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ON THE STATIONARY COMPRESSIBLE NAVIER-STOKES EQUATIONS. THE INCOMPRESSIBLE LIMIT.

CANDIDATE:
Anneliese DEFRANCESCHI

SUPERVISOR:
Prof. Hugo BEIRÃO DA VEIGA

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INDEX

1. INTRODUCTION

We intend to present the study of some topics related to the existence of solutions for the system

$$(1.1.1)\qquad \begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + \nabla p(\varrho,\zeta) = \varrho[f - (u\cdot\nabla)u]\,, \\ \operatorname{div}(\varrho u) = g \quad , \\ -\chi\Delta\zeta + c_v\varrho u\cdot\nabla\zeta + \zeta p'_\zeta(\varrho,\zeta)\operatorname{div} u = \varrho h + \Psi(u,u)\,, \text{ in }\Omega, \\ u_{|\Gamma} = 0\,, \quad \zeta_{|\Gamma} = \zeta_0 \quad , \end{cases}$$

which describes the stationary motion of a compressible, heat conducting, viscous fluid in a domain Ω of $\mathbb{R}^n$. Moreover we discuss an interesting problem involving the solutions of the stationary, compressible Navier–Stokes equations in the barotropic case: more precisely, we study the behavior of the solutions as the reciprocal of the Mach number becomes large (in this case $p = p(\varrho)$ and system (1.1.1.) is reduced to a simpler form as we shall see in chapter 5; we consider $g = 0$).

The system (1.1.1) is, in the physically interesting case $g = 0$, a particular case of $(n+2)x(n+2)$ conservation laws endowed with suitable boundary conditions (see appendix, section B).

If not otherwise specified, we assume here that Ω is a bounded open subset of $\mathbb{R}^n$, $n \geqslant 2$, and lies (locally) on one side of its boundary Γ , a C^{j+3} manifold. We indicate

$$(1.1.2)\qquad \Psi(u,u) = \chi_0 \sum_{i,j=1}^{n} (D_i u_j + D_j u_i)^2 + \chi_1(\operatorname{div} u)^2\,.$$

We denote by $u(x) = (u_1(x),\dots,u_n(x))$ the fluid velocity. $\varrho(x)$ is the density of the fluid, and since the total mass m of the fluid is given we impose the condition

$$(1.1.3)\qquad \frac{1}{|\Omega|}\int_\Omega \varrho(x)\,dx = m\,, \text{ or equivalently } \quad \frac{1}{|\Omega|}\int_\Omega \sigma(x)\,dx = 0\,,$$

where $m > 0$ and $\sigma(x)$ is defined by setting $\varrho(x) = m + \sigma(x)$.

We indicate by $\zeta(x)$ the absolute temperature. $u(x)$, $\rho(x)$ and $\zeta(x)$ are the unknowns of (1.1.1).

From a physical point of view it is not a strict assumption to impose that the coefficients $\mu > 0$, $\nu > -\mu$, c_v, $\chi_0, \chi_1, \chi > 0$ are constant. Therefore we assume also that $\zeta_0 > 0$ is constant, although a dependence of those coefficients on u, ρ, ζ would not imply substantial difficulties (the physical meaning of this coefficients is discussed in the appendix, section B).

$f(x)$ and $h(x)$ are the assigned external force field and heat sources per unit mass. Through thermodynamic considerations the scalar pressure p becomes a well-defined function of the density and the absolute temperature, i.e. $p = p(\rho, \zeta)$. We assume $p(\rho, \zeta) \in C^{j+3}([m-l, m+l] \times [\zeta_0 - l_1, \zeta_0 + l_1])$, where $0 < l \leqslant m/2$, $0 < l_1 \leqslant \zeta_0/2$. Under those hypotheses and by setting $\zeta = \zeta_0 + \alpha$, we can write

$$
(1.1.4) \qquad \begin{cases} p'_\rho(m+\sigma, \zeta_0+\alpha) = k + \omega_1(\sigma,\alpha) \\[2ex] p'_\zeta(m+\sigma, \zeta_0+\alpha) = \omega_2(\sigma,\alpha) \end{cases}
$$

where $k = p'_\rho(m, \zeta_0)$, $\omega_1(0,0) = 0$, and ω_1, ω_2 are in the space $C^{j+2}(I(l,l_1))$ with $I(l,l_1) = [-l,l] \times [-l_1, l_1]$. We assume that $k > 0$ ($k \neq 0$ would be sufficient here).

This assumptions yield the following equivalent system for (1.1.1)

$$
(1.1.5) \qquad \begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + k\nabla\sigma = F(f,u,\sigma,\alpha), \\ m \operatorname{div} u + u\cdot\nabla\sigma + \sigma \operatorname{div} u = g, \\ -\chi\Delta\alpha = H(h,u,\sigma,\alpha), \quad \text{in } \Omega, \\ u_{|\Gamma} = 0, \qquad \alpha_{|\Gamma} = 0 \end{cases}
$$

where by definition

$$
(1.1.6) \begin{cases} F(f,u,\sigma,\alpha) = (\sigma+m)[f - (u\cdot\nabla)u] - \omega_1(\sigma,\alpha)\nabla\sigma - \omega_2(\sigma,\alpha)\nabla\alpha, \\[2ex] H(h,u,\sigma,\alpha) = (\sigma+m)h - c_v(m+\sigma)u\cdot\nabla\alpha + \Psi(u,u) + \dfrac{\zeta_0+\alpha}{m+\sigma}\omega_2(\sigma,\alpha)(u\cdot\nabla\sigma - g). \end{cases}
$$

In the first part of this work we focus on certain topics related to system (1.1.1) (more precisely, to the equivalent system (1.1.5)) and present some tools needed to treat them.

Chapter 2 is concerned with an existence and regularity result for the non–homogeneous Stokes problem. We establish the smoothness of the solution of this problem applying the paper of Agmon–Douglis–Nirenberg [2] (see appendix, section A) where a priori estimates of solutions of general elliptic systems are given. Our interest in this problem is motivated by the necessity to solve the system (4.1.16) in chapter 4.

We present in chapter 3 a work of H. Beirão da Veiga [6] regarding the existence of solutions in the Sobolev spaces $W^{j,p}$, $j \geqslant -1$ for the stationary transport equations (TE). We consider the system of first order partial differential equations

$$\text{(TE)} \qquad \lambda y + (v(x)\cdot\nabla)y + A(x)y = f(x) \quad \text{in } \Omega ,$$

where λ a positive parameter, $v(x) = (v_1(x),\dots,v_n(x))$, $A(x) = (a_{ik}(x))$, $i,k = 1,\dots,N$, $f(x) = (f_1(x),\dots,f_N(x))$ are given and the vector function $y(x) = (y_1(x),\dots,y_N(x))$ is unknown. We look for $L \in \mathcal{L}(W^{j,p})$, $j \geqslant -1$ such that $y = Lf$ is a solution of (TE) for each $f \in W^{j,p}$. The theorem 3.1.1, which shows the main result for this system of equations and performs the core of this chapter, will be a powerful tool to accomplish the task of giving an existence result of system (1.1.1) in chapter 4.

In the case $N = 1$, (TE) can be seen as a particular case of a more general class of equations, namely, of the class of elliptic–parabolic equations (called also second order equations with non–negative characteristic form). It was shown by Fichera [8], [9], in this more general context that the problem above is well posed in the functional space L^p if y is assigned only on the set $\Gamma_1 = \{ x \in \Gamma : v \cdot n < 0\}$, where n indicates the unit outward normal on Γ . Let us emphasize that in chapter 3 the condition (3.1.2) ($v \cdot n = 0$ on Γ) is a necessary and sufficient condition for the existence of y in $W_0^{j,p}$ for every $f \in W_0^{j,p}$ $(j \geqslant 1)$. Other results regarding an L^p theory for the stationary transport equation are given in O.A.Oleinik [19], [20]; existence results in Sobolev spaces where previously given only in the hilbertian case $p = 2$ (see K.O.Friedrichs [10], P.D.Lax and R.S.Phillips [17], J.J.Kohn and L.Nirenberg [16]. A more complete treatment of results regarding this equation is presented by O.A.Oleinik and E.V. Radekevic in [21].

Let us now describe our main motivating problems.
The goal of chapter 4 is to give the proof of the following

Theorem A. Let $p \in]1,+\infty[$ and $j > -1$ verify $(j+2)p > n$. Assume Ω as above with $\Gamma \in C^{j+3}$ and $p(\rho,\zeta) \in C^{j+3}([m-l,m+l]x[\zeta_0-l_1,\zeta_0+l_1])$. There exist constants c'_0, c'_1 such that, if $(f,g,h) \in W^{j+1,p} \times \overline{W}_0^{j+2,p} \times W^{j+1,p}$ and

$$\|f\|_{j+1,p} + \|g\|_{j+2,p} + \|h\|_{j+1,p} \leqslant c'_0 \,, \tag{1.1.7}$$

there exists a unique solution $(u,\sigma,\alpha) \in W_{0,d}^{j+3,p} \times \overline{W}^{j+2,p} \times W_0^{j+3,p}$ of problem (1.1.5) in the ball

$$\|u\|_{j+3,p} + \|\sigma\|_{j+2,p} + \|\alpha\|_{j+3,p} \leqslant c'_1 \,. \tag{1.1.8}$$

In other words, there exists a unique solution $(u,\rho,\zeta) \in W_{0,d}^{j+3,p} \times W^{j+2,p} \times W^{j+3,p}$ of problem (1.1.1), $(1.1.3)_1$ in the ball

$$\|u\|_{j+3,p} + \|\rho - m\|_{j+2,p} + \|\zeta - \zeta_0\|_{j+3,p} \leqslant c'_1 \,.$$

In the above statement c'_0, c'_1 are suitable positive constants depending only on n, p, j, μ, ν, k, m, Ω, ζ_0, c_v, χ, χ_0, χ_1, l, l_1, T_i, S_i $(i = 1,2)$, where $T_i = \sup |\omega_i(\sigma,\alpha)|$ for $(\sigma,\alpha) \in I(l,l_1)$, and S_i is the norm of $\omega_i(\sigma,\alpha)$ in the space $C^{j+2}(I(l,l_1))$.

Theorem A was stated by H. Beirão da Veiga in reference [4] where this author gives complete proof in the case $j = -1$ and shows the main lines in the case $j > -1$. Here we present the complete calculations corresponding to the case $j > -1$. For related results we mention the papers [3], [5], [26], [27] and references given therein.

Let us briefly describe the main points treated in this chapter for proving theorem A. We emphasize that the core of chapter 4 is the study of the linear system (4.1.1) in section 4.1. We show first that this system is not an elliptic system in the sense of Agmon–Douglis–Nirenberg due to the term $v \cdot \nabla \sigma$ (except if $v(x)$ vanishes identically in Ω). and describe then the way to deal with this difficulty. In section 4.2 we give the proof of theorem A by applying theorem 4.1.1 and a fixed point argument.

Finally, in chapter 5 we turn our attention to the stationary, compressible Navier–Stokes equations in the barotropic case (see (5.1.1)) and we study the relationship between the equations of compressible and incompressible fluids when the reciprocal of the Mach number becomes large. Quite often, the limiting solution (when it exists) satisfies completely different partial differential equations. The incompressible limit of the compressible Navier–Stokes equations is a physical problem involving dissipation and it is therefore interesting to discuss such a limiting process. We consider so the solutions of the equations of the stationary , compressible fluid flow

$$(1.1.9) \qquad \begin{cases} -\mu\Delta u_\lambda - \nu\nabla \operatorname{div} u_\lambda + \nabla p_\lambda(\rho_\lambda) = \rho_\lambda[f - (u_\lambda \cdot \nabla)u_\lambda], \\ \operatorname{div}(\rho_\lambda u_\lambda) = 0, \quad \text{in } \Omega, \\ (u_\lambda)\big|_\Gamma = 0, \quad \overline{\rho_\lambda} = m. \end{cases}$$

Here $p_\lambda(\rho)$ is a one-parameter family of equations of state with $dp_\lambda/d\rho \longrightarrow +\infty$, as $\lambda \longrightarrow +\infty$. Under suitable assumptions on $p_\lambda(\rho)$, our objectives are to analyze the limit as $\lambda \longrightarrow +\infty$ of solutions of the compressible equations (1.1.9) and to discuss the convergence to solutions of the incompressible equations

$$(1.1.10) \qquad \begin{cases} -\mu\Delta u_\infty + \nabla \pi(x) = m[f - (u_\infty \cdot \nabla)u_\infty], \\ \operatorname{div} u_\infty = 0, \quad \text{in } \Omega, \\ (u_\infty)\big|_\Gamma = 0. \end{cases}$$

We give so in chapter 5 the proof of the following

Theorem B. Let $p \in]1,+\infty[$ and $j > -1$ verify $(j+2)p > n$. Let the family of state functions $p_\lambda(\rho)$ verify the assumptions done in chapter 5. Then there exist positive constants c'_8 , c'_9 depending at most on n, p, j, μ , ν , m, l, ϕ , k_0 , Ω such that if $f \in W^{j+1,p}$ and

$$\| f \|_{j+1,p} \leqslant c'_8 \quad ,$$

the following statements hold:

i) for each $\lambda \geqslant \lambda_0$, the problem (1.1.9) has a unique solution $(u_\lambda, \rho_\lambda) \in W^{j+3,p}_{0,d} \times W^{j+2,p}$ in the ball

$$\| u_\lambda \|_{j+3,p} \leqslant c'_9 \quad , \qquad \| \rho_\lambda - m \|_{j+2,p} \leqslant c'_9 / k_\lambda \ .$$

ii) If $\lim_{\lambda \to \infty} k_\lambda = +\infty$, then

$$\begin{aligned} u_\lambda &\longrightarrow u_\infty && \text{, weakly in } W^{j+3,p}_0 \ , \\ \operatorname{div} u_\lambda &\longrightarrow 0 && \text{, weakly in } W^{j+2,p}_0 \ , \\ \rho_\lambda &\longrightarrow m && \text{, strongly in } W^{j+2,p} \ , \\ \nabla p_\lambda(\rho_\lambda) &\longrightarrow \nabla\pi && \text{, weakly in } W^{j+1,p} \ , \end{aligned}$$

where (u_∞ , π) is the unique solution of the incompressible Navier–Stokes equations (1.1.10).

Our general assumptions on the family of state functions $p_\lambda(\rho)$ (see chapter 5) contain the physically interesting cases described by S.Klainerman and A.Majda [15]. To readers interested in the incompressible limit of compressible fluids we refer also to papers by A.Majda [18], S.Schochet [22].

Notations

The differential operator $\partial/\partial x_i$, $1 \leqslant i \leqslant n$, will be written D_i . In the sequel δ and η indicate multi–indexes, and $|\delta| = \delta_1 + \cdots + \delta_n$, $|\eta| = \eta_1 + \cdots + \eta_n$. D^δ will be the differential operator $D^\delta = D_1^{\delta_1} \cdots D_n^{\delta_n} = \frac{\partial^{|\delta|}}{\partial x_1^{\delta_1} \ldots \partial x_n^{\delta_n}}$.

We denote by $W^{j,p}$, j an integer, $1 < p < +\infty$, the Sobolev space $W^{j,p}(\Omega)$, endowed with the usual norm $\| \ \|_{j,p}$, and by $| \ |_p$, $1 \leqslant p \leqslant +\infty$, the usual norm in $L^p = L^p(\Omega)$. Hence, $\| \ \|_{0,p} = | \ |_p$.

Moreover, $W_0^{1,p}$ is the closure of $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ in $W^{1,p}$. $W^{-1,p'}$ is the dual space of $W_0^{1,p}$, $p' = p/(p-1)$, provided with the usual norm $\| \ \|_{-1,p'}$.

For $j \geqslant 1$ we define $W_0^{j,p} = \{v \in W^{j,p} : v = 0 \text{ on } \Gamma\}$. Note that $W_0^{j,p} = W^{j,p} \cap W_0^{1,p}$ is not the closure of $\mathcal{D}(\Omega)$.

Furthermore, $C_B^s(\Omega) = \{u \in C^s(\Omega) : |D^\delta u| \in L^\infty \text{ for } |\delta| \leqslant s\}$, where s is a non–negative integer; $C^s(\overline{\Omega}) = \{u \in C^s(\Omega) : D^\delta u$ has a continuous extension on $\overline{\Omega}$, $|\delta| \leqslant s\}$.

For convenience, these notations are also used for functional spaces whose elements are vector fields or matrices defined in Ω .

So, we use the symbol X to denote the space of vector fields v in such that $v_i \in X$, i=1,...,n, where X is a functional space, and we write $A \in X$, if $A(x) = (a_{rs}(x))$, r = 1,...,R , s = 1,...,S , where a_{rs} are real functions in Ω and $a_{rs} \in X$. We set

$$|D^l A(x)|^2 = \sum_{|\delta|=l} \sum_{r=1}^{R} \sum_{s=1}^{S} |D^\delta a_{rs}(x)|^2.$$

If $A \in W^{j,p}$, we set

$$|D^l A|_p = \left(\int_\Omega |D^l A|^p \, dx \right)^{1/p} , \quad \|A\|_{j,p} = \sum_{l=0}^{j} |D^l A|_p \ .$$

Furthermore, we set $\overline{W}^{j,p} = \{\tau \in W^{j,p} : \overline{\tau} = 0 \}$, $\overline{W}_0^{j,p} = \overline{W}^{j,p} \cap W_0^{j,p}$, $j \geqslant 1$, where in general $\overline{\varphi}$ denotes the mean value of $\varphi(x)$ in Ω.

Finally, for vector fields, we define $W^{j,p}_{o,d} = \{v \in W^{j,p}_{o} : \operatorname{div} v = 0 \text{ on } \Gamma\}$, $j \geq 2$.

Let $\xi = (\xi_1, ..., \xi_N)$, $w = (w_1, ..., w_N)$, then we define $\xi \cdot w = \sum_{i=1}^{N} \xi_i w_i$, $|\xi|^2 = \xi \cdot \xi$.

For $v = (v_1, ..., v_n)$, we define $(v \cdot \nabla)\xi = \sum_{i=1}^{n} v_i D_i \xi$, $\nabla v : \nabla^2 \xi = \sum_{i,l=1}^{n} (D_i v_l)(D_i D_l \xi)$. We use also the symbol ∇ for $D = (D_1, ..., D_n)$.

In general, if X and Y are Banach spaces, $\mathcal{L}(X,Y)$ denotes the Banach space of all bounded linear maps from X into Y. We set $\mathcal{L}(X) = \mathcal{L}(X,X)$.

If not otherwise specified we use the symbols c, c_i, $i \geq 0$ to denote positive constants depending at most on n, p, j, Ω. The symbol c may be utilized (even in the same equation) to indicate distinct constants.

Sobolev embedding theorems

We recall here briefly the Sobolev embedding theorems which will be used frequently from now on (see [1]). Under our assumption on Ω, $r \geq 1$, $p \in]1, +\infty[$, the following results hold:

$$W^{r,p} \hookrightarrow L^q, \quad \text{for } p \leq q \leq np/(n-rp), \text{ if } rp < n;$$
$$W^{r,p} \hookrightarrow L^q, \quad \text{for } p \leq q < +\infty, \text{ if } rp = n;$$
$$W^{r+s,p} \hookrightarrow C^s(\overline{\Omega}), \quad \text{if } rp > n.$$

We point out that the results given above are still valid for Ω, a bounded domain with Lipschitz continuous boundary. More generally, if Ω satisfies a uniform interior cone condition – that is, there exists a fixed cone K_Ω such that each $x \in \Omega$ is a vertex of a cone $K_\Omega(x) \subset \overline{\Omega}$ and congruent to K_Ω – then the last embedding above is reduced to the imbedding $W^{r+s,p} \hookrightarrow C^s_B(\Omega)$, if $rp > n$.

2. THE NON-HOMOGENEOUS STOKES PROBLEM

We consider the non-homogeneous Stokes equation concerning a vector function $u(x) = (u_1(x),\dots,u_n(x))$ and a scalar function $\pi(x)$

$$(2.1.1)\qquad \begin{cases} -\mu\,\Delta u + \nabla\pi = f \ , \\ \operatorname{div} u = g \quad , \ \text{in } \Omega \ , \\ u\big|_{\Gamma} = \phi \ , \end{cases}$$

where $f(x) = (f_1(x),\dots,f_n(x))$ and $\phi(x) = (\phi_1(x),\dots,\phi_n(x))$ are given vector functions and g is a given scalar function on Ω , μ a constant.

We have the following existence and regularity result:

Theorem 2.1.1. Let Ω be an open set of $\mathbb{R}^n$ with the boundary $\Gamma \in C^r$, $r = \max(l+2,2)$, l integer > -1, $p \in \,]1,+\infty[$, and let $f \in W^{l,p}$, $g \in W^{l+1,p}$, $\phi \in W^{l+2-\frac{1}{p},p}(\Gamma)$ be given satisfying the compatibility condition

$$(2.1.2)\qquad \int_{\Omega} g\,dx \ - \ \int_{\Gamma} \phi\cdot n\,d\Gamma = 0$$

where $d\Gamma$ denotes the element of surface area on Γ and n denotes the outward unit normal on Γ . Then there exist functions $u \in W^{l+2,p}$ and $\pi \in W^{l+1,p}$ which are solutions of (2.1.1). Here the function u is unique while p is unique up to an additive constant.

