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On minima of radially symmetric functionals of the gradient

Thesis submitted for the degree of
"Magister Philosophiæ"

CANDIDATE
Stefania Perrotta

SUPERVISOR
Prof. Arrigo Cellina

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Introduction.

In this thesis we consider the problems of the existence, the uniqueness and the qualitative properties (symmetry) of the minima to the problem

$$\min_{u \in W_0^{1,1}(B)} \int_B [g(|x|, |\nabla u(x)|) + h(u(x))] dx$$

where B is the unit ball of $I\!R^n$ and the map $v \to g(r, v)$ is lower semicontinuous but not necessarily convex.

Problems of this kind arise in domains as different as non-linear elasticity, fluidodynamics and shape optimization, and are considered in [B-P], [G-K-R], [K], [K-S], [M], [R], [T]. In particular, the very same problem is considered in [T].

Our existence and uniqueness results present the following features:
a) no smoothness on g or h is required: g is either a normal integrand or a lower semicontinuous function;
b) the case $h \equiv 0$ is allowed; in this case the assumption on g reduce, for the existence of solutions, to g being lower semicontinuous and growing at infinity, as is to be expected; for the uniqueness, in addition,on g^{**} being strictly increasing, as also is to be expected;
c) for the case h linear, h not zero, our Theorems yield at once existence and uniqueness of solutions with no further assumptions on g besides lower semicontinuity and growth at infinity.

Basic notations.

In what follows we shall assume that: $I\!R^n$ is endowed with the Euclidean norm $|\cdot|$; B is the unit ball, whose measure is ω_n. The (n-1)-dimensional Hausdorff measure of ∂B is $n\omega_n$.

The subgradient of a convex function g is denoted by ∂g.

A map $g : [0,1] \times [0, \infty) \to \overline{I\!R}$ is termed a normal integrand, [E-T], if

$i)$ for a.e. $r \in [0,1]$ $g(r, \cdot)$ is l.s.c. on $[0, \infty)$;

$ii)$ $\exists$ a Borel function $\tilde{g}: [0,1] \times [0,\infty) \to \bar{I\!R} : \tilde{g}(r,\cdot) \equiv g(r,\cdot)$ for a.e. $r \in [0,1]$.

Consider $g : [0,1] \times [0,\infty) \to \overline{I\!R}$. Let $\tilde{g} : B \times I\!R^n \to \overline{I\!R}$ be defined by $\tilde{g}(x,\xi) = g(|x|, |\xi|)$. Whenever the bipolar of $\tilde{g}$, $\tilde{g}^{**}$, is defined, by extension we call bipolar of g, g^{**}, the map defined by $g^{**}(|x|, |\xi|) = \tilde{g}^{**}(x,\xi)$. Remark that the map $\xi \to g^{**}(r,\xi)$ is increasing. It is known, [E-T], that $\tilde{g}^{**}$ is a normal integrand whenever so is $\tilde{g}$ and that $\tilde{g}^{**}$ satisfies the same growth assumptions as $\tilde{g}$.

Main results.

We shall consider the following problem (P):

$$(P) \qquad \min_{u\in W_0^{1,1}(B)} \int_B [g(|x|, |\nabla u(x)|) + h(u(x))]dx$$

where $B = \{x \in I\!R^n : |x| < 1\}$, $n \geq 2$, $g: [0,1] \times [0,\infty) \to \overline{I\!R}$ is a normal integrand and $h: I\!R \to I\!R$ is convex.

We shall assume throughout the following growth assumption:

$$(GA) \qquad \begin{cases} \text{there exist : a convex l.s.c. increasing function } \psi: [0,\infty) \to [0,\infty) : \\ \dfrac{\psi(s)}{s} \to \infty \text{ as } s \to \infty \text{ and } \alpha \in L^1([0,1]) \text{ satisfying} \\ g(r,s) \geq \alpha(r) + \psi(s) \text{ for all } s \in [0,\infty), \text{for a.e. } r \in [0,1]. \end{cases}$$

The following result guarantees the existence of at least one radially symmetric solution to the minimum problem (P) associated to a convex function g.

Theorem 1.

Let g be a normal integrand satisfying assumptions (GA). Assume further that $g^{**} \equiv g$. Let $h : I\!R \to I\!R$ be convex. Then problem (P) admits at least one radially symmetric solution. Moreover, if h is monotonic and either $g(r,.)$ or h is strictly monotonic, then every solution to (P) is radially symmetric.

