Spacetime Wormholes
and
Baby Universes

Thesis presented by

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for the degree of Magister Philosophiae

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S.I.S.S.A. – I.S.A.S.

Astrophysics Sector

academic year 1989 - 90

Contents

Warning!

This subject is speculative, controversial, and without experimental foundation. Thus, most of what I tell you is undoubtedly false!

Sidney Coleman

1 Introduction

One of the most ambitious challenges in the contemporary physical research is that for a well defined quantum theory for gravity and a possible grand unification model for all known forces governing the world. Unfortunately, the long time yearned for "Theory of Everything" is up to now far from being well established. Nevertheless, in the last few years, a new and interesting field has open up, that of possible topological fluctuations in a quantum theory of gravity, one of the most intriguing features being the so called "wormhole". As we will see, this possibility first aroused much enthusiasm, but also met with some old and new distressing results, which stimulated new approaches and theoretical developments: but proceed with order!

1.1 Classical and quantum gravity: an overview

A quantum approach to gravitation is essential if one wants to tackle the issue of the origin and the subsequent evolution of our universe which is compatible with its present "classical" structure.

A series of theorems by Hawking and Penrose state that any classical cosmological model with a "reasonable" matter content must start with a singularity, at which both spacetime curvature and energy density become divergent (see, for instance, [63]). As a result, all the laws of physics and, above all, the classical cosmology will break down there, and quantum effects will become dominant.

A particular model of quantum gravity is described by quantum cosmology, where the new dynamical variable is the geometry of 3-space and the matter configuration on it. The spacetime is no longer a fundamental concept: the new arena for geometrodynamics becomes "superspace".

In canonical classical cosmology, the spacetime (a 4-manifold M with metric $g_{\mu\nu}$) is the trajectory or history of 3-space (h_{ij}), which becomes the new dynamical variable along with the value of matter fields (generally assumed as scalars ϕ) on 3-surfaces of constant time.

The (Lorentzian) action for matter+gravity is:

$$I = \frac{1}{16\pi G}\int_M (R-2\Lambda)\sqrt{-g}d^4x \; + \frac{1}{8\pi G}\int_{\partial M} K\sqrt{h}d^3x \; + \int_M \mathcal{L}_M\sqrt{-g}d^4x \qquad (1)$$

the surface term being added [53] to obtain the Einstein's equations under variations of the metric such that $\delta g_{\mu\nu} = 0$ but $\delta\nabla_\mu g_{\nu\rho} \neq 0$ on the boundary ∂M ($\mathcal{L}_M$ is the matter Lagrangian).

Using the 3+1 splitting of the metric and introducing the momenta (Π_ϕ, Π_{ij}) conjugate to the dynamical variables (ϕ, h_{ij}), one then obtains the Hamiltonian:

$$H = \int (\Pi_{ij}\dot{h}_{ij} + \Pi_\phi\dot{\phi} - L_M)d^3x \; - \int (NH_o + N_iH_i)d^3x \qquad (2)$$

In order to choose the actual history of our universe, we must specify the dynamical equations and a set of initial conditions. The dynamical equation can be obtained from the Hamiltonian and are the space part of the Einstein equations and the classical equation for the field. Moreover, since the lapse and shift functions act as Lagrangean multiplier for the action, from variations of I we obtain the classical Hamiltonian and momentum constraints:

$$H_0 = +16\pi G\cdot G_{ijkl}\Pi^{ij}\Pi^{kl} - \frac{1}{16\pi G}h^{1/2}({}^3R - 2\Lambda) + h^{1/2}(h^{ij}\phi_i\phi_j + V(\phi)) + \frac{1}{2}h^{-1/2}\Pi^2_\phi = 0 \quad (3a)$$

$$H^i = -2\Pi^{ij}_{|j} + h^{ij}\phi_j\Pi_\phi = 0 \qquad (3b)$$

conserved by classical evolution ($G_{ijkl} = \frac{1}{2}h^{1/2}(h_{ik}h_{jl} + h_{il}h_{jk} - h_{ij}h_{kl})$ is the metric of superspace, with signature $- + + + ++$).

In the Shröedinger representation, the quantum state of the universe may be described by a wave functional $\Psi(h_{ij}, \phi)$ on "superspace", the space of all possible h_{ij} and ϕ that can be put on a three surface S.

To quantize the classical dynamics, we must turn classical canonical variables into quantum operators, i.e.:

$$\Pi_\phi \to -i\frac{\delta}{\delta\phi} \qquad (4a)$$

$$\Pi_{ij} \to -i\frac{\delta}{\delta h_{ij}} \qquad (4b)$$

The Hamiltonian becomes a functional differential operator. The wave functional, to describe a quantum state of the universe, must be consistent with the classical constraints, which are therefore implemented at the quantum level saying that they annihilate the wave functional. We have then the functional differential equations [19, 49, 55]:

$$\hat{H}^i\left(\phi, h_{ij}, -i\frac{\delta}{\delta\phi}, -i\frac{\delta}{\delta h_{ij}}\right)\Psi(h_{ij},\phi)=0 \tag{5a}$$

$$\hat{H}^0\left(\phi, h_{ij}, -i\frac{\delta}{\delta\phi}, -i\frac{\delta}{\delta h_{ij}}\right)\Psi(h_{ij},\phi)=0 \tag{5b}$$

The only non trivial equation is the second one, the so called Wheeler-de Witt (WdW) equation, governing the evolution of Ψ in superspace [19]. Naively speaking, if seen as a Shröedinger equation, it says that Ψ is not an explicit function of time; as an eigenvalue equation, it suggests that the total energy of the universe is zero.

There is a factor ordering ambiguity in the kinetic term for gravitation in eq. (5b): the Hawking-Page choice [65] gives the covariant laplacian in the hyperbolic superspace metric, with $h^{\frac{1}{2}}$ playing the role of time [55]. The WdW equation is thus hyperbolic and in general difficult to solve. Any solution of the WdW equation is a possible quantum state of the universe.

Another and frequently used representation of quantum dynamics of the universe is that in terms of Euclidean path integrals (EPI). Here, the wave functional is described by a path integral over a certain class C of Euclidean 4-metrics $g_{\mu\nu}$ and matter configurations (histories) on a manifold M [53, 55]:

$$\Psi(h_{ij},\phi)=\int_C [dg_{\mu\nu}][d\phi]e^{-I(g,\phi)} \tag{6}$$

I is the Euclidean action of the history $g_{\mu\nu}(x^\rho), \phi(x^\rho)$, obtained by the Lorentzian one just turning the lapse function into pure imaginary ($N \to -iN$). This choice is to make the functional integral at least formally convergent (the Lorentzian one oscillates and doesn't converge). It is argued that the residual divergence due to rapid oscillations of the conformal degrees of freedom of the metric can be removed with a peculiar choice of the contour of integration [33].

To select one definite wave functional, one has to specify the boundary conditions for the path integral. The most popular proposal is the Hartle-Hawking (HH) boundary condition, which amounts taking the class C as the class of all compact Euclidean 4-metrics and regular matter configurations on a manifold M whose only boundary is a compact 3-surface S with no boundary [49] (see fig. 1). The wave functional is then seen to satisfy, at least formally, the WdW equation and the momentum constraint (see, e.g. [55]), so it really represents a possible quantum state of the universe.

1.2 Topological fluctuations: the wormhole

The idea that one can find a wave functional for the universe from a path integral over histories for the metric $g_{\mu\nu}$ is closely related to the idea that space-time topology can "fluctuate", at least at the quantum level. This idea was probably first suggested by Wheeler [122] and later developed by many people, among which Hawking and some others [52, 54, 56, 67]. Wheeler [122] quotes: "An oscillating drop of water goes under fission. The topology changes....Before the division, the surface of the drop constituted a manifold. After the division, it's again a manifold....At the instant of division is not a manifold. But little attention does the drop pay to this distinction. It divides, despite all definitions." and "The field equations of relativity are purely local in character. They make no statements at all about global topology" .

When do topology fluctuations become important? Consider fluctuations of the metric ($g_{\mu\nu}$) about its flat background value ($\eta_{\mu\nu}$):

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \tag{7}$$

where $h_{\mu\nu}$ represents the graviton field. Dimensionally estimating the graviton propagator with a momentum cutoff M, we find for the fluctuations of the metric:

$$\sigma^2(g_{\mu\nu}) \simeq \left(\frac{M}{M_p}\right)^2 \tag{8}$$

telling us they are completely unobservable even in sophisticated particle experiments at $10\ TeV$! Only at distances of order of $10^{-33}cm$ geometry undergoes significant fluctuations. This is the central idea of the so called foam-like structure described by Wheeler [122] and Hawking [52], where lots of "ripples", "bubbles" and "handles" in the spatial geometry would appear when using an ideal quantum microscope of Planckian resolution.

1.3 The 3-D wormhole of Wheeler

One peculiar kind of these fluctuations is the so called "wormhole". This name was first introduced by Wheeler [122] to describe topology change in 3-dimensions as a possible solution to the problem of breakdown of Maxwell equations at points where charges are located. Wheeler's idea was that, while still holding to Maxwell's equations for empty space also at charge location, but giving up the assumption that 3-space must be everywhere Euclidean, one can describe the classical electric charge itself as a "..flux of lines of force trapped in the topology of space" (i.e. a wormhole). He also described the quantum concept of an eventual wormhole pinching off as "virtual" wormholes creation and annihilation while space resonates between one foam-like geometry and another.

1.4 The 4-D wormhole

Can we similarly introduce in 4-dimensions, i.e. in general relativity, the concept of a 4-D wormhole ? To answer this question we must first remember that in classical Lorentzian general relativity, on account of a Geroch no-go theorem [32, 63, 120], a globally hyperbolic manifold M cannot have any kind of topology change (M must be $R^1 \times \Sigma$, where Σ is a constant time 3-surface). This is another important reason for which one, while passing from a classical to a quantum theory of gravity allowing for topology changes, chooses to adopt the Euclidean analysis. In particular, topology change requires the existence of singularities in Lorentzian spacetime; a theorem (see [118]) states that any compact Lorentzian even dimensional spacetime has Euler character zero (for ex., a "trouser's" topology must possess at least one crutch singularity where the direction of time is ill defined, see fig. [2]). No such singularities are required when the spacetime is Euclidean.

