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PERSISTENCE OF INTERSECTIONS IN SYMPLECTIC TOPOLOGY

Thesis submitted for the degree of Magister Philosophiae

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TRIESTE

PREFACE

Everything is a
lagrangian submanifold
A. Weinstein

The aim of this work is to study some questions related to the intersection theory of lagrangian submanifolds in a symplectic manifold. This theory originates from a conjecture of V.I. Arnol'd on the number of fixed points of a symplectic diffeomorphism, but it is now a vaste subject, related to many other problems (not only in symplectic geometry).

We will focalize our attention to the more "geometric" aspects of these things, for example we shall not speak about the variational calculus arising from Arnol'd conjecture.

After recalling some basic facts, we shall study lagrangian submanifolds in a cotangent bundle and their intersections. A lagrangian submanifold in a cotangent bundle may be seen as a generalization of the notion of function (on the base of the cotangent bundle); the corresponding intersection theory may be seen as an extension of Morse-Lyusternik-Schnierelmann theory from the usual functions to this sort of generalized functions.

The main problem is then to pass from cotangent bundle to more general symplectic manifolds. We shall do this in the third chapter, in some particular (=local) situations.

In the last chapter we shall enlarge the discussion to a more general context.

We will deal always with the smooth (=C^∞) category. Function spaces, unless otherwise stated, will be given with the Whitney C^∞- topology.

Last but not least, we express our gratitude to A. Verjovsky, for stimulating conversations about this subject.

INTRODUCTION:

FROM SYMPLECTIC FIXED POINTS TO LAGRANGIAN INTERSECTIONS

We recall some definitions from symplectic geometry ([Arn-Giv], [Wein 2], [Cha]).

A *symplectic manifold* (M, ω) is a manifold M equipped with a closed, non degenerate 2-form ω. A diffeomorphism $\phi : M \to M$ is *symplectic* if it preserves ω ($\phi^*\omega = \omega$), a vector field $v : M \to TM$ is *symplectic* if $L_v\omega = 0$; equivalently, v is symplectic if $i_v\omega$ is a closed 1-form. If, moreover, $i_v\omega$ is exact, then v is said to be *hamiltonian.*

We will use the notations:

$Diff_\omega(M)$= group of symplectic diffeomorphisms

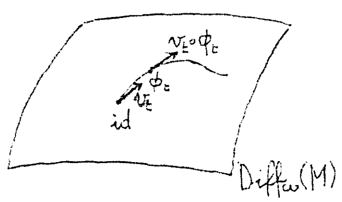
$Vec_\omega(M)$= symplectic vector fields

$Vec_H(M)$= hamiltonian vector fields

A *symplectic isotopy* $\{\phi_t\}_{t\in[0,1]}$ is a smooth curve in $Diff_\omega(M)$, $t \mapsto \phi_t \in Diff_\omega(M)$, that starts at id_M: $\phi_0 = id_M$. Here "smooth" means that $t \mapsto \phi_t(x)$ is a smooth curve on M, $\forall x \in M$. To every symplectic isotopy there is associated its *infinitesimal generator*, i.e. the smooth curve of symplectic vector fields $\{v_t\}_{t\in[0,1]}$ defined by

$$v_t(x) = \frac{d}{ds}\Big|_{s=0}\phi_{t+s}(\phi_t^{-1}(x)) = \dot{\phi}_t(x)$$

$$\forall x \in M, \quad \forall t \in [0,1]$$



Conversely, from a smooth curve of symplectic vector fields $\{v_t\}_{t\in[0,1]}$ we can construct a symplectic isotopy $\{\phi_t\}_{t\in[0,1]}$ by integrating the (non-autonomous) differential equation

$\dot{x} = v_t(x)$ (if the flow is defined up to the time 1). So, if M is compact, we have a 1-1 correspondence between curves in $Diff_\omega(M)$, starting at id_M, and curves in $Vec_\omega(M)$.

The symplectic isotopy $\{\phi_t\}_{t\in[0,1]}$ is said to be *hamiltonian* if $v_t \in Vec_H(M)\ \forall t \in [0,1]$. A symplectic diffeomorphism ϕ is called *hamiltonian* if there exists a hamiltonian isotopy $\{\phi_t\}_{t\in[0,1]}$ such that $\phi_1 = \phi$ (in other words, ϕ is the time-1-flow of a non-autonomous hamiltonian vector field).

We will write $Diff_H(M)$ for the space of hamiltonian diffeomorphisms of M. Clearly $Diff_H(M)$ is a subgroup of $Diff_\omega(M)_0$ (=connected component of $Diff_\omega(M)$ containing id_M), and a theorem of Banyaga ([Ban1]) asserts that

$$Diff_H(M) = [Diff_\omega(M)_0, Diff_\omega(M)_0]$$

If $\{v_t\}_{t\in[0,1]}$ is the infinitesimal generator of the symplectic isotopy $\{\phi_t\}_{t\in[0,1]}$, then $\int_0^1 (i_{v_t}\omega)dt$ is a closed 1-form and the following definition makes sense.

Definition:

$$I(\{\phi_t\}) = [\int_0^1 i_{v_t}\omega dt] \in H^1(M,R)$$

is the *Calabi class* of $\{\phi_t\}_{t\in[0,1]}$.

If $\{\psi_t\}$, $\{\phi_t\}$ are two homotopic symplectic isotopies with the same endpoints, then $I(\{\psi_t\}) = I(\{\phi_t\})$. If $I(\{\phi_t\}) = 0$, then there exists a hamiltonian isotopy $\{\psi_t\}_{t\in[0,1]}$ such that $\psi_1 = \phi_1$. If the symplectic form ω is exact (this is the case of cotangent bundles) then $I(\{\phi_t\})$ depends only on ϕ_1.

Arnol'd conjecture

In the middle of the sixties V.I. Arnol'd, starting from the last geometric theorem of Poincarè, stated the following conjecture on fixed points of hamiltonian diffeomorphisms ([Arn1], [Arn2]):

Conjecture 1: let (M,ω) be a compact symplectic manifold and let $\phi_t : M \to M$, $t \in [0,1]$, be a hamiltonian isotopy, then the number of fixed points of ϕ_1 is not less then the number of critical points of a function on M.

Here "number" may be understood both geometrically and algebraically (i.e. counting multiplicities).

As a consequence of this conjecture, we get that $\#Fix(\phi_1)$ satisfies Lyusternik-Schnierelmann inequality and, if the fixed points are non degenerate, Morse inequality. These estimates are sharper then those given by Lefschetz formula, valid for any diffeomorphism. In particular, it is conjectured the existence of at least one fixed point, for any compact symplectic manifold M and any hamiltonian diffeomorphism ϕ_1.

A situation in which Arnol'd conjecture is trivially true is when $\{\phi_t\}_{t\in[0,1]}$ is generated by an autonomous hamiltonian vector field v, i.e. $\{v_t\}_{t\in[0,1]}$ is constant in t: if $H : M \to R$ is the hamiltonian of v ($i_v\omega = dH$), then the critical points of H are fixed points of ϕ_1.

Using the Calabi class, it is easy to generalize conjecture 1 to symplectic diffeomorphisms which are still in $Diff_\omega(M)_0$ but not in $Diff_H(M)$:

Conjecture 1': let (M,ω) be a compact symplectic manifold and let $\phi_t : M \to M$, $t \in [0,1]$, be a symplectic isotopy, then the number of fixed points of ϕ_1 is not less then the number of zeroes of a closed 1-form on M, whose cohomology class is $I(\{\phi_t\})$.

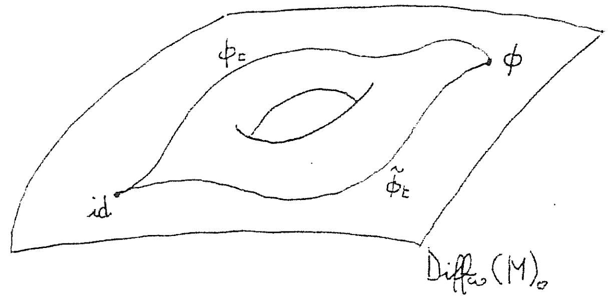
Again, conjecture 1' is trivial if $\{\phi_t\}_{t\in[0,1]}$ is generated by an autonomous vector field v: if v is a symplectic vector field and ϕ_t is its flow, then $I(\{\phi_t\}) = [i_v\omega]$ and the zeroes of the 1-form $i_v\omega$ are fixed points of ϕ_1.

From the point of view of concrete estimates of $\#Fix(\phi_1)$, conjecture 1' leads to the use of Novikov inequality instead of Morse inequality.

Remark: given $\phi \in Diff_\omega(M)_0$ we can choose different isotopies $\{\phi_t\}_{t\in[0,1]}$ such that $\phi_1 = \phi$ and (if $H^1(M,R) \neq 0$) we can obtain different classes $I(\{\phi_t\}) \in H^1(M,R)$; if we define

$$\Gamma = \{I(\{\phi_t\}) | \{\phi_t\}_{t\in[0,1]} \quad s.t. \quad \phi_1 = id_M\} \subset H^1(M,R)$$

(it is a sugroup), then the set of Calabi classes obtained from ϕ by considering all the possible isotopies ending at ϕ is of the form $\gamma+\Gamma$, for some $\gamma \in H^1(M,R)$. Hence conjecture 1' gives different estimates for the number of fixed points of the same diffeomorphism ϕ, depending on the isotopy choosen.



This unsatisfactory situation may be bypassed by considering in conjecture 1 or 1' not merely the fixed points of ϕ_1, but more precisely the only *contractible* fixed points, i.e. those fixed points x_0 of ϕ_1 such that the (closed) curve $t \mapsto \phi_t(x_0), t \in [0,1]$, is contractible in M. The number of contractible fixed points depends on $\{\phi_t\}_{t\in[0,1]}$, and not only on ϕ_1.

In fact, the results obtained up to now about Arnol'd conjecture give an estimate of the number of contractible fixed points.

Givental' conjecture

Let $\phi \in Diff_\omega(M) \setminus Diff_\omega(M)_0$, then it is not possible to apply conjecture 1,1' in order to bound $\#Fix(\phi)$; however it seems that also in this case it is possible to say something about the fixed points of ϕ. Givental' suggests ([Giv1], [Giv2]) that if ϕ is obtained by deforming with a hamiltonian isotopy some "special" $\psi \in Diff_\omega(M)$, then $\#Fix(\phi)$ can be bounded below in terms of the topology of $Fix(\psi)$. More precisely:

Conjecture 2: let (M,ω) be a compact symplectic manifold, let $\psi : M \to M$ be a periodic symplectic diffeomorphism ($\psi^k = id_M$ for some k), let $N = Fix(\psi)$; let $\phi_t : M \to M$, $t \in [0,1]$, be a hamiltonian isotopy, then the number of fixed points of $\phi_1 \circ \psi$ is not less then the number of critical points of a function on N.

Arnol'd conjecture follows from Givental' conjecture with $\psi = id_M$. Remark that Givental' conjecture is *not* trivial when $\{\phi_t\}_{t\in[0,1]}$ is generated by an autonomous hamiltonian vector field. In fact, the proof of conjecture 2 in such a situation, joined with the theorem of Banyaga that states the simplicity of $Diff_H(M)$ ([Ban1]), would imply the complete proof of conjecture 1.

A small generalization of conjecture 2 is obtained by considering a quasi- periodic symplectic diffeomorphism instead of a periodic one: $\psi : M \to M$ is called quasi-periodic if it generates a precompact subgroup of $Diff(M)$. For example, if $v \in Vec_\omega(M)$ is the generator of a S^1-action, then the time-T-flow of v, T irrational, is a quasi-periodic symplectic diffeomorphism.

The case of symplectic non hamiltonian isotopies leads to:

Conjecture 2': let (M,ω) be a compact symplectic manifold, let $\psi : M \to M$ be a periodic symplectic diffeomorphism, let $N = Fix(\psi)$; let $\phi_t : M \to M$, $t \in [0,1]$, be a symplectic isotopy, then the number of fixed points of $\phi_1 \circ \psi$ is not less then the number of zeroes of a closed 1-form on N whose cohomology class is $i^* I(\{\phi_t\})$, $i : N \to M$ being the canonical inclusion.

Remark: let $x_0 \in M$ be a fixed point of $\phi_1 \circ \psi$, then $\gamma : t \mapsto \phi_t(\psi(x_0))$ defines a curve on M starting from $\psi(x_0)$ and ending to x_0; on the manifold $M(\psi) \stackrel{def}{=} M \times [0,1]/(\psi(x),0) \sim (x,1)$ the map $\Gamma : t \mapsto (\phi_t(\psi(x_0)),t)$ defines a closed curve, and if this closed curve is homotopic to one of the form $t \mapsto (y,t)$, $y \in N$ fixed, we shall say that γ is *ψ-contractible*. For example, id_M-contractible = contractible. We shall say also that the fixed point x_0 is ψ- contractible. Probably, ψ-contractible fixed points play in conjectures 2, 2' the same rôle as contractible fixed points in conjectures 1, 1'.

Lagrangian intersections

It is useful to translate the above conjectures in the language of lagrangian intersections.

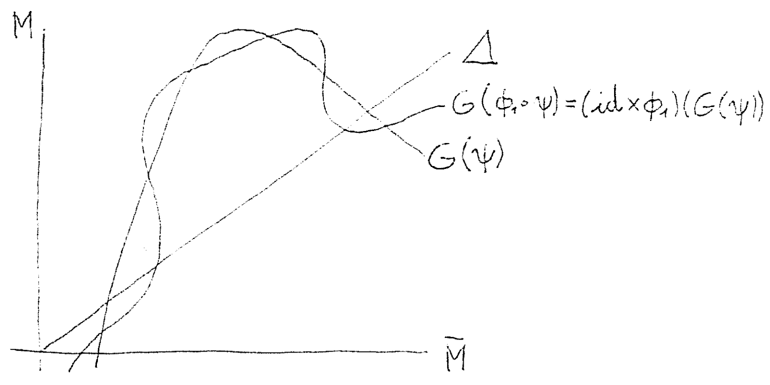
Recall that if (M, ω) is a symplectic manifold, the submanifold $L \overset{i}{\hookrightarrow} M$ is *lagrangian* if $dimL = \frac{1}{2} dimM$ and $i^*\omega = 0$.

We denote by $\overline{M}$ the symplectic manifold obtained from M by reversing the sign of ω; if (M_1, ω_1), (M_2, ω_2) are two symplectic manifolds, we will consider on $M_1 \times M_2$ the symplectic structure given by $\pi_1^*\omega_1 + \pi_2^*\omega_2$, where $\pi_j : M_1 \times M_2 \to M_j$ are the projections.

The diagonal $\Delta \subset \overline{M} \times M$ is then a lagrangian submanifold, and so is the graph $G(\phi)$ of any symplectic diffeomorphism.

If $\{\phi_t\}_{t \in [0,1]}$ is a hamiltonian (symplectic) isotopy of M, then $\{id \times \phi_t\}_{t \in [0,1]}$ is a hamiltonian (symplectic) isotopy of $\overline{M} \times M$; the graph of ϕ_1 is the deformation of Δ by $\{id \times \phi_t\}$: $G(\phi) = (id \times \phi_1)(\Delta)$; fixed points of ϕ_1 correspond to intersections between the two lagrangian submanifolds Δ and $(id \times \phi_1)(\Delta)$.

More generally, if $\psi : M \to M$ is a symplectic diffeomorphism, then the fixed points of $\phi_1 \circ \psi$ are in 1-1 correspondence with the intersections between the lagrangian submanifolds Δ and $(id \times \phi_1)(G(\psi))$. Observe that non-degenerate fixed points correspond to transverse intersections.



In this way, we are motivated to study the following problem: given $L_0, L_1 \subset M$ lagrangian submanifolds and $\phi_t : M \to M$, $t \in [0, 1]$, hamiltonian or symplectic isotopy, estimate $\phi_1(L_0) \cap L_1$ in terms of the topology of $L_0 \cap L_1$. Clearly, it is necessary to make some additional hypotheses on L_0, L_1.

