**ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES**

# Scaling behaviour of domain walls in a random ferromagnet

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Stefano Lise

SUPERVISOR

Prof. Amos Maritan

October 1996

**SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI**

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# Introduction

Minimazation principles have always played a major role in physics, both for their elegance and for their power as a calculational tool. A large number of problems involve finding the minimum of an "energy" function. For instance in thermodynamics systems at equilibrium minimize a suitable free energy.

Sometimes, however, one is not interested only in the global minimum, but rather on the whole energy landscape. This is particularly true for disordered systems[1], like for instance spin glasses and random field model. In these systems the free energy landscape is very rugged with many local free energy minima competing with the global ground state. This gives rise to a new complicated behaviour, not generally present in ideal non disordered materials. In order to describe it new theorethical tool and concept had to be devised.

Recently[2] in a problem related to the evolution of fluvial network a new concept in the field of complex systems has been introduced, the concept of feasible optimality. The evolution of river network is a problem of complex optimization where one tries to minimize a complicate cost function. It was found that imperfect optimal search procedure yield local minima which reproduce the scaling properties determined experimentally from field data.

On the other hand more refined optimization procedures and exact results give anomalous values for the scaling exponents. This result suggests that natural structures are optima accessible to the dynamics (given the initial conditions), rather than global minima.

In this thesis we wish to address the question of wether local minima with scaling properties exist in other physical systems. We consider a random bond Ising ferromagnet. This model was choosen because it has a sufficiently rich and complex physics behind and very much is known about its equilibrium properties. We indeed find that a class of sub–optimal interfaces can be defined at zero temperature. The interfaces show fractal behaviour and their scaling properties are characterized by a new, robust universality class associated with non–equilibrium states of the system.

We suggest that similar structures can be seen in real experiment on phase separation dynamics. The rest of the thesis is organized as follows. The first part is dedicated to a review and discussion of the equilibrium properties of a class of random systems. In the second part we present results about frozen, non equilibrium states of the system and test on them the idea of feasible optimality.

# 1 Equilibrium properties of a class of disordered systems

In this chapter we review some of the properties of a class of disordered systems. We consider disorder that does not lead to frustration (spin glasses are thus not included). The results we present are well established and can be found in literature [3, 4, 5].

We begin with the problem of a domain wall interface in a disordered ferromagnet. In the special case of dimension two, the interface is a line and the problem is equivalent to a polymer in a random environment. We are then led to study the polymer problem in some details and show that in any dimension it can be mapped into a problem of growing interfaces. The latter can be solved in $(1+1)$ dimensions[1], providing exact informations about the other two problems.

[1] By $(1+1)$ dimensions we mean one spatial and one temporal coordinates

## 1.1 Interfaces in random media

In the following we will discuss the equilibrium properties of a $d$ dimensional interface in a $d+1$ dimensional medium with quenched disorder, i.e. spatially varying couplings that are fixed in time. In particular we will focuse on the scaling properties, like, for example, the dipendence of the perpendicular width $W$ of the interface and fluctuations (over many realizations of the disorder) of the ground state energy on the linear size $L$ of the system. The problem is most easily understood using the language and framework provided by the Ising model. Consider a $(d+1)$ dimensional hypercubic latttice of $N = L^{(d+1)}$ sites. On each site there is an Ising variable $\sigma_i$, which is allowed to take values $\pm 1$. The spins interact according to an Hamiltonian of the form:

$$H = H_{pur} + H_{imp} \tag{1.1}$$

where

$$H_{pure} = -J \sum_{<ij>} \sigma_i \sigma_j \tag{1.2}$$

is the Hamiltonian of a pure, nonrandom ferromagnetic ($J > 0$) Ising model and

$$H_{imp} = \sum_{<ij>} \Delta J_{ij} \sigma_i \sigma_j \tag{1.3}$$

contains the effects due to the random impurities, $\Delta J_{ij}$ being independent randomly distributed variables. The sum in (1.2) and (1.3) is over nearest neighbours pair $< ij >$.
The impurities we consider here are sufficiently weak so that the system will still order fermagnetically at low enough temperatures (this includes, for instance, the case of dilution). By suitably choosing the boundary conditions it is then possible to induce an interface

separating two domains of predominantly "up" and "down" magnetized spins, respectively. Quenched disorder greatly affects the properties of this interface, since it is distorted by the random potential, so that it's shape it's much rougher than in a pure system.

The ground state energy of the system is just the sum of the energies of the "broken bonds"[2] and can thus be thought as the energy of the interface. Clearly the configuration of the interface in the ground state and its energy depend on the particular realization of the disorder. In general there will be fluctuations from sample to sample and one has therefore to consider moments of the distribution higher than the first.

Two quantities define the properties of the ground state. The pinning energy is related to the fluctuations of the ground state energy over different realization of the disorder

$$\Delta E = \left[ < (E_{GS} - < E_{GS} >)^2 > \right]^{\frac{1}{2}} \tag{1.4}$$

where the angular brackets indicate an average over the disorder. If overhangs are ignored the interface can be described by its height $z(\vec{x})$ from some reference plane, as a function of the remaining $d$ coordinates $\vec{x}$. The roughness of the interface is then defined by the root–mean–square fluctuation in the height

$$W = \left[ \overline{(z - \overline{z})^2} \right]^{\frac{1}{2}} \tag{1.5}$$

where the overbar indicates sample average, that is averages taken on a specific realization of disorder. The roughness is, in fact, a self–averaging quantity: the average over just one realization of disorder is, in the thermodinamic limit, equal to the average over many

[2] Broken bonds are bonds for which $\text{sign}(J_{ij}\sigma_i\sigma_j) = -1$

realization

$$W =< W > \tag{1.6}$$

Contrary to the pure case both the energy fluctuations and the roughness have a non trivial scaling behaviour, signaling that we are dealing with a critical phenomenon. They are bielived to scale according to power laws

$$W \sim L^{\zeta} \quad \Delta E \sim L^{\chi} \tag{1.7}$$

where L is size of the system. These relations define the two critical exponent, $\zeta$ and $\chi$, and the universality class of the problem. As usual when dealing with critical phenomena, the exponents turn out to be highly universal. They do not depend on the details of the realiztions of $H_{imp}$ as long as the disorder has only short-range correlations and is not so strong as to destroy the ferromagnetic ordering at low temperature.

