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EMBEDDINGS AND GENERAL RELATIVITY

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Summary

We review the main results on embeddings of Riemannian manifolds in higher dimensional flat spaces of interest for general relativity. Among others, the main theorems on local and global isometric embeddings and the equations determining them, the Gauss-Codazzi-Ricci equations. Then we apply the embeddings to general relativity. By means of a particular parametrisation of the embedding we are able to decompose the metric and the affine connection of space-time in their inertial and gravitational parts. The steps towards a variational formulation of this approach are sketched.

Content

0. Introduction

Embeddings of Riemannian manifolds into higher dimensional flat spaces start to play an important role both in mathematics and in physics.

In mathematics they are as old as intrinsic Riemannian geometry. In the physics of gravitation they are as old as general relativity.

In mathematics they are conceptually important since they allow to establish the equivalence between the intrinsic and the extrinsic approaches to Riemannian geometry. In physics it is expected they provide a better understanding of the physics of gravitation.

We start reviewing the main theorems on local and global isometric embeddings of Riemannian manifolds into higher dimensional flat spaces.

As a result of these theorems any Riemannian manifold $V_n(t,s)$ can be considered as a local submanifold of $\bar{E}_N(T,S)$ with $N=n(n+1)/2$, $T \geqslant t$, $S \geqslant s$.

Then we consider a set of equations describing the embedding, the Gauss-Codazzi-Ricci equations. These constitute necessary conditions for the embedding.

Then we study the cases in which one needs only one or two extra dimensions to do the embedding.

Finally, making use of the Euclidean global character of $\bar{E}_N(T,S)$ we introduce a particular parametrisation of the embedding. This consists in writing $\bar{E}_N(T,S)$ as a direct product of two manifolds, $\bar{E}_N=M_n \otimes B_{N-n}$, with M_n being a maximally symmetric space. In the case of general relativity, the space-time $V_4(1,3)$ is obtained by a dimensional reduction of $M_4 \otimes B_6$ into M_4. This dimensional reduction is in term of six fields ϕ, which behave as scalar fields both in

$M_n(t,s)$ and in $V_n(t,s)$.

With the use of this parametrisation of the embedding we are able to decompose the metric and the affine connection in their inertial and gravitational parts. The gravitational part depends only on the fields introduced previously.

Having been able to identify the fields responsible for the gravitational interaction we look for a variational formulation for them. Einstein field equations in term of these fields look structurally very similar to Maxwell ones when written in terms of $\vec{E}$ and $\vec{B}$ fields.

As an example, in the Appendix we consider the embedding of Friedman-Robertson-Walker spaces.

Part I. The Geometry of Embeddings

I.1. Introduction

The concept of an abstract Riemannian manifold arises inmathematics as the result of the evolution in mathematical attitudes. In the earlier period mathematicians thought more concretely of curved surfaces embedded in a flat Euclidean space(Gauss, 1827). The concept of an abstract Riemannian manifold defined intrinsically was first explicitly formulated by Riemann(1868). Almost inmediately after the abstract view of manifolds came into favor, a question naturally arose, the isometric embedding problem, the question of the existence of concrete realisations as submanifolds of higher dimensional Euclidean spaces of abstract Riemannian manifolds. Today, we know that the answer is yes: any intrinsically defined Riemannian manifold can be isometrically embedded, locally and globally, in an Euclidean space of appropriate dimension and signature.

The embedding problem was first considered by Schläfli (1873) just after Riemann presented his famous thesis. Schläfli discussed the local form of the embedding problem and he conjectured that a Riemannian manifold with positive defined and analytic metric can be locally and isometrically embedded as a submanifold of a Euclidean space E_N with $N=n(n+1)/2$.

In 1926 Janet described a method of proof based on a power series development, so it was limited to local results. Furthermore, he required the metric to be analytic. This proof however, as Janet himself noticed, was incomplete. He solved only the local problem for two-dimensional manifolds with analytic metric. In 1927 E. Cartan extended the Janet's proof to n-dimensional manifolds treating it as an application of his theory of Pfaffian forms. The dimensionality requirement was $N=n(n+1)/2$, as conjectured by Schläfli.

In 1931 Burstin completed the Janet's proof and also extended it to the case in which the embedding space is a given Riemannian manifold V_N with positive defined and analytic metric. In 1956 Leichtweiss gave a new proof based more substantially than Burstin in the Gauss-Codazzi-Ricci equations of Riemannian geometry. His proof is more involved than that by Burstin. In 1961 Friedman extended the theorem to Riemannian manifolds with indefinite metrics, such as that from general relativity. For semi-positive definite metrics see(Lense, 1926).

The first global isometric embedding theorem of V_n into E_N were established by Nash(1956). The results depend crucially on the compactness of V_n. For V_n compact he obtains $N=n(3n+11)/2$; for non-compact manifolds $N=n(n+1)(3n+11)/2$. The first global results for indefinite metrics were obtained by Clarke(1970) and by Greene(1970).

Therefore, any intrinsically defined Riemannian manifold has a, local and global, isometric embedding in some Euclidean space. Then one can consider the two approaches, intrinsic and extrinsic, to Riemannian geometry as completely equivalent.

In the mathematical literature there exist many results on local and global isometric embeddings, but only a few of them are useful for applications in physics, more particularly in general relativity. Our selection of topics has been guided by what people think is promising for an eventual application in physics. Since physics is normally a local affair, the study has been mainly restricted to local results. Today we know that global properties are as important as the local ones, therefore, some global results are also included.

The material of this part is arranged as follows. We start defining what an isometric embedding is. Then we consider the corresponding local and global isometric embedding theorems. We introduce the concept of class of the embedding; this is the minimal number of extra dimensions required to satisfy the Gauss-Codazzi-Ricci equations. Then, we consider some results for class one and two isometric embeddings and finally we introduce a particular parametrisation of the embedding.

The most complete reviews focused towards physics existing up to now are (Goenner, 1980; Kramer et al., 1981; Robinson and Ne'eman, 1965).

I.2. Isometric Embeddings

Let $V_n=(M_n,g)$ and $\bar{V}_N=(\bar{M}_N,\bar{G})$ be Riemannian manifolds of dimension n and $N \geqslant n$, with metric g and $\bar{G}$, respectively. A differentiable map $f:M_n \to \bar{M}_N$ is a C^k(C^∞, analytic), $k \geqslant 1$, inmersion if (Choquet-Bruhat et al., 1982)

i. f is of differentibility class C^k(C^∞, analytic).

ii. rank(df)=n at all points $p \in M_n$. Here df is the differential of f.

An inmersion is not necessarily injective, therefore, $f(M_n)$ is not necessarily a manifold. An injective inmersion is an embedding. The set $f(M_n)$ with the differential structure induced by the embedding is a manifold. If $f(M_n)$ has a submanifold structure equivalent to the manifold structure induced by the embedding, then f is regular. Thus, if f is a regular embedding, $f(M_n)$ is a submanifold of $\bar{M}_N$.

The embedding $f:V_n \to V_N$ is isometric at a point $p \in M_n$ if

$$(2.1) \quad g(U,V) = G(df \cdot U, df \cdot V)$$

for every U, V in the tangent space $T_p(M_n)$. If (2.1) holds only in an open neighbourhood of a point p -M_n, then the isometric embedding is local. If (2.1) holds for all points of M_n , then the isometric embedding is global.

Let x^μ, $\mu=0,\ldots,n-1$, and X^A, $A=0,\ldots,N-1$, be local coordinates on M_n and $\bar{M}_N$, respectively. The map f is given by

$$(2.2) \quad X^A = X^A(x^\mu)$$

where $\mathrm{rank}(X^A{}_\mu)=n$, $X^A{}_\mu=\partial_\mu X^A$. The information about the intrinsic geometry of M_n and that about its situation as a submanifold of $\bar{M}_N$, the extrinsic geometry, are both contained in $X^A=X^A(x^\mu)$. With local orthonormal bases $\{\partial_\mu\}$ and $\{\partial_A\}$ of $T_p(M_n)$ and $T_{f(p)}(\bar{M}_N)$, respectively, the components of the metric tensors of M_n and $\bar{M}_N$ are defined by

$$(2.3)\quad g(\partial_\mu,\partial_\nu)=g_{\mu\nu},\quad G(\partial_A,\partial_B)=G_{AB}.$$

Equation (2.1) in local coordinates is

$$(2.4)\quad g_{\mu\nu}=G_{AB}\,X^A{}_\mu\,X^B{}_\nu$$

with $g=\det(g_{\mu\nu})\neq 0$.