Moreover (2.1.4) is satisfied.

For the proof of the existence part of this theorem we refer to L.Cattabriga [7] when $n = 2$ or 3. For $n = 2$ one can reduce the problem to a classical biharmonic problem (see R.Temam [25]). For an arbitrarily large n the proof can be find in Y.Giga [13]. Among other authors, also V.A.Solonnikov [24] and M.Giaquinta – G.Modica [12] (for $p = 2$, without using potential theory) gave a proof of this theorem.

Note that we can define the trace on Γ of a function $u \in W^{j,p}$, $j \geqslant 1$, $p \in [1,+\infty[$, if Ω is a open set of $\mathbb{R}^n$ with smooth boundary. In other words, there exists an operator $tr \in \mathcal{L}(W^{j,p}, W^{j-\frac{1}{p},p}(\Gamma))$ such that $tr\ u = u|_\Gamma$ for every $u \in W^{j,p}$. $W^{j-\frac{1}{p},p}(\Gamma)$ is equipped with the image norm $\|\psi\|_{W^{j-\frac{1}{p},p}(\Gamma)} = \inf\limits_{u|_\Gamma=\psi} \|u\|_{j,p}$.

We give here the proof of the regularity result for the Stokes problem (2.1.1).

Theorem 2.1.2. Let Ω be an open bounded set of class C^{l+2}, l integer $\geqslant 0$, $p \in]1,+\infty[$. Let us suppose that

$$u \in W^{2,p} \quad , \quad \pi \in W^{1,p} \quad ,$$

are solutions of the generalized Stokes problem (2.1.1).

If $f \in W^{l,p}$, $g \in W^{l+1,p}$ and $\phi \in W^{l+2-\frac{1}{p},p}(\Gamma)$, then

$$u \in W^{l+2,p}, \qquad \pi \in W^{l+1,p} \tag{2.1.3}$$

and there exists a constant $c(p,l,n,\Omega)$ such that

$$\mu\|u\|_{l+2,p} + \|\pi\|_{l+1,p/\mathbb{R}} \leqslant c\,[\,\|f\|_{l,p} + \|g\|_{l+1,p} + \|\phi\|_{W^{l+2-\frac{1}{p},p}(\Gamma)}]. \tag{2.1.4}$$

Proof. This theorem results from the paper of Agmon–Douglis–Nirenberg [2], giving a priori estimates of solutions of general elliptic systems (reported in the appendix).

Let $u = (u_1,\dots,u_{n+1})$, $u_{n+1} = (1/\mu)\pi$, $F = (-f_1/\mu,\dots,-f_n/\mu, g)$.

Then the system of partial differential equations given by the equations $(2.1.1)_1$, $(2.1.1)_2$, will be represented as

$$(2.1.5) \qquad \sum_{j=1}^{n+1} l_{ij}(x,D)u_j(x) = F_i(x)\,, \qquad 1 \leqslant i \leqslant n+1\,,$$

where $l_{ij}(x,\xi)$ $[\xi = (\xi_1,\dots,\xi_n) \in \mathbb{R}^n$, $x = (x_1,\dots,x_n) \in \Omega]$ is the matrix

$$(2.1.6) \qquad \begin{cases} l_{ij}(\xi) = |\xi|^2\delta_{ij}\,, \ 1 \leqslant i,j \leqslant n & (\delta_{ij} \text{ the Kronecker symbol}) \\ l_{n+1,j}(\xi) = -l_{j,n+1}(\xi) = \xi_j & 1 \leqslant j \leqslant n \\ l_{n+1,n+1}(\xi) = 0. \end{cases}$$

According to the appendix we shall introduce two systems of integer weights s_i and t_j , $1 \leqslant i,j \leqslant n+1$, attached to the equations and to the unknowns respectively. We take $s_i = 0$, $t_i = 2$, $1 \leqslant i \leqslant n$, $s_{n+1} = -1$, $t_{n+1} = 1$. This implies that

$$(2.1.7) \qquad \deg l_{ij}(x,\xi) \leqslant s_i + t_j \ , \quad i,j = 1,\dots,n+1\,,$$

as requested in (A.2). Furthermore, (A.3) and (A.4) are satisfied. We observe that $l'_{ij}(x,\xi) = l_{ij}(x,\xi)$. By induction, an easy computation gives $L(x,\xi) \equiv$ $= \det(l'_{ij}(x,\xi)) = |\xi|^{2n}$,and so $L(x,\xi) \neq 0$ for any real $\xi \neq 0$. This ensures (condition (A.5) is satisfied) the ellipticity of the system of partial differential equations introduced above.

Let us verify the Supplementary Condition on L. From $L(x,\xi) = |\xi|^{2n}$ follows that $L(x,\xi)$ is of even degree $2n$ with respect to ξ. Furthermore, $L(x, \xi + \tau \xi') = |\xi + \tau\xi'|^{2n} = 0$ has exactly n roots with positive imaginary part and these roots are all equal to

$$\tau^{+}(x,\xi,\xi') = -\xi\cdot\xi' + i\sqrt{|\xi|^2|\xi'|^2 - |\xi\cdot\xi'|^2} \quad . \tag{2.1.8}$$

Moreover, $2n = \deg (L(x,\xi)) > 0$, and (A.7) is satisfied for $A = 1$ and $m = n$.

To apply the theorem A.1, we must finally analyze the complementing boundary conditions $(2.1.1)_3$, which are expressed as

$$\sum_{j=1}^{n+1} B_{hj}(x,D)u_j(x) = \phi_h(x) \quad \text{on } \Gamma , \quad h = 1,\ldots,n, \tag{2.1.9}$$

where $B_{hj}(x,D) = \delta_{hj}$, for $1 \leqslant h \leqslant n$, $1 \leqslant j \leqslant n+1$. Above we have assigned the system of integer weights $t_1, \ldots, t_{n+1}$ attached to the dependent variables. Furthermore, here we assign a new system $r_h = -2$ for $h = 1,\ldots,n$, attached to the equations representing the complementing boundary conditions for the elliptic system above. Then, as requested in (A.9), $\deg B_{hj}(x, \xi) \leqslant r_h + t_j$ and we have $B'_{hj}(x,\xi) = B_{hj}(x,\xi)$. From above it follows that $\tau_k^{+}(x,\xi) = \tau^{+}(x,\xi,n)$, $k = 1,\ldots,n$. Therefore,

$$M^{+}(x,\xi,\tau) = (\tau - \tau^{+}(x,\xi,n))^{n}.$$

The matrix with elements $\sum_{j=1}^{n+1} B'_{hj}(\xi)L^{jk}(\xi)$ is simply the matrix with elements $l_{hk}(\xi)$, $1 \leqslant h,k \leqslant n$, $-l_{h,n+1}(\xi)$, $1 \leqslant h \leqslant n$. A combination $\sum_{h=1}^{n} C_h \sum_{j=1}^{n+1} B'_{hj}L^{jk}$ is then equal to

$$(C_1|\xi + \tau n|^2, \ldots, C_n|\xi + \tau n|^2, \sum_{i=1}^{n} C_i(\xi_i + \tau n_i))$$

and this is zero modulo M^{+} only if $C_1 = \cdots = C_n = 0$, and the Complement Boundary Condition is satisfied.

Therefore, we can apply theorem A.1. We have $l_1 = \max(0, r_n + 1) = 0$. From our assumptions it follows that $\| u_j \|_{l+2,p}$, $1 \leqslant j \leqslant n$, and $\| \pi \|_{l+1,p}$ are finite , $l \geqslant 0$. Moreover, there exists a constant $c = c(p, l, \mu, n, \Omega)$ such that (2.1.4) is satisfied (according to the remark after theorem A.1 one can take $d_p = 0$ in (A.11) since the solutions of (2.1.1) are unique (π is unique up to an additive constant).

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3. EXISTENCE RESULTS IN SOBOLEV SPACES FOR THE STATIONARY TRANSPORT EQUATION

In this chapter let us consider the system of first order partial differential equations

$$(3.1.1)\qquad \lambda y + (v(x)\cdot\nabla)y + A(x)y = f(x) \quad , \text{ in } \Omega ,$$

where $\lambda \in \mathbb{R}^+$, $v(x) = (v_1(x),\dots,v_n(x))$, $A(x) = (a_{ik}(x))$, $i,k = 1,\dots,N$, $f(x) = (f_1(x),\dots,f_N(x))$ are given and the vector function $y(x) = (y_1(x),\dots,y_N(x))$ is unknown.

Let us point out that the study of the system (3.1.1) in this context is motivated by the necessity to solve equation (4.1.9) below (in this last case $N = 1$, $\lambda = mk/(\mu + \nu)$, $A(x) \equiv 0$, $f = G(F,g,\tau)$).

The main result of this chapter can be summarized in the following theorem:

Theorem 3.1.1. Let $p \in]1,+\infty[$ and $j \geq -1$ verify $(j+2)p > n$. Assume that $v \in W^{j+3,p}$, $A \in W^{j+2,p}$, $\Gamma \in C^{j+3}$, and

$$(3.1.2)\qquad v\cdot n = 0 \quad \text{on } \Gamma .$$

Then, if

$$(3.1.3)\qquad \lambda > \lambda_j \equiv \bar{c}(\| v\|_{j+3,p} + \| A\|_{j+2,p}),$$

there exists a bounded linear map $L \in \mathcal{L}(W^{j,p})$ such that $u = Lf$ is a solution of (3.1.1), for every $f \in W^{j,p}$.

Moreover

$$(3.1.4) \qquad (\lambda - \lambda_j)\, \|u\|_{j,p} \leqslant \hat{c}\, \|f\|_{j,p} \ .$$

Finally, if $j \geqslant 1$, then $u \in W_o^{j,p}$ if and only if $f \in W_o^{j,p}$. Here $\bar{c}$ and $\hat{c}$ are suitable positive constants depending only on n, N, p, j, Ω .

In proving theorem 3.1.1 we proceed in several steps:

3.2. PRELIMINARIES

We summarize briefly some later useful inequalities. Let Ω be an open, bounded set of $\mathbb{R}^n$ satisfying a uniform interior cone condition. Let $p \in]1,+\infty[$ be fixed.

We denote by $r = r(p)$ and $s = s(p)$ two reals such that:

$$(3.2.1) \qquad \begin{array}{llll} r = p & , \text{if } p > n \,, & s = p & , \text{if } p > n/2 \,, \\ r > n & , \text{if } p = n \,, & s > n/2 & , \text{if } p = n/2 \,, \\ r = n & , \text{if } p < n \,; & s = n/2 & , \text{if } p < n/2 \,. \end{array}$$

By using Hölder's inequality and the Sobolev's inequalities one verifies that there exist positive constants $c = c(n,N,p,r,s,\Omega)$ such that

$$(3.2.2) \qquad \| |F| \, w \|_p \leqslant c |F|_s \, \|w\|_{2,p} \quad , \quad \| |G| \, |Dw| \|_p \leqslant c |G|_r \, \|w\|_{2,p} \ ,$$

for every $w \in W_o^{2,p}$, $F \in L^s$, $G \in L^r$.

Let us proof for example $(3.2.2)_1$.

If $2p > n$, we have $W^{2,p} \hookrightarrow C^0(\Omega)$ and $s = p$. Therefore

$$\| |F| \, w \|_p \leqslant c(n,N,p,s,\Omega) \, |F|_p \, \| w \|_{2,p} \leqslant c \, |F|_s \, \| w \|_{2,p} \, .$$

If $2p < n$, we have $W^{2,p} \hookrightarrow L^q$ for $p \leqslant q \leqslant np/(n-2p)$; $s = n/2$.

Let us take $q = np/(n-2p)$. We get moreover $p/s + p/q = 1$.

Therefore $\| |F| \, w \|_p \leqslant |F|_s \, | w |_{\frac{np}{n-2p}}$. The Sobolev

embedding theorem above gives $(3.2.2)_1$.

If $2p = n$, we have $W^{2,p} \hookrightarrow L^q$ for $p \leqslant q < +\infty$; $s > n/2$.

We take $q = ns/(2s-n)$. Furthermore $p/s + p/q = 1$. As above we get $(3.2.2)_1$.

This proves $(3.2.2)_1$. ***

From (3.2.2) one deduces that there exist constants $c_i = c_i(n,N,p,r,s,\Omega)$, $i = 1,\dots,3$, such that

$$(3.2.3) \qquad \begin{aligned} |\nabla v : \nabla^2 w|_p &\leqslant c_1 |\nabla v|_\infty |\Delta w|_p \, , \\ |(\Delta v \cdot \nabla) w|_p &\leqslant c_2 |\Delta v|_r \, |\Delta w|_p \, , \\ |\Delta (A w)|_p &\leqslant c_3 \left(\| A \|_{2,s} + \| A \|_{1,r} + |A|_\infty \right) |\Delta w|_p \, , \end{aligned}$$

for every $w \in W_0^{2,p}$. Note the equivalence of the norms $|\Delta w|_p$ and $\| w \|_{2,p}$ in $W_0^{2,p}$.

3.3. THE EXISTENCE THEOREM OF (3.1.1) IN $W_0^{2,p}$.

Let us define

$$(3.3.1) \quad \vartheta_2 = (1/p) \, |\operatorname{div} v|_\infty + 2 \, c_1 \, |\nabla v|_\infty + c_2 |\Delta v|_r + c_3 (\| A \|_{2,s} + \| A \|_{1,r} + |A|_\infty).$$

One has the following result:

Theorem 3.3.1. Let $p \in]1,+\infty[$, $\Gamma \in C^4$ and

(3.3.2) $$v \in W^{2,r} \cap W^{1,\infty} , \qquad A \in W^{2,s} \cap W^{1,r} \cap L^\infty .$$

Let us assume that (3.1.2) holds.

Then, if $\lambda > \vartheta_2$ and $f \in W_o^{2,p}$ there exists a unique solution $y \in W_o^{2,p}$ of (3.1.1). Moreover

(3.3.3) $$(\lambda - \vartheta_2)|\Delta y|_p \leqslant |\Delta f|_p ,$$

and the estimates (3.4.3) ,(3.5.3) hold.

In particular $(\lambda - \vartheta_2)(|\Delta y|_p + |\nabla y|_p + |y|_p) \leqslant |\Delta f|_p + |\nabla f|_p + |f|_p$.

Note that $W^{1,\infty}$ is the Sobolev space $W^{1,\infty}(\Omega) = \{u \in L^\infty : D^\delta u \in L^\infty , |\delta| \leqslant 1\}$ endowed with the usual norm $\| \ \|_{1,\infty}$.

Proof. Let ε be a constant > 0 , and let us consider the elliptic system with Dirichlet boundary conditions

(3.3.4) $$\begin{cases} -\varepsilon \Delta y_\varepsilon + \lambda y_\varepsilon + (v \cdot \nabla) y_\varepsilon + A y_\varepsilon = f , \text{ in } \Omega , \\ (y_\varepsilon)\big|_\Gamma = 0 . \end{cases}$$

For λ sufficiently large ($\lambda > \vartheta_2$ gives an existence result for (3.3.4)), the problem (3.3.4) has a unique solution $y \in W_o^{4,p}$.

Since $y_\varepsilon = 0$ on Γ ,it follows from (3.1.2) that $(v \cdot \nabla) y_\varepsilon = 0$ on Γ . Therefore (3.3.4) yields

(3.3.5) $$\Delta y_\varepsilon = 0 \quad \text{on } \Gamma .$$

Our aim is to get the solution of (3.1.1) as the limit for $\varepsilon \longrightarrow 0^+$ of the solutions of (3.3.4). To prove the existence of that limit we are looking for estimates, uniform in ε , for the norms of the solutions of (3.3.4) in a suitable selected functional space, more precisely, in $W_0^{2,p}$.

Let us therefore consider a positive parameter @, and set $\Lambda = (@ + |\Delta y_\varepsilon|^2)^{1/2}$. By applying the Laplace operator to both sides of system $(3.3.4)_1$ and taking the scalar product on $\mathbb{R}^N$ with $\Lambda^{p-2}\Delta y_\varepsilon$ and finally integrating it over Ω, one gets

$$(3.3.6)\quad -\int_\Omega \Delta(\Delta y_\varepsilon)\cdot\Lambda^{p-2}\Delta y_\varepsilon\, dx + \lambda\int_\Omega |\Delta y_\varepsilon|^2\Lambda^{p-2}\, dx + \int_\Omega \Delta[(v\cdot\nabla)y_\varepsilon]\cdot\Lambda^{p-2}\Delta y_\varepsilon\, dx +$$
$$+\int_\Omega \Delta(Ay_\varepsilon)\cdot\Lambda^{p-2}\Delta y_\varepsilon\, dx = \int_\Omega \Delta f\cdot\Lambda^{p-2}\Delta y_\varepsilon\, dx \quad .$$

Let us consider (3.3.6) term by term.

By doing an integration by parts, and by taking in account (3.3.5), it follows

$$(3.3.7)\qquad -\varepsilon\int_\Omega \Delta(\Delta y_\varepsilon)\cdot\Lambda^{p-2}\Delta y_\varepsilon\, dx = \varepsilon\int_\Omega D(\Delta y_\varepsilon)\cdot D(\Lambda^{p-2}\Delta y_\varepsilon)\, dx \quad .$$

Without difficulties one shows that

$$D(\Delta y_\varepsilon)\cdot D(\Lambda^{p-2}\Delta y_\varepsilon) = \Lambda^{p-2}|D\Delta y_\varepsilon|^2 + \frac{(p-2)}{4}\Lambda^{p-4}|D(|\Delta y_\varepsilon|^2)|^2, \quad \text{a.e. in } \Omega .$$

Hence

$$(3.3.8)\qquad -\varepsilon\int_\Omega \Delta(\Delta y_\varepsilon)\cdot\Lambda^{p-2}\Delta y_\varepsilon\, dx = \varepsilon\int_\Omega \Lambda^{p-2}|D\Delta y_\varepsilon|^2\, dx + \varepsilon\frac{(p-2)}{4}\int_\Omega \Lambda^{p-4}|D(|\Delta y_\varepsilon|^2)|^2\, dx .$$

This proves that the left hand side of (3.3.8) is non–negative for $p \geqslant 2$.

For $p \in]1,2]$ we had to observe that

$$\begin{aligned} D(\Delta y_\varepsilon)\cdot D(\Lambda^{p-2}\Delta y_\varepsilon) &= \\ &= \Lambda^{p-2}|D(\Delta y_\varepsilon)|^2 + [(p-2)/2]\Lambda^{p-4}\, D(|\Delta y_\varepsilon|^2)\cdot[D(\Delta y_\varepsilon)]\cdot\Delta y_\varepsilon \geqslant \\ &\geqslant \Lambda^{p-2}|D(\Delta y_\varepsilon)|^2 + (p-2)\Lambda^{p-4}|D(\Delta y_\varepsilon)|^2\,|\Delta y_\varepsilon|^2 = \\ &= [@ + (p-1)|\Delta y_\varepsilon|^2]\Lambda^{p-4}|D(\Delta y_\varepsilon)|^2 . \end{aligned}$$

Hence, the left hand side of (3.3.8) is non–negative for $p \in]1,+\infty[$.