Therefore, in Euclidean quantum gravity, we can almost naturally speak about fluctuations. The literature then provides different kinds of definitions for a wormhole.

Loosely speaking, a 4-d wormhole may be seen as an Euclidean spacetime where a "tube", a little closed spatial geometry (a baby universe) splits off and rejoins a unique large Lorentzian parent universe or links two otherwise disconnected ones (see fig. [3, 4]). Quantum mechanically, it represents the topology changing fluctuation in the ground state of quantum gravity.

Often, wormholes are more precisely regarded as gravitational instantons, i.e. exact solutions (with a closed 3-geometry) of the Euclidean version of the Einstein equations. In this case, wormholes can be interpreted as dominant contribution to the vacuum to vacuum tunneling amplitude in quantum gravity. These instantons, symmetric about some minimum radius, may represent tunneling events between: a) the same connected or two disconnected asymptotically flat Lorentzian manifolds; b) one asymptotically flat spacetime and a closed Friedmann-Lemaitre-Robertson-Walker (FLRW) universe ; c) one large de Sitter space at minimum radius and a small closed FLRW universes at maximum radius. A lot of solutions of this kind have been found, such as that for gravity coupled to an axionic field [34], that for a charged massive minimally coupled scalar [16], Yang-Mills [70] etc..

Some wormholes are not solutions of the Einstein equations (for instance, the Tolman- Hawking conformally flat one) or are just end points of the action [100]. Pure gravity wormholes also exist [6, 30, 37].

Moreover, Visser [118], in debating the Lorentzian causality properties connected with this kind of topological fluctuations, stressed the distinction between transient structures (the above ones) and permanent (i.e. formed ab initio) wormholes.

1.5 The Hawking model for black hole evaporation

The first time the concept of a 4-d wormhole in linear Einstein gravity was introduced and explicitly used was in a paper by Hawking [58], and was related to the problem of black hole evaporation.

Using quantum field theory (QFT) in curved spacetime, it was discovered

[51] that a black hole should give off radiation, in particular it should emit particle-antiparticle pairs at a steady rate; one member of the pair should escape to infinity, the other should fall into the black hole. The black hole would then be expected to lose mass until it will disappear completely, leaving no remnant; it would take with it the particles that formed the black hole and the antiparticles of the emitted radiation.

But what happens to these particles? The original idea was that they might have gone off into a little closed universe of their own, i.e. a macroscopic wormhole (see fig. (5)). Later on, Coleman and Lee [16] gave support to this model showing that the charge q of their wormhole solution was (surprisingly!) just of the same order of magnitude to that expected for a black hole containing q mesons.

As already remarked and as it will be discussed in more details in the following, the possibility of branching off of little closed baby universes from their "parents" opened a controversial debate upon the introduction of eventual and strange effects such as an extra degree of uncertainty in quantum gravity (if not a real loss of quantum coherence- as claimed by [13, 35]-at least a reflection of our lack of knowledge about the initial quantum state of the universe-as claimed by [64]), and the possible final fate of the nonrenormalizability of gravity in whatever more general physical theory (such as superstrings...).

Obviously, the assumed existence of "handles" changing the connectedness of spacetime and linking distant regions in the same or different universes, also stimulated the study of the apparent existence of nonlocal and acausal effects and their significance and consistency in QFT.

1.6 The Baum-Hawking-Coleman model for setting $\Lambda = 0$

A new and exciting period in the study of these topological features arose when Coleman [14], essentially developing a previous idea by Baum [5] and Hawking [57], suggested a mechanism for the vanishing of the low energy effective cosmological constant (including the renormalizations from all interactions at all orders), a long time outstanding and fundamental problem of cosmology and particle physics.

Coleman considered a multiuniverse theory where disconnected large smooth universes may actually be connected by small wormholes, of typical size not much greater than the Planck one. Wormholes are regarded semiclassically, in the instanton approximation and with no interactions (the "dilute gas" approximation).

The idea is to integrate out wormhole fluctuations to obtain an effective field theory with a short distance cutoff given by the wormhole size. This effective theory turns out as a superposition of "superselection" sectors not communicating with each other through any local physics and labeled by an infinite set of parameters α (similar to the θ vacua angle of QCD), each for any given wormhole kind. In each sector, both bare and renormalized couplings of the effective theory are functions of α, and the superposition of α-dependent effective theories is described by a probability distribution that is sharply peaked ($\exp(\exp(\frac{3}{8G^2\Lambda}))$) at $\Lambda = 0$. In other words, the coupling constants of nature become dynamically determined quantities and are affected by an intrinsic (statistical) indeterminacy; our universe is chosen at random from an ensemble of possible universes (a priori with different values of the couplings), but whose probability distribution is peaked at a fixed set of the constants.

A lot of subsequent papers were therefore devoted to the effort of determining or at least giving reasonable bounds to the other relevant physical couplings, such as the gravitational constant G, the masses of the scalars, bosons and fermions presently known in particle physics (the so called "big fix").

There is now less enthusiasm and agreement on the actual possibility of fixing all constants of nature and, even if some models minimizing G (but preventing it from being zero, which would give gravity as a free theory!) have been proposed, other disappointing results have also been claimed ($m_\pi = 0, \theta_{QCD} = \pi$) and a lot of issues appear still unclear and debated (the normalization of the infrared divergent measure in the probability distribution, the effect of the addition of nonlinear terms in the gravitational lagrangian etc.).

On the other hand, the idea of considering wormholes as dominant topological structures and the universe as possibly made by a lot of connected and disconnected pieces was of a great stimulation in the search for a more general class of semiclassical and full quantum wormhole solutions, and for the construction of a 3^{rd} quantized multiuniverse theory allowing for topology change.

1.7 A new set of wormhole solutions

An entire chapter of this thesis will in fact be devoted to the description of a new class of semiclassical asymptotically flat instanton solutions, just found considering the Einstein equations for a homogeneous and isotropic model described by a perfect fluid equation of state [9]. These wormholes may be understood as analytical continuation of closed expanding universes at maximum radius, and they exist only if the matter source obeys the strong energy condition $\rho+3P > 0$, exactly complementary to the inflationary universes. For every classical solution in standard cosmology with closed spatial geometry and obeying this condition there is a wormhole solution, and all main known solutions are reproduced. We also constructed wormholes that are analytical continuation of closed expanding universes driven by a minimally coupled scalar field. We showed that, for the wormhole solutions to exist, it is necessary to analytically continue to the Euclidean regime by an asymmetric Wick rotation of the lapse function in the matter and gravitational part of the action. The physical relevance of this will certainly need further consideration (especially in a full quantum context). An explicit form for the potential of the scalar field has been found, leading to a periodicity in the scalar field itself and, possibly, to the notion of a finite temperature for wormholes. This may be related to the Hawking's idea about the role of wormholes in the evaporation of black holes.

A different general class of full quantum wormhole solutions (with both a discrete and continuous spectrum), subject to appropriate asymptotic conditions, has recently been found by [66] and [8].

1.8 Looking for a multiuniverse 3^{rd} quantized theory

A lot of papers appeared, on the other hand, constructing and discussing a 3^{rd} quantization Euclidean theory, pioneers being [2, 3, 7, 27, 36, 69, 72, 77, 79, 84, 89, 90, 95, 106, 111, 114]. The main idea is here to consider the many universe system as a quantum field theory in the arena of superspace , where third quantized operators, acting on the "void" (a Fock state containing no universes), create second quantized states in the field theory of a single universe. The role of particles is played by the universes, that of the interaction vertexes by topology changes. The third quantized state is an operator valued functional of the 3-metric and the other dynamical variables on a spacelike surface: it is a 2^{nd} quantized WdW wave function. Because of universes interactions, the field equation in the 3^{rd} quantized theory is nonlinear and it represents a dynamical equation for the 2^{nd} quantized spacetime couplings.

Essentially, all these models, though being still far from a clear definite and unified scheme, seem to introduce once again a kind of indetermination for the coupling constants, the 3^{rd} quantized uncertainty principle, and to essentially agree with Coleman's model and prediction about Λ.

1.9 Problems in the wormhole Euclidean theory

However, a lot of issues remain open and unclear, if not obscure and badly founded, in both original (and improved) Coleman's scheme and these 3^{rd} quantization models. First of all, in these theories one must carefully consider the problem of defining a convergent measure in the path integral, and the existence of gauge simmetries (BRST conserved charge...). Of critical importance is then the present nonexistence of a well defined Euclidean theory of gravity; as it is well known the Euclidean gravitational action is not bounded from below. The Gibbons-Hawking-Perry [33] prescription for the imaginary rotation of the conformal degrees of freedom is supposed to lead to a well defined path integral (PI) which hopefully delivers Coleman's result, but so far this prescription has been understood only in the simplest cases, and it is known that it might lead to problems in general. Moreover, as pointed out by [27], in all these theories there seems to be a bit of a confusion about the correct interpretation and use of the concept of probabilities (a priori, conditional, weighted ?), the distinction between transition amplitudes (as Coleman's path integral seems to be) and expectation values for observables in our own universe etc.. Neither the so called giant wormhole puzzle, the fact the EPI derived peak for Λ also leads to a catastrophic number of macroscopic or even cosmically large wormholes, hard to reconcile with the well tested successes of local field theory in describing low energy physics, has yet been resolved [15, 28, 100, 102]. A lot of other problems still remain, some of which will be addressed in this thesis: the actual existence of a path integral contour enclosing all wormhole saddle points and the latter's dominance in the path integral itself, the effect of wormhole interactions, the claimed existence of an additional quantum prefactor i^{D-2} (in a D-dim spacetime) destroying the Coleman's peak at $\Lambda = 0$ [101], etc..