When talking of the embedding of Riemannian manifolds we will, generally, imply the local, or global, isometric embedding of M_n into a flat space $\overline{E}_N$ with Euclidean global topology. Eventually one can consider Riemannian manifolds which are, among others, (i) of constant curvature, (ii) conformally flat, (iii) Ricci flat, etc.

I.3. Local Embeddings

Now $V_n(t,s)=(M_n, g_{\mu\nu}(x^\lambda))$ denotes a Riemannian n-dimensional manifold with analytic and non-degenerate metric $g_{\mu\nu}(x^\lambda)$ with t positive and s negative eigenvalues, respectively, t+s=n. $\bar{E}_N(T,S)$ denotes a flat space with Euclidean global topology with analytic and non-degenerate metric $\bar{G}_{AB}(X^C)$ with T positive and S negative eigenvalues, respectively, T+S=N. Finally, $V_n(n,0)=V_n$ denotes a Riemannian manifold with analytic positive definite metric.

The first result concerns positive definite metrics and is due to Janet(1926), Cartan(1927) and Burstin (1931).

Theorem. Any Riemannian manifold V_n can be analytically and isometrically embedded in E_N with N=n(n+1)/2.

The proof consists in a power series development. The generalisation to indefinite metrics is due to Friedman(1961).

Theorem. Any Riemannian manifold $V_n(t,s)$ can be analytically and isometrically embedded in $E_N(T,S)$ with $N=n(n+1)/2$, $T \geqslant t$, $S \geqslant s$.

This theorem can be stated in an equivalent and more illuminating way as follows.

Theorem. Let $g_{\mu\nu}(x^\lambda)$ be analytic functions in a neighbourhood of $x^\lambda=0$ and let $\bar{G}_{AB}(X^C)$ be analytic functions in a neighbourhood of $X^A=0$. If N=n(n+1)/2, then there exist analytic functions $X^A=X^A(x^\mu)$ in a neighbourhood of $x^\mu=0$ satisfying the conditions

$$X^A(0)=0, \quad \text{rank}\, X^A{}_{,\mu}(0)=n,$$

$$g_{\mu\nu}(x)=G_{AB}(X(x))\,X^A{}_{,\mu}(x)\,X^B{}_{,\nu}(x).$$

The proof by Friedman, as that by Burstin, is based in the general sketch by Janet but in the rest is a new proof even in the case of positive definite metrics.

Sketch of the Proof. By mean of an analytic transformation(Eisenhart, 1926) it is always possible to obtain new $\bar{G}_{AB}$'s and new coordinates X^A's which are Euclidean in the origin

$$\bar{G}_{AB}(X) = \bar{\eta}_{AB} + \frac{1}{2}\partial_{CD}G_{AB}(0)\,X^C X^D + O(X^3),$$

$$X^A(x) = X^A{}_{\mu}(0)\,x^{\mu} + \frac{1}{2}X^A{}_{\mu\nu}(0)\,x^{\mu}x^{\nu} + \frac{1}{3!}X^A{}_{\mu\nu\lambda}(0)\,x^{\mu}x^{\nu}x^{\lambda} + O(x^4),$$

$$X^A{}_{\mu}(x) = X^A{}_{\mu}(0) + X^A{}_{\mu\nu}(0)\,x^{\nu} + \frac{1}{2}X^A{}_{\mu\nu\lambda}(0)\,x^{\nu}x^{\lambda} + O(x^3),$$

$$g_{\mu\nu}(x) = \eta_{\mu\nu} + \frac{1}{2}\partial_{\lambda\rho}g_{\mu\nu}(0)\,x^{\lambda}x^{\rho} + O(x^3).$$

Comparing to order $O(x^2)$ one obtains

$$\eta_{\mu\nu} = \bar{\eta}_{AB}\,X^A{}_{\mu}(0)\,X^B{}_{\nu}(0)$$

$$0 = \bar{\eta}_{AB}\,X^A{}_{\mu\lambda}(0) + \bar{\eta}_{AB}$$

$$\frac{1}{2}\partial_{\lambda\rho}g_{\mu\nu}(0) = \frac{1}{8}\left[\bar{\eta}_{AB}\,X^A{}_{\mu\lambda}(0)\,X^B{}_{\nu\rho}(0) + \bar{\eta}_{AB}\,X^A{}_{\mu\rho}(0)\,X^B{}_{\nu\lambda}(0)\right]$$
$$+ \frac{1}{5!}\left[\bar{\eta}_{AB}\,X^A{}_{\mu}(0)\,X^B{}_{\nu\lambda\rho}(0) + \bar{\eta}_{AB}\,X^A{}_{\mu\lambda\rho}(0)\,X^B{}_{\nu}(0)\right]$$
$$+ \frac{1}{2}\partial_{CD}G_{AB}(0)\,X^C{}_{\lambda}(0)\,X^D{}_{\rho}(0)\,X^A{}_{\mu}(0)\,X^B{}_{\nu}(0).$$

The proof proceeds by induction looking at an embedding of V_n in $\bar{V}_{n+1}$, etc. The interested reader is referred to the original work(Friedman, 1961).

Thus, the theorem assures the existence of a local analytic and isometric embedding of $V_n(t,s)$ into $\bar{E}_N(T,S)$, with $N=n(n+1)/2$, $T \geq t$, $S \geq s$.

The proof by Friedman shows that the embedding is not uniquely determined and that the number of free parameters is independent of the signature of the metric tensors involved. For positive definite metrics this number of parameters was computed by Leichtweiss(1956).

Both metrics, that of $V_n(t,s)$ to be embedded and that of the embedding space $\bar{E}_N(T,S)$, can be chosen quite arbitrarily, except for some dimension and signature conditions. For axample, no additional restrictions are imposed over the signature of the remaining, non-null, N-n eigenvalues of $\bar{E}_N(T,S)$. Based on a work by Rosen (1965) and other results in the literature (Goenner, 1980) one can conjecture the following.

Conjecture. Any analytic Riemannian manifold $V_n(t,s)$ is a local submanifold of $E_N(T,S)$, with $N=n(n+1)/2$ and

$$T = t + t(t-1)/2 + s(s-1)/2,$$

$$S = s + ts.$$

For the space-time $V_4(1,3)$ of general relativity one obtains $\bar{E}_{10}(4,6)$ which fits very well with recent proposals in Kaluza-Klein theories (Aref'eva et al., 1986).

I.4. Global Embeddings

The first concrete result on global embeddings is due to Nash(1956).

Theorem. Any compact(non-compact) C^k, $k \geqslant 3$, Riemannian manifold V_n has a C^k global isometric embedding into E_N with $N_c = n(3n+11)/2$ ($N_{nc} = n(n+1)(3n+11)/2$).

In the proof of this theorem the positivity of the metric plays an indispensable role and, in fact, the proof breaks down if the metric is indefinite. The extension to indefinite metrics was given by Clarke(1970) which in the case of non-compact manifolds is also an improvement on Nash's result.

Theorem. Any C^∞ Riemannian manifold $V_n(t,s)$ with C^k, $k \geqslant 3$, Riemannian metric can be globally and C^k isometrically embedded in $E_N(T,S)$ with $N=T+S$, $S=t+1$ and

$$T = n(3n+11)/2$$

for compact $V_n(t,s)$ and

$$T = \frac{1}{3}n^3 + \frac{5}{2}n^2 + \frac{37}{6}n + 1$$

for non-compact $V_n(t,s)$.

The next result, due to Green(1970), uses a stronger differentiability assumption.

Theorem. Any C^∞ Riemannian manifold $V_n(t,s)$ with C^∞ Riemannian metric can be globally and C^∞ isometrically embedded in $E_N(T,S)$ where $N=T+S$ and

$$T = S = n(n+5)/2$$

for compact $V_n(t,s)$, and

$$T = S = 2(2n+1)(n+5)$$

for non-compact $V_n(t,s)$.

In the non-compact case the improvement over Clarke's result starts only for $n \gtrsim 20$.

All the previous numbers are only best bounds.

One unpleasant aspect of Clarke's theorem is the dependence of N on the signature of $V_n(t,s)$. Local isometric embedding theorems do not do it. By the other side, the Green's theorem introduces an artificially augmented number of time- and space-like dimensions to guarantee the possibility of hosting the time- and space-like dimensions of $V_n(t,s)$.

In many cases, N is well below the minimal value required by the local and global isometric embedding theorems. This is the reason to introduce the following definition.