Let us now consider the third term on the left hand side of (3.3.6). Straightforward calculations prove the identity

$$\Delta[(v\cdot\nabla)y_\varepsilon] = (v\cdot\nabla)\Delta y_\varepsilon + 2\nabla v : \nabla^2 y_\varepsilon + (\Delta v\cdot\nabla)y_\varepsilon .$$

Since $D_i(\Delta y_\varepsilon)\cdot\Delta y_\varepsilon \Lambda^{p-2} = (1/2)\Lambda^{p-2} D_i(|\Delta y_\varepsilon|^2) = (1/p)D_i\Lambda^p$, an integration by parts implies

$$\int_\Omega (v\cdot\nabla)\Delta y_\varepsilon\cdot\Lambda^{p-2}\Delta y_\varepsilon \, dx = -(1/p)\int_\Omega (\operatorname{div} v)\Lambda^p \, dx .$$

This yields

$$(3.3.9) \qquad \int_\Omega \Delta[(v\cdot\nabla)y_\varepsilon]\cdot\Lambda^{p-2}\Delta y_\varepsilon \, dx = -(1/p)\int_\Omega (\operatorname{div} v)\Lambda^p \, dx + \\ + 2\int_\Omega (\nabla v : \nabla^2 y_\varepsilon)\Lambda^{p-2}\Delta y_\varepsilon \, dx + \int_\Omega [(\Delta v\cdot\nabla)y_\varepsilon]\Lambda^{p-2}\Delta y_\varepsilon \, dx.$$

From our considerations above it follows therefore

$$(3.3.10) \qquad -\varepsilon\int_\Omega \Delta(\Delta y_\varepsilon)\cdot\Lambda^{p-2}\Delta y_\varepsilon \, dx + \lambda\int_\Omega |\Delta y_\varepsilon|^2\Lambda^{p-2} \, dx - (1/p)\int_\Omega (\operatorname{div} v)\Lambda^p \, dx \leqslant \\ \leqslant 2\int_\Omega |\nabla v : \nabla^2 y_\varepsilon||\Delta y_\varepsilon|\Lambda^{p-2} \, dx + \int_\Omega |(\Delta v\cdot\nabla)y_\varepsilon||\Delta y_\varepsilon|\Lambda^{p-2} \, dx + \\ + \int_\Omega |\Delta(Ay_\varepsilon)||\Delta y_\varepsilon|\Lambda^{p-2} \, dx + \int_\Omega |\Delta f||\Delta y_\varepsilon|\Lambda^{p-2} \, dx .$$

Since $0 \leqslant |\Delta y_\varepsilon|\Lambda^{p-2} \leqslant \Lambda^{p-1}$, the Lebesgue's dominated convergence theorem applies as $@ \longrightarrow 0^+$. By taking in account that the first term on the left hand side of (3.3.10) is non–negative, and by passing to the limit as $@\longrightarrow 0^+$, one gets

$$(3.3.11) \qquad \lambda\int_\Omega |\Delta y_\varepsilon|^p \, dx - (1/p)\int_\Omega (\operatorname{div} v)|\Delta y_\varepsilon|^p \, dx \leqslant \\ \leqslant \int_\Omega (2|\nabla v : \nabla^2 y_\varepsilon| + |(\Delta v\cdot\nabla)y_\varepsilon| + |\Delta(Ay_\varepsilon)| + |\Delta f|)|\Delta y_\varepsilon|^{p-1} \, dx.$$

By taking in account (3.2.3) and (3.3.1) we get in particular $(\lambda - \vartheta_2)|\Delta y_\varepsilon|_p \leqslant |\Delta f|_p$. Since the y_ε are uniformly bounded in $W_0^{2,p}$,there exists a subsequence y_ε (we use the same notation for the sequence and the subsequence) and a vector function $y \in W_0^{2,p}$ such that y_ε converges weakly in $W_0^{2,p}$ to y (the convergence of all y follows from the uniqueness of the solution of (3.1.1) which will be proved below). Clearly y verifies (3.3.3). From the estimate above it follows also that $\varepsilon\Delta y_\varepsilon \longrightarrow 0$ strongly in L^p , as $\varepsilon \longrightarrow 0^+$. By passing to the limit in $(3.3.4)_1$ as $\varepsilon \longrightarrow 0^+$, it follows that y is a solution of (3.1.1).

The uniqueness of solution of (3.1.1) and the estimates (3.4.3) and (3.5.3) will be proved in the sequel. ***

3.4. A PRIORI ESTIMATE IN L^p AND UNIQUENESS RESULT OF (3.1.1).

Let m_A and M_A be constants such that for every $\xi \in \mathbb{R}^N$, $|\xi| = 1$, the estimates

(3.4.1) $$m_A \leqslant \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \quad , \quad \sum_{i,j=1}^{N} |a_{ij}(x)\xi_i\xi_j| \leqslant M_A \; ,$$

hold a.e. in Ω. Define

(3.4.2) $$\vartheta_0 = (1/p)\,|\operatorname{div} v|_\infty - m_A \; .$$

Clearly, $\vartheta_0 \leqslant (1/p)\,|\operatorname{div} v|_\infty + M_A \leqslant c(\|v\|_{1,\infty} + |A|_\infty)$,

for a suitable constant c. We have so the following result:

Theorem 3.4.1. Let $p \in]1,+\infty[$, $\Gamma \in C^1$, and

$$v \in W^{1,\infty} \quad , \qquad A \in L^\infty , \qquad f \in L^p .$$

Let $y \in W_0^{1,p}$ be a solution of (3.1.1) [resp. $y \in W^{1,p}$ be a solution of (3.1.1), under the assumption (3.1.2) for the coefficient v].

If $\lambda > \vartheta_0$, then

$$(3.4.3) \qquad (\lambda - \vartheta_0)|y|_p \leqslant |f|_p .$$

In particular from (3.4.3) follows that the solution, if it exists, is unique.

Proof: Let us define now $\Lambda = (@ + |y|^2)^{1/2}$. By taking the scalar product on $\mathbb{R}^N$ of (3.1.1) with $\Lambda^{p-2}y$, by integrating in Ω , and by taking in account $(3.4.1)_1$,one gets

$$(3.4.4) \quad \lambda\int_\Omega |y|^2\Lambda^{p-2}dx + (1/p)\int_\Omega (v.\nabla)\Lambda^p \; dx + m_A\int_\Omega |y|^2\Lambda^{p-2}dx \leqslant \int_\Omega |f||y|\Lambda^{p-2}\,dx .$$

Since $0 \leqslant |y|\Lambda^{p-2} \leqslant \Lambda^{p-1}$, the Lebesgue's dominated convergence theorem applies as $@ \longrightarrow 0^+$. For $y \in W_0^{1,p}$ (if $y \in W^{1,p}$, we take in account (3.1.2)) we get

$$(1/p)\int_\Omega (v\cdot\nabla)|y|^p \; dx = -(1/p)\int_\Omega (\operatorname{div} v)|y|^p \; dx \; .$$

Therefore,

$$(\lambda - (1/p)\,|\operatorname{div} v|_\infty + m_A)|y|_p^p \leqslant |f|_p |y|_p^{p-1} .$$

and hence (3.4.3). ***

3.5. THE EXISTENCE THEOREM OF (3.1.1) IN $W_0^{1,p}$.

Let $A \in W^{1,r} \cap L^\infty$. Recall (3.2.1). By using Hölder's inequality and the Sobolev's inequalities one verifies as in the proof of (3.2.2) that there exists a constant $c = c(n,N,p,r,\Omega)$ such that $\|DA\|\,|w|_p \leq c\,\|A\|_{1,r}\,\|w\|_{1,p}$, for every $w \in W_0^{1,p}$. Poincare's inequality furthermore implies $\|w\|_{1,p} \leq c|Dw|_p$, for every $w \in W_0^{1,p}$, and a suitable constant $c = c(n,N,p,\Omega)$. One easily verifies that

$$(3.5.1) \qquad |D(Aw)|_p \leq c_4\,(\,\|A\|_{1,r} + |A|_\infty\,)|Dw|_p \quad , \text{ for every } w \in W_0^{1,p} \ ,$$

where $c_4 = c_4\,(n,N,p,r,\Omega)$. We set

$$(3.5.2) \qquad \vartheta_1 = (1/p)|\operatorname{div} v|_\infty + |[\nabla v]|_\infty + c_4\,(\,\|A\|_{1,r} + |A|_\infty\,)\,,$$

where $[\nabla v] = \max\limits_{\substack{\xi \in \mathbb{R}^N \\ |\xi| = 1}} |\sum_{i,l=1}^{n} (D_l v_i)\,\xi_i \xi_l|$.

One has the following

<u>*Theorem*</u> <u>3.5.1.</u> Let $p \in]1,+\infty[$, $\Gamma \in C^4$, and

$$v \in W^{2,r} \cap W^{1,\infty} \quad , \qquad A \in W^{1,r} \cap L^\infty.$$

Let us assume that (3.1.2) holds.

Then, if $\lambda > \vartheta_1$ and $f \in W_0^{1,p}$ there exists a unique solution $y \in W_0^{1,p}$ of (3.1.1). Moreover

$$(3.5.3) \qquad (\lambda - \vartheta_1\,)|\nabla y|_p \leq |\nabla f|_p \quad ,$$

and the estimate (3.4.3) holds.

In particular $(\lambda - \vartheta_1\,)\|y\|_{1,p} \leq \|f\|_{1,p}$.

Proof:

i) Let us first assume that $A(x) \equiv 0$, and denote by $\overline{\vartheta}_2$ the right hand side of (3.3.1) for $A \equiv 0$. Assume that $\lambda > \overline{\vartheta}_2$. Let $f_m \in W_0^{2,p}$ be a sequence such that $f_m \longrightarrow f$ in $W_0^{1,p}$ as $m \longrightarrow +\infty$, and denote by $y_m \in W_0^{2,p}$ the solution of $\lambda y + (v\cdot\nabla)y = f_m$, given by theorem 3.3.1. Set for convenience, $\Lambda = (@ + |Dy_m|^2)^{1/2}$, for a positive parameter @. One has immediately, for $v\cdot n = 0$ on Γ,

$$(3.5.4)\qquad \int_\Omega \sum_{i=1}^n [(v\cdot\nabla)D_i y_m]\Lambda^{p-2} D_i y_m \, dx = (1/p)\int_\Omega (v\cdot\nabla)\Lambda^p \, dx = -(1/p)\int_\Omega (\operatorname{div} v)\Lambda^p \, dx.$$

By taking the scalar product on $\mathbb{R}^n$ of both sides of the equation $\lambda D_i y_m + (v\cdot\nabla)D_i y_m + [(D_i v)\cdot\nabla]y_m = D_i f_m$ with $\Lambda^{p-2}D_i y_m$, by adding for $i = 1,\ldots,n$, by integrating on Ω, and taking in account (3.5.4), one gets

$$(3.5.5)\qquad \lambda\int_\Omega |Dy_m|^2\Lambda^{p-2}\,dx - (1/p)\int_\Omega (\operatorname{div} v)\Lambda^p\,dx + \int_\Omega \sum_{i=1}^n [(D_i v)\cdot\nabla]y_m\, D_i y_m\, \Lambda^{p-2}\,dx =$$
$$= \int_\Omega \sum_{i=1}^n D_i f_m\, D_i y_m\, \Lambda^{p-2}\,dx$$

Hence

$$(3.5.6)\qquad \lambda\int_\Omega |Dy_m|^2\Lambda^{p-2}\,dx - (1/p)\,|\operatorname{div} v|_\infty \int_\Omega \Lambda^p\,dx \leqslant$$
$$\leqslant \int_\Omega |[\nabla v]||Dy_m|^2\Lambda^{p-2}\,dx + \int_\Omega |Df_m||Dy_m|\Lambda^{p-2}\,dx\,.$$

By passing to the limit as $@ \longrightarrow 0^+$, we show that $(\lambda - \overline{\vartheta}_1)\,|Dy_m|_p \leqslant |Df_m|_p$, where $\overline{\vartheta}_1$ denotes the right hand side of (3.5.2) for $A \equiv 0$. For m sufficiently large we get so a uniform bound in the L^p-norm for Dy_m. Since $W_0^{1,p}$ is a reflexive Banach space, we get a solution $y \in W_0^{1,p}$ of $\lambda y + (v\cdot\nabla)y = f$ as a weak limit in $W_0^{1,p}$ of a subsequence of y_m (the uniqueness result above guaranties that the whole sequence y_m converges to y).

Furthermore $(\lambda - \overline{\vartheta}_1)\,|Dy|_p \leqslant |Df|_p$.

ii) We extend now the result proved in part i) to the case in which $A(x) \equiv 0$ and $\lambda > \overline{\vartheta}_1$. Fix $\overline{\lambda} > \overline{\vartheta}_2$, and denote by $y = Tw$ the solution of problem $\overline{\lambda} y + (v \cdot \nabla) y = f + (\overline{\lambda} - \lambda) w$ for an arbitrary $w \in W_0^{1,p}$ (the existence of such y is shown in i)). If $\overline{y} = T\overline{w}$,one has $(\overline{\lambda} - \overline{\vartheta}_1) |D(y-y)|_p \leqslant (\overline{\lambda} - \lambda) |D(w-\overline{w})|_p$. Since $(\overline{\lambda} - \lambda)/(\overline{\lambda} - \overline{\vartheta}_1) < 1$, T is a contraction in $W_0^{1,p}$. Therefore, there exists a unique fixed point $y \in W_0^{1,p}$,$y = Ty$. y is the solution of $\lambda y + (v \cdot \nabla) y = f$, and one gets $(\overline{\lambda} - \overline{\vartheta}_1) |Dy|_p \leqslant |D(f + (\overline{\lambda} - \lambda) y)|_p$. This implies $(\lambda - \overline{\vartheta}_1) |Dy|_p \leqslant |Df|_p$.

iii) Finally, we assume that $A(x) \not\equiv 0$ and that $\lambda > \vartheta_1$. Clearly $\lambda > \overline{\vartheta}_1$. Let $w \in W_0^{1,p}$, and denote by $y = Tw$ the solution of $\lambda y + (v \cdot \nabla) y = f - Aw$ (by taking in account (3.5.1), we get $Aw \in W_0^{1,p}$,and the existence of y is shown in ii)). Let $\overline{y} = T\overline{w}$. One has $(\lambda - \overline{\vartheta}_1) |D(y-\overline{y})|_p \leqslant |D[A(w-\overline{w})]|_p \leqslant c_4 (\|A\|_{1,r} + |A|_\infty) |D(w-\overline{w})|_p$. Since $c_4 (\|A\|_{1,r} + |A|_\infty) = \vartheta_1 - \overline{\vartheta}_1$, and $\lambda > \vartheta_1$, we get that T is a contraction in $W_0^{1,p}$. The fixed point $y = Ty$ is a solution of (3.1.1). Moreover $(\lambda - \overline{\vartheta}_1) |Dy|_p \leqslant |D(f-Ay)|_p \leqslant |Df|_p + c_4 (\|A\|_{1,r} + |A|_\infty) |Dy|_p$. This proves (3.5.3). ***

Remark. The assumption $v \in W^{2,r}$ in the theorem 3.5.1 can be dropped. In fact, let Γ , A, f, λ , be as in the theorem above, and $v \in W^{1,\infty}$, $v \cdot n = 0$ on Γ . Let $v_m \in W^{2,r} \cap W^{1,\infty}$, $v_m \cdot n = 0$ on Γ ,and $v_m \longrightarrow v$ strongly in $W^{1,\infty}$. Define $\vartheta_1^m = (1/p) |\mathrm{div}\, v_m|_\infty + |[\nabla v_m]|_\infty + c_4 [\|A\|_{1,r} + |A|_\infty]$. Let $\varepsilon_0 = \lambda - \vartheta_1$, and fix $\varepsilon_0 > \varepsilon > 0$. Then there exists $M \in \mathbb{N}$ such that $\vartheta_1^m < \vartheta_1 + \varepsilon < \lambda$, for $m \geqslant M$. The theorem 3.5.1 shows that there exists $y_m \in W_0^{1,p}$ such that $\lambda y_m + (v_m \cdot \nabla) y_m + A y_m = f$ and $(\lambda - \vartheta_1^m) \| y_m \|_{1,p} \leqslant \| f \|_{1,p}$, for $m \geqslant M$. It follows $(\lambda - \vartheta_1 - \varepsilon) \| y_m \|_{1,p} \leqslant \| f \|_{1,p}$. Therefore, we get a solution $y \in W_0^{1,p}$ of (3.1.1) as a weak limit in $W_0^{1,p}$ of y_m ,as $m \to +\infty$. It follows also $(\lambda - \vartheta_1 - \varepsilon) \| y \|_{1,p} \leqslant \| f \|_{1,p}$. Since ε has been fixed arbitrarily we get (3.5.3).

3.6. THE EXISTENCE THEOREM OF (3.1.1) IN L^p.

Let us first explain what we mean by saying that $u \in L^p$ is a solution of (3.1.1).

Definition 3.6.1. Let $v \in W^{1,\infty}$, $A \in L^\infty$, and $f \in L^p$.
We say that $y \in L^p$ is a weak solution of (3.1.1) if y satisfies

$$(3.6.1) \qquad \lambda(y,\varphi) - (\sum_{i=1}^{n} D_i(v_i\varphi),y) + (Ay,\varphi) = (f,\varphi) \qquad \forall\, \varphi \in W_0^{1,p'},$$

where $\varphi = (\varphi_1,...,\varphi_N)$. We recall that $p' = p/(p-1)$.

Note that $(u,w) = \int_\Omega \sum_{i=1}^{N} u_i w_i \,dx$. One has the following result:

Theorem 3.6.1. Let $p \in]1,+\infty[$, $\Gamma \in C^4$, and
$v \in W^{2,r} \cap W^{1,\infty}$, $\qquad A \in W^{1,r} \cap L^\infty$.
Let us assume that (3.1.2) holds.

Then, if $\lambda > \vartheta_1$, there exists a bounded linear map $L \in \mathcal{L}(L^p)$ such that $y = Lf$ is a weak solution of (3.1.1) for every $f \in L^p$.

Moreover (3.4.3) holds.

Proof: Let $f_m \in W_0^{1,p}$ be a sequence which converges to f in L^p. The theorem 3.5.1 shows the existence of the solution $y_m \in W_0^{1,p}$ of $\lambda y + (v\cdot\nabla)y + Ay = f_m$. From (3.4.3) it follows that $(\lambda - \vartheta_1)\,|y_m - y_n|_p \leqslant |f_m - f_n|_p$, and therefore $\{y_m\}$ is a Cauchy sequence in L^p. Hence the sequence $\{y_m\}$ converges strongly to a function y in L^p, as $m \longrightarrow +\infty$. Since $\sum_{i=1}^{n} D_i(v_i\varphi) \in L^{p'}$, $Ay_m \in L^p$, by passing to the limit in $\lambda(y_m,\varphi) - (\sum_{i=1}^{n} D_i(v_i\varphi),y_m) + (Ay_m,\varphi) = (f_m,\varphi)$, we get that y is the desired solution. The proof of (3.4.3) is immediate. ***

Remark. The assumptions $v \in W^{2,r}$ and $A \in W^{1,r}$ can be dropped in theorem 3.6.1. Moreover, the map L exists for every $\lambda > \vartheta_0$. Finally, from the existence theorem for the adjoint problem $\lambda\varphi - (v\cdot\nabla)\varphi - (\operatorname{div} v)\varphi + A^*\varphi = g$ (see below) it follows an uniqueness result for the above solution y, at least for sufficiently large values of λ.

3.7. THE EXISTENCE THEOREM OF (3.1.1) IN $W^{-1,p'}$.

For the sake of completeness we give here also an existence theorem for (3.1.1) in the space $W^{-1,p'}$, $p' \in]1,+\infty[$.
This theorem is needed for the study of equation (4.1.9) in the case j = −1, see [4].

Denote by < , > the duality pairing between $W_0^{1,p}$ and the dual space $W^{-1,p'}$.

Definition 3.7.1. Let $v \in W^{1,\infty}$, $A \in L^\infty$, and $f \in W^{-1,p'}$.
We say that $y \in W^{-1,p'}$ is a weak solution of (3.1.1) if y satisfies

$$(3.7.1) \qquad \langle \lambda\varphi - \sum_{i=1}^{n} D_i(v_i\,\varphi) + A^*\varphi, y\rangle = \langle \varphi, f\rangle \qquad \forall \varphi \in \mathcal{D}(\Omega),$$

where $\varphi = (\varphi_1, \dots, \varphi_N)$ and A^* is the transpose matrix of A.

We proof the following statement:

Theorem 3.7.1. Let $p' \in]1,+\infty[$, $p = p'/(p'-1)$. Let $r(p)$ be as defined in (3.2.1), $\Gamma \in C^4$, and

$$v \in W^{2,r} \cap W^{1,\infty} \quad , \qquad A \in W^{1,r} \cap L^\infty.$$

Let us assume that (3.1.2) holds.

Then, if $\lambda > \vartheta_1^*$, defined by (3.7.3), there exists a bounded linear map $B_p^* \in \mathcal{L}(W^{-1,p'})$ (as defined below) such that $y = B_p^* f$ is a weak solution of (3.1.1) for every $f \in W^{-1,p'}$ (actually, (3.7.7) holds).

Moreover

(3.7.2) $$(\lambda - \vartheta_1^*) \| y \|_{-1,p'} \leqslant \| f \|_{-1,p'} .$$

Proof. By applying theorem 3.5.1 it follows that there exists a positive constant $c_5 = c_5(n,N,p,r,\Omega)$ such that

(3.7.3) $$\lambda > \vartheta_1^* \equiv c_5 \left(\| v \|_{2,r} + \| v \|_{1,\infty} + \| A \|_{1,r} + |A|_\infty \right)$$

implies the existence of a bounded linear map $B_p \in \mathcal{L}(W_0^{1,p})$ such that $\varphi = B_p g$ is the (unique) solution of the equation

(3.7.4) $$\lambda\varphi - (v \cdot \nabla)\varphi - (\mathrm{div}\, v)\varphi + A^*\varphi = g$$

for every $g \in W_0^{1,p}$ (note that $(\mathrm{div}\, v) \in W^{1,r} \cap L^\infty$, and can therefore be treated as A^*). Moreover

(3.7.5) $$(\lambda - \vartheta_1^*) \| \varphi \|_{1,p} \leqslant \| g \|_{1,p} .$$

The operator B_p is invertible and we set $A_p = B_p^{-1}$. We denote by $D(A_p)$ the domain of A_p, i.e. the range of B_p.