Proof.

Let u be a solution to problem (P). Consider the function $\bar{u}$ defined by

$$(1) \qquad \bar{u}(x) = \frac{1}{n\omega_n} \int_{|\omega|=1} u(\omega|x|)d\omega$$

where $n\omega_n$ is the $(n-1)$-dimensional Hausdorff measure of ∂B.

It is our purpose to show that $\bar{u}$ is a radially symmetric solution to (P). The symmetry comes from the very definition.

a) We claim that

$$(2) \qquad \begin{cases} \nabla \bar{u}(x) = \dfrac{1}{n\omega_n} \dfrac{x}{|x|} \displaystyle\int_{|\omega|=1} \langle \nabla u(\omega|x|), \omega \rangle d\omega, \quad x \neq 0 \\ \nabla \bar{u}(0) = 0. \end{cases}$$

First remark that the above is true when u is of class C^1. In this case, when $|x| \neq 0$ one can differentiate with respect to the parameter x to obtain

$$\frac{\partial \bar{u}}{\partial x_i}(x) = \frac{1}{n\omega_n} \int_{|\omega|=1} \langle \nabla u(\omega|x|), \omega \rangle \frac{x_i}{|x|} d\omega$$

while, for $x = 0$

$$\begin{aligned}\frac{\partial \bar{u}}{\partial x_i}(0) &= \lim_{h_i \to 0} \frac{1}{h_i}\left(\frac{1}{n\omega_n}\int_{|\omega|=1}(u(\omega h_i) - u(0))d\omega\right) = \\ &= \lim_{h_i \to 0} \frac{1}{h_i}\left(\frac{1}{n\omega_n}\int_{|\omega|=1}(h_i\langle \omega, \nabla u(0)\rangle + h_i|\omega|\epsilon(h_i|\omega|)d\omega\right) = \\ &= 0.\end{aligned}$$

To show the validity of the above formula for any u in $W^{1,1}(B)$ let us consider a sequence $\{u_h\}$, each u_h of class C^1 and $u_h \to u$ in $W^{1,1}(B)$. Hence, from the previous result, $\nabla \bar{u}_h$ satisfy (2). We are going to show first that the functions $\bar{u}_h$ defined by (1) convege to $\bar{u}$ strongly in L^1.

Set w to be

$$w(x) = \begin{cases} \frac{1}{n\omega_n}\frac{x}{|x|}\int_{|\omega|=1}\langle \nabla u(\omega|x|), \omega\rangle d\omega, & x \neq 0 \\ 0, & x = 0. \end{cases}$$

To prove the claim hence it will be left to show that $\nabla \bar{u}_k$ converges to w strongly in $L^1(B)$. We have

$$\begin{aligned}\| \bar{u} - \bar{u}_k \|_1 &\leq \int_B \left(\frac{1}{n\omega_n}\int_{|\omega|=1} |u(\omega|x|) - u_k(\omega|x|)|\, d\omega\right) dx = \\ &= \frac{1}{n\omega_n}\int_{|\omega|=1}\left(\int_B |u(\omega|x|) - u_k(\omega|x|)|\, dx\right) d\omega = \\ &= \frac{1}{n\omega_n}\int_{|\omega|=1}\left(n\omega_n \int_0^1 |u(\omega r) - u_k(\omega r)|\, r^{n-1} dr\right) d\omega = \\ &= \int_B |u(x) - u_k(x)|\, dx = \| u - u_k \|_1 .\end{aligned}$$

Through the same steps,

$$\begin{aligned}&\| w - \nabla \bar{u}_k \|_1 = \\ &= \int_B \frac{1}{n\omega_n}\left|\frac{x}{|x|}\int_{|\omega|=1}\langle \nabla u(\omega|x|) - \nabla u_k(\omega|x|), \omega\rangle d\omega\right| dx \leq \\ &\leq \int_B \frac{1}{n\omega_n}\int_{|\omega|=1} |\nabla u(\omega|x|) - \nabla u_k(\omega|x|)|\, d\omega dx\end{aligned}$$

and, by applying again Fubini's Theorem,

$$\| w - \nabla \bar{u}_k \|_1 \leq \| \nabla u - \nabla u_k \|_1 .$$

The claim is proved.

