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FUNCTIONS OF A QUATERNIONIC VARIABLE

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INTRODUCTION

The richness of the theory of functions over the complex field makes it natural to investigate a similar theory for Clifford algebras(§ 2.4) and in particular for the only other non trivial real associative field, namely the quaternions. Such a theory was not developed until nearly a century after Hamilton's discovery of quaternions. Hamilton ([**Ha**])himself and his principal followers Tait ([**Ta**]) and Joly ([**J**]) only developed the theory of functions of a quaternionic variable as far as it could be taken by the general methods of the theory of functions of several real variables. They did not delimit a special class of regular functions among quaternionic - valued functions of a quaternionic variable, analogous to the regular functions of a complex variable. This may have been because neither of the two fundamental definitions of a holomorphic function of a complex variable has interesting consequences when adapted to the quaternions; one is too restrictive, the other not restrictive enough: the functions of a quaternionic variable which have quaternionic derivatives, in the obvious sense, are just the constant and linear functions and the functions which can be represented by quaternionic power series are just those which can be represented by power series in four real variables(§ 2.2). In spite of this, the zero set of a quaternionic power series with real coefficients assumes an independent interest; we follow in § 2.3 Datta and Nag who proved in [**D - N**] that this set is a union of 2-spheres.
In 1935 Fueter [**F**] proposed a definition of regularity for quaternionic functions by means of the analogue of the Cauchy-Riemann equations. He showed that this definition led to close analogues of Cauchy's Theorem and Cauchy's Integral Formula.
Sudbery [**Su**], in 1978 gave a self-contained account of the main stream of quaternionic analysis by using the exterior calculus and the definition of the quaternionic differential forms dq and Dq (§ 3.2) to clarify the relationship between quaternionic analysis and complex analysis.
In particular, Sudbery also pointed out some of the difficulties of the quaternionic theory, such as, for example, the fact that the identity map is not regular and that pointwise multiplication and composition of maps do not mantain regularity.
In § 3.1 a few classes of examples of regular functions are exhibited.
§ 3.3 is devoted to a quaternionic version of Cauchy's Theorem and Cauchy's Integral Formula.

If the function f is continuously differentiable and satisfies

$$\frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0$$

then

$$\int_{\partial U} Dqf = 0$$

where U is any smooth domain . Moreover if q_0 lies inside U then

$$f(q_0) = \frac{1}{2\pi^2}\int_{\partial U} \frac{(q-q_0)^{-1}}{|q-q_0|^2} Dqf(q).$$

The above Cauchy's Integral Formula yields, analogously to what happens in complex analysis, that if f is regular in an open set U then it has a power series expansion about each point of U(§ 3.4).

We have a certain number of new results concerning domains of convergence of quaternionic regular series:

we prove in § 3.5 that if Ω is a Reinhardt domain (which is simply a ball in the one dimensional case) then the series expansion about a point of Ω converges normally on Ω and in § 3.8 we give examples which show that, opposite to what happens in the complex case, (Abel's Lemma), a domain of convergence is not necessarily a Reinhardt domain.

The notions of domain of regularity and of regularly convex domain are defined in § 3.6 : these notions are completely similar to the more familiar ones of domain of holomorphy and holomorphically convex domain.

In [P] Pertici proves that any open subset of **H** is a domain of regularity regularly convex(§ 3.6). As a consequence of the analogous result holding in the complex case, Dolbeault's theorem (§1.4) ensures that any cohomology group of the sheaf of holomorphic functions vanishes on any open subset of **C**: in § 3.7 we define the sheaf of regular functions and prove that the first cohomology group of the sheaf of regular functions vanishes on any open subset of **H**.

In § 3.9 we are concerned with a reflection principle: a regular function defined in the upper part of a domain Ω , symmetric with respect to the purely imaginary space, is (under certain conditions of symmetry and continuity) the restriction of a function which is regular in all of Ω.

CONTENTS

1. HOLOMORPHIC FUNCTIONS

§ 1.1. Preliminaries

The following standard notation is used:
R denotes the field of real numbers,
C denotes the field of complex numbers,
$(x_1, ..., x_n)$ is an element of $\mathbf{R}^n$ and
$(z_1, ..., z_n) = (x_1+iy_1, ..., x_n+iy_n)$ is an element of $\mathbf{C}^n$.
The partial differential operators on $\mathbf{C}^n$, $\frac{\partial}{\partial z_j}$ and $\frac{\partial}{\partial \bar{z}_j}$ are given by:

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right) \qquad j = 1, ..., n ;$$

$$\frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right) \qquad j = 1, ..., n.$$

Correspondingly, we have the *differentials*

$$dz_j = dx_j + idy_j , \qquad j = 1, ..., n;$$

$$d\bar{z}_j = dx_j - idy_j \qquad j = 1, ..., n.$$

If $z^0 \in \mathbf{C}^n$ and $r > 0$, let us define the *open ball* as the set

$$B(z^0,r) = \{z \in \mathbf{C}^n : | z - z^0 | < r \}$$

and the *open polydisc* as

$$P^n(z^0,r) = \{ z \in \mathbf{C}^n : | z_j - z_j^0 | < r, j = 1, ... n.\}.$$

We also need a notation for a multi - index.
Let $N = \{0, 1, 2, ...\}$. A *multi - index* α is an element of N^j where j is usually understood from the context. If α is a multi - index and if $z = (z_1, ..., z_n)$, set

$$z^\alpha = z_1^{\alpha_1}...z_n^{\alpha_n} ,$$

$$\frac{\partial^\alpha}{\partial z^\alpha} = \frac{\partial^{\alpha_1}}{\partial z_1^{\alpha_1}}...\frac{\partial^{\alpha_n}}{\partial z_n^{\alpha_n}}$$

$$\frac{\partial^\alpha}{\partial \bar z} = \frac{\partial^{\alpha_1}}{\partial \bar z_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial \bar z_n^{\alpha_n}}$$

Also $\alpha! = \alpha_1!\ldots\alpha_n!$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i \geq 0$,

$dz^\alpha = dz_{\alpha_1} \wedge \ldots \wedge dz_{\alpha_n}$ and $d\bar z^\alpha = d\bar z_{\alpha_1} \wedge \ldots \wedge d\bar z_{\alpha_n}$.

DEFINITION 1.1.1. *Let $\Omega \subseteq C^n$ be an open set. A function*

$$f: \Omega \to C$$

is holomorphic in Ω if and only if it is holomorphic in each variable separately , that is f is continuosly differentiable in each complex variable separately on Ω and

$$\frac{\partial f}{\partial \bar z_j} = 0 \text{ on } \Omega , \; j = 1, \ldots, n.$$

$\theta(\Omega)$ will denote the C-algebra of holomorphic functions in Ω.

The following theorem shows that there are many others different definitions.

THEOREM 1.1.2(Hartogs). *Let Ω be an open subset of C and* $f: \Omega \to C^n$ *be a function. The following statements are equivalent:*

i) f is holomorphic in Ω ;

ii) for each $z^0 \in \Omega$ there is a positive real number r so that $P^n(z^0,r) \subseteq \Omega$ and f can be written as an absolutely convergent power series

$$f(z) = \sum_{\alpha=0}^{\infty} a_\alpha (z - z^0)^\alpha \quad \text{for } z \in P^n(z^0,r) \; ;$$

iii) f is continuous in each variable separately and locally bounded. For each $z^0 \in \Omega$ there is a positive real number r so that $P^n(z^0,r) \subseteq \Omega$ and

$$f(z) = \frac{1}{(2\pi i)^n} \int_{\partial_0 P^n} \frac{f(z_1,\ldots,z_n)}{(\zeta_1 - z_1)\ldots(\zeta_n - z_n)} d\zeta_1 \ldots d\zeta_n$$

where

$$\partial_0 P^n = \{(\zeta_1,\ldots,\zeta_n) \in P^n(z^0,r) : |\zeta_i - z_i| = r \; , \; i = 1, \ldots, n.\}.$$

§ 1.2. Regions of holomorphy and differential forms

DEFINITION 1.2.1. *An open subset Ω of C^n is a region of holomorphy if the following property holds:*

there do not exist two nonempty open sets Ω_1, Ω_2 such that :

i) Ω_2 is connected ,

ii) $\Omega_1 \subset \Omega_2 \cap \Omega$,

iii) $\Omega_2 \not\subset \Omega$,

so that for every function u that is holomorphic in Ω there exists a function u_2 holomorphic in Ω_2 such that $u = u_2$ on Ω_1 .

We see from the definition that an open set Ω is not a region of holomorphy if each analytic function on Ω can be analytically continued to some slightly larger open set (and this set does not depend on u).

DEFINITION 1.2.2. *An open subset Ω of C^n is called holomorphically convex if, for each relatively compact subset K of Ω (we will indicate it by $K \subset\subset \Omega$), the holomorphically convex hull of K in Ω:*

$$\hat{K}_\Omega = \{ z \in \Omega : \forall f \in \theta(\Omega) , | f(z) | \leq \sup_{w \in K} | f(w) | \}$$

is relatively compact in Ω.

The concept of holomorphical convexity can be subsumed under the following general scheme, which illuminates the similarity to elementary convexity: for $K \subset\subset \Omega$ and a family $\mathcal{F}$ of continuous functions in Ω ,

$$\hat{K}_\mathcal{F} = \{ z \in \Omega : |f(z)| \leq \sup_{w \in K} |f(w)| , \forall f \in \mathcal{F} \}$$

is called the $\mathcal{F}$ - convex hull of K in Ω and Ω is called $\mathcal{F}$ - convex if $\hat{K}_\mathcal{F}$ is relatively compact in Ω, for each $K \subset\subset \Omega$.

As an example if $\mathcal{F}$ denotes the set of all real affine functions defined in C^n then $\mathcal{F}$- convexity is equivalent to convexity.

THEOREM 1.2.3 (Cartan-Thullen). *Let Ω be open in C^n. The following conditions are equivalent :*

i) Ω is a region of holomorphy ;

ii) Ω is holomorphically convex ;

iii) For each $z \in \partial\Omega$ there exists an holomorphic function in Ω that is not holomorphically extendible at z.

As a corollary, any open subset Ω of C is a region of holomorphy.

In fact, for any $\xi \in \partial\Omega$ the holomorphic function in Ω f_ξ defined by:

$$f_\xi(z) = \frac{1}{z-\xi}$$

is not holomorphically extendible at ξ.

If Ω is an open subset of C^n and ω is a differential form on Ω, then ω is a sum of terms of the form

$$\omega_{\alpha\beta}\, dz^\alpha \wedge d\bar{z}^\beta$$

where α and β are multi-indices with $|\alpha| \leq n$, $|\beta| \leq n$ and $\omega_{\alpha\beta}$ is a smooth function.

If $0 \leq p, q \leq n$ and

$$\omega = \sum_{|\alpha|=p, |\beta|=q} \omega_{\alpha\beta}\, dz^\alpha \wedge d\bar{z}^\beta$$

then ω is called a differential form of type (p,q). For the same ω define

$$\partial\omega = \sum_{\alpha,\beta,j} \frac{\partial \omega_{\alpha\beta}}{\partial z_j} dz_j \wedge dz^\alpha \wedge d\bar{z}^\beta$$

$$\bar{\partial}\omega = \sum_{\alpha,\beta,j} \frac{\partial \omega_{\alpha\beta}}{\partial \bar{z}_j} d\bar{z}_j \wedge dz^\alpha \wedge d\bar{z}^\beta \;;$$

The following important theorem is due to Hörmander:

THEOREM 1.2.5. *Let Ω be a domain of holomorphy (i.e. a connected region of holomorphy) and f be a (p,q+1) form on Ω satisfying*

$\bar{\partial}f = 0$ (i.e. f is a $\bar{\partial}$-closed form).

Then there exists a (p,q) form ω on Ω such that

$\bar{\partial} w = f$ (i.e. f is a $\bar{\partial}$-exact form).

§ 1.3. Reinhardt domains and power series

A domain is a connected open set.

DEFINITION 1.3.1. *Let Ω be a domain in C^n. Ω is a Reinhardt domain if:*

$$\forall (z_1, \ldots, z_n) \in \Omega, \forall \alpha_i \in C \;:\; |\alpha_i| = 1, i = 1, \ldots, n \Rightarrow (\alpha_1 z_1, \ldots, \alpha_n z_n) \in \Omega.$$

Consider a power series in C^n about the origin:

$$P(z) = \sum_{\alpha} a_\alpha z^\alpha$$

DEFINITION 1.3.2. *The domain of convergence associated with P is the set of all points in a neighbourhood of which P converges absolutely and uniformly.*

THEOREM 1.3.3 (Abel's lemma). *If the power series P is convergent at a point $z = (z_1, \ldots, z_n)$ such that $z_1 z_2 \ldots z_n \neq 0$ then P converges normally in*

$$P^1(0, |z_1|) \times \ldots \times P^1(0, |z_n|).$$

Therefore :

THEOREM 1.3.4. *A domain of convergence of a power series is a Reinhardt domain.*

The connection between Reinhardt domains of convergence associated with a power series and domains of holomorphy is given by the following equivalence:

THEOREM 1.3.5. *A Reinhardt domain containing the origin is the domain of convergence of a power series if ,and only if , it is a domain of holomorphy.*

§ 1.4. Dolbeault's isomorphism

In complex analysis one frequently has to deal with the question of the existence of a "global" function having a prescribed "local behaviour". For instance, a classical problem is the construction of a meromorphic function having prescribed principal parts or poles. The notion of a *sheaf* was given as a suitable formal setting to handle this situation ([G]).

SHEAVES

Suppose X is a topological space and Π is the family of open sets in X.

DEFINITION 1.4.1. *A presheaf of abelian groups (rings,vector spaces,...) on X is a pair* $(\mathcal{F},\rho)$ *consisting of:*

i) a family $\mathcal{F} = (\mathcal{F}(U))_{U\in\Pi}$ *of abelian groups (resp. rings,...) ;*

ii) a family $\rho = (\rho^U_V)_{U;V\in\Pi}$ *of group (resp. rings,...) homomorphisms*

$$\rho^U_V : \mathcal{F}(U) \to \mathcal{F}(V)\,,\ \ V\subset U$$

with the following properties :

a) $\forall\, U\in \Pi:\ \ \rho^U_U = \mathbf{id}_U$;

b) $\forall\, \Omega \subset V\subset U : \rho^V_W \circ \rho^U_V = \rho^U_W$.

Generally one writes $\mathcal{F}$ instead of ($\mathcal{F}$, ρ). The homomorphisms ρ^U_V are called restriction homomorphisms and instead of ρ^U_V (f) one just writes $f_{/V}$, for any $f\in \mathcal{F}(U)$.

A presheaf $\mathcal{F}$ on X is a *sheaf* if for every open subset U of X and every family of open subsets $U_i \subset U$, $i\in I$ such that $U = \bigcup_{i\in I} U_i$ the following conditions , called the *Sheaf Axioms* , are satisfied:

AI. If $f,g \in \mathcal{F}(U)$ are such that for every $i\in I$, $f_{/U_i} = g_{/U_i}$,then f = g;

AII. Given elements $f_i \in \mathcal{F}(U_i)$, $(i\in I)$ such that $f_{i/U_i\cap U_j} = f_{j/U_i\cap U_j}$,for all i and j in I , then there exists f in $\mathcal{F}(U)$ such that $f_{/U_i} = f_i$ for every i belonging to I.