The operator A_p is closed in $W_0^{1,p}$ (in fact, since $D(B_p) = W_0^{1,p}$ is closed in $W_0^{1,p}$ and B_p is a bounded and linear operator on $W_0^{1,p}$, it follows that B_p is closed in $W_0^{1,p}$. From the existence of $A_p = B_p^{-1}$ follows that A_p is closed in $W_o^{1,p}$). Moreover

(3.7.6) $$D(A_p) = \{ \varphi \in W_o^{1,p} : (v \cdot \nabla)\varphi \in W_o^{1,p} \}.$$

In particular one has $\mathcal{D}(\Omega) \subset D(A_p)$, and therefore $D(A_p)$ is dense in $W_o^{1,p}$. Denote by A_p^* the adjoint of A_p. Since $A_p^{-1} = B_p \in \mathcal{L}(W_o^{1,p})$, we get $(A_p^*)^{-1} = B_p^* \in \mathcal{L}(W^{-1,p'})$ (this follows from a result on Functional Analysis which will be recalled below) and $\| B_p \|_{\mathcal{L}(W_o^{1,p})} = \| B_p^* \|_{\mathcal{L}(W^{-1,p'})}$.

Consequently, the equation $A_p^* \, y = f$ has a unique solution $y = B_p^* \, f$, for every $f \in W^{-1,p'}$. This equation is equivalent to $\langle A_p \varphi, y \rangle = \langle \varphi, f \rangle$, for each $\varphi \in D(A_p)$; hence it is equivalent to

(3.7.7) $$\langle \lambda\varphi - (v \cdot \nabla)\varphi - (\operatorname{div} v)\varphi + A^* \varphi, y \rangle = \langle \varphi, f \rangle \quad , \quad \forall \varphi \in D(A_p).$$

In conclusion, $y = B_p^* f$ is a weak solution of (3.1.1).
Moreover $\| y \|_{-1,p'} = \| B_p^* f \|_{-1,p'} \leqslant \| B_p^* \|_{\mathcal{L}(W^{-1,p'})} \| f \|_{-1,p'} = \| B_p \|_{\mathcal{L}(W_o^{1,p})} \| f \|_{-1,p'}$.
In the other hand (3.7.5) can be written $(\lambda - \vartheta_1^*) \| B_p \, g \|_{1,p} \leqslant \| g \|_{1,p}$.

These last two inequalities imply (3.7.2). ***

Remark. We have the uniqueness of the weak solution of (3.1.1) in $W^{-1,p'}$ if it is defined by using the equation (3.7.7) instead of (3.7.1).

Let us recall the following theorem used in the proof above:

Theorem. Let X and Y be two normed linear spaces and T a closed operator from X into Y. Assume $\overline{D(T)} = X$. If T^{-1} exists and $T^{-1} \in \mathcal{L}(Y,X)$, then $(T^*)^{-1}$ exists and $(T^*)^{-1} \in \mathcal{L}(X^*, Y^*)$. Moreover $(T^*)^{-1} = (T^{-1})^*$.

In the other hand, if Y is complete and $(T^*)^{-1}$ exists and $(T^*)^{-1} \in \mathcal{L}(X^*, Y^*)$, then T^{-1} exists, $T^{-1} \in \mathcal{L}(X,Y)$, and $(T^*)^{-1} = (T^{-1})^*$.

Note that $\| T \|_{\mathcal{L}(X,Y)} = \sup\limits_{\substack{x \in X \\ \|x\|_X \leq 1}} \| Tx \|_Y$.

3.8. A PRIORI ESTIMATE AND EXISTENCE THEOREM OF (3.1.1) IN $W^{j,p}$.

In this section we prove theorem 3.1.1. We start from the main a priori bound.

Theorem 3.8.1. Let $p \in]1,+\infty[$ and $j > -1$ verify $(j+2)p > n$. Let $v \in W^{j+3,p}$ satisfy (3.1.2), $A \in W^{j+2,p}$ and $f \in W^{j,p}$.

There exists a positive constant $\bar{c} = c(n,N,p,j,\Omega)$ such that if λ verifies (3.1.3) and $y \in W^{j+1,p}$ is a solution of (3.1.1), then

(3.8.1)
$$(\lambda - \lambda_j)\, \|y\|_{j,p} \leq \|f\|_{j,p} \, .$$

Proof. Let $\delta = (\delta_1, \dots, \delta_n)$ be a multi-index such that $|\delta| = j$. By applying the operator D^δ to both sides of (3.1.1) one gets

(3.8.2)
$$\lambda D^\delta y + (v \cdot \nabla) D^\delta y + \sum_{i=1}^{n} \sum_{1 \leq |\eta| \leq j} \binom{\delta}{\eta} D^\eta v \, D^{\delta-\eta} (D_i y) + \\ + \sum_{0 \leq |\eta| \leq j} \binom{\delta}{\eta} D^\eta A \, D^{\delta-\eta} y = D^\delta f \quad , \text{ in } \Omega .$$

We define here $\Lambda = (@ + |D^j y|^2)^{1/2}$, where $@$ is a positive parameter, and $|D^j y|^2 = \sum_{|\delta|=j} \sum_{i=1}^{N} |D^\delta y_i|^2$. Similarly to some previous proof, we take the scalar product on $\mathbb{R}^N$ of (3.8.2) with $\Lambda^{p-2} D^\delta y$. By adding then side by side for all indexes δ such that $|\delta| = j$, by integrating in Ω, and by taking in account that the third term (resp. last term) on the left hand side of (3.8.2) is bounded by $c\|v\|_{j+3,p} \|y\|_{j,p}$ (resp. $c\|A\|_{j+2,p} \|y\|_{j,p}$), it follows that

(3.8.3) $$\lambda\int_{\Omega}|D^j y|^2 \Lambda^{p-2}\, dx \leqslant (1/p)\,|\mathrm{div}\, v|_\infty |\Lambda|_p^p \;+$$

$$+\, c(\|v\|_{j+3,p} + \|A\|_{j+2,p})\|y\|_{j,p}|\Lambda|_p^{p-1} + |D^j f|_p |\Lambda|_p^{p-1} \;.$$

The Lebesgue's dominated convergence theorem can be applied as @ $\longrightarrow 0^+$. It follows

(3.8.4) $$\lambda|D^j y|_p \leqslant \bar{c}(\|v\|_{j+3,p} + \|A\|_{j+2,p})\|y\|_{j,p} + |D^j f|_p \;.$$

Since (3.8.4) holds for every integer j_o , $0 \leqslant j_o \leqslant j$, by adding side by side all that estimates for $j_o = 0,1,...,j$, one gets (3.8.1). ***

Now we are able to proof the main result of this chapter.

Proof of theorem 3.1.1:

i) We prove first the statement of theorem 3.1.1 for $-1 \leqslant j \leqslant 2$.

Let I be a fixed ball such that $\bar{\Omega} \subset I$, and define maps $T_1 \in \mathcal{L}(W^{j+3,p}, W_o^{j+3,p}(I))$, $T_2 \in \mathcal{L}(W^{j+2,p}, W^{j+2,p}(I))$ such that $(T_1\, v)|_\Omega = v$, $(T_2\, A)|_\Omega = A$. If $j = 0$ [resp. $j = -1$], let $T_3 \in \mathcal{L}(L^p, L^p(I))$ [resp. $T_3 \in \mathcal{L}(W^{-1,p}, W^{-1,p}(I))$] such that $(T_3\, f)|_\Omega = f$ {in the case $j = 0$ we define $(T_3\, f)(x) = f(x)$, for $x \in \Omega$, $(T_3\, f)(x) = 0$ for $x \notin \Omega$; therefore $|f|_{p,\Omega} = |T_3\, f|_{p,I}$ }. For $j = 1$ and $j = 2$ we define $T_3 \in \mathcal{L}(W^{j,p}, W_o^{j,p}(I))$ such that $(T_3\, f)|_\Omega = f$. We set $\tilde{v} = T_1\, v$, $\tilde{A} = T_2\, A$, $\tilde{f} = T_3\, f$. Without difficulties one proves using Sobolev's inequalities that the coefficients $\tilde{v}$ and $\tilde{A}$ verify in the ball I the assumptions of theorem 3.7.1 for $j = -1$ (we have to use the statement of theorem 3.7.1 with the roles of p and p' exchanged), of theorem 3.6.1 for $j = 0$, of theorem 3.5.1 for $j = 1$, and of theorem 3.3.1 for $j = 2$.

Let us give for example the proof in the case $j = 0$. Our hypotheses are $p \in]n/2, +\infty[$, $\tilde{v} \in W_o^{3,p}(I)$, $\tilde{A} \in W^{2,p}(I)$. Since $2p > n$, we get $\tilde{v} \in W^{1,\infty}(I)$ [resp. $\tilde{A} \in L^\infty(I)$].

If $p > n$, we have immediately $\tilde{v} \in W^{2,r}(I)$ [resp. $\tilde{A} \in W^{1,r}(I)$] .
If $p = n$, we have $r > n$. Since $W^{1,p} \hookrightarrow L^q$, $p \leqslant q < +\infty$, it follows $\tilde{v} \in W^{2,r}(I)$ [resp. $\tilde{A} \in W^{1,r}(I)$].
If $p < n$, we have $r = n$. Since $W^{1,p} \hookrightarrow L^q$, $p \leqslant q \leqslant np/(n-p)$, and $q = r$ is allowed , we get $\tilde{v}$ and $\tilde{A}$ as above. From $\tilde{v} \in W_0^{1,p}(I)$ follows obviously (3.1.2).

Then by applying one of the theorems above , depending on the value of j, there exists a constant $c = c(n,N,p,j,\Omega)$ such that if $\lambda > \tilde{\lambda}_j \equiv c(\|\tilde{v}\|_{j+3,p} + \|\tilde{A}\|_{j+2,p})$ and $\tilde{f} \in W^{j,p}(I)$ we get a solution $\tilde{y} \in W^{j,p}(I)$ of $\lambda y + (\tilde{v}\cdot\nabla)y + \tilde{A}y = \tilde{f}$ in I, and $(\lambda - \tilde{\lambda}_j)\|\tilde{y}\|_{j,p} \leqslant \|\tilde{f}\|_{j,p}$. It follows that $y = \tilde{y}|_\Omega$ is a solution of (3.1.1) in Ω .

ii) We prove now the statement of theorem 3.1.1 under the assumption $j \geqslant -1$ and $p \in]n,+\infty[$.

For $j \leqslant 2$ this was proved in part i). Let us proceed by induction on j. Hence, we assume that the thesis hold for a value $j \geqslant 1$, and we will prove it for the value j+1.

Let $v \in W^{j+4,p}$, $v \cdot n = 0$ on Γ , $A \in W^{j+3,p}$, $f \in W^{j+1,p}$ and assume that λ verifies (3.1.3). By the induction hypothesis there exists a unique solution $y \in W^{j,p}$ of (3.1.1). Moreover (3.1.4) holds . By applying the differential operator D_i to both sides of (3.1.1), we get for each $i = 1,\ldots,n$

$$\lambda D_i y + (v\cdot\nabla)D_i y + AD_i y + [(D_i v)\cdot\nabla]y = D_i f - (D_i A)y \quad , \text{in } \Omega . \tag{3.8.5}$$

Note that $D_i v \in W^{j+3,p}$ and can therefore be regard as A. Moreover $(D_i A)y \in W^{j,p}$. (3.8.5) is therefore again a system of type (3.1.1) on the nN unknowns $D_i y_j$. By the induction hypothesis there exist $\bar{c} = c(nN,n,j,p,\Omega)$ and $\hat{c}(nN,n,j,p,\Omega)$ such that if $\lambda > \bar{\lambda}_j \equiv \bar{c}(\|v\|_{j+3,p} + \|A\|_{j+2,p})$ one has $H = (h_{ji})$, $1 \leqslant j \leqslant N$, $1 \leqslant i \leqslant n$, $H \in W^{j,p}$, satisfying (3.8.5) and $(\lambda - \bar{\lambda}_j)\|H\|_{j,p} \leqslant \hat{c}(\|Df\|_{j,p} + \|A\|_{j+1,p}\|y\|_{j,p})$. In the other hand, also $D_i y_j \in W^{j-1,p}$ is a solution of (3.8.5) for each $i = 1,\ldots,n$. From the previous theorem it follows $H = Dy$. Therefore $y \in W^{j+1,p}$ and

$$(\lambda - \bar{\lambda}_j)\|Dy\|_{j,p} \leqslant \hat{c}(\|Df\|_{j,p} + \|A\|_{j+1,p}\|y\|_{j,p}). \tag{3.8.6}$$

The estimate (3.1.4) for j+1 follows from the estimate (3.1.4) for j together with (3.8.6).

Set for instance $\bar{c} = c(N,n,j,p,\Omega) = \max\{c(N,n,j,p,\Omega),\ c(nN,n,j,p,\Omega)\} + \hat{c}(nN,n,j,p,\Omega)$, and $\hat{c}(N,n,j,p,\Omega) = \max\{\hat{c}(N,n,j,p,\Omega), \hat{c}(nN,n,j,p,\Omega)\}$.

iii) Finally, let $j \geqslant 1$ and $p \in]n/(j+2),+\infty[$ be fixed (the cases $k = -1$ and $k = 0$ are already proved). Fix a real $p_o > p$, $p_o > n$, and let $v \in W^{j+4,p_o}$ verify (3.1.2) and $A \in W^{j+3,p_o}$. Let $\{f_m\}$ be a sequence of functions belonging to W^{j+1,p_o}, such that $f_m \to f$ in $W^{j,p}$. Let λ be such that

$$(3.8.7) \qquad \lambda > \mu \equiv c(N,n,j+1,p_o,\Omega)(\|v\|_{j+4,p_o} + \|A\|_{j+3,p_o})$$

and let $y_m \in W^{j+1,p_o}$ (hence $y_m \in W^{j+1,p}$) be the solution of $\lambda y + (v\cdot\nabla)y + Ay = f_m$, whose existence was proved in ii). We can apply theorem 3.8.1. We get from (3.8.1) that $\{y_m\}$ is a Cauchy sequence in $W^{j,p}$, and converges so in $W^{j,p}$ to a function y. The function y satisfies (3.1.1) and verifies (3.1.4).

If $\lambda \in]\lambda_j, \mu]$, let us fix a value $\bar{\lambda} > \mu$. We consider the problem $\bar{\lambda}y + (v\cdot\nabla)y + Ay = f + (\bar{\lambda} - \lambda)w$, $w \in W^{j,p}$. From above follows that there exists a solution y of this problem. Let us denote it by $y = Tw$. If $\bar{y} = T\bar{w}$, from (3.1.4) follows $(\bar{\lambda} - \lambda_j)\|y-\bar{y}\|_{j,p} \leqslant (\bar{\lambda} - \lambda)\|w-\bar{w}\|_{j,p}$. Since T is a contraction in $W^{j,p}$ there exists a unique fixed point $y = Ty$ in $W^{j,p}$; y is then a solution of (3.1.1) and satisfies (3.1.4).

Let us now suppose $v \in W^{j+3,p}$, $v\cdot n = 0$ on Γ, $A \in W^{j+2,p}$. We consider sequences $\{v_m\} \in W^{j+4,p_o}$, $v_m\cdot n = 0$ on Γ, $\{A_m\} \in W^{j+3,p_o}$ such that $v_m \to v$ in $W^{j+3,p}$ and $A_m \to A$ in $W^{j+2,p}$, as $m \to +\infty$. From above follows that if $\lambda > \bar{c}(\|v_m\|_{j+3,p} + \|A_m\|_{j+2,p})$ (note that from our assumption (3.1.3) this follows for m sufficiently large) there exists a solution y_m of the problem $\lambda y + (v_m\cdot\nabla)y + A_m y = f$ in Ω; moreover $(\lambda - \lambda_j)\|y_m\|_{j,p} \leqslant \hat{c}\|f\|_{j,p}$. Let y be the weak limit in $W^{j,p}$ of a subsequence of the sequence $\{y_m\}$ and y is the desired solution of problem (3.1.1). ***

4. THE STATIONARY, COMPRESSIBLE NAVIER–STOKES EQUATIONS: EXISTENCE OF SOLUTIONS (PROOF OF THEOREM A)

4.1. THE LINEARIZED SYSTEM

In this section we consider the linear system

$$(4.1.1) \qquad \begin{cases} -\mu \Delta u - \nu \nabla \operatorname{div} u + k \nabla \sigma = F(x) , \\ m \operatorname{div} u + v \cdot \nabla \sigma + \sigma \operatorname{div} v = g , \quad \text{in } \Omega , \\ u\big|_{\Gamma} = 0 . \end{cases}$$

where $F(x) = (F_1(x),\dots,F_n(x))$, $v(x) = (v_1(x),\dots,v_n(x))$ are given vector functions and g is a given scalar function on Ω. Let m, k, μ, ν are constants as defined in the introduction. We are looking for a pair $(u = (u_1(x),\dots,u_n(x)), \sigma)$ solution of (4.1.1).

It should be emphasized that (4.1.1) is not an elliptic system in the sense of Agmon–Douglis–Nirenberg (see appendix), except if $v(x)$ vanishes identically in Ω. In fact, let us consider the system (4.1.1), where for convenience we assume $\mu = m = k = 1$ and $\nu = 0$. We have

$$(4.1.2) \qquad \begin{cases} -\Delta u + \nabla \sigma = F , \\ \operatorname{div} u + v \cdot \nabla \sigma + \sigma \operatorname{div} v = g , \quad \text{in } \Omega , \\ u\big|_{\Gamma} = 0 . \end{cases}$$

The system of partial differential equations $(4.1.2)_1$ and $(4.1.2)_2$ can be written as a system of type (A.1), where $l_{ij}(x,\xi)$, $x = (x_1,\dots,x_n) \in \Omega$, is the matrix

$$\begin{aligned} l_{ij}(x,\xi) &= |\xi|^2 \delta_{ij} , && 1 \leqslant i,j \leqslant n \\ l_{n+1,j}(x,\xi) &= -l_{j,n+1}(x,\xi) = \xi_j , && 1 \leqslant j \leqslant n \\ l_{n+1,n+1}(x,\xi) &= v(x) \cdot \xi + a(x) . \end{aligned}$$

We indicate $a(x) = \operatorname{div} v(x)$. As in the proof of theorem 2.1.2we introduce two systems of integer weights s_i and t_j . Let us distinguish two cases:

If $v(x) \neq 0$, we assume $s_i = 0$ for $1 \leqslant i \leqslant n+1$ and $t_j = 2$ for $1 \leqslant j \leqslant n$, $t_{n+1} = 1$. In this case it follows that $l'_{ij}(x,\xi) = l_{ij}(x,\xi)$ for $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant n+1$, and $l'_{n+1,j}(x,\xi) = 0$ for $1 \leqslant j \leqslant n$, $l'_{n+1,n+1}(x,\xi) = v(x)\cdot\xi$. Therefore $L(x,\xi) = (v(x)\cdot\xi)|\xi|^{2n}$.

If $v(x) = 0$, we assume $s_i = 0$ for $1 \leqslant i \leqslant n$, $s_{n+1} = -1$ and $t_j = 2$ for $1 \leqslant j \leqslant n$, $t_{n+1} = 1$. In this case it follows that $l'_{ij}(x,\xi) = l_{ij}(x,\xi)$ for $1 \leqslant i \leqslant n+1$, $1 \leqslant j \leqslant n$, and $l'_{n+1,n+1}(x,\xi) = l_{n+1,n+1}(x,\xi) = a(x)$. It follows that $L(x,\xi) = (1 + a(x))|\xi|^{2n}$.

Consequently, the elliptic condition $L(x,\xi) \neq 0$ for real $\xi \neq 0$ (see (A.5)) can not be satisfied unless $v(x) = 0$ for all $x \in \Omega$.

If $v(x) = 0$ for all $x \in \Omega$ and if a is small (for instance, if $|a|_\infty < 1$), the system is elliptic. However, this last property can not be used to treat the term $v\cdot\nabla\sigma$ as a perturbation term. In fact, by assuming that $\sigma \in W^{1,p}$, the term $v\cdot\nabla\sigma$ belongs just to L^p . In this situation the equation $(4.1.2)_2$ yields $\operatorname{div} u \in L^p$, and equation $(4.1.2)_1$ gives $u \in W^{1,p}$, $\sigma \in L^p$. Hence, from the point of view of regularity we lose one derivative.

Let us explain so the ideas used to solve the linear system (4.1.1) and we state the main result of this section :

Theorem 4.1.1. Let $p \in]1,+\infty[$ and $j > -1$ verify $(j+2)p > n$. Let $F \in W^{j+1,p}$, $g \in \overline{W}_0^{j+2,p}$. There exist positive constants c, c_2 and $\gamma = \gamma(n,p,j,\mu,\nu,m,\Omega)$, defined by equation (4.1.30), such that if $v \in W_{0,d}^{j+3,p}$ verifies the condition

$$\|v\|_{j+3,p} \leqslant \gamma k , \tag{4.1.3}$$

then there exists a unique solution $(u,\sigma) \in W_{0,d}^{j+3,p} \times \overline{W}^{j+2,p}$ of problem (4.1.1).