I.5. The Embedding Class

The local and global isometric embedding of $V_n(t,s)$ in some $E_N(T,S)$ produces arithmetic invariants characterising the intrinsic geometry of $V_n(t,s)$. One of these is the embedding class defined as follows. First, one looks for a minimal embedding, i.e., a local or global isometric embedding in a E_{N_0} with the minimal possible dimension, $N \geqslant N_0 \geqslant n$, and appropriate signature. The embedding class of $V_n(t,s)$ is defined as the number N_0-n. Therefore, the embedding class runs from zero to N-n. For the local isometric embedding of a space-time $V_4(1,3)$ the embedding class runs from zero to six.

The invariance of the embedding class gives rise to a classification scheme of Riemannian manifolds which is on the same footing with the classifications with respect to groups of motion or Petrov types in general relativity.

The equations constituting necessary conditions for an isometric embedding $f: V_n(t,s) \to \overline{E}_N(T,S)$ are the Gauss-Codazzi-Ricci(GCR) equations. The embedding class e is the minimum number of extra dimensions enabling to satisfy the GCR equations for a given $V_n(t,s)$.

I.6. The Gauss-Codazzi-Ricci Equations

Now, we describe a set of equations constituting necessary conditions for an isometric embedding $f: M_n \to \bar{M}_N$. The space V_n is considered as already embedded in $\bar{V}_N$, with N sufficiently large, not necessarily flat. The curvature tensor $\bar{R}(\bar{U},\bar{V})\bar{W} \in T(\bar{M})$ with $\bar{U},\bar{V} \in T(\bar{M})$ restricted to $T(M)$ is denoted by $R(U,V)W \in T(M)$.

The covariant derivative $\bar{\nabla}_{\bar{X}}\bar{Y}$ on $\bar{M}$ is restricted to $\bar{X}=X$ and $\bar{Y}=Y$ in $T(M)$. Then the decomposition of $\bar{\nabla}_X$ into tangential and normal(to V_n) components

(6.1) $\bar{\nabla}_X Y = \nabla_X Y + \Omega(X,Y)$

defines the symmetric, bilinear map $\Omega: T(M)xT(M) \to N(M)$, called the second fundamental form of $M \subset \bar{M}$; $N(M)$ is the normal complement of $T(M) \subset T(\bar{M})$ restricted to M. The condition $\bar{G}(\Omega(X,Y),Z)=0$ is equivalent to $\nabla_Z g(X,Y)=0$. $\nabla_X Y$ is a linear connection in the tangent bundle $T(M)$, the usual Riemannian connection. If $\xi \in N(M)$ a similar decomposition of $\bar{\nabla}_X \xi$ leads to

(6.2) $\bar{\nabla}_X \xi = D_X \xi + A(X,\xi)$

$D_X \xi$ is the covariant differentiation of a linear connection in the normal bundle of M.

The linear map $A: T(M)xN(M) \to T(M)$ provides the tangential(to V_n) component of $\bar{\nabla}_X \xi$ and is connected to Ω through

(6.3) $\bar{G}(A(X,\xi),Y) + \bar{G}(\Omega(X,Y),\xi) = 0.$

In local coordinates

(6.4a) $\nabla_U V = u^\mu (V^\nu{}_{,\mu} + \Gamma^\nu{}_{\mu\lambda} v^\lambda)\partial_\nu$

(6.4b) $$D_U \xi = u^\mu \left(N^a{}_{,\mu} + t_\mu{}^a{}_b N^b\right)\hat{\xi}_a$$

where

(6.5) $$U = u^\mu \partial_\mu\,,\quad V = v^\mu \partial_\mu\,,\quad \xi = N^a \hat{\xi}_a$$

For what follows it will be necessary the following result (Eisenhart, 1926).

Theorem. If the metric tensor of V_n is non-degenerate, then the $\hat{\xi}_a$ canbe chosen as non-null vectors.

This theorem was originally established only for N=n+1 but, by recurrence, it can be generalised to an arbitrary $N > n$. Therefore, the $\hat{\xi}_a$ are a local, orthomormal basis of $N_p(M)$ and

(6.6a) $$\bar{G}\left(\hat{\xi}_a, \hat{\xi}_b\right) = \bar{\eta}_{ab} = \mathrm{diag}\,(\pm, \ldots, \pm)$$

(6.6b) $$\bar{G}\left(\hat{\xi}_a, \partial_\mu\right) = 0$$

With

(6.7) $$\Omega(\partial_\mu, \partial_\nu) = -\Omega^a{}_{\mu\nu}\hat{\xi}_a$$

it follows the expression

(6.8) $$A\left(\partial_\mu, \hat{\xi}_a\right) = \Omega_{a\mu}{}^\nu \partial_\nu$$

$\Omega^a{}_{\mu\nu}$ are the components of N-n symmetric tensors in $T_p(M)$ which are also called second fundamental forms, or Gauss tensors; they are the generalisation to higher dimensions of the Gauss tensor of the quadratic second fundamental form of a V_n embedded in a $\bar{V}_{n+1}$ used in the theory of hypersurfaces. $t^\mu{}_{ab}$ are the components of $\binom{N-n}{2}$ antisymmetric vectors in $T_p(M)$. They correspond to the normal part of the connection in M and are called torsion vectors, because $t^\mu{}_{(ab)}=0$.

The tensors $\Omega^a{}_{\mu\nu}$ and $t^\mu{}_{ab}$ cannot be prescribed arbitrarily, they must satisfy the Gauss-Codazzi-Ricci(GCR) equations, the integrability conditions for the embedding. These are obtained as follows.

By combining the connections $\nabla_X Y$ in T(M) and $D_X\xi$ in N(M) the covariant derivative of Ω and A with respect to the connection in T(M)+N(M) is defined

(6.9) $$\tilde{\nabla}_X \Omega(Y,Z) = D_X \Omega(Y,Z) - \Omega(\nabla_X Y, Z) - \Omega(Y, \nabla_X Z)$$

(6.10) $$\bar{\nabla}_X A(Y,\xi) = \nabla_X A(Y,\xi) - A(\nabla_X Y, \xi) - A(Y, D_X \xi).$$

Also, the curvature tensor $K(X,Y)\xi \in N(M)$ is introduced by

(6.11) $$K(X,Y)\xi = (D_{[X} D_{Y]} - D_{[X,Y]})\xi.$$

Now, making use of the vanishing of the torsions of M and $\bar{M}$ the projections parallel and normal to the tangent space T(M) of $\bar{R}(X,Y)\bar{Z}$, $\bar{Z}=Z+\xi$, $Z\in T(M)$, $\xi\in N(M)$, are obtained. The result can be expressed by the maps

(6.12a) $$G: T\times T\times T \to T$$

(6.12b) $$C: T\times T\times N \to T$$

(6.12c) $$\mathcal{E}: T\times T\times T \to N$$

(6.12d) $$K: T\times T\times N \to N$$

defined as follows

(6.13a) $$G(X,Y,Z) = \bar{R}(X,Y)Z|^{\shortparallel} - R(X,Y)Z - A(X,\Omega(Y,Z)) + A(Y,\Omega(X,Z))$$

(6.13b) $$C(X,Y,\xi) = \bar{R}(X,Y)\xi|^{\shortparallel} - \tilde{\nabla}_X A(Y,\xi) + \tilde{\nabla}_Y A(X,\xi)$$

(6.13c) $\mathcal{E}(X,Y,Z) = \bar{R}(X,Y)Z|^{\perp} - \tilde{\nabla}_X \Omega(Y,Z) + \tilde{\nabla}_Y \Omega(X,Z)$,

(6.13d) $K(X,Y,\xi) = \bar{R}(X,Y)\xi|^{\perp} - K(X,Y)\xi - \Omega(X,A(Y,\xi)) + \Omega(Y,A(X,\xi))$.

Then, the decomposition of $\bar{R}(X,Y)\bar{Z}$ leads to

(6.14a) $G(X,Y,Z) = 0$

(6.14b) $\mathcal{E}(X,Y,Z) = 0$

(6.14c) $K(X,Y,Z) = 0$

These equations are called the Gauss-Codazzi-Ricci(GCR) equations. The map C does not appear in these equations because C and $\mathcal{E}$ are equivalent due to

(6.15a) $\bar{G}(\bar{R}(X,Y)\xi|^{\parallel}, Z) + \bar{G}(\bar{R}(X,Y)Z|^{\perp}, \xi) = 0$,

(6.15b) $\bar{G}(\tilde{\nabla}_X A(Y,\xi), Z) + \bar{G}(\tilde{\nabla}_X \Omega(Y,Z), \xi) = 0$.