The submanifolds $\Delta, G(\psi) \subset \overline{M} \times M$ are fixed points of antysymplectic involutions of $\overline{M} \times M$: $\Delta = Fix(\sigma_0)$, $\sigma_0(x,y) = (y,x)$, $G(\psi) = Fix(\sigma_1)$, $\sigma_1(x,y) = (\psi^{-1}(y), \psi(x))$. The diffeomorphism ψ is periodic (quasi-periodic) if and only if σ_0, σ_1 generate in $Diff(\overline{M} \times M)$ a finite (precompact) subgroup. This fact leads Givental' to state the following general conjecture ([Giv1]).

Conjecture 3: let (M,ω) be a symplectic manifold, $L_0, L_1 \subset M$ two lagrangian submanifolds of fixed points of antisymplectic involutions σ_0, σ_1, generating a precompact subgroup in $Diff(M)$; let $\phi_t : M \to M$, $t \in [0,1]$, be a hamiltonian isotopy, then the number of points in $\phi_1(L_0) \cap L_1$ is not less then the number of critical points of a function on $L_0 \cap L_1$.

As conjectures 1 and 2, also conjecture 3 has a symplectic version.

Firstly, it should be remarked that if $\phi_t : M \to M$, $t \in [0,1]$, is a symplectic isotopy and $N \subset M$ is a submanifold, then

$$i^*I(\{\phi_t\}) = j^*I(\{id \times \phi_t\})$$

where $i : N \to M$, $j : N \to \overline{M} \times M$ are the canonical inclusions $(j(x) = (i(x), i(x)), \forall x \in N)$. Hence if $N = Fix(\psi)$, ψ periodic symplectic diffeomorphism, then $i^*I(\{\phi_t\})$ (appearing in conjecture 2) is equal to the restriction of $I(\{id \times \phi_t\})$ to the intersection $\Delta \cap G(\psi)$ $(\simeq N)$.

Conjecture 3': let M, L_0, L_1 be as in conjecture 3 and let $\phi_t : M \to M$, $t \in [0,1]$, be a symplectic isotopy, then the number of points in $\phi_1(L_0) \cap L_1$ is not less then the number of zeroes of a closed 1-form on $L_0 \cap L_1$, whose cohomology class is $i^*I(\{\phi_t\})$, $i : L_0 \cap L_1 \to M$ being the canonical inclusion.

Some results

Arnol'd conjecture was proven by Eliashberg in 1978 in the case M = surface, with a method which seems not generalizable to higher dimension ([Eli]).

In 1983 Conley and Zehnder proved the same conjecture in the case $M = T^{2n}$, using a variational approach (a Lyapunov-Schmidt reduction of the hamiltonian action, whose critical points are the fixed points of the hamiltonian diffeomorphism, followed by an application of Conley's theory on isolated invariant sets of flows on locally compact spaces). The ideas of Conley and Zehnder, joined with those of Chaperon about "broken geodesics", lead to the results of Laudenbach and Sikorav ([Lau-Sik], [Sik1], [Sik2], [Sik3]) about conjectures 3, 3' in the case M = cotangent bundle, $L_0 = L_1$ = null section. As a corollary, Sikorav extended the result of Conley and Zehnder to the case M = compact symplectic symmetric manifolds with non-positive curvature ([Sik3]).

In all this works the main step of the proofs is a finite dimensional reduction of a variational problem, in order to apply Conley theory to estimate critical points of a function defined on a non-compact manifold but with a nice behaviour at infinity.

But since 1987 Floer developed a suitable Morse-Conley theory for the hamiltonian action, starting from the fact that the space of *bounded* trajectories of the gradient of this functional "is" a finite dimensional compact manifolds, and avoiding in this way any reduction. This fact is related to the compactness theorem of [Gro2]. In [Flo1], [Flo2] (see also [Sal]) Floer proves Morse-estimate and cup-lenght-estimate for $\phi(L) \cap L$, where ϕ is a hamiltonian diffeomorphism and $L \subset M$ is a compact lagrangian submanifold such that $\pi(M, L) = 0$. It should be remarked that there is not the hypothesis on antisymplectic involutions.

Other results on $\phi(L) \cap L$ are due to Hofer ([Hof1], [Hof2]), again with a variational approach.

Conjecture 2 is proven in [Giv2] for the case M = surface. In the same paper there is a proof of conjecture 3 in the case $M = T^{2n}$, L_0, L_1 = affine lagrangian submanifolds, and $M = CP^n$, $L_0 = L_1 = RP^n$.

Other contributions of Arnol'd school can be found in the survey [Arn2].

In the next chapters we shall be concerned with local versions of the above conjectures. Related results are due to Weinstein ([Wein1], [Wein3]), Laudenbach and Sikorav, Givental' ([Givl]); for example, if ϕ_1 is small in the C^1 - topology then conjecture 3 with $L_0 = L_1$ is a trivial consequence of a classical theorem of Weinstein (corollary: Arnol'd conjecture is true if ϕ is a hamiltonian diffeomorphism C^1- near to the identity).

There are not many results about the non hamiltonian case; we don't know other then [Sik3]. In some situations (for example, the local ones) it is easy to pass from hamiltonian to symplectic results, but in other situations it is not clear how to do this.

GENERATING FUNCTIONS AND FORMS

Results about intersections between lagrangian submanifolds of a cotangent bundle can be proven using generating functions or generating forms.

Let $\Sigma \subset T^*M$ be the null section; T^*M is equipped with the canonical symplectic structure $\omega_M = d\lambda_M$, $\lambda_M =$ Liouville form; $\pi_M : T^*M \to M$ will denote the canonical projection.

If $\phi : T^*M \to T^*M$ is a diffeomorphism C^1 - near to the identity, then $\phi(\Sigma) \subset T^*M$ is the image of some 1-form $\beta : M \to T^*M$, and if ϕ is hamiltonian then β is exact, $\beta = df$. Hence $\phi(\Sigma) \cap \Sigma \simeq$ critical points of f, and points of transverse intersection correspond to Morse critical points.

However, if ϕ is not small in the C^1 - topology then $\phi(\Sigma)$ may be not the image of an (exact) 1-form and the above arguments fails.

Definition: let $L \subset T^*M$ be a lagrangian submanifold; a *generating form* for L is a closed 1-form

$$\alpha : M \times R^N \to T^*(M \times R^N) \simeq T^*M \times T^*R^N$$

such that:

a) α is transverse to $T^*M \times R^N$ (R^N is identified with the null section of T^*R^N)

b) if $\alpha_1 : M \times R^N \to T^*M$, $\alpha_2 : M \times R^N \to T^*R^N$ are the components of α along T^*M and T^*R^N:

$$L = \{\xi \in T^*M | \exists \lambda \in R^N : \alpha_2(\pi_M(\xi), \lambda) \in R^N, \quad \xi = \alpha_1(\pi_M(\xi), \lambda)\}$$

if α is exact, $\alpha = dS$, then $S : M \times R^N \to R$ will be called *generating function* for L.

A more general definition of generating functions and forms can be found in [Wein2], [Gir2], but we will not need such a generalization.

Explicitly, if S is a generating function for L:

$$L = \{\xi \in T^*M | \exists \lambda \in R^N : \frac{\partial S}{\partial \lambda}(\pi_M(\xi), \lambda) = 0, \quad \xi = d(S(\cdot, \lambda))(\pi_M(\xi)\}$$

L is the image of the differentials of $S(\cdot, \lambda)$ in those points (x, λ) which are critical along the fibre $\{x\} \times R^N$.

Example: if $L = df(M)$, then $f : M \to R$ is a generating function for L.

If $L \subset T^*M$ has a generating form $\alpha \in \Lambda^1(M \times R^N)$, then there is a 1-1 correspondence between (Morse) zeroes of α and (transverse) intersections of L with Σ; this correspondence is given by the projection from $M \times R^N$ to $\Sigma \simeq M$.

Moreover, let $X \subset M$ be a submanifold and let $R_X \subset T^*M$ be the conormal bundle:

$$R_X = \{\xi \in T^*M | \pi_M(\xi) \in X, \quad \xi = 0 \ on \ T_{\pi_M(\xi)}X\}$$

R_X is a lagrangian submanifold of T^*M (remark that $R_X \cap \Sigma \simeq X$) and we have:

Proposition 1: if $L \subset T^*M$ has a generating form $\alpha \in \Lambda^1(M \times R^N)$ and $X \subset M$ is a submanifold, then there is a 1-1 correspondence between points in $L \cap R_X$ and zeroes of $i^*\alpha \in \Lambda^1(X \times R^N)$, $i : X \times R^N \to M \times R^N$ being the canonical inclusion; points of transverse intersection correspond to zeroes of Morse type.

The proof is straightforward. $\square$

The above proposition is not useful if we are not able to give an estimate of the number of zeroes of $i^*\alpha$. In general a closed (or exact) 1-form on $X \times R^N$ can have no zeroes, unless we impose some growth condition at infinity.

Definition: let M be a manifold, a 1-form $\alpha \in \Lambda^1(M \times R^N)$ is said to be *quadratic at infinity* if $\alpha = p^*(\beta) + dS$ where $\beta \in \Lambda^1(M)$, $p : M \times R^N \to M$ is the projection, $S : M \times R^N \to R$ is a function equal on each fibre $\{x\} \times R^N$ and outside a compact set to a fixed non-degenerate quadratic form.

Proposition 2([Cha-Zeh], [Sik3]): let M be a compact manifold;
a) if $\alpha \in \Lambda^1(M \times R^N)$ is an exact 1-form quadratic at infinity, then the number of zeroes of α, $\#Z(\alpha)$, satisfies the following inequalities:

$$\#Z(\alpha) \geq CL(M) + 1 \qquad CL(M) = \text{cup} - \text{lenght of } M$$

$$\#Z(\alpha) \geq r(H_*(M, Z)) + 2q(H_*(M, Z))$$

if all the zeroes are of Morse type, where $r(H_*(M,Z))$ and $q(H_*(M,Z))$ are the rank and the torsion of the Z-module $H_*(M,Z)$.
b) if $\alpha \in \Lambda^1(M \times R^N)$ is a closed 1-form quadratic at infinity, $\alpha = p^*(\beta) + dS$, with $[\beta] \in H^1(M,Q)$, and if all the zeroes $Z(\alpha)$ of α are of Morse type, then $\#Z(\alpha)$ satisfy:

$$\#Z(\alpha) \geq r(H_*(M,[\beta])) + 2q(H_*(M,[\beta]))$$

here $r(H_*(M,[\beta]))$ and $q(H_*(M,[\beta]))$ are the rank and the torsion of the module $H_*(M,[\beta])$ defined by Novikov ([Nov], [Sik3]) in his generalization of Morse inequalities. □

Remark 1: the meaning of this proposition is that a closed or exact 1-form on $M \times R^N$, quadratic at infinity, is like a closed or exact 1-form on M from the point of view of concrete estimates of $\#Z(\alpha)$.

Remark 2: if in b) β is not rational, we can replace β with a sufficiently near $\tilde{\beta}$ rational and, by transversality, we obtain the same inequality with $\tilde{\beta}$ instead of β ($\tilde{\alpha}$ near $\alpha \Rightarrow \#Z(\tilde{\alpha}) = \#Z(\alpha)$).

The property a) was proven in [Cha-Zeh] using Conley ideas in order to avoid the non compactness of $M \times R^N$: the set $B \subset M \times R^N$ of bounded trajectories of the gradient of a function quadratic at infinity is an isolated invariant set and Chaperon and Zehnder proved (for example) that a connected component B_0 of B satisfies $CL(B_0) \geq CL(M)$; the restriction to B_0 of the gradient flow is a flow of gradient-type and hence it has at least $CL(B_0) + 1$ zeroes. This idea is recognizable also in Floer's work, in a infinite dimensional context: the hamiltonian action is something like a function quadratic at infinity. It is remarkable that in [Cha-Zeh] there is no use of Morse - Lyusternik - Schnierelmann inequalities.

Property b) was proven in [Sik3] using a compactification of $M \times R^N$ and an extension of α to this compactification, thanks to the quadraticity at infinity of α (see the appendix). If α is exact ($[\beta] = 0$) then $H_*(M, [\beta]) \simeq H_*(M, Z)$ and b) $\Rightarrow$ the second inequality of a).

Corollary: if $L \subset T^*M$ has a generating form $\alpha \in \Lambda^1(M \times R^N)$ quadratic at infinity ($\alpha = p^*(\beta) + dS$) and $X \subset M$ is a closed submanifold, then $\#(L \cap R_X)$ satisfy the following inequalities:

a) if β is exact:

$$\#(L \cap R_X) \geq CL(X) + 1$$

b) if β is exact and all intersections are transverse:

$$\#(L \cap R_X) \geq r(H_*(X, Z)) + 2q(H_*(X, Z))$$

c) if all intersections are transverse and $[\gamma] \stackrel{def}{=} j^*[\beta]$, $j : X \to M$ inclusion:

$$\#(L \cap R_X) \geq r(H_*(X, [\gamma])) + 2q(H_*(X, [\gamma]))$$

Proof:

let $i : X \times R^N \to M \times R^N$ be the inclusion ($i = j \times id$) and let $q : X \times R^N \to X$ be the projection, then $i^*(\alpha) = q^*(\gamma) + d(S \circ i)$ is a closed 1-form quadratic at infinity and the proof follows from propositions 1 and 2. □

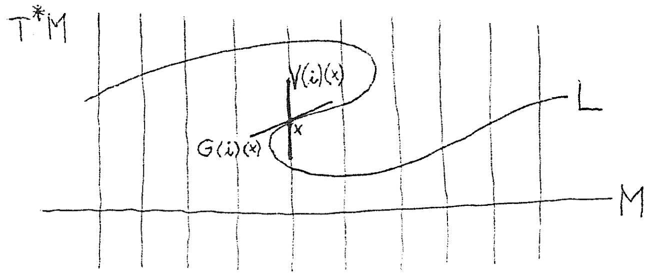
The main result of this chapter will be the proof that the existence of a generating form quadratic at infinity is a property invariant under symplectic isotopies, i.e. if $\phi_t : T^*M \to T^*M$ is a symplectic isotopy and L has such a generating form, then so has $\phi_1(L)$.

As a consequence, we will obtain estimates of $\#(\phi_1(\Sigma) \cap R_X)$ in terms of the topology of $X \simeq \Sigma \cap R_X$.

Existence of generating forms

A necessary and sufficient condition for a lagrangian submanifold to have a generating form has been recently stated by Giroux ([Girl], [Gir2]). He considers the more general case of lagrangian immersions $i : L \to T^*M$ (the image $i(L)$ may have selfintersections); the definition of generating form is the same as in the case where i is an embedding.

For any symplectic manifold N let $\Lambda(N) \to N$ be the fibre bundle whose fibre over $x \in N$ is formed by the lagrangian planes of the symplectic vector space $T_x N$. A lagrangian immersion $i : L \to T^*M$ induces the following sections of the bundle $i^*\Lambda(T^*M) \to L$:



1) the Gauss map:

$$G(i) : L \to i^*\Lambda(T^*M), \quad x \mapsto (T_x i)(T_x L)$$

2) the vertical map:

$$V(i) : L \to i^*\Lambda(T^*M), \quad x \mapsto Ker(T_{i(x)}\pi_M)$$

If $k \in N$, there is a natural extension $i_k : L \to T^*(M \times R^k)$, by means of which it is possible to define two sections $G_k(i)$, $V_k(i)$: $L \to i_k^*\Lambda(T^*(M \times R^k))$. The maps $G(i)$, $V(i)$ are said to be *stably homotopic* if for some k $G_k(i)$, $V_k(i)$ are homotopic.