To gain some understanding of the problem we first consider a continuum model of a domain wall in absence of disorder. In a uniform ferromagnet the interface separating domains can be described by the Hamiltonian of a free elastic surface

$$H_c = \int d^d\vec{x}\frac{1}{2}\sigma|\nabla z|^2 \tag{1.8}$$

where $\sigma$ is the effective domain–wall stiffness. Note that the system is invariant under translation of the position of the interface, since it makes no difference where we define $z = 0$.

At nonzero temperature the domain wall is rough for $d \leq 2$ because of thermal fluctuations. In fact the height-difference correlation function $G(\vec{x}_1 - \vec{x}_2)$ is given, in the limit

$|\vec{x}_1 - \vec{x}_2| \to \infty$, by

$$G(\vec{x}_1 - \vec{x}_2) = < [z(\vec{x}_1) - z(\vec{x}_2)]^2 >_T \propto \begin{cases} |\vec{x}_1 - \vec{x}_2|^{2-d} & \text{for } d < 2 \\ \ln |\vec{x}_1 - \vec{x}_2| & \text{for } d = 2 \\ \text{constant} & \text{for } d > 2 \end{cases} \tag{1.9}$$

where $<>_T$ denote a thermal avarage. It's possible to obtain the relations (1.9) through dimensional analysis. In fact, given that $z \sim x^\alpha$ and $|\nabla z|^2 \sim x^{2(\alpha-1)}$, it follows that

$$H_c \sim x^d x^{2(\alpha-1)} \tag{1.10}$$

Since the system is "pure" we don't expect any fluctuations in the energy,which implies

$$d + 2\alpha - 2 = 0 \tag{1.11}$$

or

$$\alpha = \frac{2-d}{2} \tag{1.12}$$

If $d > 2$ (1.12) would predict $\alpha < 0$. However by definition $\alpha \geq 0$ and thus $\alpha = \max(\frac{2-d}{2}, 0)$. This is equivalent to say that the Hamiltonian (1.8) is a fixed point of a renormalization–group rescaling under which the coordinate $\vec{x}$ and $z$ rescale to

$$\vec{x}' = \vec{x}/b \quad z' = z/b^\alpha \tag{1.13}$$

In fact

$$H_c = \int d^d\vec{x} \frac{1}{2}\sigma|\nabla z|^2 = \int d^d\vec{x}' \frac{1}{2}\sigma|\nabla z'|^2 = H_c' \tag{1.14}$$

The presence of impurities makes the analysis of the system radically different. A random potential, which depends on the local impurity density, must be added to (1.8)

$$H_c = \int d^d\vec{x} \left[\frac{1}{2}\sigma|\nabla z|^2 + V(\vec{x}, z(\vec{x}))\right] \tag{1.15}$$

where $V(\vec{x}, z)$ is the local domain-wall energy and it has only short-range correlations

$$< V(\vec{x_1}, z_1)V(\vec{x_2}, z_2) >= \Delta\delta(\vec{x_1} - \vec{x_2})\delta(z_1 - z_2) \tag{1.16}$$

The impurities break the translational symmetry of the system and will tend to pin the domain wall in certain favorable locations where the exchange couplings are weaker. Moreover they alter abrubtly the scaling behaviour of the domain wall for $\frac{2}{3} < d < 4$. Indeed by looking at the long-distance behaviour of the domain wall, through the following rescaling

$$\vec{x} = b\vec{x}' \tag{1.17}$$

$$z = b^{\alpha}z' \tag{1.18}$$

where $b \gg 1$. The gradient square term will scale accordingly to

$$|\nabla z|^2 = b^{2(\alpha-1)}|\nabla z'|^2 \tag{1.19}$$

and the potential to[3]

$$V = b^{-\frac{d+\alpha}{2}}V' \tag{1.20}$$

Comparing (1.19) and (1.20) we see the potential is relevant if

$$b^{-\frac{d+\alpha}{2}} >> b^{2(\alpha-1)} \tag{1.21}$$

or, equivalently

$$4 - d > 5\alpha \tag{1.22}$$

[3]This relation comes from $< V(l\vec{x_1}, l^{\alpha}z_1)V(l\vec{x_2}, l^{\alpha}z_2) >= \Delta\delta^{(d)}(l\vec{x_1} - l\vec{x_2})\delta(l^{\alpha}z_1 - l^{\alpha}z_2) = \Delta l^{-d}l^{-\alpha}\delta^{(d)}(\vec{x_1} - \vec{x_2})\delta(z_1 - z_2)$

If we start close to the fixed point (1.8) we can use as zeroth approximation for $\alpha$ the value

$$\alpha = \max(\frac{2-d}{2}, 0) \tag{1.23}$$

which gives

$$d < 4 \tag{1.24}$$

$$d > \frac{2}{3} \tag{1.25}$$

For these values of $d$, the potential become stronger under iterative rescaling; in other words the effective potential becomes important in the large scale limit so we are not allowed to treat it as a small perturbation and, generally, one expect a completely different scaling behaviour from the "free" case.

The estimate of the exponent $\zeta$ and $\chi$ relies mostly on numerical simulations. Only for $d = 1$ is possible to give an analytical solution. In this case the exponent are exactly $\zeta = \frac{2}{3}$ and $\chi = \frac{1}{3}$, as was shown by Huse, Henely and Fisher[4].

In higher dimensions the most accurate numerical work has been done by Middleton[6]. The values he founds for the exponents are $\zeta = 0.41 \pm 0.01, 0.22 \pm 0.01$ and $\chi = 0.84 \pm 0.03, 1.45 \pm 0.04$ in $d = 2$ and $d = 3$, respectively.