The above arguments defining $\bar{u}$ out of u are similar to those employed in [C-F] and [F] for a problem involving the Laplacian.

b) From the convexity of h we obtain

$$\int_B h(u(x))dx \geq \int_B h(\bar{u}(x))dx.$$

In fact

$$\int_B h(\bar{u}(x))dx \leq \int_B \frac{1}{n\omega_n}\int_{|\omega|=1} h(u(\omega|x|))d\omega dx =$$
$$= \int_{|\omega|=1}\int_0^1 h(u(\omega r))r^{n-1}drd\omega = \int_B h(u(x))dx.$$

In particular the same computation in the case $h(u) = u$ yields

$$\int_B u(x)dx \geq \int_B \bar{u}(x)dx. \tag{3}$$

To prove that $\bar{u}$ is a solution it is left to show that

$$\int_B g(|x|, |\nabla\bar{u}(x)|)dx \leq \int_B g(|x|, |\nabla u(x)|)dx.$$

Since

$$|\nabla\bar{u}(x)| \leq \frac{1}{n\omega_n}\int_{|\omega|=1} |\nabla u(\omega|x|)|\, d\omega$$

and $s \mapsto g(r,s)$ is monotonic,

$$\int_B g(|x|, |\nabla\bar{u}(x)|)dx \leq \int_B g\left(|x|, \frac{1}{n\omega_n}\int_{|\omega|=1} |\nabla u(\omega|x|)|\, d\omega\right) dx.$$

By the convexity of $s \mapsto g(r,s)$ and Jensen's inequality,

$$\int_B g\left(|x|, \frac{1}{n\omega_n}\int_{|\omega|=1} |\nabla u(\omega|x|)|\, d\omega\right) dx \leq$$
$$\leq \int_B \left(\frac{1}{n\omega_n}\int_{|\omega|=1} g\left(|x|, |\nabla u(\omega|x|)|\right) d\omega\right) dx =$$
$$= \int_{|\omega|=1}\int_0^1 g\left(r, |\nabla u(\omega r)|\right) r^{n-1}drd\omega = \int_B g(|x|, |\nabla u(x)|)dx.$$

Hence $\bar{u}$ is a radially symmetric solution to (P).

c) Assume now that h is monotonic and let us prove the result first in the case h monotonic increasing. The map $v \to g(r,v)$ is non decreasing; by assumption, either g or h is strictly increasing. Let us show that every solution is radially symmetric.

Let u be any solution and set $\bar{u}$ to be

$$\bar{u}(x) = \frac{1}{n\omega_n} \int_{|\omega|=1} u(\omega|x|)d\omega.$$

From the above, $\bar{u}$ is a solution. Consider the average $w = \frac{1}{2}(u+\bar{u})$. By the convexity of the problem, w is a solution. In particular,

$$\int_B g(|x|, |\nabla w(x)|)dx = \frac{1}{2}\int_B g(|x|,|\nabla u(x)|)dx + \frac{1}{2}\int_B g(|x|,|\nabla \bar{u}(x)|)dx. \tag{4}$$

By the monotonicity and the convexity of g, we have

$$g(|x|,|\nabla w(x)|) \leq g\left(|x|, \frac{1}{2}\left(|\nabla u(x)| + |\nabla \bar{u}(x)|\right)\right) \leq \frac{1}{2}g(|x|,|\nabla u(x)|) + \frac{1}{2}g(|x|,|\nabla \bar{u}(x)|)$$

so that, by (4), equality holds:

$$\begin{aligned} g(|x|,|\nabla w(x)|) &= g\left(|x|, \frac{1}{2}\left(|\nabla u(x)| + |\nabla \bar{u}(x)|\right)\right) = \\ &= \frac{1}{2}g(|x|,|\nabla u(x)|) + \frac{1}{2}g(|x|,|\nabla \bar{u}(x)|). \end{aligned} \tag{5}$$

Set $T(r)$ to be $T(r) = \sup\{v : g(r,v) - g(r,0) = 0\}$. The supremum is actually a maximum.

We wish to show that for almost every x, $|\nabla w(x)| \geq T(|x|)$. In the case g strictly increasing, $T = 0$ and there is nothing to prove. Assume that h is strictly increasing. Remark that $v \to g(r,v)$ is strictly increasing for $v \geq T(r)$. Hence, notice the following property of w to be used later: for those x such that the gradient exists, whenever $|\nabla w(x)| \geq T(|x|)$, there exists $\lambda(x) \geq 0$ such that $\nabla u(x) = \lambda(x)\nabla \bar{u}(x)$. In fact in this case the first equality in (5) implies that

$$|\nabla w(x)| = \frac{1}{2}(|\nabla u(x)| + |\nabla \bar{u}(x)|)$$

and by the strict convexity of the euclidean norm, this is true only if $\nabla u(x) = \lambda(x)\nabla \bar{u}(x)$.