Theorem 1 ([Gir2]): let L be a compact manifold and let $i : L \to T^*M$ be a lagrangian immersion, then L has a generating form if and only if $G(i)$ and $V(i)$ are stably homotopic. □

A *regular homotopy of lagrangian immersions* is a smooth deformation of a lagrangian immersion in the space of lagrangian immersions).

Corollary: the property to have a generating form is invariant by regular homotopy of lagrangian immersions . □

This corollary was previously found by Laudenbach ([Lau1]). It is *not* true if we replace "generating form" by "generating form quadratic at infinity".

Remark: an *isotopy of lagrangian embeddings* is a smooth deformation of a lagrangian embedding in the space of lagrangian embeddings; such an isotopy may be always induced by a symplectic isotopy of the ambient space ([Cha]), hence the theorems of the next section will ensure that the property to have a generating form quadratic at infinity is invariant by isotopy of lagrangian embeddings.

Existence of generating forms quadratic at infinity

We first consider the exact case.

Theorem 2 ([Sik1]): let M be a compact manifold and let $L \subset T^*M$ be a lagrangian submanifold with a generating function $S : M \times R^N \to R$ quadratic at infinity; let $\phi_t : T^*M \to T^*M$, $t \in [0,1]$, be a hamiltonian isotopy, then $\phi_1(L)$ has a generating function quadratic at infinity.

The idea of Sikorav's proof relies on the fact that the hamiltonian action may be considered, at least formally, as a generating function of a lagrangian submanifold (cfr. also [Lau-Sik], [Sik2], [Vit]).

In order to clarify this idea, suppose that $L = \Sigma$, so we can choose $S \equiv 0$. Let $\{v_t\}_{t\in[0,1]}$ be the infinitesimal generator of $\{\phi_t\}_{t\in[0,1]}$ and let $\{H_t\}_{t\in[0,1]}$ be the corresponding family of hamiltonians ($i_{v_t}\omega_M = dH_t$).

On the space

$$\Gamma = \{\gamma \in C^\infty([0,1], T^*M) | \gamma(0) \in \Sigma\}$$

we consider the projection $p : \Gamma \to M$, $\gamma \mapsto \pi_M(\gamma(1))$ and the functional $S : \Gamma \to R$ given by

$$S(\gamma)=\int_0^1[\lambda_M(\gamma(t))(\dot\gamma(t))-H_t(\gamma(t))]dt$$

If $\delta\gamma\in T_\gamma\Gamma$ (i.e. $\delta\gamma:[0,1]\to T(T^*M)$), $(\delta\gamma)(t)\in T_{\gamma(t)}T^*M\forall t\in[0,1]$, $(\delta\gamma)(0)\in T_{\gamma(0)}\Sigma$), a simple calculation shows:

$$dS(\gamma)(\delta\gamma)=\lambda_M(\gamma(1))((\delta\gamma)(1))+\int_0^1\omega_M(\gamma(t))((\delta\gamma)(t),\dot\gamma(t)-v_t(\gamma(t)))dt$$

If $\delta\gamma$ is tangent to the fibre of p throught γ ($\Longleftrightarrow$ $(\delta\gamma)(1)$ tangent to the fibre of T^*M throught $\gamma(1)$), then the first term is zero; in particular:

$$dS(\gamma)(\delta\gamma)=0\quad\forall\delta\gamma\in T_\gamma(p^{-1}(p(\gamma)))\iff\dot\gamma(t)=v_t(\gamma(t))\forall t\in[0,1]$$

If γ is such a point then $\gamma(1)=\phi_1(\gamma(0))\in\phi_1(\Sigma)$, and if $\delta\gamma\in T_\gamma\Gamma$ then $dS(\gamma)(\delta\gamma)=\lambda_M(\gamma(1))((\delta\gamma)(1))$, i.e. the component along T^*M of $dS(\gamma)$ is $\gamma(1)$.

This shows very roughly speaking that S generate $\phi_1(\Sigma)$.

The problem is then the construction of a finite dimensional reduction of S; this is done by the method of "broken geodesics".

A technical difficulty of Sikorav's proof is due to the nonlinearity of M, or, in others words, to the fact that $\Gamma\xrightarrow{p}M$ is not a vector bundle. We will give a proof of theorem 2 which avoids this difficulty.

In the non exact case, the same trick used in [Sik3] leads to:

Theorem 3 ([Sik1]): let M be a compact manifold and let $L\subset T^*M$ be a lagrangian submanifold with a generating form $\alpha\in\Lambda^1(M\times R^N)$ quadratic at infinity; let $\phi_t:T^*M\to T^*M$, $t\in[0,1]$, be a symplectic isotopy, then $\phi_1(L)$ has a generating form $\tilde\alpha\in\Lambda^1(M\times R^{\tilde N})$ quadratic at infinity. Moreover, if $\alpha=p^*(\beta)+dS$ and $\tilde\alpha=\tilde p^*(\tilde\beta)+d\tilde S$, then

$$[\tilde\beta]=[\beta]+j^*I(\{\phi_t\})$$

where $j:M\to T^*M$ is the null section.

Remark 1: in theorems 2, 3, we may suppose that L is only a lagrangian immersion.

Remark 2: theorems 2 and 3 give sufficient conditions for the existence of generating forms or functions quadratic at infinity; it seems an open problem to find necessary and sufficient conditions for the existence of such generating functions or forms. A (obvious) necessary condition is that L intersects every fibre of T^*M (because $S(q,\cdot)$ has at least 1 critical point $\forall q \in M$). Another one is that L is lagrange-cobordant (see [Arn4]) to a section of T^*M.

Reduction of theorem 3 to theorem 2

If $\phi_t : T^*M \to T^*M$ is an isotopy:

$$\phi_t^*\lambda_M - \lambda_M = \int_0^t \frac{d}{ds}(\phi_s^*\lambda_M)ds = \int_0^t \phi_s^*(L_{\dot{\phi}_s}\lambda_M)ds =$$
$$= \int_0^t \phi_s^*(i_{\dot{\phi}_s}\omega_M)ds + \text{exact form}$$

and so:

a) ϕ_t symplectic $\iff$ $\phi_t^*\lambda_M - \lambda_M$ closed $\forall t \in [0,1]$, and in this case

$$I(\{\phi_t\}) = [\phi_1^*\lambda_M - \lambda_M], \quad j^*I(\{\phi_t\}) = [j^*\phi_1^*\lambda_M]$$

b) ϕ_t hamiltonian $\iff$ $\phi_t^*\lambda_M - \lambda_M$ exact $\forall t \in [0,1]$

Let $\alpha_t = j^*\phi_t^*\lambda_M$ (a closed 1-form on M), let $\tilde{\phi}_t : T^*M \to T^*M$ be the symplectic isotopy defined by

$$\tilde{\phi}_t(\xi) = \xi - \alpha_t(\pi_M(\xi)) \quad \forall \xi \in T^*M$$

and let $\psi_t : T^*M \to T^*M$ be the composition $\phi_t \circ \tilde{\phi}_t$.

From $\tilde{\phi}_t^*\lambda_M = \lambda_M - \pi_M^*\alpha_t$ and $\phi_t^*\lambda_M = \lambda_M + \pi_M^*\alpha_t +$ exact form, we obtain:

$$\psi_t^*\lambda_M = \tilde{\phi}_t^*(\lambda_M + \pi_M^*\alpha_t) + \text{exact form}$$
$$= \lambda_M - \pi_M^*\alpha_t + \tilde{\phi}_t^*\pi_M^*\alpha_t + \text{exact form} = \lambda_M + \text{exact form}$$

hence ψ_t is hamiltonian. In this way, we have decomposed ϕ_t into the product $\psi_t \circ \tilde{\phi}_t$ of a hamiltonian isotopy and a "simple" symplectic isotopy (= a translation along the fibres).

Let $p^*(\beta) + dS \in \Lambda^1(M \times R^N)$ be the generating form of L. Then $\tilde{\phi}_1^{-1}(L)$ has the generating form $p^*(\beta + \alpha_1) + dS$. Define:

$$\bar{\phi} : T^*M \to T^*M \qquad \bar{\phi}(\xi) = \xi - (\beta + \alpha_1)(\pi_M(\xi)) \quad \forall \xi \in T^*M$$

$\bar{\phi}$ is symplectic and $\bar{\phi}(\tilde{\phi}_1^{-1}(L))$ has the generating function S.

Write

$$\phi_t(L) = (\bar{\phi}^{-1} \circ \bar{\phi} \circ \psi_t \circ \bar{\phi}^{-1} \circ \bar{\phi} \circ \tilde{\phi}_t^{-1})(L)$$

$\gamma_t \stackrel{def}{=} \bar{\phi} \circ \psi_t \circ \bar{\phi}^{-1}$ is again a hamiltonian isotopy, hence by theorem 2 $\gamma_1(\bar{\phi}(\tilde{\phi}_1^{-1}(L))$ has a generating function $\tilde{S} : M \times R^{2\tilde{N}} \to R$, quadratic at infinity, and $\phi_1(L) = \bar{\phi}^{-1}(\gamma_1 \circ \bar{\phi} \circ \tilde{\phi}_1^{-1})(L)$ has the generating form $\tilde{p}^*(\beta + \alpha_1) + d\tilde{S}$, quadratic at infinity; moreover $[\beta + \alpha_1] = [\beta] + j^*I(\{\phi_t\})$. $\square$

Proof of theorem 2

The main step is a reduction to $M = R^k$ (cfr. also the "Chekanov trick" in [Gir2]). The non compactness of R^k will require some precaution with the supports of diffeomorphisms.

Let N be any manifold and let $i : M \to N$ be any embedding; we fix on N any riemannian metric $<,>$ and on M the metric induced by i, so i is an isometric embedding.

The tangent map of i induces an embedding

$$Ti : TM \to TN$$

which maps isometrically T_xM into $T_{i(x)}N$, $\forall x \in M$. On the other hand, the metrics on M and N induce isomorphisms

$$f : TM \to T^*M, \quad g : TN \to T^*N$$

Lemma 1:

$$j : T^*M \stackrel{g \circ Ti \circ f^{-1}}{\longrightarrow} T^*N$$

is a symplectic embedding.

Proof:

let $\tilde{\lambda}_M \in \Lambda^1(TM)$ defined by

$$\tilde{\lambda}_M(v)(\eta) = < v, (T_v\tilde{\pi}_M)(\eta) > \qquad \forall v \in TM, \quad \forall \eta \in T_v(TM)$$

where $\tilde{\pi}_M : TM \to M$ is the projection, $<,>=$ scalar product; recalling the explicit expression of the Liouville form $\lambda_M \in \Lambda^1(T^*M)$:

$$\lambda_M(\xi)(\zeta) = (\xi, (T_\xi\pi_M)(\zeta)) \qquad \forall \xi \in T^*M, \quad \forall \zeta \in T_\xi(T^*M)$$

where $\pi_M : T^*M \to M$ is the projection and $(,)$ is the duality product, we see easily that

$$f^*(\lambda_M) = \tilde{\lambda}_M$$

and similarly

$$g^*(\lambda_N) = \tilde{\lambda}_N$$

A tedious computation shows that

$$(Ti)^*(\tilde{\lambda}_N) = \tilde{\lambda}_M$$

hence $j^*\lambda_N = f_*(Ti)^*g^*\lambda_N = \lambda_M$ and, a fortiori, $j^*\omega_N = \omega_M$. □

Remark that $j(T^*M)$ is contained in $T^*N\big|_{i(M)}$.

We want now to extend hamiltonian isotopies and lagrangian submanifolds from T^*M to T^*N.

Lemma 2: let $\phi_t : T^*M \to T^*M$, $t \in [0,1]$, be a hamiltonian isotopy, then there exists a hamiltonian isotopy $\psi_t : T^*N \to T^*N$, $t \in [0,1]$, such that:

1) $j \circ \phi_t = \psi_t \circ j$, $\forall t \in [0,1]$

2) ψ_t leaves $(T^*N)\big|_{i(M)}$ invariant $\forall t \in [0,1]$

moreover, if $V \subset N$ is a compact neighborhood of $i(M)$ we can choose ψ_t with support in $(T^*N)\big|_V$.

Proof:

there is a natural decomposition

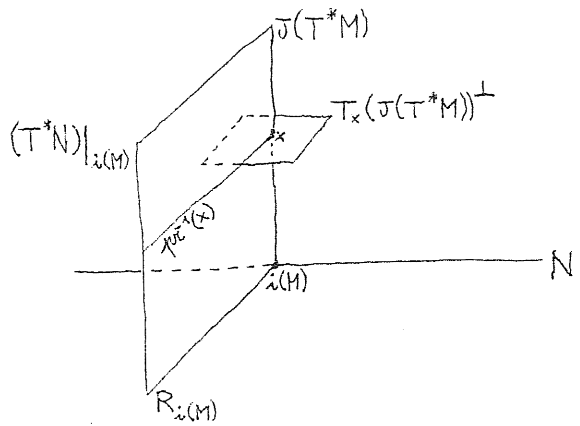
$$(T^*N)\big|_{i(M)} = j(T^*M) \oplus R_{i(M)}$$

and hence a projection $pr : (T^*N)\big|_{i(M)} \to j(T^*M)$; if $k : (T^*N)\big|_{i(M)} \to T^*N$ is the injection, then $k^*\omega_N = pr^* j_* \omega_M$ and this property, joined together a counting of dimensions, imply:

$$I) \quad \forall x \in (T^*N)\big|_{i(M)} \qquad T_x((T^*N)\big|_{i(M)})^\perp = T_x(pr^{-1}(pr(x)))$$

($\perp$= antiortogonal complement in T^*N)

i.e. the fibres of pr are the characteristic leaves of the characteristic foliation of $(T^*N)\big|_{i(M)}$.



If $x \in j(T^*M)$ then $T_x(j(T^*M))^\perp$ is transverse to $T_x(j(T^*M))$, because $j(T^*M)$ is a symplectic submanifold of T^*N. Moreover:

$$\forall x \in j(T^*M) \qquad T_x(j(T^*M))^\perp \cap T_x((T^*N)\big|_{i(M)}) = T_x(pr^{-1}(x))$$

and this suffices to ensure the existence of a tubular neighborhood $U \xrightarrow{l} j(T^*M)$ such that

$$II) \quad \forall x \in j(T^*M) \qquad (l^{-1}(x)) \cap (T^*N)\big|_{i(M)} = (pr^{-1}(x)) \cap U$$

Let $H_t : T^*M \to R$, $t \in [0,1]$, be the family of hamiltonians of $v_t = \dot{\phi}_t$. Define $K_t : U \cup (T^*N)\big|_{i(M)} \to R$ by:

$$K_t(x) = \begin{cases} H_t(j^{-1}(pr(x))) & if\ x \in (T^*N)\big|_{i(M)} \\ H_t(j^{-1}(l(x))) & if\ x \in U \end{cases} \quad \forall t \in [0,1]$$

this is a good definition, thanks to II), and we can extend the family K_t to a smooth family of functions (denoted again by K_t) on T^*N. Choosing U sufficiently small, we may suppose that *supp* $K_t \subset (T^*N)\big|_V$.

Let $\{\psi_t\}_{t\in[0,1]}$ be the hamiltonian isotopy of T^*N generated by $\{K_t\}_{t\in[0,1]}$: $\dot{\psi}_t = w_t$, $i_{w_t}\omega_N = dK_t$. The property I) implies $\omega_N(x)(w_t(x),\zeta) = dK_t(x)(\zeta) = 0 \ \forall x \in (T^*N)\big|_{i(M)} \ \ \forall\zeta \in T_x((T^*N)\big|_{i(M)})^\perp$, i.e.

$$w_t(x) \in T_x((T^*N)\big|_{i(M)}) \qquad \forall x \in (T^*N)\big|_{i(M)}$$

Similarly, property II) implies

$$w_t(x) \in T_x(j(T^*M)) \qquad \forall x \in j(T^*M)$$

and, moreover, $w_t(x) = (j_*v_t)(x)$.