REMARK.The element f, whose existence is assured by **AII,** is by **AI** uniquely determined.

EXAMPLES.

◊For any open subset U of $\mathbf{R}^n$, let $\mathcal{C}^\infty(U)$ be the ring of the $\mathcal{C}^\infty$ functions in U. Taking the usual restriction mapping one gets the sheaf $\mathcal{C}^\infty$ of $\mathcal{C}^\infty$ - functions.

◊For an open subset U of $\mathbf{C}^n$,$\theta(U)$ is the ring of the holomorphic functions defined on U. Taking the usual restriction mapping one gets the sheaf θ of holomorphic functions.

◊The presheaf of meromorphic functions $\mathcal{M}$ is defined by setting $\mathcal{M}(U)$ to be the quotient field of $\theta(U)$ i.e. $\mathcal{M}(U) = \{ \frac{f}{g} : f, g \in \theta(U) \}$.

Suppose $\mathcal{F}$ is a presheaf of sets on a topological space X and $a \in X$ is a point.

On the disjoint union of the $\mathcal{F}(U)$'s ,taken over all the open neighbourhoods U of a, let us introduce an equivalence relation $\underset{a}{\sim}$ as follows:

for $f \in \mathcal{F}(U)$, $g \in \mathcal{F}(V)$:

$$f \underset{a}{\sim} g \Leftrightarrow \text{there exists an open set } W \subset U \cap V,\ a \in W, \text{ such that } f_{/W} = g_{/W}.$$

The set $\mathcal{F}_a$ of all equivalence classes is the inductive limit of $\mathcal{F}(U)_{U \in \Pi}$ is called the *stalk* of $\mathcal{F}$ at a. If $\mathcal{F}$ is a presheaf of abelian groups (rings,...) then the stalk $\mathcal{F}_a$ with the operation naturally induced on the equivalence classes is also an abelian group (resp. ring,...).

COHOMOLOGY GROUPS

Let $\mathcal{U} = (U_i)_{i \in I}$ be an open covering of X. For q = -1, 0, 1, 2, ... define the *qth-cochain group of* $\mathcal{F}$ with respect to $\mathcal{U}$ as

$$C^{-1}(\mathcal{U}, \mathcal{F}) = \{0\};$$

$$C^q(\mathcal{U}, \mathcal{F}) = \prod_{i_0 \dots i_q \in I} \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_q}) \quad \text{for } q \geq 0.$$

The elements of $C^q(\mathcal{U}, \mathcal{F})$ are called *q-cochains* . Thus a q-cochain is a family

$(f_{i_0 \dots i_q})_{i_0 \dots i_q \in I}$ such that $(f_{i_0 \dots i_q}) \in \mathcal{F}(U_{i_0} \cap \dots \cap U_{i_q})$ for $q \geq 0$.

For every $q \geq -1$ define the coboundary operator

$\delta_{-1} : C^{-1}(\mathcal{U}, \mathcal{F}) = \{0\} \to C^0(\mathcal{U}, \mathcal{F})$ to be the zero map,

$\delta_q : C^q(\mathcal{U}, \mathcal{F}) \to C^{q+1}(\mathcal{U}, \mathcal{F})$ for $q \geq 0$ to be such that

$$(\delta_q f)_{i_0 \ldots i_{q+1}} = \sum_{j=0}^{q+1} (-1)^j f_{i_0 \ldots \hat{i}_j \ldots i_{q+1} / U_{i_0} \cap \ldots \cap U_{i_{q+1}}}$$

where " $\hat{\alpha}$ " means that α is omitted.

The coboundary operator is a group (resp. ring,...) homomorphism. For $q \geq -1$ we define the following sub-groups (resp. sub-rings,...) of $C^q(\mathcal{U}, \mathcal{F})$:

$Z^q(\mathcal{U}, \mathcal{F}) = \operatorname{Ker} \delta_q$, its elements are called *q-cocycles*;

$B^q(\mathcal{U}, \mathcal{F}) = \operatorname{Im} \delta_{q-1}$, its elements are called *q-coboundaries.*

Thus, by definition, a 1-cochain (f_{ij}) is a cocycle if and only if

$f_{ik} = f_{ij} + f_{ik}$ on $U_i \cap U_j \cap U_k$ for all i,j,k. (these are called the *cocycle relations*)

Remark that (taking i=j=k)

$f_{ii} = 0$ for every i ;

and (taking i=k)

$f_{ij} = -f_{ji}$ for any i, j .

An easy computation shows that

$\delta_q \circ \delta_{q-1} = 0.$

As a consequence, every coboundary is a cocycle.

Thus a 1-cocycle (f_{ij}) is a coboundary if and only if there is a 0-cochain (g_i) such that

$f_{ij} = g_i - g_j$ on $U_i \cap U_j$.

The quotient group

$H^q(\mathcal{U}, \mathcal{F}) = Z^q(\mathcal{U}, \mathcal{F}) / B^q(\mathcal{U}, \mathcal{F})$

is the *qth-cohomology group* with coefficients in $\mathcal{F}$ with respect to the covering $\mathcal{U}$. Its elements are called *cohomology classes* and two cocycles which belong to the same cohomology class are called *cohomologous* . Thus two cocycles are cohomologous if and only if their difference is a coboundary.

We study now the effect on $H^q(\mathcal{U}, \mathcal{F})$ of refining the covering $\mathcal{U}$. We shall be interested only in locally finite coverings of X.

Let $\mathcal{V} = \{V_j\}_{j \in J}$ be a locally finite covering of X such that $\mathcal{V}$ is a refinement of $\mathcal{U}$. Then each V_j is contained in some $U_i \in \mathcal{U}$. We make an arbitrary choice of the U_i. This arbitrary choice defines a map

$$\phi : J \to I$$

having the property that

$$\forall j \in J : V_j \subset U_{\phi(j)}.$$

Then ϕ defines a map

$$\phi^* : C^q(\mathcal{U}, \mathcal{F}) \to C^q(\mathcal{V}, \mathcal{F}).$$

For $f \in C^q(\mathcal{U}, \mathcal{F})$ let us assign the section

$$f_{\phi(j_o)\ldots\phi(j_q)} \text{ to } U_{\phi(j_o)} \cap \ldots \cap U_{\phi(j_q)} \supseteq V_{j_o} \cap \ldots \cap V_{j_q}$$

then $\phi^*(f)$ is defined to be the cochain which assigns the restriction

$$f_{\phi(j_o)\ldots\phi(j_q)/V_{j_o} \cap \ldots \cap V_{j_q}} \text{ to } V_{j_o} \cap \ldots \cap V_{j_q}.$$

It is easy to show that ϕ^* has the following properties:

1) ϕ^* is a group (resp. ring...) homomorphism;

2) $\phi^* \circ \delta_q = \delta_q \circ \phi^*$ (with a standard notation);

3) for any two coverings $\mathcal{U}$ and $\mathcal{V}$ of X :

$\phi^*(Z^q(\mathcal{U}, \mathcal{F})) \subset Z^q(\mathcal{V}, \mathcal{F})$ and $\phi^*(B^q(\mathcal{U}, \mathcal{F})) \subset B^q(\mathcal{V}, \mathcal{F})$.

Hence ϕ^* induces a homomorphism of the cohomology groups:

$$\Phi^{\mathcal{U}}_{\mathcal{V}} : H^q(\mathcal{U}, \mathcal{F}) \to H^q(\mathcal{V}, \mathcal{F}).$$

It can be shown ([K] or [H]) that $\Phi^{\mathcal{U}}_{\mathcal{V}}$ does not depend on ϕ.

Let Λ be the collection of all locally finite open coverings of X. The collection Λ is directed by refinement, i.e.

a) $\mathcal{U} \leq \mathcal{V}$ means that $\mathcal{V}$ is a refinement of $\mathcal{U}$;

b) given $\mathcal{U}_1$ and $\mathcal{U}_2$ there exists $\mathcal{V}$ such that $\mathcal{U}_1 \leq \mathcal{V}$ and $\mathcal{U}_2 \leq \mathcal{V}$.

Whenever $\mathcal{U} \leq \mathcal{V}$ we have the homomorphism $\Phi^{\mathcal{U}}_{\mathcal{V}}$ having the properties:

1) $\Phi^{\mathcal{U}}_{\mathcal{U}}$ is the identity on $H^q(\mathcal{U}, \mathcal{F})$;

2) For $\mathcal{U} \leq \mathcal{V} \leq \mathcal{W}$: $\Phi^{\mathcal{V}}_{\mathcal{W}} \circ \Phi^{\mathcal{U}}_{\mathcal{V}} = \Phi^{\mathcal{U}}_{\mathcal{W}}$.

Let us consider $\bigcup_{\mathcal{U} \in \Lambda} H^q(\mathcal{U}, \mathcal{F})$ and partition it into equivalence classes by defining the following equivalence relation:

$h_1 \in H^q(\mathcal{U}_1, \mathcal{F})$ and $h_2 \in H^q(\mathcal{U}_2, \mathcal{F})$ are equivalent if,
for some $\mathcal{U}_1 \leq \mathcal{V}$ and $\mathcal{U}_2 \leq \mathcal{V}$: $\Phi_{\mathcal{V}}^{\mathcal{U}_1}(h_1) = \Phi_{\mathcal{V}}^{\mathcal{U}_2}(h_2)$.

The set of all equivalence classes, $\mathbf{H}^q(X, \mathcal{F})$ is the direct limit on Λ of the groups (resp.rings,...) $H^q(\mathcal{U}, \mathcal{F})$.

For an element f in $H^q(\mathcal{U}, \mathcal{F})$ we denote by $\Phi_{\mathcal{U}}(f)$ the equivalence class of f in $\mathbf{H}^q(X, \mathcal{F})$. Then

$$\Phi_{\mathcal{U}} : H^q(\mathcal{U}, \mathcal{F}) \rightarrow \mathbf{H}^q(X, \mathcal{F})$$

is a group (resp. ring,...) homomorphism.

REMARKS.

1) If $\mathcal{U} \leq \mathcal{V}$ then $\Phi_{\mathcal{V}}^{\mathcal{U}} : H^0(\mathcal{U}, \mathcal{F}) \rightarrow H^0(\mathcal{V}, \mathcal{F})$ is an isomorphism; therefore each $H^0(\mathcal{U}, \mathcal{F})$ is isomorphic to $\Gamma(X, \mathcal{F})$, the group (resp. ring,...) of *global sections.*

2) If $\mathcal{U} \leq \mathcal{V}$ then $\Phi_{\mathcal{V}}^{\mathcal{U}} : H^1(\mathcal{U}, \mathcal{F}) \rightarrow H^1(\mathcal{V}, \mathcal{F})$ is injective. As a consequence, $\Phi_{\mathcal{U}} : H^1(\mathcal{U}, \mathcal{F}) \rightarrow \mathbf{H}^1(X, \mathcal{F})$ is injective. Hence $\mathbf{H}^1(X, \mathcal{F}) = 0$ if and only if $H^1(\mathcal{U}, \mathcal{F}) = 0$ for every $\mathcal{U} \in \Lambda$.

PROPOSITION 1.4.1. *Let X be an open subset of $\mathbf{R}^n$, $\mathcal{C}^\infty$ be the sheaf of the $\mathcal{C}^\infty$-functions. Then*

$$\forall p \in \{1,2,...\} \; ; \quad \mathbf{H}^p(X, \mathcal{C}^\infty) = 0.$$

The proof of this classical result is based on the existence of a partition of unity on every locally finite covering of X (**[H]** , **[K]**).

DOLBEAULT'S ISOMORPHISM

Let Ω be a domain and θ be the sheaf of holomorphic functions on Ω.The $\bar{\partial}$ operator and differential (p, q) - forms have been defined in § 1.2.

Let $\Lambda^{(0,p)}(\Omega)$ be the group of differential (0, p) - forms,

$Z^{(0,p)}(\Omega) = \{ \omega \in \Lambda^{(0,p)}(\Omega) : \bar{\partial}\omega = 0 \}$ be the subgroup of $\bar{\partial}$ - closed forms;

$B^{(0,p)}(\Omega) = \{ \bar{\partial}\omega : \omega \in \Lambda^{(0,p-1)}(\Omega)\}$ be the subgroup of $\bar{\partial}$-exact forms.

Since $\bar{\partial} o \bar{\partial} = 0$ we have:

$B^{(0,p)}(\Omega) \subset Z^{(0,p)}(\Omega)$

The following theorem illustrates the relation between the $\bar{\partial}$ - resolution problem and cohomology groups of the sheaf of holomorphic functions.

THEOREM 1.4.2 (Dolbeault). *Let Ω be a domain in C^n, and $p > 0$. Then*

$$H^p(\Omega, \theta) \cong Z^{(0,p)}(\Omega) / B^{(0,p)}(\Omega).$$

For a proof of this result, see ([H] , [K]).

Since, from theorem 1.2.5., in any domain of holomorphy, every closed form is exact, we have the following corollary :

COROLLARY 1.4.5. *Let $\Omega \subset C^n$ be a domain of holomorphy, and $p > 0$. Then*

$$H^p(\Omega, \theta) = 0.$$

2.
FUNCTIONS OF A QUATERNIONIC VARIABLE

§ 2.1. Quaternions

Let 1, i, j, k denote the elements of the standard basis for $\mathbf{R}^4$. The *quaternionic product* is the $\mathbf{R}$-bilinear product

$\mathbf{R}^4 \times \mathbf{R}^4 \to \mathbf{R}^4$:

$(a, b) \to ab$,

with unit 1, defined by the formulae:

$i^2 = j^2 = k^2 = -1$ and

$ij = -ji = k$; $jk = -kj = i$; $ki = -ik = j$.

The space $\mathbf{R}^4$ with such a product is a real algebra $\mathbf{H}$ called *algebra of quaternions* . The quaternionic product is associative, but not commutative ($ij \neq ji$).

We will identify $\mathbf{R}$ with $1\mathbf{R} = \{ x1 : x \in \mathbf{R} \}$ and $\mathbf{R}^3$ with

$\mathbf{R}i + \mathbf{R}j + \mathbf{R}k = \{ x_1 i + x_2 j + x_3 k : x_1 , x_2 , x_3 \in \mathbf{R} \}$(its elements are called pure quaternions or purely imaginary quaternions).

Each quaternion q is expressible in a unique way in the form

$q = x_0 + i\, x_1 + jx_2 + kx_3$ where $x_0, x_1, x_2, x_3 \in \mathbf{R}$

We will usually denote 1, i, j, k by i_0, i_1, i_2, i_3 respectively.

PROPOSITION 2.1.1. *A quaternion is real if and only if it commutes with every quaternion.*

PROOF. Obviously, any real number commutes with all quaternions. Conversely, let $q = x_0 + i\, x_1 + jx_2 + kx_3$ be a quaternion commuting with i and j. We have:

$iq = qi \Rightarrow x_2 = x_3 = 0$;

$jq = qj \Rightarrow x_1 = 0$.

Hence, $q = x_0 \in \mathbf{R}$.