We claim that the map $r \to T(r)$ is measurable. Fix $\epsilon > 0$. Since $g(r,v)$ is a normal integrand, by Theorem 1.1, p.232 of [E-T], there exists a compact K_ϵ in $[0,1]$, $m([0,1]\backslash K_\epsilon) < \epsilon$, such that the restriction of g to $K_\epsilon \times I\!R$ is lower semicontinuous and the restrictions of $r \to g(r,0)$ and of $r \to \alpha(r)$ to K_ϵ are continuous. In particular, there exists $M_\epsilon : g(r,0) - \alpha(r) \leq M_\epsilon$ in K_ϵ. Hence, for every v satisfying $g(r,v) - g(r,0) = 0$ for some r in K_ϵ we have $\psi(v) \leq g(r,0) - \alpha(r) \leq M_\epsilon$ that implies $|v| \leq V_\epsilon$ for some V_ϵ . Consider a sequence (r_n) in K_ϵ converging to r^* and set T^* to be the limsup of $T(r_n)$. By taking a subsequence we can assume that $g(r_n, T(r_n))$ converges to y satisfying

$y - g(r^*, 0) = 0$; hence $g(r^*, T^*) - g(r^*, 0) \leq 0$. Being g non decreasing in v, we have $g(r^*, T^*) - g(r^*, 0) = 0$, hence $T(r^*) \geq T^*$, i.e. the restriction to K_ϵ of $r \to T(r)$ is upper semicontinuous. By Lusin's Theorem this proves the claim.

Let us first show that $|\nabla \bar{u}(x)| \geq T(|x|)$. In fact set $A = \{|x| : |\nabla \bar{u}(x)| < T(|x|)\}$ and assume that $m(A) > 0$. Define $v: [0,1] \to I\!R$ by $v(|x|) = \bar{u}(x)$. Remark that $r \to v(r)$ is locally absolutely continuous in $(0,1]$. In fact apply the change of variable formula to the transformation from Cartesian to polar coordinates, and obtain for the map $\bar{v}(r,\theta) = u(\phi(r,\theta))$ that for almost every $\bar{\theta}$, the map $v : r \to \bar{v}(r,\bar{\theta})$ is locally absolutely continuous in $(0,1]$ and

$$v'(r) = \frac{\partial \bar{v}}{\partial r}(r,\bar{\theta}) = \langle \frac{\partial \phi}{\partial r}, \nabla u(\phi(r,\bar{\theta}))\rangle.$$

Since $\bar{u}$ is radially symmetric, $|\langle \nabla \bar{u}(x), \frac{x}{|x|}\rangle| = |\nabla \bar{u}(x)|$; by the chain rule, $v'(x) = \langle \nabla \bar{u}(x), \frac{x}{|x|}\rangle$, hence

$$|v'(x)| = |\langle \nabla \bar{u}(x), \frac{x}{|x|}\rangle|$$

Set $\tilde{v}(r)$ to be $\int_1^r \left(v'(s)\chi_{\mathcal{C}(A)}(s) + T(s)\chi_A(s)\right) ds$ and $\bar{v}: B \to I\!R$ to be $\bar{v}(x) = \tilde{v}(|x|)$. Then, from the very definition, $\bar{v}(x) \leq \bar{u}(x)$ and the strict inequality holds on a subset of B of positive measure, hence

$$\int_B h(\bar{v}(x))dx < \int_B h(\bar{u}(x))dx.$$

On the other hand

$$g(|x|, |\nabla \bar{v}(x)|) = g(|x|, |\nabla \bar{u}(x)|),$$

so that $\bar{u}$ cannot be a minimum. Hence $|\nabla \bar{u}(x)| \geq T(|x|)$ a.e..