These properties of the infinitesimal generator $\{w_t\}_{t\in[0,1]}$ ensure the announced properties of ψ_t. The statement on the support of ψ_t follows from *supp* $K_t \subset (T^*N)\big|_V$. □

Lemma 3: let $L \subset T^*M$ be a (immersed) lagrangian submanifold with a generating function quadratic at infinity, then there exists an immersed lagrangian submanifold $\tilde{L} \subset T^*N$ such that:

1) $\tilde{L} \cap (T^*N)\big|_{i(M)} = j(L)$

2) $\tilde{L}$ has a generating function quadratic at infinity

moreover, if $V \subset N$ is a compact neighborhood of $i(M)$ we can choose $\tilde{L}$ such that $\tilde{L} = \Sigma_N$ (= null section) outside $(T^*N)\big|_V$.

Proof:

let $S : M \times R^N \to R$ be the generating function of L and let $U \xrightarrow{l} i(M)$ be a tubular neighborhood of $i(M)$ such that $\forall x \in i(M)$ $T_x(l^{-1}(x))$ is ortogonal to $T_x(i(M))$.

Define $\tilde{S} : U \times R^N \to R$ by

$$\tilde{S}(x,\mu) = S(i^{-1}(l(x)),\mu)$$

and extend $\tilde{S}$ to all $N \times R^N$, preserving the quadraticity at infinity (this is possible by standard differential topology). Choosing U sufficiently small, we can suppose that $\tilde{S}$ is equal to a quadratic form outside $V \times R^N$. A transversality argument show that we may arrange that $\tilde{S}$ is a generating function (quadratic at infinity) of a lagrangian (immersed) submanifold $\tilde{L} \subset T^*N$, and the behaviour of $\tilde{S}$ in U ensures that

$$\tilde{L} \cap (T^*N)\big|_{i(M)} = j(L)$$

The behaviour of $\tilde{S}$ outside $V \times R^N$ ensures that $\tilde{L} = \Sigma_N$ outside $(T^*N)\big|_V$. □

Conversely:

Lemma 4: let $L \subset T^*M$, $\tilde{L} \subset T^*N$ be immersed lagrangian submanifolds such that $\tilde{L} \cap (T^*N)\big|_{i(M)} = j(L)$; if $\tilde{S} : N \times R^N \to R$ is a generating function for $\tilde{L}$ (quadratic at infinity), then $S : M \times R^N \to R$ defined by

$$S(x, \mu) = \tilde{S}(i(x), \mu)$$

is a generating function for L (quadratic at infinity).

Proof:

by definition:

$$\tilde{L} = \{\xi \in T^*N | \exists \mu \in R^N : \quad \frac{\partial \tilde{S}}{\partial \mu}(\pi_N(\xi), \mu) = 0, \quad \xi = d(\tilde{S}(\cdot, \mu))(\pi_N(\xi))\}$$

$j(L) = \tilde{L} \cap (T^*N)\big|_{i(M)}$ implies:

$$j(L) = \{\xi \in T^*N | \exists \mu \in R^N : \quad \frac{\partial \tilde{S}}{\partial \mu}(\pi_N(\xi), \mu) = 0, \ \xi = d(\tilde{S}(\cdot, \mu))(\pi_N(\xi)), \ \pi_N(\xi) \in i(M)\}$$

if $\xi \in j(L)$ then $\exists! \eta \in L$ s.t. $\xi = j(N)$ and the hypotheses of the lemma imply that η is given by $d(S(\cdot, \mu))(\pi_M(\eta))$, where μ is such that $\frac{\partial \tilde{S}}{\partial \mu}(\pi_N(\xi), \mu) = 0$, i.e. $\frac{\partial S}{\partial \mu}(\pi_M(\eta), \mu) = 0$. It is now clear that

$$L = \{\eta \in T^*M | \exists \mu \in R^N : \quad \frac{\partial S}{\partial \mu}(\pi_M(\eta), \mu) = 0, \quad \eta = d(S(\cdot, \mu))(\pi_M(\eta))\}$$

that is, S generates L. $\square$

From now on we specialize $N = R^k$.

If $S : M \times R^N \to R$ generates L and is quadratic at infinity:

$$S(x,\mu) = S_0(x) + Q(\mu) \qquad \text{for } \|\mu\| > R_0$$

then

$$\hat{S}(x,\mu) \stackrel{def}{=} \rho(\frac{\|\mu\|}{R_1})[S(x,\mu) - Q(\mu)] + Q(\mu)$$

with $\rho : [0,+\infty) \to [0,1]$ s.t. $\rho(t) = 1\ \forall t \leq 1$, $\rho(t) = 0\ \forall t \geq 2$ and with R_1 sufficiently big, is again a generating function for L and moreover

$$\hat{S}(x,\mu) = Q(\mu) \qquad \text{for } \|\mu\| > 2R_1$$

Let $\psi_t : T^*R^k \to T^*R^k$, $t \in [0,1]$, $\tilde{L} \subset T^*R^k$ be extensions of ϕ_t, L as in lemmas 2,3. The generating function $\tilde{S} : R^k \times R^N \to R$ constructed in the proof of lemma 3 starting from $\hat{S}$ has the property:

$$\tilde{S}(x,\mu) = Q(\mu) \qquad \text{outside a compact set.}$$

The submanifold $\psi_t(\tilde{L})$ intersects $(T^*R^k)|_{i(M)}$ along $j(\phi_t(L))$, so, by lemma 4, in order to prove the theorem it is sufficient to prove that $\psi_1(\tilde{L})$ has a generating function quadratic at infinity.

We "compactify" the isotopy $\{\psi_t\}_{t\in[0,1]}$, i.e. we replace $\{\psi_t\}_{t\in[0,1]}$ by a hamiltonian isotopy $\{\tilde{\psi}_t\}_{t\in[0,1]}$ with compact support, such that $\psi_t(\tilde{L}) = \tilde{\psi}_t(\tilde{L})$. This is always possible by the properties of $\tilde{L}$, ψ_t stated in lemmas 2 and 3.

Let $(p,q) = (p_j,q_j)_{j=1..k}$ be the canonical coordinates on T^*R^k ($\lambda_{R^k} = \sum p_j dq_j$, $\omega_{R^k} = \sum dp_j \wedge dq_j$).

If ψ_1 is sufficiently small in the C^1- topology then there exists a function $F : R^k \times R^k \to R$ such that:

$$(P,Q) = \psi_1(p,q) \iff P = p + \frac{\partial F}{\partial Q}(Q,p), \quad q = Q + \frac{\partial F}{\partial p}(Q,p)$$

F is called *generating function of* ψ_1 and, using a suitable symplectic diffeomorphism $\overline{T^*R^k} \times T^*R^k \xrightarrow{\Phi} T^*R^{2k}$, it can be understood as a generating function of the lagrangian submanifold $\Phi(graph\ \psi_1)$ (if ψ_1 is C^1- small, then $graph\ \psi_1$ is C^1- near to the diagonal and $\Phi(graph\ \psi_1)$ is C^1- near to the null section).

If ψ_1 is not C^1- small, we can decompose it as a product of small hamiltonian diffeomorphisms $\bar{\psi}_j$, $(j = 1...l)$ with compact support and with generating function. The proof of the existence of a generating function quadratic at infinity for $\psi_1(\tilde{L})$ (and hence the proof of theorem 2) will be completed by an iteration of

Lemma 5: let $\tilde{L} \subset T^*R^k$ be an immersed lagrangian submanifold with generating function $\tilde{S} : R^k \times R^N \to R$ equal to a quadratic form $Q(\mu)$ outside a compact set and let $\psi : T^*R^k \to T^*R^k$ be a hamiltonian diffeomorphism with compact support and with a generating function $F : R^k \times R^k \to R$; then $\psi(\tilde{L})$ has a generating function equal to a quadratic form $\hat{Q}(\hat{\mu})$ outside a compact set.

Proof:

using the above coordinates (p, q) on T^*R^k:

$$\tilde{L} = \{(p,q) \in T^*R^k | \exists \mu \in R^N : \quad \frac{\partial \tilde{S}}{\partial \mu}(q,\mu) = 0, \ p = \frac{\partial \tilde{S}}{\partial q}(q,\mu)\}$$

$$\begin{aligned}\psi(\tilde{L}) =& \{(p,q) \in T^*R^k | \exists (\bar{p},\bar{q}) \in L : \ \psi(\bar{p},\bar{q}) = (p,q)\} = \\ &= \{(p,q) \in T^*R^k | \exists (\mu,\bar{p},\bar{q}) \in R^N \times R^k \times R^k : \\ &\frac{\partial \tilde{S}}{\partial \mu}(\bar{q},\mu) = 0, \ \bar{p} = \frac{\partial \tilde{S}}{\partial q}(\bar{q},\mu), \ p = \bar{p} + \frac{\partial F}{\partial q}(q,\bar{p}), \ \bar{q} = q + \frac{\partial F}{\partial \bar{p}}(q,\bar{p})\}\end{aligned}$$

define $T : R^k \times R^N \times R^k \times R^k \to R$ by

$$T(q,\mu,\xi,\eta) = F(q,\xi) + \tilde{S}(q+\eta,\mu) - \xi \cdot \eta$$

then:

$$\frac{\partial T}{\partial q}(q,\mu,\xi,\eta) = \frac{\partial F}{\partial q}(q,\xi) + \frac{\partial \tilde{S}}{\partial q}(q+\eta,\mu)$$

$$\frac{\partial T}{\partial \mu}(q,\mu,\xi,\eta) = \frac{\partial \tilde{S}}{\partial \mu}(q+\eta,\mu)$$

$$\frac{\partial T}{\partial \xi}(q,\mu,\xi,\eta) = \frac{\partial F}{\partial \xi}(q,\xi) - \eta$$
$$\frac{\partial T}{\partial \eta}(q,\mu,\xi,\eta) = \frac{\partial \tilde{S}}{\partial q}(q+\eta,\mu) - \xi$$

and

$$\psi(\tilde{L}) = \{(p,q) \in T^*R^k | \exists(\mu,\xi,\eta) \in R^N \times R^k \times R^k : \frac{\partial T}{\partial \mu}(q,\mu,\xi,\eta) = 0,$$
$$\frac{\partial T}{\partial \xi}(q,\mu,\xi,\eta) = 0,\ \frac{\partial T}{\partial \xi}(q,\mu,\xi,\eta) = 0,\ p = \frac{\partial T}{\partial q}(q,\mu,\xi,\eta)\}$$

that is, T is a generating function for $\psi(\tilde{L})$, with $N+2k$ parameters.

The compactness of *supp* ψ allows to choose $F \equiv 0$ outside a compact set.

Let $\rho : [0,+\infty) \to [0,1]$ such that $\rho(t) = 1\ \forall t \leq 1$, $\rho(t) = 0\ \forall t \geq 2$ and define

$$\hat{T}(q,\mu,\xi,\eta) = \rho(\frac{\|\mu\|}{K_1})\rho(\frac{\|\eta\|}{K_2})F(q,\xi) + \rho(\frac{\|q\|}{K_3})\rho(\frac{\|\xi\|}{K_4})[\tilde{S}(q+\eta,\mu) - Q(\mu)] + Q(\mu) - \xi \cdot \eta$$

then, choosing appropriatly the constants $K_1, K_2, K_3, K_4 > 0$, we see easily that $\hat{T}$ is again a generating function for $\psi(\tilde{L})$, and moreover $\hat{T}$ is equal to the non degenerate quadratic form $Q(\mu) - \xi \cdot \eta$ outside a compact set.

Remark: if $\{\psi_t\}_{t\in[0,1]}$ is a C^1- small hamiltonian isotopy, then there exists a smooth family of functions $F_t : R^k \times R^k \to R$, $t \in [0,1]$, such that F_t is a generating function for ψ_t. Decomposing a (big) hamiltonian isotopy into the product of small hamiltonian isotopies we see that (under the same hypotheses of theorem 2)) there exists a smooth family of functions $\{S_t\}_{t\in[0,1]}$ such that S_t is a generating function for $\phi_t(L)$.

A last remark on generating functions

The above proof constructs a generating function for $\psi_1(L)$ with more parameters then the generating function for L. However, there is a situation in which it is not necessary to increase the number of parameters.

Definition ([Arn-Var-Gou]): two lagrangian submanifolds L_0, $L_1 \subset T^*M$ are said to be *lagrangian equivalent* if there exists a symplectic diffeomorphism $\psi : T^*M \to T^*M$ such that:

1) $\psi(L_0) = L_1$

2) ψ maps fibres of T^*M into fibres.

A symplectic diffeomorphism of T^*M preserving the fibration $T^*M \stackrel{\pi_M}{\to} M$ may always be written in the form:

$$\psi(\xi) = (T^*\phi)(\xi) + \beta(\pi_M((T^*\phi)(\xi))) \qquad \forall \xi \in T^*M$$

where $\phi : M \to M$ is some diffeomorphism, $T^*\phi : T^*M \to T^*M$ is its cotangent map (which is a symplectic diffeomorphism), β is a closed 1-form on M.

Proposition: let $L_0, L_1 \subset T^*M$ be two lagrangian submanifolds and let $\psi : T^*M \to T^*M$ be a symplectic diffeomorphism realizing a lagrangian equivalence between L_0, L_1, $\psi(\xi) = (T^*\phi)(\xi) + \beta(\pi_M((T^*\phi)(\xi))) \quad \forall \xi \in T^*M$; if $\alpha_0 \in \Lambda^1(M \times R^N)$ is a generating form for L_0, then

$$\alpha_1 \stackrel{def}{=} (\phi \times id_{R^N})^*(\alpha_0) + p^*(\beta) \in \Lambda^1(M \times R^N)$$

($p : M \times R^N \to M$ is the projection) is a generating form for L_1.

The proof is a simple globalization of the analogous local statement contained in [Arn-Var-Gou]. □

Observe that $[\alpha_1] = (\phi \times id_{R^N})^*[\alpha_0] + [\beta]$ and in particular if β is exact (e.g. if ψ is hamiltonian) and L_0 has a generating function, then so has L_1. Remark also that α_0 quadratic at infinity $\Rightarrow$ α_1 quadratic at infinity.

A lagrangian submanifold $L \subset T^*M$ is *stable* if every lagrangian submanifold $\tilde{L} \subset T^*M$ sufficiently near to L (with respect to a suitable topology, cfr. [Arn-Var-Gou]) is lagrangian equivalent to L.

Corollary: if $L \subset T^*M$ is stable and has a generating function (quadratic at infinity), then $\psi(L)$ has a generating function (quadratic at infinity) with the same number of parameters, for every $\psi \in Diff_H(T^*M)$ sufficiently small. □

The method of generating functions and forms may be used to study lagrangian intersections on any symplectic manifold if we are able to translate our problem on a cotangent bundle.

The main results of this chapter will be the following.

Definition: let M be a manifold and let $L_0, L_1 \subset M$ be submanifolds; we shall say that L_0, L_1 have *clean intersection* $L = L_0 \cap L_1$ (or L_0, L_1 intersects cleanly along L) if L is a submanifold of M, $L = \bar{L}_0 \cap \bar{L}_1$, and

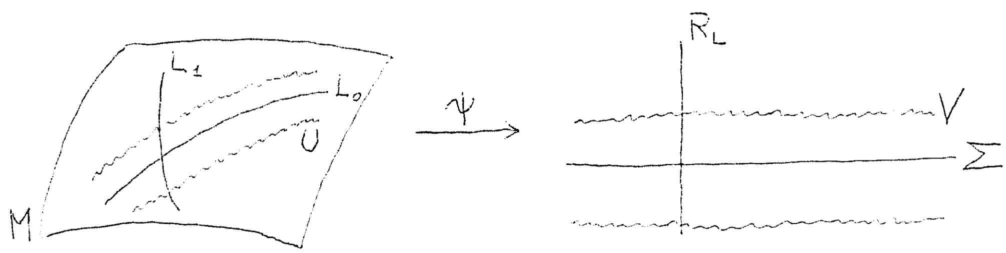
$$\forall x \in L \qquad T_xL = T_xL_0 \cap T_xL_1$$

Example: the null section and a conormal bundle in a cotangent bundle.