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PROPOSITION 2.1.2. *A quaternion is pure if and only if its square is a non positive real number.*

PROOF. Remark that if $q = t + ix + jy + kz$ is a quaternion,then $q^2 = 2tq - t^2 - x^2 - y^2 - z^2$. It is clear that, if $t = 0$ then $q^2 = -x^2 - y^2 - z^2 \leq 0$. Conversely, suppose that $q^2 \in \mathbf{R}$ and $q^2 \leq 0$. Then :
$q^2 + x^2 + y^2 + z^2 + t^2 \in \mathbf{R} \Rightarrow 2tq \in \mathbf{R}$. If $t \neq 0$ then q is real and $q^2 = t^2 > 0$, a contradiction. Hence $t = 0$ and q is a pure quaternion.

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The *conjugate* $\overline{q}$ *of a quaternion* $q = t + ix + jy + kz$ is defined by

$$\overline{q} = t - ix - jy - kz$$

Conjugation is an algebra anti -involution, that is, for $q, w \in \mathbf{H}$, $\lambda \in \mathbf{R}$:

$$\overline{q+w} = \overline{q} + \overline{w};$$

$$\overline{\lambda q} = \lambda \overline{q};$$

$$\overline{qw} = \overline{w}\,\overline{q}.$$

Moreover,

$$q \in \mathbf{R} \Leftrightarrow q = \overline{q} \text{ and } q \in \mathbf{R}^3 \Leftrightarrow \overline{q} = -q.$$

The conjugate $\overline{q}$ can be expressed by a **H**-bilinear combination of q. In fact

$$\overline{q} = -\frac{1}{2}(q + iqi + jqj + kqk) \qquad (2.1.1).$$

For $q = t + ix + jy + kz$ the non negative number

$$|q| = (q\overline{q})^{\frac{1}{2}} = \sqrt{t^2+x^2+y^2+z^2}$$

is the *norm* of q. Remark that

$$|qw| = |q||w|$$

because conjugation is anti -involutive.

PROPOSITION 2.1.3. *Each non zero quaternion q is invertible and*

$$q^{-1} = \frac{\overline{q}}{|q|^2}.$$

Remark that if q and w are different from zero then

$$(qw)^{-1} = w^{-1}q^{-1}.$$

For $q = t + ix + jy + kz$ the following relations hold :

$$t = \frac{1}{4}(q - iqi - jqj - kqk) \qquad (2.1.2)$$

$$x = \frac{1}{4}i(-q + iqi - jqj - kqk) \qquad (2.1.3)$$

$$y = \frac{1}{4}j(-q - iqi + jqj - kqk) \qquad (2.1.4)$$

$$z = \frac{1}{4}k(-q - iqi - jqj + kqk) \qquad (2.1.5)$$

In what follows we will discuss about a way in which the analogue of a holomorphic function can be defined in the (non commutative) field **H**.

Let Ω be an open subset of **H** and

$$f : \Omega \rightarrow \mathbf{H}$$

be a function. A first attempt to extend the definition of a holomorphic mapping could be requiring the function f to be **H**- Fréchet differentiable at every point $q \in \Omega$.

§ 2.2. H-Fréchet differentiable functions

Let Ω be an open set.

DEFINITION 2.2.1. *A function $f : \Omega \rightarrow H$ is quaternionic differentiable (on the left) at q if the limit*

$$\frac{df}{dq} = \lim_{h \to 0} h^{-1}(f(q+h) - f(q))$$

exists.

PROPOSITION 2.2.2. *Suppose the function f is defined and quaternion differentiable on the left throughout a connected open set Ω. Then, on Ω, f has the form:*

$$f(q) = a+qb$$

for some quaternionic constants a and b.

PROOF([SU]). It follows from the definition that if f is quaternion-differentiable at q, it is real-differentiable at q and that

$$\frac{\partial f}{\partial x_0} = \lim_{\substack{h \to 0 \\ h \in \mathbf{R}}} h^{-1}(f(q+h) - f(q))$$

$$= \lim_{\substack{h \to 0 \\ h \in \mathbf{R}}} (i_\lambda h)^{-1}(f(q+i_\lambda h) - f(q))$$

$$= -i_\lambda \frac{\partial f}{\partial x_\lambda}, \qquad \lambda \in \{1,2,3\}. \qquad (2.2.1)$$

Put $q = v + jw$, where $v = x_0 + ix_1$ and $w = x_2 - ix_3$ and let

$$f(q) = g(v,w) + jh(v,w)$$

where g and h are two complex valued functions of the two complex variables v and w. The equalities (2.2.1) imply:

$$\frac{\partial g}{\partial x_0} = -i\frac{\partial g}{\partial x_1} = \frac{\partial h}{\partial x_2} = i\frac{\partial h}{\partial x_3};$$

$$\frac{\partial h}{\partial x_0} = i\frac{\partial h}{\partial x_1} = -\frac{\partial g}{\partial x_2} = i\frac{\partial g}{\partial x_3};$$

In terms of complex derivatives, these equalities can be written as:

$$\frac{\partial g}{\partial \bar{v}} = \frac{\partial h}{\partial \bar{w}} = \frac{\partial h}{\partial v} = \frac{\partial g}{\partial w} = 0 \qquad (1)$$

$$\frac{\partial g}{\partial v} = \frac{\partial h}{\partial w} \qquad (2)$$

$$\frac{\partial h}{\partial \bar{v}} = - \frac{\partial g}{\partial \bar{w}} \qquad (3)$$

Equation (1) shows that g is a complex analytic function of v and $\bar{w}$, and h is a complex-analytic function of $\bar{v}$ and w.

It follows from Hartog's Theorem (Theorem 1.1.2) that f and g have continuous partial derivatives of all orders.

From (1) and (2) we have:

$$\frac{\partial^2 g}{\partial v^2} = \frac{\partial}{\partial v}\left(\frac{\partial h}{\partial w}\right) = \frac{\partial}{\partial w}\left(\frac{\partial h}{\partial v}\right) = 0,$$

and using the same techniques :

$$\frac{\partial^2 g}{\partial \bar{w}^2} = \frac{\partial^2 h}{\partial \bar{v}^2} = \frac{\partial^2 h}{\partial w^2} = \frac{\partial^2 g}{\partial v \partial \bar{w}} = \frac{\partial^2 h}{\partial \bar{v} \partial w} = 0.$$

Then we can deduce that g is linear in v and $\bar{w}$ and h is linear in $\bar{v}$ and w. Thus

$g(v,w) = \alpha + \beta v + \gamma \bar{w}$,

$h(v,w) = \varepsilon + \zeta \bar{v} + \eta w$ where the greek letters are complex constants.

It follows from (2) and (3) that $\beta = \eta$ and $\zeta = -\gamma$. Then

$$f(q) = \alpha + \beta v + \gamma \bar{w} + j(\varepsilon - \gamma \bar{v} + \beta w) = (\alpha + j\varepsilon) + (v + jw)(\beta - j\gamma)$$

since, for every complex number c, the equality $cj = j\bar{c}$ holds.

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The above Proposition shows that the requirement that a function of a quaternionic variable should have a quaternionic derivative is too strong to have interesting properties.

A second attempt could be to consider the class of functions which can be expanded as a quaternionic power series at any point $q \in \Omega$. That is to require that, for instance, in a neighbourhood of $0 \in \Omega$, f is a sum of terms of type $a_0 q a_1 q \ldots a_{n-1} q a_n$ with q the quaternionic variable and $a_0, \ldots, a_n \in H$ the coefficients. If we set $q = t + i x + jy + kz$ it turns out from (2.1.2 - 2.1.5) that t,x,y,z are polynomials in q. Therefore the space of $\mathbf{H}$ -analytic functions coincides with the space of real analytic functions from $\mathbf{R}^4$ to $\mathbf{R}^4$. In the following paragraph we will be concerned with the set of zeros of analytic functions with real coefficients .

§ 2.3. Quaternionic analytic functions with real coefficients

Let $\Omega \subset \mathbf{C}$ be a domain in the upper half plane $\Pi_+ = \{x+iy \in \mathbf{C} : y > 0\}$.Suppose that the map $\phi : \Omega \rightarrow \mathbf{C}$ defined by $\phi(x+iy) = \xi(x,y) + i\eta(x,y)$ is holomorphic .
Let us define the open domain $F(\Omega)$ of $\mathbf{H}$ as
$F(\Omega) = \{ x_0 + i x_1 + jx_2 + kx_3 : (x_0, \sqrt{x_1^2+x_2^2+x_3^2}) \in \Omega\}$.

The Fueter transform $F(\phi)$ of ϕ is the map from $F(\Omega)$ into $\mathbf{H}$ defined by

$$F(\phi)(x_0 + i x_1 + jx_2 + kx_3) = \xi(x_0,x) + \frac{ix_1+jx_2+kx_3}{x}\eta(x_0,x)$$

where $x = \sqrt{x_1^2+x_2^2+x_3^2}$.

LEMMA 2.3.1. *Let ϕ_1 and ϕ_2 be two holomorphic mappings from Ω into $\mathbf{C}$ and λ be a real number. Then:*

i) $F(\phi_1+\phi_2) = F(\phi_1)+F(\phi_2)$;

ii) $F(\lambda\phi_1) = \lambda F(\phi_1)$;

iii) $F(\phi_1\phi_2) = F(\phi_1)F(\phi_2)$;

iv) $F(z) = q$.

PROOF. i), ii), iv) are obvious.iii): let $\phi_1 = \xi_1+i\eta_1$ and $\phi_2= \xi_2+i\eta_2$.

Let $q = x_0+i x_1+jx_2+kx_3 \in F(\Omega)$, $x = \sqrt{x_1^2+x_2^2+x_3^2} > 0$, $\sigma(q) = \dfrac{ix_1+jx_2+kx_3}{x}$.

Then $F(\phi_1)(q) = \xi_1(x_0,x) + \sigma(q)\eta_1(x_0,x)$ and $F(\phi_2)(q) = \xi_2(x_0,x) + \sigma(q)\eta_2(x_0,x)$. $\sigma^2(q) = -1$. Therefore

$$F(\phi_1)F(\phi_2)(q) = (\xi_1\xi_2 - \eta_1\eta_2)(x_0,x) + \sigma(q)(\xi_1\eta_2 + \xi_2\eta_1)(x_0,x) .$$

On the other hand

$$\phi_1\phi_2(x,y) = (\xi_1\xi_2 - \eta_1\eta_2)(x,y) + i(\xi_1\eta_2 + \xi_2\eta_1)(x,y).$$

The proof follows from the definition of F.

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As a corollary we have:

PROPOSITION 2.3.2. *Let $\phi: \Omega \to C$ be a holomorphic mapping having a power series expansion with real coefficients around real centres, that is:*

$$\phi(z) = \sum_n a_n(z-c)^n \qquad \text{with } a_i,\ c \in \mathbf{R}.$$

Then

$$F(\phi)(q) = \sum_n a_n(q-c)^n$$

q being a quaternionic variable.The two series have the same radius of convergence.

In their paper ([D - N]) Datta and Nag are concerned with the set of zeros of these quaternionic analytic functions with real coefficients. We summarize here their results.

THEOREM 2.3.3 (Datta - Nag). *Let θ be a quaternionic analytic power series with real coefficients, convergent in $B(c, R) = \{ q \in \mathbf{H} : |q - c| < R\}$, $c \in \mathbf{R}$, namely*

$$\theta(q) = \sum_n a_n(q-c)^n$$

The zero set of this function ,$\{q \in \mathbf{H} : \theta(q) = 0 \}$, is
$F(\{z: \phi(z) = 0\} \cup \{ t \in \mathbf{R} : \phi(t) = 0 \}$where the function

$$\phi(z) = \sum_n a_n(z-c)^n$$

is defined in $\{ z \in C : |z - c| < R\}$.

PROOF. Let $\phi = \xi + i\eta$, $q = x_0 + i x_1 + jx_2 + kx_3$, $x = \sqrt{x_1^2 + x_2^2 + x_3^2} > 0$,

$$\sigma(q) = \frac{ix_1 + jx_2 + kx_3}{x} .$$

As $\theta(q) = F(\phi)(q) = \xi(x_0, x) + \sigma(q)\eta(x_0, x)$ it is clear that
$\theta(q) = 0 \Leftrightarrow$ q is a real zero of ϕ or

$$\xi(x_0, x) = \eta(x_0, x) \Leftrightarrow \phi(x_0, x) = 0 \Leftrightarrow q \in F(\{z \in C : f(z) = 0\}.$$

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Remark that for α and β real numbers, the set

$$F(\{\alpha + i\beta\}) = \{ \alpha + ix + jy + kz \in \mathbf{H} : x^2 + y^2 + z^2 = \beta^2 \}$$

is a 2-sphere.

For polynomials, we have, as a corollary

COROLLARY 2.3.4. *Let $\{\alpha_1 \pm i\beta_1, \dots, \alpha_m \pm i\beta_m, \gamma_1, \dots, \gamma_k\}$ (α_i, β_i, γ_i real numbers) be set of complex roots of the polynomial equation*

$$a_n q^n + \dots + a_1 q + a_0 = 0 , \ a_j \in \mathbf{R}. \qquad (2.3.1)$$

Then the quaternionic roots of (2.3.1) form the set

$$\bigcup_{j=1}^{m} S_{\alpha_j,\beta_j} \cup \{\gamma_1,\dots, \gamma_k\}$$

where $S_{\alpha_j,\beta_j} = F(\{\alpha_j + i\beta_j\})$

Now, we are able to solve $\theta(q) = a \in \mathbf{R}$. When we wish to solve $\theta(q) = A \in \mathbf{H}$ we use the following lemma:

LEMMA 2.3.5. *Let* $\sigma \in S^2$, $\sigma = i\,\sigma_1 + j\sigma_2 + k\sigma_3$, $(\sigma_1)^2 + (\sigma_2)^2 + (\sigma_3)^2 = 1$, $C_\sigma = \{x_0 + x\sigma \ : \ x_0, x \in R\}$ *and* $\phi\colon \Omega \subset \Pi_+ \to C$ *be a holomorphic mapping. Then*

$$F(\phi)(C_\sigma) \subset C_\sigma .$$

PROOF . Let $\phi = \xi + i\eta$, $q = x_0 + i\,x_1 + jx_2 + kx_3 \in C_\sigma$ i.e. $q = x_0 + x\sigma$ where

$$x = \sqrt{x_1^2 + x_2^2 + x_3^2} > 0 , \quad \sigma = \sigma(q) = \frac{ix_1 + jx_2 + kx_3}{x} .$$

Then $F(\phi)(q) = \xi(x_0, x) + \sigma\eta(x_0, x)$ hence $F(\phi)(q) \in C_\sigma$.

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THEOREM 2.3.6. *The root set of* $\theta(q) = a_0 + i\,a_1 + ja_2 + ka_3 \in H \setminus R$ *is the set*

$$S = \{\ \alpha + \beta \frac{ia_1 + ja_2 + ka_3}{a}) : \ \phi(\alpha + i\beta) = a_0 + ia\ , a = \sqrt{a_1^2 + a_2^2 + a_3^2} > 0\ \}.$$

PROOF. Let $\sigma = \dfrac{ia_1 + ja_2 + ka_3}{a}$. From the previous Lemma, $\theta(q) = a \in C_\sigma$ implies $q \in C_\sigma$.

Now let $q = x_0 + x\sigma \in C_\sigma$.