Moreover

$$\mu\|u\|_{j+3,p} + k\|\sigma\|_{j+2,p} \leqslant c\left(1 + \frac{\mu+|\nu|}{\mu+\nu}\right)\|F\|_{j+1,p} + c_2\frac{\mu+|\nu|}{m}\|g\|_{j+2,p} . \tag{4.1.4}$$

Proof: By applying the divergence operator to both sides of equation $(4.1.1)_1$, and by applying the Laplace operator to both sides of equation $(4.1.1)_2$ one gets

(4.1.5) $$-(\mu+\nu)\Delta \operatorname{div} u + k\Delta\sigma = \operatorname{div} F \ ,$$

and

(4.1.6) $$m\,\Delta \operatorname{div} u + v\cdot\nabla\Delta\sigma = \Delta g - [2\,\nabla v : \nabla^2\sigma + \Delta v\cdot\nabla\sigma + \Delta(\sigma \operatorname{div} v)] \ ,$$

respectively in Ω. We recall that $\nabla v : \nabla^2\sigma = \sum_{i,j=1}^{n} (D_i v_j)(D_i D_j \sigma)$. By adding side by side these two equations one obtains

(4.1.7) $$\frac{mk}{\mu+\nu}\Delta\sigma + v\cdot\nabla\Delta\sigma = G(F,g,\sigma), \quad \text{in } \Omega \ ,$$

where by definition

(4.1.8) $$G(F,g,\sigma) = \Delta g + \frac{m}{\mu+\nu} \operatorname{div} F - [2\,\nabla v : \nabla^2\sigma + \Delta v\cdot\nabla\sigma + \Delta(\sigma \operatorname{div} v)] \ , \text{in } \Omega .$$

In order to be able to solve equation (4.1.7) for $\Delta\sigma$ we replace $G(F,g,\sigma)$ by $G(F,g,\tau)$ where $\tau \in \overline{W}^{j+2,p}$ is an arbitrary function. We consider so the linear equation

(4.1.9) $$\frac{mk}{\mu+\nu}\, y + v\cdot\nabla y = G(F,g,\tau) \quad , \quad \text{in } \Omega \ ,$$

where formally y "should be regarded as $\Delta\sigma$ " (actually, $y = \Delta\sigma$ if $\tau = \sigma$ is a solution of (4.1.1)).

Note that G is a linear map from $W^{j+1,p} \times \overline{W}_0^{j+2,p} \times \overline{W}^{j+2,p}$ into $W^{j,p}$ and by using Sobolev's inequalities one gets

(4.1.10) $$\| G \|_{j,p} \leqslant \| g \|_{j+2,p} + \frac{m}{\mu+\nu} \| F \|_{j+1,p} + c\, \| v \|_{j+3,p} \| \tau \|_{j+2,p} \ .$$

We recall that under our assumptions on j and Ω, $W^{j+2,p}$ is a Banach algebra.

From chapter 3, theorem 3.1.1, it follows that there exist positive constants c and c_1 such that if $\|v\|_{j+3,p} \leq c_1\, mk/(\mu+\nu)$, then there exists a bounded linear map $L \in \mathcal{L}(W^{j,p})$ such that $y = LG$ is a solution of problem (4.1.9) and one has

$$\frac{mk}{2(\mu+\nu)} \|y\|_{j,p} \leq c \|G\|_{j,p} . \tag{4.1.11}$$

(Note that in this case $\lambda_j = mk/2(\mu+\nu)$, $\bar{c} = 1/2c_1$, $\hat{c} = c$)

Recall that if $j = 0$, y is a weak solution of problem (4.1.9). In other words, y satisfies

$$\left(\frac{mk}{\mu+\nu} y,\varphi\right) - (\operatorname{div}(v\varphi),y) = (G,\varphi) \qquad \text{, for every } \varphi \in W_0^{1,p'} .$$

Using (4.1.10) in (4.1.11) one gets

$$\frac{mk}{\mu+\nu} \|y\|_{j,p} \leq c\left(\frac{m}{\mu+\nu}\|F\|_{j+1,p} + \|g\|_{j+2,p} + \|v\|_{j+3,p}\|\tau\|_{j+2,p}\right) . \tag{4.1.12}$$

Now let $\vartheta \in W_0^{j+2,p}$ (see [14]) be the solution of the Dirichlet problem

$$\begin{aligned} (\mu+\nu)\Delta\vartheta &= k\,y - \operatorname{div} F , \quad \text{in } \Omega , \\ \vartheta\big|_\Gamma &= 0 . \end{aligned} \tag{4.1.13}$$

Moreover

$$\begin{aligned} (\mu+\nu)\|\vartheta\|_{j+2,p} &\leq c\,\|F\|_{j+1,p} + ck\,\|y\|_{j,p} \leq \\ &\leq c\,\|F\|_{j+1,p} + c\,\frac{\mu+\nu}{m}\,[\,\|g\|_{j+2,p} + \|v\|_{j+3,p}\|\tau\|_{j+2,p}\,]. \end{aligned} \tag{4.1.14}$$

We point out that the function ϑ should be regarded as div u. Furthermore, the boundary condition $(4.1.13)_2$ is suggested by the property $(\operatorname{div} u)|_\Gamma = 0$, and this follows from $(1.15)_2$. (This property allows us to impose the condition $(\operatorname{div} v)|_\Gamma = 0$ on the coefficient v appearing in equation $(4.1.1)_2$).

Let us define

$$\vartheta_0(x) = \vartheta(x) - \overline{\vartheta} \,. \tag{4.1.15}$$

Obviously, the function ϑ_0 verifies the estimate $\|\vartheta_0\|_{j+2,p} \leqslant c\, \|\vartheta\|_{j+2,p}$. Since $\overline{\vartheta_0} = 0$, the theorem 2.1.1 implies the existence of a unique solution $(u,\sigma) \in W_0^{j+3,p} \times \overline{W}^{j+2,p}$ of the non–homogeneous linear Stokes problem

$$\begin{cases} -\mu\, \Delta u + k\nabla\sigma \;=\; F + \nu\, \nabla\vartheta_0 \quad , \\ \operatorname{div} u = \vartheta_0 \qquad , \text{ in } \Omega \quad , \\ u\big|_\Gamma = 0 \,. \end{cases} \tag{4.1.16}$$

Moreover (u,σ) verifies the estimate

$$\mu\, \| u\|_{j+3,p} + k\, \|\sigma\|_{j+2,p} \leqslant c\, [\, \|F\|_{j+1,p} + (\mu + |\nu|\,)\|\vartheta_0\|_{j+2,p} \,]. \tag{4.1.17}$$

Putting together this last estimate with the estimate (4.1.14) one gets

$$\begin{aligned} \mu\, \| u\|_{j+3,p} + k\, \|\sigma\|_{j+2,p} \leqslant c\left(1 + \frac{\mu + |\nu|}{\mu + \nu}\right) \|F\|_{j+1,p} \; + \\ + \, c_2\, \frac{\mu + |\nu|}{m} \;[\, \|g\|_{j+2,p} + \|v\|_{j+3,p}\, \|\tau\|_{j+2,p} \,]. \end{aligned}$$

The inequalities (4.1.3) and (4.1.30) imply

$$\mu\|u\|_{j+3,p} + k\|\sigma\|_{j+2,p} \leqslant (k/2)\|\tau\|_{j+2,p} + c\left(1 + \frac{\mu+|\nu|}{\mu+\nu}\right)\|F\|_{j+1,p} + c_2\, \frac{\mu+|\nu|}{m}\|g\|_{j+2,p} \,. \tag{4.1.18}$$

We call now attention to the sequence of linear maps

$$(F,g,\tau) \longrightarrow (F,G) \longrightarrow (F,y) \longrightarrow (F,\vartheta) \longrightarrow (F,\vartheta_0) \longrightarrow (u,\sigma)$$

where F is left unchanged, and the elements G, y, ϑ, ϑ_0, (u,σ) are defined by equations (4.1.8), (4.1.9), (4.1.13), (4.1.15), (4.1.16), respectively. The product map is linear and continuous by (4.1.18). Hence, if (u_1, σ_1) is the solution corresponding to the data

(F, g, τ_1) it follows that $(u - u_1, \sigma - \sigma_1)$ is the solution corresponding to the data $(0,0, \tau - \tau_1)$.

Therefore the estimate (4.1.18) implies $\|\sigma - \sigma_1\|_{j+2,p} \leqslant (1/2)\|\tau - \tau_1\|_{j+2,p}$ and the map $\tau \longrightarrow \sigma$ is a contraction in $\overline{W}^{j+2,p}$. Hence it has a (unique) fixed point $\sigma = \tau$.

Finally, we prove that the pair (u, σ) corresponding to the fixed point $\tau = \sigma$ is a solution of (4.1.1). We note that $(4.1.1)_1$ and $(4.1.1)_3$ follow immediately from (4.1.16). The main point is to prove $(4.1.1)_2$.

From (4.1.13) it follows

$$y = \frac{\mu + \nu}{k} \operatorname{div} u + \frac{1}{k} \operatorname{div} F \ , \quad \text{in } \Omega . \tag{4.1.19}$$

The equations (4.1.15) and $(4.1.16)_2$ yield $\Delta \vartheta = \Delta \operatorname{div} u$. By applying the divergence operator to both sides of equation $(4.1.16)_1$ and using (4.1.19) one gets $y = \Delta\sigma$.

Let us first suppose $j > 0$. Let y be replaced by (4.1.19) in the first term on the left hand side of (4.1.9) and by $\Delta\sigma$ in the second term on the left hand side of (4.1.9). This yields

$$\Delta\ (m \operatorname{div} u + v\cdot\nabla\sigma + \sigma \operatorname{div} v - g + m\overline{\vartheta}\,) = 0\ , \quad \text{in } \Omega . \tag{4.1.20}$$

On the other hand, since $(\operatorname{div} u)|_\Gamma = -\overline{\vartheta}$, $v|_\Gamma = 0$, $(\operatorname{div} v)|_\Gamma = 0$, $g|_\Gamma = 0$, one has $(m \operatorname{div} u + v\cdot\nabla\sigma + \sigma\operatorname{div} v - g)|_\Gamma = - m\overline{\vartheta}$.

Therefore taking in account (4.1.20) it follows that

$$m \operatorname{div} u + \operatorname{div}(\sigma v) - g + m\overline{\vartheta} = 0 \ , \quad \text{in } \Omega .$$

By integrating in Ω both sides of this equation we obtain $\overline{\vartheta} = 0$. This proves $(4.1.1)_2$ in the case $j > 0$.

Let us now suppose $j = 0$. y in this case satisfies the equation

$$\left(\frac{mk}{\mu+\nu}\, y, \varphi\right) - (\operatorname{div}(v\varphi), y) = (G, \varphi) \ , \quad \forall \varphi \in W_0^{1,p'} . \tag{4.1.21}$$

We replace y by the right hand side of (4.1.19) in the first term on the left hand side of the last expression and by $\Delta\sigma$ in the second term on the left hand side of the last expression. This yields for every $\varphi \in W_0^{1,p'}$

(4.1.22) $(m\,\Delta \operatorname{div} u, \varphi) - (\operatorname{div}(v\varphi), \Delta\sigma) = (\Delta g - 2\nabla v : \nabla^2\sigma - \Delta v \cdot \nabla\sigma - \Delta(\sigma \operatorname{div} v), \varphi)$.

We claim that for every $\varphi \in W_0^{1,p'}$

(4.1.23) $$-(\operatorname{div}(v\varphi), \Delta\sigma) + (2\nabla v : \nabla^2\sigma + \Delta v \cdot \nabla\sigma, \varphi) = (\Delta(v \cdot \nabla\sigma), \varphi).$$

Let $\sigma_n \in C^3(\Omega)$, $\sigma_n \to \sigma$ in $W^{2,p}$ as $n \longrightarrow +\infty$. The identity (4.1.23) follows for the functions σ_n from the formula $\Delta(v \cdot \nabla\sigma_n) = v \cdot \nabla\Delta\sigma_n + 2\nabla v : \nabla^2\sigma_n + \Delta v \cdot \nabla\sigma_n$. By passing to the limit as $n \longrightarrow +\infty$ we prove (4.1.23) for σ. From (4.1.22) and (4.1.23) we get

(4.1.24) $$\Delta(m \operatorname{div} u + m\overline{\vartheta} + v \cdot \nabla\sigma + \sigma \operatorname{div} v - g) = 0 \quad , \text{ in } \Omega .$$

As above we can immediately conclude that

(4.1.25) $$m \operatorname{div} u + m\overline{\vartheta} + v \cdot \nabla\sigma + \sigma \operatorname{div} v - g = 0 \quad , \text{ in } \Omega ,$$

since the left hand side of (4.1.25) is harmonic in Ω and vanishes on Γ. Then by integrating in Ω both sides of this equation we conclude $\overline{\vartheta} = 0$. This proves $(4.1.1)_2$ in the case $j = 0$.

(Note that in the case $j = -1$ this argument has to be carefully handled since the function $\nabla\sigma$ has not a trace on Γ. See reference [4].)

We have so proved the existence part of theorem 4.1.1.

We start now by proving the uniqueness of the solution of the linear system (4.1.1) valid under our assumption (4.1.3).

Let (u_1, σ_1), (u_2, σ_2) be two solutions of (4.1.1) with data F and g. Then $(u, \sigma) = (u_1 - u_2, \sigma_1 - \sigma_2)$ is a solution of (4.1.1) with data $F = 0$, $g = 0$.

By multiplying both sides of system (4.1.1) by mu and of equation (4.1.1) by $k\sigma$, by integrating over Ω and by adding side by side the two equations one gets

$$(4.1.26) \qquad (\mu+\nu)m \int_\Omega |Du|^2\,dx = -(k/2)\int_\Omega \sigma^2 \mathrm{div}\, v\,dx \quad .$$

Therefore by putting $\mu_0 = \min\{\mu , \mu+\nu\}$ it follows

$$(4.1.27) \qquad \|u\|^2_{1,2} \leqslant c(k/2m\mu_0)\,|\mathrm{div}\, v|_\infty\, |\sigma|^2_2 \quad .$$

Moreover from $\bar\sigma = 0$ and $(4.1.1)_2$ it follows that

$$(4.1.28) \qquad k|\sigma|_2 \leqslant ck\,\|\nabla\sigma\|_{-1,2} \leqslant c(\mu+\nu)\,\|u\|_{1,2} \quad .$$

Hence

$$(4.1.29) \qquad \|u\|^2_{1,2} \leqslant c[(\mu+|\nu|)^2/mk\mu_0]\,\|v\|_{j+3,p}\,\|u\|^2_{1,2} \quad .$$

This implies that for a suitable constant c_0 the solution of (4.1.1) is unique whenever $\|v\|_{j+3,p} \leqslant (mk\mu_0)/[c_0(\mu+|\nu|)^2]$. For convenience we set

$$(4.1.30) \qquad \gamma = \min\left\{\frac{c_1 m}{\mu+\nu} , \frac{m}{2c_2(\mu+|\nu|)} , \frac{m\mu_0}{c_0(\mu+|\nu|)^2}\right\}.$$

Then inequality (4.1.30) includes all the assumptions on $\|v\|_{j+3,p}$ utilized in proving theorem 4.1.1. ***

4.2. THE NON-LINEAR PROBLEM

In this section we prove the existence and uniqueness theorem A in the case $j > -1$. Recalling (1.1.4) we set

$$T_i = \sup_{I(l,l_1)} |\omega_i(\sigma,\alpha)| \ , \qquad S_i = \|\omega_i(\sigma,\alpha)\|_{\mathscr{C}^{j+2}(I(l,l_1))} \ , \ i = 1,2.$$

We denote by c_3 a positive constant such that

$$(4.2.1) \qquad |\tau|_\infty \leqslant c_3 \|\tau\|_{j+2,p} \ , \qquad |\beta|_\infty \leqslant c_3 \|\beta\|_{j+2,p} \ , \forall \tau \in \overline{W}^{j+2,p} \ , \forall \beta \in W_0^{j+2,p} \ .$$

For convenience, in this section we denote by c', c'_i , $i \geqslant 0$ positive constants depending at most on n, p, j, Ω , μ , ν , k, m, ζ_0 , c_v , χ , χ_0 , χ_1 , l, l_1 , T_i , S_i (i=1,2). The symbol c' may be utilized even in the same equation to denote distinct constants.

In order to prove theorem A for $j > -1$ we prove the following statement:

Lemma 4.2.1. Let $p \in]1,+\infty[$ and $j > -1$ verify $(j+2)p > n$. Let $\tau \in \overline{W}^{j+2,p}$, $\beta \in W_0^{j+2,p}$ verify the assumptions

$$(4.2.2) \qquad \|\tau\|_{j+2,p} \leqslant l/c_3 \ , \qquad \|\beta\|_{j+2,p} \leqslant l_1/c_3 \ ,$$

and $v \in W^{j+3,p}$, $(f,g,h) \in W^{j+1,p} \times \overline{W}_0^{j+2,p} \times W^{j+1,p}$. Then

$$(4.2.3) \left\{ \begin{aligned} &\|F(f,v,\tau,\beta)\|_{j+1,p} \leqslant c'(\|f\|_{j+1,p} + c'\|v\|^2_{j+3,p}) + c'S_1\|\tau\|^2_{j+2,p} + c'S_2\|\beta\|^2_{j+2,p} \\ &\|H(h,v,\tau,\beta)\|_{j+1,p} \leqslant c'\|h\|_{j+1,p} + c'c_v m\|v\|_{j+2,p}\|\beta\|_{j+2,p} + c'(\chi_0+\chi_1)\|v\|^2_{j+3,p} \\ &\qquad + c'\frac{\zeta_0 S_2}{m}(\|v\|_{j+2,p}\|\tau\|_{j+2,p} + \|g\|_{j+1,p}). \end{aligned} \right.$$

The verification of this result is based on the following lemma:

Lemma 4.2.2. Let $p \in]1,+\infty[$ and $j > -1$ verify $(j+2)p > n$. Let Ω be a domain in $\mathbb{R}^n$ having the cone property. Assume $u \in W^{j+2,p}$, $v \in W^{j+1,p}$.

Then the product uv, defined pointwise a.e. in Ω , belongs to $W^{j+1,p}$ and there exists a constant $K = K(n,p,j,\Omega)$ such that

$$\|uv\|_{j+1,p} \leqslant K \|u\|_{j+2,p} \|v\|_{j+1,p} \quad . \tag{4.2.4}$$

Proof of lemma 4.2.2. We recall that Sobolev embedding theorems for $w \in W^{r,p}$, r a non-negative integer, imply that for every index δ, $|\delta| \leqslant r$ there exists a constant $K(\delta) = K(\delta,n,p,r,\Omega)$ such that

$$|D^{\delta} w(x)| \leqslant K(\delta) \|w\|_{r,p} \quad \text{a.e. on } \Omega \text{, if } (r-|\delta|)p > n; \tag{4.2.5}$$

$$\int_{\Omega} |D^{\delta} w(x)|^{q} \, dx \leqslant K(\delta) \|w\|^{q}_{r,p} \ , \quad \begin{array}{l} \text{for } p \leqslant q \leqslant np/[n-(r-|\delta|)p], \text{ if } (r-|\delta|)p < n; \\ \text{for } p \leqslant q < +\infty \text{ , if } (r-|\delta|)p = n. \end{array} \tag{4.2.6}$$

In order to establish (4.2.4) it is sufficient to show that, if $|\eta| \leqslant j+1$, then

$$\int_{\Omega} |D^{\eta}(uv)|^{p} \, dx \leqslant K_{\eta} \|u\|^{p}_{j+2,p} \|v\|^{p}_{j+1,p} \quad ,$$

where $K_{\eta} = K_{\eta}(n,p,j,\Omega)$.

Let us assume for the moment that $v \in C^{\infty}(\Omega)$. By Leibniz's rule for distributional derivatives, that is,

$$D^{\eta}(uv) = \sum_{\delta \leq \eta} \binom{\eta}{\delta} D^{\delta} u D^{\eta-\delta} v,$$

it is sufficient to show that for any $\delta \leqslant \eta$, $|\eta| \leqslant j+1$, we have

$$\int_\Omega |D^\delta u(x) D^{\eta-\delta} v(x)|^p \, dx \leqslant K_{\eta,\delta} \, \| u \|^p_{j+2,p} \, \| v \|^p_{j+1,p} ,$$

where $K_{\eta,\delta} = K_{\eta,\delta}(n,p,j,\Omega)$.