1.10 Towards a more complete theory?

All these disappointing issues therefore lead some people to start building a new quantum mechanical theory of gravity consistent with the classical general relativity constraints, based on a Minkowskian 3^{rd} quantization scheme: the quantum mechanics of the Googolplexus [27].

This theory is essentially the Lorentzian analogue of the Euclidean ones (a nonlinear WdW equation for a functional operator acting on a Fock space of universes and replacing the 2^{nd} quantized wave function, coupling constants turned into dynamical variables with a probability distribution...) but the first estimated results appear quite different. For instance, it was found [27] that the probability of a given Λ should be flat with no peak at zero and, even if the mean number of universes is exponentially peaked at $\Lambda = 0$, all these appear to be cold and uninteresting. Much work is still to be done, however, and it is clearly too soon for giving a definite judgement.

1.11 Summary of chapters' content

The main lines of the thesis will proceed as follows. In chapter 2 we will essentially enumerate all known fundamental wormhole solutions classified as asymptotically and non asymptotically flat instantons, end point solutions and quantum solutions.

In chapter 3 we will introduce a new general class of semiclassical wormhole solutions, considering both the bulk matter model and the scalar field model. For the latter, a new set of rules for the Euclidean path integral are defined and explained. Old known solutions are recovered and the quantum relevance for transition amplitudes discussed for all models. We also discuss the evidence that wormholes may have a finite temperature.

In chapter 4 we give a detailed description of the argument for the vanishing of Λ and comment on Coleman's original semiclassical approach, including the idea of the "big fix" for the constants of nature (G, masses, etc.).

In chapter 5 we give a discussion of the main problems of the theory, such as the contour problem, the regulation of the infrared divergent measure in the E.P.I., the correct probability interpretation, the large wormhole problem.

In chapter 6 we discuss some of the main features of the wormhole, such as the construction of its vertex for the effective lagrangian, the problems of loss of coherence, causality and that of wormholes interactions.

Chapter 7 gives more relevance to the cosmological aspects of the theory, such as a possible "time" dependent model for Λ, the simultaneous presence of universes of different sizes, etc..

Chapter 8 is essentially devoted to a schematic presentation of the 3^{rd} quantization Euclidean theories, their results and difficulties and finally their Lorentzian analogue.

Chapter 9 closes with a discussion about future perspectives of the wormhole theory.

2 Known wormhole solutions

2.1 Historical remarks

Historically, the first known examples of 4-D asymptotically Euclidean gravitational instanton solutions can be found in the papers by Horowitz, Perry and Strominger [115] and by Strominger [113]. They considered the model of a conformally invariant quantum theory of gravity and for which a set of nontrivial topological configurations was recovered.

However, the first papers introducing an explicit wormhole solution in "canonical" Einstein gravity is due to Hawking [58] and Giddings and Strominger [34]. In the last 2 years, then, a certain amount of new papers dealing with new solutions or slight modifications of old ones has been published. Semiclassical gravitational instantons have been found, in minisuperspace models, joining two asymptotically flat manifolds [1, 16, 34, 58, 74, 75, 85], an asymptotically flat space with a closed FRLW universe [9, 84, 107], and a de Sitter space with a closed FLRW or another de Sitter space [6, 17, 18, 30, 37, 43, 70, 91, 96, 97, 104]. Some theorems have also been quoted and demonstrated about the conditions for the existence of gravitational instantons. More recently, Hawking and Page [66] and Campbell and Garay [8] started a detailed investigation about the existence and properties of full quantum wormhole solutions subject to some asymptotic boundary conditions.

The main goal of this chapter is to give a systematic description of the fundamental wormhole solutions provided by the literature.

2.2 (Non)existence theorems

Th.: There are no asymptotically flat solutions of the Einstein equations with zero energy or Ricci flat except flat space (Shoen and Yau [110]).

This theorem excludes, for example, processes such as that from flat R^3 to any connected but topologically nontrivial 3-manifold N: the tunnelling conserves energy, and N must have zero energy as does flat R^3. Also the tunneling $R^3 \rightarrow R^3$, eventually accompanied by topological changes of a spin structure, is not allowed: this is a Ricci flat solution of the Einstein equations.

An important theorem was then stated by Giddings and Strominger [34] using the result of a previous work by Cheeger and Grommol [10]:

Th.: Given an asymptotically flat 4-geometry with $n > 1$ compact interior boundaries with vanishing extrinsic curvature, the Ricci tensor always has some negative eigenvalue somewhere.

One may now think of an instanton describing a tunnelling from R^3 to the disconnected $R^3 \oplus S^3$ (see fig. [6]). It will be shown (see section 2.6.2) that the S^3 boundary is a minimal surface (i.e. it has vanishing extrinsic curvature). This rules out, then, such instantons in theories for a pure gravity case or for gravity minimally coupled to a scalar field for which $R_{\mu\nu} = \nabla_\mu \phi \nabla_\nu \phi$ (i.e. $V(\phi) = 0$). However, for instantons provided by antisymmetric tensor fields, such as the axion and the electromagnetic ones, a solution is expected to exist [21, 34, 66].

Another interesting paper by Jungman and Wald [73] states some nonexistence results for Euclidean instantons with gravity and matter fields satisfying appropriate fall-off conditions (matter fields going to zero at infinity at a sufficient rate to ensure that the matter action is finite and the boundary term of matter action asymptotically vanishes). With these assumptions, it is shown that no instanton solutions exist for conformally or scale invariant matter fields. Moreover, also for nonconformally invariant cases, the matter equations alone rule out solutions for a scalar field ϕ with potential V satisfying the condition $\phi \frac{\partial V}{\partial \phi} > 0$, such as a free massless minimally coupled Klein-Gordon field.

2.3 A brief introductory survey on the known solutions

The following sections describe some of the main wormhole solutions produced so far in the literature. The first work is that by Hawking [58], who found an asymptotically flat wormhole solution for a metric that does not satisfy the Einstein equations and whose relevance in quantum gravity (i.e., in the evaluation of the EPI) is still not so clear (see section 3.2).

The Hawking wormhole has recently been reproduced by Gonzales-Diaz [37], but in a slightly different context (see section 2.6.1). The interesting idea is to consider pure gravity with a cutoff in the scale factor, essentially motivated by the expected impossibility of measuring the position or size of any object with infinite precision in a quantum context [99].

In chapter 3, we will show how one can reproduce the same wormhole from a simple classical solution in standard cosmology with the closed spatial geometry and equation of state $p = \frac{\rho}{3}$.

Other wormhole solutions have been found for different matter contents. Giddings and Strominger [34] and Myers [96, 97], for example, studied the case of gravity minimally coupled to an antisymmetric tensor field representing an axion. The case of a minimally coupled charged scalar field has been considered by Lee [85], Coleman and Lee [16], Abbott and Wise [1] and Midorikawa [91]. Hosoya and Ogura [70] and Rey [104] described the case of a SU(2) Yang-Mills field and Halliwell and Laflamme [43] that of a conformally coupled scalar field. Dowker [21] studied the case of a wormhole driven by an electromagnetic field. Non linear gravity wormhole instantons have been studied by Fukutaka, Ghoroku and Tanaka [30], Bertolami [6] and, recently, by Coule and Maeda [17] in the context of a theory also containing scalar and axion fields. Some other works then described instanton solutions for a combination of the previous cases (see, for example, the very interesting and cosmologically relevant papers by Lavrelashvili, Rubakov and Tinyakov [84] and Rubakov and Tinyakov [107], for gravity coupled to an axion and a scalar field).

One of the most striking features shared by almost all of these works quite evidently appears to be the following: the presence of some conservation law (i.e. a current, a charge) seems to be crucial for the existence of such instanton solutions. It is essential, for instance, in the axion model presented by [34], the existence of a conserved axion current flowing down the wormhole which locally defines a pseudoscalar field and which, in its turn, gives a stress tensor with negative eigenvalues. In particular, in order to get the "appropriate" Euclidean equations of motion, it is essential to a priori properly fix the sign of the axion field strength in the action and, only after continuation to the Euclidean space, substitute it by the pseudoscalar field. If one does the opposite, the instanton solution does not exist.

Also for the case of the charged scalar field treated in [85] and [16] the existence of a conserved charge is determinant for the existence (and the stability) of the wormhole solution. While in [85] this is used directly in the variational principle, in [16] it is used to project the transition amplitude given by the EPI on states of definite charge and to fix appropriate boundary conditions. In [16], moreover, in order to get a real stationary point in the EPI, the phase of the field is rotated into an imaginary value (this, en passant, should be related to the problem of choosing a good contour of integration for the determination of the EPI). In both cases, however, the conservation equation leads once again to a change in a relative sign in the Euclidean action (see section 2.5.2), which is fundamental to get a solvable Euclidean classical equation. In effect, it has also been shown that the instantons found by [16] represent nothing but a more general class of wormhole solutions which includes the axion model of [34].

Essentially, the same conditions (a proper a priori choice of the initial sign of the kinetic term in the Euclidean action, the existence of a conservation equation for the gauge field) ensure the existence of the Yang-Mills wormholes. All these models show in addition the presence of a stress energy tensor with at least one negative eigenvalue, and therefore satisfy the condition stated in section 2.2 for the existence of wormhole instantons. This also holds for the nonlinear gravity and conformal scalar field theories.

A lot of these works (with perhaps the exception of [34] and [43]), appear not to pay a detailed attention to the problems connected with the analytic continuation of the equations for the wormhole instantons into the Lorentzian space. As we will show in the next chapter, devoted to the discussion of a new, more general, class of solutions recently found [9], this is actually not a simple task, and might be connected in the end with the problem of giving a correct definition of the Euclidean formalism and of the methods of integration for the EPI. In the same context, we will also show that the set of wormhole solutions found up to now should be easily reproduced without the need of any conservation law and the imposition of the related boundary conditions. It is just in this sense that our solutions might appear, if confirmed, more general.

