which is therefore unconstrained, to run over the entire hyperelliptic torus. This gives (4.28). We remark that in the limit (4.22) this does not contraddict our assumption that ν_{34} "lives" on the lower sheet; in fact, z_3 is fixed, by Möbius invariance, to stay on the lower sheet and z_4 does never cross the upper sheet.

Then we fix a "fundamental region" for a_2 and a_4 which encloses a_1 and a_3 respectively:

$$0 \leq \left|\frac{a_{12}}{a_{13}}\right| \leq 1 \leftrightarrow 0 \leq x \leq \infty$$

$$0 \leq \left|\frac{a_{34}}{a_{13}}\right| \leq 1 \leftrightarrow 0 \leq y \leq \infty$$

Setting finally:

$$\nu_{34} \equiv \frac{\alpha + i\beta}{2\pi} \tag{4.29}$$

we get:

$$A_1 \cong \frac{N}{t}\left(\int_0^\infty dx \int_0^\infty dy\right)_{x+y\geq\pi} \frac{1}{(x+y)^4}\int_{-2x}^{2y} d\beta \quad \beta^2 \mathrm{e}^{-\dfrac{\beta^2}{x+y}}.$$
$$\int_{-\pi}^{\pi} d\alpha(\cosh\beta - \cos\alpha)^2 \quad \mathrm{e}^{t\left(\dfrac{2xy}{x+y} - \dfrac{\beta}{2}\dfrac{x-y}{x+y} - \dfrac{1}{2}\ln(2(\cosh\beta - \cos\alpha))\right)} \tag{4.30}$$

Let us comment further on the coefficient of t in the exponential of eq. (4.30). The first term goes to infinity as $x, y \to \infty$ while the other two terms remain finite; therefore they contribute sensitively to the exponential only if $\ln(\cosh\beta - \cos\alpha) \to \pm\infty$ and $\beta \to \pm\infty$. The first condition is satisfied if the second one holds or if $\alpha \to 0$ *and* $\beta \to 0$. However, in this last case (corresponding to $z_4 \to z_3$, see eq. (4.29)) the factor in front of the exponential in the second row of eq. (4.30) vanishes; the limit $z_4 \to z_3$ would correspond to the diagram of Fig.5, where there appears a one–loop two–point function between massless states, and, although the non renormalization theorem does not hold for the three–point function we are considering, it forbids still a non vanishing two–point function for massless states. We should therefore evaluate the exponential in eq. (4.30) in the relevant integration region $\beta \to \pm\infty$:

$$-\frac{1}{2}\ln(2(\cosh\beta - \cos\alpha)) - \frac{\beta}{2}\frac{x-y}{x+y} \to \begin{cases} -\dfrac{\beta x}{x+y} & \beta \to +\infty \\ +\dfrac{\beta y}{x+y} & \beta \to -\infty \end{cases}$$

In this limit we can also integrate over α to get:

$$\begin{aligned} A_1 \cong & \frac{N}{t}\left(\int_0^\infty dx \int_0^\infty dy\right)_{x+y\geq\pi} \frac{1}{(x+y)^4}\int_0^{2x} d\beta \quad \beta^2 e^{-\dfrac{\beta^2}{x+y}}. \\ & (2+\cosh(2\beta)) \quad e^{t\left(\dfrac{2xy}{x+y} - \dfrac{\beta y}{x+y}\right)} \\ \equiv & \frac{8N}{t} I(t) \end{aligned} \tag{4.31}$$

Eq. (4.31) is our final result. Let us study first the convergence properties of I(t) and its behavior as $t \to 0$. To this order the following change of variables is convenient:

$$x = \xi L$$

$$y = (1-\xi)L$$

$$\beta = 2\alpha L$$

So we get:

$$I(t) = \int_0^1 d\xi \int_\pi^\infty dL \int_0^\xi d\alpha \quad \alpha^2(2+\cosh(4\alpha L))e^{-4\alpha^2 L} e^{2tL(1-\xi)(\xi-\alpha)} \tag{4.32}$$

The convergence of I(0) can be established integrating, by analytic continuation, over L (the need for analytic continuation is due to the positive exponential in cosh(4αL), which in section 4.5 will be seen to correspond to a Feynman diagram where the massive particle disintegrates into two massless ones):

$$I(0) = \frac{1}{8}\int_0^1 d\xi \int_0^\xi d\alpha \quad e^{-4\pi\alpha^2}\left(4 - \frac{\alpha}{1-\alpha}e^{4\pi\alpha} + \frac{\alpha}{\alpha+1}e^{-4\pi\alpha}\right) \tag{4.33}$$

Eq. (4.33) is easily seen to be finite.

We show now that the small t behavior of I(t) is as follows:

$$I(t) \to I(0) + a\sqrt{|t|} \qquad \text{as} \quad t \to 0$$

This amounts to show that

$$\lim_{t \to 0} 2\left(\sqrt{|t|}\frac{dI(t)}{dt}\right) = a \neq 0 \tag{4.34}$$

Taking the derivative of (4.32) and integrating over L, the term proportional to cosh(4αL) gives an integrand in α which is regular for $\alpha \to 0$ even for t=0 and provides a contribution to dI(t)/dt which is regular for $t \to 0$. Therefore it gives a vanishing contribution to (4.34). By computing the remaining part, making use of the rescalings $\alpha \to |t|^{\frac{1}{2}}\alpha$, $L \to |t|^{-1}L$, we get:

$$a = -\frac{\sqrt{2}}{4}\pi^2. \tag{4.35}$$

Restoring now the dimensionful constants we get for the subdiagram (Fig.1), we considered in our computation in the limit $t \to 0$ and $s \to \infty$,

$$A_1 \cong c(\varepsilon_1 \cdot \varepsilon_2)(\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2)\left(2\sqrt{2}\bar{I}(0)\frac{(Gs)^2}{\alpha'|t|} - \pi^2\frac{\sqrt{2}}{4}M\frac{(Gs)^2}{|t|^{\frac{1}{2}}} + \frac{o(t)}{t}\right) \tag{4.36}$$

Let us now comment on this result. First of all we observe that the first term in (4.36) would correspond to a finite one–loop renormalization of Newton's constant. However, our pinching limit computation was designed to look for the $1/|t|^{1/2}$ behavior of the diagram of Fig.1 and therefore the numerical constant I(0) of (4.33) represents only a part of the contribution. Accordingly, we corrected it to $\bar{\mathrm{I}}(0)$ in (4.36). By looking at eq. (4.19) at $t \to 0$ one sees formally at the right hand side a total derivative in z_0, which would vanish when integrated in z_0, giving $\bar{\mathrm{I}}(0) = 0$. But the integrand in z_0 has poles for $z_0 \to z_{3,4}$ corresponding to the diagrams of Fig.3, i.e. the above recalled singularities due to the fact that the massive external lines are on the mass shell. The $t \to 0$ limit of (4.19) is therefore a total derivative plus diverging terms proportional to $\delta^{(2)}(z_0 - z_{3,4})$. In order to discuss properly the computation of $\bar{\mathrm{I}}(0)$ one should then require a careful treatment of the amputation of the singular external on–shell propagators, a problem which is outside the present investigation and which is probably not yet fully understood in String Theory. However, let us observe that, due to the fact that $\bar{\mathrm{I}}(0)$ receives contributions only from the pinching limit corresponding to Fig.3, which would provide only a finite shift of the mass and of the wave function normalization of the massive states, one would expect that it cancels with a proper normalization of the external states.

The second term in (4.36) is our main result and corresponds to a genuine $o(G^2)$ Schwarzschild–like correction to the Newton potential. In other words, recalling that the standard Einstein deflection of a photon (here a graviton) by a massive body comes from the first order gravitational potential GM/r, corresponding to the 1/t singularity of the one graviton exchange in the tree diagram, the contribution which we find at one loop corresponds to a correction proportional to $(\mathrm{GM/r})^2$, like the first correction to Newton's potential resulting from an expansion of the Schwarzschild metric.

4.5 Comparison with Feynman diagrams

In this section we compare the computation from String Theory with Quantum Gravity Feynman diagrams.

We analysed three possible Feynman diagrams: in Fig.6 the massive particle scatters by graviton exchanges at one loop, in Fig.7 the intermediate states in the loop are made out only of gravitons and in Fig.8 the massive external particle gets excited to a state belonging to the second excited level of the string with mass $\alpha' M^2 = 8$. Of course, the Feynman diagrams are ill–defined due to UV divergences, and we will only pay attention to their IR behavior. We find qualitatively the behavior which we obtain in string theory, even though the details, giving rise to subleading terms, are different.

The $1/|t|^{1/2}$ behavior, which is our main concern, here is seen in the diagram of Fig.6, whereas the diagrams of Fig.7 and Fig.8 appear to give contributions similar to the ones coming respectively from the two exponentials $e^{4\alpha L}$ and $e^{-4\alpha L}$ of the $\cosh(4\alpha \mathrm{L})$ in (4.32), which is subleading. The diagram in Fig.6 corresponds to an "elastic" scattering where the lower massive line itself gets not excited but retains its mass. This shows that the dominant contributions for $t \to 0$ and $s \to \infty$ are the "elastic" ones. This is precisely what happens also in the rescattering diagrams of four–graviton scattering (see chapter 5).

To make the comparison more precise we write our result (4.32) in the form:

$$\mathrm{I(t) = I_1(t) + I_2(t) + I_3(t)}$$

with

$$I_1(t) = \frac{1}{2}\int_0^1 d\xi \int_{4\pi}^{\infty} dL \int_0^{\xi} d\alpha\alpha^2 \mathrm{e}^{-\alpha^2 L_{\mathrm{e}}(t/2) L(1-\xi)(\xi-\alpha)} \tag{4.37}$$

$$I_2(t) = \frac{1}{8}\int_0^1 d\xi \int_{4\pi}^{\infty} dL \int_0^{\xi} d\alpha\alpha^2 \mathrm{e}^{\alpha(1-\alpha) L_{\mathrm{e}}(t/2) L(1-\xi)(\xi-\alpha)} \tag{4.38}$$

$$I_3(t) = \frac{1}{8}\int_0^1 d\xi \int_{4\pi}^{\infty} dL \int_0^{\xi} d\alpha\alpha^2 \mathrm{e}^{-\alpha(1+\alpha) L_{\mathrm{e}}(t/2) L(1-\xi)(\xi-\alpha)} \tag{4.39}$$

We will see that $I_1(t)$, $I_2(t)$ and $I_3(t)$ can be viewed as "stringy" versions of the diagrams in Figs.6, 7 and 8 respectively.