Then $\theta(q) = F(\phi)(q) = \xi(x_0, x) + \sigma\eta(x_0, x)$. Hence $\theta(q) = a_0 + \sigma a$ implies $\xi(x_0, x) = a_0$ and $\eta(x_0, x) = a$.

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Notice that the Fueter transform of an holomorphic mapping is not necessarily harmonic (for instance, $\Delta F(z) = \Delta q^2 \neq 0$). This explains why quaternionic analytic functions with real coefficients do not represent a satisfactory theory.

§ 2.4. Monogenic functions

Clifford Algebras.

The complex Clifford algebra over $\mathbf{C}^m$ $(m \geq 1)$ is defined as follows:

$$\mathcal{A} = \Big\{ \sum_{A \subset \{1,\dots,m\}} a_A e_A : a_A \in \mathbf{C} \Big\},$$

where $e_A = e_{\alpha_1} \dots e_{\alpha_h}$, $A = \{\alpha_1, \dots, \alpha_h\}$, $\alpha_1 < \dots < \alpha_h$.

Moreover,

$$\begin{array}{ll} e_0 = 1 \text{ , } e_{\{k\}} = e_k & k = 1,\dots,m \\ e_i e_j = - e_j e_i & j \neq k = 1,\dots,m, \\ e_j^2 = -1 & j = 1,\dots,m. \end{array}$$

For $m = 2$, $\mathcal{A}$ is called the algebra of biquaternions.

As $\mathcal{A}$ is isomorphic to $\mathbf{C}^{2m}$, we provide it with the $\mathbf{C}^{2m}$ norm

$$|a| = \Big(\sum_{A \subset \{1,\dots,m\}} |a_A|^2 \Big)^{\frac{1}{2}} \quad ;$$

the inequality

$$|ab| \leq 2^{\frac{m}{2}} |a| \, |b| \quad \text{holds.}$$

The elements

$(x_0, \mathbf{x}) = (x_0, x_1, \dots, x_m) \in \mathbf{R}^{m+1}$ and

$\mathbf{z} = (z_1, \dots, z_m) \in \mathbf{C}^m$

will be identified respectively with the Clifford numbers

$$x_0 + \mathbf{x} = x_0 + \sum_{j=1}^{m} x_j e_j \qquad \text{and}$$

$$\mathbf{z} = \sum_{j=1}^{m} z_j e_j$$

In this way $\mathbf{R}^{m+1}$ and $\mathbf{C}^m$ are imbedded in $\mathcal{A}$.

For ξ and $\mathbf{z}$ elements of $\mathbf{C}^m$, the bilinear form $\sum_{j=1}^{m} \xi_j z_j$ is denoted by $< \xi , \mathbf{z} >$.

Monogenic functions.

Let Ω be an open subset of $\mathbf{R}^{m+1}$ and f be a continuously differentiable function in Ω.

DEFINITION 2.4.1. *The function f is left (resp.right) monogenic in Ω if*

$$Df = \sum_{j=0}^{m} e_j \frac{\partial f}{\partial x_j} = 0 \qquad \Big(\text{resp. } \sum_{j=0}^{m} \frac{\partial f}{\partial x_j} e_j = 0 \Big) \text{ in } \Omega.$$

$D = \sum_{j=0}^{m} e_j \frac{\partial}{\partial x_j}$ stands for the generalized Cauchy-Riemann operator.

The right $\mathcal{A}$-module of left monogenic functions is denoted by $\mathcal{M}_l(\Omega, \mathcal{A})$. Several properties concerning these functions can be found in **[B-D-S]** (There exists a Cauchy representation theorem, any monogenic function is harmonic, ...).

There is a canonical way to associate a monogenic function with a holomorphic function.

THEOREM 2.4.2 (Sommen). *Let f be a holomorphic function admitting the Taylor series expansion for $|z| < \rho$:*

$$f(z) = \sum_k C_k z^k$$

Then, for z fixed in $\mathbf{C}^m$, the series

$$F(u, z) = \sum_k C_k(< \mathbf{u}, \mathbf{z} > - u_0 z)^k \quad , \quad u_0 + \mathbf{u} = u \in \mathbf{R}^{m+1}$$

is monogenic in u and normally convergent as a multiple Taylor series of $u_0, ..., u_m$ in the subdomain D of $\mathbf{R}^{m+1}$ determined by the inequality:

$$|u_0| \left(\sum_{j=1}^{m} |z_j|^2 \right)^{\frac{1}{2}} + \sum_{j=1}^{m} |u_j z_j| < \rho \quad (2.4.1)$$

Notice that when m =1, by the identification $e_1 = -i$ and $u = w = u_1 + iu_0$ we have $F(u, \mathbf{z}) = f(wz)$ $(w, z \in \mathbf{C})$

PROOF. As

$$(< \mathbf{u}, \mathbf{z} > - u_0 \mathbf{z})^k = \sum_{k_0 + ... + k_m = k} \frac{k!}{k_0! ... k_m!} (-u_0 \mathbf{z})^{k_0} \prod_{j=1}^{m} (u_j z_j)^{k_j}$$

the domain of absolute convergence of the Taylor series under consideration is determined by the condition

$$\sum_{k_0 + ... + k_m = k} |C_k| \frac{k!}{k_0! ... k_m!} (|u_0| \, |z|)^{k_0} \prod_{j=1}^{m} (|u_j z_j|)^{k_j} < \infty \qquad (2.4.2)$$

We have:

for $k_0 = 2s$, $s \in \mathbf{N}$: $$z^{k_0} = (-1)^s \left(\sum_{j=1}^{m} |z_j|^2\right)^s$$

for $k_0 = 2s + 1$, $s \in \mathbf{N}$: $$z^{k_0} = (-1)^s \left(\sum_{j=1}^{m} |z_j|^2\right)^s \mathbf{z}$$

Hence, for any $k_0 \in \mathbf{N}$

$$|z^{k_0}| \leq \left(\sum_{j=1}^{m} |z_j|^2\right)^{\frac{k_0}{2}}.$$

Then , for every k,

$$|C_k| \frac{k!}{k_0! \ldots k_m!} (|u_0|\,|z|)^{k_0} \prod_{j=1}^{m} (|u_j z_j|)^{k_j} \leq$$

$$\leq |C_k| \frac{k!}{k_0! \ldots k_m!} |u_0|^{k_0} \left(\sum_{j=1}^{m} |z_j|^2\right)^{\frac{k_0}{2}} \prod_{j=1}^{m} (|u_j z_j|)^{k_j}$$

$$= |C_k| \left(|u_0| (\sum_{j=1}^{m} |z_j|^2)^{\frac{1}{2}} + \sum_{j=1}^{m} |u_j z_j| \right)^k .$$

As ρ is the radius of convergence of f then (2.4.2) is satisfied as soon as (u, **z**) fulfills the inequality (2.4.1) and the series converges uniformly on every compact subset of D. Furthermore, an easy computation shows that, for every natural integer k, the function

$$C_k(<\mathbf{u}, \mathbf{z}> - u_0 \mathbf{z})^k$$

is left monogenic in u. Hence any finite sum of terms of the series F(u, **z**) is left monogenic in u (the uniform convergence theorem holds for monogenic functions, see **[B-D-S]**).

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In Sommen`s paper **[S]** the function

$$P(u, \mathbf{z}) = \sum_k (<\mathbf{u}, \mathbf{z}> - u_0 \mathbf{z})^k$$

plays the role of a kernel.

Let Ω be a domain of holomorphy (see § 1.2) and $\mathcal{H}_l(\Omega, \mathcal{A})$ be the left $\mathcal{A}$-module of $\mathcal{A}$-valued holomorphic functions in Ω endowed with the topology of uniform convergence on compact sets. Its dual module $\mathcal{H}_l'(\Omega, \mathcal{A})$ consists of all left $\mathcal{A}$-linear analytic functionals in Ω. Let $T \in \mathcal{H}_l'(\Omega, \mathcal{A})$. We set

$$\mathcal{P}(T)(u) = < T_z , P(u, z) >,$$

u belonging to a suitable open subset Ω^* of $\mathbf{R}^{m+1}$, depending on Ω, defined by

$$\Omega^* = \left\{ u \in \mathbf{R}^{m+1}: \ \forall z \in \Omega , \ |u_0|(\sum_{j=1}^{m} |z_j|^2)^{\frac{1}{2}} + \sum_{j=1}^{m} |u_j z_j| < 1 \right\}.$$

Notice that if $\mathcal{P}(T)$ is well-defined, then it is left monogenic in Ω^*.

Let

$$\Lambda(R_1,...,R_m) = \{z \in \mathbf{C}^m: |z_j| < R_j , j = 1,...m\};$$

$$\Pi(R_1,...,R_m) = \left\{ u \in \mathbf{R}^{m+1}: \ |u_0|\left(\sum_{j=1}^{m} |R_j|^2\right)^{\frac{1}{2}} + \sum_{j=1}^{m} |u_j R_j| < 1 \right\}$$

It follows from Theorem 2.4.2 that if T is an analytic functional in $\Lambda(R_1,...,R_m)$ then $\mathcal{P}(T)$ is a left monogenic function in $\Pi(R_1,...,R_m)$. Furthermore, we have:

$$\mathcal{P}(T)(u) = \sum_{k=0}^{\infty} < T_z, (< \mathbf{u}, \mathbf{z} > - u_0 z)^k >) =$$

$$= \sum_{k_0+...+k_m=k} \frac{k!}{k_0!...k_m!} u_0^{k_0}...u_m^{k_m} < T_z, (-z)^{k_0} z_1^{k_1}...z_m^{k_m} >.$$

Let $\bar{\Lambda}(R_1,...,R_m)$ the topological closure of $\Lambda(R_1,...,R_m)$.

THEOREM 2.4.3 (Sommen, [S]). *Let $T \in \mathcal{H}_l'(\bar{\Lambda}(R_1,...,R_m), \mathcal{A})$ .Then the multiple Taylor series of the left-monogenic function $\mathcal{P}(T)$ converges absolutely in $\Pi(R_1,...,R_m)$. Conversely, let f be a left monogenic function in a neighbourhood of the origin such that its multiple series converges absolutely in $\Pi(R_1,...,R_m)$. Then $f = \mathcal{P}(T)$ for some $T \in \mathcal{H}_l'(\bar{\Lambda}(R_1,...,R_m), \mathcal{A})$.*

It turns out that monogenic functions, in their construction, do not involve the "complex" or "quaternionic" structure of the domain on which they are defined. Thus, they do not constitute a satisfactory generalization of the idea of a complex holomorphic function.

§ 2.5. Fueter regular functions.

In the case of a complex function f: $\Omega \to C$ one has that the fact of being holomorphic can be expressed in one of the two equivalent ways:

i) $\bar{\partial} f = 0$ in Ω;

ii) $d(fdz) = 0$ in Ω.

Equation ii) leads (via the Stokes's Theorem) to the Cauchy representation Formula.

DEFINITION 2.5.1. *Let $\Omega \subset H$ and $f: \Omega \to H$ be an $\mathbf{R}$-differentiable function in Ω. f is left (resp. right)-regular in Ω if*

$$\bar{\partial}_l f = \frac{\partial f}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0 \quad \text{in } \Omega;$$

$$(\text{resp. } \bar{\partial}_r f = \frac{f\partial}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1}i + \frac{\partial f}{\partial x_2}j + \frac{\partial f}{\partial x_3}k = 0) \quad \text{in } \Omega.$$

f is left (resp.right)-anti regular if

$$\partial_l f = \frac{\partial f}{\partial q} = \frac{\partial f}{\partial x_0} - i\frac{\partial f}{\partial x_1} - j\frac{\partial f}{\partial x_2} - k\frac{\partial f}{\partial x_3} = 0 \quad \text{in } \Omega;$$

$$(\text{resp. } \partial_r f = \frac{f\partial}{\partial q} = \frac{\partial f}{\partial x_0} - \frac{\partial f}{\partial x_1}i - \frac{\partial f}{\partial x_2}j - \frac{\partial f}{\partial x_3}k = 0) \quad \text{in } \Omega.$$

Let us remark that any left (resp.right) regular function is harmonic. In fact:

$$\Delta = \frac{\partial}{\partial \bar{q}} \circ \frac{\partial}{\partial q} = \frac{\partial}{\partial q} \circ \frac{\partial}{\partial \bar{q}}.$$

Clearly, the theory of left-regular functions will be entirely equivalent to the theory of right-regular functions. For the sake of simplicity, we will only consider left-regular functions, which we will call simply *regular*.

The operator $\bar{\partial} = \bar{\partial}_l$ is called the Cauchy-Fueter operator and corresponds to the Dirac operator of $\mathbf{H} \cong \mathbf{C} + j\mathbf{C}$.

Let $f = f_0 + i f_1 + jf_2 + kf_3$.

The Cauchy-Fueter equation $\bar{\partial} f = 0$ is equivalent to the following scalar equations :

$$\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1} - \frac{\partial f_2}{\partial x_2} - \frac{\partial f_3}{\partial x_3} = 0 \qquad (2.5.1)$$

$$\frac{\partial f_0}{\partial x_1} + \frac{\partial f_1}{\partial x_0} + \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} = 0 \qquad (2.5.2)$$

$$\frac{\partial f_0}{\partial x_2} + \frac{\partial f_2}{\partial x_0} + \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} = 0 \qquad (2.5.3)$$

$$\frac{\partial f_0}{\partial x_3} + \frac{\partial f_3}{\partial x_0} + \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} = 0 \qquad (2.5.4)$$

A relation between regulars functions and holomorphy is illustrated by the following :

PROPOSITION 2.5.2. *Let $f : \Omega \subset \mathbf{H} \cong \mathbf{C} + j\mathbf{C} \to \mathbf{C}$ be an $\mathbf{R}$-differentiable function . Then*

f is a regular function if and only if f is holomorphic.

PROOF. Let $f = f_0 + i\, f_1$, $q = z_1 + jz_2 \in \Omega$ where

$z_1 = x_0 + ix_1$, $z_2 = x_2 - ix_3$. From equations (2.5.1-4), regularity of f is equivalent to:

$$\frac{\partial f_0}{\partial x_0} = \frac{\partial f_1}{\partial x_1} \qquad (2.5.5)$$

$$\frac{\partial f_0}{\partial x_1} = -\frac{\partial f_1}{\partial x_0} \qquad (2.5.6)$$

$$\frac{\partial f_0}{\partial x_2} = -\frac{\partial f_1}{\partial x_3} \qquad (2.5.7)$$

$$\frac{\partial f_0}{\partial x_3} = \frac{\partial f_1}{\partial x_2} \qquad (2.5.8)$$

(2.5.5) and (2.5.6) are equivalent to

$$\frac{\partial f}{\partial \bar{z}_1} = 0 \ ,$$

(2.5.7) and (2.5.8) are equivalent to

$$\frac{\partial f}{\partial \bar{z}_2} = 0 \, .$$

#

If f and g : $\Omega \to \mathbf{H}$ are two differentiable functions the following equalities hold:

$$\frac{\partial(f+g)}{\partial\bar{q}} = \frac{\partial f}{\partial\bar{q}} + \frac{\partial g}{\partial\bar{q}} \qquad (2.5.9)$$

$$\frac{\partial(fg)}{\partial\bar{q}} = \frac{\partial f}{\partial\bar{q}}g + \sum_{j=0}^{3} i_\lambda f\frac{\partial g}{\partial x_\lambda} \qquad (2.5.10)$$

Hence, if a is a quaternionic constant then

$$\frac{\partial(ga)}{\partial\bar{q}} = \frac{\partial g}{\partial\bar{q}}a \qquad (2.5.11)$$

and if f is a real valued function then

$$\frac{\partial(fg)}{\partial\bar{q}} = \frac{\partial f}{\partial\bar{q}}g + f\frac{\partial g}{\partial\bar{q}} \qquad (2.5.12)$$

Let $\mathcal{R}_l(\Omega)$ be the set of all left-regular functions in $\dot{\Omega}$. It is clear, from (2.5.9) and (2.5.11) that $\mathcal{R}_l(\Omega)$ is an H-right module.