As in standard critical phenomena it is possible to relate these exponents through a scaling relation. Consider for semplicity the $d = 1$ case. Let $E_0(L, z)$ be the ground state energy of a domain wall running from $(0, 0)$ to $(L, z)$ and $< E_0(L, z) >$ be its average over the disorder. By assuming the domain wall energy per unit lenght is an analytical function

of $z/L$ and expanding around zero, we obtain

$$< E_0(L,z) >=< E_0(L,0) > + \frac{1}{2} \left. \frac{\partial^2 \frac{<E_0(L,0)>}{L}}{\partial(\frac{z}{L})^2} \right|_{\frac{z}{L}=0} \frac{z^2}{L} + O(\frac{z^4}{L^3}) \qquad (1.26)$$

Since we are dealing with fluctuations of the energy, one can reasonably suspect that they are controlled by the same exponents $\chi$ as in (1.7) and write for large $L$

$$< E_0(L,z) > - < E_0(L,0) >= L^{\chi} f(\frac{z}{L^{\zeta}}) \qquad (1.27)$$

where $f(x)$ is a scaling function. Taking $z$ of the order of $L^{\zeta}$ and[4] equating (1.26) and (1.27) yields $2\zeta - \chi = 1$. In higher dimension this relation generalizes to $2\zeta - \chi = 2 - d$, a relation which is certainly consistent with the analytical solution in $d = 1$ and with numerical foundings.

It is possible to consider the exponents $\zeta$ and $\chi$ also at nonzero temperature. In this case the roughness of the interface is calculated by averaging over all the allowed configurations, with their corresponding Boltzmann weight. The fluctuations in the ground state energy are instead substituted by fluctuations of the free energy.
Numerical simulations in $d = 1$ done by Kardar[5] show that indeed the same exponents $\zeta$ and $\chi$ also apply at $T > 0$.

## 1.2 Polymers in a disordered environment

The problem of polymers in a random environment can be stated as follows. Consider for semplicity a 2 dimensional discrete lattice, whose horizontal axis is the "time" $t$ and vertical

[4] This implies $\zeta < 1$ in order for the expansion (1.26) to make sense

axis is $x$. The polymer is modelized as a self-avoiding walk (SAW) which starts at $t = 0$ and $x = 0$ and moves in discrete steps, up to a maximum horizontal displacement $t$. To each bond $< i, j >$ of the lattice is associated a random energy $E_{ij}$ taken from a probability distribution with variance $\sigma$. The total energy of the polymer is the sum of the energies along the polymer lenght. Accordingly the energy of a path $\Gamma$ is

$$H_\Gamma = \sum_{bonds \in \Gamma} E_{i,j} \tag{1.28}$$

and the partition function of the model is

$$Z(\beta) = \sum_\Gamma e^{-\beta H_\Gamma} \tag{1.29}$$

In the absence of disorder ($\sigma = 0$) the ground state of the polymer is a straight line of $t$ steps. The introduction of a weak disorder ($\sigma << 1$) makes the polymer wander in order to take advantage of the randomly distributed low energy bonds. Nonetheless overhangs in the path are very rare, since they cost two much energy and they are likely to be absent on long lenght scale. The partition function is consequently dominated by directed paths, which are referred to as directed polymers

$$Z(\beta) \simeq \sum_{dir.paths} e^{-\beta H(\Gamma)} \tag{1.30}$$

The quantities of interest are defined in a similar way to the domain wall problem. The fluctuation in the ground state energy is $\Delta E = [< E_{GS}^2 > - < E_{GS} >^2]^{\frac{1}{2}}$ and scales as $t^\chi$. The transverse displacement of the polymer from its average position is $W = \left[\overline{\left(\vec{x} - \overline{\vec{x}}\right)^2}\right]^{\frac{1}{2}}$ and scales with system size as $t^\zeta$. The directed polymer problem and the domain wall problem are equivalent for $d = 2$. In higher dimesion this is not true. In the domain wall

case, we have a $d$–dimensional interface separating two $d+1$–dimensional domains. The higher dimensional generalization of the directed polymers problem, instead, deals with a one–dimensional line following a directed path in a $d+1$–dimensional space.

It is convenient to turn to a continuum description of the directed polymer. In this case the polymer is described by a single-valued function, $\vec{x}(t)$. Let $\epsilon(\vec{x},t)$ be an energy density per unit lenght. The energy of the polymer can then be written as

$$H_\Gamma = \int_0^t d\tau \epsilon(\vec{x}(\tau),\tau)\sqrt{1+\left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|^2} \tag{1.31}$$

where $\sqrt{1+\left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|^2}$ is the infinitesimal arc lenght of the curve $\Gamma$.

The random energy density can be expresed as

$$\epsilon(\vec{x},t) = \epsilon_0 + \eta(\vec{x},t) \tag{1.32}$$

where $\epsilon_0$ is constant and $\eta(\vec{x},t)$ is the fluctuating part. Expanding the lenght element in power of $\left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|$ and stopping to the first non zero term, we get

$$\sqrt{1+\left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|^2} \sim 1+\frac{1}{2}\left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|^2 + \ldots \tag{1.33}$$

Combining (1.32) and (1.33) the expresssion (1.31) for the energy can be approximated by

$$H_\Gamma = \epsilon_0 t + \frac{\epsilon_0}{2}\int_0^t d\tau \left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|^2 + \int_0^t d\tau \eta(\vec{x}(\tau),\tau) + \int_0^t d\tau \eta(\vec{x}(\tau),\tau)\left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|^2 + \ldots \tag{1.34}$$

The first term on the *r.h.s.* of (1.34) is just a constant and ensure that the energy is an extensive quantity. It does not affect the scaling properties we are interested in and can be neglected without loss of generality. The remaining terms account for the fluctuation around the mean value of the energy. It is possible to show that the last term is actually

subleading on long lenght scale and can henceforth be droped. In fact the transverse width of the polymer scale as $t^{\zeta}$ and $\left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right| \sim t^{\zeta-1}$ implying that

$$\eta\left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|^2 \sim t^{-\frac{d}{2}\zeta-\frac{1}{2}+2(\zeta-1)} << \left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|^2 \tag{1.35}$$