Here we give the local form of GCR equations only for the special case of $\bar{R}(X,Y)\bar{Z}=0$, i.e., for a Euclidean embedding space $\bar{E}_N$

(6.16a) $R_{\lambda\mu\nu\rho} = \bar{\eta}_{ab}(\Omega^a{}_{\lambda\nu}\Omega^b{}_{\mu\rho} - \Omega^a{}_{\lambda\rho}\Omega^b{}_{\mu\nu})$,

(6.16b) $\Omega^a{}_{\mu\nu;\lambda} - \Omega^a{}_{\mu\lambda;\nu} = (t^{ab}{}_{\lambda}\Omega_{b\mu\nu} - t^{ab}{}_{\nu}\Omega_{b\mu\lambda})$,

(6.16c) $t^{ab}{}_{\mu;\nu} - t^{ab}{}_{\nu;\mu} = \bar{\eta}_{cd}(t^{ac}{}_{\mu}t^{bd}{}_{\nu} - t^{ac}{}_{\nu}t^{bd}{}_{\mu})$

$+(\Omega^a{}_{\mu\lambda}\Omega^{b\lambda}{}_{\nu} - \Omega^a{}_{\nu\lambda}\Omega^{b\lambda}{}_{\mu})$

A set of real $\Omega^a{}_{\mu\nu}$, $t^{\mu}{}_{ab}$, solving the GCR equations for given metrics g and $\bar{G}$ is called an implicit embedding of $V_n(t,s)$ into $\bar{E}_N(T,S)$.

In an explicit embedding the functions $X^A(x^\mu)$ solving (2.4) are known while the second fundamental forms and torsion vectors are calculated by the use of

$$(6.18a) \quad \Omega^a_{\mu\nu} = \bar{G}\left(\hat{\xi}^a, \Omega(\partial_\mu, \partial_\nu)\right) = \bar{G}\left(\hat{\xi}^a, \bar{\nabla}_{\partial_\mu} \partial_\nu\right)$$

$$= \bar{G}_{AB} N^A_a \left(X^B_{\mu;\nu} + \bar{\Gamma}^B_{CD} X^C_\mu X^D_\nu\right)$$

$$(6.18b) \quad t_{\mu a b} = \bar{G}\left(\hat{\xi}_a, D_{\partial_\mu} \hat{\xi}_b\right) = \bar{G}\left(\hat{\xi}_a, \bar{\nabla}_{\partial_\mu} \hat{\xi}_b\right)$$

$$= \bar{G}_{AB} N^A_a \left(N^B_{b,\mu} + \bar{\Gamma}^B_{CD} N^C_b X^D_\mu\right)$$

Here

$$(6.19) \quad \hat{\xi}_a = N^A_a \partial_A, \quad \partial_\mu = X^A_\mu \partial_A$$

and $\bar{\Gamma}(x)$ is the connection of $T_{f(p)}(\bar{M})$ restricted to $T_p(M)$.

In order to obtain eqs.(6.18) one introduces N vectors in $\bar{E}_N$, in any point of V_n; these vectors are X^A_μ, N^A_a. By considering the change of these vectors along V_n, i.e., their covariant derivatives with respect to x^μ, the eqs.(6.18) are obtained.

If V_n is of class e, then it must admit e tensors $\Omega^a_{\mu\nu}$ and e(e-1)/2 vector fields t^μ_{ab} satisfying the GCR equations. For $e > 1$, the tensors $\Omega^a_{\mu\nu}$ and the vectors t^μ_{ab} are not defined uniquely by the embedding, due to the possibility of doing pseudo-rotations at every point of V_n, of the vectors N^A_a orthogonal to V_n. These degrees of freedom can be used to simplify the GCR equations in some special cases.

If one consider $V_n(t,s)$ as a submanifold of $\bar{E}_N(T,S)$ with $N=n(n+1)/2$ then, the existence of a solution of the GCR equations is guaranteed. Then we must only use the Gauss equation defining the Riemanntensor in terms of the

Gauss tensors. The other two equations, the Codazzi and Ricci ones, appearing as integrability conditions for the embedding, do not bring new information.

The GCR equations are not completely independent among themselves. In fact, they are interrelated by the Bianchi identities for $\bar{R}(\bar{X},\bar{Y})\bar{Z}\in T(\bar{M})$ and $R(X,Y)Z\in T(M)$, involving derivatives of the Gauss tensors. As a consequence of these identities part of the Codazzi equations are trivially satisfied. These identities are

(6.20a) $\sigma\{\tilde{\nabla}_W G(X,Y,Z) - C(X,Y,\Omega(W,Z)) + A(W,\mathcal{E}(X,Y,Z))\} = 0$

(6.20b) $\sigma\{\tilde{\nabla}_W C(X,Y,\xi) - G(X,Y,A(W,\xi)) + A(W,K(X,Y,\xi))\} = 0$

(6.20c) $\sigma\{\tilde{\nabla}_W \mathcal{E}(X,Y,Z) - K(X,Y,\Omega(W,Z)) + \Omega(W,G(X,Y,Z))\} = 0$

(6.20d) $\sigma\{\tilde{\nabla}_W K(X,Y,\xi) - \mathcal{E}(X,Y,A(W,\xi)) + \Omega(W,C(X,Y,\xi))\} = 0$

where σ denotes cyclic permutation of X, Y and W.

Because of

(6.21) $\bar{G}(C(X,Y,\xi),Z) + \bar{G}(\mathcal{E}(X,Y,Z),\xi) = 0$

the equations (6.20b) and (6.20c) are, again, equivalent. Then, we can rewrite

(6.22a) $\sigma\{\bar{G}(\tilde{\nabla}_W G(X,Y,Z),U) - \bar{G}(C(X,Y,\Omega(W,Z)),U) + \bar{G}(C(X,Y,\Omega(W,U)),Z)\} = 0,$

(6.22b) $\sigma\{\bar{G}(\tilde{\nabla}_W C(X,Y,\xi),Z) - \bar{G}(\Omega(W,G(X,Y,Z)),\xi) + \bar{G}(A(W,K(X,Y,Z)),Z)\} = 0,$

(6.22c) $\sigma\{\bar{G}(\tilde{\nabla}_W K(X,Y,\xi),\eta) + \bar{G}(C(X,Y,\eta),A(W,\xi)) - \bar{G}(C(X,Y,\xi),A(W,\eta))\} = 0.$

In local coordinates these equations were given by Blum (1955). In some exceptional cases it is only necessary to satisfy the Gauss equations to assure the embedding property of a given V_n. A particular application of these identities to the case in which the rank of one of the N-n second fundamental forms is $\geqslant 3$ is found in (Goenner, 1977; Gupta and Goel, 1975). The results include the Thomas's theorem for class one embeddings, cf. sec. I.7.

In the general case, for arbitrary class, a practical algorithm to solve the GCR equations, either in the sense of determining all the solutions of Einstein field equations of a given class, or in the sense of determining the class of a given metric, is not known. The tradition is to look for necessary algebraic conditions replacing the GCR equations. The purposes are to determine the class by means of algebraic manipulations with the metric, the curvature tensor and, if necessary, its covariant derivatives.

Up to now the progress made has resulted in

i. The connection between class and other properties of the metric, e.g., special vector and tensor fields, groups of motion, etc.

ii. Exact solutions of class one and two.

iii. Explicit embedding of certain metrics or kinds of metrics.

In what follows we are going to present the few general results concerning the class of embedding of V_n into $\bar{E}_N$. It is trivial to check if e=0 or not, because one must only check if the Riemann tensor is or is not identically zero.

The embeddings of class one and two are the subject of the next sections.

I.7. Class One Embeddings

Class one embeddings provides the simplest case to start. Furthermore, they are of interest for cosmology since several cosmological solutions of Einstein field equations, in particular the Friedman-Robertson-walker metric, are of class one.

$V_n(t,s)$ is of class one if and only if there exists a symmetric tensor satisfying the Gauss

(7.1) $$R_{\lambda\mu\nu\rho} = \eta(\Omega_{\lambda\nu}\Omega_{\mu\rho} - \Omega_{\lambda\rho}\Omega_{\mu\nu})$$

and Codazzi

(7.2) $$\Omega_{\mu\nu;\lambda} - \Omega_{\mu\lambda;\nu} = 0$$

equations, where $\eta = \pm 1$ is conveniently chosen. The Ricci tensor is given by

(7.3) $$R_{\mu\nu} = \eta(\Omega\Omega_{\mu\nu} - \Omega_{\mu\lambda}\Omega^{\lambda}{}_{\nu})$$

where $\Omega = \Omega_{\mu}{}^{\mu} = \mathrm{tr}\,\Omega_{\mu\nu}$.