To write the amplitude corresponding to Fig.6 we have to decide what vertices we will choose for the three graviton interaction, V_{ABC}, and for the emission of a massless particle from a massive scalar V_ε. For the latter we choose the minimal coupling (for factorized polarizations and incoming momenta):

$$V_\varepsilon = [(k_1 - k_2)\cdot\varepsilon][(k_1 - k_2)\cdot\bar{\varepsilon}] \tag{4.40}$$

while for the former we choose the string induced vertex [GSW]:

$$
\begin{aligned}
V_{ABC} = \big[&(\varepsilon_A \cdot \varepsilon_B)\varepsilon_C \cdot (k_A - k_B) + (\varepsilon_B \cdot \varepsilon_C)\varepsilon_A \cdot (k_B - k_C) + \\
&(\varepsilon_C \cdot \varepsilon_A)\varepsilon_B \cdot (k_C - k_A)\big]\,\big[right\big]
\end{aligned} \tag{4.41}
$$

In (4.41) we invoked Bose symmetry to get rid of the fact that string theory produces only on–shell vertices. Choosing for the massless particles propagators in a Feynman-like gauge (the trace part, corresponding to a dilaton, gives a subleading contribution in s):

$$
D_{\mu\nu,\rho\sigma} = \frac{g_{\mu\rho}g_{\nu\sigma}}{p^2} \tag{4.42}
$$

we get for the numerator of the one–loop integral representing Fig.6 ($s \gg M^2, s \gg t, M^2 \gg t$):

$$
N_1(k) = G^2(\varepsilon_1 \cdot \varepsilon_2)(\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2)\left\{(k \cdot k_1)(4M^2 - 2k \cdot k_3) + s(k^2 - 2k \cdot k_3)\right\}^2 \tag{4.43}
$$

The diagram of Fig.6 becomes then ($k_0^2 = t$):

$$
F_1 = \frac{1}{t}\int \frac{N_1(k)d^4k}{k^2(k+k_0)^2((k-k_3)^2 - M^2)} \tag{4.44}
$$

Exponentiating in the usual way the propagators:

$$
\frac{1}{p^2} = \int_0^\infty \mathrm{e}^{-up^2}\, du
$$

we obtain:

$$
\begin{aligned}
F_1 = \frac{1}{t}\int d^4k \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \quad & N_1\left(k \to k - \frac{yk_0 - zk_3}{x+y+z}\right) \\
\mathrm{e}^{-(x+y+z)k^2}\,\mathrm{e}^{\left[\frac{-z^2M^2 + txy}{x+y+z}\right]}&
\end{aligned} \tag{4.45}
$$

Of the numerator N_1 (with shifted argument) in eq. (4.45) we have to extract those contributions which give a factor of s^2 and get the leading t–behavior from the integration region of large x, y and z. With rescalings analogous to the ones which led to eq. (4.35) we deduce that those contributions are of the form:

$$
\begin{aligned}
N_1\left(k \to k - \frac{yk_0 - zk_3}{x+y+z}\right)_{\substack{s^2 \\ t \to 0}} = &G^2s^2(\varepsilon_1 \cdot \varepsilon_2)(\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2)\cdot \\
&\left\{c_1M^2k^2 + c_2M^4\left(\frac{z}{x+y+z}\right)^2\right\}
\end{aligned} \tag{4.46}
$$

where c_1 and c_2 are numerical constants. Rescaling x, y and z by a factor of $\frac{1}{M^2}$ and changing then also here variables:

$$x = (1-\xi)L$$
$$y = (\xi - \alpha)L$$
$$z = \alpha L$$

we get:

$$\begin{aligned} F_1 =& \pi^2 \frac{G^2 s^2 M^2 (\varepsilon_1 \cdot \varepsilon_2)(\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2)}{t} \int_0^1 d\xi \int_0^\infty dL \int_0^\xi d\alpha \cdot \\ & \left(c_1 \frac{2}{L} + c_2 \alpha^2 \right) \mathrm{e}^{-\alpha^2 L} \mathrm{e}^{(t/M^2) L (1-\xi)(\xi - \alpha)} \end{aligned} \tag{4.47}$$

Substituting $M^2 = 2$ in (4.47) we can compare this result with (4.37): The two different pieces (in c_1 and c_2 respectively) of (4.47) coming from the numerator (4.43) lead to the same $\frac{1}{\sqrt{t}}$ behavior, coming from the region of large L, as (4.37). The structure of the propagators, that is, of the exponentials of (4.47), is exactly reproduced by the string in the pinching limit while the string seems to use a different numerator as Q.F.T. (at least in our very simple "Q.F.T. model" represented by eqs. (4.40)–(4.42): in fact, an explicit computation gives $c_1 \neq 0, c_2 = 0$, whereas in the previous string computation $c_1 = 0, c_2 \neq 0$).

We note also the presence of an ultraviolett cut-off 4π in $I_1(t)$, which appears naturally in the string computation, and which is absent in F_1, i.e. in the field theoretic counterpart of $I_1(t)$. It is a remarkable fact that the string reproduces in the infrared region almost precisely the pattern of ordinary Feynman diagrams: the role of the Feynman parameters is played by the logarithms of differences of moduli and by the imaginary parts of the ν variables. We will check this fact ones again in chapter 5 in the case of a rescattering diagram.

We comment now briefly on the diagrams of Fig.7 and Fig.8. Fig.7 describes the virtual decay of the massive external scalars in massless particles. Proceeding in the same manner as before we get now:

$$F_2 = \frac{1}{t}\int d^4k \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \quad N_2\left(k \to k - \frac{yk_0 - zk_3}{x+y+z}\right)$$
$$e^{-(x+y+z)k^2} e^{\left[\frac{-z(x+y)M^2 + txy}{x+y+z}\right]} \tag{4.48}$$

Instead of investigating a precise Q.F.T. model for this diagram, as we did before, we examine the effect of a contribution to N_2 of the form:

$$\Delta N_2(k) = cG^2(\varepsilon_1 \cdot \varepsilon_2)(\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2)M^4(k \cdot k_1)^2 \tag{4.49}$$

which is allowed by dimensional reasoning. Then, as before, we get:

$$\Delta N_2\left(k \to k - \frac{yk_0 - zk_3}{x+y+z}\right)_{\substack{s^2 \\ t \to 0}} = \frac{c}{4}G^2 s^2(\varepsilon_1 \cdot \varepsilon_2)(\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2)M^4\left(\frac{z}{x+y+z}\right)^2$$

Scaling and changing variables as in the previous case we obtain:

$$F_2 = \frac{c\pi^2}{4}\frac{G^2 s^2 M^2(\varepsilon_1 \cdot \varepsilon_2)(\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2)}{t}\int_0^1 d\xi \int_0^\infty dL \int_0^\xi d\alpha \cdot$$
$$\alpha^2 e^{\alpha(1-\alpha)L} e^{(t/M^2)L(1-\xi)(\xi-\alpha)} \tag{4.50}$$

which reproduces (4.38) modulo the differences we noted before.

Finally in Fig.8 the double line represents a particle with mass $m^2 = 2M^2$ belonging to the second excited level of the superstring spectrum as noted before. With this value for the mass we get:

$$F_3 = \frac{1}{t}\int d^4k \int_0^\infty dx \int_0^\infty dy \int_0^\infty dz \quad N_3\left(k \to k - \frac{yk_0 - zk_3}{x+y+z}\right)$$
$$e^{-(x+y+z)k^2} e^{\left[\frac{-z(x+y+2z)M^2 + txy}{x+y+z}\right]} \tag{4.51}$$

and choosing a possible contribution $\Delta N_3(k)$ as in (4.49) we get then:

$$F_3 = \frac{c\pi^2}{4}\frac{G^2 s^2 M^2(\varepsilon_1 \cdot \varepsilon_2)(\bar{\varepsilon}_1 \cdot \bar{\varepsilon}_2)}{t}\int_0^1 d\xi \int_0^\infty dL \int_0^\xi d\alpha \cdot$$
$$\alpha^2 e^{-\alpha(1+\alpha)L} e^{(t/M^2)L(1-\xi)(\xi-\alpha)} \tag{4.52}$$

This expression reproduces (4.39) and the comments we made above are in order also here.

5. Four–graviton scattering with low space–time supersymmetry

5.1 Supersymmetry and the cosmological constant

In this chapter we will investigate the four-graviton amplitude in a type II model in 4 dimensions with low space-time supersymmetry.

The model we chose is the one described by G_3 given in (2.14). Although its defining set of generators G_3 is not symmetric between the interchange of left- and right-chirality fields, the model itself is indeed left-right symmetric. This can be seen noting that:

$$\bar{b} = bs\bar{s} \tag{5.1}$$

where the composition law is given by (2.3). Eq. (5.1) implies that the set Σ_3, generated by G_3, is left-right symmetric, giving thus a model which exhibits this symmetry too. As a consequence also the general form of the correlators (2.24) has this symmetry, which will provide some simplification in the computations that we will do in short and, moreover, the space-time supersymmetry of this model can be only $N = 4, 2$ or 1.

So let us first determine its space-time supersymmetry counting the number of (massless) gravitini which survive the generalized GSO– projections imposed by the sum over spin structures.

We note that the physical Hilbert space is a direct sum of sectors of the kind $R^{\alpha}N^{F\alpha}$ [SC,ABKW] on which all other sets $\beta \in \Sigma_3$ act as GSO-projectors. In particular, the corresonding vacuum $|0\rangle_{\alpha}$ must represent the Clifford algebra of zero-modes of all fermions in α. Here "R" and "N" refer to Ramond and Neveu-Schwarz boundary conditions respectively. The mass–shell conditions (coming from the zeroth–momentum Virasoro conditions) are given by:

$$
\begin{aligned}
M_L^2 &= \frac{n_L(\alpha)}{16} - \frac{1}{2} + \sum_{\text{left-movers}} (oscillators) \\
M_R^2 &= \frac{n_R(\alpha)}{16} - \frac{1}{2} + \sum_{\text{right-movers}} (oscillators)
\end{aligned}
\tag{5.2}
$$

and $M_L^2 = M_R^2$. Eqs. (5.2) imply that massless states (and in particular gravitini) can arise only in a sector $R^\alpha N^{F\alpha}$ such that $n_L(\alpha) = 0$ or 8 *and* $n_R(\alpha) = 0$ or 8 separately. It is easily seen that the only sets in Σ_3 which fulfill these requirements are s and $\bar{s}$. We will now concentrate on s, the discussion about $\bar{s}$ beeing completely analogous. In s we have potential gravitini of the form $\Psi^\mu = \bar{\psi}^\mu_{-\frac{1}{2}}|0\rangle_s$ which fulfill (5.2). Recalling the definition of s:

$$s = \{\psi^1, \psi^2, x^1, \ldots, x^6\}$$

we see that the vacuum $|0\rangle_s$ is a four-dimensional Weyl spinor and carries furthermore an internal SO(6) spinor index. So the states Ψ^μ describe 8 Weyl spin 3/2 particles (in addition to 8 Weyl spin 1/2 particles). However, we have still to perform the GSO projections for all $\alpha \in \Sigma_3$. These projections assume on the states in $R^s N^{Fs}$ the form [ABKW]:

$$(-)^\alpha = C(s|\alpha)\delta(s) \equiv \gamma = -C(s|\alpha) \tag{5.3}$$

where the α parity operator $(-)^\alpha$ is defined as $(-)^\alpha = \prod_{f \in \alpha \cap \beta} f_0$ if $\alpha \cap \beta \neq \emptyset$ where f_0 is the zero mode (i.e. Dirac-matrix part) of the fermion f. If $\alpha \cap \beta = \emptyset$ we must keep only those states built on all components of the vacuum $|0\rangle_s$ with an even (when $\gamma = 1$) or odd (when $\gamma = -1$) number of α–oscillators. It is sufficient to perform the projections imposed by the set of generators G_3 in (2.14). So half of the states Ψ^μ are always eliminated by the projector $(-)^s = \prod_{f \in s} f_0 = \delta(s)C(s|s)$ itself (corresponding to the usual chirality projection in 10 dimensions), leaving us therfore with 4 gravitini. The remaining non trivial projection can be seen to come from $(-)^b = \psi_0^3 \psi_0^4 x_0^1 x_0^2 = -C(s|b)$ which truncates ones again half of the

Ψ^μ, leaving 2 massless gravitini in each chirality sector and giving, therefore, an $N = 4$ space-time supersymmetry.