Finally, equation (1.34) reduces to

$$H_{\Gamma} = \frac{\epsilon_0}{2}\int_0^t d\tau \left[(\frac{\partial \vec{x}(\tau)}{\partial \tau})^2 + \eta(\vec{x}(\tau),\tau)\right] \tag{1.36}$$

A scaling relation can at this point be deduced. The fluctuations of the energy should scale as $t^{\chi}$. On the other hand the term $\left|\frac{\partial \vec{x}(\tau)}{\partial \tau}\right|$ scales as $t^{\zeta-1}$, and the overall term in the integral in (1.36) scales as $t^{2\zeta-1}$. Therefore the exponents are related by the relation

$$\chi = 2\zeta - 1 \tag{1.37}$$

Polymers in a disordered environment are formally equivalent to a problem of growing interfaces. To establish this connection we consider the total weight $Z(\vec{x},t)$ of paths connecting the origin to $(\vec{x},t)$, given by

$$Z(\vec{x},t) = \int_{(0,0)}^{(\vec{x},t)} D[\vec{x}(\tau)]e^{-\beta H(\vec{x}(\tau)} \tag{1.38}$$

where the symbol $D[\vec{x}(\tau)]$ stands for an integral over all the paths $\vec{x}(\tau)$. This path integral representation has a corresponding Schroedinger equation, given by

$$\frac{\partial Z(\vec{x},t)}{\partial t} = \frac{1}{2\epsilon_0\beta}\nabla^2 Z(\vec{x},t) + \beta\eta(\vec{x},t)Z(\vec{x},t) \tag{1.39}$$

By performing the change of variable

$$F(\vec{x},t) = -\frac{1}{\beta}\ln Z(\vec{x},t) \tag{1.40}$$

equation (1.39) becomes

$$\frac{\partial F(\vec{x},t)}{\partial t} = \frac{1}{2\epsilon_0\beta}\nabla^2 F(\vec{x},t) - \frac{1}{2\epsilon_0}(\nabla F(\vec{x},t))^2 - \eta(\vec{x},t) \tag{1.41}$$

This equation is now in the form of the Kardar-Parisi-Zhang equation, an equation that describes the growth of an interface. Many interesting properties of (1.41) can be deduced. Among them the exact solution for $d = 1$. This will be the subject of next paragraph.

## 1.3 The Kardar–Parisi–Zhang equation

The Kardar–Parisi–Zhang (KPZ) equation is a continuum stochastic equation, which describes the growth of a surface under the influence of an external flux of particle. In terms of the coarse–grained surface height $h(\vec{x},t)$ as a function of time $t$ and substrate coordinates $\vec{x}$, the equation takes the form

$$\frac{\partial h(\vec{x},t)}{\partial t} = \nu\nabla^2 h(\vec{x},t) + \frac{\lambda}{2}(\nabla h(\vec{x},t))^2 + \eta(\vec{x},t) \tag{1.42}$$

This is exactly the same as eq. (1.41) with identification of $F = -h$, $\nu = \frac{1}{2\beta\epsilon_0}$ and $\lambda = \frac{1}{\epsilon_0}$. The noise $\eta(\vec{x},t)$ satisfies

$$\langle \eta(\vec{x},t) \rangle = 0 \tag{1.43}$$

and

$$\langle \eta(\vec{x},t)\eta(\vec{x}',t') \rangle = 2D\delta(\vec{x}-\vec{x}')\delta(t-t') \tag{1.44}$$

As before, $\langle\ \rangle$ denotes average over the noise distribution.

We want to understand how the roughness of the interface $W$ depends on time. Suppose at time zero the interface is flat ($W = 0$) and that subsequently it evolves according to (1.42).

Initially there is a transient regime ($t \ll t_{sat}$), in which the width increases as a power of time

$$W(L,t) \sim t^{\beta} \quad t \ll t_{sat} \tag{1.45}$$

The exponent $\beta$ is called the growth exponent. After the transient regime ($t \gg t_{sat}$) the width of the interface reaches a saturation value. This saturation width depends on the linear size of the system, $L$, through a power law

$$W_{sat}(L) \sim L^{\alpha} \quad t \gg t_{sat} \tag{1.46}$$

The exponent $\alpha$ is called the roughness exponent. The saturation time $t_{sat}$ itself depend on the system size as

$$t_{sat} \sim L^{z} \tag{1.47}$$

where $z$ is called the dynamic exponent.

The three relation (1.45), (1.46) and (1.47) can be groupped together by the scaling relation

$$W(L,t) \sim L^{\alpha} f(\frac{t}{L^{z}}) \tag{1.48}$$

where $f(x)$ is a scaling function with the following properties

$$f(x) = \begin{cases} x^{\beta} & \text{if } x \ll 1 \\ \text{const} & \text{if } x \gg 1 \end{cases} \tag{1.49}$$

The scaling form (1.48) of the roughness implies that $\alpha$, $\beta$ and $z$ satisfies the relation

$$\alpha = \beta z \tag{1.50}$$

We can now understand how the exponents in the KPZ equation are connected to the exponents in the directed polymer problem. The height $h(\vec{x},t)$ in a point $\vec{x}$ at the time $t$

in the KPZ equation is linked to the free energy of a path connecting the origin to $(\vec{x}, t)$. The roughness of the interface scale with time as $t^{\beta}$. The fluctuation of the free energy should scale as $t^{\chi}$, hence $\beta = \chi$. The time for the interface to build up correlations up to a distance $L$ is $t \sim L^{z}$. Correspondingly the time needed to a polymer to have a transverse displacement $L$ is $t = L^{\frac{1}{\zeta}}$. We can therefore conclude $z = \frac{1}{\zeta}$. Consequently relation (1.37) becomes

$$\alpha + z = 2 \tag{1.51}$$

To study the properties of (1.42), we can carry out a scaling analysis. Under the scale transformation $\vec{x} \to l\vec{x}$, $t \to l^{z}t$, and $h \to l^{\alpha}h$, equation (1.42) transforms to

$$l^{\alpha - z}\frac{\partial h(\vec{x}, t)}{\partial t} = \nu l^{\alpha-2}\nabla^2 h(\vec{x}, t) + \frac{\lambda}{2}l^{2\alpha-2}(\nabla h(\vec{x}, t))^2 + l^{\frac{-d-z\alpha}{2}}\eta(\vec{x}, t) \tag{1.52}$$