For positive definite metrics Thomas(1936) and Rosenson(1940, 1941, 1943) gave an algebraic criterion to determine whether or not the embedding class is one. While the work by Thomas is non-covariant, Rosenson derived necessary and sufficient conditions in the form of tensor equations. This criterion involves long calculations, namely, the evaluation of a big number of determinants. This criterion is extended with minor changes to indefinite metrics. In (O'Neill,1959) a modern and generalised version of Thomas's result is given. Thus, always there exists a sure, but long, criterion to check if e=1 or not. However, there exist some examples where one can apply a necessary and sufficient condition to check if e=1 or not.

The first result on this line is due to Kasner (1921b) and involves a necessary condition.

Theorem. There are no vacuum solutions of class one.

All the components of the Ricci tensor cannot be zero at the same time, and if $R_{\mu\nu}=0$, then $e\neq 1$. This theorem was also derived independently and contemporarily by Schouten and Struik(1921). However, the proofs given by Kasner and by Schouten and Struik depend on the diagonalisation of a symmetric tensor through a local rotation, an invalid procedure for indefinite metrics. The correct proof for this theorem for normal hyperbolic manifolds was given by Szekeres(1966). Even more, the Szekeres's proof fails to be valid if the signature of the metric has more than one plus sign. The Szekeres's proof makes use of the method sketched in the next section.

For general relativity, $n=4$, there are no non-flat vacuum space-times having an embedding in five dimensions. The embedding of a non-vacuum metric in five dimensions is minimal; the embedding of a non-flat vaccum metric in six dimensions is minimal.

The next result is due to Eisenhart(1926).

Theorem. Any $V_n(t,s)$ of constant scalar curvature has class one embedding.

Therefore, any $V_n(t,s)$ of constant curvature is locally isometric to a portion of a hypersphere in $E_{n+1}(T,S), T\geqslant t,\ S\geqslant s$.

The next result is due to Collinson(1968).

Theorem. There are no solutions of the Einstein-Maxwell equations of class one with non-null electromagnetic field.

The following result, concerning the relation between tha Gauss and Codazzi equations, is due to Thomas (1936).

Theorem. If $V_n(t,s)$ is embedded in $E_{n+1}(T,S)$ and if rank $a_{\mu\nu}\geqslant 4$, then the Codazzi equations follow from the Gauss ones.

The proof makes use of the Gauss equations and of the Bianchi identities(Goenner, 1977). For $n\geqslant 4$, the condition on the rank of $a_{\mu\nu}$ can be expressed intrinsically as

(7.4) $C_{\lambda\mu\nu\rho}C^{\lambda\mu\nu\rho} - 2R_{\mu\nu}R^{\mu\nu} + \frac{2}{3}R^2 \neq 0.$

In general relativity, $n=4$, the theorem is applicable only if the equality holds, rank$\Omega_{\mu\nu}=4$, i.e., det det$\Omega_{\mu\nu}\neq 0$. Therefore, for general relativity one can establish the following result.

Corollary. If there exists a non-singular symmetric tensor $\Omega_{\mu\nu}$ satisfying the Gauss equations then the space-time is of class one.

Another, quite interesting result is the following.

Theorem. Any Riemannian manifold V_n with analytic metric, locally, can be isometrically and analitycally embedded into a certain(unique) Ricci-flat space $\overline{V}_{n+1}$.

This theorem was proved by Campbell(1926). Its extension to arbitrary signature seems trivial.

I.7.1 Canonical Forms of the Gauss Tensor

Because of the algebraic simplicity of the Gauss equations for class one embedding, all possible Gauss tensors corresponding to a given Ricci tensor can be determined starting with an adequate tetrad representation of $R_{\mu\nu}$ and $\Omega_{\mu\nu}$. This method is due to Szekeres(1966) and it gives also a criterion to know if a space-time is of class one or not.

We assume the space-time to be of class one, then (7.3) is valid and $[R,\Omega]=0$. Then $R_{\mu\nu}$ and $\Omega_{\mu\nu}$ admit the same set of eigenvectors. Let $(v_{(\alpha)}{}^{\mu})$ be this set, $\alpha=0,1,2,3$. Let $(\mu_{(\alpha)})$ and $(\lambda_{(\alpha)})$ be the corresponding eigenvectors.

(7.5) $$R_{\mu\nu}\, V_{(\alpha)}{}^{\nu} = \mu_{(\alpha)}\, V_{(\alpha)\mu},$$

(7.6) $$\Omega_{\mu\nu}\, V_{(\alpha)}{}^{\nu} = \lambda_{(\alpha)}\, V_{(\alpha)\mu}.$$

The set of simultaneous eigenvectors of $R_{\mu\nu}$ and $\Omega_{\mu\nu}$ provides a tetrad for the space-time. If all the eigenvectors $(v_{(\alpha)}{}^{\mu})$ are non-null, then they can be normalised to $\sigma_{(\alpha)}=v_{(\alpha)}{}^{\mu}v_{(\alpha)\mu}=\pm1$, if they are time- or spce-like, respectively, and in this way they constitute an orthonormal tetrad. The generalisation to the case in which one or more of the eigenvectors are null presents only technical difficulties. The metric, the Gauss and the Ricci tensors can be written, in terms of the tetrad, as

$$g_{\mu\nu} = \sum_{\alpha} \sigma_{(\alpha)}\, V_{(\alpha)\mu}\, V_{(\alpha)\nu}$$

(7.7) $$\Omega_{\mu\nu} = \sum_{\alpha} \sigma_{(\alpha)}\, \lambda_{(\alpha)}\, V_{(\alpha)\mu}\, V_{(\alpha)\nu}$$

$$R_{\mu\nu} = \sum_{\alpha} \sigma_{(\alpha)}\, \mu_{(\alpha)}\, V_{(\alpha)\mu}\, V_{(\alpha)\nu}$$

The factor $\sigma_{(\alpha)}$ is included to give account of the time- or space-like character of $v_{(\alpha)}{}^{\mu}$. The trace of the Gauss tensor is

(7.8) $\Omega = tr\,\Omega_{\mu\nu} = \sum_\alpha \lambda_{(\alpha)}.$

Then, eq;(7.3) can be written as

(7.9) $\kappa\, \lambda_{(\alpha)}\left(\sum_\beta \lambda_{(\beta)} - \lambda_{(\alpha)}\right) = \mu_{(\alpha)}.$

It may happens that these equations have not a solution at all, then $e>1$. If a solution exists, this is in general not unique. The correct solution is selected by putting back the solution in the Gauss equation (7.1). This may also lead to no solution, then $e>1$. If a solution survives, then we must check the condition $\det\Omega_{\mu\nu}\neq 0$, but in terms of the eigenvalues this is

(7.10) $\prod_\alpha \lambda_{(\alpha)} \neq 0.$

Therefore, if one of the eigenvalues is zero the Codazzi equations (7.2) must also be used. Again it may happens not to have a solution, then $e>1$. If a solution manages to survive, then $e=1$.

Equation (7.9) can also be used to determine the class one solutions of Einstein field equations when the energy-momentum tensor is given. For perfect fluid and Maxwell type energy-momentum tensors cf.(Stephani, 1967).

I.8. Class Two Embeddings

The concrete results for class two embeddings are not so abundant as for class one.

For space-times of class two there are some algebraic relations needing to be hold. The first result is due to Yakupov(1968a,b).

Theorem. A necessary condition for a space-time to be of class two is

$$(8.1)\quad {}^*R^{*\lambda\mu\nu\rho}\,{}^*R_{\sigma\tau\nu\rho}R_{\lambda\mu}{}^{\sigma\tau}=0.$$

The left and right duals are defined by

$$(8.2)\quad \begin{aligned} {}^*R_{\alpha\beta\gamma\delta} &= \tfrac{1}{2}\sqrt{-g}\,\varepsilon_{\alpha\beta\mu\nu}R^{\mu\nu}{}_{\gamma\delta}, \\ R^*_{\alpha\beta\gamma\delta} &= \tfrac{1}{2}\sqrt{-g}\,\varepsilon_{\gamma\delta\mu\nu}R_{\alpha\beta}{}^{\mu\nu}. \end{aligned}$$

The next result is due to Matsumoto(1950).

Theorem. A necessary condition for algebraically special space-times to be of class two is

$$(8.3)\quad {}^*R^{\lambda\mu\nu\rho}R_{\lambda\mu\nu\rho}=0.$$

If two spaces are conformally related, then their classes are also related. The next result is due to Brinkmann(1923).