We conclude this section by noting the interesting role of the $\bar{s}$ projection. It gives in fact:

$$(-)^{\bar{s}} = -C(s|\bar{s}) = -1 \tag{5.4}$$

The relation (5.4) means that only those states survive which are built with an odd number of $\bar{s}$–oscillators acting on $|0\rangle_s$. So the gravitini states we found, and therfore supersymmetry, survive thanks to our choise for $C(s|\bar{s})$ made in eq. (2.22). We remember that it was precisley this choise which assured also the vanishing of the cosmological constant. Choosing $C(s|\bar{s}) = -1$ would break completely supersymmetry and produce a non zero cosmological constant. We regard this fact as a signal of the deep relation existing between the two subjects which is still under investigation in string theory.

5.2 Rescattering revisited

We write now down the one–loop contribution to the four–graviton amplitude which is our main concern in this chapter. Reading the structure of the sum over spin structures from formula (2.26), taking into account also (2.23), one can write down the expression for this amplitude as follows:

$$\begin{aligned} A_1 = G^2 \int \frac{\prod_{i=1}^4 d^2 a_i}{|P(a)|^4 T^3 dV_{pr}} \Bigg\{ & \left(\sum_{i,j=2}^{4} \eta_i \eta_j Q_i \bar{Q}_j \left\langle \prod_{l=1}^{4} O_l \right\rangle_{ij} \right) \left(\sum_{k=2}^{4} |Q_k|^3 \right) \\ & + \sum_{i<j}^{4} |Q_i Q_j|^2 \left(|Q_i| + |Q_j|\right) \left(\left\langle \prod_{l=1}^{4} O_l \right\rangle_{ii} - \left\langle \prod_{l=1}^{4} O_l \right\rangle_{ij} \right. \\ & \left. - \left\langle \prod_{l=1}^{4} O_l \right\rangle_{ji} + \left\langle \prod_{l=1}^{4} O_l \right\rangle_{jj} \right) \Bigg\} \equiv A_1^{N=8} + A_1^{N=4} \end{aligned} \tag{5.5}$$

This time all vertex operators $O_j = \int V_j(z,\bar{z})d^2z$, $1 \le j \le 4$, are of the form

$$\begin{aligned} V^j(z_j,\bar{z}_j) =&(\varepsilon^j \cdot \partial X(j) + iA^j_{\alpha\beta}\psi^\alpha(j)\psi^\beta(j))(\bar{\varepsilon}^j\bar{\partial}X(j) + i\bar{A}^j_{\alpha\beta}\bar{\psi}^\alpha(j)\bar{\psi}^\beta(j))\cdot \\ &e^{ik^j \cdot X(j)} \end{aligned} \tag{5.6}$$

where $A^j_{\alpha\beta}$ is defined in (4.6). The on-shell and irreducibility conditions are:

$$\varepsilon_j \cdot k_j = \bar{\varepsilon}_j \cdot k_j = k_j^2 = \varepsilon_j^{[\alpha}\bar{\varepsilon}_j^{\beta]} = \bar{\varepsilon}_j \cdot \varepsilon_j = 0 \tag{5.7}$$

The contribution in the first line in (5.5), $A_1^{N=8}$, is present also in models with highest $N = 8$ space-time supersymmetry, for example in the model we used to analyze the Compton-like scattering in Chapter 4, and has been extensively studied by various authors (see [ACV] and references therein). In particular in the limit $s \to \infty$ and $t \to 0$ the leading contributions are seen to correspond to rescattering terms [ACV2], see Fig.9, in which the external massless energetic lines k_1, k_3 exchange gravitons (vertical lines) among themself and get excited into massive states (horizontal lines); the horizontal lines, however, may remain also massless. Actually, the contributions with massless horizontal lines are the ones which dominate in the above limit. The corresponding expressions, like eq. (14) in Ref. [ACV1], contain momentum integrations $\int d^{D-2}k$ over the transverse $D-2$ dimensional phase space which appear typically in eikonal approximations. In this section we will extract a Feynman parameter like representation of a rescattering term in the limit of small momentum transfer $t \to 0$, represented by a suitable region in the integration domain of Koba-Nielsen variables and moduli (i.e. branch points). It is generally believed that string theories reduce to ordinary Q.F.T's at low energies which is usually expressed saying that $\alpha' \to 0$. Interpreting the "low energies" as small transferred energies we check this believe at one loop, showing that our Feynman diagram like representation reproduces actually, in the just mentioned integration region (modulo some difference) a true Feynman diagram. Of course, for $s \to \infty$ the obtained expression reduces to the eikonal approximation of graviton scattering.

In the next section, instead, we will investigate $A_1^{N=4}$, represented by the second and third lines in (5.5), which represents genuine new features.

The sum over spin structures in $A_1^{N=8}$ can easily been carried out using (4.16) and (3.27) and noting that among the various correlators which appear in $A_1^{N=8}$, only the ones with eight ψ's survive [NNS]:

$$
\begin{aligned}
A_1^{N=8} = G^2 K(k_i) \int \frac{\prod_{i=1}^{4} d^2 a_i}{|P(a)|^2 T^3 dV_{pr}} \left(\sum_{k=2}^{4} |Q_k|^3 \right) \prod_{i=1}^{4} \frac{d^2 z_i}{|y(i)|^2} \\
e^{-\frac{t}{2} \langle (X(1)-X(3))(X(2)-X(4)) \rangle} \\
e^{-\frac{s}{2} \langle (X(1)-X(2))(X(3)-X(4)) \rangle}
\end{aligned}
\tag{5.8}
$$

The non renormalization theorem is reflected in this ampitude in the fact that the integrand does not contain any pole in the Koba Nielsen variables z_i (a part from the ones in the exponentials): this implies, in particular, that diagrams like the one in Fig.1 (where now also the lower line is massless) are not present. In (5.8) $K(k_i)$ is the usual kynematical factor which appears also at tree level [GSW] and at two loops [IZ] with the leading s-behavior:

$$K(k_i) \to s^4 Tr(\varepsilon_1 \cdot \varepsilon_2) Tr(\varepsilon_3 \cdot \varepsilon_4) \tag{5.9}$$

Before going on we remark that the integrand in (5.8) depends on the z_i (besides the $|y(i)|^2$ factors in the measure) only through the ν_{ab}–variables as can be seen from (3.24). Thus this integrand can be arranged to depend, for example, only on ν_{12}, ν_{13} and ν_{14}. So we switch from the variables z_2, z_3 and z_4 to these ν–variables noting that:

$$\frac{d^2 z_2}{|y(2)|^2} = |K(a)|^2 d^2 \nu_{12} \tag{5.10}$$

with analogous relations for z_3 and z_4. Then we can integrate over $d^2 z_1/|y(1)|^2$ with the result that one power of T in (5.8) gets canceled. At this point we can fix z_1 in the exponentials of (5.8) arbitrarily. This is clearly a consequence of the translational invariance of the torus. Moreover, due to the fact that the ν

variables are Möbius invariant, the amplitude is still $PSL(2,C)$ invariant and we fix the projective volume as in (4.21). Putting everthing together we get:

$$\begin{aligned} A_1^{N=8} = G^2 K(k_i) \int & \frac{d^2 a_2 d^2 a_4 d^2 \nu_{12} d^2 \nu_{14} |K(a)|^6}{|y(3)|^2 |P(a)|^2 T^2} \\ & \left(\sum_{k=2}^{4} |Q_k|^3 \right) |(z_3 - a_1)(z_3 - a_3) a_{13}|^2 \\ & e^{-\frac{t}{2} \langle (X(1) - X(3))(X(2) - X(4)) \rangle} \\ & e^{-\frac{s}{2} \langle (X(1) - X(2))(X(3) - X(4)) \rangle} \end{aligned} \tag{5.11}$$

We are searching for rescattering configurations in which the two massless external lines 1,3 exchange gravitons. As seen in section 4.3 this situation is represented by the limit $a_2 \to a_1, a_4 \to a_3$. The integration region in the space of Koba–Nielsen variables which gives rise to the dominant contributions is given by $z_2 \to a_1$ and $z_4 \to a_1$ (or $z_2 \to a_3$ and $z_4 \to a_3$) which represents the "direct" diagram, Fig.10, and by $z_2 \to a_1$ and $z_4 \to a_3$ (or viceversa) which represents the diagram where k_1 and k_2 are exchanged, Fig. 11, as we will see in the following. We concentrate here on the first diagram: $z_2, z_4 \to a_1$. In this case the lines $z_{2,4}$ are attached to the same exchanged graviton. Keeping again $z_{1,2}$ on the upper sheet and $z_{3,4}$ on the lower sheet, and fixing z_1 and z_3 such that $f(1) = f(3) = 0$ this means that $a \equiv \mathrm{Im}\nu_{12} \to +\infty$ and $-b \equiv \mathrm{Im}\nu_{34} \to -\infty$. In these limits we have (see(3.36)):

$$\begin{aligned} \langle (X(1) - X(3))(X(2) - X(4)) \rangle &\to -\frac{4xy}{x+y} + \frac{4\pi y(a+b)}{(x+y)} \\ \langle (X(1) - X(2))(X(3) - X(4)) \rangle &\to -\frac{4\pi^2 ab}{x+y} \end{aligned} \tag{5.12}$$

The second relation is valid also a and b remain finite, while the term proportional to (a+b) in the first row goes to zero if a and b are kept finite. We will see in short that the term in (a+b) does actually not contribute if $s \to \infty$. Here we kept it to make our comparison with the Feynman diagram more precise. In (5.11) we can change integration variable from ν_{14} to ν_{34} substituting simply $d^2\nu_{14}$ with $d^2\nu_{34}$.