Under such scaling the parameter of eq. (1.42) are transformed to

$$\begin{aligned} \nu &\to l^{z-2}\nu \\ \lambda &\to l^{\alpha+z-2}\lambda \\ D &\to l^{z-2\alpha-d}D \end{aligned} \tag{1.53}$$

In the absence of the non linear term ($\lambda = 0$) eq. (1.42) reduces to the so called Edwards-Wilkinson equation. This is a linear equation which can be solved and whose exponents are known exactly. They can be also inferred from (1.53), since, for $\lambda = 0$ the equation is made scale invariant upon the choice of $z_0 = 2$, and $\alpha_0 = (2 - d)/2$. Close to this linear fixed point, $\lambda$ scales to $l^{z_0+\alpha_0-2}\lambda = l^{(2-d)/2}\lambda$. Since the non–linearity grows larger under scaling it cannot be ignored in dimensions $d < 2$.

A perturbative dynamic renormalization group analysis suggests that in $d < 2$ even a small

$\lambda$ leads to a scaling behaviour different from $\lambda = 0$. On the contrary for $d > 2$ there exists a $\lambda_c$ such that when $\lambda < \lambda_c$ the scaling behaviour is like $\lambda = 0$ (and this is consistent with the heuristic analysis above) and when $\lambda > \lambda_c$ the non-linearity leads to different critical behaviour.

In the special case $d = 1$ the exponents of the KPZ equation can be determined exactly. The Fokker–Plank equation associated with (1.42) is

$$\frac{\partial P([h(\vec{x})],t)}{\partial t} = -\int d\vec{x} \frac{\delta}{\delta h(\vec{x})} \left[ \left( \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 \right) P - D \frac{\delta P}{\delta h(\vec{x})} \right] \tag{1.54}$$

where $P([h(\vec{x})],t)$ is the probability of a surface configuration $h(\vec{x})$ at time $t$. The above equation admits a time–independent (stationary) solution for $d = 1$; specifically

$$P_o[h(\vec{x})] \propto \exp\left[ -\frac{\nu}{2D} \int d\vec{x} (\nabla h)^2 \right] \tag{1.55}$$

This stationary distribution is also the distribution of a free elastic surface at a temperature proportional to $D/\nu$. From (1.9) we subsequently deduce the roughness exponent, $\alpha = \frac{1}{2}$. The other two exponents are provided by scaling relations (1.50) and (1.51), $z = \frac{3}{2}$ and $\beta = \frac{1}{3}$.

Going back to the polymer problem we obtaine

$$\zeta = \frac{1}{z} = \frac{2}{3} \quad \lambda = \beta = \frac{1}{3} \tag{1.56}$$

# 2 Feasible optimality in a 2–D random ferromagnet

In the previous chapter we investigated the scaling properties of a domain wall in a 2 dimensional random ferromagnet. In particular we deduced that the roughness exponent in the ground state is 2/3.

In this chapter we want to address the question of wether local minima, rather than the absolute minimum, have themselves scaling properties. Indeed we will find that local minima have scaling properties but are completely different from that determined in the ground state. This is a paradigm of the concept of feasible optimality.

## 2.1 Scaling properties of sub–optimal interfaces

Consider a 2D random exchange Ising ferromagnet with Hamiltonian

$$H = \sum_{\langle ij \rangle} J_{ij}\sigma_i\sigma_j \tag{2.1}$$

where the sum $\langle ij \rangle$ is over nearest neighbors and the $J_{ij}$ are random quenched variable, taken, for instance, from a uniform distribution between 0 and 1. Anti–periodic boundary

conditions are imposed in one direction so that the top line of spins is fixed to $+1$ and the bottom line to $-1$. Free boundary conditions are instead set in the other direction. The ground state of the system results in a self–affine interface, whose roughness exponent is $2/3$.

Suppose now we are at infinite temperature. The spins are therefore completely uncorrelated. Half of them will be pointing up and the other half down. If the system is cooled slowly enough. it should be able in principle to relax to the ground state at $T = 0$. On the contrary if the system undergoes an instantaneous quenching to $T = 0$ it is likely to get trapped in a local minimum configuration, rather than reaching the absolute minimum. In the following we will study the latter situation and we will try to understand wether the system retains some scaling properties, carachterized by some universal exponent.

To simulate the instantaneous quenching we have implemented a Monte Carlo algorithm. The initial configuration is a random spin configuration. The system then evolves according to zero temperature Glauber dynamics. Every Monte Carlo time step (MCS) we attempt to flip all single spins. The order of the proposed moves is choosen randomly each MCS and we flip a spin only if it lowers the total energy. This procedure is repeated until no further spin flips are possible. The resulting state corresponds to a local energy minimum, accessible via the dynamics from the given initial condition.