To show that the same inequality holds for w, remark that by the definitions one also has

$$\bar{u}(x) = \frac{1}{n\omega_n} \int_{|\omega|=1} w(\omega|x|)d\omega$$

so that

$$\nabla \bar{u}(x) = \frac{1}{n\omega_n} \frac{x}{|x|} \int_{|\omega|=1} \langle \nabla w(\omega|x|), \omega\rangle d\omega$$

and, by the definition and (3),

$$\int_B \bar{u}(x)dx \leq \int_B w(x)dx.$$

Assume that there exists a subset S of B, $m(S) > 0$, such that $|\nabla w(x)| < T(|x|)$ for x in S. Since $\nabla w = \frac{1}{2}\nabla u + \frac{1}{2}\nabla \bar{u}$, on S we must have $|\nabla u(x)| < T(|x|)$; moreover $|\nabla \bar{u}(x)|$ must be equal to $T(|x|)$. In fact, if it is not so,

$$g(|x|, |\nabla w(x)|) < \frac{1}{2}g(|x|, |\nabla u(x)|) + \frac{1}{2}g(|x|, |\nabla \bar{u}(x)|)$$

contradicting (5).

Set

$$S_r = \{\omega \ : \ |\nabla w(\omega r)| < T(r)\}$$

$$m(S) = \int_0^1 r^{n-1}\left(\int_{|\omega|=1} \chi_S(\omega r) d\omega\right) dr = \int_0^1 r^{n-1} \int_{|\omega|=1} \chi_{S_r}(\omega) d\omega dr$$

so that for r in a subset $E \subset [0,1]$ of positive measure,

$$\int_{|\omega|=1} \chi_{S_r}(\omega) d\omega > 0.$$

Consider one such r in E. Since, for $|x| = r$,

$$T(r) = |\nabla \bar{u}(x)| \leq \frac{1}{n\omega_n} \int_{|\omega|=1} |\nabla w(\omega |x|)| \, d\omega,$$

on a subset of $\mathcal{C}(S_r)$ of a positive measure we must have $|\nabla w(\omega r)| > T(r)$. Again for r in E,

$$\begin{aligned}
&\int_{|\omega|=1} g(r, |\nabla w(\omega r)|) d\omega = \\
&= \int_{S_r} g(r, T(r)) d\omega + \int_{\mathcal{C}(S_r)} g(r, |\nabla w(\omega r)| d\omega > \\
&> \int_{|\omega|=1} g(r, T(r)) d\omega = \int_{|\omega|=1} g(r, |\nabla \bar{u}(\omega r)|) d\omega.
\end{aligned}$$

Set X to be $\{x = \omega r \ : \ |\omega| = 1, r \in E\}$. Then

$$\begin{aligned}
&\int_X g(|x|, |\nabla w(x)|) \, dx = n\omega_n \int_0^1 r^{n-1} \int_{|\omega|=1} \chi_X(\omega r) g(r, |\nabla w(\omega r)|) d\omega dr = \\
&= n\omega_n \int_0^1 r^{n-1} \chi_E(r) \left(\int_{|\omega|=1} g(r, |\nabla w(\omega r)|) d\omega\right) dr > \\
&> n\omega_n \int_0^1 r^{n-1} \chi_E(r) \left(\int_{|\omega|=1} g(r, |\nabla \bar{u}(\omega r)|) d\omega\right) dr = \\
&= \int_X g(|x|, |\nabla \bar{u}(x)|) dx.
\end{aligned}$$

For x not in X, by a previous remark, $\nabla u(x) = \lambda(x) \nabla \bar{u}(x)$, so that $\nabla w(x) = (\lambda(x) + 1)\nabla \bar{u}(x)$; hence on $B \backslash X$, w itself is radially symmetric, i.e. w coincides with $\bar{u}$. By Lemma 7.7 in [G-T], $\nabla w = \nabla \bar{u}$ a.e. in $B \backslash X$. Hence

$$\begin{aligned}
\int_B g(|x|, |\nabla w(x)|) dx &= \int_B g(|x|, |\nabla w(x)|)[\chi_X(x) + \chi_{B\backslash X}(x)] dx > \\
&> \int_B g(|x|, |\nabla \bar{u}(x)|) dx,
\end{aligned}$$

a contradiction, since w is a solution and

$$\int_B h(w(x))dx \geq \int_B h(\bar{u}(x))dx.$$

Then $m(S) = 0$, i.e. for almost every x in B, $|\nabla w(x)| \geq T(|x|)$. Hence, for almost every x, $\nabla u(x) = \lambda(x)\nabla\bar{u}(x)$.

This proves the Theorem in the case h increasing.

d) Finally, notice that the case of a decreasing function h can be reduced to the previous one by setting

$$\tilde{h}(\xi) = h(-\xi), \quad \xi \in \mathbb{R}.$$

Now, $\tilde{h}$ is increasing and each solution to the problem

$$(\tilde{P}) \qquad \min_{u \in W_0^{1,1}(B)} \int_B [g(|x|, |\nabla u(x)|) + \tilde{h}(u(x))]dx$$

is radially symmetric.