In the case under investigation we choose for the integration limits of $\mathrm{Im}\nu_{12}$, $\mathrm{Im}\nu_{34}$ the following ones:

$$
\begin{aligned}
0 \leq & \mathrm{Im}\nu_{12} \leq \frac{x}{2\pi} \\
0 \leq & -\mathrm{Im}\nu_{34} \leq \frac{x}{2\pi}
\end{aligned}
\tag{5.13}
$$

which are dictated by the fact that z_2 and z_4 have to live on opposite sheets. Substituting the above relations and (3.30) we get for the contribution we studied:

$$
\begin{aligned}
\Delta A_1^{N=8} \cong const.G^2 K(k_i) \left(\int \frac{dx dy}{(x+y)^2} \right)_{x+y \geq \pi} & \int_0^{x/2} da \int_0^{x/2} db \\
& \mathrm{e}^{s \frac{2ab}{x+y} + t \frac{2y(x-(a+b))}{x+y}}
\end{aligned}
\tag{5.14}
$$

This can also be rewritten as follows:

$$
\begin{aligned}
\Delta A_1^{N=8} \cong const.G^2 K(k_i) \int_{2\pi}^{\infty} L dL \int_0^1 d\xi & \int_0^{\xi/2} d\alpha \int_0^{\xi/2} d\beta \\
& \mathrm{e}^{L(s\alpha\beta + t(1-\xi)(\xi - (\alpha+\beta)))}
\end{aligned}
\tag{5.15}
$$

Eq. (5.15) is the Feynman–parameter–like representation of a rescattering term we searched for. In order to obtain the eikonal approximation of rescattering one should add also the contribution in which $z_2 \to a_1$ and $z_4 \to a_3$ (or viceversa). This would provide for one of the variables α, β an integration region of negative values, such that, after performing the rescalings $u = \alpha\sqrt{s}, v = \beta\sqrt{s}$ and sending s to ∞ the integration region would extend over the entire real axis giving us back the eikonal approximation of graviton scattering at one loop [ACV,GIZ] with amplitude $A \cong G^2 s^3/t$; also the Coulomb infrared divergence [ACV] is recovered in this way. We remark that under the above rescalings the term proportional to $\alpha + \beta$ in the exponential of eq. (5.15) goes to zero as $s \to \infty$ and is therefore unessential in this limit as asserted previously. The addition of the just mentioned contribution corresponds, in a Q.F.T. language, to add the diagram in which k_1 and k_2 get interchanged, Fig.11. This sends $s \to -s - t \cong -s$ and $t \to t$ and extends the relevant integration regions also in this case over the entire real axis.

To conclude this section we write down the expression for the genuine Feynman diagram, Fig.10, which should correspond to the contribution (5.15), we extracted from (5.11). We give here simply the result written in terms of Feynman parameters:

$$A_R = \int d^4k \int_0^\infty LdL \int_0^1 d\xi \left(\int_0^1 d\alpha \int_0^1 d\beta\right)_{\alpha+\beta\leq\xi} N\left(\frac{k}{L^{1/2}} \to \frac{k}{L^{1/2}} - Q\right) e^{-k^2}$$
$$e^{L(s\alpha\beta + t(1-\xi)(\xi-(\alpha+\beta)))} \tag{5.16}$$

where Q is defined as $Q = k_2(\xi - \alpha - \beta) + k_1(\xi - \beta - 1) - k_3\beta$. Of course, also in this expression the $(\alpha + \beta)$–term in the exponential is unessential for large s. Here N(k) is a numerator whose explicit form in conventional Quantum Gravity is very complicated and will not be given here; in fact, it would result from the contraction of four three–graviton vertices (and in a "consistent" quantization of gravity it should contain also contributions of ghosts), but for $s \to \infty$ it is just $s^4\mathrm{Tr}(\varepsilon_1\varepsilon_2)\mathrm{Tr}(\varepsilon_3\varepsilon_4)$. Eq. (5.16) can be compared with (5.15): as differences we note the presence of the usual ultraviolet cut–off in (5.15) and the slightly different integration region for α, β in the two expressions. Similar differences are encountered also in the comparison of the diagram with the lines 1 and 2 interchanged. We conclude saying that, although the string admits at low energies a "classical" unitarity interpretation in terms of Feynman diagrams and Feynman loops and particles circulating there, the details which arise in string theory are different from those arising in Q.F.T. As a rule, the details where they differ give non leading contributions for $s \to \infty$ and $t \to 0$: it is in this limit that String Theory is equivalent to Quantum Field Theory.

In the next section we investigate $A_1^{N=4}$ which, with respect to the rescattering terms of the present section, will give rise to genuinely new features due to the breakdown of the non renormalzation theorem which holds, on the other side, for $A_1^{N=8}$. We will again be able to give to those features, in the asymptotic limit of small t and large s, a "classical" Q.F.T.–like interpretation.

5.3 New effects from low space–time supersymmetry

We study in the last section of this thesis the supersymmetry (N=8) and non renormalization theorem breaking contribution to the one–loop amplitude (5.5) which we repeat here for definiteness:

$$
\begin{aligned}
A_1^{N=4} = G^2 \int \frac{\prod_{i=1}^4 d^2 a_i}{|P(a)|^4 T^3 dV_{pr}} \sum_{i<j}^{4} |Q_i Q_j|^2 \left(|Q_i| + |Q_j|\right) \cdot \\
\left\langle \left(\prod_{l=1}^{4} (O_l^L)_i - \prod_{l=1}^{4} (O_l^L)_j \right) \cdot \left(\prod_{l=1}^{4} (O_l^R)_i - \prod_{l=1}^{4} (O_l^R)_j \right) \right\rangle
\end{aligned}
\tag{5.17}
$$

We wrote here the vertices in the factorized form as anticipated in (2.26). O^L and O^R are respectively the left– and right–parts of the vertices (5.6) where the exponentials $e^{ikX(z)}$ are understood to belong to one of $O^{R,L}$. In (5.17) there are now non vanishing contributions also from correlators of 4 and 6 ψ's besides the ones with 8 ψ's which we encountered in (5.8). To carry out the sum over spin structures we need now generalized identities analogous to the ones of (4.16). These identities assume now, partly, a more complicated form. We give them here for completeness. Defining

$$
\begin{aligned}
H(13|24)_i &\equiv \left(\frac{u(1)u(3)}{u(2)u(4)} + \frac{u(2)u(4)}{u(1)u(3)} \right)_i \\
G(1|2)_i &\equiv \left(\frac{u(1)}{u(2)} + \frac{u(2)}{u(1)} \right)_i
\end{aligned}
\tag{5.18}
$$

and taking a look at (5.17) we see that we have to compute differences like $H_i - H_j$ and $G_i - G_j$. After some simple (but tedious) algebra we can verify the following two identities:

$$
\begin{aligned}
H(13|24)_2 - H(13|24)_3 = & \frac{Q_4}{\prod_{k=1}^4 y(k)} \cdot \Big\{ z_{12}z_{14}z_{32}z_{34}\Big(\sum_i a_i \sum_j z_j - \sum_i z_i^2 \\
& - (z_1+z_3)(z_2+z_4) - 2\sum_{i<j} a_i a_j + (a_1+a_4)(a_2+a_3)\Big) \\
& + z_{34}z_{32}y^2(1) + z_{41}z_{43}y^2(2) + z_{14}z_{12}y^2(3) + z_{21}z_{23}y^2(4)\Big\} \\
= & \frac{Q_4}{\prod_{k=1}^4 y(k)} \cdot \{z_{12}[(z_3-a_2)(z_4-a_3) + (z_3 \leftrightarrow z_4)] \\
& + z_{34}[z_3 \leftrightarrow z_1, z_2 \leftrightarrow z_4]\} \cdot \{a_1 \leftrightarrow a_2, a_3 \leftrightarrow a_4\}
\end{aligned}
\tag{5.19}
$$

The second identity exhibits explicitly the invariance under Möbius transformations of the left hand side, which follows from (3.28), while in the first identity the symmetry properties of $H_2 - H_3$ under interchange of the z_i variables is manifest. Furthermore, setting in the first identity $z_3 = z_4$ we get directly the other identity we need:

$$
G(1|2)_2 - G(1|2)_3 = Q_4 \frac{(z_{12})^2}{y(1)y(2)} \tag{5.20}
$$

Completely analogous relations hold for $H_2 - H_4$ and $H_3 - H_4$, and consequently for $G_2 - G_4$ and $G_3 - G_4$, which exhibit on the right hand side a factor of Q_3 and Q_2 respectively, instead of the Q_4 in (5.19) and (5.20). In addition, in (5.19) we have to substitute the term $(a_1 + a_4)(a_2 + a_3)$ with $(a_2 + a_4)(a_1 + a_3)$ in the first case and $(a_1 + a_2)(a_3 + a_4)$ in the second case. Eqs. (5.19) and (5.20) suffice to "evaluate" $A_1^{N=4}$. In particular, we note that the Q_k–factors appearing in the above formulae, one for the left sector and one for the right sector, combine with the term $|Q_iQ_j|^2$ in (5.17) to give $|P(a)|^2$, which cancels with a corresponding factor in the denominator. This resembles much what happened in our Compton–like scattering and also in the contribution to 4–graviton scattering with high space–time supersymmetry (5.8). One of the main differences between (5.17) and (5.8) is that the unique kynematical factor $K(k_i)$ which appears in (5.8) is spoiled in (5.17), i.e., while in eq. (5.8) we have *one* kynematical factor which multiplies *one* multiple integral, in (5.17) we have various kynematical factors, $\mathrm{tr}(A_1A_2A_3A_4)$, $\mathrm{tr}(A_1A_2)\mathrm{tr}(A_3A_4)$ etc. which multiply *different* multiple integrals and can thus not be combined to give $K(k_i)$. The origin of this fact is twofold: first, the 8–ψ correlators, when summed over spin structures, are now not all equal among them and, second, we have now also non vanishing correlators involving X-fields too, and thus less then 8 $\psi's$, which, in turn, give kynematical factors which are not made out only of traces. This spoiling of the factor $K(k_i)$ can be linked to the supersymmetry breakdown from $N = 8$ to $N < 8$ (although we are considering only bosonic amplitudes and have thus no direct control over space–time supersymmetry). In fact, the ten-dimensional supersymmetric origin of $K(k_i)$ can be inferred by noting that a model with an $N = 8$ SUSY in four dimensions corresponds basically to a dimensionally reduced $N = 2$ Supergravity model in ten dimensions. In an effective ten–dimensional bosonic Lagrangian the term $K(k_i)$ (or rather, its ten dimensional counterpart of which it is a dimensional reduction) corresponds to a contribution which is quartic in a (generalized) Riemann curvature (see ref. [GRS] for the tree-level contribution and ref. [ST] for the one–loop contribution):

$$\Delta L = R_{abcd}R_{efgh}R_{ijkl}R_{mnop}f^{abcdefghijklmnop} \tag{5.21}$$

where $f^{abc\cdots}$ is a particular (and rather complicated) $D = 10$ Lorentz invariant tensor, made out of Kroneckers δ's, which is fixed by $K(k_i)$. The supersymmetrizability of (5.21) relies crucially on the fact that, remarkably enough, the ten-dimensional $N = 1$ superspace integral of the dilaton superfield (see [LP] and references therein) $\int \Phi(x,\theta)d^{16}\theta$ yields a contribution which is proportional to (5.21) where the invariant tensor $f^{abc\cdots}$ is precisely the one coming out from $K(k_i)$. This supersymmetrization procedure is no longer valid for our "spoiled" kynematical factors, i.e. for their ten dimensional counterparts, the superspace integral of the dilaton superfield giving just *one* fixed particular combination of R^4. This shows that our model can not be viewed as a dimensional reduction of a supersymmetric ten dimensional theory. Actually, we showed the breakdown of the $N = 1$, $D = 10$ SUSY, but of course this implies also the breakdown of the $N = 2$, $D = 10$ and therefore of the $N = 8$, $D = 4$ SUSY.