Let k be the largest integer such that $(j+2-k)p > n$. Since $(j+2)p > n$ we have that $k \geqslant 0$

Let us first assume $k = 0$. Hence $(j+2)p > n$, $(j+1)p \leqslant n$.

a) If $|\delta| = 0$, then applying (4.2.5) we obtain

$$\int_\Omega |D^\delta u(x) D^{\eta-\delta} v(x)|^p \, dx = \int_\Omega |u(x) D^\eta v(x)|^p \, dx \leqslant K(n,p,j,\Omega) \| u \|^p_{j+2,p} \, | D^\eta v |^p_p \leqslant$$

$$\leqslant K(n,p,j,\Omega) \| u \|^p_{j+2,p} \, \| v \|^p_{j+1,p} .$$

b) If $|\eta - \delta| = 0$, then

$$\int_\Omega |D^\delta u(x) D^{\eta-\delta} v(x)|^p \, dx = \int_\Omega |D^\delta u(x) v(x)|^p \, dx .$$

Now, if $|\delta| = 0$, we are in the case a). Otherwise $|\delta| \geqslant 1$ so that $(j+2-|\delta|)p \leqslant n$. Moreover

$$\frac{n-(j+2-|\delta|)p}{n} + \frac{n-(j+1)p}{n} = 2 - \frac{(2j+3-|\delta|)p}{n} \leqslant 2 - \frac{(j+2)p}{n} < 1 .$$

Hence there exist positive numbers r',r" with $(1/r') + (1/r'') = 1$ such that

$$p \leqslant r'p \leqslant np/[n-(j+2-|\delta|)p] , \qquad p \leqslant r''p \leqslant np/[n-(j+1)p].$$

Thus by Hölder's inequality and (4.2.6) we have

$$\int_\Omega |D^\delta u(x) v(x)|^p \, dx \leqslant \left(\int_\Omega |D^\delta u(x)|^{r'p} \, dx \right)^{1/r'} \left(\int_\Omega |v(x)|^{r''p} \, dx \right)^{1/r''} \leqslant$$

$$\leqslant [K(\delta)]^{1/r'} [K(\eta-\delta)]^{1/r''} \| u \|^p_{j+1,p} \, \| v \|^p_{j+1,p} \leqslant$$

$$\leqslant [K(\delta)]^{1/r'} \; [K(\eta-\delta)]^{1/r''} \; \| u \|^p_{j+2,p} \; \| v \|^p_{j+1,p} \; .$$

c) Now if $|\delta| > 0$ and $|\eta - \delta| > 0$, we have $(j+2-|\delta|)p \leqslant n$ and $(j+1-|\eta-\delta|)p \leqslant n$. Moreover it follows

$$\frac{n-(j+2-|\delta|)p}{n} + \frac{n-(j+1-|\eta-\delta|)p}{n} = 2 - \frac{(2j+3-|\eta|)p}{n} \leqslant 2 - \frac{(2j+3-j-1)p}{n} < 1 .$$

Hence there exist positive numbers r', r" con $(1/r') + (1/r'') = 1$ such that

$$p \leqslant r'p \leqslant np/[n-(j+2-|\delta|)p] \; , \qquad p \leqslant r''p \leqslant np/[n-(j+1-|\eta-\delta|)p.$$

As above we have

$$\int_\Omega |D^\delta u(x) D^{\eta-\delta} v(x)|^p \, dx \leqslant \left(\int_\Omega |D^\delta u(x)|^{r'p} \, dx \right)^{1/r'} \left(\int_\Omega |D^{\eta-\delta} v(x)|^{r''p} \, dx \right)^{1/r''} \leqslant$$

$$< [K(\delta)]^{1/r'} \; [K(\eta-\delta)]^{1/r''} \; \| u \|^p_{j+2,p} \, \| v \|^p_{j+1,p} \; .$$

Let us now assume $k > 0$. Hence we have $(j+2-k)p > n$, $(j+1-k)p \leqslant n$. Let us also here distinguish different cases:

a') If $|\delta| < k$, then $(j+2-|\delta|)p > n$, so (4.2.5) implies

$$\int_\Omega |D^\delta u(x) D^{\eta-\delta} v(x)|^p \, dx \leqslant |D^\delta u|^p_\infty \; |D^{\eta-\delta} v|^p_p \leqslant [K(\delta)]^p \, \| u \|^p_{j+2,p} \| v \|^p_{j+1,p} \; .$$

b') Consider now the case $|\eta - \delta| \leqslant k$.

If $|\eta - \delta| \leqslant k-1$, then $(j+1-|\eta-\delta|)p > n$; using again (4.2.5) we have

$$\int_\Omega |D^\delta u(x) D^{\eta-\delta} v(x)|^p \, dx \leqslant [K(\eta-\delta)]^p \, \| u \|^p_{j+2,p} \, \| v \|^p_{j+1,p} \; .$$

If $|\eta-\delta| = k$, then $|\delta| = |\eta| - k \leqslant j+1-k$. If $|\delta| \leqslant k$ we are in the case a'). Otherwise $|\delta| > k$. Since

$$\frac{n-(j+2-|\delta|)p}{n} + \frac{n-(j+1-|\eta-\delta|)p}{n} < 1 .$$

(4.2.4) follows in the same way as in c).

c') Finally, if $|\delta| > k$ and $|\eta - \delta| > k$, then $(j+2-|\delta|)p \leqslant n$ and $(j+1-|\eta-\delta|)p \leqslant (j+1-k)p \leqslant$ $\leqslant n$.

The proof of (4.2.4) follows then as in c).

This completes the proof of (4.2.4) for $u \in W^{j+2,p}$, $v \in C^\infty(\Omega)$.

If $v \in W^{j+1,p}$, then there exists a sequence $\{v_n\}$ of $C^\infty(\Omega)$ functions converging to v in $W^{j+1,p}$. Then by the above argument $\{uv_n\}$ is a Cauchy sequence in $W^{j+1,p}$, so it converges to an element w of that space. Since $(j+2)p > n$, u may be assumed continuous and bounded on Ω . Thus

$$\| w-uv \|_{j+1,p} \leqslant \|w-uv_n\|_{j+1,p} + \|(v-v_n)u\|_{j+1,p} \leqslant \|w-uv_n\|_{j+1,p} + |u|_\infty \| v-v_n\|_{j+1,p} \longrightarrow 0$$

as $n \longrightarrow +\infty$. Hence $w = uv$ in L^p and so $w = uv$ in the sense of distributions.

Therefore $w = uv$ in $W^{j+1,p}$ and $\| uv \|_{j+1,p} \leqslant \limsup_{n\to+\infty} \| uv_n \|_{j+1,p} \leqslant K \|u\|_{j+2,p} \| v\|_{j+1,p}$.

This completes the proof of the lemma 4.2.2. ***

Proof of lemma 4.2.1. By recalling (116) we have to estimate

$F(f,v,\tau,\beta) = (\tau+m)[f - (v\cdot\nabla)v] - \omega_1(\tau,\beta)\nabla\tau - \omega_2(\tau,\beta)\nabla\beta$,

$H(h,v,\tau,\beta) = (\tau+m)h - c_v(m+\tau)v\cdot\nabla\beta + \Psi(v,v) + \frac{\zeta_0+\beta}{m+\tau}\,\omega_2(\tau,\beta)(v\cdot\nabla\tau - g)$,

in the $W^{j+1,p}$ norm. Using lemma 4.2.2 and (4.2.2) we have

$\|(\tau+m)f\|_{j+1,p} \leqslant c' \|f\|_{j+1,p}$. Being $W^{j+2,p}$ a Banach algebra and taking in account (4.2.2) we obtain $\|(\tau+m)(v\cdot\nabla)v\|_{j+1,p} \leqslant c' \|v\|^2_{j+3,p}$. Finally,

$\|\omega_1(\tau,\beta)\nabla\tau\|_{j+1,p} \leqslant cS_1(\|\nabla\tau\|_{j+1,p} + \|\nabla\beta\|_{j+1,p})\|\nabla\tau\|_{j+1,p}$, and

$\|\omega_2(\tau,\beta)\nabla\beta\|_{j+1,p} \leqslant cS_2(\|\nabla\tau\|_{j+1,p} + \|\nabla\beta\|_{j+1,p})\|\nabla\beta\|_{j+1,p}$; this proofs $(4.2.3)_1$.

Again using lemma 4.2.2 and (4.2.2) we have

$\|(\tau+m)h\|_{j+1,p} \leqslant c' \|h\|_{j+1,p}$. From the fact that $W^{j+2,p}$ is a Banach algebra follows $(\tau+m)v \in W^{j+2,p}$. Lemma 4.2.2 then gives $\|c_v(\tau+m)v\cdot\nabla\beta\|_{j+1,p} \leqslant c_v c' \|v\|_{j+2,p} \|\beta\|_{j+2,p}$.

From (112) we get immediately $\Psi(v,v) \leqslant 4\chi_0 \sum_{i,j=1}^{n}(D_i v_j)^2 + \chi_1(\operatorname{div} v)^2$.

Since $W^{j+2,p} \hookrightarrow C^0$, the following estimate holds:

$$\|\psi(v,v)\|_{j+1,p} \leqslant \chi_0 c \|\nabla v\|^2_{j+1,p} + \chi_1 c \|\nabla v\|^2_{j+2,p} \leqslant c(\chi_0+\chi_1)\|v\|^2_{j+3,p} .$$

Finally by taking again in account lemma 4.2.2 we get

$$\left\|\frac{\zeta_0+\beta}{m+\tau}\,\omega_2(\tau,\beta)(v\cdot\nabla\tau - g)\right\|_{j+1,p} \leqslant \frac{c'\zeta_0 S_2}{m}\left(\|v\|_{j+2,p}\|\tau\|_{j+2,p} + \|g\|_{j+1,p}\right).$$

This proofs $(4.2.3)_2$. ***

Theorem 4.1.1, lemma 4.2.1, and classical results for the Dirichlet problem (4.2.8) yield the following statement:

Theorem 4.2.2. Assume that the hypotheses in lemma 4.2.1 hold. Let $v \in W^{j+3,p}_{0,d}$ verify (4.1.3). Then there exists a unique solution $(u,\sigma) \in W^{j+3,p}_{0,d} \times \overline{W}^{j+2,p}$ of problem

$$(4.2.7)\qquad \begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + k\nabla\sigma = F(f,v,\tau,\beta) , & \\ m \operatorname{div} u + v\cdot\nabla\sigma + \sigma \operatorname{div} v = g , & \text{in } \Omega , \\ u\big|_\Gamma = 0 & \end{cases}$$

and a unique solution $\alpha \in W^{j+3,p}_0$ of the problem

$$(4.2.8)\qquad \begin{cases} -\chi\Delta\alpha = H(h,v,\tau,\beta) , & \text{in } \Omega , \\ \alpha\big|_\Gamma = 0 . & \end{cases}$$

Moreover

$$(4.2.9)\ \mu\|u\|_{j+3,p} + k\|\sigma\|_{j+2,p} \leqslant c'\left(1+\frac{\mu+|\nu|}{\mu+\nu}\right)\left(\|f\|_{j+1,p} + \|v\|^2_{j+3,p} + \|\tau\|^2_{j+2,p} + \|\beta\|^2_{j+2,p}\right)$$

$$+ c_2\frac{\mu+|\nu|}{m}\|g\|_{j+2,p} ,$$

$$(4.2.10)\quad \chi\|\alpha\|_{j+3,p} \leqslant c'\|h\|_{j+1,p} + c'c_v\|v\|_{j+2,p}\|\beta\|_{j+2,p} + c'(\chi_0+\chi_1)\|v\|^2_{j+3,p} +$$
$$+ \frac{c'\zeta_0 S_2}{m}\left(\|v\|_{j+2,p}\|\tau\|_{j+2,p} + \|g\|_{j+1,p}\right).$$

In the sequel the solution (u, σ, α) of problems (4.2.7), (4.2.8) with data (v, τ, β) is denoted by $(u, \sigma, \alpha) = T(v, \tau, \beta)$.

Let us write (4.2.9), (4.2.10) in the abbreviated form

$$\begin{aligned} &\|u\|_{j+3,p} + \|\sigma\|_{j+2,p} \leqslant c_2'(\|f\|_{j+1,p} + \|g\|_{j+2,p} + \|v\|^2_{j+3,p} + \\ &\qquad\qquad + \|\tau\|^2_{j+2,p} + \|\beta\|^2_{j+2,p}), \\ &\|\alpha\|_{j+3,p} \leqslant c_2'(\|h\|_{j+1,p} + \|g\|_{j+2,p} + \|v\|^2_{j+3,p} + \|\tau\|^2_{j+2,p} + \|\beta\|^2_{j+2,p}). \end{aligned} \tag{4.2.11}$$

Set

$$r_0 = \min\left\{ \gamma k, \frac{1}{c_3}, \left(\frac{l_1}{5c_2'c_3}\right)^{1/2}, \frac{1}{5c_2'} \right\}. \tag{4.2.12}$$

Let $r \in]0, r_0]$ and assume that

$$\|f\|_{j+1,p} \leqslant r^2, \quad \|h\|_{j+1,p} \leqslant r^2, \quad \|g\|_{j+2,p} \leqslant r^2, \tag{4.2.13}$$

and that

$$\|v\|_{j+3,p} \leqslant r, \quad \|\tau\|_{j+2,p} \leqslant r, \quad \|\beta\|_{j+2,p} \leqslant 5c_2'r^2. \tag{4.2.14}$$

Under this assumptions the conditions (4.1.3), (4.2.2) are satisfied. So from (4.2.11) we obtain for $(u, \sigma, \alpha) = T(v, \tau, \beta)$ the following estimates

$$\|u\|_{j+3,p} \leqslant r, \quad \|\sigma\|_{j+2,p} \leqslant r, \quad \|\alpha\|_{j+2,p} \leqslant 5c_2'r^2. \tag{4.2.15}$$

Consequently $T(B_r) \subset B_r$, where B_r is the subset of $W^{j+3,p}_{0,d} \times \overline{W}^{j+2,p} \times W^{j+3,p}_0$ defined by equations (4.2.14).
To complete the proof of theorem A we show that T is a contraction in B_r respect to the norm $W^{1,2}_0 \times L^2 \times W^{1,2}_0$, if r is sufficiently small.

Let $(u, \sigma, \alpha) = T(v, \tau, \beta)$, $(u_1, \sigma_1, \alpha_1) = T(v_1, \tau_1, \beta_1)$, $F = F(f, v, \tau, \beta)$, $F_1 = F(f, v_1, \tau_1, \beta_1)$, $H = H(h, v, \tau, \beta)$, $H_1 = H(h, v_1, \tau_1, \beta_1)$. One has

$$(4.2.16)\quad \begin{cases} -\mu\Delta(u-u_1)-\nu\nabla\,\mathrm{div}(u-u_1)+k\nabla(\sigma-\sigma_1)=F-F_1 \quad , \\ m\,\mathrm{div}(u-u_1)+v_1\cdot\nabla(\sigma-\sigma_1)+(v-v_1)\cdot\nabla\sigma+\sigma_1\,\mathrm{div}(v-v_1)+(\sigma-\sigma_1)\mathrm{div}\,v=0 \end{cases}$$

in Ω, and

$$(4.2.17)\qquad -\chi\Delta(\alpha-\alpha_1)=H-H_1 \quad , \text{in } \Omega \ .$$

Here we use the notation $\| \ \|_{k,2} = \| \ \|_k$. By multiplying both sides of equation $(4.2.16)_1$ by $m(u-u_1)$, both sides of equation $(4.2.16)_2$ by $k(\sigma-\sigma_1)$, by integrating the resulting equations in Ω and by adding side by side we show

$$(4.2.18)\quad m(\mu-|\nu|)\|\nabla(u-u_1)\|_0^2 \leqslant m\|F-F_1\|_{-1}\ \|u-u_1\|_1 + \\ + ck\|\sigma_1\|_{j+2,p}\ \|v-v_1\|_1\ \|\sigma-\sigma_1\|_0 + ck\|v\|_{j+3,p}\|\sigma-\sigma_1\|_0^2 .$$

Note that in proving (4.2.18) we use the Sobolev embedding theorems , in particular, $W^{1,2}\hookrightarrow L^{2^*}$, where $2^* = 2n/(n-2)$. Furthermore, $w\in L^{\frac{np}{n-(j+1)p}}$, if $(j+1)p \leqslant n$, and then $1/2 + 1/2^* + [n-(j+1)p]/np < 1$.

Arguing as on proving inequality (3.12) in reference [3] we show that

$$(4.2.19)\quad \|u-u_1\|_1^2 + c_4'(1-c_5'\|v\|_{j+3,p})\|\sigma-\sigma_1\|_0^2 \leqslant \\ \leqslant c'\|\sigma_1\|_{j+2,p}^2\ \|v-v_1\|_1^2 + c'\|F-F_1\|_{-1}^2 \quad .$$

Moreover

$$(4.2.20)\qquad \|\alpha-\alpha_1\|_1 \leqslant c'\|H-H_1\|_{-1} \quad .$$

We will establish on the end of this section the following estimates (4.2.21), (4.2.22) :

$$(4.2.21)\quad \|F-F_1\|_{-1} \leqslant c'(\|f\|_{j+1,p} + \|v_1\|_{j+3,p}^2)\|\tau-\tau_1\|_0 + \\ +c'(1+\|\tau\|_{j+2,p})(\|v\|_{j+3,p} + \|v_1\|_{j+3,p})\|v-v_1\|_1 + \\ +c'(\|\tau\|_{j+2,p} + \|\tau_1\|_{j+2,p} + \|\beta\|_{j+2,p} + \|\beta_1\|_{j+2,p})(\|\tau-\tau_1\|_0 + \|\beta-\beta_1\|_0)$$

and

$$(4.2.22)\quad \|H-H_1\|_{-1} \leqslant c'\,(\|v\|_{j+3,p} + \|v_1\|_{j+3,p} + \|\tau\|_{j+2,p} + \|\beta\|_{j+2,p})\,\|v-v_1\|_1 +$$
$$+c'\,[\|h\|_{j+1,p} + \|g\|_{j+2,p} + (\|\beta\|_{j+2,p} + \|\tau\|_{j+2,p})\|v\|_{j+2,p} +$$
$$+ (1 + \|\tau_1\|_{j+2,p} + \|\beta_1\|_{j+2,p} + \|\tau_1\|^3_{j+2,p} + \|\beta_1\|^3_{j+2,p})\|v_1\|_{j+2,p}]\,\|\tau-\tau_1\|_0 +$$
$$+c'(\|g\|_{j+2,p} + \|v\|_{j+2,p}\|\tau\|_{j+2,p} + \|v_1\|_{j+2,p}\|\tau_1\|_{j+2,p} +$$
$$+ \|v_1\|_{j+3,p})\,\|\beta-\beta_1\|_0 \quad .$$

Now by using (4.2.13), (4.2.14) it follows that the coefficients of $\|v-v_1\|_1$, $\|\tau-\tau_1\|_0$, $\|\beta-\beta_1\|_0$ in the right hand sides of equations (4.2.21), (4.2.22) are polynomials on r, vanishing for r = 0, and with coefficients which are positive constants of type c'.

Also the coefficients $c'_5\|v\|_{j+3,p}$ and $c'\|\sigma_1\|^2_{j+2,p}$ which appear in the inequality (4.2.19) are polynomials on r of the above type. Since the exact form of these polynomials is not important here we denote them by the symbol $\varepsilon = \varepsilon(r)$.

Using this notation the estimates (4.2.19), (4.2.20), (4.2.21), (4.2.22), yield

$$(4.2.23)\quad \|u-u_1\|^2_1 + (c'-\varepsilon)\|\sigma-\sigma_1\|^2_0 \leqslant \varepsilon\|v-v_1\|^2_1 + \varepsilon\|\tau-\tau_1\|^2_0 + \varepsilon\|\beta-\beta_1\|^2_0 \ ,$$

$$(4.2.24)\quad \|\alpha-\alpha_1\|^2_1 \leqslant \varepsilon\|v-v_1\|^2_1 + \varepsilon\|\tau-\tau_1\|^2_0 + \varepsilon\|\beta-\beta_1\|^2_0 \quad .$$

By adding this results one gets

$$(4.2.25)\quad \|u-u_1\|^2_1 + (c'-\varepsilon)\|\sigma-\sigma_1\|^2_0 + \|\alpha-\alpha_1\|^2_1 \leqslant$$
$$\leqslant \varepsilon\|v-v_1\|^2_1 + \varepsilon\|\tau-\tau_1\|^2_0 + \varepsilon\|\beta-\beta_1\|^2_0 \quad .$$

Hence for a sufficiently small value of r, T is a contraction in B_r with respect to a suitable norm in $W_0^{1,2} \times L^2 \times W_0^{1,2}$.

To complete the proof of the theorem it remains to proof the estimates (4.2.21), (4.2.22).