Theorem 1: let L_0, $L_1 \subset M$ be compact lagrangian submanifolds of the symplectic manifold (M, ω) with clean intersection $L = L_0 \cap L_1$, then there exist a neighborhood U of L_0 in M, a neighborhood V of Σ in T^*L_0 and a symplectic diffeomorphism $\psi : U \to V$ such that:

a) $\psi(L_0) = \Sigma$

b) $\psi(L_1 \cap U) = R_L \cap V$



Corollary: let M, L_0, L_1, L as in theorem 1 and let $\phi_t : M \to M$, $t \in [0,1]$, be a symplectic isotopy; let $i : L \to M$ be the inclusion and let $\gamma = i^* I(\{\phi_t\}) \in H^1(L, R)$, then:

a) if $\{\phi_t\}$ is sufficiently small in the C^1- topology, then points in $\phi_1(L_0) \cap L_1$ are in 1-1 correspondence with zeroes of a closed 1-form β on L s.t. $[\beta] = \gamma$; transverse intersections correspond to Morse zeroes

b) if $\{\phi_t\}$ is sufficiently small in the C^0- topology, then points in $\phi_1(L_0) \cap L_1$ are in 1-1 correspondence with zeroes of a closed 1-form $\alpha = p^*(\beta) + dS$ on $L \times R^N$ quadratic at infinity, s.t. $[\beta] = \gamma$; transverse intersections correspond to Morse zeroes.

Proof:

by theorem 1 and by the smallness of $\{\phi_t\}$ we may suppose that $M = T^* L_0$, $L_0 = \Sigma$, $L_1 = R_L$

in case a) $\phi_1(\Sigma) = \bar{\beta}(L_0)$ for some $\bar{\beta} \in \Lambda^1(L_0)$ closed, and if $j_0 : L_0 \to T^* L_0$, $j_1 : L \to L_0$ are the canonical inclusions:

$$[\bar{\beta}] = j_0 I(\{\phi_t\}) \qquad [j_1^* \bar{\beta}] = (j_0 \circ j_1)^* I(\{\phi_t\}) = j^* I(\{\phi_t\})$$

the (transverse) intersections between $\phi_1(\Sigma)$ and R_L correspond to (Morse) zeroes of $\beta = j_1^* \bar{\beta}$

in case b) $\phi_1(\Sigma)$ has a generating form $\bar{\alpha} = \bar{p}^*(\bar{\beta}) + d\bar{S} \in \Lambda^1(L_0 \times R^N)$ quadratic at infinity, for some $\bar{\beta} \in \Lambda^1(L_0)$ closed, and again we have

$$[j_1^* \bar{\beta}] = (j_0 \circ j_1)^* I(\{\phi_t\}) = j^* I(\{\phi_t\})$$

the (transverse) intersections between $\phi_1(\Sigma)$ and R_L correspond now to (Morse) zeroes of the closed 1-form, quadratic at infinity, $\alpha = (j_1 \times id_{R^N})^*(\bar{\alpha}) = p^*(\beta) + dS \in \Lambda^1(L \times R^N)$.□

Remark 1: proposition 2, chapter 2, gives explicit estimates of $\phi_1(L_0) \cap L_1$.

Remark 2: the compactness of L_0 and L_1 is not really necessary, it is sufficient the compactness of L.

Remark 3: in b) the hypothesis on the C^0- smallness of ϕ_t means that $\phi_t(L_0)$ $\forall t \in [0,1]$ is contained in the neighborhood U of theorem 1; in some cases, this neighborhood may be "large" and so ϕ_t may be "big" (cfr [Sik1]).

Related results to the corollary were obtained by Weinstein ([Wein3]).

Proof of theorem 1

The proof of theorem 1 is a simple extension of a classical result of Weinstein about tubular neighborhoods of lagrangian submanifolds. If $L_0 = L_1$ theorem 1 is contained in [Wein1]. The proof relies on the *homotopy method* (or *méthode du chemin*) of Weinstein-Moser ([Wein1], [Arn-Giv], [Gro1]). In a first step we shall prove the existence of a diffeomorphism which satisfy a) and b) and which is "infinitesimally symplectic", i.e. its derivatives at points of L_0 are symplectic. Next we shall "symplectify" this diffeomorphism, preserving properties a) and b). In this second step the main ingredient is:

Proposition (*generalized Poincarè lemma*, [Wein1]): let $E \xrightarrow{\pi} L$ be a vector bundle, $j : L \to E$ its null section, $\pi_t : E \to E$, $t \in [0,1]$, the multiplication by t along the fibres; if Ω is a closed p-form on a π_t-invariant neighborhood U of $j(L) \subset E$ such that $j^*\Omega = 0$, then

$$\Omega = d\omega$$

where

$$\omega = \int_0^1 \pi_t^*(i_{\dot{\pi}_t}\Omega)dt$$

is a $(p-1)$-form on the same neighborhood U.

Proof:

it is, essentially, Cartan's homotopy formula: $\forall t \in (0,1]$ $\pi_t : E \to E$ is a diffeomorphism and the vector field $\dot{\pi}_t$ is well defined, so (on U):

$$\begin{aligned}\Omega = \Omega - \pi^* j^* \Omega = \pi_1^* \Omega - \pi_0^* \Omega = \\ = \int_0^1 \frac{d}{dt}(\pi_t^* \Omega) dt = \int_0^1 \pi_t^* L_{\dot{\pi}_t} \Omega dt = \\ = \int_0^1 \pi_t^* d i_{\dot{\pi}_t} \Omega dt = d \int_0^1 \pi_t^* (i_{\dot{\pi}_t} \Omega) dt \quad . \square\end{aligned}$$

Remark that $\forall x \in j(L) \quad \omega(x) = 0$ ($\omega(x)(\xi) = 0 \ \forall \xi \in T_x E$), i.e. $\omega|_{(TE)|_{j(L)}} = 0$; this is much more then $j^* \omega = 0$.

Corollary: if $L \overset{j}{\hookrightarrow} M$ is a submanifold of the manifold M and $\Omega \in \Lambda^p(M)$ is closed and such that

$$j^* \Omega = 0$$

then in a neighborhood of L

$$\Omega = d\omega$$

for some $(p-1)$-form ω such that

$$\omega|_{(TM)|_{j(L)}} = 0 \ . \qquad \square$$

We can now start the proof of theorem 1.

Definitions:

$\Lambda(n) \overset{def}{=}$ space (manifold) of lagrangian subspaces of R^{2n}

$Sp(n) \overset{def}{=}$ (Lie) group of symplectic linear automorphisms of R^{2n}

if $L_0 \in \Lambda(n)$ is fixed, and $N \subset L_0$ is a fixed subspace:

$\Lambda(L_0; N) \overset{def}{=} \{L \in \Lambda(n) | L \cap L_0 = N\}$

$Sp(L_0) \overset{def}{=} \{\mathcal{A} \in Sp(n) | \mathcal{A}|_{L_0} = 1\}$.

The structure of $\Lambda(n)$ and $\Lambda(L_0; \{0\})$ is well known ([Arn3]).

Let $Mat(k,l)$ be the space of $k \times l$ matrices (on R) and $Sym(k)$ the space of symmetric $k \times k$ matrices.

Lemma 1: $\Lambda(L_0;N)$ is diffeomorphic to $Sym(n-k)$, $k = dim\ N$; $Sp(L_0)$ is diffeomorphic to $Sym(n)$; $Sp(L_0)$ acts transitively on $\Lambda(L_0;N)$, with isotropy groups diffeomorphic to $Sym(k) \times Mat(k, n-k)$.

Proof:

we may choose on R^{2n} linear symplectic coordinates $(q_1..q_n, p_1..p_n)$ such that, denoting $p = (p_1..p_k)$, $P = (p_{k+1}..p_n)$, $q = (q_1..q_k)$, $Q = (q_{k+1}..q_n)$:

$$L_0 = \{p = 0, P = 0\} \qquad N = \{p = 0, P = 0, Q = 0\}$$

if $L \in \Lambda(L_0;N)$, then:

$$L \cap L_0 = N \Rightarrow L \supset N \Rightarrow L^\perp \subset N^\perp = \{p = 0\}$$

but $L = L^\perp$, so L is transverse to the plane generated by p and Q and we may express L as:

$$L : \begin{cases} p = & A \cdot q & + & B \cdot P \\ Q = & C \cdot q & + & D \cdot P \end{cases}$$

$A, B, C, D =$ matrices of suitable dimensions

now

$$N \subset L \Rightarrow A = 0, \quad C = 0$$

$$L \text{ lagrangian} \Rightarrow B = 0, \quad D \in Sym(n-k)$$

conversely, $\{p = 0, Q = D \cdot P\}$ defines an element of $\Lambda(L_0;N)$ for every symmetric D, so

$$\Lambda(L_0;N) \simeq Sym(n-k)$$

If $\mathcal{A} \in Sp(L_0)$, then in the above basis (q, Q, p, P) $\mathcal{A}$ is expressed by a matrix of the type $\begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix}$, $A \in Sym(n)$, and clearly

$$Sp(L_0) \simeq Sym(n) \qquad \text{(as Lie groups)}$$

If $A = \begin{pmatrix} A_1 & A_2 \\ A_2^T & A_3 \end{pmatrix}$, $A_1 \in Sym(k)$, $A_3 \in Sym(n-k)$, $A_2 \in Mat(k, n-k)$, then

$$\mathcal{A}(\{p = 0, Q = -A_3 \cdot P\}) = \{p = 0, Q = 0\}$$

and this complete the proof. $\square$

Remark: the important fact in lemma 1 is the contractibility of spaces in discussion.

Lemma 2: let M, L_0, L_1, L as in theorem 1, then there exist a neighborhood U of L_0 in M, a neighborhood V of Σ in $T^*L_)$, and a diffeomorphism $\psi : U \to V$ such that:

a) $\psi(L_0) = \Sigma$ and here $\psi \simeq id$

b) $\psi(L_1 \cap U) = R_L \cap V$

c) $T_x\psi : T_xM \to T_x(T^*L_0)$ is symplectic, $\forall x \in L_0$

N.B.= we will identify frequently $L_0 \subset M$ with the null section $\Sigma \subset T^*L_0$.

Proof:

define $E \to L_0$ as the bundle whose fibre over $x \in L_0$ is formed by the linear symplectic maps from T_xM to $T_x(T^*L_0)$, which map identically T_xL_0 to $T_x\Sigma$; lemma 1 asserts the contractibility of the fibre, hence the existence of a smooth family of linear symplectic operators

$$A_x : T_xM \to T_x(T^*L_0) \qquad x \in L_0$$

such that

$$A_x|_{T_xL_0} \simeq \mathbf{1}$$

moreover, lemma 1 again guarantees that we may choose A_x such that

$$A_x(T_xL_1) = T_xR_L \qquad \forall x \in L$$

this is possible because, for every choice of A_x, $A_x(T_xL_1)$ is a lagrangian plane of $T_x(T^*L_0)$ which intersects $T_x\Sigma$ along T_xL, and so is T_xR_L.

Hence we have a symplectic bundle map

$$A : TM|_{L_0} \to T(T^*L_0)|_\Sigma$$

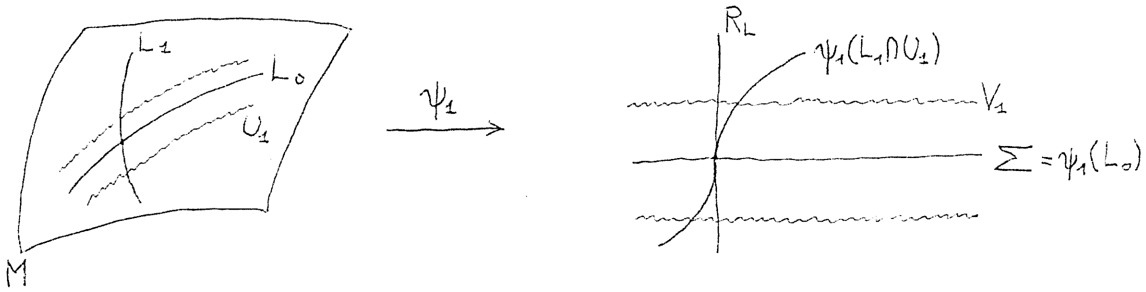
such that

$$A|_{TL_0} \simeq id \qquad A(TL_1|_L) = (TR_L)|_L$$

Let $F_1 \to L_0$ be a subbundle of $TM|_{L_0}$ transverse to TL_0, and let $F_2 \to \Sigma$ be the subbundle of $T(T^*L_0)|_\Sigma$ image of F_1 by A: $(F_2)_x = A_x((F_1)_x \forall x \in L_0$. Using exponential maps from F_1, F_2 to neighborhoods U_1, V_1 of L_0, Σ, we may construct a diffeomorphism $\psi_1 : U_1 \to V_1$ such that

$$\psi_1(x) = x \quad \text{and} \quad T_x\psi_1 = A_x \qquad \forall x \in L_0$$

(ψ_1 is simply the composition of $A : F_1 \to F_2$ with the exponential maps). In particular, $\psi_1(L_0) = \Sigma$ and $\psi_1(L_1 \cap U_1)$ is tangent to R_L along L.



Let now $\psi_2 : V_2 \to V_3$ be a diffeomorphism of neighborhoods of Σ such that

$$\psi_2|_\Sigma = id, \quad T_x\psi_2 = \mathbf{1} \qquad \forall x \in \Sigma$$

and

$$\psi_2(\psi_1(L_1 \cap U_1) \cap V_2) = R_L \cap V_3$$

then the composition $\psi = \psi_2 \circ \psi_1$, defined from a suitable U to a suitable V, satisfies trivially a), b) and satisfies c) because $\forall x \in L_0$ $T_x\psi = T_x\psi_1 = A_x$. □

Let us define on V the symplectic form $\tilde{\omega} = \psi_*\omega$: $\tilde{\omega}$ is equal to the canonical symplectic form ω_{L_0} of T^*L_0 on $T(T^*L_0)|_\Sigma$, thanks to c). Moreover, $R_L \cap V$ is a lagrangian submanifold also for $\tilde{\omega}$, because $R_L \cap V = \psi(L_1 \cap U)$ and L_1 is lagrangian for ω.

Lemma 3: let ω_0, ω_1 be symplectic forms on a neighborhood of $\Sigma \subset T^*L_0$ such that:

i) $\omega_0 = \omega_1$ on $T(T^*L_0)|_\Sigma$

ii) R_L is lagrangian both for ω_0 and ω_1

then there exists a diffeomorphism $\phi : \tilde{U} \to \tilde{V}$ (neighborhoods of Σ) such that:

a) $\phi^*\omega_1 = \omega_0$

b) $\phi(\Sigma) = \Sigma, \qquad \phi|_\Sigma = id$

c) $\phi(R_L \cap \tilde{U}) = R_L \cap \tilde{V}$.