After this digression we will now come back to our main task, i.e. the investigation of (5.17). Also in this case rescattering terms are depicted by the same integration region of Koba–Nielsen variables and moduli as in the case of (5.8) and, looking at particular terms, they are seen to be there. However, in order to establish definitively their presence, one should take care of all the various correlators appearing in (5.17), compute the asymptotic behavior of their kynematical factors and look at possible cancellations. We did not the tedious (even if straightforward) calculations but we think that rescattering terms can be present also in (5.17).

Here we will not consider them any more and jump directly to the study of new effects.

The first new effect is depicted by Fig. 12. The lines z_1 and z_2 collide on the upper sheet emitting a graviton and the lines z_3 and z_4 collide on the lower sheet emitting a graviton too. What remains is a one–particle–reducible diagram containing a graviton one–loop two–point function: this two–point function is now different from zero due to the fact that the non renormalization theorem does not hold in this case. In fact, the right hand side of (5.20) is now different from zero, in contrast to the first eq. of (4.16) which, in turn, implies the vanishing of two– and three-point functions for gravitons in the model considered in the preceding chapter.

In summary, we have to evaluate V(1)V(2) and V(3)V(4) in the limit of $\Delta_b \to 0$ and $\Delta_c \to 0$ where we defined: $z_1 = z_b + \frac{\Delta_b}{2}$, $z_2 = z_b - \frac{\Delta_b}{2}$, $z_3 = z_c + \frac{\Delta_c}{2}$, $z_4 = z_c - \frac{\Delta_c}{2}$ This has already been done in (4.14). We repeat here the results:

$$V(1)V(2) \to -\frac{1}{|z_{12}|^2} e^{-\frac{t}{2}\langle X(1)X(2)\rangle_{\Delta_b\to 0}} e^{ik_0X(b)}$$
$$(\varepsilon^b\cdot\partial X(b)+iA^b_{\alpha\beta}\psi^\alpha(b)\psi^\beta(b))(\bar\varepsilon^b\cdot\bar\partial X(b)+i\bar A^b_{\alpha\beta}\bar\psi^\alpha(b)\bar\psi^\beta(b))$$
$$\equiv V(b)$$

$$V(3)V(4) \to -\frac{1}{|z_{34}|^2} e^{-\frac{t}{2}\langle X(3)X(4)\rangle_{\Delta_c\to 0}} e^{-ik_0X(c)}$$
$$(\varepsilon^c\cdot\partial X(c)+iA^c_{\alpha\beta}\psi^\alpha(c)\psi^\beta(c))(\bar\varepsilon^c\cdot\bar\partial X(c)+i\bar A^c_{\alpha\beta}\bar\psi^\alpha(c)\bar\psi^\beta(c))$$
$$\equiv V(c) \tag{5.22}$$

Here $\varepsilon^b \equiv \varepsilon$ and $A^b_{\alpha\beta} \equiv A_{\alpha\beta}$ are kynematical factors which are given in eq. (4.12) and ε^c and $A^c_{\alpha\beta}$ are obtained from the formers by the substitution $1\to 3, 2\to 4, k_0\to -k_0$. According to (5.17) we have to evaluate

$$W_{ij} \equiv \left\langle \left(\left(V^L(b)V^L(c)\right)_i - \left(\left(V^L(b)V^L(c)\right)_j\right) \cdot (\mathrm{Right}) \right\rangle \tag{5.23}$$

The unique correlator which survives in (5.23) is the one involving 4 ψ's. This is essentially due to the fact that the cosmological constant is zero in our model. For this correlator we get from (5.20):

$$\langle A^b_{\alpha\beta}\psi^\alpha(b)\psi^\beta(b))A^c_{\gamma\delta}\psi^\gamma(c)\psi^\delta(c))\rangle_i - \langle\mathrm{same}\rangle_j = \frac{\mathrm{Tr}(A^bA^c)}{2}\frac{Q_k}{y(a)y(b)} \tag{5.24}$$

and an analogous formula for the (bared) fermions in the other chirality sector. With the help of (5.24) we get:

$$W_{ij} = \frac{|Q_k|^2}{4}\frac{\mathrm{Tr}(A^bA^c)\mathrm{Tr}(\bar A^b\bar A^c)}{|y(b)y(c)|^2|\Delta_b\Delta_c|^2} e^{-\frac{t}{2}G(1,2,3,4)_{\Delta_{b,c}\to 0}} \tag{5.25}$$

where G(1,2,3,4) has been defined in (4.24). For the combination appearing in (5.17) we get then:

$$\sum_{i<j}^{4}|Q_iQ_j|^2\left(|Q_i|+|Q_j|\right)W_{ij} = \frac{1}{2}|P(a)|^2\left(\sum_{i=2}^{4}|Q_i|\right)\frac{\mathrm{Tr}(A^bA^c)\mathrm{Tr}(\bar A^b\bar A^c)}{|y(b)y(c)|^2|\Delta_b\Delta_c|^2} e^{-\frac{t}{2}G(1,2,3,4)_{\Delta_{b,c}\to 0}} \tag{5.26}$$

Fixing again the projective and translational invariances in the usual way (4.21), with $f(a) = f(b) = 0$, (see 3.37) the amplitude becomes now an integral over the remaining moduli $a_{2,4}$ and over $\Delta_{b,c}$ (z_a is fixed by translational invariance and z_b, a_1 and a_3 are fixed by projective invariance):

$$A_{2-pt} \cong G^2 \mathrm{Tr}(A^b A^c)\mathrm{Tr}(\bar{A}^b \bar{A}^c) \frac{\int d^2a_2 d^2a_4 d^2\Delta_b d^2\Delta_c |a_{13}(z_b - a_1)(z_b - a_3)|^2}{|P(a)|^2 T^2 |\Delta_b \Delta_c|^2 y(b)|^2} \cdot$$
$$\left(\sum_{i=2}^{4} |Q_i|\right) \mathrm{e}^{-\frac{t}{2} G(1,2,3,4)_{\Delta_{b,c}\to 0}} \tag{5.27}$$

The leading behavior of this contribution for $s \to \infty$ and $t \to 0$ is again given by the pinching limit (4.22). In this case (3.36) becomes simply:

$$G(1,2,3,4) \to -\frac{4xy}{x+y} + ln|\Delta_b \Delta_c|^2 \tag{5.28}$$

So we can integrate over $\Delta_{b,c}$ to get a factor of $\frac{1}{t^2}$ which represents, of course, the propagation of the two gravitons emitted respectively from $z_{1,2}$ and $z_{3,4}$ in Fig.12. Again we can use (3.30) to evaluate the remaining formula, clearly $\sum_{i=2}^{4} |Q_i| \to 2|a_{13}|^2$:

$$A_{2-pt} \cong \left(\frac{G}{t}\right)^2 \mathrm{Tr}(A^b A^c)\mathrm{Tr}(\bar{A}^b \bar{A}^c) \left(\int \frac{dx dy}{(x+y)^2}\right)_{x+y \geq \pi} \mathrm{e}^{\frac{2txy}{x+y}} \tag{5.29}$$

The kynematical factors can be evaluated by their very definition (4.12) and in our limit of small t and large s we get:

$$\mathrm{Tr}(A^b A^c) \to \frac{st}{4}(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4) \tag{5.30}$$

Switching also here from the variables (x, y) to (ξ, L) with $x = \xi L,\ y = (1-\xi)L$ we get our final result:

$$\begin{aligned} A_{2-pt} &\cong (Gs)^2 \mathrm{Tr}(\varepsilon_1 \cdot \varepsilon_2)\mathrm{Tr}(\varepsilon_3 \cdot \varepsilon_4) \int_0^1 d\xi \int_{2\pi}^{\infty} \frac{dL}{L} \mathrm{e}^{t\xi(1-\xi)L} \\ &\equiv (Gs)^2 \mathrm{Tr}(\varepsilon_1 \cdot \varepsilon_2)\mathrm{Tr}(\varepsilon_3 \cdot \varepsilon_4) G(t) \end{aligned} \tag{5.31}$$

First we observe that G(t) is finite and has, in particular, no infrared divergence. In fact, the integral over L converges for $\xi \neq 0,1$, while, for example, for $\xi \to 0$ the integral over L goes as $\ln\xi$, which has a finite integral in ξ. Moreover

$$\frac{dG}{dt} = -\frac{1}{t}\int_0^1 d\xi \mathrm{e}^{2\pi t\xi(1-\xi)} \tag{5.32}$$

and therefore for $t \to 0$

$$A_{2-pt} \cong (Gs)^2 \ln(\alpha' t) \mathrm{Tr}(\varepsilon_1 \cdot \varepsilon_2) \mathrm{Tr}(\varepsilon_3 \cdot \varepsilon_4) \tag{5.33}$$

We note in particular that the t–pole appearing in the tree–level amplitude Gs^2/t gets not renormalized by the self–energy of the graviton, although we are considering a case in which there is no a priori non–renormalization theorem. That the contribution, we found, can be interpreted as a self–interaction of the graviton, can indeed be seen writing down the contribution to the amplitude of the Feynman diagram in Fig.13. This time we get:

$$F_{2-pt} = \frac{1}{t^2}\int_0^1 d\xi \int_0^\infty \frac{dL}{L} \mathrm{e}^{t\xi(1-\xi)L} \int d^4k e^{-k^2} N\left(k/L^{1/2} + \xi k_0\right) \tag{5.34}$$

It is interesting to note that the string cancels both of the 1/t poles appearing in (5.34). We remark that, in order to reproduce the expression (5.31), Quantum Gravity should necessarily provide a contribution to the numerator of the form $\Delta N(k) = (Gst)^2 \mathrm{Tr}(\varepsilon_1 \cdot \varepsilon_2) \mathrm{Tr}(\varepsilon_3 \cdot \varepsilon_4)$ from the beginning, because otherwise the powers of L and ξ would not match. Choosing the three–graviton vertices, appearing in Fig.13, as in (4.41) and using propagators as in (4.42) this is actually seen to be the case.