2.4 The Hawking wormhole

The first known explicit wormhole solution is the Tolman or Hawking one [58]. This is characterized by the conformally flat metric:

$$ds^2 = \left[1 + \frac{b^2}{(x^\mu - x_o^\mu)^2}\right]^2 dx^\mu dx_\mu \tag{2.1}$$

which looks like an asymptotically Euclidean metric with a singularity at the point x_o. Actually, this is a mere coordinate singularity, since the line element is invariant under the transformation:

$$x'^\mu - x'^\mu_o = \frac{b^2 O(x^\mu - x_o^\mu)}{(x^\mu - x_o^\mu)^2} \tag{2.2}$$

such that the region $x^2 < b^2$ has the same geometry as that with $x^2 > b^2$ (O is an orthogonal rotation matrix). Therefore, this metric describes two asymptotically flat regions connected by a neck (see fig. (7, 8)) at the 3-sphere with radius $2b$, known as baby universe.

This metric, as introduced by Hawking, is not a solution of the Einstein equations ($R_{\mu\nu} \neq 0$, though it has $R = 0$). Even for $\Lambda = 0$, however, the gravitational action is not zero, because of the boundary term (see next chapter):

$$S_b = -\frac{1}{8\pi G}\int d^3x\sqrt{h}(K - K_o) = \frac{3\pi b^2}{G} \tag{2.3}$$

where K and K_o are the trace of the extrinsic curvature of the boundary and of the boundary embedded in flat space. Therefore, the e^{-S} factor in the path integral suppresses the effects of such wormholes of size much greater than the Planck length.

2.5 Semiclassical asymptotically flat instantons

2.5.1 The Giddings-Strominger axionic wormhole

The first exact instanton solution that may be interpreted as a wormhole was found by Giddings and Strominger [34], using an axion field minimally coupled to gravity. The Euclidean action is:

$$S = \frac{M_p^2}{16\pi} \int d^4x \sqrt{g}(-R + H^2) + (top. \quad and \quad bound. \quad terms) \tag{2.4}$$

where the 3-form $H \doteq dB$ is the axion strength and satisfies $dH = 0$. The equations of motion are:

$$\begin{cases} G_{\mu\nu} = (3H_{\mu\alpha\beta}H_\nu^{\alpha\beta} - \frac{1}{2}g_{\mu\nu}H_{\alpha\beta\gamma}H^{\alpha\beta\gamma}) \\ \nabla_{[\mu}H_{\alpha\beta\gamma]} = 0 \end{cases} \tag{2.5a, b}$$

One can work in the spherically symmetric ansatz:

$$\begin{cases} dS^2 = dt^2 + a^2 d\Omega_3^2 \\ H_{ijk} = b^2 \epsilon_{ijk} \end{cases} \tag{2.6}$$

(the other components of H all vanish; ϵ_{ijk} is the volume element normalized to integrate to $2\pi^2$ on surfaces of constant a).

The model conserves the axion current H^μ, and we can define an axionic charge flowing down the wormhole:

$$q = M_p^2 \int_\Sigma H = \frac{2\pi^2 b^2}{G} \tag{2.7}$$

if the 3-surface Σ encloses the origin $t = 0$. Grinstein [38] showed that this charge should actually be quantized. The existence of such a charge is essential to obtain the following structure of the time-time component of the Einstein equation (2.5a):

$$\left[\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2}\right] = -\frac{3b^4}{a^6} \tag{2.8}$$

which is seen to be satisfied by:

$$\begin{cases} a^2 = b^2 \cosh 2x \\ t = b \int \sqrt{\cosh 2x}\, dx \end{cases} \tag{2.9}$$

The metric (2.9) is invariant under $x \to -x$, and represents a geometry with two asymptotically flat regions as $|x| \to \infty$, connected by a throat whose cross sections are S^3 and with minimum radius b (at $x = 0$). The S^3 boundary at $x = 0$ has vanishing K (see section 2.6.2) and time components of H: the solution therefore describes a tunneling between an initial 3-surface Σ_i, topologically R^3, to a final surface Σ_f , topologically $R^3 \oplus S^3$ (see fig. (9)). We have the nucleation (annihilation) of a closed baby FLRW universe at its maximum radius (see also next chapter). Note that the Ricci tensor is $R_{\mu\nu} = -6^* H^*_\mu H_\nu$ (where $^*H_\mu = \frac{1}{3!}\epsilon_\mu^{\nu\alpha\beta} H_{\nu\alpha\beta}$ is the Hodge dual of H), which is negative definite, just as required by the theorem stated in [34].

It is critical for the description of a tunnelling process that the fields and their first time derivatives on Σ_i and Σ_f are all real when analytically continued back to the Lorentzian regime. This is obvious for the R^3 regions. On the S^3 portion, the time derivative of the metric vanishes because it is a minimal surface, and the time components of H vanish in the assumed ansatz. The instanton thus appears to obey the right boundary conditions. Actually, on virtue of the current conservation, one can locally substitute the axion tensor field by a pseudoscalar, defined as $H =^* da$. In this case, it is not clear whether the analytical continuation of the corresponding equations of motion can be properly done or not. Giddings and Strominger themselves point out that the result will depend on the order in which one proceeds in the operations of the substitution $H =^* da$ and of the analytical continuation, which do not commute (if one first substitutes H with a, he will find a canonical massless scalar field model and no instanton solutions). We will suggest a way out of this problem in chapter 3. For this instanton, using Einstein equations to eliminate R and integrating over H^2 for $t \in (-\infty + \infty)$, the action becomes (no contribution from gravity boundary terms):

$$S = \frac{3|q|}{8} \tag{2.10}$$

Once again, the nucleation of "babies" large relative to the Planck size is highly suppressed.

A similar analysis is applied to the case of a superstring (SST) model where a massless dilaton field is coupled to the antisymmetric tensor H. The action can be written in the form:

$$S = \int d^4x\sqrt{\hat{g}}\left(-R(\hat{g}) + \frac{1}{2}(\nabla\phi)^2 + e^{\beta\phi}H^2\right) \tag{2.11}$$

where $\hat{g} = ge^{\phi}$. A wormhole solution similar to that previously described apparently exists. However, this action diverges for $\beta > 2/\sqrt{3}$, and the "coupling" $e^{-\beta\phi} \to 0$. Since in SST it is expected (at least for the bosonic string) that $\beta = 2$, the conclusion is that this tunnelling is in fact suppressed.

2.5.2 The massive charged scalar field wormhole

A set of wormhole solutions for gravity minimally coupled to a charged scalar field has been discussed by Lee [85], Kim [75] (for the case of spontaneously broken U(1) symmetry), and by Abbott and Wise [1] and Coleman and Lee [16] (for the case of unbroken symmetry). We will essentially describe the main features of the last approach.

Consider gravity plus a massive charged scalar field $\psi = \frac{1}{\sqrt{2}}fe^{-i\theta}$ with no other interactions and with action:

$$S = \int d^4x\sqrt{g}\left(\frac{-M_p^2}{16\pi}R + \partial_\mu\psi^*\partial^\mu\psi + m^2\psi^*\psi\right) + surf. \quad terms \tag{2.12}$$

Since this theory conserves a charge (see below), before computing any transition amplitude between initial and final states $(|I,F> = |f_{I,F}(x),\theta_{I,F}(x)>)$, it is necessary to project these states into charge density eigenstates with eigenvalues $\rho_{I,F}(x)$, i.e. multiplying them by delta functions of the charge density J_o, and obtaining the following result (initially assume no gravity):

$$< F|\delta(\rho_F - J_o)e^{-H(t_F-t_I)}\delta(\rho_I - J_o)|I > = e^{i\int d^3x(\rho_I\theta_I - \rho_F\theta_F)}\int [df]f[d\theta]e^{-(S+\sigma)} \tag{2.13}$$

with $\sigma = i\int d^3x(\rho_F(x)\theta(x,t_F) - \rho_I(x)\theta(x,t_I))$, and $\theta(x,t_{I,F})$ are the initial and final values of the integration field. This functional has no nontrivial real stationary points (S is real and σ is imaginary); the cure is to look for stationary points at the rotated $\phi = i\theta$. For incidence, even though it has not been stressed by the authors, this choice somehow hides the problem of finding an appropriate contour of integration for the determination of the EPI (see also section 3.3).

The U(1) symmetry of the theory gives the current conservation equation (making everything generally covariant):

$$\partial_\mu J^\mu = 0 \qquad with \quad J^\mu = \sqrt{g} f^2 g^{\mu\nu} \partial_\nu \phi \tag{2.14}$$

The associated conserved charge is $q = \int d\Sigma_\mu J^\mu$, where the integral is over a 3-sphere containing the wormhole mouth.

Using the boundary condition $J^0(x, t_{I,F}) = \rho_{I,F}(x)$, we obtain:

$$S + \sigma = \int d^4x \sqrt{g} \left[\frac{1}{2} f^2 \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} \partial_\mu f \partial^\mu f + \frac{m^2 f^2}{2} \right] + ... \tag{2.15}$$

This model can be shown to be equivalent to the axionic solution of [34] in the particular case $f = const$ (massless Goldstone boson). The main idea is to introduce an extra integration field J^μ in the path integral:

$$\int [df] f [d\theta] e^{-S} = \int [df][d\theta][dJ_\mu] e^{-\int d^4x \sqrt{g}(iJ_\mu \partial_\mu \theta + \frac{1}{2} f^{-2} J_\mu J^\mu + ...)} \tag{2.16}$$

Because of the boundary conditions on J_o, we have:

$$\int d^4x J_\mu \partial^\mu \theta = -\int d^4x \theta \partial^\mu J_\mu + \int d^3x (\rho_F \theta_F - \rho_I \theta_I) \tag{2.17}$$

which gives:

$$< F | \delta(\rho_F - J_o) e^{-H(t_F - t_I)} \delta(\rho_I - J_o) | I >= e^{i \int d^3x (\rho_I \theta_I - \rho_F \theta_F)} \int [df][dJ^\mu][d\theta] \delta(\partial_\mu J^\mu) e^{-S_1} \tag{2.18}$$

with the effective action:

$$S_1 = \int d^4x \sqrt{g} \frac{f^{-2}}{2} J_\mu J^\mu + ... \tag{2.19}$$

This is the same as in the axionic theory, since a vector density is equivalent to a 3-form, $J_\mu = \epsilon_\mu^{\nu\rho\sigma} H_{\nu\rho\sigma}$. Once again it was essential to have a conserved current in the theory.