We deduce, from (2.5.10), that $\mathcal{R}_l(\Omega)$ is not an algebra. We can proove it by taking, for instance,

$f(q) = x_0 i - x_1$, $g(q) = x_0 j - x_2$, where $q = x_0 + i\,x_1 + jx_2 + kx_3$:

f and g are (left) regular in H but

$$fg(q) = x_1x_2 - x_0x_2 i - x_0x_1 j + x_0^2 k$$

is not regular. In fact

$$\frac{\partial(fg)}{\partial\bar{q}} = 2x_0 k \neq 0 .$$

Moreover, the product of a regular function with itself is not necessarily regular:

$f(q) = ix_2 + jx_1$ is regular

but

$f^2(q) = -x_1^2 - x_2^2$ is not.

Non trivial affine functions are not regular. More precisely:

PROPOSITION 2.5.3. *The only **H**- linear regular function is the zero function.*

PROOF. From (2.5.11), a non zero right linear function qa is regular if and only if the identity is regular.But

$$\frac{\partial q}{\partial \bar{q}} = -2 \neq 0.$$

From (2.5.10) , for a ≠ 0 :

$$\frac{\partial (aq)}{\partial \bar{q}} = \sum_{j=0}^{3} i_\lambda a \frac{\partial q}{\partial x_\lambda} = a + iai + jaj + kak = -2\bar{a} \neq 0.$$

(the last equality follows from (2.1.1)).

#

Remark that in the case of complex functions, one chooses to call holomorphic the functions f which satisfy the equation $\bar{\partial} f = 0$, instead of those, having similar properties, which satisfy $\partial f = 0$.

The choice of one of the two operators corresponds to the choice of one point in the unit imaginary sphere of **C**. The situation in the case of **H** is similar, but there are many more possibilities: for every choice of a point in the imaginary sphere S^3 one can define a class of " regular functions " for the corresponding operator. Of course these classes are strictly related. In particular one of the choices correspond to the operator

$$\bar{\partial}_1 = \frac{\partial}{\partial x_0} + i\frac{\partial}{\partial x_1} - j\frac{\partial}{\partial x_2} + k\frac{\partial}{\partial x_3}$$

for which the class of " regular functions " contains the identity.

We end this section with a negative result concerning composition of regular functions.

PROPOSITION 2.5.4. *Let Ω be an open subset of H , and $f\colon \Omega \rightarrow H$ be a differentiable function in Ω. The following conditions are equivalent :*

i) $f(q) = aq + b$;

ii) for any regular function g defined in a neighbourhood of $f(A)$, gof is regular in Ω.

See [P] for a proof.

In spite of these negative results, Fueter regular functions appear as the natural generalization of holomorphic functions to the case of the field H of the quaternions, as we will explain in the next Chapter.

3.
FUETER REGULAR FUNCTIONS

§ 3.1. Some examples of regular functions

PROPOSITION 3.1.1. *Let Ω_1, Ω_2, Ω_3 be open sets in* **C** *and f, g, h be three holomorphic functions*

$$f : \Omega_1 \to \mathbf{C} \qquad f = f_0 + if_1 ,$$
$$g : \Omega_2 \to \mathbf{C} \qquad g = g_0 + ig_2 ,$$
$$h : \Omega_3 \to \mathbf{C} \qquad h = h_0 + ih_3 ,$$

where $f_0, f_1, g_0, g_2, h_0, h_3$ are real-valued functions. Let Ω be the open set defined by
$\Omega = \{q = x_0 + i\,x_1 + jx_2 + kx_3 \in \mathbf{H} : x_0 + i\,x_1 \in \Omega_1 , x_0 + ix_2 \in \Omega_2 , x_0 + ix_3 \in \Omega_3\}$.
Then the function

$F : \Omega \to \mathbf{H}$ *defined by*

$$F(x_0 + i\,x_1 + jx_2 + kx_3) =$$
$$= f_0(x_0, x_1) + g_0(x_0, x_2) + h_0(x_0, x_3) + if_1(x_0, x_1) + jg_2(x_0, x_2) + kh_3(x_0, x_3)$$

is regular in Ω.

PROOF. For any $q = x_0 + i\,x_1 + jx_2 + kx_3 \in \Omega$ we have :

$$F(q) = (f_0 + if_1)(x_0, ix_1) + (g_0 + jg_2)(x_0, x_2) + (h_0 + kh_3)(x_0 ,x_3).$$

Since f, g, h are holomorphic in their domains the following relations hold:

$$\frac{\partial f_0}{\partial x_0} = \frac{\partial f_1}{\partial x_1} \ ; \ \frac{\partial f_1}{\partial x_0} = - \frac{\partial f_0}{\partial x_1} \qquad (3.1.1)$$

$$\frac{\partial g_0}{\partial x_0} = \frac{\partial g_2}{\partial x_2} \ ; \ \frac{\partial g_2}{\partial x_0} = - \frac{\partial g_2}{\partial x_0} \qquad (3.1.2)$$

$$\frac{\partial h_0}{\partial x_0} = \frac{\partial h_3}{\partial x_3} \ ; \ \frac{\partial h_3}{\partial x_0} = - \frac{\partial h_0}{\partial x_3} \qquad (3.1.3)$$

We have

$$\frac{\partial F}{\partial x_0} = \frac{\partial f_0}{\partial x_0} + i\frac{\partial f_1}{\partial x_0} + \frac{\partial g_0}{\partial x_0} + j\frac{\partial g_2}{\partial x_0} + \frac{\partial h_0}{\partial x_0} + k\frac{\partial h_3}{\partial x_0} \ ;$$

$$\frac{\partial F}{\partial x_1} = \frac{\partial f_0}{\partial x_1} + i\frac{\partial f_1}{\partial x_1} \ ;$$

$$\frac{\partial F}{\partial x_2} = \frac{\partial g_0}{\partial x_2} + j\frac{\partial g_2}{\partial x_2} \ ;$$
$$\frac{\partial F}{\partial x_3} = \frac{\partial h_0}{\partial x_3} + k\frac{\partial h_3}{\partial x_3} \ .$$

Hence

$$\frac{\partial F}{\partial \bar{q}} = \frac{\partial F}{\partial x_0} + i\frac{\partial F}{\partial x_1} + j\frac{\partial F}{\partial x_2} + k\frac{\partial F}{\partial x_3}$$
$$= \left(\frac{\partial f_0}{\partial x_0} - \frac{\partial f_1}{\partial x_1}\right) + i\left(\frac{\partial f_1}{\partial x_0} + \frac{\partial f_0}{\partial x_1}\right) + \left(\frac{\partial g_0}{\partial x_0} - \frac{\partial g_2}{\partial x_2}\right) + j\left(\frac{\partial g_0}{\partial x_2} + \frac{\partial g_2}{\partial x_0}\right) + \left(\frac{\partial h_0}{\partial x_0} - \frac{\partial h_3}{\partial x_3}\right) +$$
$$+ k\left(\frac{\partial h_3}{\partial x_0} + \frac{\partial h_0}{\partial x_3}\right)$$

The result follows from (3.1.1) - (3.1.2) - (3.1.3) .

#

Roughly speaking, proposition 3.1.1 asserts that a " disjoint sum" of holomorphic functions is a regular function.

It has been showed, in §2.3, how to construct analytic functions with real coefficients from holomorphic mappings.

We describe in what follows how to obtain regular functions by a slightly different construction.

The Fueter transform of a holomorphing mapping (§2.3) is not necessarily regular. For instance $F(z) = q$ (see Lemma 2.3.1).

Let $\phi = \xi + i\eta$ be a holomorphic function defined in an open subset Ω of the upper half plane Π_+.

For $q = x_0 + i\,x_1 + jx_2 + kx_3$, let $x = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $\sigma(q) = \frac{ix_1+jx_2+kx_3}{x}$ as usual.

Then

$$\Delta F(\phi) = 2\left(-\frac{\partial}{\partial x_0}\left(\frac{\eta}{x}\right) + \frac{\sigma}{x}\frac{\partial}{\partial x}\left(\frac{\eta}{x}\right)\right)$$

PROPOSITION 3.1.2. *$\Delta F(\phi)$ is a regular function from $F(\Omega)$ into H.*

PROOF.([D]) We have :

$$\frac{\partial F(\phi)}{\partial \bar{q}} = \frac{\partial \xi}{\partial x_0} + \frac{\sigma}{x}\frac{\partial \eta}{\partial x_0} + \frac{\sigma}{x}\frac{\partial \xi}{\partial x} + \left(\frac{\partial \eta}{\partial x}\right)\sigma^2 - \frac{3\eta}{x} - \frac{\eta}{x}\sigma^2$$

Since $\sigma^2 = -1$ (Proposition 2.1.2) and the Cauchy - Riemann conditions

$$\frac{\partial\xi}{\partial x_0} = \frac{\partial\eta}{\partial x}$$

$$\frac{\partial\xi}{\partial x} = -\frac{\partial\eta}{\partial x_0}$$

are satisfied , we have

$$\frac{\partial F(\phi)}{\partial\bar{q}} = -2\frac{\eta}{x}$$

Furthermore,

$$\Delta\left(\frac{\eta}{x}\right) = \frac{1}{x}\left(\frac{\partial^2\eta}{\partial x_0^2} + \frac{\partial^2\eta}{\partial x^2}\right) = 0$$

because η is harmonic.

The Laplace operator Δ is a real operator, hence

$$\Delta \circ \frac{\partial}{\partial\bar{q}} = \frac{\partial}{\partial\bar{q}} \circ \Delta.$$

As a consequence, Δf is regular.

#

Remark that the " transform" of the Cauchy kernel $\frac{1}{z}$ is the function given by

$$\Delta F\left(\frac{1}{z}\right)(q) = -4\frac{q^{-1}}{|q|^2}$$

The function

$$G(q) = \frac{q^{-1}}{|q|^2} .$$

is called the Cauchy-Fueter kernel and plays in this theory a role similar to that played by the Cauchy kernel in complex analysis.

PROPOSITION 3.1.3. *Let $v(q) = (aq + b)(cq + d)^{-1}$ with $a^{-1}b \neq c^{-1}d$ be a conformal transformation of the one point compactification of $\mathbf{H}$ and $f : \mathbf{H} \to \mathbf{H}$ be a regular function. Then the function F defined by*

$$F(q) = \frac{(cq+d)^{-1}}{|cq+d|^2} f(v(q)) \quad \textit{is regular.}$$

(see **[Su]** for a proof.)

An application of the Proposition for $v(q) = q^{-1}$ gives the

COROLLARY 3.1.4. *Let* f: **H** $\to$ **H** *be a regular function. Then the function defined in $\mathbf{H} \setminus \{0\}$ by*

$$G(q)f(q^{-1})$$

is regular.

§ 3.2. Quaternion valued forms

A quaternion-valued p-form ϕ in $\Omega \subset \mathbf{H}$ is defined by:

$$\phi = \phi_0 + i\,\phi_1 + j\phi_2 + k\phi_3$$

where the ϕ_i are real-valued p-forms with $\mathcal{C}^\infty$coefficients.
A p-form ϕ can be expressed as

$$\phi = \sum_{0 \leq i_1 < \ldots < i_p \leq 3} a_{i_o \ldots i_p} dx_{i_1} \wedge \ldots \wedge dx_{i_p}$$

where the $a_{i_o \ldots i_p}$ are $\mathcal{C}^\infty$ functions.
Let $\mathcal{E}^p_{\mathbf{H}}(\Omega)$ be the $\mathbf{H}$ left module of the quaternion-valued p-forms.

We define an exterior product Λ in a natural way

$$\Lambda : \mathcal{E}^p_{\mathbf{H}}(\Omega) \times \mathcal{E}^q_{\mathbf{H}}(\Omega) \rightarrow \mathcal{E}^{p+q}_{\mathbf{H}}$$

and a differential :

$$d : \mathcal{E}^p_{\mathbf{H}}(\Omega) \rightarrow \mathcal{E}^{p+1}_{\mathbf{H}}(\Omega)$$

such that
$d^2 = 0$
$d(\omega^p \wedge \omega^q) = d\omega^p \wedge \omega^q + (-1)^p \omega^p \wedge d\omega^q$ for $\omega^p \in \mathcal{E}^p_{\mathbf{H}}(\Omega)$ and $\omega^q \in \mathcal{E}^p_{\mathbf{H}}(\Omega)$.