Proof:

we look for a family of diffeomorphisms ϕ_s, $s \in [0,1]$, such that

$$\phi_s^*\omega_s = \omega_0$$

where

$$\omega_s = \omega_0 + s(\omega_1 - \omega_0)$$

so ϕ_1 will satisfy a) (we omit the domains of definitions of ϕ_s and other objects, it is understood that we work on neighborhoods of Σ)

a derivation shows that $L_{\dot{\phi}_s}\omega_s + \frac{\partial \omega_s}{\partial s} = 0$, i.e.

$$di_{\dot{\phi}_s}\omega_s = \omega_0 - \omega_1$$

$(\omega_0 - \omega_1)$ is a closed 2-form which vanishes on $T\Sigma$, and by the generalized Poincarè lemma:

$$\omega_0 - \omega_1 = d\int_0^1 \pi_t^*(i_{\dot{\pi}_t}(\omega_0 - \omega_1))dt$$

so we look for a family of vector fields v_s such that

$$i_{v_s}\omega_s = \int_0^1 \pi_t^*(i_{\dot{\pi}_t}(\omega_0 - \omega_1))dt \stackrel{def}{=} \beta$$

For every $s \in [0,1]$ ω_s is a symplectic form (thanks to i)), so v_s exists and is unique; moreover $s \mapsto v_s$ is smooth;

$$\beta|_{T(T^*L_0)|_\Sigma} = 0 \;\Rightarrow\; v_s(x) = 0 \;\forall x \in \Sigma,\; \forall s \in [0,1]$$

Let $x \in R_L$ and $\xi \in T_xR_L$, then $\dot{\pi}_t(\pi_t(x))$, $(T_x\pi_t)(\xi) \in T_{\pi_t(x)}R_L$ $\forall t \in [0,1]$, hence

$$(\pi_t^* i_{\dot{\pi}_t}(\omega_0 - \omega_1))(x)(\xi) = (\omega_0 - \omega_1)(\pi_t(x))(\dot{\pi}_t(\pi_t(x)), (T_x\pi_t)(\xi)) = 0 \quad \forall t \in [0,1]$$

by the lagrangianity of R_L with respect to ω_0 and ω_1; this means that

$$\omega_s(x)(\xi, v_s(x)) = \beta(x)(\xi) = 0 \quad \forall s \in [0,1], \ \forall x \in R_L \ \forall \xi \in T_x R_L$$

and the lagrangianity of R_L with respect to every ω_s implies that

$$v_s(x) \in T_x R_L \qquad \forall x \in R_L, \ \forall s \in [0,1]$$

Finally, let ϕ_s be the flow of v_s ($\frac{d}{du}|_{u=0}\phi_{s+u} = v_s \circ \phi_s$), then the above properties of v_s guarantee that ϕ_1 satisfy a), b), c); ϕ_1 is defined on a sufficiently small neighborhood of Σ because $v_s = 0$ on Σ. □

The proof of theorem 1 is now achieved by a composition of lemmas 2 and 3. □

A remark on degenerate lagrangian intersections

Let $L_0, L_1 \subset M$ be lagrangian submanifolds and let $x \in L_0 \cap L_1$ be a point of non-transverse intersection; $k = dim\ (T_x L_0) \cap (T_x L_1) > 0$.

The Darboux - Weinstein theorem ([Arn-Giv], [Wein1]) and linear symplectic geometry show that there exist symplectic coordinates (p, q) on a neighborhood U of x, centered at x, such that

$$L_0 \cap U = \{p = 0\}$$

$$L_1 \cap U \text{ tangent to } \{p_j = 0, j = 1..k, \ q_l = 0, l = k+1..n\} \text{ at } x \equiv 0$$

consequently, for a suitable function S with vanishing derivatives at 0 up to order 2:

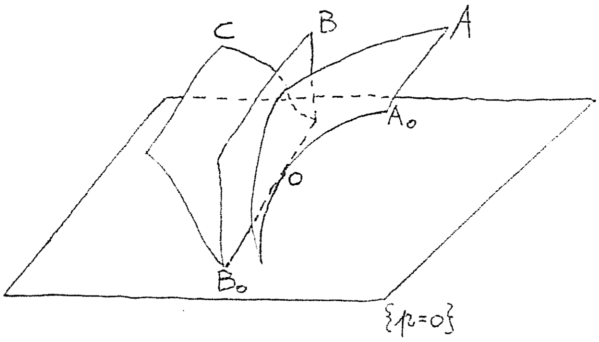
$$L_1 \cap U = \{p_j = \frac{\partial S}{\partial q_j}(q_1..q_k, p_{k+1}..p_n), \ q_l = -\frac{\partial S}{\partial p_l}(q_1..q_k, p_{k+1}..p_n), \ j = 1..k, l = k+1..n\}$$

We want to show that we may choose (p, q) in such a way that, moreover, S is independent on p_l, $l = k+1..n$. This is equivalent to say that $L_1 \cap U$ is contained in $\{q_l = 0, \ l = k+1..n\}$, hence it is sufficient to prove that there exists a symplectic diffeomorphism $\psi : (V, 0) \to (\tilde{V}, 0)$, $V, \tilde{V}$ neighborhoods of 0 in R^{2n}, such that $T_0\psi = \mathbf{1}$ and:

1) ψ preserves $\{p=0\}$

2) ψ maps $A = \{q_l = -\frac{\partial S}{\partial p_l}(q_1..q_k, p_{k+1}..p_n), l = k+1..n\}$ into $B = \{q_l = 0, l = k+1..n\}$.

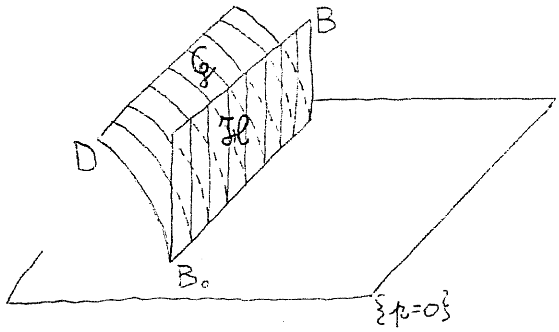
Define $A_0 = A \cap \{p=0\}$, $B_0 = B \cap \{p=0\}$; clearly, there exists a diffeomorphism of the q-plane $\{p=0\}$ tangent to id at 0 and sending A_0 to B_0, and its cotangent map is a symplectic diffeomorphism of $T^*R^n \simeq R^{2n}$, tangent to id at 0, preserving $\{p=0\}$, sending A to some submanifold C such that $C \cap \{p=0\} = B_0$.



Remark that C is coisotropic ($\forall x \in C : T_xC^\perp \subset T_xC$) and $T_0C = T_0B$, but in general $T_xC \neq T_xB$ for $x \in B_0$, $x \neq 0$; in any case,

$$T_xC \cap T_x\{p=0\} = T_xB_0 = T_xB \cap T_x\{p=0\} \qquad \forall x \in B_0$$

i.e. T_xC and T_xB are both transverse to $T_x\{p=0\}$, with the same intersection. An argument of linear symplectic algebra similar to that used in theorem 1 allows us to construct a diffeomorphism near $0 \in R^{2n}$ and tangent to id at 0, which is the identity on $\{p=0\}$, maps C to a submanifold D s. t. $T_xD = T_xB \;\; \forall x \in B_0$, and is symplectic along $\{p=0\}$. This diffeomorphism maps the characteristic foliation $\mathcal{F}$ of C (given by integrals of the integrable distribution $TC^\perp \subset TC$) to some foliation $\mathcal{G}$ of D, whose leaves are tangent along B_0 to the leaves of the characteristic foliation $\mathcal{H}$ of B.



Now it is easy to construct a diffeomorphism, tangent to id along $\{p = 0\}$ and mapping D to B, $\mathcal{G}$ to $\mathcal{H}$.

Collecting this facts, we ensure the existence of a diffeomorphism

$$\psi_1 : (U_1, 0) \to (\tilde{U}_1, 0)$$

$U_1, \tilde{U}_1 \subset R^{2n}$, with the following properties:

1) $\psi_1(\{p = 0\}) = \{p = 0\}$ (near 0)

2) $\psi_1(A) = B$ (near 0)

3) ψ_1 is tangent to id at 0

4) ψ_1 is symplectic along $\{p = 0\}$

As in theorem 1, we must "symplectify" ψ_1.

Let $\tilde{\omega} = \psi_{1*}\omega$, then $\tilde{\omega}$ is a symplectic form equal to ω on $TR^{2n}|_{\{p=0\}}$ by 4) and moreover:

5) B is coisotropic both for ω and $\tilde{\omega}$

6) the characteristic foliation of B with respect to ω and $\tilde{\omega}$ is the same.

As in lemma 3, theorem 1, we may construct a diffeomorphism ψ_2 which sends $\tilde{\omega}$ to ω and which preserves $\{p = 0\}$ and B. The definition of the vector fields $\{v_t\}$ that generate ψ_2 is as in lemma 3, the only difference is that we must verify that v_t are tangent to B. This follows from properties 5), 6) and from the fact that π_t ($\pi_t(p.q) = (tp, q)$) maps B into itself preserving its characteristic foliation (hence $T\pi_t$ preserves TB *and* $TB^{\perp}$):

if $x \in B$ and $\xi \in T_x B^\perp$, then

$$\omega_t(x)(v_t(x),\xi) = \int_0^1 (\omega - \tilde{\omega})(\pi_t(x))(\dot{\pi}_t(\pi_t(x)),(T_x\pi_t)(\xi))dt = 0$$

because $\dot{\pi}_t(\pi_t(x)) \in T_{\pi_t(x)}B$, $(T_x\pi_t)(\xi) \in T_{\pi_t(x)}B^\perp$, so $v_t(x) \in T_x B$ (remark: the characteristic foliation of B is the same w.r. to all ω_t).

The composition $\psi_2 \circ \psi_1$ gives finally the announced result:

Theorem 2: let $L_0, L_1 \subset M$ be lagrangian submanifolds and let $x \in L_0 \cap L_1$ be a point of non transverse intersection, $k = dim\ T_x L_0 \cap T_x L_1$. Then there exist symplectic coordinates (p,q) centered at x such that, locally:

$$L_0 = \{p = 0\}$$
$$L_1 = \{p_j = \frac{\partial S}{\partial q_j}(q_1..q_k),\ j = 1..k, \quad q_l = 0,\ l = k+1..n\}$$

for some $S : R^k \to R$ with vanishing derivatives at 0 up to order 2. □

This theorem is probably not very useful; it is stated here because its proof may clarify some specific features of the homotopy method in symplectic geometry that, perhaps, are not evident in the proof of theorem 1. See, for example, the rôle of characteristic foliations.

Fixed manifolds of symplectic diffeomorphisms

Theorem 1 and its corollary may be useful to study the decomposition of a manifold of fixed points of a symplectic diffeomorphism.

Definition: a diffeomorphism $\phi : M \to M$ of a manifold M is said to be a *Bott diffeomorphism* if its graph $G(\phi) \subset M \times M$ has clean intersection with the diagonal Δ.

The definition is given in analogy with the case of functions: a Bott function is a function $f : M \to R$ such that $df(M) \subset T^*M$ has clean intersection with the null section Σ. A Bott critical manifold is a critical manifold L such that f is Bott near L.

If $\phi : M \to M$ is a Bott symplectic diffeomorphism, then $G(\phi) \subset \bar{M} \times M$ is a lagrangian submanifold that intersects cleanly Δ along $\{(x,y) \in \bar{M} \times M | x = y \in N\}$, $N = Fix(\phi) \subset M$. The arguments given in chapter 1 and the corollary to theorem 1 give:

Proposition: let $\psi : M \to M$ be a Bott symplectic diffeomorphism, with $N = Fix(\psi)$ compact; if $\phi_t : M \to M$, $t \in [0,1]$, is a symplectic isotopy and if $\gamma = j^* I(\{\phi_t\}) \in H^1(N, R)$, $j : N \to M$ canonical inclusion:

a) $\{\phi_t\}$ C^1-small $\Rightarrow Fix(\phi_1 \circ \psi)$ is in 1-1 correspondence with zeroes of a closed 1-form $\beta \in \Lambda^1(N)$, s.t. $[\beta] = \gamma$

b) $\{\phi_t\}$ C^0-small $\Rightarrow Fix(\phi_1 \circ \psi)$ is in 1-1 correspondence with zeroes of a closed 1-form $p^*(\beta) + dS \in \Lambda^1(N \times R^N)$ quadratic at infinity, s.t. $[\beta] = \gamma$

Proof:

recall that if $i : N \to \bar{M} \times M$ is the inclusion ($i = (j, j)$):

$$i^* I(\{id \times \phi_t\}) = j^* I(\{\phi_t\})$$

and apply the corollary to $L_0 = G(\psi)$, $L_1 = \Delta$ ($L_0 \cap L_1 = i(N)$), $id \times \phi_t : \bar{M} \times M \to \bar{M} \times M$.□

A particular class of Bott symplectic diffeomorphisms appears as flows of vector fields $v \in Vec_\omega(M)$.

Let $v \in Vec_\omega(M)$ and let ϕ =time-1-flow. Suppose that ϕ is a Bott diffeomorphism and that a compact connected component N of $Fix(\phi)$ is formed by non trivial closed orbits of v (i.e. N does not contain zeroes of v). Let $\tilde{v} \in Vec_\omega(M)$ C^0-near to v and such that $v - \tilde{v}$is hamiltonian, then $\tilde{\phi}$ =time-1-flow of $\tilde{v}$ is C^1- near to ϕ and the above proposition tell us that $Fix(\tilde{\phi})$ is, near N, in 1-1 correspondence with critical points of a function f on N. Fixed points of $\tilde{\phi}$ are closed orbits (with period 1) of $\tilde{v}$ and the hypotheses on N imply that these closed orbits are not trivial near N; as a consequence, the fixed points of $\tilde{\phi}$ near N are not isolated, but are "organized" in circles. The same is true for the critical points of f.

Suppose that the perturbation $v \mapsto \tilde{v}$ is "generic", then f will be a function with k critical circles transversally non degenerate (= transversally of Morse type), i.e. a Bott function with k critical circles. A lower bound for k is given by Morse-Bott theory:

$$k \geq \frac{1}{2}\{\text{sum of Betti numbers of } N\}$$

Remark: Morse critical points correspond to non degenerate fixed points, Bott critical manifolds to transversally non degenerate fixed manifolds.

On a theorem of Moser

Let $N \subset M$ be a coisotropic submanifold ($TN \supset TN^{\perp}$) and let $\mathcal{F}$ be its characteristic foliation. Theorem 1 has as a simple corollary the following theorem of Moser ([Mos], [Ban2]).

Theorem 3: if N is compact and if $\psi : M \to M$ is a hamiltonian diffeomorphism sufficiently C^1-small, then $\#\{x \in N | \psi(x) \in \mathcal{F}_x = \text{leaf throught } x\}$ is not less then the number of critical points of a function on N.

Remark 1: we leave to the reader the C^0-small and the symplectic versions of this theorem.

Remark 2: if $N = M$ then $\mathcal{F}_x = \{x\}$ $\forall x \in N$ and the theorem estimetes $\#Fix(\psi)$; if N is lagrangian then $\mathcal{F}_x = N$ $\forall x \in N$ and the theorem estimates $\#\psi(N) \cap N$.

Proof:

define

$$L = \{(x,y) \in \bar{M} \times M | x \in N \text{ and } y \in \mathcal{F}_x\}$$

$$\Psi = (id, \psi) : \bar{M} \times M \to \bar{M} \times M$$

then

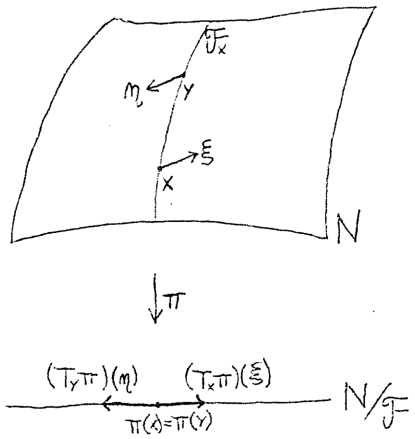
$$\{x \in N | \psi(x) \in \mathcal{F}_x\} \simeq \Psi(\Delta) \cap L$$

moreover $\Delta \cap L \simeq N$ and the intersection is clean; in order to prove the theorem, it is sufficient to verify that L is lagrangian (this is not precise, because L may be not a submanifold, but only an immersed manifold; however it is clear how bypass this little problem).