A set of wormhole solutions is found in the spherically symmetric ansatz $ds^2 = dt^2 + r^2(t) d\Omega_3^2$ and with a homogeneous scalar field. The ϕ, f and Einstein equations are:

$$\begin{cases} q = 2\pi^2 r^3 f^2 \dot{\phi} \\ \ddot{f} + 3\frac{\dot{r}}{r}\dot{f} + f\dot{\phi}^2 - m^2 f = 0 \\ \dot{r}^2 = 1 + \frac{4r^2}{3M_P^2}(\dot{f}^2 - f^2\dot{\phi}^2 - m^2 f^2) \end{cases} \tag{2.20a, b, c}$$

The Giddings-Strominger wormhole is obtained for $f = const$ and (to avoid a cosmological constant) $m = 0$. Since ϕ is fixed by the current conservation, we get:

$$\dot{r}^2 = 1 - \left(\frac{L}{r}\right)^4 \tag{2.21}$$

with

$$L = \left(\frac{q}{M_p f\sqrt{3\pi}}\right)^{\frac{1}{2}} \tag{2.22}$$

being the wormhole size. This is the energy equation for a particle moving in 1 dimension in a repulsive r^{-4} potential. The particle moves in from infinity until it bounces off the barrier at the turning point $r = L$, to return to infinity. This is a geometry of wormhole type.

The other explicit wormhole solutions exist in the "large" and "small" wormhole limits, for a non spontaneously broken U(1) symmetry. The "large" solution is for $r \gg L \gg \frac{1}{m}$, and is obtained in the WKB approximation for the scalar field equation:

$$\frac{d^2A}{dy^2} - k^2A = 0 \tag{2.23}$$

where A stands for $A_\pm = fe^{\pm\phi}$ or $A_{0,1} = f\cosh(\sinh)\phi$ and $dt = r^3dy$, $k = mr^3$. The two independent WKB solutions (fixed by the equation for q and the condition that f^2 vanishes at large t) are:

$$A_\pm = \sqrt{\frac{q}{2\pi^2mr^3}}e^{\pm mt} \tag{2.24}$$

The Einstein equation is then:

$$\dot{r}^2 = 1 - \frac{L}{r} \tag{2.25}$$

with

$$L = \frac{4qm}{3\pi M_p^2} \tag{2.26}$$

being the wormhole size. The "small" solution is for $r \ll \frac{1}{m}$. Neglecting m in the equation of motion, the scalar field equations imply that $\frac{\dot{A}_1}{A_0}$ is a constant. The solution for which f vanishes as r goes to infinity can be chosen to be that of constant A_0 (fixed, by the condition $A_\pm(\mp\infty) = 0$, at $A_0 = \sqrt{3\pi}/8M_p$). Charge conservation says that:

$$\dot{A}_1 = \frac{q}{2\pi^2A_0r^3} \tag{2.27}$$

and the Einstein equation is:

$$\dot{r}^2 = 1 - \left(\frac{L}{r}\right)^4 \tag{2.28}$$

with

$$L = \left(\frac{8q}{3\pi^2 M_p^2}\right)^{\frac{1}{2}} \tag{2.29}$$

being the wormhole size.

A numerical analysis of the general wormhole confirms the results for the charge-size relation (see fig. (10)).

In a certain sense, charge classically plays the same role as nonzero angular momentum does for the dynamics of a particle in an attractive potential; as the latter keeps the particle from falling into the origin, the former's flow down the wormhole keeps it from pinching off.

Moreover, there appears a very nice connection between wormholes and black holes, where the former can simulate the formation and decay of the latter. The idea is to make q mesons (of mass m) collapsing in a black hole of size $L \simeq L_s = \frac{M}{M_p^2}$; if q is large, the mesons are nonrelativistic, $M = mq$ and $q \propto \frac{M_p^2 L}{m}$ (large wormhole case); if q is small, we have to squeeze the system to get relativistic, $M \propto \frac{q}{L}$ and $q \propto (M_p L)^2$ (small wormhole limit).

To estimate the action corresponding to the insertion of one wormhole end into a region of constant background field (f_{bg}), one may imagine to cut a 3-sphere of radius r_f out of the background and replace it by the wormhole cut off at a radius r_f for which $f = f_{bg}$. Applying this patch procedure in the intermediate case $L \ll r \ll m^{-1}$ (where we neglect m), the action is found:

$$S = -q \ln \frac{4f}{\sqrt{\pi} M_p} \tag{2.30}$$

where only the difference between the gravity surface terms for the wormhole and flat space and the σ term contribute (the Hilbert action vanishes because it is scale invariant- see [73]).

A more general and precise computation, describes the insertion of two wormhole ends of opposite charge at the antipodes of a de Sitter sphere. This corresponds to an exact solution of the field equations with cosmological constant $a \propto \Lambda^{-\frac{1}{2}}$, and one has not to worry about the best patching procedure.

To calculate the action difference between the wormhole solution and the de Sitter sphere one can compute the derivative of the action with respect to q and then integrate. For this theory, the infinitesimal change in the Hilbert and Gibbons-Hawking surface term is:

$$d(S+\Sigma) = -\frac{3\pi}{2} M_p^2 r\dot{r}dr + 2\pi^2 r^3 \dot{f} df - q d\phi \big|_I^F \tag{2.31}$$

and there is also the contribution:

$$d\sigma = q d\phi + \phi dq \big|_I^F \tag{2.32}$$

In this case, the initial (I) and final (F) surfaces are the join of the severed wormhole ends. At the join $r, \dot{r}, f, \dot{f}$ and q are all continuous, while $\phi|_I^F \simeq \pi m a$ [16]. Therefore, the action contribution becomes:

$$S \simeq \pi q m a \tag{2.33}$$

Reference [1] considers a very similar analysis for the case of a charged scalar field with a U(1) invariant potential $V = \frac{1}{2}f^2m^2 + \frac{\lambda}{4}f^4 + V_o$. The main assumptions are $m \ll M_p$ (low energy physics) and $\lambda \ll 1$ (perturbative theory), but large enough to dominate the scalar mass term near the wormhole.

Three different regions are investigated: $t < t_n$ (near the wormhole, where $m \simeq 0$ and the dynamics of both gravity and the scalar field are important), $t_n < t < m^{-1}$ (the massless region, where gravity is ignored, $r = t$) and $t > m^{-1}$ (the far region, where gravity is ignored but m is relevant). $t_n \simeq r(0)$ is some measure of the wormhole size. In the third region, the behaviour of the action S depends on the sign of m^2. If the U(1) symmetry is spontaneously broken ($m^2 < 0$), $f \rightarrow \frac{m}{\sqrt{\lambda}}$ exponentially and the action is finite. If the symmetry is not broken ($m^2 > 0$), $f \rightarrow 0$ like $t^{-\frac{3}{2}}$ and the total action S diverges as $S \simeq qmt$. This case is actually shown to be still relevant, since the effect of a wormhole carrying charge q is to insert a finite local U(1) noninvariant term (see section 6.3) in the low energy effective theory.

2.5.3 Axion + Electromagnetic field

Another version of the axionic solution was found by Keay and Laflamme [74] including a 2-index antisymmetric (electromagnetic) tensor field F for monopole configurations on a space with 3-surfaces of topology $S^1 \times S^2$.

The Euclidean action is:

$$S = \int d^4x\sqrt{g}\left(-\frac{R}{16\pi G} + \frac{f^2H^2}{3!} + \frac{e^2F^2}{2!}\right) - \int d^3x\sqrt{h}\frac{K}{8\pi G} \tag{2.34}$$

(f and e are the couplings for the 3 and 2 form fields).

The metric is chosen homogeneous and of the form:

$$ds^2 = \sigma^2(N^2d\tau^2 + a^2dr^2 + b^2d\Omega_2^2) \tag{2.35}$$

where $\sigma^2 = \frac{G}{2\pi}, r \in [0, 2\pi]$. The investigated monopole forms for the fields are:

$$\begin{cases} H = \frac{\sigma Q_a}{4\pi f}\sin\theta dr \wedge d\theta \wedge d\phi \\ F = \frac{\sigma Q_m}{4\pi e}\sin\theta d\theta \wedge d\phi \end{cases} \tag{2.36}$$

(only a magnetic field is present).

Once again, the conservation laws ensuring the constancy of Q_m and Q_a are essential for the existence of the wormhole solutions.

Two solutions are found, one corresponding to a wormhole in the S^2 direction, fixed radius of S^1 and action πQ_a, and another corresponding to a wormhole in the S^1 direction, keeping S^2 constant, and zero action. No solutions have both a and b going simultaneously to infinity, even if including a homogeneous electric field. These solutions can also be seen as dimensional slices of 5-D spaces with horizons.

2.6 Non asymptotically flat instantons

2.6.1 The Gonzales-Diaz wormhole

Gonzales-Diaz [37] showed that it is possible to reproduce the Tolman-Hawking wormhole solution in a pure gravity minisuperspace model with a positive cosmological constant, provided a cutoff in the scale factor is introduced. This cutoff is essentially motivated invoking a sort of indeterminacy principle (see [99]), according to which it would be impossible to measure the position or size of any object with a precision greater than M_p^{-1}.