We start by proving (4.2.21). Recalling (116) one has

$$\|F-F_1\|_{-1} \leqslant \|(\tau-\tau_1)f\|_{-1} + \|(\tau_1-\tau)(v\cdot\nabla)v_1\|_{-1} + \|\tau[(v_1-v)\cdot\nabla]v_1\|_{-1} + \\ + \|\tau(v\cdot\nabla)(v_1-v)\|_{-1} + \|m[(v_1-v)\cdot\nabla]v_1\|_{-1} + \|m(v\cdot\nabla)(v_1-v)\|_{-1} + \\ + \|\omega_1(\tau_1,\beta_1)\nabla\tau_1 - \omega_1(\tau,\beta)\nabla\tau\|_{-1} + \|\omega_2(\tau_1,\beta_1)\nabla\beta_1 - \omega_2(\tau,\beta)\nabla\beta\|_{-1}.$$

Recall that the Sobolev embedding theorems imply $\tau, \tau_1 \in C^0(\Omega)$. Then $\tau, \tau_1 \in L^2$ hold. We have also from those theorems for $w \in W^{j+1,p}$ that $w \in L^\infty$, if $(j+1)p > n$; $w \in L^{\frac{np}{n-(j+1)p}}$, if $(j+1)p < n$; in the last case it follows also $1/2 + 1/2^* + [n-(j+1)p]/np < 1$.

Since $f \in W^{j+1,p}$, using Hölder's inequality one gets

$\|(\tau-\tau_1)f\|_{-1} \leqslant c\|\tau-\tau_1\|_0 \, \|f\|_{j+1,p}$.

Under our assumptions the Sobolev embedding theorems imply v, $v_1 \in C^1(\Omega)$. Therefore v, $v_1 \in W_0^{1,2} \hookrightarrow L^{2^*}$.

Hence making a suitable use of Hölder's inequality we have

$$\|(\tau_1-\tau)(v_1\cdot\nabla)v_1\|_{-1} \leqslant c\|v_1\|^2_{j+3,p}\|\tau-\tau_1\|_0 \quad ;$$
$$\|\tau[(v_1-v)\cdot\nabla]v_1\|_{-1} \leqslant c\|\tau\|_{j+2,p}\|v_1\|_{j+3,p}\|v-v_1\|_1 \quad ;$$
$$\|\tau(v\cdot\nabla)(v_1-v)\|_{-1} \leqslant c\|\tau\|_{j+2,p}\|v\|_{j+3,p}\|v-v_1\|_1 \quad ;$$
$$\|m[(v_1-v)\cdot\nabla]v_1\|_{-1} \leqslant c'\|v_1\|_{j+3,p}\|v-v_1\|_1 \quad ;$$
$$\|m(v\cdot\nabla)(v_1-v)\|_{-1} \leqslant c'\|v\|_{j+3,p}\|v_1-v\|_1 \quad ;$$

Taking in account $\beta, \beta_1 \in L^2$, it follows

$$\|\omega_1(\tau_1,\beta_1)\nabla\tau_1 - \omega_1(\tau,\beta)\nabla\tau\|_{-1} \leqslant$$
$$\leqslant \left\|\int_{\tau(x)}^{\tau_1(x)}\omega_1(y,\beta_1)\,dy\right\|_0 + \|\omega_1(\tau,\beta_1)\nabla\tau - \omega_1(\tau,\beta)\nabla\tau\|_{-1} \leqslant$$
$$\leqslant c'(\|\tau\|_{j+2,p} + \|\tau_1\|_{j+2,p})\|\tau_1-\tau\|_0 + c'\|\tau\|_{j+2,p}\|\beta-\beta_1\|_0 \quad .$$

In this last estimate we have used the fact that $\nabla\tau \in W^{j+1,p}$.

Finally, proceeding in a similar way as above one gets

$$\|\omega_2(\tau_1,\beta_1)\nabla\beta_1 - \omega_2(\tau,\beta)\nabla\beta\|_{-1} \leqslant c'(\|\tau-\tau_1\|_0 + \|\beta-\beta_1\|_0)\|\beta_1\|_{j+2,p} +$$

$$+ \sup_{\substack{\varphi\in W_0^{1,2}\\ \|\varphi\|_1\leq 1}} \int_\Omega |\nabla\omega_2(\tau,\beta)(\beta-\beta_1)\varphi|\,dx + \sup_{\substack{\varphi\in W_0^{1,2}\\ \|\varphi\|_1\leq 1}} \int_\Omega |\omega_2(\tau,\beta)(\beta-\beta_1)\nabla\varphi|\,dx \leqslant$$

$$\leqslant c'(\|\tau-\tau_1\|_0 + \|\beta_1-\beta\|_0)\|\beta_1\|_{j+2,p} +$$

$$+ c'(\|\tau\|_{j+2,p} + \|\beta\|_{j+2,p})\|\beta-\beta_1\|_0 \quad .$$

This proofs (4.2.21).

Now we proof (4.2.22). One has

$$\|H-H_1\|_{-1} \leqslant \|(\tau-\tau_1)h\|_{-1} + c_v\|(\tau-\tau_1)v\cdot\nabla\beta\|_{-1} + \|(m+\tau_1)(v-v_1)\cdot\nabla\beta\|_{-1} +$$
$$+ \|(m+\tau_1)v_1\cdot\nabla(\beta-\beta_1)\|_{-1} + \|\Psi(v,v) - \Psi(v_1,v_1)\|_{-1} +$$
$$+ \|(\beta-\beta_1)(m+\tau)^{-1}\omega_2(\tau,\beta)(v\cdot\nabla\tau - g)\|_{-1} +$$
$$+ \|(\zeta_0+\beta_1)(\tau_1-\tau)(m+\tau)^{-1}(m+\tau_1)^{-1}\omega_2(\tau,\beta)(v\cdot\nabla\tau - g)\|_{-1} +$$
$$+ \|(\zeta_0+\beta_1)(m+\tau_1)^{-1}[\omega_2(\tau,\beta) - \omega_2(\tau_1,\beta_1)](v\cdot\nabla\tau - g)\|_{-1} +$$
$$+ \|(\zeta_0+\beta_1)(m+\tau_1)^{-1}\omega_2(\tau_1,\beta_1)(v-v_1)\cdot\nabla\tau\|_{-1} +$$
$$+ \|(\zeta_0+\beta_1)(m+\tau_1)^{-1}\omega_2(\tau_1,\beta_1)v_1\cdot\nabla(\tau-\tau_1)\|_{-1} \quad .$$

Recall that under our assumptions in this section one has $|\tau|_\infty \leqslant l$, $|\tau_1|_\infty \leqslant l$, $|\beta|_\infty \leqslant l_1$, $|\beta_1|_\infty \leqslant l_1$. Moreover recall the Sobolev embedding theorems mentioned in the beginning and the observations leading (4.2.21). So similarly to above one has immediately

$$\|(\tau-\tau_1)h\|_{-1} \leqslant c\|h\|_{j+1,p}\|\tau_1-\tau\|_0 \quad ;$$
$$c_v\|(\tau-\tau_1)v\cdot\nabla\beta\|_{-1} \leqslant c_v\, c\,\|v\|_{j+2,p}\|\beta\|_{j+2,p}\|\tau-\tau_1\|_0 \quad ;$$
$$\|(m+\tau_1)(v-v_1)\cdot\nabla\beta\|_{-1} \leqslant c(m+l)\|\beta\|_{j+2,p}\|v-v_1\|_0 \quad .$$

Furthermore, one gets

$$\|(m+\tau_1)v_1\cdot\nabla(\beta-\beta_1)\|_{-1} \leqslant \sup_{\substack{\varphi\in W_0^{1,2}\\ \|\varphi\|_1\leqslant 1}} \int_\Omega |(\nabla\tau_1)\cdot v_1\ (\beta-\beta_1)\varphi|\ dx \ +$$

$$+ \sup_{\substack{\varphi\in W_0^{1,2}\\ \|\varphi\|_1\leqslant 1}} \int_\Omega |(m+\tau_1)(\mathrm{div}\ v_1)(\beta-\beta_1)\varphi|\ dx \ +$$

$$+ \sup_{\substack{\varphi\in W_0^{1,2}\\ \|\varphi\|_1\leqslant 1}} \int_\Omega |(m+\tau_1)v_1\cdot(\nabla\varphi)(\beta-\beta_1)|\,dx \ \leqslant$$

$$\leqslant c\,\|\tau_1\|_{j+1,p}\|v_1\|_{j+2,p}\|\beta-\beta_1\|_0 \ + \ c(m+l)(\|v_1\|_{j+3,p}+\|v_1\|_{j+2,p})\|\beta-\beta_1\|_0 \ .$$

Using again the fact that $v,\ v_1\in C^1(\Omega)$, $W_0^{1,2}\hookrightarrow L^{2^*}$, the Hölder's inequality implies

$$\|\Psi(v,v)-\Psi(v_1,v_1)\|_{-1} \leqslant c(\chi_0+\chi_1)(\|v\|_{j+3,p}+\|v_1\|_{j+3,p})\|v-v_1\|_1 \ .$$

Besides we have

$$\|(\beta-\beta_1)(m+\tau)^{-1}\,\omega_2(\tau,\beta)(v\cdot\nabla\tau-g)\|_{-1} \leqslant \frac{cT_2}{m}(\|v\|_{j+2,p}\|\tau\|_{j+2,p}+\|g\|_{j+2,p})\|\beta-\beta_1\|_0 \ ;$$

$$\|(\zeta_0+\beta_1)(\tau-\tau_1)(m+\tau)^{-1}(m+\tau_1)^{-1}\,\omega_2(\tau,\beta)(v\cdot\nabla\tau-g)\|_{-1} \leqslant$$
$$\leqslant \frac{c(\zeta_0+l_1)T_2}{m^2}(\|v\|_{j+2,p}\|\tau\|_{j+2,p}+\|g\|_{j+2,p})\|\tau-\tau_1\|_0 \ ;$$

$$\|(\zeta_0+\beta_1)(m+\tau_1)^{-1}[\omega_2(\tau,\beta)-\omega_2(\tau_1,\beta_1)](v\cdot\nabla\tau-g)\|_{-1} \leqslant$$
$$\leqslant \frac{c(\zeta_0+l_1)S_2}{m}(\|v\|_{j+2,p}\|\tau\|_{j+2,p}+\|g\|_{j+2,p})(\|\tau-\tau_1\|_0+\|\beta-\beta_1\|_0);$$

$$\|(\zeta_0+\beta_1)(m+\tau_1)^{-1}\,\omega_2(\tau_1,\beta_1)(v-v_1)\cdot\nabla\tau\|_{-1} \leqslant \frac{c(\zeta_0+l_1)T_2}{m}\|\tau\|_{j+2,p}\|v-v_1\|_0 \ ;$$

Finally it follows

$$\|(\zeta_0+\beta_1)(m+\tau_1)^{-1}\,\omega_2(\tau_1,\beta_1)\mathbf{v}_1\cdot\nabla(\tau-\tau_1)\|_{-1} \leqslant$$

$$\leqslant \sup_{\substack{\varphi\in W_0^{1,2}\\ \|\varphi\|_1\leq 1}} \int_\Omega |\nabla\beta_1\cdot\mathbf{v}_1\,(m+\tau_1)^{-1}\,\omega_2(\tau_1,\beta_1)(\tau-\tau_1)\varphi|\,\,dx +$$

$$+ \sup_{\substack{\varphi\in W_0^{1,2}\\ \|\varphi\|_1\leq 1}} \int_\Omega |(\zeta_0+\beta_1)\frac{(\nabla\tau_1)\cdot\mathbf{v}_1}{(m+\tau_1)^2}\,\omega_2(\tau_1,\beta_1)(\tau-\tau_1)\varphi|\,\,dx +$$

$$+ \sup_{\substack{\varphi\in W_0^{1,2}\\ \|\varphi\|_1\leq 1}} \int_\Omega |(\zeta_0+\beta_1)(m+\tau_1)^{-1}\,\nabla\omega_2(\tau_1,\beta_1)\cdot\mathbf{v}_1\,(\tau-\tau_1)\varphi|\,\,dx +$$

$$+ \sup_{\substack{\varphi\in W_0^{1,2}\\ \|\varphi\|_1\leq 1}} \int_\Omega |(\zeta_0+\beta_1)(m+\tau_1)^{-1}\,\omega_2(\tau_1,\beta_1)(\tau-\tau_1)\mathbf{v}_1\cdot\nabla\varphi|\,\,dx \leqslant$$

$$\leqslant c(\zeta_0+l_1+\|\beta_1\|_{j+2,p})(1/m+(\|\tau_1\|_{j+2,p})/m)\cdot$$
$$\cdot(T_2+S_2\|\tau_1\|_{j+2,p}+S_2\|\beta_1\|_{j+2,p})\|\mathbf{v}_1\|_{j+2,p}\cdot\|\tau_1-\tau\|_0 \,.$$

This proofs (4.2.22).

The proof of theorem A is accomplished. ***

5. THE INCOMPRESSIBLE LIMIT (PROOF OF THEOREM B)

In this chapter we study the incompressible limit of the compressible Navier–Stokes equations in the case $W^{j,p}$, $j > -1$ (see theorem B in the introduction).

As in references [3],[4], we study the system

$$\text{(5.1.1)} \qquad \begin{cases} -\mu \Delta u - \nu \nabla \operatorname{div} u + \nabla p(\rho) = \rho[f - (u \cdot \nabla)u] \,, \\ \operatorname{div}(\rho u) = 0 \ , \qquad \text{in } \Omega \ , \\ u\big|_{\Gamma} = 0 \,, \end{cases}$$

describing the barotropic motion of a compressible, viscous fluid. We are interested to study the limit of the solution u when $k \equiv p'_\rho(m) \longrightarrow +\infty$.

Therefore it is necessary to state an existence result for problem (5.1.1) in which:

i) the dependence of some suitable structural constants of the state function $p(\rho)$ in terms of k is given;

ii) the dependence on k of the constants appearing on the estimates is shown. Assume that $k \gg k_0 > 0$ (the constant k_0 has no special meaning since we let $k \longrightarrow +\infty$).

Let $p(\rho) \in C^{j+3}([m-l/k, m+l/k])$, where $0 < l/k \leqslant m/2$, and write $p'_\rho(\rho)$ in the form $p'_\rho(m+\sigma) = k + \omega(\sigma)$. We assume that there exists $\alpha \in (0,1]$ such that

$$\text{(5.1.2)} \qquad \|\omega(\sigma)\|_{\mathscr{C}^{j+2}} \leqslant S|\sigma|^{\alpha} \quad , \quad \forall \sigma \in [-l/k, l/k],$$

and we denote by ϕ a positive constant for which

$$\text{(5.1.3)} \qquad S \leqslant \phi k^{1+\alpha} \quad .$$

We show so the following theorem:

Theorem 5.1.1. Let $p \in]1,+\infty[$ and $j > -1$ verify $(j+2)p > n$. Let the above assumptions on the state function $p(\rho)$ hold, let $v \in W^{j+3,p}_{0,\alpha}$ verify (4.1.3) and let

$$\|k\tau\|_{j+2,p} \leqslant 1/c_3 \;. \tag{5.1.4}$$

If $f \in W^{j+1,p}$, there exists a unique solution $(u,\sigma) \in W^{j+3,p}_{0,\alpha} \times \overline{W}^{j+2,p}$ of the problem

$$\begin{cases} -\mu\Delta u - \nu\nabla \operatorname{div} u + k\nabla\sigma = F(f,v,\tau) \\ m \operatorname{div} u + v\cdot\nabla\sigma + \sigma \operatorname{div} v = 0 \qquad , \text{ in } \Omega \,, \\ u\big|_{\Gamma} = 0 \end{cases} \tag{5.1.5}$$

where $F(f,v,\tau) = (\tau+m)[f - (v\cdot\nabla)v] - \omega(\tau)\nabla\tau$.

Moreover

$$\|u\|_{j+3,p} + \|k\sigma\|_{j+2,p} \leqslant c'_1 \|f\|_{j+1,p} + c'_2 \|v\|^2_{j+2,p} + c'_3\, \phi\, \|k\tau\|^{1+\alpha}_{j+2,p} \tag{5.1.6}$$

where $c'_i = c'_i(n,p,j,\mu,\nu,m,\Omega)$, $i = 1,\dots,3$.

Proof. From our assumptions it follows immediately that $F \in W^{j+1,p}$. Hence theorem 4.1.1 implies the existence and uniqueness of the solution $(u,\sigma) \in W^{j+3,p}_{0,\alpha} \times \overline{W}^{j+2,p}$ of problem (5.1.5). Moreover, using lemma 4.2.2 and our assumptions above one gets

$$\|(\tau+m)f\|_{j+1,p} \leqslant c(3/2)m\, \|f\|_{j+1,p} \;,$$
$$\|(\tau+m)(v\cdot\nabla)v\|_{j+1,p} \leqslant c(3/2)m\, \|v\|_{j+2,p} \|\nabla v\|_{j+1,p} \leqslant c(3/2)\, \|v\|^2_{j+2,p} \;,$$
$$\|\omega(\tau)\nabla\tau\|_{j+1,p} \leqslant S\,|\tau|^{\alpha}_{\infty} \|\nabla\tau\|_{j+1,p} \leqslant c\, \phi\, \|k\tau\|^{1+\alpha}_{j+1,p} \;.$$

The estimate (4.1.4) then implies (5.1.6). This completes the proof of the theorem. ***

As in section 4.2 we denote by $(u,\sigma) = T(v,\tau)$ the solution of system (5.1.5). We define

$$r_0 = \min\left\{ \gamma k, \frac{1}{c_3}, \frac{1}{3c_2'}, \left(\frac{1}{3c_3'\phi}\right)^{1/\alpha} \right\}, \tag{5.1.7}$$

and assume that $f \in W^{j+1,p}$ is fixed and verifies

$$\| f \|_{j+1,p} \leqslant r/3c_1' \qquad , \text{ for } r \in]0,r_0]. \tag{5.1.8}$$

Under this assumptions using (5.1.6) one easily verifies $T(B_r) \subset B_r$, where B_r is defined by the inequalities

$$\| v \|_{j+3,p} \leqslant r \quad , \qquad \| k\tau \|_{j+2,p} \leqslant r . \tag{5.1.9}$$

Taking in account that $k \geqslant k_0$ and proceeding as in proving inequality (4.2.18) there exist positive constants c', c_4', c_5', depending at most on n, p, j, μ, ν, m, k_0, Ω, such that

$$\| u-u_1 \|_1^2 + c_4'(1-c_5' \| v \|_{j+3,p}) \| k\sigma - k\sigma_1 \|_0 \leqslant \tag{5.1.10}$$
$$\leqslant c' \| k\sigma_1 \|_{j+2,p}^2 \;\| v-v_1 \|_1^2 + c' \| F-F_1 \|_{-1}^2 ,$$

where $(u_1, \sigma_1) = T(v_1, \tau_1)$, $F = F(f,v,\tau)$, $F_1 = F(f,v_1, \tau_1)$. Again from our assumptions and the fact that $k \geqslant k_0 > 0$, arguing as in proving (4.2.21), it follows

$$\| F-F_1 \|_{-1} \leqslant c' (\| f \|_{j+1,p} + \| v \|_{j+3,p}^2) \| k\tau - k\tau_1 \|_0 + \tag{5.1.11}$$
$$+ c' (1 + \| k\tau \|_{j+2,p})(\| v \|_{j+3,p} + \| v_1 \|_{j+3,p}) \| v-v_1 \|_1 +$$
$$+ c_3^\alpha \phi (\| k\tau \|_{j+2,p}^\alpha + \| k\tau_1 \|_{j+2,p}^\alpha) \| k\tau - k\tau_1 \|_0 .$$

Using (5.1.9)–(5.1.11) and arguing as in section 4.2 one verifies that there exists a positive constant $r_1 \leqslant r_0$, depending only on n, p, j, μ, ν, m, l, ϕ, k_0, Ω, such that T is a contraction in B_r for $r \leqslant r_1$ with respect to a suitable norm in $W_0^{1,2} \times L^2$

What we have seen above we can summarize in the following theorem:

Theorem 5.1.2. Let $p \in]1,+\infty[$ and $j > -1$ verify $(j+2)p > n$. Let the above assumptions on the state function $p(\varrho)$ hold. Then there exist positive constants c'_6, c'_7, depending only on n, p, j, μ, ν, m, l, ϕ, Ω and k_o, such that if $f \in W^{j+1,p}$ and

$$\| f \|_{j+1,p} \leqslant c'_6 \quad ,$$

there exist a unique solution $(u, \varrho) \in W^{j+3,p}_{o,d} \times W^{j+2,p}$ of problem (5.1.1), $(1.3)_1$ in the ball

$$\| u \|_{j+3,p} \leqslant c'_7 \quad , \qquad \| \varrho - m \|_{j+2,p} \leqslant c'_7 / k .$$

Next we will prove theorem B. The assumptions on the family of state functions $p_\lambda(\varrho)$, $\lambda \in [\lambda_o,+\infty[$, are the following:

Let $Dp_\lambda(\varrho)$ denote the derivative $dp_\lambda(\varrho)/d\varrho$. We set $k_\lambda = Dp_\lambda(m)$, we assume that $k_\lambda \geqslant k_o > 0$, and we suppose that $p_\lambda(\varrho) \in C^{j+3}([m-l/k_\lambda, m+l/k_\lambda])$, where $0 < l \leqslant k_o m/2$. Moreover we assume that $\| Dp_\lambda(\varrho) - Dp_\lambda(m) \|_{\mathcal{C}^{j+2}} \leqslant S_\lambda |\varrho - m|^\alpha$ for a fixed $\alpha \in [0,1]$. Hence, by setting $Dp_\lambda(m+\sigma) = k_\lambda + \omega_\lambda(\sigma)$ one has $\omega_\lambda(0) = 0$ and

(5.1.12) $$\| \omega_\lambda(\sigma) \|_{\mathcal{C}^{j+2}} \leqslant S_\lambda |\sigma|^\alpha \quad , \forall \sigma \in [-l/k_\lambda, l/k_\lambda].$$

We assume that there exists a positive constant ϕ such that

(5.1.13) $$S_\lambda \leqslant \phi\, k_\lambda^{1+\alpha} \qquad \forall \lambda \geqslant \lambda_o .$$

Proof of theorem B. By applying theorem 5.1.2 we get immediately part i) of the theorem.