The last object of investigation of this thesis is a contribution to the amplitude like the one in Fig.14, where a graviton one–loop three–point function appears. This will allow us also to compare this contribution with the results of Chapter 4.

Again we have to take the product V(1)V(2) in the limit in which $|z_1 - z_2| \to 0$. We return here to the notation of (4.14) since no confusion should arise. What we have to evaluate now is:

$$V_{ij} \equiv \left\langle \left(\left(V^L(0)V^L(3)V^L(4)\right)_i - \left(V^L(0)V^L(3)V^L(4)\right)_j \right) (\mathrm{Right}) \right\rangle \tag{5.35}$$

In addition to the correlators of four ψ's, which we encountered already in the previous case, we have now also to evaluate correlators of six ψ's. This can again be done with the aid of the identity (5.20), which gives now, for general polarizations A,B,C:

$$\begin{aligned}&\left\langle A_{\alpha\beta}\psi^{\alpha}(1)\psi^{\beta}(1))B_{\gamma\delta}\psi^{\gamma}(2)\psi^{\delta}(2))C_{\mu\nu}\psi^{\mu}(3)\psi^{\nu}(3))\right\rangle_i - \langle\text{same}\rangle_j = \\ &\frac{Q_k \mathrm{Tr}(ABC)}{z_{12}z_{23}z_{13}y(1)y(2)y(3)}\left((z_{12})^2y(3)+(z_{23})^2y(1)+(z_{13})^2y(2)\right)\end{aligned} \tag{5.36}$$

and an analogous formula for the other chirality sector. We stress ones more that this formula has the right transformation properties under Möbius transformations; in fact, under these transformations the one–loop fermion correlators transform in the same way as the tree–level correlators. We remember also that the $y(i)$, appearing in (5.36), are understood, as always, to appear together with their signes, depending on whether the z_i lie on the upper or on the lower sheet. Taking into account also the contributions from the correlators of four ψ, and hence using ones more (5.24), we get:

$$\begin{aligned}V_{ij} = &-\frac{1}{|z_{12}|^2}\frac{|Q_k|^2}{4\,|y(0)y(3)y(4)|^2}\mathrm{e}^{-\frac{t}{2}\langle X(1)X(2)\rangle_{\Delta\to 0}}\cdot \\ &\left\langle\left\{\frac{2i^3\mathrm{Tr}(AA_3A_4)}{z_{03}z_{04}z_{34}}\left((z_{03}^2y(4)+(z_{04})^2y(3)+(z_{34})^2y(0)\right)\right.\right. \\ &+i^2\mathrm{Tr}(AA_3)y(4)\varepsilon_4\cdot\partial X(4)+i^2\mathrm{Tr}(AA_4)y(3)\varepsilon_3\cdot\partial X(3) \\ &\left.\left.+i^2\mathrm{Tr}(A_3A_4)y(0)\varepsilon\cdot\partial X(0)\right\}\cdot\left\{\text{Right}\right\}\mathrm{e}^{ik_0X(0)}\mathrm{e}^{ik_3X(3)}\mathrm{e}^{ik_4X(4)}\right\rangle\end{aligned} \tag{5.37}$$

In (5.37) we have evaluated all fermionic correlators while the X–correlators have still to be performed. The asymptotic behavior of the kynematical factors is as follows:

$$\begin{aligned}\mathrm{Tr}(AA_3A_4) &\to -\frac{1}{16}st(\varepsilon_1\cdot\varepsilon_2)(\varepsilon_3\cdot\varepsilon_4) \\ \mathrm{Tr}(AA_3) &\to -\frac{s}{4}k_4\cdot\varepsilon_3 \\ \mathrm{Tr}(AA_4) &\to \frac{s}{4}k_3\cdot\varepsilon_4 \\ \mathrm{Tr}(A_3A_4) &= \frac{1}{2}\left(\varepsilon_3\cdot k_4\varepsilon_4\cdot k_3-\frac{t}{2}\varepsilon_3\cdot\varepsilon_4\right)\end{aligned} \tag{5.38}$$

In deriving (5.38) we took into account the following behavior of the contractions between polarizations and momenta: in one case we have $\varepsilon_1 \cdot k_2 = \varepsilon_1 \cdot k_0 \to \sqrt{t}$ etc. while the ones of the kind $\varepsilon_1 \cdot k_3$ can be sent to zero by suitable gauge–choices. This can, in fact, be seen by going to the center–of–mass frame where $k_1 = (k, k, 0, 0)$, $k_3 = (k, -k, 0, 0)$ and $\varepsilon_1 = (0, 0, a_1, b_1)$, $\varepsilon_3 = (0, 0, a_3, b_3)$. From these behaviors we can deduce that, after performing also the X-contractions, *all* the kynematical factors in front of the various terms in (5.37) become of the leading order (st) which becomes $(st)^2$ when we combine both chirality sectors. One can see that also the integrals, which multiply these kynematical factors, behave, under the usual pinching limit that we will consider in short, essentially in the same way. For simplicity we will here present the study of only a class of contributions, probably the most significant one, and explain at the end which are the details for which the other contributions differ from the ones we are going to study.

The class we consider here is the one that exhibits the "direct" combination of polarizations:

$$\mathrm{Tr}(\varepsilon_1 \varepsilon_2)\mathrm{Tr}(\varepsilon_3 \varepsilon_4) \tag{5.39}$$

which multiplies also the leading contributions of the rescattering terms (5.9) and of the diagram in Fig.12. This means that we have to take of eq. (5.37) the term in the second row and the first term in the fourth row. We perform now for this contributions also the X-contractions:

$$\begin{aligned} V_{ij} = \frac{1}{16^2} \frac{|Q_k|^2}{|z_{12}|^2} \frac{(st)^2 \mathrm{Tr}(\varepsilon_1 \varepsilon_2)\mathrm{Tr}(\varepsilon_3 \varepsilon_4)}{|y(0)y(3)y(4)|^2} & \left\{ \frac{(z_{03})^2 y(4) + (z_{04})^2 y(3) + (z_{34})^2 y(0)}{z_{03} z_{04} z_{34}} \right. \\ & \left. + y(0) \langle \partial X(0) X(3) - \partial X(0) X(4) \rangle \right\} \cdot \\ & \left\{ \text{Right} \right\} e^{-\frac{t}{2} G(1,2,3,4)_{\Delta \to 0}} \end{aligned} \tag{5.40}$$

To derive this formula we used the asymptotic relations $\varepsilon \cdot k_3 = -\varepsilon \cdot k_4 \to s/2(\varepsilon_1 \cdot \varepsilon_2)(\varepsilon_3 \cdot \varepsilon_4)$ and substituted the kynematical factors of eqs. (5.38). We should also note that in eq. (5.40) we neglected a term coming from a (mixed) contraction between the left and the right sector:

$$\mathrm{Tr}(A_3 A_4)\mathrm{Tr}(\bar{A}_3 \bar{A}_4)(\varepsilon \cdot \varepsilon)\left\langle \partial X(0)\bar{\partial} X(0)\right\rangle \tag{5.41}$$

Although this looks like a selfcontraction which should not be kept, we have to remember that z_0 results from the collision between z_1 and z_2. Therefore in (5.41) we have δ–function contributions, which go to zero under analytic continuation on external momenta, and a zero mode part (3.20) which survives. Actually (5.41) is suppressed for kynematical reasons, indeed, $(\varepsilon \cdot \varepsilon) \cong t$, which gives alltogether t^3 and no power of s.

In the pinching limit (4.22), with the usual distribution of the z_i variables on the two sheets (Fig.4), we can apply (3.38) and use the relation:

$$\begin{aligned}(z_{03}^2 y(4) + (z_{04})^2 y(3) + (z_{34})^2 y(0) = \\ -z_{04} z_{03}\left((z_3 - a_1)(z_4 - a_3) + (z_3 - a_3)(z_4 - a_1)\right)\end{aligned} \tag{5.42}$$

to evaluate further (5.40). The relation (5.42) is a consequence of the fact that we keep z_0 and $z_{3,4}$ on different sheets. This implies that the correlator (5.36) has not to have poles for $z_0 \to z_3$ or $z_0 \to z_4$. As a consequence the numerator in (5.36), which in the pinching limit becomes polinomial, must have one zero for $z_0 = z_3$ and one for $z_0 = z_4$. The form of the "residue" in (5.42) is fixed, a part from a constant, by Möbius covariance and by the requirement of symmetry under $a_1 \leftrightarrow a_3$. Substituting (5.42) and (3.38) in (5.40) we can now compute the combination which appears in (5.17):

$$\begin{aligned}&\sum_{i<j}^{4} |Q_i Q_j|^2 \left(|Q_i| + |Q_j|\right) V_{ij} \to |P(a)|^2 \left(\sum_{i=2}^{4} |Q_i|\right) \cdot \\ &\frac{|Q_k|^2}{|z_{12}|^2} \frac{(st)^2 \mathrm{Tr}(\varepsilon_1\varepsilon_2)\mathrm{Tr}(\varepsilon_3\varepsilon_4)}{|y(0)y(3)y(4)|^2} \mathrm{e}^{-\frac{t}{2}G(1,2,3,4)_{\Delta\to 0}} \cdot \\ &\left|\frac{(z_3 - a_1)(z_4 - a_3) + (z_3 - a_3)(z_4 - a_1)}{z_{43}} + \pi a_{13} \frac{\mathrm{Im}\nu_{34}}{x+y}\right|^2\end{aligned} \tag{5.43}$$

Inserting this in (5.17) we can apply (3.30) and proceed in the usual manner to obtain:

$$A_{3-pt} \cong \mathrm{Tr}(\varepsilon_1\varepsilon_2)\mathrm{Tr}(\varepsilon_3\varepsilon_4)(Gst)^2 \int \frac{d^2a_2 d^2a_4}{|a_{12}a_{34}a_{13}|^2(x+y)^2} \int \frac{d^2z_{12}}{|z_{12}|^2} \int d^2\nu_{34}$$
$$\mathrm{e}^{-\frac{t}{2}G(1,2,3,4)_{\Delta\to 0}} \left| \frac{(z_3-a_1)(z_4-a_3)+(z_3-a_3)(z_4-a_1)}{z_{43}} + \pi a_{13}\frac{\mathrm{Im}\nu_{34}}{x+y} \right|^2 \tag{5.44}$$