With the metric $ds^2 = N^2 d\tau^2 + a^2 d\Omega_3^2$, the action is:

$$I = -\frac{1}{16\pi G} \int d\tau N a \left(1 + \frac{\dot{a}^2}{N^2} - a^2 \lambda\right) \tag{2.37}$$

The cutoff is introduced transforming $a^2 \to a^2 - m^2$ (m constant), equivalent to having a minimum radius for the Euclidean 3-sphere ($a > m$); moreover, time is transformed as $d\tau' = (1 - \frac{m^2}{a^2})^{\frac{1}{2}} d\tau$. The equations of motion become, using a conformal time $d\eta = \frac{d\tau}{a}$:

$$\begin{cases} \frac{1}{2}a'^2 + U(a,m) = 0 \\ a'' = -\frac{\partial U}{\partial a} \end{cases} \tag{2.38}$$

with $a' = \frac{da}{d\eta}$ and representing the motion of a particle of zero energy in the potential $U = \frac{1}{2}[m^2(1+m^2\lambda) - (1+2m^2\lambda)a^2 + \lambda a^4]$.

In the $\lambda = 0$ case, the solution is the Tolman-Hawking wormhole of radius m:

$$a = (m^2 + \tau^2)^{\frac{1}{2}} \tag{2.39}$$

The $\lambda > 0$ solutions are:

$$\begin{cases} a = \lambda^{-\frac{1}{2}}(m^2\lambda + \sin^2(\lambda^{\frac{1}{2}}\tau))^{\frac{1}{2}} \\ a = \lambda^{-\frac{1}{2}}(m^2\lambda + \cos^2(\lambda^{\frac{1}{2}}\tau))^{\frac{1}{2}} \end{cases} \tag{2.40}$$

in the region $a_- < a < a_+$ with $a_\pm = (\frac{1+2m^2\lambda \pm 1}{2\lambda})^{\frac{1}{2}}$. The meaning of these solutions and of the corresponding Lorentzian ones is described in the section of the conformally invariant solutions (2.6.5).

2.6.2 Axionic

A more general axionic instanton solution with the inclusion of a cosmological constant and extended to a spacetime of arbitrary dimension D was introduced by Myers [96, 97].

The action is:

$$S = -\frac{1}{16\pi G} \int_M d^D x \sqrt{g}(R - 2\Lambda) - \frac{1}{8\pi G} \int_{\partial M} d^{D-1} x \sqrt{h}(K - K_o) + \int_M d^D x \sqrt{g} A^2 \tag{2.41}$$

A is a D-1 form ($A = dB$) such that $dA = 0$, $A^2 = A_{\mu\nu\rho\ldots}A^{\mu\nu\rho\ldots}$. The equations of motion are:

$$\begin{cases} G_{\mu\nu} + \Lambda g_{\mu\nu} = 16\pi G((D-1)A^2_{\mu\nu} - \frac{1}{2}g_{\mu\nu}A^2) \\ d^*A = 0 \end{cases} \tag{2.42}$$

with $^*A$ the Hodge dual of A and $A^2_{\mu\nu} = A_{\mu\alpha\beta\ldots}A_\nu^{\alpha\beta\ldots}$. $^*A$ is the usual conserved axionic current as in ref. [34]. An explicit wormhole solution is found in the ansatz:

$$\begin{cases} ds^2 = g^{-2}dr^2 + r^2 d\Omega^2_{D-1} \\ A = m\epsilon \end{cases} \tag{2.43}$$

with $\int \epsilon = A_{D-1} = \frac{2\pi^{\frac{D}{2}}}{\Gamma(\frac{D}{2})}$ on a surface of constant r and m is a constant.

The nontrivial Einstein equation is:

$$g^2 = 1 - \hat{\Lambda}r^2 - \frac{\hat{m}^2}{r^{2D-4}} \tag{2.44}$$

with $\hat{\Lambda} = \frac{2\Lambda}{(D-1)(D-2)}$ and $\hat{m}^2 = 16\pi G m^2 (D-3)!$.

For $\Lambda = 0$, the apparent singularity at $r^{2D-4} = r^{2D-4}_{min} = \hat{m}^2$ is actually seen to be a mere coordinate singularity. It can be removed by the appropriate change of coordinates $r^{D-2} = \frac{|\hat{m}|}{2}\left((\frac{u}{u_o})^{D-2} + (\frac{u_o}{u})^{D-2}\right)$ (u_o is an arbitrary scale) which gives:

$$ds^2 = r^2\left(\frac{du^2}{u^2} + d\Omega^2_{D-1}\right) \tag{2.45}$$

This metric is invariant under the transformation $\tilde{u} = \frac{u_o^2}{u}$, and for large u the instanton is asymptotically flat. As $r \to r_{min}$ or $u \to u_o$, the instanton has a neck of minimum radius as shown in fig. (11). In the range $u \in [0, u_o]$ or $\tilde{u} \in [u_o, \infty]$, the geometry is repeated and the instanton connects to a second asymptotically flat region. Dividing the instanton at its minimal neck, this wormhole appears exactly as the D dimensional analogue of the Giddings-Strominger tunneling process ($R^{D-1} \to R^{D-1} \oplus S^{D-1}$), with the same analytic continuation properties (and related problems).

For $\Lambda = 0$, we have $R = -16\pi G A^2$ and the action reduces to $S = 2\int d^D x \sqrt{g} A^2 = \frac{(D-1)A_{D-1}}{16} \cdot (\frac{r_{min}}{l_{Pl}})^{D-2}$ (with $l_{Pl}^{D-2} = G$; the boundary term is zero since it is calculated on a minimal surface).

For $\Lambda > 0$ (the de Sitter case), a valid Euclidean solution ($g^2 > 0$) of eq. (2.44) exists only for $\hat{m}^2 < \frac{1}{(D-1)}\left(\frac{D-2}{(D-1)\hat{\Lambda}}\right)^{D-2}$. The singularities at these end points are once again coordinate ones. Analytic solutions exist only for D=3,4,5. The extrinsic curvature $K = rg\tilde{g}_{ij}$ ($\tilde{g}_{ij}$ is the metric of the unit (D-1)-sphere) is zero on the spheres $r = r_{min}$ ($g^2 = 0$), which are then minimal surfaces. Assuming this is a sufficient condition to analytically continue to Lorentz space, these instantons may represent tunneling between a de Sitter space at the minimum radius and a baby closed FLRW universe at the maximum radius, the birth (death) of a de Sitter and a "baby" from (into) nothing or (patching the end surfaces of several reproductions of the same instanton) tunneling between two de Sitter or two "babies" (see fig. (12, 13)). The instanton action can be calculated exactly if, for instance, we assume that "babies" are small compared to the parent de Sitter ($\hat{m}^2 \ll 1$):

$$S(m=0) = -\frac{\pi^{\frac{D-1}{2}}}{4\Gamma\left(\frac{D-1}{2}\right) G\hat{\Lambda}^{\frac{D-2}{2}}} \tag{2.46}$$

The $\Lambda < 0$ case is not semiclassically interesting since, due to the infinite volume of the solution, the cosmological constant contribution alone makes the action divergent.

2.6.3 Yang-Mills wormholes

A spherically symmetric classical wormhole solution of a SU(2) Yang Mills (magnetic) field coupled to gravity with a cosmological constant was firstly discovered by Hosoya and Ogura [70]. A version with time dependent magnetic and electric fields has been studied by Rey [104], which we will describe here.

The Euclidean action is:

$$S = \int d^4x\sqrt{g}\left(\Lambda - \frac{R}{16\pi G} + \frac{1}{16\pi\alpha_g}F^a_{\mu\nu}F^{a\mu\nu}\right) - \int d\Sigma\sqrt{h}\frac{1}{8\pi G}(K - K_o) \tag{2.47}$$

A SO(4) symmetric wormhole solution is looked for in the ansatz:

$$\begin{cases} ds^2 = \sigma^2(N^2 d\tau^2 + a^2\sigma^a \otimes \sigma^a) \\ F^a_{\mu\nu} = 2\partial_{[\mu}A^a_{\nu]} + \epsilon^{abc}A^b_\mu A^c_\nu \\ A^a_\mu dx^\mu = (1 + H(\tau))\sigma^a \qquad \text{in the } (A_o = 0 \text{ gauge}) \\ d\sigma^a + \epsilon^{abc}\sigma^b \wedge \sigma^c = 0 \end{cases} \tag{2.48}$$

The field strength is, therefore, found:

$$F^a = \frac{1}{2}F^a_{\mu\nu}dx^\mu \wedge dx^\nu = \dot{H}d\tau \wedge \sigma^a + \frac{(H^2-1)}{2}\epsilon^{abc}\sigma^b \wedge \sigma^c \tag{2.49}$$

Explicitly computing $F^a_{\mu\nu}$, in the ansatz (2.48) the action S becomes:

$$S = \frac{1}{2}\int d\tau N a^3 \left[\lambda - \left(\left(\frac{\dot{a}}{Na}\right)^2 + \frac{1}{a^2}\right) + \frac{r_o^2}{a^4}\left((H^2-1)^2 + \left(\frac{\dot{H}a^2}{N}\right)^2\right)\right] \tag{2.50}$$

with $\lambda = \frac{16\pi G^2\Lambda}{9}$ and $r_o^2 = \frac{3\pi}{2\alpha}$. Then we get the equation of motion for H (in the conformal gauge $N = a$):

$$\ddot{H} - 2H(H^2-1) = 0 \tag{2.51}$$

whose first integral is $\dot{H}^2 - (H^2-1)^2 = -E$, and the constraint equation:

$$2(2\lambda\dot{a}^2) + (2\lambda a^2 - 1)^2 = 1 - 4r_o^2 E\lambda \tag{2.52}$$

These equations represent a particle moving in a double well potential $V \propto (2\lambda a^2 - 1)^2$ (see fig. (14)). The classical motion is possible only if $-1 \le -E \le 0$ (H must be bounded) and $0 \le 1 - 4r_o^2 E\lambda \le 1$ (a must be positive), and is qualitatively shown in the phase diagrams (fig. (15)). A periodic behaviour in a is found in the range $R_- \le a \le R_+$, with $R_\pm = \frac{1}{2\lambda}(1 \pm \sqrt{1 - 4r_o^2 E\lambda})$. For $\lambda \to 0$, $R^2_{max} \doteq \sigma^2 R_+^2 \simeq \frac{2G}{3\pi\lambda} \to \infty$ while $R^2_{min} \simeq \frac{EG}{\alpha}$, defining the wormhole neck. H oscillates within $-h_- \le H \le h_-$ with $h_\pm = \sqrt{1 \pm \sqrt{E}}$. The interpretation of such solutions as wormholes follows essentially from a discussion similar to that of Myers [96, 97]. Outside of these ranges solutions still exist but are not interpretable as wormholes. The explicit analytic solutions are given by elliptic integrals, and a discrete set of wormhole solutions is found for appropriate boundary conditions. As a final consideration, we would like to stress once again how much the choice of the sign for the F term in the action and the implied existence of a conserved energy density (E) of the gauge field are critical and make this set of solutions similar to the prototype of [16] and [34].