Theorem. Any conformally flat V_n has embedding class at most two.

An example of a class one conformally flat space are the Friedman-Robertson-Walker spaces.

A stronger result is due to Szekeres(1966).

Theorem. If V_n is of class e and $\hat{V}_n$ is conformally related to V_n then $\hat{e}\leq e+2$.

From Brinkmann's explicit embedding of a conformally flat manifold a slightly more general result is obtained (Goenner, 1980).

Theorem. If V_n is of class e and $\hat{V}_n\subset E_N$ is conformally related to V_n, then $\hat{e}\leq e+2$.

Further results for class two embeddings are related with Petrov types. The interested reader is referred to the literature(Goenner, 1980; Kramer et al., 1981).

Some examples of space-times of class two are, among others, the interior(Fronsdal, 1959) and exterior(Kasner, 1921a) Schwarzschild solutions, the Reissner-Nordstrøm and the Reissner-Weyl(charged particle) solutions.

Eisland(1925) proved $e \leqslant 2$ for any spherically symmetric Riemannian manifold and gave a necessary and sufficient condition for class one formulated covariantly by Takeno(1966) and rediscovered by Karmarkar(1948) and by Kohler and Chao(1965).

I.9. A Parametrisation of the Embedding

The maximal mobility of the flat embedding space allows the introduction of a particular parametrisation of the embedding. Since $\bar{E}_N(T,S)$ has Euclidean global topology, it can be decomposed as

$$E_N(T,S) = M_n(t,s) \otimes B_{N-n}(T-t, S-s) \tag{9.1}$$

with $M_n(t,s)$ being a maximally symmetric space. The reason for this choice will be done clear in the section II.2. This parametrisation corresponds to choose Gaussian coordinates in the embedding space. The first n coordinates X^A are internal coordinates in $V_n(t,s)$. The coordinates in $V_n(t,s)$ are denoted by x^μ, those in B_{N-n} by ϕ^a, in such a way that the coordinates in $E_N(T,S)$ can be written as $X^A=(x^\mu,\phi^a)$. For $X^A{}_\mu$ one obtains

$$X^A{}_\mu = (\delta^\nu_\mu, \phi^a{}_\mu) \tag{9.2}$$

this is not a tensor index

with $\phi^a{}_\mu = \partial_\mu \phi^a$.

In general relativity the maximally symmetric spaces are Minkowski, de Sitter and anti-de Sitter. We deal separately with each case.

The Minkowski case

The first results in this direction are partially due to Krause(1973, 1975, 1976). In this case M_n and B_{N-n} are both Minkowski spaces. The metric tensor of $\bar{E}_N$ can be written as

(9.3) $$\bar{G}_{AB} = \left(\begin{array}{c|c} \mathring{g}_{\mu\nu}(x) & 0 \\ \hline 0 & \bar{n}_{ab} \end{array}\right)$$

with Rie($\mathring{g}$)=0. Quantities with a ° are associated with M_n. The metric tensor of V_n can be written as

(9.4) $$g_{\mu\nu} = \mathring{g}_{\mu\nu} + n_{ab}\,\phi^a{}_\mu \phi^b{}_\nu .$$

The affine connection takes also a simple form

(9.5) $$\Gamma_{\lambda,\mu\nu} = \mathring{\Gamma}_{\lambda,\mu\nu} + n_{ab}\,\phi^a{}_\lambda \phi^b{}_{\mu\nu}$$

with $\phi^a{}_{\mu\nu} = \partial_\mu \partial_\nu \phi^a$. The vectors N^{aA} are given by

(9.6) $$N^a{}_A = (-N^a{}_c \phi^c{}_\mu , N^a{}_b)$$

with $N^a{}_b$ satisfying

(9.7) $$(\bar{n}^{cd} + \mathring{g}^{\mu\nu}\phi^c{}_\mu \phi^d{}_\nu) N^a{}_c N^b{}_d = \bar{n}^{ab}.$$

For the Gauss tensors one obtains

(9.8) $$\Omega^a{}_{\mu\nu} = N^a{}_b \mathring{\nabla}_\mu \phi^b{}_\nu = N^a{}_b (\phi^b{}_{\mu\nu} - \mathring{\Gamma}^\lambda{}_{\mu\nu}\phi^b{}_\lambda)$$

The de Sitter case

In this case M_n is the de Sitter, or anti-de Sitter, space and B_{N-n} is of the form $\mathbb{R}^+ \otimes B_{N-n-1}$, such that

(9.9) $$E_N(T,S) = M_n(t,s) \otimes \mathbb{R}^+(n) \otimes B_{N-n-1}(T-t-A(n), S-s-A(-n))$$

with A(1)=1, A(-1)=0. The metric tensor of $\bar{E}_N$ can be written as

$$(9.10)\quad \bar{G}_{AB} = \left(\begin{array}{c|c} \left(\frac{\phi}{\ell}\right)^2 \hat{g}_{\mu\nu}(x;\ell) & 0 \\ \hline 0 & \bar{\eta}_{ab} \end{array}\right)$$

where $Rie(\hat{g}) = \left(\frac{\kappa}{\ell^2}\right)(\hat{g}\hat{g} - \hat{g}\hat{g})$; $\hat{g}$ is the metric of a pseudo-sphere of radius ℓ, $\kappa=1$ is the de Sitter space, $\kappa=-1$ is the anti-de Sitter one. ℓ is a radial-like coordinate, and $\eta_{ab} = (\kappa \pm \ldots \pm)$. The index associated with the radial-like coordinate is denoted by $a=r$. Careted quantities are associated with M_n. The metric tensor of V_n can be written as

$$(9.11)\quad g_{\mu\nu} = \left(\frac{\phi}{\ell}\right)^2 \hat{g}_{\mu\nu}(\ell) + \bar{\eta}_{ab}\,\phi^a{}_\mu \phi^b{}_\nu$$

The affine connection takes the form

$$(9.12)\quad \Gamma_{\lambda,\mu\nu} = \left(\frac{\phi}{\ell}\right)^2 \hat{\Gamma}_{\lambda,\mu\nu} + \frac{\phi}{\ell^2}\left(\phi_\mu \hat{g}_{\nu\lambda} + \phi_\nu \hat{g}_{\mu\lambda} - \phi_\lambda \hat{g}_{\mu\nu}\right) + \bar{\eta}_{ab}\,\phi^a{}_\lambda \phi^b{}_{\mu\nu}$$

The vectors $N^a{}_A$ are given by

$$(9.13)\quad N^a{}_A = \left(-\left(\frac{\phi}{\ell}\right)^{-2} N^a{}_c\, \phi^c{}_\mu,\ N^a{}_b\right)$$

with $N^a{}_b$ satisfying

$$(9.14)\quad \left[\bar{\eta}^{cd} + \left(\frac{\phi}{\ell}\right)^{-2} \hat{g}^{\mu\nu} \phi^c{}_\mu \phi^d{}_\nu\right] N^a{}_c N^b{}_d = \bar{\eta}^{ab}.$$

For the Gauss tensors one obtains

$$(9.15)\quad \Omega^a{}_{\mu\nu} = N^a{}_b\left[\phi^b{}_{\mu\nu} - \hat{\Gamma}^\lambda{}_{\mu\nu}\phi^b{}_\lambda - \phi^{-1}\left(\phi^b{}_\mu \phi_\nu + \phi^b{}_\nu \phi_\mu\right) - \frac{\phi}{\ell^2}\eta^{br}\hat{g}_{\mu\nu}\right]$$

In both cases, Minkowski and de Sitter, the quantities ϕ^a behave like scalar fields both in V_n and in M_n. Therefore, the decompositions of the metric and of the affine connection are invariant.

For class one embeddings the Gauss tensor is given by

$$(9.16)\quad \Omega_{\mu\nu} = \left[1 + \kappa\, g^{\mu\nu} \phi_\mu \phi_\nu\right]^{-1/2} \mathring{\nabla}_\mu \phi_\nu$$

for the Minkowski case, and

$$\Omega_{\mu\nu} = \left[1 + \hbar\left(\frac{\phi}{\ell}\right)^{-2} \hat{g}^{\mu\nu}(\ell)\, \phi_\mu \phi_\nu\right]^{-1/2} \times \left[\hat{\nabla}_\mu \phi_\nu - 2\phi^{-1}\phi_\mu\phi_\nu - \hbar\frac{\phi}{\ell^2}\hat{g}_{\mu\nu}\right] \tag{9.17}$$

for the de Sitter case.