The leading graviton exchange corresponds here to the integration regions $z_4 \to a_1$ or $z_4 \to a_3$, which can be seen to give the same contribution; so we multiply by two and compute only one of them, say the one for $z_4 \to a_3$. Setting $\nu_{34} = \frac{1}{\pi}(\alpha + i\beta)$ (3.36) gives:

$$G(1,2,3,4)_{\Delta\to 0} \to \ln|z_{12}|^2 + \frac{4x(\beta - y)}{x+y}$$

and the modulus squared in the second row of (5.44) becomes:

$$|a_{13}|^2\left(1 + \frac{\beta}{x+y}\right)^2$$

The integration over z_{12} cancels just one power of t and we get:

$$A_{3-pt} \cong \mathrm{Tr}(\varepsilon_1\varepsilon_2)\mathrm{Tr}(\varepsilon_3\varepsilon_4)G^2s^2t\int\left(\frac{dxdy}{(x+y)^2}\right)_{x+y\geq\pi}$$
$$\int_0^y d\beta\left(1+\frac{\beta}{x+y}\right)\mathrm{e}^{\frac{2tx(y-\beta)}{x+y}} \tag{5.45}$$

This can be rewritten as:

$$A_{3-pt} \cong \mathrm{Tr}(\varepsilon_1\varepsilon_2)\mathrm{Tr}(\varepsilon_3\varepsilon_4)G^2s^2t\int_{2\pi}^{\infty}\frac{dL}{L^{\frac{D}{2}-2}}$$
$$\int_0^1 d\xi \int_0^{\xi} d\alpha (1+\alpha)^2 \mathrm{e}^{t(1-\xi)(\xi-\alpha)L} \tag{5.46}$$
$$\equiv \mathrm{Tr}(\varepsilon_1\varepsilon_2)\mathrm{Tr}(\varepsilon_3\varepsilon_4)G^2s^2tH(t)$$

Eq. (5.45) is actually ill defined and so we performed a "dimensional" regularization in (5.46) with $D = 4 + 2\varepsilon$. This corresponds essentially to substitute in (5.17) T^3 with $T^{\frac{D}{2}+1}$ with similar modifications in the powers of the Q's in order to preserve Möbius invariance. Although the factor $T^{\frac{D}{2}+1}$ appears always in the amplitudes of D–dimensional models, it is not clear how one can really extend our model to $D \neq 4$. However, when comparing with the corresponding Feynman diagram, we will see that our regularization procedure reflects indeed the ordinary dimensional regularization of Q.F.T.

The terms which we did not consider in our computation, i.e. the ones with kynematical factors which differ from $\mathrm{Tr}(\varepsilon_1\varepsilon_2)\mathrm{Tr}(\varepsilon_3\varepsilon_4)(Gst)^2$, even if they are of the same order in s,t, change in (5.46) the "numerator" $(1+\alpha)^2$ by a polynomial in α, but the conclusions we will reach are not sensitive to these details. Scaling in eq. (5.46) $L \to L/(-t)$ we get:

$$H(t) = \frac{t^{\varepsilon}}{t}\int_{2\pi t}^{\infty}\frac{dL}{L^{\frac{D}{2}-2}}\int_0^1 d\xi\int_0^{\xi} d\alpha(1+\alpha)^2 \mathrm{e}^{-(1-\xi)(\xi-\alpha)L} \tag{5.47}$$

which for $t \to 0$ becomes:

$$H(t) \to \frac{1}{t}\left(c_1(\ln t)^2 + c_2\ln t + c_3\frac{\ln t}{D-4} + c_4\frac{1}{D-4} + c_5\frac{1}{(D-4)^2}\right) \tag{5.48}$$

where the c_i are numerical constants. Thus one gets for the finite part of the amplitude in our limit:

$$A^{fin}_{3-pt} \cong \mathrm{Tr}(\varepsilon_1\varepsilon_2)\mathrm{Tr}(\varepsilon_3\varepsilon_4)G^2s^2\left(c_1(\ln(\alpha' t))^2 + c_2\ln(\alpha' t)\right) \tag{5.49}$$

Also in this case we note that, due to the fact that the 1/t poles have been cancelled, the three–graviton "vertex" $\sqrt{\mathrm{G}}\mathrm{s}$ gets not renormalized.

Needless to say that (5.46) can be represented in terms of Feynman diagrams by Fig.15:

$$\begin{gathered} F_{3-pt} = \frac{1}{t}\int_0^{\infty}\frac{dL}{L^{\frac{D}{2}-2}}\int_0^1 d\xi\int_0^{\xi} d\alpha \mathrm{e}^{t(1-\xi)(\xi-\alpha)L}. \\ \int d^4k \mathrm{e}^{-k^2} N\left(\frac{k}{L^{1/2}} - (\xi-\alpha)k_0 + \alpha k_3\right) \end{gathered} \tag{5.50}$$

The $\frac{1}{D-4}$ poles appearing in (5.48) have to be interpreted as infrared singularities; they result, in fact, from the integration region of large L. In the corresponding

Feynman diagram, Fig.15, they would result from $k \to 0$, where k is the momentum of the horizontal propagator in the loop of Fig. 15. For these values of the loop momentum also the other two propagators become singular.

The infrared nature of the singularities appearing in the string one–loop amplitude corresponding to Fig.14 can be inferred also observing that they are absent in the diagram of Fig.1 (as we saw in the preceding chapter) where the lower line is massive. However, in our opinion, a more accurate analysis is necessary to see if these infrared divergences are only artifacts of our approximation scheme, and so are ficticious, or if they are true infrared divergences, and so, hopefully, of the Bloch–Nordsieck type [W]. This question is still under investigation.

Finally we observe that in the case of four–graviton scattering with low space–time SUSY no Schwarzschild-like correction to Newton's potential appeares. This last one would have been of the kind $(Gs)^2\sqrt{\frac{s}{t}}$, i.e. the true Schwarzschild–like correction where the mass M has been replaced by $\sqrt{s}$. In particular, the $\frac{1}{\sqrt{t}}$ behavior, found in the preceding chapter, which represents a longe range correction to Newton's potential, is absent here. Although the tree–level amplitudes, corresponding to the two kinds of scattering that we studied in chapters 4 and 5, i.e. the Comtpon–like scattering and four–graviton scattering respectively, have the same asymptotic behavior for high energies (and therefore the presence of a mass is not felt) this is no longer true at one–loop level for the one–particle–reducible contributions that we studied in this thesis. In particular (5.49) has no "classical" counterpart [AS].

6. Concluding remarks

In this thesis we were concerned mainly with situations, at one–loop level in string perturbation theory, in which the non renormalization theorem does not hold. In the presence of such a theorem the non abelian nature of the gravitational interaction is revealed only from two loops onwards, while in the absence of such a theorem it is manifest already at one loop, see Figs.13,15. We evaluated the effects of such contributions in the case of a scattering between a massive scalar and a graviton and in the case of four–graviton scattering at low space–time supersymmetry in type II models.

While basically there is no real *classical* counterpart of a four graviton scattering (beyond tree level), and thus it is not so meaningful to speak about post–Newtonian corrections to this scattering, there exists a *par excellence* example of a classical gravitational interaction between a massive body and a massless particle, which is in fact Einstein's deflection of light rays which pass near the sun. The situation is described classically by the Schwarzschild metric and we were able to compare our results with the predictions made by it. Massive scattering turns out to be more appropriate for the comparison between classical and "stringy" gravitational effects.

One of the drawbacks of massive scattering remains that the amplitudes are ill–defined due to the presence of the singular on–shell propagators of the external legs. Nevertheless, it is possible to compute finite quantum corrections. It would, in particular, be interesting to fix the absolute normalization of the Schwarzschild–correction we found, employing unitarity. To this order it is sufficient to compute the overall normalization of the amplitude. This can most conveniently be done by evaluating the rescattering term in the eikonal approximation and imposing the exponentiation of the eikonal phase, adding tree-level and one–loop contributions.

Another interesting question would be if the breakdown of supersymmetry, from $N = 8$ down to $N = 4$, implies a further renormalization of the

Schwarzschild–term. In this case there would be a non vanishing contribution also from the diagram in Fig.5. This diagram could also be used to check further the non renormalization of the 1/t pole discussed in section 5.3.

Finally, we note that the hyperelliptic language can be readily extended to two–loop level [GIS,IZ], the unique essential technical complication being supercurrent insertions. However, in the case of absence of non renormalization theorems, the sum over spin structures would assume a rather involved form. Nevertheless, when investigating one–particle–reducible contributions to the two–loop amplitude, in the relevant pinching limit(s) several simplifications are expected to occur [GIZ]. This time two–loop Schwarzschild–like corrections would be represented by $G^3 s^2 M^2 \ln t$. It would also be interesting to see whether the Q.F.T.–structure exhibited by the string at one–loop for large s and small t persists also at the much richer two–loop level.

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Figure Captions

1. A one–loop stringy three point function with massive external legs.

2. The hyperelliptic torus: the two non–contractible loops are shown.

3. One particle reducible diagrams with singular on shell propagators.

4. The positions of the Koba–Nielsen variables on the two sheets of the hyperelliptic torus in our "pinching limit".

5. In string models with highest spacetime supersymmetry this contribution is zero.

6. The leading Feynman diagram for $t \to 0$ and $s \to \infty$.

7. A diagram in which the massive particles decay virtually into massless ones.

8. A diagram in which in the loop a state of the second excited level of the string appears.

9. "Elastic" and "anelastic" contributions to four–graviton rescattering.

10. Leading contributions to rescattering: the "direct" diagram.

11. The leading "flipped" rescattering diagram.

12. A one particle reducible diagram with a stringy one–loop graviton two–point function.

13. Graviton self–energy in "conventional" Quantum Gravity.

14. A stringy one–loop graviton three–point function.

15. A Feynman diagram, which is infrared divergent due to soft gravitons which are exchanged between k_3 and k_4 through the horizontal propagator.