Let $(x,y) \in L$ and let $\pi : U \to \frac{U}{\mathcal{F}}$ be the projection, where U is a domain of a foliated chart of N containing x and y, then

$T_{(x,y)}L =$

$= \{(\xi,\eta) \in T_{(x,y)}(\bar{M} \times M) \simeq T_x\bar{M} \times T_yM | \xi \in T_xN,\ \eta \in T_yN,\ (T_x\pi)(\xi) + (T_y\pi)(\eta) = 0\}$



$(\bar{\xi},\bar{\eta}) \in T_{(x,y)}L^{\perp}$ means:

$$-\omega(x)(\xi,\bar{\xi}) + \omega(y)(\eta,\bar{\eta}) = 0 \qquad \forall(\xi,\eta) \in T_{(x,y)}L$$

in particular:

$$\omega(x)(\xi,\bar{\xi}) = 0 \qquad \forall\xi \in T_x\mathcal{F}_x$$

$$\omega(y)(\eta,\bar{\eta}) = 0 \qquad \forall\eta \in T_y\mathcal{F}_x$$

$\Rightarrow \quad \bar{\xi} \in T_xN,\ \bar{\eta} \in T_yN$

if ω_R is the projection of ω on $\frac{U}{\mathcal{F}}$ (ω is projectable because it is constant along the leaves, cfr. [Wein2]), then

$$-\omega_R(\pi(x))((T_x\pi)(\xi),(T_x\pi)(\bar{\xi})) + \omega_R(\pi(y))((T_y\pi)(\eta),(T_y\pi)(\bar{\eta})) = 0 \quad \forall(\xi,\eta) \in T_{(x,y)}L$$

$$\omega_R(\pi(x))((T_x\pi)(\xi),(T_x\pi)(\bar{\xi}) + (T_y\pi)(\bar{\eta})) = 0 \quad \forall\xi \in T_xN$$

and the symplecticity of ω_R ([Wein2]) implies:

$$(T_x\pi)(\bar{\xi}) + (T_y\pi)(\bar{\eta}) = 0$$

hence $(\bar{\xi},\bar{\eta}) \in T_{(x,y)}L$, i.e. $T_{(x,y)}L = T_{(x,y)}L\perp\ \forall(x,y) \in L$ and L is lagrangian. □

OTHER INTERSECTION PROBLEMS

The above results show a certain *rigidity* in moving lagrangian submanifolds with symplectic isotopies. This rigidity appears as a global phenomenon, because locally symplectic diffeomorphisms acts very freely, at least on submanifolds of positive codimension ([Gro1], [Ben]); it is a rigidity very different from that appearing in riemannian geometry or complex geometry.

It is natural to ask if such a rigidity manifests also on other classes of submanifolds, and in particular on submanifolds of dimension strictly less than the half-dimension of the ambient symplectic manifold.

Laudenbach has proved the following nice result, which states that on submanifolds of small dimension symplectic isotopies acts with many *flexibility*, from the C^0 point of view.

Theorem 1 ([Lau2]): let (M^{2n}, ω) be a symplectic manifold and let $N \subset M$ be a compact submanifold of dimension $< n$; let $\phi_t : M \to M$, $t \in [0,1]$, be a (differentiable) isotopy and let $U \subset M$ be a neighborhood of $\phi_1(N)$, then there exists a hamiltonian isotopy $\psi_t : M \to M$, $t \in [0,1]$, such that $\psi_1(N) \subset U$. □

In order to clarify this theorem we consider the following construction.

Let $L_0 \subset M$ be an isotropic ($TL_0^\perp \supset TL_0$) compact submanifold, $dim\ L_0 = k < n$. Let $E \to L_0$ be the subbundle of $TM|_{L_0}$ defined by $E_x = \frac{T_x L_0^\perp}{T_x L_0}$; E_x has a natural symplectic (algebraic) form, and E is a symplectic vector bundle.

On $E \oplus T^* L_0$ we may introduce a symplectic form $\tilde{\omega}$, given by the "sum" of the symplectic form on T^*L_0 and the symplectic forms on the fibres on E (this requires the choice of a connection on E). Weinstein ([Wein2]) proved that a neighborhood of L_0 in M is symplectically diffeomorphic to a neighborhood of the null section Σ of $E \oplus T^*L_0$.

Let now $L_1 \subset M$ be a coisotropic ($TL_1^\perp \subset TL_1$) compact submanifold of M, $dim\ L_1 = 2n-k$. If $L = L_0 \cap L_1$, we suppose:

$$T_xL = T_xL_0 \cap T_xL_1 \quad \forall x \in L$$

$$T_xL = T_xL_0^\perp \cap T_xL_1^\perp \quad \forall x \in L$$

The second condition is equivalent to

$$T_xL^\perp = T_xL_0 + T_xL_1 \quad \forall x \in L$$

and from $T_xL \subset T_xL_0$ we deduce $T_xL_0^\perp \subset T_xL_0 + T_xL_1$.

There is a natural decomposition

$$T_xM = E_x \oplus F_x \qquad \forall x \in L_0$$

where F_x is also (as E_x) a symplectic vector space, naturally isomorphic to $T_xL_0 \oplus T_x^*L_0$; the symplectic form $\omega(x)$ on T_xM is the direct sum of the symplectic forms on E_x and F_x; from $T_xL_0^\perp \subset T_xL_0 + T_xL_1$ it follows that

$$T_xL_1 = E_x \oplus R_x \qquad \forall x \in L$$

where $R_x \subset F_x$ is lagrangian because T_xL_1 is coisotropic and $dim\ R_x = \frac{1}{2} dim\ F_x$.

On the other hand, under the same decomposition $T_xM = E_x \oplus T_xL_0 \oplus T_x^*L_0$, $x \in L_0$, we have also

$$T_xL_0 = \{0\} \oplus T_xL_0 \oplus \{0\}$$

$$T_xL = \{0\} \oplus T_xL \oplus \{0\} \quad \text{if } x \in L$$

the hypothesis of clean intersection implies $T_xL \oplus \{0\} = R_x \cap (T_xL_0 \oplus \{0\}$, i.e. $R_x \subset F_x$ is a lagrangian plane that intersects the lagrangian plane $T_xL_0 \oplus \{0\} \subset F_x$ along $T_xL \oplus \{0\}$.

The linear symplectic algebra of chapter 3 shows that there exists a smooth field of linear symplectic operators

$$L \ni x \mapsto A_x : F_x \to F_x$$

such that $A_x|_{T_xL_0 \oplus \{0\}} = \mathbf{1}$, $A_x(R_x) = T_xL \oplus S_x$ $(x \in L)$, where $S_x \subset T_x^*L_0$ is the annihilator of T_xL; hence the linear operators

$$\mathbf{1} \oplus A_x : E_x \oplus F_x \to E_x \oplus F_x$$

are the identity on T_xL_0 and map T_xL_1 to $E_x \oplus T_xL \oplus S_x$, for $x \in L$.

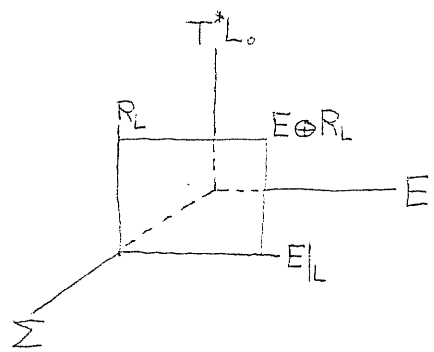
These calculations ensure the existence of a diffeomorphism from a neighborhood U of L_0 in M to a neighborhood V of Σ in $E \oplus T^*L_0$ such that:

1) ψ maps L_0 to Σ, and here is the identity $(\Sigma \simeq L_0)$

2) ψ maps $L_1 \cap U$ to $(E \oplus R_L) \cap V$, where

$$E \oplus R_L \stackrel{def}{=} \{(e,x) \in E \oplus T^*L_o | x \in R_L\}$$

3) $\forall x \in L_0 \quad T_x\psi$ is symplectic.



The homotopy method allows to obtain:

Theorem 2: let L_0, L_1 be compact submanifold of the symplectic manifold (M, ω) such that:

1) L_0 is isotropic, $dim\ L_0 = k$

2) L_1 is coisotropic, $dim\ L_1 = 2n - k$

3) $L = L_0 \cap L_1$ is a clean intersection and

$$T_x L = T_x L_0^{\perp} \cap T_x L_1^{\perp} \qquad \forall x \in L$$

then there exists a symplectic diffeomorphism $\psi : U \to V$, $U =$ neighborhood of L_0 in M, $V =$ neighborhood of Σ in $E \oplus T^* L_0$, such that:

a) $\psi(L_0) = \Sigma$

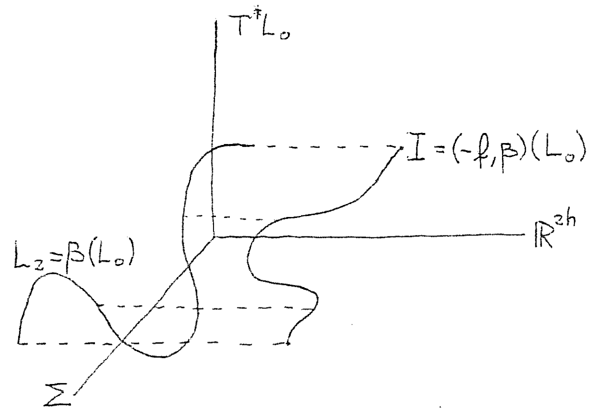
b) $\psi(L_1 \cap U) = (E \oplus R_L) \cap V$. □

This normal form theorem may be used to study the decomposition of $L_0 \cap L_1$ under symplectic isotopies.

Suppose that $\phi : E \oplus T^* L_0 \to E \oplus T^* L_0$ is a symplectic diffeomorphism, then $\phi(\Sigma) \cap (E \oplus R_L)$ is in 1-1 correspondence with $\pi(\phi(\Sigma)) \cap R_L$, where $\pi : E \oplus T^* L_0 \to T^* L_0$ is the projection. But now $\pi(\phi(\Sigma))$ is *not* a lagrangian submanifold of $T^* L_0$, and we can not deduce some estimates of Morse-type or of cup-lenght-type on $\pi(\phi(\Sigma)) \cap R_L$.

More specifically, suppose that $E \oplus T^* L_0 \simeq R^{2h} \times T^* L_0$ $(h = n - k)$, and let $L_2 = \pi(\phi(\Sigma))$ such that its projection on $\Sigma_0 =$ null section of $T^* L_0$ is everywhere non singular, so $L_2 = \beta(L_0)$ for some 1-form $\beta \in \Lambda^1(L_0)$ and $\pi(\phi(\Sigma)) \cap R_L \simeq$ zeroes of the pull-back of β on L.

A simple calculation gives $d\beta = f^*(\omega_0)$, where $\omega_0 = \sum_{j=1}^{h} dx_j \wedge dy_j$ is the canonical symplectic form on R^{2h} and $f : L_0 \to R^{2h}$ is some map. If h is sufficiently big, then every 1-form on L_0 is of this type. Conversely, if $\beta \in \Lambda^1(L_0)$ is of the type $f^*(\omega_0)$, then $L_2 \overset{def}{=} \beta(L_0)$ is the projection of some isotropic submanifold $I \subset R^{2h} \times T^* L_0$ (I is the image of the embedding $L_0 \overset{(-f,\beta)}{\longrightarrow} R^{2h} \times T^* L_0$), which is, moreover, isotopic to Σ in the space of isotropic submanifolds.



Consequently, $I = \phi(\Sigma)$ for some symplectic ϕ, and we have:

$$\beta(L_0) = \pi(\phi(\Sigma)),\ \beta \in \Lambda^1(L_0),\ \phi \text{ symplectic}$$

$$\iff d\beta = f^*(\omega_0),\ \text{for some } f : L_0 \to R^{2h}$$

Similarly, if $\gamma_0 = dz - \sum_{j=1}^h y_j dx_j$ is the standard contact form on R^{2h+1}:

$$\beta(L_0) = \pi(\phi(\Sigma)),\ \beta \in \Lambda^1(L_0),\ \phi \text{ hamiltonian}$$

$$\iff d\beta = g^*(\gamma_0),\ \text{for some } g : L_0 \to R^{2h+1}$$

In conclusion, if h is sufficiently big (with respect to $k = dim\ L_0$) then the image of every 1-form on L_0 is of the type $\pi(\phi(\Sigma))$ with ϕ symplectic or hamiltonian, and there is no hope to obtain interesting estimates on $\phi(\Sigma) \cap (R^{2h} \times R_L)$. The theorem of Laudenbach suggest that this remains true $\forall h \geq 1$.

Generic decomposition of clean intersections

Let M^n be a manifold and let $L_o^k, L_1^{n-k} \subset M$ be two submanifolds with compact clean intersection $L^l = L_0 \cap L_1 = \bar{L}_0 \cap \bar{L}_1$. Let $\phi_t : M \to M$, $t \in [0,1]$, be an isotopy and let $v = v_0 \in Vec(M)$ be its infinitesimal generator at $t = 0$: $v = \frac{d}{dt}|_{t=0}\phi_t$.

The vector field v defines, after restriction and projection, a section w of $E \to L$, where

$$E = \frac{(TM)|_L}{(TL_0)|_L + (TL_1)|_L}$$

observe that the dimension of the fibre of E is equal to the dimension of L.

Proposition: if $w : L \to E$ is transversal to the null section Σ of E, then $\exists \epsilon > 0$ with the following property:

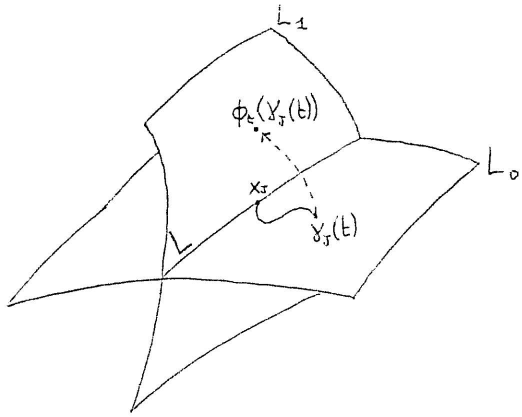
there exist smooth curves $\gamma_j : [0, \epsilon) \to L_0$, $j = 1..r$,such that

1) $\gamma_j(0) = x_j \in L \ \forall j = 1..r$, where $x_1..x_r$ are the zeroes of w

2) $\phi_t(\gamma_j(t)) \in L_1 \ \forall j = 1..r$ and $\forall t \in [0, \epsilon)$

3) $\forall t \in [0, \epsilon)$ $\phi_t(L_0)$ is transversal to L_1 and $\phi_t(L_0) \cap L_1 = \cup_{j=1}^{r}\{\phi_t(\gamma_j(t))\}$

in particular $\#[\phi_t(L_0) \cap L_1]$ is not less then the absolute value of the Euler-Poincaré characteristic of E.



Proof:

let $N \to L_0$ be the normal bundle of L_0 in M, with respect to some metric, and define

$$A = (TL_1)|_L \cap N|_L \quad \text{(a vector bundle over } L)$$

$$B = \text{ a complementary of } A \text{ in } N|_L$$

so $(TM)|_L$ is decomposed as

$$(TM)|_L = (TL_0)|_L \oplus A \oplus B$$

$(TL_0)|_L \oplus A = (TL_0)|_L \oplus (TL_1)|_L$, and we may identify E and B.