A quaternion-valued p-form can be regarded as a mapping from $\mathbf{H}$ to the space of alternating $\mathbf{R}$-multilinear maps from $\mathbf{H} \times \ldots \times \mathbf{H}$ (p times) to $\mathbf{H}$.
The exterior product of two differential forms acts as follows:

$$(\omega^p \wedge \omega^q)(h_1,\ldots, h_{p+q}) = \frac{1}{p!q!}\sum_\rho \varepsilon(\rho)\omega^p(h_{\rho(1)}\ldots h_{\rho(p)})\omega^q(h_{\rho(p+1)}\ldots h_{\rho(p+q)})$$

for all $h_1,\ldots, h_{p+q}$ in $\mathbf{H}$, where the sum is taken over all permutations ρ of p+q objects and $\varepsilon(\rho)$ is the sign of ρ.
The differential of the identity function is

$$dq = dx_0 + i\,dx_1 + jdx_2 + kdx_3$$

the canonical real 4-form is

$$\theta = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$$

so that

$$\theta(1, i, j, k) = 1.$$

Define Dq as the 3-form which satisfies

$$< h_1, Dq(h_2, h_3, h_4) > = \theta(h_1,h_2, h_3, h_4)$$

for all $h_1,h_2, h_3, h_4 \in \mathbf{H}$, where < , > stands for the usual scalar product of $\mathbf{R}^4$. Geometrically, Dq(a,b,c) is a quaternion which is perpendicular to a, b, c and has magnitude equal to the volume of the 3-dimensional parallepiped whose edges are a,b,c. namely, we have :

$$Dq = dx_1\wedge dx_2\wedge dx_3 - i dx_0\wedge dx_2\wedge dx_3 + j dx_0\wedge dx_1\wedge dx_3 - k dx_0\wedge dx_1\wedge dx_2 .$$

If f and g are two differentiable functions from an open subset of **H** into **H** then

$$df = \frac{\partial f}{\partial x_0}dx_0 + \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3 \qquad (3.2.1)$$

$$Dq\wedge df = -\frac{\partial f}{\partial \bar{q}}\theta \qquad (3.2.2)$$

$$df\wedge Dq = \frac{f\partial}{\partial \bar{q}}\theta \qquad (3.2.3)$$

Hence

$$d(gDqf) = \left(\frac{g\partial}{\partial \bar{q}} f + g\frac{\partial f}{\partial \bar{q}}\right)\theta \qquad (3.2.4)$$

and (for g = 1)

$$d(Dqf) = \frac{\partial f}{\partial \bar{q}}\theta \qquad (3.2.5)$$

Furthermore,

$$d\bar{q}\wedge Dq = -Dq\wedge d\bar{q} = -4\theta . \qquad (3.2.6)$$

PROPOSITION 3.2.1. *Let a be a quaternion. For any* $h_1 , h_2 , h_3 \in H$ *:*

$$Dq(ah_1, ah_2, ah_3) = a\, |a|^2\, Dq(h_1,h_2,h_3).$$

PROOF([**Su**]). For any unit quaternion u, the map $q \rightarrow uq$ is an orthogonal transformation of **H** with determinant 1. Hence

$$Dq(uh_1, uh_2, uh_3) = uDq(h_1,h_2,h_3).$$

Take $u = a|a|^{-1}$. Using the **R**-trilinearity of Dq, we obtain

$$\begin{aligned} Dq(ah_1, ah_2, ah_3) &= Dq(|a|uh_1, |a|uh_2, |a|uh_3) \\ &= |a|^3 Dq(uh_1, uh_2, uh_3) \\ &= |a|^3 u Dq(h_1,h_2,h_3) \\ &= a\,|a|^2\, Dq(h_1,h_2,h_3). \end{aligned}$$

#

We conclude with a Proposition, whose proof is the same as in the complex case.
PROPOSITION 3.2.2([Su]). *If f and g are continuous functions in H and $C:[0,1]^3 \to H$ is rectifiable, then*

$$\left| \int_C fDqg \right| \leq \left(\max_C |f|\right)\left(\max_C |g|\right) \int_C |Dq| .$$

§ 3.3. The Cauchy-Fueter integral formula

The following Theorem extend Morera's theorem to regular functions. Its proof can be found in [Su] .
THEOREM 3.3.1. *Let $f : \Omega \to H$ be a continuous function such that*

$$\int_{\partial U} Dqf = 0.$$

for every relatively compact domain U in Ω having a differentiable boundary. Then f is a regular function.
THEOREM 3.3.2. *Let f and g be two continuously differentiable functions in a domain Ω and $U \subset \Omega$ be a relatively compact domain having a differentiable boundary. If f is left regular in Ω and g is right regular in Ω then*

$$\int_{\partial U} gDqf = 0.$$

PROOF. From Stokes's Theorem, we have:

$$\int_{\partial U} gDqf = \int_U d(gDqf)$$

Then, applying (3.2.4)

$$\int_U d(gDqf) = \int_U \left(\frac{g\partial}{\partial \bar{q}} f + g \frac{\partial f}{\partial \bar{q}}\right)\theta = 0$$

because

$$\frac{\partial f}{\partial \bar{q}} = \frac{g\partial}{\partial \bar{q}} = 0.$$

#

As a consequence of the last two Theorems, a function $f : \Omega \rightarrow \mathbf{H}$ is regular if and only if

$$\int_{\partial U} Dqf = 0$$

for every relatively compact domain U in Ω having a differentiable boundary.

The Cauchy-Fueter kernel G has already been introduced in §3.1. It is defined by:

$$G(q) = \frac{q^{-1}}{|q|^2} = \frac{\bar{q}}{|q|^4} = \frac{\partial}{\partial q}\left(-\frac{1}{2|q|^2}\right)$$

Since $-\frac{1}{2|q|^2}$ is harmonic in $\mathbf{H}\backslash\{0\}$ and $\Delta = \bar{\partial}_l \circ \partial_l = \bar{\partial}_r \circ \partial_r$ then G is left and right regular in $\mathbf{H}\backslash\{0\}$.

We can now prove the Cauchy-Fueter integral formula which is one of the reasons of interest in Fueter regularity.

THEOREM 3.3.3(Cauchy-Fueter integral formula).
Let f be a continuously differentiable function from an open subset Ω of $\mathbf{H}$ into $\mathbf{H}$. Let $D \subset\subset \Omega$ be a relatively compact subset of Ω, having a differentiable boundary.
For any q_0 in D we have

$$f(q_0) = \frac{1}{2\pi^2}\int_{\partial D} G(q-q_0)Dqf(q).$$

PROOF. Let $\varepsilon > 0$ be such that

$$\bar{B}(q_0,\varepsilon) = \{q \in \mathbf{H} : |q - q_0| \leq \varepsilon\} \subset D.$$

Let $U = D \setminus \bar{B}(q_0,\varepsilon)$. Theorem 3.3.2 implies that

$$\int_{\partial D} G(q-q_0)Dqf(q) = \int_{|q-q_0|=\varepsilon} G(q-q_0)Dqf(q) = \frac{1}{\varepsilon^4}\int_{|q-q_0|=\varepsilon} (\bar{q}-\bar{q}_0)Dqf(q).$$

Stokes's Theorem , (3.2.2) and (3.2.6) , imply

$$\frac{1}{\varepsilon^4}\int_{|q-q_0|=\varepsilon} (\bar{q}-\bar{q}_0)Dqf(q) = \frac{1}{\varepsilon^4}\int_{|q-q_0|<\varepsilon} d\bar{q} \wedge Dqf(q) = \frac{4}{\varepsilon^4}\int_{|q-q_0|<\varepsilon} \theta f(q).$$

Since $\mathrm{Vol}(B(q_0,\varepsilon)) = \frac{\pi^2\varepsilon^4}{2}$ we have $\lim_{\varepsilon\to 0} \frac{4}{\varepsilon^4}\int_{|q-q_0|<\varepsilon} \theta f(q) = 2\pi^2 f(q_0)$.

#

It follows that any $\mathcal{C}^1$ regular function is a $\mathcal{C}^\infty$ function . As in complex analysis, however, the condition on f can be weakened by using Goursat's dissection argument ([P]).

§ 3.4. Regular power series

Analogously to what happens for holomorphic functions, regular functions can be expanded in series with respect to homogeneous regular polynomials.
Let ν be an unordered set of n integers $\{\lambda_1, ..., \lambda_n\}$ where $1 \le \lambda_i \le 3$; ν can also be specified by three integers n_1, n_2, n_3 ($n_1+n_2+n_3 = n$) where n_i is the number of i's in ν. We'll write $\nu = [n_1, n_2, n_3]$. There are $\frac{1}{2}(n+1)(n+2)$ of such ν's. We denote the set of all of them by σ_n. They are to be used as labels; when n=0, so that $\sigma_n=\varnothing$ we use the suffix 0 instead of $\varnothing$.
We write for the n-th order differential operator

$$\partial_\nu = \frac{\partial^\nu}{\partial x_1^{n1} \partial x_2^{n2} \partial x_3^{n3}}$$

If G is the Cauchy-Fueter kernel, let us set

$$G_\nu(q) = \partial_\nu G(q) ;$$

moreover, let us define

$$P_\nu(q) = \frac{1}{n!} \sum (x_0 i_{\lambda_1} - x_{\lambda_1}) ... (x_0 i_{\lambda_n} - x_{\lambda_n}) \quad \text{where } q = x_0 + i\, x_1 + j x_2 + k x_3,$$

where the sum is taken over all $\frac{n!}{n_1!\ n_2!\ n_3!}$ different orderings of n_1 1's, n_2 2's, n_3 3's.
The polynomial P_ν is homogeneous of degree n and G_ν is homogeneous of degree -n-3. Let U_n be the right quaternionic module of homogeneous (left) regular functions of degree n.
PROPOSITION 3.4.1. *The polynomials P_ν ($\nu \in \sigma_n$) are regular and form a basis for U_n. If $f \in U_n$ then*

$$\forall q \in H \ : \ f(q) = \sum_{\nu \in \sigma_n} (-1)^n P_\nu(q) \partial_\nu f(q).$$

We need two Lemmas.
LEMMA 3.4.2. *Let f be harmonic and homogeneous of degree n over R. Then f is a polynomial.*
LEMMA 3.4.3. dim $U_n = \frac{1}{2}(n+1)(n+2)$.
A proof of these Lemmas can be found in [Su].

PROOF of Theorem 3.4.1([Su]).Let f be a regular homogeneous polynomial of degree n. Since f is regular

$$\frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0$$

and since it is homogeneous,

$$x_0\frac{\partial f}{\partial x_0} + x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + x_3\frac{\partial f}{\partial x_3} = nf.$$

Hence

$$nf(q) = \sum_{\lambda=1}^{3} (x_\lambda - x_0 i_\lambda)\frac{\partial f}{\partial x_\lambda}.$$

$\frac{\partial f}{\partial x_\lambda}$ is also regular and homogeneous (of degree n-1), so we can repeat the argument; after n steps we obtain

$$f(q) = \frac{1}{n!}\sum_{\lambda_1,\dots,\lambda_n} (-x_0 i_{\lambda_1}+x_{\lambda_1})\dots(-x_0 i_{\lambda_n}+x_{\lambda_n})\frac{\partial^n f}{\partial x_{\lambda_1}\dots\partial x_{\lambda_n}}(q)$$

$$= \sum_{\nu\in\sigma_n} (-1)^n P_\nu(q)\partial_\nu f(q).$$

Since f is a polynomial, $\partial_\nu f$ is a constant, thus any regular homogeneous polynomial is a linear combination of the P_ν's.

Since, from Lemma 3.4.1, the elements of U_n are polynomials, the P_ν's span U_n and hence they form a basis of U_n . In fact, by Lemma 3.4.2, there are ($\dim U_n$) of such polynomials.

#

The "mirror image" of this argument proves that the P_ν's are also right regular and that if f is a homogeneous right regular function then

$$f(q) = \sum_{\nu\in\sigma_n} (-1)^n \partial_\nu f(q)P_\nu(q).$$

In the case of one complex variable, we have

$$(1-q)^{-1} = \sum_{n=0}^{\infty} q^n \qquad \text{for } |q| < 1 ,$$

and the series converges absolutely and uniformly in any ball $|q| \leq r < 1$.

This gives rise to an expansion of G(p-q) in powers of $p^{-1}q$; identifying it with the Taylor series of G about p, we obtain the

PROPOSITION 3.4.4. *The expansions*

$$G(p-q) = \sum_{n=0}^{\infty} \sum_{v \in \sigma_n} G_v(p)P_v(q) = \sum_{n=0}^{\infty} \sum_{v \in \sigma_n} P_v(q)G_v(p)$$

are valid for $|q| < |p|$. The series converges uniformly in any region $\{(p,q) : |q| \leq r|p|\}$ of H^2 with $r < 1$.

PROOF.

$$G(p-q) = \frac{(p-q)^{-1}}{|p-q|^2}$$

$$= \frac{(1-p^{-1}q)^{-1}p^{-1}|(1-p^{-1}q)^{-2}|}{|p|^2}$$

$$= \sum_{n=0}^{\infty} f^n(q)$$

where

$$f^n(q) = \sum_{v=[n_1,n_2,n_3]\in\sigma_n} \frac{(p^{-1}q)^{n_1}\, p^{-1}\, (p^{-1}q)^{n_2}\, (\overline{p^{-1}q})^{n_3}}{|p|^2}$$

The series is uniformly convergent for $|p^{-1}q| \leq r < 1$, and f^n are homogeneous polynomials of degree n. Furthermore,

$$\frac{\partial f^n}{\partial \overline{q}} \text{ and } \frac{f^n \partial}{\partial \overline{q}}$$

are homogeneous of degree n-1 and hence they are zero for $|q| < |p|$ since G(p-q) is left and right regular in q. Proposition 3.4.1 implies that

$$f^n(q) = \sum_{v\in\sigma_n} (-1)^n \partial_v f^n(q) P_v(q)$$

$$= \sum_{v\in\sigma_n} (-1)^n P_v(q) \partial_v f^n(q)$$

Clearly, $(-1)^n \partial_v f^n(q) = \partial_v G(p)$ so the proof is complete.

#

Now, classic arguments of complex analysis adapted to this case and Proposition 3.2.2 give:

THEOREM 3.4.5. *Suppose f is regular in a neighbourhood of 0. Then there is a ball B with centre 0 in which f(q) is represented by a uniformly convergent series*

$$f(q) = \sum_{n=0}^{\infty} \sum_{v\in\sigma_n} P_v(q) a_v$$

where the coefficients a_v are given by:

$$a_v = \frac{1}{2\pi^2} \int_{\partial B} G_v(q) Dq f(q) = (-1)^n \partial_v f(0) \quad \text{if } v \in \sigma_n.$$

§ 3.5. Regular series in Reinhardt domains

DEFINITION 3.5.1. A domain $\Omega \subset H^n$ is called a left-Reinhardt domain if

$$\forall (q_1, ..., q_n) \in \Omega, \forall (t_1, ..., t_n) \in \mathbf{H}^n, \ |t_i| = 1 \ (i = 1, ..., n) \Rightarrow (q_1 t_1, ..., q_n t_n) \in \Omega$$

.

REMARK. In dimension 1, the only Reinhardt domains are the open balls. We conjecture that the following result, proved here in dimension one, holds in several quaternionic variables.

THEOREM 3.5.2. *Let Ω be a left-Reinhardt domain containing the origin and let $f \in \mathcal{R}_l(\Omega)$. Then the series of f at the origin*

$$f(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} P_\nu(q) a_\nu \quad ; \qquad a_\nu = (-1)^n \partial_\nu f(0) \ \textit{if} \ \nu \in \sigma_n .$$

is normally convergent in Ω.

PROOF. Write $\Omega = \bigcup_{\varepsilon > 0} \Omega_\varepsilon$ where

$$\Omega_\varepsilon = \{ q \in \Omega : \mathrm{dist}(q, \partial\Omega) > \varepsilon |q| \}.$$

Notice that

(1) $\Omega_\varepsilon \subset \Omega_\eta$ if $\eta < \varepsilon$,

(2) $0 \in \Omega_\varepsilon$,

(3) Ω_ε is open for any $\varepsilon > 0$.

Let Ω'_ε be the connected component of Ω_ε that contains the origin. Since Ω is connected, we have

(4) $\Omega = \bigcup_{\varepsilon > 0} \Omega'_\varepsilon$.

Remark that

(5) $q \in \Omega'_\varepsilon \Rightarrow q(1+\varepsilon) \in \Omega$,

(6) $q \in \Omega'_\varepsilon \Rightarrow (\forall t \in \mathbf{H}, \ |t| = 1+\varepsilon \Rightarrow qt \in \Omega)$.