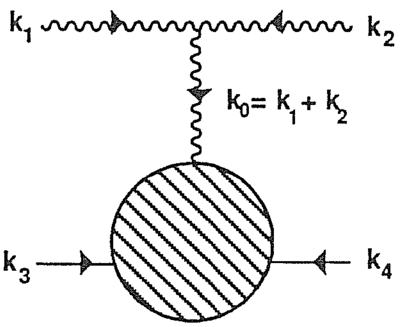

Fig. 1

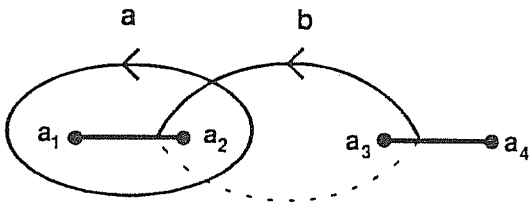

Fig. 2

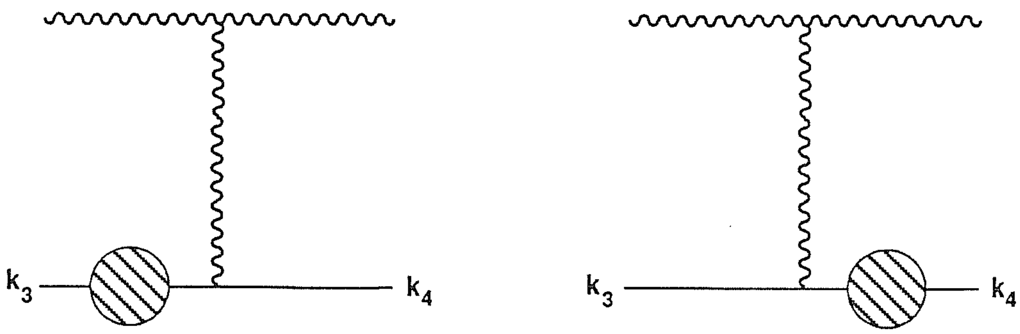

Fig. 3

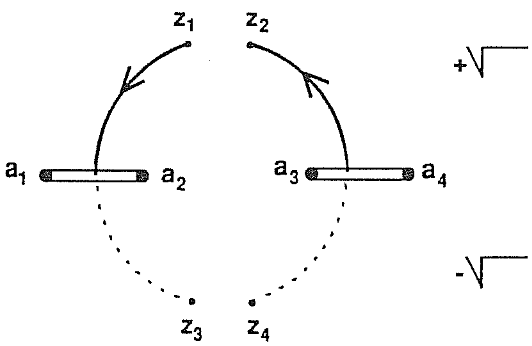

Fig. 4

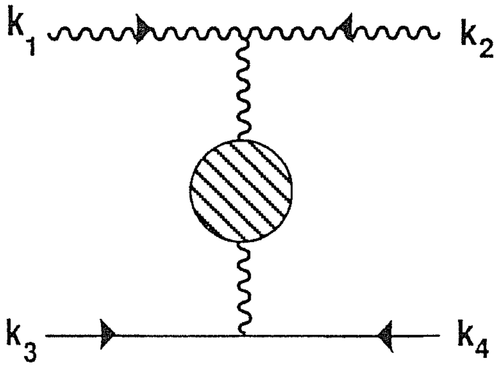

Fig. 5

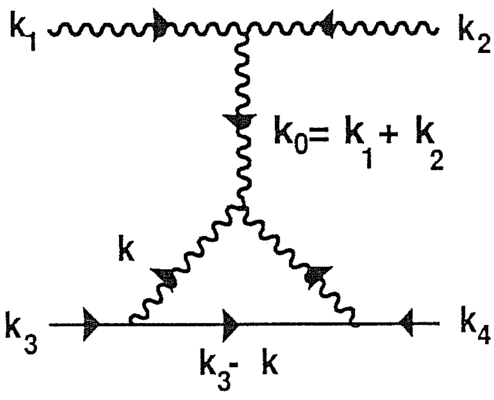

Fig. 6

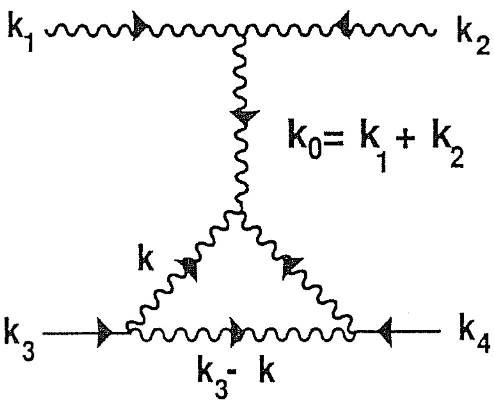

Fig. 7

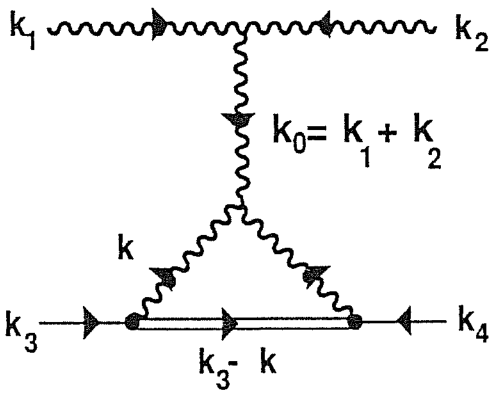

Fig. 8

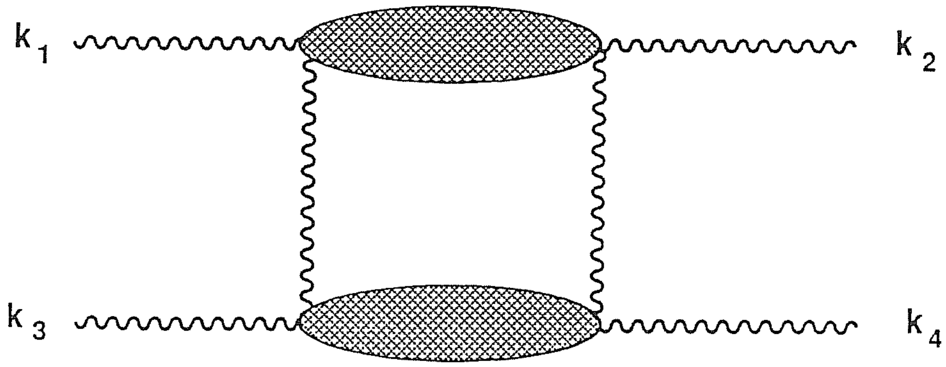

Fig. 9

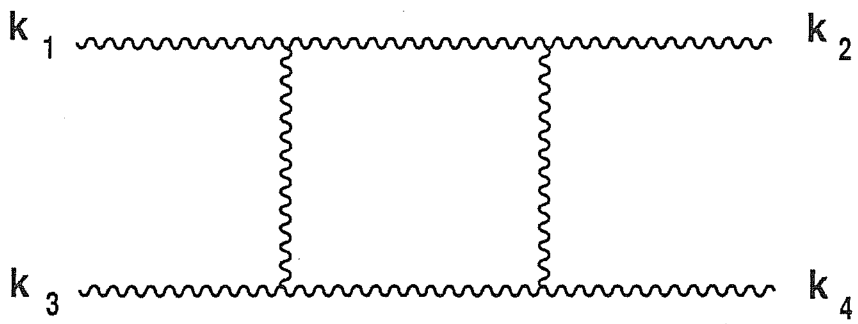

Fig. 10

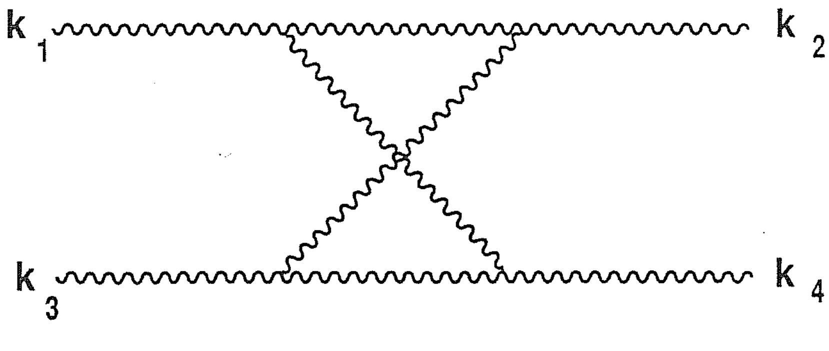

Fig. 11

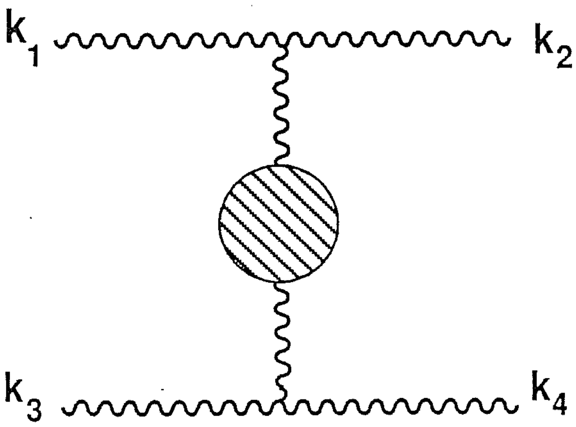

Fig. 12

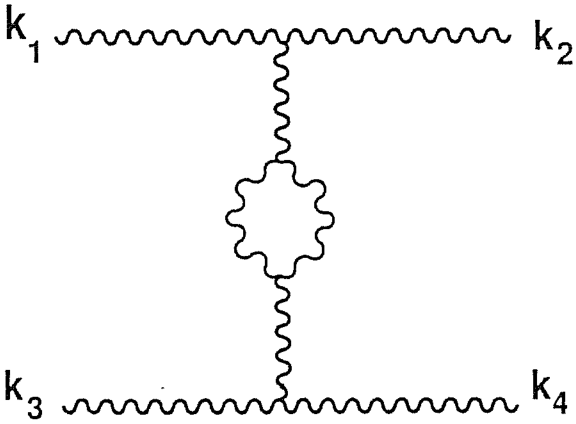

Fig. 13

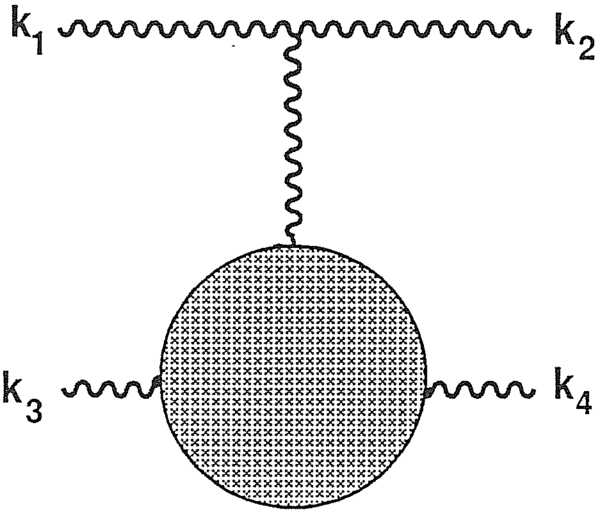

Fig. 14

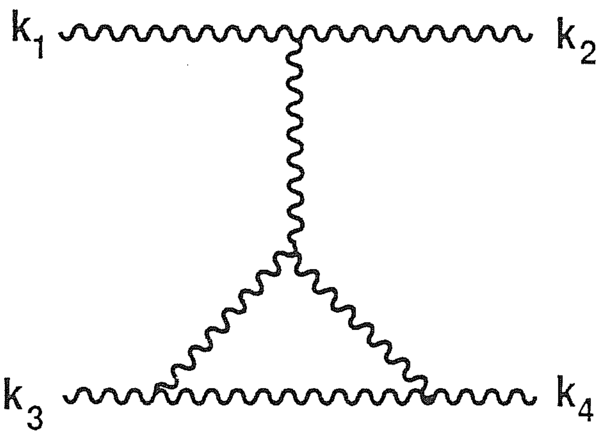

Fig.15