Part II. Embeddings in General Relativity

II.1. Introduction

The use of embeddings in general relativity is as old as general relativity itself. In fact, just after Einstein presented the final version of general relativity(Einstein, 1915), Kasner(1921a,b) and Schouten and Struik(1921) used embeddings in their studies of general relativity.

General relativity is quite reasonable in terms of the intrinsic curvilinear coordinates of the Riemannian space-time itself. Nevertheless, embeddings have played some role in it. In fact, from time to time some new interesting results are obtained using embedding techniques. An example of this is provided by the work by Fronsdal(1959) on the complete analytic extension of the Schwarzschild solution through its embedding. The same result was contemporarily derived and published after a short time by Kruskal(1960) making use only of intrinsic differential geometry techniques.

Until now the main application of embedding techniques in general relativity has been as a mathematical tool. The stronger developments have been in finding exact solutions which cannot be obtained by means of other methods.

Besides considering the embedding techniques only as a mathematical tool, there exists the possibility of giving it a physical interpretation or derive new physical properties of the embedded space-time. What it is lacking now is to find a reasonable physical interpretation for the degrees of freedom inherent to the additional dimensions and for the geometrical objects induced by the embedding on the space-time. This is the task we are going to take in the next section.

II.2. The Foundations of General Relativity and Beyond

In 1905 Einstein established the foundations of the special theory of relativity. In this, the role of the Minkowski metric is to guarantee the covariance of the physical laws.

By 1908 Einstein attempts to sep up a special relativistic theory of gravitation adopting the Equivalence Principle, the statement that all bodies fall with the same acceleration in a gravitational field, as a fundamental criterion that any sensible theory of gravitation must satisfy. The main difficulty here was to see that gravitation has a very special relationship to inertia. In fact, gravitation and inertia are basically equal, they manifest in the same way and produce the same effects. They are practically indistinguishable.

By 1913 Einstein concludes that no scalar generalisation of Newton's theory will suffice and that the gravitational interaction must be described by a non-flat four-dimensional metric tensor. The main difficulty here was to see that a non-degenerate four-dimensional metric tensor with Minkowski signature represents not only the space-time structure(in some sense the inertial frames), but also provides the potentials from which the gravitational field can be derived. More revolutionary than the introduction of a non-flat space-time structure was the dynamising of this structure: the metric tensor not only acts upon matter, but is acted upon by it. Thus, the role of the metric becomes double: by one hand it serves to guarantee the covariance of the physical laws, and by the other one, it plays the role of a dynamical field.

Since the affine connection is not a tensor, it correctly embodies the gravitational-cum-inertial nature of the metric field. Einstein often emphasized that there is not a unique decomposition of the affine connection into an inertial connection plus a gravitational tensor. We will see, using embedding techniques, that this is not the case.

The following statement is independent of what was said in the previous paragraph(a free quotation of Einstein (1950)): "...what characterises the existence of a gravitational field is the impossibility of finding an inertial frame in which all the components of the affine connection $\Gamma_{\lambda,\mu\nu}$ vanish, not the non-vanishing of the Riemann tensor $R_{\lambda\mu\nu\rho}$." We will see, using again embeddings, that this statement needs also a modification.

As a direct consequence of the Equivalence Principle the fundamental field describing the gravitational interaction is the metric of the space-time. The main criticism against this is the fact that the metric is not a dynamical object at all.

At this point it seems not only advisable, but conceptually strictly necessary, to have an invariant decomposition of the metric in an inertial part, guaranteeing the covariance of the physical laws and giveng account of the inertia, plus a dynamical part giving account of the gravitational interaction.

Now it comes the problem of how to obtain an invariant decomposition of the metric in an inertial plus a gravitational part.

We start writing Einstein field equations with a cosmological term

$$(2.1) \quad R_{\mu\nu} - \frac{1}{2}\left(R - \frac{\Lambda}{2}\right) g_{\mu\nu} = T_{\mu\nu}$$

and the Riemann tensor as

(2.2) $$R_{\lambda\mu\nu\rho} = C_{\lambda\mu\nu\rho} - \frac{R}{6}\left(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu}\right) + \frac{1}{2}\left(R_{\lambda\nu}g_{\mu\rho} + R_{\mu\rho}g_{\lambda\nu} - R_{\lambda\rho}g_{\mu\nu} - R_{\mu\nu}g_{\lambda\rho}\right)$$

The inertial part of the metric is obtained by switching off all what have something to do with gravitation.

Since the Weyl tensor is not determined by the interaction of gravitation with matter, it represents the pure gravitational contribution. The first step is to put $C_{\lambda\mu\nu\rho}=0$,

(2.3) $$R_{\lambda\mu\nu\rho} = \frac{1}{2}\left(R_{\lambda\nu}g_{\mu\rho} + R_{\mu\rho}g_{\lambda\nu} - R_{\lambda\rho}g_{\mu\nu} - R_{\mu\nu}g_{\lambda\rho}\right) - \frac{R}{6}\left(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu}\right).$$

No gravitation means no matter in the game, therefore, for the Ricci tensor one obtains

(2.4) $$R_{\mu\nu} = \frac{\Lambda}{4} g_{\mu\nu}$$

or equivalently, for the Riemann tensor

(2.5) $$R_{\lambda\mu\nu\rho} = \frac{\Lambda}{12}\left(g_{\lambda\nu}g_{\mu\rho} - g_{\lambda\rho}g_{\mu\nu}\right)$$

Therefore, quite formally, a maximally symmetric space-time defines the inertial part of the metric: Minkowski ($\Lambda=0$), de Sitter($\Lambda>0$) and anti-de Sitter($\Lambda<0$). The rest of the metric corresponds to the dynamical, or gravitational part. Any maximally symmetric space-time corresponds to the absence of dynamical fields, i.e., only kinematics. Any deviation of the metric from that corresponding to a maximally symmetric space-time is to be interpreted as the presence of gravitation. In this way the maximally symmetric space-times play the role of the ground state of the geometry.

This gives the criterion to choose $M_n(t,s)$ in sec.I.9.

$M_n(t,s)$ must be a maximally symmetric space.

From eqs.(9.4) and (9.8), and (9.11) and (9.15), one sees that $Rie=(\frac{\Lambda}{12})(gg-gg)$ implies, for each case, that the ϕ^a's are constant, and viceversa. Furthermore, since the ϕ^a's behave as scalar fields, both in $V_n(t,s)$ and in $M_n(t,s)$, the decomposition of the metric and of the affine connection are invariant. Therefore, we have been able to decompose the metric and the affine connection in their inertial and gravitational parts.

The procedure is then the following. Take your metric and imposse $Rie=(\frac{\Lambda}{12})(gg-gg)$. This will give some conditions on the original metric fixing quite uniquely the value of Λ (its sign). This is the part of the metric corresponding to $M_n(t,s)$. Then, with the use of embeddings, cf. eqs.(9.4) and (9.11), it is guaranteed that one can write the rest of the metric in terms of six functions ϕ^a.

It could seem a little bit artificial to consider Einstein field equations with a cosmological term. This is justified by considering the following fact. If we put $\Lambda=0$, then the bagkground topology is Minkowski. However, Minkowski space cannot be the ground state geometry for topologies not diffeomorphic to it. Proceeding as before one introduces a change in the signature of the metric of space-time. The solution is, of course, to enlarge the set of spaces admissible for the ground state of the geometry. But this is already done when considering a cosmological term in the Einstein field equations.

Summarising, the metric of space-time has been decomposed in a maximally symmetric part which is kinematic in character corresponding to $M_4(1,3)$, and a genuinly gravitational part of dynamical character coming from the dimensional reduction of $M_4 \otimes B_6$ into M_4. Then, the dynamics of the gravitational field, or equivalently of the space-time, appears only from B_6.

The coordinates of B_6 written in terms of the coordinates of M_4, i.e., the fields ϕ^a, can be considered as truly dynamical fields responsible for the gravitational interaction.

Now the components of the metric are no longer the basic variables but rather derived objects constructed from the functions ϕ^a describing the embedding of V_4 into $\bar{E}_{10}$. According to this, the metric of the space-time is not a fundamental, but more an effective field produced by the gravitational interaction.

The impossibility, up to now, of obtaining a consistent quantisation of general relativity can be traced back to the fact of considering the metric as the fundamental field to be quantised. In this case one is, at the same time, quantising dynamical and kinematic quantities. Quantisation of kinematic quantities is a non-sense.