A typical final configuration is shown in figure 2.1. It is possible to single out two large cluster of spin up and down respectively. An up (down) spin belongs to the "white" ("black") cluster if it is connected to the upper (lower) boundary. The two domains in

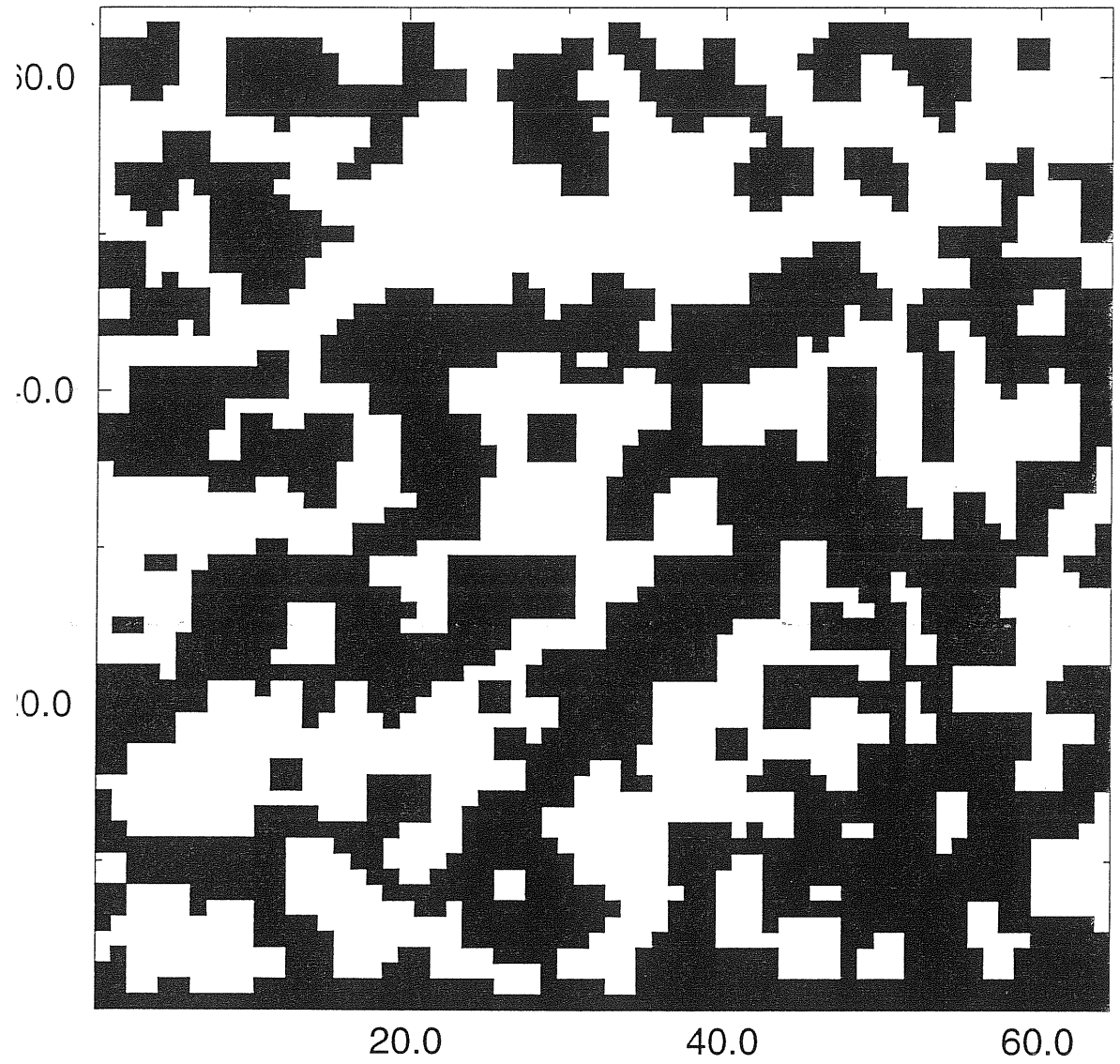


Figure 2.1: A typical non–equilibrium spin configuration reached by the dynamics. The black and white regions denote regions of opposite magnetization

general intersect along a line, defining in this way an interface. The interface corresponding to the spin configuration shown in figure is plotted in figure 2.2. It might happen from time to time that the two domains do not define precisely a single interface. This is very rare, but it's still possible to give a criterion on how to choose an interface[1]. However this is not an essential point, since any selection procedure for distinguishing between degenerate interfaces does not change the scaling properties of the system. The interfaces have many overhanging segments on all lenght scales and are thus not self affine. This new feature is in contrast to the self affine interfaces that would result if the true ground state had been reached. Furthermore, we have verified that a uniform ferromagnet with a dynamics which allow zero energy–change spin flips, the system relaxes to the true ground state with a single, flat interface. Thus, the novel scaling properties are an intrinsic feature of the local energy minima present in disordered systems that are accessible to the dynamics from the random initial condition.

The systems we have studied are Ising models on square lattices of size $L \times L$, with $L$ ranging from 16 to 256. We have performed quenching on $10^4$ realizations of the disorder and measured on each of them the interfacial lenght $I$. The average $\langle I \rangle$ as a function of system size is reported in figure 2.3. It scales with system size as $L^{d_f}$, with $d_f \sim 1.6$. This indicates the interface is self–similar with a fractal dimension given by $d_f$. This behaviour is the signature of a new universality class for interfaces in disordered systems, being distinct from the ground state.

[1] For instance we could take the one with shortest lenght or with minimum energy

Moreover we have measured the full distribution function of $I$ for different system sizes, $P(I, L)$. We find that $P(I, L)$ has a scaling form

$$P(I, L) = \frac{1}{\langle I \rangle} f(\frac{I}{\langle I \rangle}) \tag{2.2}$$

as shown in figure 2.4.

The fractal dimension $d_f = 1.6$ is robust and independent of the lattice structure and the discrete nature of the spin variables.The same result is obtained by adopting either a single or two spin flip dynamics, by repeting the analysis for an Ising model on a triangular lattice or for a continuum Langevin equation.

## 2.2 Conclusions

The fact that interfaces frozen in non equilibrium configurations possess well defined scaling properties is quite remarkable. The infinite temperature state in fact has no correlations and the quenched state is reached only through local spin flip. Feasible optimality refers exactly to the scaling properties of local minima which are accessible through the dynamics. These scaling properties form a new universality class and are independent of the true ground state properties of the system. This result suggest that the concept of feasible optimality maybe apllicable to a wide range of physical systems in which scaling behaviour arises from optimazation processes. Many of the fractal structures and diversity observed in nature might not be true equilibrium properties.