For any $\varepsilon > 0$ let $T_\varepsilon = \{ t \in \mathbf{H} : |t| < 1+\varepsilon \}$. Remark that the function $q \to f(qt)$ is well defined for $t \in \partial T_\varepsilon$ and $q \in \Omega'_\varepsilon$. Hence the Cauchy integral

$f_\varepsilon : \Omega'_\varepsilon \to \mathbf{H}$ given by

$$f_\varepsilon(q) = \frac{1}{2\pi^2} \int_{\partial T_\varepsilon} G(t-1) Dq f(qt)$$

is defined.

i) *f_ε agrees with f (restricted to Ω'_ε) in a neighbourhood of the origin.*

We have

$$f_\varepsilon(0) = \frac{1}{2\pi^2}\int_{\partial T_\varepsilon} G(t-1)Dqf(0) = f(0)$$

since from the Cauchy-Fueter representation formula (Theorem 3.3.2)

$$\frac{1}{2\pi^2}\int_{\partial T_\varepsilon} G(t-1)Dq = 1.$$

Let δ be such that

$$B(0, \delta) \subset\subset \Omega'_\varepsilon$$

and let

$$0 \neq q \in \Omega'_\varepsilon$$

be such that

$$|q| < \frac{\delta}{1+\varepsilon}.$$

Let $Q = qt$ in the integral representation formula of f_ε. It follows from Proposition 3.2.1 that

$$f_\varepsilon(q) = \frac{1}{2\pi^2}\int_{|Q|=(1+\varepsilon)|q|} G(q^{-1}Q-1)q^{-1}|q^{-1}|^2\, DQf(Q)$$

Since

$$G(q^{-1}Q-1)q^{-1}|q^{-1}|^2 = G(Q-q)$$

then

$$f_\varepsilon(q) = \frac{1}{2\pi^2}\int_{|Q|=(1+\varepsilon)|q|} G(Q-q)DQf(Q).$$

Since, for the choice of δ, we have

$$(1+\varepsilon)|q| < \delta$$

we can apply Theorem 3.3.2 for $U = \{q \in \Omega : (1+\varepsilon)|q| < |Q| < \delta\}$. Hence

$$f_\varepsilon(q) = \frac{1}{2\pi^2}\int_{|Q|=\delta} G(Q-q)DQf(Q)$$

It follows from the Cauchy-Fueter integral formula (Theorem 3.3.3) that

$$f_\varepsilon(q) = f(q) \quad \text{for } |q| < \delta.$$

ii) *f_ε agrees with f (restricted to Ω'_ε) in a neighbourhood of any point $q_0 \neq 0$.*

For $q_0 \neq 0$ let δ be such that

$B(q_0,\delta) \subset\subset \Omega'_\varepsilon$;

$0 \notin B(q_0,\delta)$;

$\delta < \frac{\varepsilon}{2+\varepsilon} |q_0|$.

Let $q \in B(q_0,\delta)$.

By the same argument used in i) we have

$$f_\varepsilon(q) = \frac{1}{2\pi^2} \int_{|Q|=(1+\varepsilon)|q|} G(Q-q)DQf(Q)$$

Remark that

$B(q_0,\delta) \subsetneqq B(0, (1+\varepsilon)|q|)$.

In fact, $w \in B(q_0,\delta)$ implies $|q_0| - \delta < |w| < |q_0| + \delta$. Since $\delta < \frac{\varepsilon}{2+\varepsilon}|q_0|$ then

$$|w| < \frac{2+2\varepsilon}{2+\varepsilon}|q_0| = (1+\varepsilon)\left(|q_0| - \frac{\varepsilon}{2+\varepsilon}|q_0|\right) < (1+\varepsilon)(|q_0| - \delta) < (1+\varepsilon)|q|.$$

Again, Theorem 3.3.2 gives, for $U = \{w \in \Omega'_\varepsilon : |w| < (1+\varepsilon)|q| \text{ and } |w - q_0| > \delta \}$,

$$f_\varepsilon(q) = \frac{1}{2\pi^2} \int_{|Q-q_0|=\delta} G(Q-q)DQf(Q) \quad .$$

Hence, from the integral representation formula (Theorem 3.3.3), $f_\varepsilon(q) = f(q)$ for $q \in B(q_0,\delta)$.

iii) *f_ε ($= f$) has a normally convergent regular series expansion in Ω'_ε* .

We have

$$G(t-1) = \frac{(t-1)^{-1}}{|t-1|^2}$$

$$= \sum_{n_1,n_2,n_3} (\overline{t^{-1}})^{n_1}(t^{-1})^{n_2+n_3+1} |t|^{-2}$$

with normal convergence when $t \in \partial T_\varepsilon$.

Since qt belongs to a compact set in Ω if q belongs to a compact set in Ω'_ε and $t \in \partial T_\varepsilon$ we have, using the same techniques as in complex analysis ([V]) and Proposition 3.3.2,

$$f(q) = \sum_{n=0}^{\infty} g_n(q)$$

where

$$g_n(q) = \frac{1}{2\pi^2} \int_{\partial T_\varepsilon} \sum_{\nu=[n_1,n_2,n_3]\in\sigma_n} (\overline{t^{-1}})^{n_1}(t^{-1})^{n_2+n_3+1} |t|^{-2} Dqf(qt) \text{ if } \nu = [n_1, n_2, n_3].$$

Let $Q = qt$ ($q \neq 0$). We have

$$g_n(q) = \frac{1}{2\pi^2} \int_{Q=(1+\varepsilon)|q|} \sum_{\nu=[n_1,n_2,n_3]\in\sigma_n} (\overline{Q^{-1}q})^{n_1}(Q^{-1}q)^{n_2+n_3} Q^{-1}|Q^{-1}|^2 DQf(Q)$$

Remark that , for Q fixed ,

$$\phi_Q^n(q) = \sum_{\nu=[n_1,n_2,n_3]\in\sigma_n} (\overline{Q^{-1}q})^{n_1}(Q^{-1}q)^{n_2+n_3} Q^{-1} |Q^{-1}|^2$$

is the homogeneous polynomial of degree n in q in the expansion

$$G(Q - q) = \sum_{n=0}^{\infty} \sum_{\nu\in\sigma_n} P_\nu(q)G_\nu(Q).$$

Hence , for any n,

$$g_n(q) = \frac{1}{2\pi^2} \int_{|Q|=(1+\varepsilon)|q|} \phi_Q^n(q) DQf(Q)$$

is regular in Ω'_ε (just apply the definition of regular function).

iv) *the series expansions of f and f_ε agree in a neighbourhood of the origin.*

Using a similar argument concerning series expansions of homogeneous functions we see that, for a fixed q in Ω'_ε , $\phi_Q^n(q)$ is a right-regular function in Q. Then by theorem 3.3.2 we have , for a small δ and $|q| < \delta$:

$$g_n(q) = \frac{1}{2\pi^2} \int_{|Q|=\delta} \phi_Q^n(q) DQf(Q)$$

$$= \frac{1}{2\pi^2} \int_{|Q|=\delta} \sum_{\nu\in\sigma_n} P_\nu(q)G_\nu(Q)DQf(Q)$$

$$= \sum_{v \in \sigma_n} P_v(q) \frac{1}{2\pi^2} \int_{|Q|=\delta} G_v(Q) DQ f(Q)$$

$$= \sum_{v \in \sigma_n} P_v(q)(-1)^n \partial_v f(0) \quad \text{(Theorem 3.4.5).}$$

Since, from step iii) all g_n are regular in the domain Ω'_ε, the last equality holds in Ω'_ε by the identity principle.

#

§ 3.6. Regions of regularity

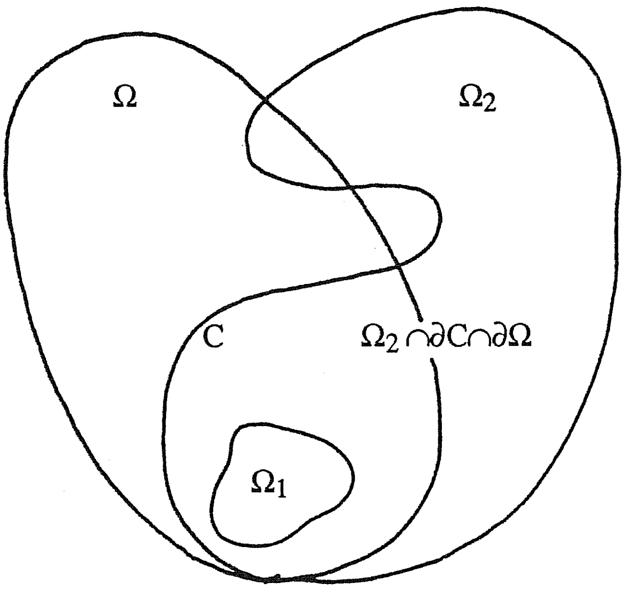
DEFINITION 3.6.1. *An open subset Ω of H is called a region of regularity if the following property holds:*
There do not exist two nonempty open sets Ω_1 and Ω_2 such that:

i) Ω_2 is connected,

ii) $\Omega_1 \subset \Omega_2 \cap \Omega$,

iii) $\Omega_2 \not\subset \Omega$,

so that for every function u that is regular on Ω there exists a function u_2 regular in Ω_2 and such that $u = u_2$ on Ω_1.



PROPOSITION 3.6.2.([C], III.2.3) *Let Ω be an open subset of H, Ω_1 and Ω_2 be two domains such that $\Omega_1 \subset \Omega_2 \cap \Omega$ and $\Omega_2 \not\subset \Omega$. If C denotes the connected component in $\Omega_2 \cap \Omega$ containing Ω_1 then*

$$\Omega_2 \cap \partial C \cap \partial\Omega \neq \varnothing .$$

DEFINITION 3.6.3. *An open subset Ω of H is regularly convex if, for each relatively compact subset K of Ω, the regularly convex hull of K in Ω*

$$\hat{K}_\Omega = \{q \in \Omega : \forall f \in \mathcal{R}_1(\Omega) \ , \ |f(q)| \leq \sup_{w \in K} |f(w)| \}$$

is relatively compact in Ω.

We prove that any open subset of H is both regularly convex and a region of regularity.

PROPOSITION 3.6.4. *Any open subset of H is regularly convex.*

PROOF. Let $K \subset\subset \Omega$ be a relatively compact subset of Ω.

i) $\hat{K}_\Omega$ is bounded.

Let $\rho^2 = \sup_{x_0+ix_1+jx_2+kx_3 \in K} \left(\max_{i=1,2,3} (x_0^2+x_i^2) \right)$

and

$$P_\lambda(q) = x_0 i_\lambda - x_\lambda \ \text{ for } \lambda = 1, 2, 3.$$

Since $\rho = \sup_{\lambda=1,2,3, w \in K} |f_\lambda(w)|$ we have :

$$q \in \hat{K}_\Omega \Rightarrow |q| \leq |P_1(q)| + |P_2(q)| + |P_3(q)| \leq 3\rho.$$

ii) cl $(\hat{K}_\Omega) \subsetneq \Omega$.

Without loss of generality, suppose that K is closed. Following [K], let

$$r = \text{dist}(K, \partial\Omega).$$

Suppose that $w \in \Omega$ satisfies

$$dist(w, \partial\Omega) < r.$$

Choose $w' \in \partial\Omega$ such that $|w - w'| = \text{dist}(w, \partial\Omega)$.

Consider the regular function defined in Ω by

$$f(q) = G(q - w') = \frac{\overline{q-w'}}{|q-w'|^4} \quad (\S\ 3.1).$$

Since, for every q in Ω,

$$|f(q)| = \frac{1}{|q-w'|^3}$$

it is clear that

$$q \in K \Rightarrow |q-w'| \geq r \Rightarrow |f(q)| \leq \frac{1}{r^3} \ ,$$

hence

$$\sup_{q \in K} |f(q)| \leq \frac{1}{r^3},$$

whereas

$$|f(w)| = \frac{1}{|w-w'|^3} > \frac{1}{r^3} \ . \text{ Then } w \notin \hat{K}_\Omega.$$

#

PROPOSITION 3.6.5. *Any open subset of H is a region of regularity.*

PROOF. Let Ω be an open set which is not a region of regularity. There exist two domains Ω_1 and Ω_2 , $\Omega_1 \subset \Omega_2 \cap \Omega$ and $\Omega_2 \not\subset \Omega$ so that for every function u that is regular in Ω there is a function u_2, regular in Ω_2 such that $u = u_2$ on Ω_1. Let C be the connected component in $\Omega_2 \cap \Omega$ containing Ω_1 and $q_0 \in \Omega_2 \cap \partial C \cap \partial\Omega (\neq \emptyset$ by Proposition 3.6.2).

The Fueter Cauchy-kernel $G(q-q_0)$ (which has been defined in § 3.1) is regular (in q) on Ω. Any regular function G_2 in Ω_2 such that $G = G_2$ on Ω_1 should satisfy the equalities:

$$|G_2(q_0)| = \lim_{\substack{q\in\Omega \\ q\to q_0}} |G(q)| = \infty .$$ The assertion follows.

#

If an open set Ω is a region of holomorphy (§ 1.2) there exists a function f, holomorphic in Ω that is singular at any point of the boundary of Ω ([K]). The proof of this statement is based on the fact that f^n is holomorphic in Ω for every positive integer n. Since, as we showed in § 2.5, this property does not hold for regular functions,we are not yet able to prove a similar statement.

§ 3.7. A vanishing cohomology theorem

We defined $\mathcal{R}_l(\Omega)$ to be the set of regular functions in Ω. Taking the usual restriction mapping one gets the sheaf $\mathcal{R}$ of regular functions.

It follows from Theorem 3.4.5 that any element of the stalk of $\mathcal{R}$ at a point q_0 is given by a regular series,

$$\sum_{n=0}^{\infty} \sum_{\nu\in\sigma_n} P_\nu(q-q_0)(-1)^n\partial_\nu f(q_0)$$

uniformly convergent in a neighbourhood of q_0.

In §3.6 we proved that any domain of **H** is a domain of regularity regularly convex.

In the complex case, analogous properties (§ 1.2) ensure that all the positive cohomology groups vanish (§1.4).

We present a similar result, concerning the first cohomology group of $\mathcal{R}$.

THEOREM 3.7.1. *Let Ω be a domain . Then*

$$H^1(\Omega, \mathcal{R}) = 0.$$

LEMMA 3.7.2. *Let U be an open subset of H. Let $f \in C^k(U)$ be an H-valued function. There exists $g \in C^k(U)$ such that*

$$\bar{\partial} g = f \quad \textit{in } U.$$

The proof of the above Lemma (which can be found in [P]) is an adjustement of the well-known proof concerning holomorphic functions: it is first shown that the equation can be solved if f has a compact support, then U is approximated with a sequence of compact sets.

PROOF (of Theorem 3.7.1). Let $\mathcal{U} = (U_i)_{i\in I}$ be a locally finite covering of Ω and $g = (g_{ij})_{i,j\in I}$ be a cocycle in $C^1(\mathcal{U}, \mathcal{R})$, that is, for every i, j, k in I:

$$g_{ij} + g_{jk} = g_{ik}\,;$$

g_{ij} is regular on $U_i \cap U_j$.

As it is evident, g is a cocycle in $C^1(\mathcal{U}, C^\infty)$. Hence, by Proposition 1.4.1 and by the remark preceding it, g is a coboundary in $C^1(\mathcal{U}, C^\infty)$. It follows that there exists , for every $i\in I$, a C^∞ function ϕ_i defined in U_i such that

$$g_{ij} = \phi_i - \phi_j \quad \text{on } U_i \cap U_j, \text{ for } i, j \in I.$$

Since, for every i, j;

$$0 = \bar{\partial} g_{ij} = \bar{\partial}\phi_i - \bar{\partial}\phi_j \quad \text{on } U_i \cap U_j$$

then the functions

$$(\bar{\partial}\phi_i)_{i\in I}$$

define a regular function h in Ω which agrees with $\bar{\partial}\phi_i$ on U_i.