Since $\tilde{u}$ is a solution to $(\tilde{P})$ if and only if $-\tilde{u}$ is a solution to (P), we see that every solution to (P) is radially symmetric.

Remark. To see how the assumptions of the previous Theorem are sharp, consider the case $h \equiv 0$ (h is monotonic, but not strictly monotonic) and $g = g(v)$ to be the indicator of the interval $[0, 1]$ (g is convex and monotonic, but not strictly monotonic). Clearly there exist non radially symmetric solutions to (P).

The next results are concerned with existence and uniqueness of solutions for the minimum problem (P) when g is (possibly) non convex. We shall assume that g is independent on $|x|$, i.e. $g(r, \xi) = g(\xi)$.

Theorem 2.

Assume that: $g(r, \xi) = g(\xi)$ is l.s.c. and satsfies (GA); $h : \mathbb{R} \to \mathbb{R}$ is convex and monotonic.

Then the minimum problem (P) admits at least a solution u in $W_0^{1,1}(B)$.

Proof.

a) As in the proof of the Theorem 1, it is enough to consider the case where h is monotonic decreasing.

Let $\tilde{u}$ be a radially symmetric solution to the convexified problem

$$(P^{**}) \qquad \min_{u \in W_0^{1,1}(B)} \int_B [g^{**}(|\nabla u(x)|) + h(u(x))]dx$$

Define $u:[0,1] \to I\!R$ by $u(|x|) = \tilde{u}(x)$. Remark that the map $r \to u(r)$ is locally absolutely continuous on $(0,1]$, and $u'(|x|) = \langle \frac{x}{|x|}, \nabla \tilde{u}(x) \rangle$.

We are going to show that $|u'|$ cannot take its values (for r in a set of positive measure) on any interval (a,b) where g^{**} is affine. This in particular shows that $|u'|$ takes its values where g and g^{**} coincide, hence proves that u is a solution to the original problem.

Let $h'_+(u)$ be the right derivative of h at u; the map $r \to h'_+(u(r))$ is positive and bounded; consider H defined by

$$H(r) = \int_0^r s^{n-1} h'_+(u(s)) ds.$$

Assume $g^{**\prime} = \alpha$ on (a,b). Consider first the case $\alpha = 0$; in this case $(a,b) = (0,T)$, and consider those r such that $|u'(r)| \le T$. The same reasoning as in Theorem 1, point c), imply that there exists another radial solution v such that $|v'| \ge T$. Hence we can as well assume $\alpha > 0$. Set $E_\sigma = \{r : |u'(r)| \in (a+\sigma, b-\sigma)\}$ and assume that for some σ, $m(E_\sigma) > 0$. Call E this E_σ. Consider $A(r) = (\alpha r^{n-1} - H(r))\chi_E(r)$. Let r_1 and r_2, $r_1 < r_2$, be points of density for E and Lebesgue points for A, so that

$$\frac{1}{\delta}\int_{r_1}^{r_1+\delta} A(r) dr \to A(r_1) = (\alpha r_1^{n-1} - H(r_1)).$$

$$\frac{1}{\delta}\int_{r_2}^{r_2+\delta} A(r) dr \to A(r_2) = (\alpha r_2^{n-1} - H(r_2)).$$

Since A is monotonic, $A(r_2) - A(r_1) = 2\lambda > 0$. Set $\delta^* = \delta^*(\delta)$ to be $m[(r_2-\delta, r_2) \cap E]$. Since r_2 is of density for E, $\delta^*/\delta \to 1$, i.e. $\delta^* > 0$. Let $\bar{\delta}$ be such that $\delta < \bar{\delta}$ implies

$$\left| \frac{1}{\delta}\int_{r_1}^{r_1+\delta} A(r) dr - A(r_1) \right| < \frac{\lambda}{4};$$

$$\left| \frac{1}{\delta}\int_{r_2-\delta}^{r_2} A(r) dr - A(r_2) \right| < \frac{\lambda}{4}.$$

Set η' to be

$$\eta' = \frac{1}{\delta}\chi_{[r_1, r_1+\delta^*]\cap E} - \frac{1}{\delta^*}\chi_{[r_2-\delta, r_2]\cap E}$$

and define η as $\eta(r) = \int_1^r \eta'(s) ds$ so that $\eta(1) = 0$ and, by the choice of δ^*, $\eta(r) \ge 0$. Let us remark that $u'(r) + \epsilon\eta'(r)$ coincides with $u'(r)$ for r not in E, and, for r in E, is in (a,b) whenever ϵ is sufficiently small. Hence $g^{**}(|u'(r) + \epsilon\eta'(r)|)$ is well defined and integrable.