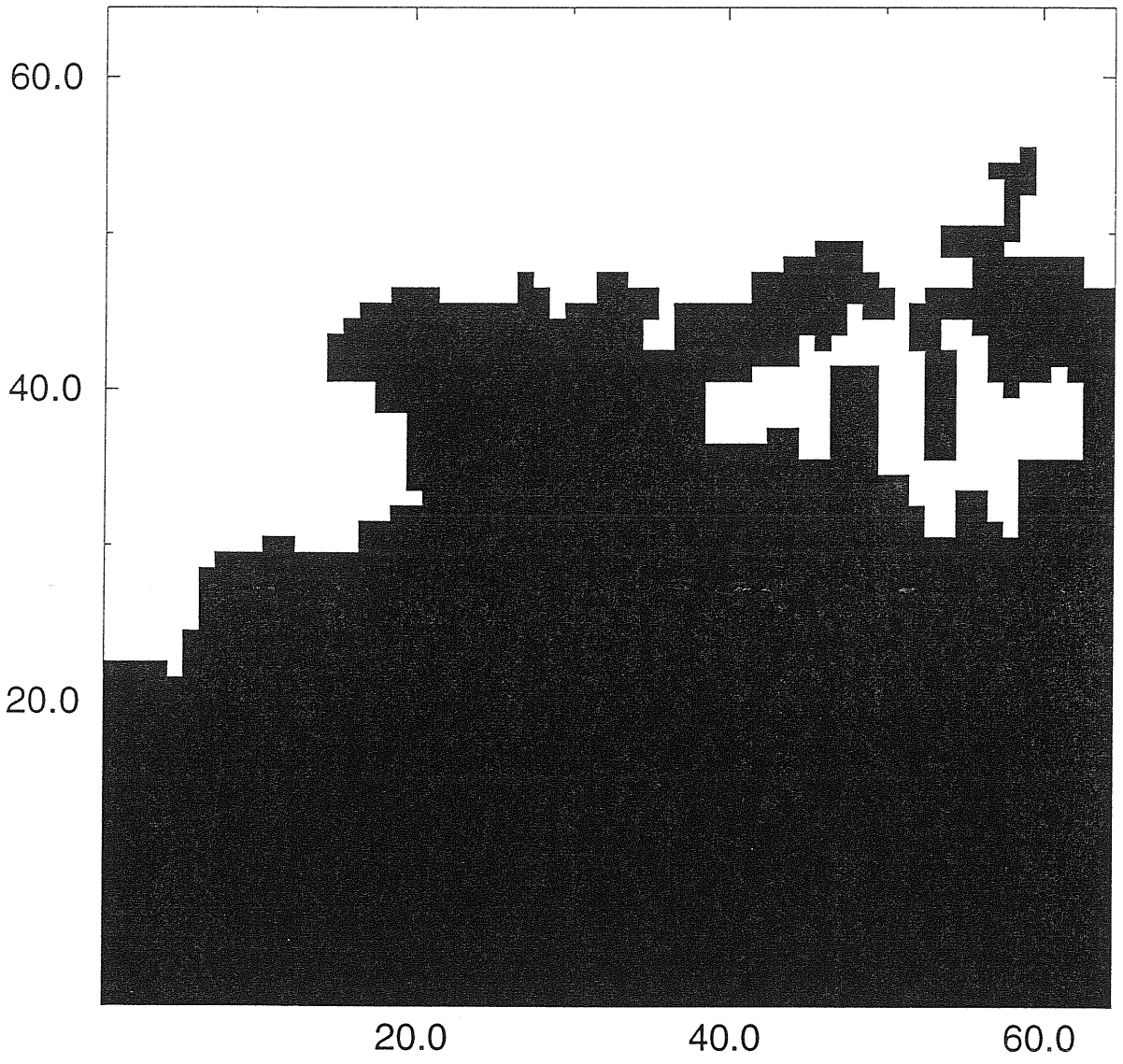


Figure 2.2: The interfacial configuration that results from figure 1

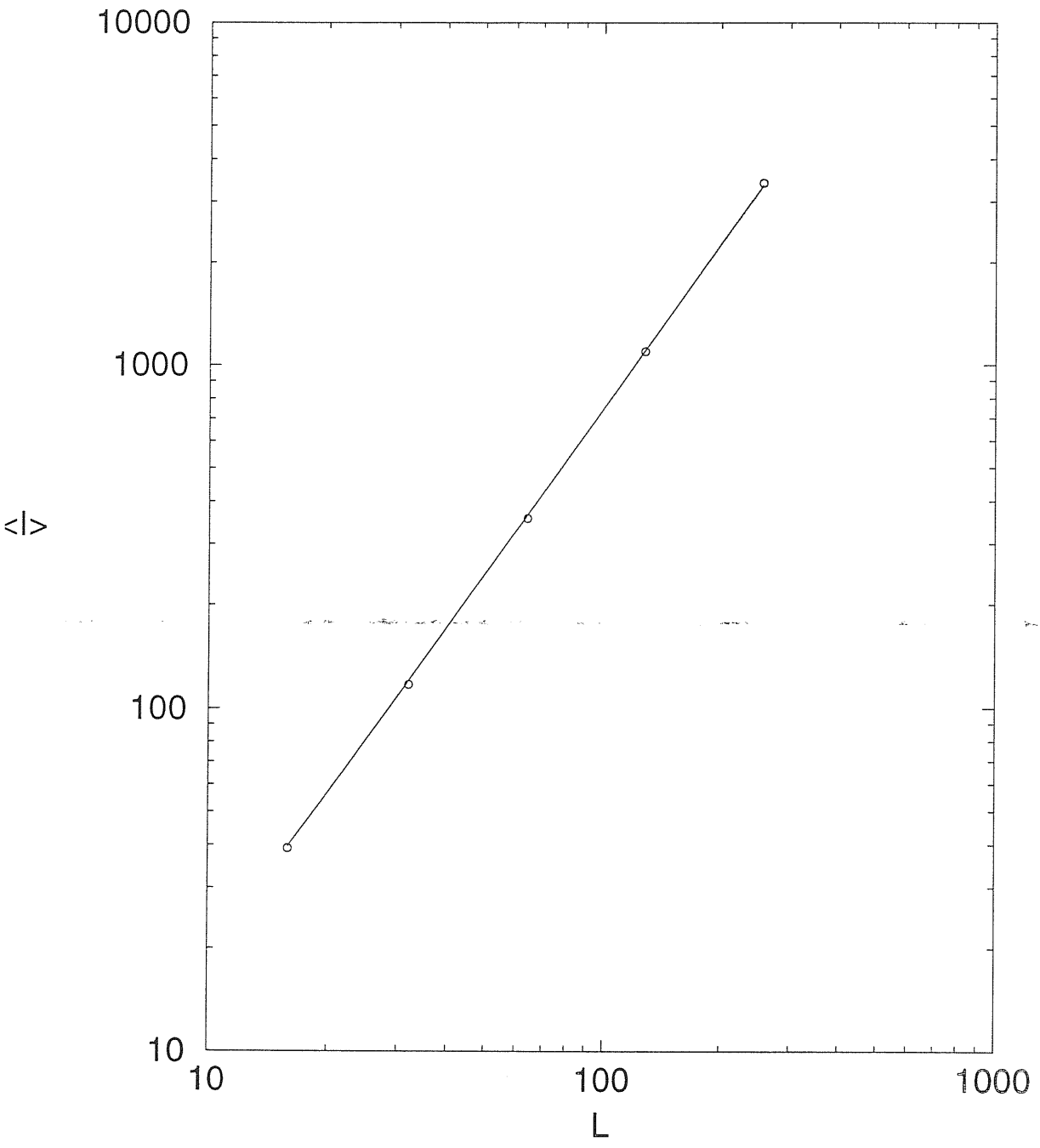


Figure 2.3: Scaling plots of the average interfacial lenght $< I >$ for increasing system size $L$. The line has slope 1.6