2.6.4 Yang-Mills + Axion

An improved version of the two previous solutions has been given by Das-Maharana [18], which considered Einstein gravity coupled to a 3-form and a SU(2) field. An instanton solution is shown to exist only for a certain relation between the axion coupling and the string coupling constant.

2.6.5 Conformal scalar field wormholes

Halliwell and Laflamme [43] solved the Einstein equations with a cosmological constant and a conformally coupled scalar field in a minisuperspace context. The action is:

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{g}[(1-8\pi G\xi\phi^2)R - 2\Lambda] + \frac{1}{2}\int d^4x\sqrt{g}(\nabla\phi)^2 \\ -\frac{1}{8\pi G}\int d^3x\sqrt{h}(1-8\pi G\xi\phi^2)K \qquad (2.53)$$

Working in the ansatz $ds^2 = \sigma^2(N^2d\tau^2 + a^2d\Omega_3^2)$, setting $\phi = (2\pi^2\sigma^2)^{-\frac{1}{2}}\frac{\chi}{a}$ and $\lambda = \frac{1}{3}\sigma^2\Lambda$, the field and constraint equations are found:

$$\begin{cases} a'' - a + 2\lambda a^3 = 0 \\ \chi'' - \chi = 0 \\ a'^2 - a^2 + \lambda a^4 = \chi'^2 - \chi^2 \end{cases} \qquad (2.54a,b,c)$$

in the conformal gauge $d\eta = \frac{d\tau}{a}$ and $N = 1$.

The $\Lambda = 0$ solution for a is of the form:

$$a(\eta) = R_o(\sin\alpha\sinh\eta + \sin\beta\cosh\eta) \qquad (2.55)$$

(R_o, α, β constants). If $\sin\alpha < \sin\beta$, a has a minimum and the solution becomes asymptotically flat as $|\eta| \to \infty$. For instance, for $\sin\alpha = 0$ and $\sin\beta = 1$, we have $a = (R_o^2 + \tau^2)^{\frac{1}{2}}$, representing two asymptotically flat regions connected by a Hawking wormhole of radius R_o.

For the case $\Lambda > 0$, the a equation may be rewritten as:

$$\begin{cases} \frac{a'^2}{2} + U(a) = 0 \\ a'' = -\frac{dU}{da} \end{cases} \qquad (2.56)$$

representing the motion of a zero energy particle in the potential $U = \frac{1}{2}(k - a^2 + \lambda a^4)$, with $k = R_o^2(\sin^2\beta - \sin^2\alpha)$. Euclidean solutions can only exist if $U < 0$ somewhere, which gives $4\lambda k < 1$; solutions with $k < 0$ are singular. A phase diagram describes them in fig. (16). Periodic wormhole solutions (with the same interpretation as given by Myers) lie in the region $a_- < a < a_+$ with $a_\pm = (2\lambda)^{-\frac{1}{2}}(1 \pm (1-4\lambda k)^{\frac{1}{2}})^{\frac{1}{2}}$ and have the explicit form:

$$a = (2\lambda)^{-\frac{1}{2}}(1 - (1-4\lambda k)^{\frac{1}{2}}\cos 2\lambda^{\frac{1}{2}}\tau)^{\frac{1}{2}} \qquad (2.57)$$

The corresponding solution of the Lorentzian equations of motion (obtained from eq. (2.54) by turning $\eta \to \pm i\eta$) is (for $\Lambda = 0$):

$$a(\eta) = R_1(\cosh\gamma \sin\eta + \sinh\epsilon \cos\eta) \tag{2.58}$$

(R_1, ϵ, η constants). This gives, for $\epsilon = \gamma = 0$, a Tolman universe bouncing between a singularity and a maximum radius R_1: $a = (R_1^2 - t^2)^{\frac{1}{2}}$. Since $\dot{a}(0) = 0$, it may be joined into the Euclidean corresponding solution (2.55), which then describes an instanton connecting two flat spaces with a spatially disconnected Tolman "baby". A similar result has also been recovered in the more general context described in chapter 3.

The $\Lambda > 0$ case is described by the same energy equation as (2.56), with $U \to -U$ and $k = R_1^2(\sinh^2\gamma + \cosh^2\epsilon)$; the phase plane of solutions is given in fig. (17). When $0 < 4\lambda k < 1$, Lorentzian solutions exist only for $a < a_-$ or $a > a_+$ and are:

$$a = (2\lambda)^{-\frac{1}{2}}(1 \pm (1 - 4\lambda k)^{\frac{1}{2}} \cosh 2\lambda^{\frac{1}{2}} t)^{\frac{1}{2}} \tag{2.59}$$

respectively representing a Tolman universe with maximum size a_- ($-$) and a de Sitter universe with minimum size a_+ ($+$). The instanton describes the tunnelling between these two classically allowed regions. Analysis of the ϕ equation shows that $\phi' \neq 0$ when $a' = 0$, suggesting it is not appropriate thinking of ϕ as also undergoing tunnelling. In this case the role of the conserved quantity is played by the "energy" of the conformal scalar field (given by the right hand side of eq. (2.54c)), which is to be fixed on the boundary.

Unfortunately, the physical significance of this simple model is actually threatened by a serious problem: numerical calculations show that these solutions may have $\phi > 1$, which would give the undesirable result of a negative effective gravitational constant ($\tilde{G} = (1 - \phi^2)^{-1}G$). The hope is that, in a more detailed analysis, including also anisotropies, there exist at least some bounded regions for which $G = const > 0$ (as, for example, suggested by [112]).

2.6.6 Nonlinear gravity

Gravitational instanton solutions in the Einstein theory for a nonlinear Lagrangian with a general polynomial dependence on the scalar curvature and with a cosmological constant (which is conformally equivalent to Einstein gravity coupled with a scalar matter field) have been given by Fukutaka, Ghoroku and Tanaka [30], Coule and Maeda [17] and by Bertolami [6]. We will concentrate on this latter. The effective action is assumed to be:

$$\Gamma = \int d^4x \sqrt{g} F(R) \tag{2.60}$$

with $F(R) = \sum_{m=0}^{n} \lambda_m R^m$, λ_o is a cosmological constant and $\lambda_1 = -k^2 = -\frac{1}{16\pi G}$.

If one studies physics at an energy $M < M_p$, making the conformal transformation $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ (with $\Omega^2 \doteq \frac{F'}{M^2}$) and introducing a scalar field $\phi = k \ln \Omega^2$, the Einstein equation reads:

$$G_{\mu\nu} = \frac{1}{2k^2}(3\nabla_\mu\phi\nabla_\nu\phi - \frac{3}{2}g_{\mu\nu}\nabla_\lambda\phi\nabla^\lambda\phi - g_{\mu\nu}V) \tag{2.61}$$

implemented by the ϕ equation of motion:

$$(\partial_\mu\phi)^2 - \frac{1}{6}\frac{\partial V}{\partial\phi} = 0 \tag{2.62}$$

with $V = \Omega^{-4}(RF' - F)$.

For the case of a quadratic Lagrangian ($n = 2$), we have $V = \Lambda + \alpha e^{-\frac{\phi}{k}} + \beta e^{-\frac{2\phi}{k}}$ ($\Lambda = \frac{M^4}{4\lambda_2}, \alpha = \frac{k^2M^2}{2\lambda_2}, \beta = \frac{k^4}{4\lambda_2} - \lambda_o$). Tracing the Einstein equation and using the theorem stated in [34], instanton solutions exist only for $\lambda_2 < 0$ and/or $\lambda_o > 0$ and sufficiently large to overcome the ϕ kinetic term.

For the large curvature limit, $V \simeq \gamma e^{-(\frac{n-2}{n-1})\frac{\phi}{k}}$ ($\gamma = (n-1)n^{-\frac{n}{n-1}}\lambda_n^{-\frac{1}{n-1}}M^{\frac{2n}{n-1}}$); gravitational instantons may exist only for $\lambda_n < 0$ and n even.

A peculiar set of solutions is found in the usual O(4) ansatz, in which the equations of motion become:

$$\begin{cases} k^2(\dot{a}^2 - 1) = \frac{1}{2}\dot{\phi}^2 a^2 - \frac{1}{6}Va^2 \\ \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{1}{6}\frac{\partial V}{\partial\phi} = 0 \end{cases} \tag{2.63}$$

with the asymptotic conditions: $\dot{a}^2 = 1 + g(a)$ (for $a \gg R_o$, with R_o the instanton radius) and $\dot{a} \simeq 1 + \delta a^\epsilon$ (for a small). An additional assumption is $\dot{\phi} = \frac{A}{a^q}$ (q, ϵ are integers, A and δ constants).

For the quadratic theory two solutions ($q = 2$, $\epsilon = -1$) are found, corresponding to a gravitational instanton connecting a small closed universe of Planck radius with an asymptotically de Sitter universe (or, for $\Lambda = 0$, flat space).