b) Let us consider the family $\{u + \epsilon\eta\}$. By the mean value Theorem there exists: $\xi_1(r) \in [0,\epsilon]$ and $g_\epsilon^{**\prime}(r) \in \partial g^{**}(|u'(r) + \xi_1(r)\eta(r)|)$ such that

$$g^{**}(|u'(r) + \epsilon\eta'(r)|) - g^{**}(|u'(r)|) = \epsilon g_\epsilon^{**\prime}(r)\eta'(r).$$

Moreover, since the subgradient of h is monotonic, there exists $\xi_2(r) \in [0,\epsilon]$ such that

$$h(u(r) + \epsilon\eta(r)) - h(u(r)) \le \epsilon h'_+(u(r) + \xi_2(r)\eta(r))\eta(r).$$

Then

$$\Delta_\epsilon = \int_0^1 r^{n-1}[g^{**}(u'+\epsilon\eta') + h(u+\epsilon\eta)]dr - \int_0^1 r^{n-1}[g^{**}(u') + h(u)]dr \leq$$

$$\leq \epsilon \int_0^1 r^{n-1}[g_\epsilon^{**\prime}(r)\eta'(r) + h'_+(u(r) + \xi_2(r)\eta(r))\eta(r)]dr.$$

Set H_ϵ to be

$$H_\epsilon(r) = \int_0^r s^{n-1} h'_+(u(s) + \xi_2(s)\eta(s))ds.$$

Since $H_\epsilon(0) = 0$ and $\eta(1) = 0$, we have

$$\int_0^1 r^{n-1} h'_+(u(r) + \xi_2(r)\eta(r))\eta(r)dr = -\int_0^1 H_\epsilon(r)\eta'(r)dr.$$

Hence

$$\Delta_\epsilon \leq \epsilon \int_0^1 [r^{n-1} g_\epsilon^{**\prime}(r) - H_\epsilon(r)]\eta'(r)dr =$$

$$= \epsilon\frac{1}{\delta}\int_{r_1}^{r_1+\delta^*} [r^{n-1} g_\epsilon^{**\prime}(r) - H_\epsilon(r)]\chi_E(r)dr - \epsilon\frac{1}{\delta^*}\int_{r_2-\delta}^{r_2} [r^{n-1} g_\epsilon^{**\prime}(r) - H_\epsilon(r)]\chi_E(r)dr.$$

For every fixed s, $u(s) + \xi_2(s)\eta(s)$ converges to $u(s)$ from the right, so that, being the subdifferential of a convex function monotonic, we have $h'_+(u(s) + \xi_2(s)\eta(s)) \to h'_+(u(s))$ as $\epsilon \to 0$. Moreover since h is finite on $I\!R$, there exists M that bounds all the values of $|h'_+(v)|$ for v in a neighborhood of the image of the solution u. Hence $H_\epsilon(r) \to H(r)$ pointwise and is dominated by a constant.

Moreover, for every $r \in E$, $g_\epsilon^{**\prime}(r) = \alpha$, for every ϵ sufficiently small. By integrating we obtain that, for every ϵ sufficiently small,

$$\left| \frac{1}{\delta}\int_{r_1}^{r_1+\delta^*} [r^{n-1} g_\epsilon^{**\prime}(r) - H_\epsilon(r)]\chi_E(r)dr - \frac{1}{\delta}\int_{r_1}^{r_1+\delta^*} A(r)dr \right| < \frac{\lambda}{4}$$

$$\left| \frac{1}{\delta^*}\int_{r_2-\delta}^{r_2} [r^{n-1} g_\epsilon^{**\prime}(r) - H_\epsilon(r)]\chi_E(r)dr - \frac{1}{\delta^*}\int_{r_2-\delta}^{r_2} A(r)dr \right| < \frac{\lambda}{4}.$$