There exists a diffeomorphism ψ from a neighborhood U of L in M to a neighborhood V of L in N ($L \subset L_0$, $L_0 \simeq$ null section of N) such that:

1) $\psi(L_0 \cap U) = L_0 \cap V$

2) $\psi(L_1 \cap U) = A \cap V$

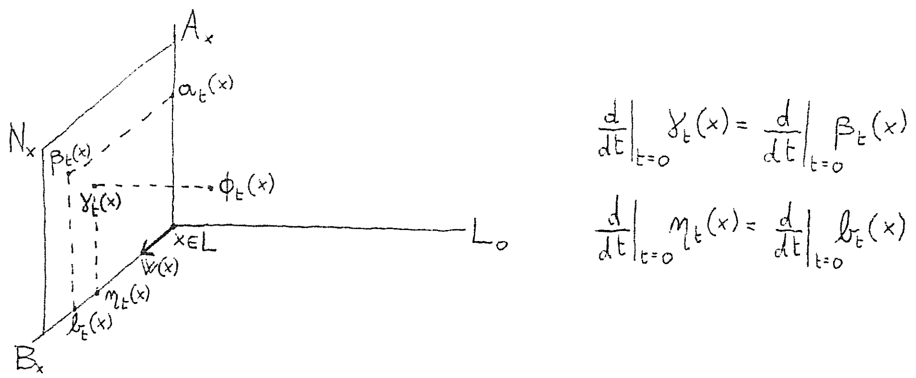
(ψ "push" L_1 on A, which is its linearization along L).

For t small, $\psi(\phi_t(L_0) \cap U) = \beta_t(L_0) \cap V$, where $\beta_t : L_o \to N$ is a suitable section; if $x \in L$ then $\beta_t(x) = (a_t(x), b_t(x))$, where $a_t : L \to A$, $b_t : L \to B$ (recall that $N|_L = A \oplus B$).

For t small, $\phi_t(L_0) \cap L_1 \subset U$ and $\psi(\phi_t(L_0) \cap L_1) = \beta_t(L_0) \cap A \cap V \simeq$ zeroes of b_t.

From $v = \frac{d}{dt}|_{t=0}\phi_t$ and from $w =$ projection of $v|_L$ on $E \simeq B$, it follows that $w(x) = \frac{d}{dt}|_{t=0} b_t(x)$, $\forall x \in L$ (identifing $B_x \simeq T_0 B_x$), i.e.

$$b_t = tw + O(t^2)$$



The transversality hypothesis on w ensures the existence of the curves $\gamma_1 ... \gamma_r$. □

The above proposition has some consequence on the decomposition of clean intersection in symplectic geometry.

Lemma: let (M^{2n}, ω) be a symplectic manifold, let $L_0^k, L_1^{2n-k} \subset M$ be submanifolds with compact clean intersection L^l such that

$$T_x L = T_x L_0^{\perp} \cap T_x L_1^{\perp} \qquad \forall x \in L$$

let $v \in Vec_H(M)$ and let w be the corresponding section of $\frac{(TM)|_L}{(TL_0)|_L + (TL_1)|_L}$, then the (transversal) zeroes of w are the (Morse) critical points of $H|_L$, $H =$ hamiltonian of v.

Proof:

$w(x) = 0 \iff v(x) \in T_x L_0 + T_x L_1 \iff v(x) \perp (T_x L_0 + T_x L_1)^{\perp} \iff$

$\omega(x)(v(x), \xi) = 0 \ \forall \xi \in T_x L \iff dH(x)(\xi) = 0 \forall \xi \in T_x L.$ □

Corollary: under the same hypotheses, if $\phi_t : M \to M$, $t \in [0,1]$, is an isotopy such that $\frac{d}{dt}|_{t=0}\phi_t = v$ and w is transversal to the null section, then for t sufficiently small $\phi_t(L_0) \cap L_1$ is in 1-1 correspondence with the critical points of a Morse function on L. □

This means that if $\{\phi_t\}$ is a "generic" hamiltonian isotopy, then $\#[\phi_t(L_0) \cap L_1]$ satisfy Morse inequality on L, for small t. It should be remarked that this *does not means* that if ϕ is a small "generic" hamiltonian diffeomorphism then $\#[\phi(L_0) \cap L_1]$ satisfy Morse inequality, unless we do some additional hypotheses on L_0 and L_1.

A pictorial way to understand this phenomenon is the following: let

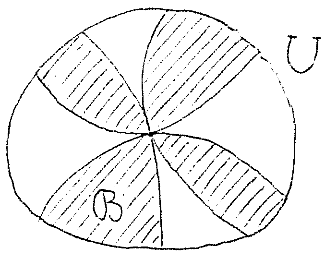
$$\mathcal{B} = \{\phi \in Diff_H(M) | \phi(L_0) \pitchfork L_1 \text{ and } \#[\phi(L_0) \cap L_1] \geq SB(L)\} \cup id$$

$(SB(L) =$ sum of Betti numbers of $L)$,

then for every neighborhood U of id

in $Diff_H(M)$ $\mathcal{B} \cap U$ is not generic in U,

but $T_{id}\mathcal{B}$ is generic in $Vec_H(M)$.



Example (cfr. also the discussion on isotropic-coisotropic intersections):

$M = T^*S^1 \times R^2 = S^1 \times R \times R^2$, $\omega = d\theta \wedge dp + dx \wedge dy$

$L_0 = S^1 \times \{0\} \times \{(0,0)\}$ (isotropic)

$L_1 = S^1 \times \{0\} \times R^2$ (coisotropic)

$L_0 \subset L_1$, L_0 is a characteristic leaf of L_1; $L = L_0$

let ϕ_t be generated by the hamiltonian $H(\theta, p, x, y) = x\cos\theta + y\sin\theta$, then

$$\phi_t(L_0) = \{(\theta, \frac{1}{2}t^2, -t\sin\theta, t\cos\theta) | \theta \in S^1\}$$

$$\phi_t(L_0) \cap L_1 = \emptyset \qquad \text{for } t \neq 0$$

fix t as small as you want, then (by transversality, because $\phi_t(L_0) \pitchfork L_1$!) there exists a neighborhood U of ϕ_t s.t. $\psi \in U \Rightarrow \psi(L_0) \cap L_1 = \emptyset$, so there exists an open set $V \subset Diff_H(M)$ s.t. $\psi \in V \Rightarrow \psi(L_0) \cap L_1 = \emptyset$ and $id \in \bar{V}$. On the other hand, $v = (y\cos\theta - x\sin\theta)\frac{\partial}{\partial p} - \sin\theta\frac{\partial}{\partial x} + \cos\theta\frac{\partial}{\partial y}$, so $w \simeq p$-component of $v|_{L_0}$ is identically zero and non transversal to the null section. The section $b_t = tw + \mathcal{O}(t^2)$ that appears in the proof of the proposition is identificable with the 1-form $\frac{1}{2}t^2 d\theta \in \Lambda^1(S^1)$, whose image is equal to the projection of $\phi_t(L_0)$ on $T^*S^1 \times \{0\}$. Because the transversality condition does not holds, there is no relation between the zeroes of w ($\simeq L_0$) and the zeroes of b_t for t small ($= \emptyset$).

APPENDIX: MORSE-NOVIKOV INEQUALITIES AND QUADRATICITY AT INFINITY

We sketch here, following Sikorav, the proof that a closed 1-form on $M \times R^N$ quadratic at infinity satisfy the same Morse-Novikov inequality as a closed 1-form on M.

Firstly, we recall Novikov's theory.

Let β be a closed 1-form on the compact manifold M, such that $[\beta] \in H^1(M, Z)$ (i.e. the periods of β are integers). The class $[\beta]$ defines a map F from $\pi_1(M)$ to $\pi_1(S^1) \simeq Z$, by integration on C^1 representatives of elements of $\pi_1(M)$. In other words,

$$\beta = f^*(d\theta)$$

for some $f : M \to S^1$ and

$$F = \pi_1(f) : \pi_1(M) \to \pi_1(S^1)$$

This map F defines a cyclic covering $\hat{M} \xrightarrow{\pi} M$ in the following way: $\pi_1(M)$ acts on the universal covering $\tilde{M}$ of M ("deck transformation"), so we have also an action of the normal subgroup $Ker\ F \subset \pi_1(M)$ on $\tilde{M}$:

$$Ker\ F \times \tilde{M} \to \tilde{M}$$

$$(\gamma, \tilde{x}) \mapsto \gamma \cdot \tilde{x}$$

$\hat{M}$ is the quozient of $\tilde{M}$ by this action; $\pi : \hat{M} \to M$ is induced by the obvious projection $\tilde{M} \to M$. The tipical fibre of this covering is $\frac{\pi_1(M)}{Ker\ F}$.

For example, if β is exact ($\iff$ f is homotopic to zero $\iff$ $F(\gamma) = 0\ \forall \gamma \in \pi_1(M)$) then $\hat{M} = M$. Otherwise, if β is not exact then $\frac{\pi_1(M)}{Ker\ F} \simeq Z$ and $\hat{M}$ is a Z-covering of M (remark that it is not compact).

The pull-back $\hat{\beta} = \pi^*(\beta) \in \Lambda^1(\hat{M})$ is now an exact form, because closed paths $\hat{\Gamma}$ on $\hat{M}$ projects by π to those closed paths Γ on M such that $\int_\Gamma \beta = 0$; $\hat{M}$ is the "minimal" covering of M such that the pull-back of β is exact.

Define

$$\Lambda = Z[t][t^{-1}] = \{\sum_{j=j_0}^{j_1} a_j t^j, \quad a_j \in Z, j_0, j_1 \in Z\}$$

$$\hat{\Lambda} = Z[t][[t^{-1}]] = \{\sum_{j=-\infty}^{j_1} a_j t^j, \quad a_j \in Z, j_1 \in Z\}$$

The above map $F : \pi_1(M) \to Z$ defines actions of $\pi_1(M)$ on Λ and $\hat{\Lambda}$: $\gamma \in \pi_1(M)$ acts as a multiplication by $t^{F(\gamma)}$. This defines systems of local coefficients Λ_F, $\hat{\Lambda}_F$ and allows to define the homology groups $H_*(M, \Lambda_F)$ (a Λ-module) and $H_*(M, \hat{\Lambda}_F)$ (a $\hat{\Lambda}$-module). These modules enjoy the following properties:

a) $H_*(M, \Lambda_F) \simeq H_*(\hat{M}, Z)$ (as Λ-modules)

b) $H_*(M, \hat{\Lambda}_F) \simeq H_*(M, \Lambda_F) \otimes_\Lambda \hat{\Lambda} \simeq H_*(\hat{M}, Z) \otimes_\Lambda \hat{\Lambda}$.

For instance, if $[\beta] = 0$ then $\Lambda_F = \Lambda$, $\hat{\Lambda}_F = \hat{\Lambda}$, $H_*(M, \Lambda_F) \simeq H_*(M, Z) \otimes_Z \Lambda$ and $H_*(M, \hat{\Lambda}_F) \simeq H_*(M, Z) \otimes_Z \hat{\Lambda}$.

Define:

$$H_*(M, [\beta]) = H_*(M, \hat{\Lambda}_F)$$

$$r(H_*(M, [\beta])) = \text{rank (over}\hat{\Lambda}) \text{ of } H_*(M, \hat{\Lambda}_F)$$

$$q(H_*(M, [\beta])) = \text{torsion (over}\hat{\Lambda}) \text{ of } H_*(M, \hat{\Lambda}_F)$$

Remark that if $[\beta] \in H^1(M, Q)$ then $\lambda[\beta] \in H^1(M, Z)$ for some $\lambda \neq 0$ and the zeroes of β coincide with the zeroes of $\lambda\beta$. We may define $H_*(M, [\beta]) = H_*(M, \lambda[\beta])$, this is a good definition (it does not depend on $\lambda \neq 0$).

Theorem([Nov]): suppose that β is a rational closed 1-form of Morse-type on the compact manifold M, then the number of its zeroes, $\#Z(\beta)$, satisfies the inequality:

$$\#Z(\beta) \geq r(H_*(M, [\beta])) + 2q(H_*(M, [\beta])) \, . \quad \square$$

Let now $\alpha \in \Lambda^1(M \times R^N)$ closed and quadratic at infinity, $\alpha = p^*(\beta) + dS$, where p is the projection on M. Modulo a suspension by a quadratic form we may suppose that the signature at infinity of S is $(2k, 2k)$ for some $k \in N$ (this suspension does not change the number of zeroes).

Lemma: $\forall k \in N$ there exist a compact manifold V^{4k} of dimension $4k$ and a map $p_{4k} : V^{4k} \to S^1$ such that:

1) p_{4k} has only one critical point, of Morse type and with index $2k$

2) p_{4k} induces an isomorphism between $\pi_1(V^{4k})$ and $\pi_1(S^1) \simeq Z$. $\square$

Remark that $p_{4k}^*(d\theta)$ defines a rational closed 1-form on V^{4k}, of Morse type.

We want to construct a closed 1-form on the compact manifold $M \times V^{4k}$, whose zeroes are in bijection with the zeroes of α.

By definition, there exists $R > 0$ s.t.

$$S(x, \lambda) = Q(\lambda) \qquad \|\lambda\| > R$$

where $Q(\lambda)$ is a quadratic form of signature $(2k, 2k)$. For every A sufficiently big there exists a diffeomorphism $\chi_A : U_A \to D_{2R}$, $U_A =$ neighborhood in V^{4k} of the critical point of p_{4k}, $D_{2R} =$ disk of radius $2R$ in R^{4k}, such that

$$A\tilde{p}_{4k} = Q \circ \chi_A + constant$$

where $\tilde{p}_{4k} : U_A \to R$ is a lifting of $p_{4k}|_{U_A} : U_A \to S^1$ (Morse lemma).

Let $p_1 : M \times V^{4k} \to M$, $p_2 : M \times V^{4k} \to V^{4k}$ be the projections, and define on $M \times V^{4k}$ the closed 1-form

$$\Omega_A = \begin{cases} (id \times \chi_A)^* \alpha & \text{on } M \times U_A \\ p_1^*(\beta) + A p_2^*(p_{4k}^*(d\theta)) & \text{on } M \times (V^{4k} \setminus U_A) \end{cases}$$

this is a good definition, because on $M \times (U_A \setminus \chi_A^{-1}(D_R))$ we have $A p_2^*(p_{4k}^*(d\theta)) = (id \times \chi_A)^*(dQ) = (id \times \chi_A)^*(dS)$. We may think Ω_A as an exact deformation on $M \times U_A$ of the closed 1-form $p_1^*(\beta) + Ap_2^*(p_{4k}^*(d\theta))$, in the same sense as α is an exact deformation of $p^*(\beta) + dQ$ on $M \times D_R$.

Observe that Ω_A has no zeroes on $M \times (V^{4k} \setminus U_A)$, because p_{4k} has no critical points outside U_A, so the (Morse) zeroes of Ω_A are in 1-1 correspondence, via $(id \times \chi_A)$, with the (Morse) zeroes of α. Remark that α (or β) is rational if and only if Ω_A is rational, and moreover

$$[\Omega_A] = p_1^*[\beta] + p_2^*[Ap_{4k}^*(d\theta)]$$

In order to estimate the number of zeroes of Ω_A we may use Novikov's theory, and we must evaluate the module $H_*(M \times V^{4k}, [\Omega_A])$.

Proposition: let $\alpha = p^*(\beta) + dS \in \Lambda^1(M \times R^{4k})$ closed, rational and quadratic at infinity, let Ω_A defined as above, then

$$H_*(M \times V^{4k}, [\Omega_A]) \simeq H_*(M, [\beta])$$

as $\hat{\Lambda}$-modules.

The proof follows from Kunneth formula and from the fact that, if $F_1 : \pi_1(M) \to Z$, $F_2 : \pi_1(V^{4k}) \to Z$, $F_3 : \pi_1(M \times V^{4k}) \to Z$ are the maps induced by $\lambda[\beta] \in H^1(M, Z)$, $\lambda[Ap_{4k}^*(d\theta)] \in H^1(V^{4k}, Z)$, $\lambda[\Omega_A] \in H^1(M \times V^{4k}, Z)$, then

$$\hat{\Lambda}_{F_3} = \hat{\Lambda}_{F_1} \otimes \hat{\Lambda}_{F_2}$$

(of course, it is essential also the property 2) of p_{4k} in the lemma). □

Corollary: if α is of Morse-type, the numbers of its zeroes, $\#Z(\alpha)$, satisfies:

$$\#Z(\alpha) \geq r(H_*(M, [\beta])) + 2q(H_*(M, [\beta])) \ . \ \square$$

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