By Lemma 3.7.2 there exists a C^1 function f in Ω such that

$$\bar{\partial} f = h\,.$$

Let $f_i = \phi_i - f$. We have

$$\bar{\partial}(f_i) = \bar{\partial}\phi_i - \bar{\partial} f = 0 \text{ on } U_i\,, i\in I$$

hence the functions f_i are regular on U_i.

Now,

$$f_j - f_i = \phi_j - \phi_i = g_{ij} \quad \text{on } U_i \cap U_j \qquad \text{for } i, j \in I,$$

hence g is a coboundary in $C^1(\mathcal{U}, \mathcal{R})$. It follows that $H^1(\mathcal{U}, \mathcal{R}) = 0$ for any covering $\mathcal{U}$ of Ω .This implies that $H^1(\Omega, \mathcal{R}) = 0$.

#

§ 3.8. Domains of convergence

Since a regular function can be expanded in a series, the domain of convergence of these series become of natural interest.

DEFINITION 3.8.1. *The domain of convergence associated with a regular series*

$$S(q) = \sum_{n=0}^{\infty} \sum_{n \in \sigma_n} P_\nu(q) a_\nu$$

is the set of all points in a neighbourhood of which S converges absolutely and uniformly.

In § 1.3.3 we stated the classical Abel's Lemma for holomorphic functions. A good Abel's Lemma in the case of **H** should ensure that if a regular series converges at a point q then it converges normally on a domain whose boundary contains q.

In [T] we have:

PROPOSITION 3.8.2. *Let $q^* = ix^* + jy^* + kz^*$ be a purely imaginary quaternion such that $x^*y^*z^* \neq 0$. Suppose there exists $M \geq 0$ such that*

$$|P_\nu(q^*)|\,|a_\nu| \leq M \quad \text{for any } \nu.$$

Then the series

$$f(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} P_\nu(q) a_\nu$$

converges normally in the open set

$$\{q = x_0 + i\,x_1 + jx_2 + kx_3 \in \mathbf{H} : x_0^2 + x_1^2 \leq |x^*|^2,\ x_0^2 + x_2^2 \leq |y^*|^2,\ x_0^2 + x_3^2 \leq |z^*|^2\}.$$

As it is shown in [T], this is the best possible result with that kind of domain.

It seems also natural to ask weather an Abel-type Lemma holds with respect to others domains, for example for the balls associated with the two norms:

$$|\,t + i\,x + jy + kz\,| = (t^2 + x^2 + y^2 + z^2)^{\frac{1}{2}} \text{ and}$$

$$\|\,t + i\,x + jy + kz\,\| = \max\{(t^2 + x^2)^{\frac{1}{2}}, (t^2 + y^2)^{\frac{1}{2}}, (t^2 + z^2)^{\frac{1}{2}}\}.$$

The two following examples give a negative answer in this direction:

EXAMPLE 3.8.3. *Abel's lemma does not hold for the $|\,.\,|$ norm on **H**.*

Let $\Delta_1 = \{q = t+i\,x+jy+kz \in \mathbf{H} :\ t^2 + x^2 < 1\}$ and $f : \Delta_1 \to \mathbf{H}$ defined by

$$f(t+i\,x+jy+kz) = \frac{1}{1-(ti-x)}\ .$$

f is a regular function in Δ_1 . Its series expansion

$$f(q) = \sum_{n=0}^{\infty} (ti-x)^n$$

converges normally in Δ_1.

Now, take $q^* = \frac{1}{2} + i\frac{1}{2} + j\frac{1}{2} + 10k$ and $q^- = 10$.

Since $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 < 1$ then

f converges at q^*.

Furthermore,

$/q^-/ < /q^*/$,

but $10 > 1$, hence

the series of f does not converge at q^-.

#

EXAMPLE 3.8.4. *Abel's lemma does not hold for the $\|\ .\ \|$ norm on **H**.*

Let $\Delta = \{\ t+i\,x+jy+kz \in \mathbf{H} : t^2 + x^2 + y^2 + z^2 < 1\}$, q^* and q^- be such that:

$q^* \in \Delta$ (1)

$q^- \notin \Delta$ (2)

$\| q^- \| < \| q^* \|$ (3)

(For instance, $q^* = \frac{3}{4} + i\frac{1}{2} + j\frac{1}{4} + k\frac{1}{4}$ and $q^- = \frac{3}{4} + i\frac{9}{20} + j\frac{9}{20} + k\frac{9}{20}$).

Now, let $\Omega = \Omega_1 = \Delta$ and $\Omega_2 = \Delta \cup \{q \in \mathbf{H} : \| q \| < \| q^* \|\}$.Then

i) Ω_2 is connected ,

in fact Δ and $\{q \in \mathbf{H} : \| q \| < \| q^* \|\}$ are both arcwise connected and contain the origin;

ii) $\Omega_1 \subset \Omega_2 \cap \Omega$;

iii) $\Omega_2 \not\subset \Omega$,

in fact , from (2) and (3), $q^- \in \Omega_2 \setminus \Omega$.

Δ is a domain of regularity (Proposition 3.6.5): there exists a function g, regular in Δ which cannot be extended to Ω_2. Since Δ is a Reinhardt domain (Definition 3.5.1) and Theorem 3.5.2 holds then the series expansion of g at the origin

$$g(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} P_\nu(q)(-1)^n \partial_\nu g(0)$$

which is normally convergent in Δ converges at q^*. Suppose that the series of g converges normally for $\| q \| < \| q^* \|$:

then the series of g would converge normally in Ω_2. A contradiction.

#

As we remarked in § 1.3, as a consequence of the classical Abel`s Lemma any domain of convergence of a complex power series is a Reinhardt domain. The analagous theorem does not hold in the regular case over quaternions:

PROPOSITION 3.9.5. *The domain of convergence of a regular series expansion is not necessarily a Reinhardt domain.*

PROOF. Let $\Delta_1 = \{q = t+i x+jy+kz \in \mathbf{H} : t^2 + x^2 < 1\}$ and $f : \Delta_1 \to \mathbf{H}$ defined by

$$f(t+i x+jy+kz) = \frac{1}{1-(ti-x)} .$$

The function f is a regular function in Δ_1 which is the domain of convergence of its series expansion

$$f(q) = \sum_{n=0}^{\infty} (ti-x)^n .$$

Hence the series expansion of f converges at

$$q^* = \frac{1}{2} + i\frac{1}{2} + 10j$$

but doesn't at

$$q' = q^*j = -10 + j\frac{1}{2} + k\frac{1}{2} .$$

#

A few words about the zero-set of a regular function.

Opposite to what happens in the complex case or in the case of quaternionic analytic functions with real coefficients (§ 2.3) this set seems to be not "regular".

A regular function whose zero-set has an accumulation point is not necessarily zero: let

$$f(x_0 + i x_1 + jx_2 + kx_3) = x_1x_2 - x_0x_2i - x_0x_1j .$$

Its zero-set, Z(f) is the union of the three 2-planes

$$\{x_1 = x_2 = 0\} \cup \{x_0 = x_2 = 0\} \cup \{x_1 = x_0 = 0\}.$$

The zero-set of a regular function can have dimension 1. This is the case of

$$f(t+i x+jy+kz) = -t - x + it - yj$$

whose zero set is the line $\{t = x = y = 0\}$.

A regular function can have isolated zeros, for instance the regular function

$$t+ix-jy+kz$$

has an only zero .

§ 3.9. A Reflection principle

The analogous of the Schwarz Reflection Principle holds for regular functions over quaternions:

THEOREM 3.9.1. *Let Ω be a domain, symmetric with respect to the hyperplane generated by $\{i, j, k\}$ and to the three 2-planes generated by $\{j,k\}$, $\{i,k\}$, $\{i,j\}$; that is:*

$$q = t + ix + jy + kz \in \Omega \Rightarrow -\bar{q} = -t + ix + jy + kz \in \Omega \quad (3.9.1)$$
$$\Rightarrow i\bar{q}i = -t + ix - jy - kz \in \Omega \quad (3.9.2)$$
$$\Rightarrow j\bar{q}j = -t - ix + jy - kz \in \Omega \quad (3.9.3)$$
$$\Rightarrow k\bar{q}k = -t - ix - jy + kz \in \Omega \quad (3.9.4)$$

Let Ω_+ be the part of Ω contained in the upper half hyperplane, i.e.

$$\Omega_+ = \{q = t + ix + jy + kz \in \Omega : t > 0\}$$

and let Σ be the set defined by

$$\Sigma = \{q = t + ix + jy + kz \in \Omega : t = 0\}.$$

Then, a function $f = f_0 + if_1 + jf_2 + kf_3 : \Omega_+ \cup \Sigma \to H$ (where f_0, f_1, f_2, f_3 are real valued) which is regular in Ω_+, continuous in $\Omega_+ \cup \Sigma$ and real valued on Σ can be extended to a regular function $F = F_0 + iF_1 + jF_2 + kF_3$ in Ω in such a way that, for any q in Ω,

$$F(q) = F_0(-\bar{q}) - iF_1(i\bar{q}i) - jF_2(j\bar{q}j) - kF_3(k\bar{q}k).$$

PROOF. Let $\Omega_- = \Omega \setminus \Omega_+ \cup \Sigma = \{q = t + ix + jy + kz \in \Omega : t < 0\}$ and let $F : \Omega \to H$ be the function defined by

$$F(q) = f(q) = f_0(q) + if_1(q) + jf_2(q) + kf_3(q) \quad \text{if } q \in \Omega_+ \cup \Sigma,$$

$$F(q) = f_0(-\bar{q}) - if_1(i\bar{q}i) - jf_2(j\bar{q}j) - kf_3(k\bar{q}k) \quad \text{otherwise.}$$

Remark that this definition makes sense, since the symmetric relations (3.9.1-4) hold. As a consequence of being real valued in Σ, F is continuous in Ω. Furthermore, the Cauchy-Fueter conditions (2.5.1-4) are satisfied in $\Omega_+ \cup \Omega_-$. We only have to check regularity of F in Σ. Let U be the interior of a parallelepiped properly contained in Ω. If $U \cap \Sigma = \varnothing$ then $U \subset \Omega_+$ or $U \subset \Omega_-$, hence

$$\int_{\partial U} DqF = 0 \quad (1)$$

By Theorem 3.3.2. Otherwise, let

$$U = (a,b) \times I_1 \times I_2 \times I_3$$

where $a < 0 < b$ and I_j are open real intervals, $j = 1, 2, 3$.

Let

$$\Pi = U \cap \Sigma = \{0\} \times I_1 \times I_2 \times I_3$$

and for $n = 1, 2, \ldots$ let

$$U_n = U_n^+ \cup U_n^-$$

where

$$U_n^+ = \{q = t + ix + jy + kz \in U : t > \frac{1}{n}\}$$

and

$$U_n^- = \{q = t + ix + jy + kz \in U : t < -\frac{1}{n}\}$$

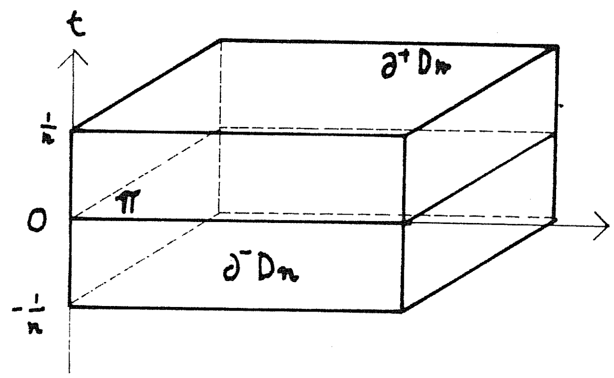
From Theorem 3.3.2 we have:

$$\int_{\partial U_n} DqF = \int_{\partial U_n^+} DqF + \int_{\partial U_n^-} DqF = 0 \quad (2)$$

since F is regular in $\Omega_+ \cup \Omega_-$.

We wish to prove that $\int_{\partial U} DqF = 0.$

Let $D_n = (-\frac{1}{n}, \frac{1}{n}) \times I_1 \times I_2 \times I_3$.



Then, for n = 1, 2, ... we have:

$$\int_{\partial U} DqF(q) - \int_{\partial U_n} DqF(q) = \int_{\partial D_n} DqF(q) = \int_{\partial^+ D_n} DqF(q) + \int_{\partial^- D_n} DqF(q) + \int_{C_n} DqF(q)$$

where

$$\partial^+ D_n = \{\frac{1}{n}\} \times I_1 \times I_2 \times I_3,$$

$$\partial^- D_n = \{-\frac{1}{n}\} \times I_1 \times I_2 \times I_3,$$

$$C_n = \partial D_n \setminus (\partial^+ D_n \cup \partial^- D_n).$$

Therefore, for n = 1, 2, ...

$$\left| \int_{\partial U} DqF(q) - \int_{\partial U_n} DqF(q) \right| \leq \left| \int_{\partial^+ D_n} DqF(q) + \int_{\partial^- D_n} DqF(q) \right| + \left| \int_{C_n} DqF(q) \right| \quad (3).$$

It is clear, by using an absolute continuity measure argument and the continuity of F, that

$$\lim_{n\to\infty} \int_{C_n} DqF(q) = 0 \quad (4).$$

Remark that $\partial^+ D_n$ and $\partial^- D_n$ have opposite orientations. Suppose Π has the same orientation as $\partial^+ D_n$. We have, by a change of variables:

$$\int_{\partial^+ D_n} DqF(q) = \int_{\Pi} DqF(q+\frac{1}{n})$$

and

$$\int_{\partial^- D_n} DqF(q) = -\int_{\Pi} DqF(q-\frac{1}{n})$$

Hence

$$\int_{\partial^+ D_n} DqF(q) + \int_{\partial^- D_n} DqF(q) = \int_{\Pi} Dq\left(F(q+\frac{1}{n}) - F(q-\frac{1}{n})\right).$$

Remark that F, being continuous in Ω, is uniformly continuous in the closure of U. Therefore, for any positive number ε there exists n_ε such that

$$\forall n \geq n_\varepsilon : \left|F(q+\frac{1}{n}) - F(q-\frac{1}{n})\right| \leq \frac{\varepsilon}{\left|\int_{\Pi} Dq\right| \max_{\Pi}|F|} \quad \text{for any q in } \Sigma.$$

Then, Proposition 3.2.2 implies

$$\forall n \geq n_\varepsilon : \left|\int_{\Pi} Dq\left(F(q+\frac{1}{n}) - F(q-\frac{1}{n})\right)\right| \leq \varepsilon .$$

This proves that

$$\lim_{n\to\infty} \int_{\Pi} Dq\left(F(q+\frac{1}{n}) - F(q-\frac{1}{n})\right) = 0 \quad (5).$$

Now, it follows from (1), (2), (3), (4) and (5) that

$$\int_{\partial U} DqF = \lim_{n\to\infty} \int_{\partial U_n} DqF(q) = 0 \text{ for any parallelepiped strictly contained in } \Omega.$$

With a classic argument, one can extend this conclusion to any domain $V \subset \Omega$, apply Morera's Theorem 3.3.1 and obtain the assertion.

#

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