Now that we have been able to identify the fields responsible for the dynamics of the gravitational interaction we must set up a variational formulation for them. This is the subject of the next section.

II.3. The Embedding Approach to General Relativity

The first attempt to give a physical interpretation to the geometry of embeddings in general relativity is due to Regge and Teitelboim(1976). These authors use an analogy with the string model of elementary particles. Rather than the components of the metric tensor, the basic fields of the formalism are taken to be the functions describing the embedding of the four-dimensional space-time in the ten-dimensional flat space. Here we consider only the Minkowski case. Other interesting approaches are being followed by Maia(1986) and Pavšič (1986).

The Einstein theory of gravitation takes as its starting point the Hilbert action

(3.1) $$S = \int R \sqrt{-g}\, d^4x$$

which, regarded as a functional of the metric tensor $g_{\mu\nu}$, under variation gives the vacuum Einstein field equations

(3.2) $$G_{\mu\nu} = 0.$$

For the action, Regge and Teitelboim use the same action but regarded this time as a functional of the fields ϕ. By requiring the action to be stationary under variation of the fields ϕ one does not obtain Einstein field equations, but rather the weaker set

(3.3) $$G^{\mu\nu} \Omega^{a}{}_{\mu\nu} = 0.$$

As a way out to this difficulty, Deser et al.(1976) proposed to find another action functional $I[\phi]$ whose Euler-Lagrange equations be equivalent to Einstein ones, (3.2). However, under variation $I[\phi]$ gives rise to six equations instead of the ten ones of Einstein. Therefore, the equivalence is hardly established without introducing some arbitrariness in the game.

As a first step towards a correct solution of this problem we rewrite the Einstein field equations, in the form $R_{\mu\nu}=0$, in term of the fields ϕ

(3.4) $$R_{\mu\nu} = g^{\lambda\rho} M_{ab} \left(\phi^a{}_{\lambda\rho}\, \phi^b{}_{\mu\nu} - \phi^a{}_{\mu\lambda}\, \phi^b{}_{\nu\rho} \right) = 0$$

where $M_{ab} = \eta_{cd} N^c{}_a N^d{}_b$. A 1+3 decomposition of these equations gives

(3.5a) $$R_{00} = g^{ij} M_{ab} \left(\phi^a{}_{00}\, \phi^b{}_{ij} - \phi^a{}_{0i}\, \phi^b{}_{0j} \right) = 0,$$

(3.5b) $$R_{0i} = -g^{0j} M_{ab} \left(\phi^a{}_{00}\, \phi^b{}_{ij} - \phi^a{}_{0i}\, \phi^b{}_{0j} \right) + g^{jk} M_{ab} \left(\phi^a{}_{0i}\, \phi^b{}_{jk} - \phi^a{}_{0j}\, \phi^b{}_{ik} \right) = 0,$$

(3.5c) $$R_{ij} = g^{00} M_{ab} \left(\phi^a{}_{00}\, \phi^b{}_{ij} - \phi^a{}_{0i}\, \phi^b{}_{0j} \right) + g^{0k} M_{ab} \left(2\phi^a{}_{ij}\, \phi^b{}_{0k} - \phi^a{}_{0i}\, \phi^b{}_{jk} - \phi^a{}_{0j}\, \phi^b{}_{ik} \right) + g^{kl} M_{ab} \left(\phi^a{}_{ij}\, \phi^b{}_{kl} - \phi^a{}_{ik}\, \phi^b{}_{jl} \right) = 0.$$

Only equations (3.5c) are truly dynamical ones. The other four are constraints and can be rewritten as

(3.6a) $$2 g^{0k} g^{ij} M_{ab} \left(\phi^a{}_{ij}\, \phi^b{}_{0k} - \phi^a{}_{0i}\, \phi^b{}_{jk} \right) \cancel{=0} + g^{ij} g^{kl} M_{ab} \left(\phi^a{}_{ij}\, \phi^b{}_{kl} - \phi^a{}_{ik}\, \phi^b{}_{jl} \right) = 0,$$

(3.6b) $$\left(g^{00} g^{jk} - g^{0j} g^{0k} \right) M_{ab} \left(\phi^a{}_{0i}\, \phi^b{}_{jk} - \phi^a{}_{0j}\, \phi^b{}_{ik} \right) + g^{0j} g^{kl} M_{ab} \left(\phi^a{}_{ij}\, \phi^b{}_{kl} - \phi^a{}_{ik}\, \phi^b{}_{jl} \right) = 0.$$

These equations correspond to a constrained Hamiltonian system; what characterises a constrained Hamiltonian system is the fact of having more equations than unknowns, not the order of the equations. For a Lagrangian system the number of equations and the number of unknowns are the same. The strategy to solve eqs.(3.5) and (3.6) is quite the same as for Maxwell equations when written in term of $\vec{E}$ and $\vec{B}$ fields. The aim is to reduce the previous equations to a Lagrangian system. This guarantee the existence of a variational formulation for them. The fields ϕ are analogous to $\vec{E}$ and $\vec{B}$ fields and one must look for the fields analogous to A_μ.

This is the status of the art of the embedding approach to general relativity;

Concluding Remarks

We feel that embeddings techniques may become a powerful tool to understand the geometry and physics of the most current theories in physics. In fact, for example, string theories are dealt with, mostly, in an extrinsic way.

Even when we have not been able to obtain more concrete results in the application of embeddings to general relativity, we believe that any new approach must be welcome in physics.

Appendix. Embedding of Friedman-Robertson-Walker Spaces

Friedman-Robertson-Walker(FRW) spaces are a quite simple and interesting example to start with.

First of all it is convenient to study the embedding geometry of maximally symmetric space-times. The interest is due to the key role they play in the embedding approach to general relativity.

Maximally symmetric space-times are of class one. Their Gauss tensor are given by

$$\Omega_{\mu\nu} = \frac{1}{\ell^2} g_{\mu\nu} \tag{A.1}$$

where $1/\ell^2 = k\Lambda/12$, and Λ is the(constant) value of the scalar curvature. If $\Lambda=0$, the space-time is flat with Euclidean global topology. For $\Lambda\neq 0$ the solution of eq.(A.1) is, in polar coordinates

$$\phi^2 = \ell^2 - k(t^2 - r^2) \tag{A.2}$$

$k=1$, $\Lambda>0$, is the de Sitter space described by the hyperboloid $\phi^2+t^2=\ell^2+r^2$. $k=-1$, $\Lambda<0$, is the anti-de Sitter space described by the hyperboloid $\phi^2+r^2=\ell^2+t^2$. In both cases ℓ is the constant value of the radial-like coordinate introduced in sec.I.9. Therefore, the field ϕ_ℓ corresponding to the radial-like coordinate is $\phi_\ell^2=\phi^2+k(t^2-r^2)$.

FRW spaces are of class one(Robertson, 1929,1933). Their line element can be written as

$$ds^2 = d\tau^2 - S^2(\tau)\left[(1-k\rho^2/\ell^2)^{-1}d\rho^2 + \rho^2 d\Omega^2\right] \tag{A.3}$$

$S(\tau)$ is the cosmic scale factor; k takes the three discrete values ± 1, 0. It is convenient to choose the background metric as that corresponding to polar coordinates. This is done by the transformation of coordinates $(\tau,\rho)\rightarrow(t,r)$

defined by

(A.4a) $\tau = \tau(t, r)$,

(A.4b) $\rho = r / S(\tau(t,r))$.

Then, using $g_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + \kappa \phi_\mu \phi_\nu$ we arrive to a set of differential equations for ϕ and τ having the following solutions:

a. $k \neq 0$. Then one can choose $\tau_1 = 0$, and

(A.5) $$\tau_0^2 = \frac{k}{k + l^2 S^2},$$

with the additional relation $-\kappa k > 0$. For ϕ one obtains

(A.6) $$\phi^2 = l^2 S^2 - k r^2.$$

b. $k=0$. Then $\kappa = -1$. τ and ϕ are given by

(A.7) $$t = \frac{S(\tau)}{2l} \rho^2 + \frac{l}{2} S(\tau) + \frac{1}{2l} \int \frac{d\tau}{\dot{S}(\tau)},$$

(A.8) $$\phi = l S - t.$$

The fields ϕ_l describing the gravitational interaction are, $k=1$:

(A.9) $$\phi_l = \sqrt{l^2 S^2 - t^2} - l$$

$k=-1$:

(A.10) $$\phi_l = \sqrt{l^2 S^2 + t^2} - l$$

$k=0$:

(A.11) $$\phi_l = l S - t$$

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