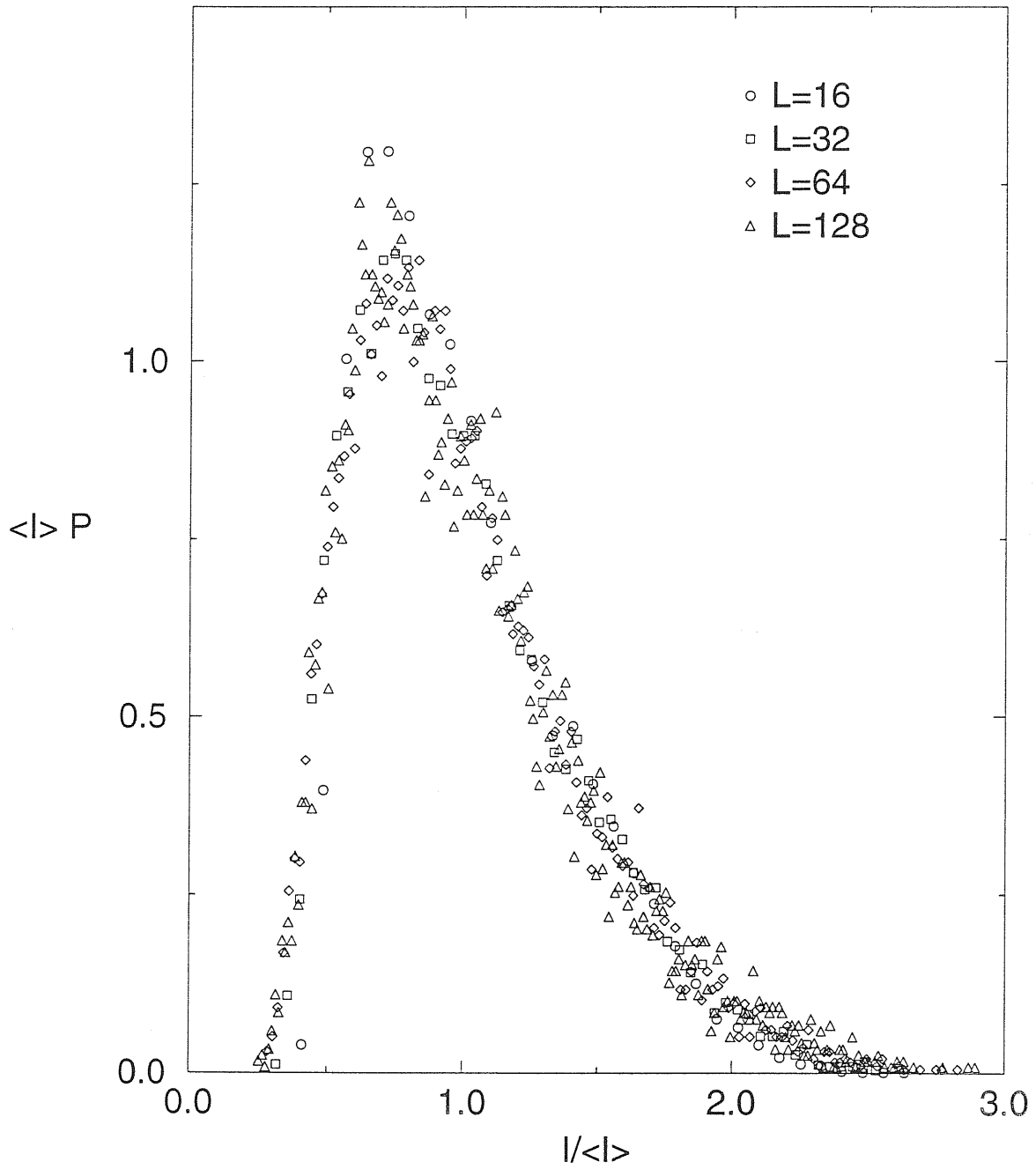


Figure 2.4: The probability distribution $P(I, L)$ showing the scaling form discussed in the text. System sizes $L = 16$, 32, 64 and 128 are shown

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