Let us prove part ii). Recall that $W^{j+3,p}_{o,d}$ is a reflexive Banach space for $1 < p < +\infty$. Therefore from the estimate $\| u_\lambda \|_{j+3,p} \leqslant c'_9$ in part i), we have that there exist a subsequence u_λ (we use the same notation for the sequence and the subsequence) and a function $u_\infty \in W_{0,d}^{j+3,p}$ such that $u_\lambda \longrightarrow u_\infty$, weakly in $W_{0,d}^{j+3,p}$. The convergence of all u_λ follows from the uniqueness of the solution of (1.1.10) which holds if c'_8 is sufficiently small.

From equation (1.1.9) ,using lemma 4.2.2 and the estimates in theorem 5.1.2 we get $\| m \operatorname{div} u_\lambda \|_{j+1,p} \leqslant \| \sigma_\lambda \operatorname{div} u_\lambda \|_{j+1,p} + \| u_\lambda \cdot \nabla \sigma_\lambda \|_{j+1,p} \leqslant c c'_8 c'_9 / k_\lambda$.
This implies $\operatorname{div} u_\lambda \longrightarrow 0$, strongly in $W^{j+1,p}$.

Since $\| \operatorname{div} u_\lambda \|_{j+2,p} \leqslant c'_9$, one gets the weak convergence in $W^{j+2,p}$. From these last two convergences we get $\operatorname{div} u_\infty = 0$ in Ω.

From the estimate in theorem 5.1.2, $\| \rho_\lambda - m \|_{j+2,p} \leqslant c'_9 / k_\lambda$ it follows $\rho_\lambda \longrightarrow m$, strongly in $W^{j+2,p}$.

Finally, we pass to the limit in equation $(1.1.9)_1$. One has from the convergences above that $\nabla p_\lambda(\rho_\lambda) \longrightarrow \mu \Delta u + m[f - (u_\infty \cdot \nabla) u_\infty]$, weakly in $W^{j+1,p}$. It follows also

$$
\begin{aligned}
\| \rho_\lambda (u_\lambda \cdot \nabla) u_\lambda - m (u_\infty \cdot \nabla) u_\infty \|_{j+1,p} \leqslant{} & \| \rho_\lambda - m \|_{j+2,p} \| u_\lambda \|^2_{j+2,p} + m \| u_\lambda - u_\infty \|_{j+2,p} \| u_\lambda \|_{j+2,p} + \\
& + m \| (u_\infty \cdot \nabla)(u_\lambda - u_\infty) \|_{j+1,p}
\end{aligned}
$$

and therefore $\rho_\lambda (u_\lambda \cdot \nabla) u_\lambda \longrightarrow m (u_\infty \cdot \nabla) u_\infty$, strongly in $W^{j+1,p}$.

Recall that $L^p = X_p \oplus G_p$, where $X_p = \overline{\{u \in C_0^\infty(\Omega) : \operatorname{div} u = 0\}}^{L^p}$, and G_p is the closed subspace of L^p such that $u \in L^p$, $u = \nabla g$ with $g \in W^{1,p}$ (see reference [11]).

Therefore there exists a function $\pi \in \overline{W}^{j+2,p}$ such that $\nabla p_\lambda(\rho_\lambda) \longrightarrow \nabla \pi$, and so (1.1.10) is satisfied. ***

APPENDIX

A. A PRIORI ESTIMATES OF SOLUTIONS OF GENERAL ELLIPTIC SYSTEMS.

We report here a theorem regarding a study of estimates pertaining to solutions of boundary problems for elliptic systems (for an equal number of dependent variables) (see theorem 10.5 in [2]).

i) The differential equations considered.

The systems of partial differential equations considered will be represented as

$$\text{(A.1)} \qquad \sum_{j=1}^{N} l_{ij}(x,D)u_j(x) = F_i(x) \qquad i = 1,..,N,$$

where the $l_{ij}(x,D)$, linear differential operators, are polynomials in $D = (D_1, ..., D_n)$ with coefficients depending on $x = (x_1, ..., x_n)$ over some domain Ω in $\mathbb{R}^n$. The orders of these operators will be assumed to depend on two systems of integer weights $s_1, ..., s_N$ and $t_1, ..., t_N$ attached to the equations and to the unknowns, respectively, s_i corresponding to the i–th equation and t_j to the j–th dependent variable u_j. The manner of dependence is expressed by the inequality

$$\text{(A.2)} \qquad \deg l_{ij}(x,\xi) \leqslant s_i + t_j \quad , \qquad i,j = 1,...,N,$$

"deg" referring of course to the degree in $\xi = (\xi_1, ..., \xi_n)$. [We use the symbol ξ^δ to abbreviate monomials $\xi_1^{\delta_1} \cdots \xi_n^{\delta_n}$ of degree $|\delta| = \delta_1 + \cdots + \delta_n$]. It is to be understood that $l_{ij} = 0$ if $s_i + t_j < 0$.

Adding a suitable constant to one system of weights and subtracting the constant from the other we readily achieve, as a normalization, the condition

(A.3) $$s_i \leqslant 0.$$

Since, for fixed j, not all l_{ij} vanish and since $s_i + t_j \geqslant 0$ for non-vanishing l_{ij}, we thus also have

(A.4) $$0 \leqslant t_j \leqslant t',$$

t' being the maximum of the t_j .

Only elliptic systems of the form (A.1) will be considered. These are defined as systems for which

(A.5) $$L(x,\xi) = \det\,(l'_{ij}(x,\xi)) \neq 0 \quad \text{for real } \xi \neq 0$$

where $l'_{ij}(x,\xi)$ consists of the terms in $l_{ij}(x,\xi)$ which are just of the order $s_i + t_j$.

We must assume a

Supplementary condition on L : $L(x,\xi)$ is of even degree 2m (with respect to ξ). For every pair of linearly independent real vectors ξ , ξ' , the polynomial $L(x, \xi + \tau\xi')$ in the complex variable τ has exactly m roots with positive imaginary part.

This condition is actually used only at points x of the boundary Γ of Ω with ξ a tangent and ξ' the normal to Γ at x .

If m = 0 , (A.1) may be explicitly solve for the u_k in terms of F_i and their derivatives; therefore we require here that

(A.6) $$2m = \deg\,(L(x,\xi)) > 0 .$$

Uniform ellipticity will be required in the sense that there is a positive constant A such that

$$\text{(A.7)} \qquad A^{-1}|\xi|^{2m} \leqslant |L(x,\xi)| \leqslant A|\xi|^{2m}$$

for every real vector $\xi = (\xi_1, ..., \xi_n)$ and for every point x in the closure of the domain Ω.

We require from to now that our systems (A.1) satisfy the conditions (A.2)–(A.7) and the above Supplementary Condition on L.

ii) The Complementing Boundary Conditions.

The boundary conditions we consider refer to a regular portion Γ of Ω. They are expressed as

$$\text{(A.8)} \qquad \sum_{j=1}^{N} B_{hj}(x,D)u_j(x) = \phi_h(x) \quad \text{on } \Gamma \ , \ h = 1,...,m,$$

where $B_{hj}(x,\xi)$ are polynomials in ξ with complex coefficients depending on x. The orders of the boundary operators depend also here on two systems of integer weights. In this case the system $t_1, ..., t_N$ is already attached to the dependent variables. The new system $r_1, ..., r_m$ corresponds to the conditions in (A.8), h = 1,...,m; in other words, r_h is associated to the h–condition in (A.8), h = 1,...,m. The exact dependence is expressed by the inequality

$$\text{(A.9)} \qquad \deg B_{hj}(x,\xi) \leqslant r_h + t_j \ ,$$

it being understood that $B_{hj} = 0$ when $r_h + t_j < 0$.

Let $B'_{hj}(x,\xi)$ consist of the terms in $B_{hj}(x,\xi)$ which are just of the order $r_h + t_j$.

The boundary conditions must "complement" the differential equations in presenting a well-posed problem for both. To do this, an algebraic criterion involving the $l'_{ij}(x,\xi)$ and the $B'_{hj}(x,\xi)$ must be satisfied. We describe this below.

At any point x of Γ let n denote the outward normal and $\xi \neq 0$ any tangent to Γ (ξ, in particular, is real). Denote by $\tau^+_k(x,\xi)$, $k = 1,\dots,m$, the m roots (in τ) with positive imaginary part of the characteristic equation $L(x, \xi + \tau n) = 0$. The existence of this roots is assured by the Supplementary Condition on L. Set

$$M^+(x,\xi,\tau) = \prod_{h=1}^{m}(\tau - \tau^+_h(x,\xi)).$$

Let $(L^{jk}(x, \xi + \tau n))$ denote the matrix adjoint to $(l'_{ij}(x, \xi + \tau n))$. The above mentioned criterion for the boundary problem (A.1), (A.8) to be coercive is that the following algebraic condition is satisfied.

Complementing boundary condition: For any $x \in \Gamma$ and any real, non-zero vector ξ tangent to Γ at x let us regard $M^+(x,\xi,\tau)$ and the elements of the matrix

$$\sum_{j=1}^{N} B'_{hj}(x, \xi + \tau n) L^{jk}(x, \xi + \tau n) \tag{A.10}$$

as polynomials in the indeterminate τ . The rows of the latter matrix are required to be linearly independent modulo $M^+(x,\xi,\tau)$, i.e.,

$$\sum_{h=1}^{m} C_h \sum_{j=1}^{N} B'_{hj} L^{jk} \equiv 0 \pmod{M^+}, \quad k = 1,\dots,m$$

only if the constants C_h are all zero.

We report now a theorem which gives global estimates for solutions of a system of partial differential equations (A.1) that satisfy the Complementing Boundary Conditions (A.8).

We consider a bounded, sufficiently regular domain Ω. We assume that Γ is coverable by a finite number of n-dimensional open set U_β such that each intersection $\overline{U_\beta} \cap \overline{\Omega}$ is the image under a 1-1 mapping T_β of the closure of a n-dimensional hemisphere $\Sigma_{R\beta}$,

$R_\beta \leqslant 1$, the flat face of the hemisphere corresponding to $\overline{U}_\beta \cap \Gamma$; each T_β and its inverse have continuous derivatives of orders up to $l + t'$ bounded by a constant $\varkappa$.

Let $l_1 = \max(0, r_h + 1)$, $h = 1,\dots,m$, and let l denote a fixed integer $\geqslant l_1$.

Theorem A.1. Let Ω be a bounded open set in $\mathbb{R}^n$ as above; let $\|F_i\|_{l-s_i}$, $\|\phi_h\|_{W^{l-r_h-\frac{1}{p},p}(\Gamma)}$ be finite for $i = 1,\dots,N$, $h = 1,\dots,m$.
If $\|u_j\|_{l_1+t_j,p}$ is finite for $j = 1,\dots,N$, then $\|u_j\|_{l+t_j,p}$ also is finite and there exists a constant c such that

$$\text{(A.11)} \qquad \|u_j\|_{l+t_j,p} \leqslant c\,\Big(\sum_{i=1}^{N} \|F_i\|_{l-s_i,p} + \sum_{h=1}^{m} \|\phi_h\|_{W^{l-r_h-\frac{1}{p},p}(\Gamma)} + \sum_{j=1}^{N} |u_j|_p \Big).$$

where c is dependent on p, l, n, N, t', r_h , A, k (which denotes here a bound in the norm $W^{l-s_i,p}$ of the coefficients in the differential equations and a bound in the norm $W^{l-r_h-\frac{1}{p},p}(\Gamma)$ of the coefficients in the boundary conditions), on $\varkappa$, on the domain Ω ,and the modulus of continuity of the leading coefficients in the l_{ij} .

Remark. The term $\sum_{j=1}^{N}|u_j|_p$ on the right may be replaced by $\sum_{j=1}^{N}|u_j|_1$ or may be omitted altogether when the solution in question is the only one for which u_j has L^p derivatives up to $l_1 + t_j$, $j = 1,\dots,N$.

B. SOME PHYSICAL CONSIDERATIONS

Let us present in a concise way the physical considerations which bring us to the system (1.1.1) (for more details we refer to [23]).

It is rather well known that the equation

$$\text{(B.1)} \qquad \rho \frac{du}{dt} = \rho f + \operatorname{div} T$$

is the mathematical formulation of the fundamental principle of the dynamics of fluid motion in a domain Ω, namely, of the *principle of conservation of the linear momentum* for a fluid (= the rate of change of linear momentum of a material volume of fluid equals the resultant force acting on the volume) with $\frac{d}{dt} = \frac{\partial}{\partial t} + \sum_{i=1}^{n} u_i D_i$.

We suppose here that the fluid possesses a density function $\rho = \rho(x,t)$, where $x \in \Omega$, and t is the time variable. $u(x,t)$ denotes the velocity field of the fluid, $f(x,t)$ the external force field per unit mass. We indicate by T the stress tensor of the fluid associated to the stress principle of Cauchy (each component of T has a simple physical interpretation, namely, T^{ij} is the j–component of the force on the surface element with outer normal in the i–direction).

The *equation of motion* (B.1) is valid for any fluid regardless of the form which the stress tensor may take. In order to complete this equation in the case of a compressible and viscous fluid, which is our case, it is necessary to express the stress tensor T in terms of other kinematic and thermodynamic variables. Such a relation between T and other flow quantities is called a constitutive equation. For a compressible ,viscous fluid some physical considerations get the classical constitutive equation (Cauchy–Poisson law)

$$\text{(B.2)} \qquad T^{ij} = [\, -p + (\xi - \tfrac{2}{3}\mu) \operatorname{div} u]\delta_{ij} + 2\mu D_{ij} \quad ,$$

where ξ , μ are the viscosity coefficients ($\xi > 0$, $\mu > 0$) and D is the deformation tensor $D_{ij} = (1/2)(D_i u_j + D_j u_i)$.

For a compressible fluid p is the thermodynamic pressure and can be formulated as

(B.3) $$p = p(\varrho, \tau) \; ,$$

where τ denotes the absolute temperature. The dynamical equation which results from inserting the Cauchy–Poisson law in (B.1) is known as the Navier–Stokes equation for a compressible fluid

(B.4) $$\varrho \frac{du}{dt} = \varrho f - \nabla p + \nabla [(\xi - \tfrac{2}{3}\mu) \operatorname{div} u] + \operatorname{div}(2\mu D) \; .$$

If ξ , μ are constants (this is not a strong assumption) this equations finds the following formulation

(B.5) $$\varrho \frac{du}{dt} = \varrho f - \nabla p + (\xi + \tfrac{1}{3}\mu) \; (\operatorname{div} u) + \mu \Delta u \; .$$

Taking $\nu = \xi + \frac{4}{3}\mu$ we get for stationary, compressible and viscous fluid the following equation of motion

(B.6) $$\varrho (u \cdot \nabla) u = \varrho f - \nabla p(\varrho, \tau) + \nu \nabla (\operatorname{div} u) + \mu \Delta u \quad , \text{ in } \Omega ,$$

which corresponds to $(1.1.1)_1$.

The equation $(1.1.1)_2$ for $g = 0$ is the mathematical formulation of the *principle of conservation of mass* (= the mass of fluid in a material volume does not change as the volume moves with the fluid) in the stationary case. $(1.1.1)_2$ for $g = 0$ is called the *equation of continuity* for a stationary fluid. The case $g \neq 0$ is only interesting from a mathematical point of view.

The number of unknown quantities in the equations $(1.1.1)_1$ – $(1.1.1)_2$ is greater than the number of equations, so that these equations are not themselves sufficient for a complete description of fluid motion. This situation is remedied by the introduction of a *total energy equation* based on the principles of classical thermodynamics, and later by the use of certain constitutive equations. Let us so deduce $(1.1.1)_3$ in the physically significant case when $g = 0$.

We define the total energy of a material volume V as the sum of its kinetic energy and its internal energy e . We postulate that the total energy is conserved, i.e.,

(B.7) $$\rho\frac{de}{dt} - \rho h = T{:}D - \operatorname{div} q ,$$

where h is the assigned external heat source per unit mass, q the heat conduction vector. We indicate by $T{:}D = \sum_{i,j=1}^{n} T^{ij} D_{ij}$. The equation (B.7) follows from the energy transfer equation (it describes the physical principle that the rate of change of kinetic energy of a moving material volume is equal to the rate at which work is being done on the volume by external forces, diminished by a "dissipation" term involving the interaction of stress and deformation) together with the first law of thermodynamics ($Q = \Delta e + W$, where Q is the total external heat supplied to the fluid during a process in which the physical system passes from one state to another, Δe is the increment of internal energy and W is the amount of work done by the physical system during this process).

In order to complete the system of equations of classical fluid mechanics it is necessary to give a constitutive equation for the heat conduction vector q. We follow the commonly accepted formulation and postulate that q is an isotropic function of the temperature gradient and thermodynamic state. From the condition of isotropy it follows that q must be parallel to $\nabla\zeta$,whence follows the classical Newton–Fourier law

(B.8) $$q = -\nabla(\chi\zeta) .$$

χ is the coefficient of heat conduction and can be assumed constant. The thermodynamical condition $q\cdot\nabla\zeta \leqslant 0$ implies $\chi \geqslant 0$. Taking in account (B.8) we get from (B.7)

(B.9) $$\rho\left[\frac{\partial e}{\partial t} + u\cdot\nabla e - h\right] = T{:}D + \chi \operatorname{div}\nabla\zeta =$$
$$= -p(\operatorname{div} u) + (\xi - \tfrac{2}{3}\mu)(\operatorname{div} u)^2 + \frac{\mu}{2}\sum_{i,j=1}^{n}(D_i u_j + D_j u_i)^2 + \chi\Delta\zeta .$$

On the other hand, for a compressible fluid we assume that e is a thermodynamical state variable $e = E(\rho, \zeta)$ satisfying the relation

(B.10) $$\zeta\, dS = de - \frac{p}{\rho^2} d\rho \ ,$$

where S is the specific entropy (it follows from the second law of thermodynamics) ,and therefore

(B.11) $$\zeta\, dS = \left(\frac{\partial E}{\partial \rho} - \frac{p}{\rho^2}\right) d\rho + \frac{\partial E}{\partial \zeta} d\zeta \ .$$

It follows

(B.12) $$\frac{\partial E}{\partial \rho} = \frac{1}{\rho^2}\left(p - \zeta \frac{\partial p}{\partial \zeta}\right) \ .$$

Hence

(B.13) $$\rho\left[\frac{\partial e}{\partial t} + u\cdot\nabla e\right] = \rho\left[\frac{\partial E}{\partial \rho}\dot{\rho} + \frac{\partial E}{\partial \zeta}\dot{\zeta} + \frac{\partial E}{\partial \rho} u\cdot\nabla\rho + \frac{\partial E}{\partial \zeta} u\cdot\nabla\zeta\right] =$$

$$= -\rho^2 \frac{\partial E}{\partial \rho} \operatorname{div} u + \rho\frac{\partial E}{\partial \zeta}\left[\dot{\zeta} + u\cdot\nabla\zeta\right] \ .$$

by taking in account the continuity equation $\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho u) = 0$.
By comparing equation (B.13) with (B.9) and by using (B.12) together with $c_V = \left(\frac{\partial E}{\partial \zeta}\right)_V$ (c_V is the heat capacity) we get in the stationary case the following equation

$$-\chi\Delta\zeta + c_V\rho u\cdot\nabla\zeta + \zeta\frac{\partial p}{\partial \zeta}\operatorname{div} u = \rho h + \frac{\mu}{2}\sum_{i,j=1}^{n}(D_i u_j + D_j u_i)^2 + (\xi - \tfrac{2}{3}\mu)(\operatorname{div} u)^2 ,$$

which corresponds to $(1.1.1)_3$ when $\chi_0 = \frac{\mu}{2}$, $\chi_1 = (\xi - \frac{2}{3}\mu)$. This equation together with $(1.1.1)_1 - (1.1.1)_2$ then gives a complete description of fluid motion.

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