Finally, for some positive ϵ,

$$\Delta_\epsilon \leq \epsilon \left\{ \frac{\lambda}{4} + \frac{1}{\delta}\int_{r_1}^{r_1+\delta^*} A(r)dr + \frac{\lambda}{4} - \frac{1}{\delta^*}\int_{r_2-\delta}^{r_2} A(r)dr \right\} =$$

$$= \epsilon \left\{ \frac{\lambda}{2} + \frac{\delta^*}{\delta}\frac{1}{\delta^*}\int_{r_1}^{r_1+\delta^*} A(r)dr - \frac{\delta}{\delta^*}\frac{1}{\delta}\int_{r_2-\delta}^{r_2} A(r)dr \right\} \leq$$

$$\leq \epsilon \left\{ \frac{\lambda}{2} + \frac{1}{\delta^*}\int_{r_1}^{r_1+\delta^*} A(r)dr - \frac{1}{\delta}\int_{r_2-\delta}^{r_2} A(r)dr \right\} \leq$$

$$\leq \epsilon \left\{ \frac{\lambda}{2} + A(r_1) + \frac{\lambda}{4} - A(r_2) + \frac{\lambda}{4} \right\} \leq -\lambda\epsilon < 0$$

i.e. for some positive ϵ, the function $\tilde{u}_\epsilon$ defined by $\tilde{u}_\epsilon(x) = u(|x|) + \epsilon\eta(|x|)$ yields a value for the integral in (P^{**}) less than the value computed at $\tilde{u}$, a contradiction.

Remark. In the case $h \equiv 0$, the assumptions of the above Theorem reduce to lower semicontinuity and growth at infinity for g, as it is to be expected.

Theorem 3.

Let g and h satisfy the same assumption as in Theorem 2; in addition assume that either g^{**} or h is strictly monotonic. Then problem (P) admits one and only one solution.

Proof.

Assume that $u, v : [0,1] \to \mathbb{R}$ are such that the maps $x \to u(|x|)$ and $x \to v(|x|)$ are two distinct solutions to (P). There exists an interval (r_1, r_2) such that: $u(r) > v(r), r \in (r_1, r_2)$; $u(r_2) = v(r_2)$ and, when $r_1 > 0$, $u(r_1) = v(r_1)$. Set w to be $\frac{1}{2}(u+v)$. The map $x \to w(|x|)$ is a further solution to (P^{**}) so that

$$\frac{1}{2}\int_{r_1}^{r_2} r^{n-1}[g^{**}(|v'(r)|) + h(v(r))]dr + \frac{1}{2}\int_{r_1}^{r_2} r^{n-1}[g^{**}(|u'(r)|) + h(u(r))]dr =$$
$$= \int_{r_1}^{r_2} r^{n-1}[g^{**}(|w'(r)|) + h(w(r))]dr.$$

The convexity of both g^{**} and h implies, in particular, that

$$\int_{r_1}^{r_2} r^{n-1} h(w(r))dr = \int_{r_1}^{r_2} r^{n-1} h(u(r))dr.$$

Being h monotonic we infer

$$h(w(r)) = h(u(r)), \; \forall r \in (r_1, r_2). \tag{6}$$

The above is a contradiction to the existence of u and v in the case h is strictly monotonic.

Assume now g^{**} strictly monotonic. Since h is convex there exists at most an interval I on which h is constant and attains its minimum. Set r^* to be $\sup\{r \le 1 : u(r) \in I\}$ and consider the map u^* defined by

$$u^*(x) = \begin{cases} u(r^*) & \text{for } r \le r^* \\ u(r) & \text{for } r > r^*. \end{cases}$$

Then both

$$\int_0^1 r^{n-1} h(u^*(r))dr \le \int_0^1 r^{n-1} h(u(r))dr$$

and

$$\int_0^1 r^{n-1} g^{**}(|u^{*\prime}(r)|)dr \le \int_0^1 r^{n-1} g^{**}(|u'(r)|)dr$$

hold, and the last inequality is strict in the case $|u^{*\prime}|$ differs from $|u'|$ for r on a set of positive measure. Since u is a minimum, $u'(r)$ must be 0 on $(0, r^*)$. From (6) one has that $w(r) \in I$ if and only if $u(r) \in I$. Since w is a solution, the above reasoning implies that also $w'(r) = 0$ on $(0, r^*)$ i.e. $u'(r) = w'(r)$ on $(0, r^*)$. The case $r^* > r_1$ would violate the assumptions on r_1, r_2, u, v. Hence on (r_1, r_2) the inequality $w(r) > u(r)$ implies the inequality $h(w(r)) > h(u(r))$ a contradiction to (6).

Remark. Whenever h is linear, h not zero, uniqueness (besides existence) is guaranteed simply by the lower semicontinuity and growth at infinity of g.

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