In the large curvature limit, using the same ansatz, a solution is found for $q = 2$, describing a Planck sized instanton connecting two asymptotically flat regions. Once again, no detailed study of the analytical continuation procedure is tackled.

2.6.7 Wormholes in nonlinear gravity with scalars and axions

Another interesting set of wormhole instanton solutions which generalizes the works by [6, 30, 34, 107] has been recently considered by Coule and Maeda [17]. The idea is to consider a theory for an antisymmetric tensor (axion) field H coupled to a scalar field ψ with arbitrary potential U. The Euclidean action is given by:

$$S = -\int d^4x\sqrt{g}\left(\frac{R}{2k^2} - \frac{1}{2}(\nabla\psi)^2 - U(\psi) - g^2 e^{\beta\psi} H^2\right) \tag{2.64}$$

where $k^2 = \frac{1}{16\pi G}$, g is the Peccei-Quinn scale coupling and β an arbitrary coupling whose value will turn out to be critical for the existence of wormhole solutions. The field equations which one recovers from this action are:

$$\begin{cases} G_{\mu\nu} = k^2\left(\nabla_\mu\psi\nabla_\nu\psi - \frac{1}{2}g_{\mu\nu}(\nabla\psi)^2 - U(\psi)g_{\mu\nu} + 6g^2e^{\beta\psi}(H_{\mu\alpha\beta}H_\nu^{\alpha\beta} - \frac{1}{6}g_{\mu\nu}H^2)\right) \\ \nabla^2\psi - \frac{\partial U}{\partial\psi} - \beta g^2 e^{\beta\psi}H^2 = 0 \end{cases} \tag{2.65}$$

The axion current H^μ is conserved as usual, and it defines a quantized charge n (see section 2.5.1). In the minisuperspace ansatz:

$$\begin{cases} ds^2 = dt^2 + a^2 d\Omega_3^2 \\ H_{ijk} = \frac{n}{g^2a^3}\epsilon_{ijk} \end{cases} \tag{2.66}$$

the Hamiltonian constraint turns out as:

$$\left(\frac{\dot{a}}{a}\right)^2 - \frac{1}{a^2} = \frac{k^2}{3}\left[\frac{1}{2}\dot{\psi}^2 - U - 6e^{\beta\psi}\frac{n^2}{g^2a^6}\right] \tag{2.67}$$

while the ψ equation of motion is:

$$\ddot{\psi} + 3\frac{\dot{a}}{a}\dot{\psi} - \frac{\partial U}{\partial\psi} - 6\beta e^{\beta\psi}\frac{U'}{g^2a^6} = 0 \tag{2.68}$$

The analysis is restricted to the case where $\frac{\partial U}{\partial \psi} \simeq 0$. Adopting a new time variable τ defined by $d\tau = a^{-3}dt$, the general solution of eq. (2.68) is:

$$e^{-\beta\psi} = \begin{cases} K_o^{-1}\cos^2[(\frac{1}{2}K_oD)^{1/2}\beta\tau] \\ K_o^{-1}D\beta^2\tau^2 \\ K_o^{-1}\sinh^2[(\frac{1}{2}K_oD)^{1/2}\beta\tau] \end{cases} \tag{2.69}$$

depending on the signature of a first integration constant. K_o is a positive arbitrary integration constant and $D \doteq 6n^2/g^2$. Substituting this solution into eq. (2.67), one finds:

$$\left(\frac{\dot{a}}{a}\right)^2 - a^4 + \frac{1}{3}k^2U_oa^6 = \frac{1}{3}k^2K_oD\begin{cases} -1 \\ 0 \\ 1 \end{cases} \tag{2.70}$$

where $U_o \doteq U(\psi) \simeq const.$. A wormhole solution for eq. (2.70) exists in the first (−1) case.

For $U_o = 0$, it is given by:

$$a = \frac{a_o}{\sqrt{\cos 2a_o^2\tau}} \tag{2.71}$$

and the scale factor at $\tau = 0$ is $a_o^2 = k(\frac{1}{3}K_oD)^{1/2}$. This represents a wormhole with minimum radius a_o at $\tau = 0$ which connects two flat regions ($a = t$) as $|t| \to \infty$ (see fig. (18a)). Using eq. (2.71) in eq. (2.69), one can write:

$$\exp\left(-\frac{1}{2}\beta\psi\right) = \exp\left(-\frac{1}{2}\beta\psi_o\right)\cos\left[\frac{\beta}{\beta_{cr}}\left[\cos^{-1}\left(\frac{a_o^2}{a^2}\right)\right]\right] \tag{2.72}$$

where $\beta_{cr} = 2\sqrt{\frac{2}{3}}k$ and $\psi_o = \beta^{-1}\ln K_o$ is the value of ψ at $\tau = 0$. As already discussed by [34], if $\beta > \beta_{cr}$, the field ψ diverges before a reaches the asymptotic region, and there cannot be a regular wormhole solution.

For $U_o \neq 0$ and $a_o \ll a_{DS} = \sqrt{\frac{3}{k^2U_o}}$, the solution is a wormhole which connects two asymptotically de Sitter spaces with radius a_{DS} (see fig. (18b)).

Similar solutions are then shown to exist in the cases of some generalized Einstein theories for gravity. For instance, a higher derivative gravity minimally coupled to an axion has the action:

$$S = -\int d^4x\sqrt{g}\left(\frac{1}{2k^2}f(R) - g^2H^2\right) \tag{2.73}$$

with $f(R)$ an arbitrary function in the scalar curvature R. This action can be easily shown to be conformally ($\tilde{g}_{\mu\nu} = e^{\frac{2k\psi}{\sqrt{6}}}g_{\mu\nu}$) equivalent to that of eq. (2.64), with a potential $U = \frac{\exp\left(-2\sqrt{\frac{2}{3}}k\psi\right)}{2k^2} \cdot \left(R\frac{\partial f}{\partial R} - f\right)$ (see also [6]), and $\beta = \sqrt{\frac{2}{3}}k < \beta_{cr}$. Then wormhole solutions exist provided $\frac{\partial U}{\partial \psi} \simeq 0$. In particular, for $f = R + \alpha R^2$, U approaches $U_o = \frac{1}{8\alpha k^2}$, and if $\alpha \gg \left(\frac{|n|k}{12\sqrt{2}g}\right)\exp\left(\frac{\psi_o}{2}\right)$ (i.e., $a_o \ll a_{DS}$), the wormhole connects two asymptotically de Sitter spaces.

Also a theory for a scalar field conformally coupled with gravity is shown to be conformally equivalent (with an additional appropriate redefinition of the scalar field) to the theory described by eq. (2.64). The new coupling β is still less than β_{cr} and wormhole solutions can exist. Similar results hold for a Jordan-Brans-Dicke theory and for an induced gravity theory.

The extension of eq. (2.64) to a theory containing a lot of scalar fields, described by the action:

$$S = -\int d^4x\sqrt{g}\left(\frac{R}{2k^2} - \frac{1}{2}\sum_{i=1}^{n}(\nabla\phi_i)^2 - g^2\exp(\sum_i \beta_i\phi_i)H^2\right) \tag{2.74}$$

is equivalent to that of eq. (2.64) if one defines $\psi_1 = (\sum_i \beta_i^2)^{-1/2}\sum_i \beta_i\phi_i$ and $\beta = (\sum_i \beta_i^2)^{1/2}$. Thus, the constraint $\beta < \beta_{cr}$ says that the chance of having wormhole solutions decreases with the number of the scalar fields.

Wormhole solutions are then also found for the case of a non minimally coupled scalar in an effective theory derived form string theory.

2.6.8 Massive charged scalar field wormholes

Essentially analyzing the same U(1) scalar field model introduced by Lee [85] and many others, Midorikawa [91] numerically found a new double periodic wormhole solution if the potential $V(\phi)$ has a local maximum at a finite value of the scalar field.

The equations of motion are the same as those described in section (2.5.2); the form of the potential studied is $V = v^2H^2[h(\frac{f^2}{v^2} - \alpha)(\frac{f^2}{v^2} - \beta)\frac{f^2}{v^2} + c]$, with $H = \frac{8\pi G V(v)}{3}$ (Hubble constant of the de Sitter space) and $f = v$ at stationary points (h, α, β and c are arbitrary constants). The boundary conditions are chosen to be $\dot{r} = \dot{f} = 0$ and $f = v$ at $\tau = 0$.

The numerical analysis shows the existence of single period wormhole solutions connecting two universes of the same size, and confirms (see [1]) that also non spontaneously broken symmetry wormholes, besides the Goldstone boson one (for $f = v = const$), exist.

A new wormhole solution of double period is found for V as in fig. (19), and it connects two universes of different sizes. To see this, one first notes that the maximum size of the scale factor is determined (at first order in r from eq. (2.20c), with $\dot{r} = \dot{f} = 0$) as $r_{max} = (\frac{3}{8\pi GV})^{\frac{1}{2}}$. The scalar field ϕ behaves like a particle moving in the potential $-V$.

In the first "cycle", the scalar field ϕ stays near v and $r_{max} \simeq \frac{1}{H}$. Since this is not a stable point, then ϕ rolls slowly towards the local minimum of $-V$, and r_{max} in the second cycle gets smaller. Finally, it goes back to v (see fig. (20)). After quantum tunnelling, ϕ rolls down from the top of $+V$, according to the Lorentzian field equations, determining an exponential inflationary expansion. This double periodic solution implies the creation of a "hot" universe with a large cosmological constant from a "cold" one with small cosmological constant and viceversa.

The existence of universes with different Λ's may be cosmologically relevant and useful for a universe like ours to have undergone its evolution. The idea is that, even if the small Λ is shifted to zero by wormhole effects (see chapter 4), the large Λ may stay finite. The Λ in the "hot" universe, then, may be driven to zero by the "cold" universes lying outside our universe.

2.6.9 Axion + Scalar field wormholes and expanding universes

A new and very interesting type of gravitational instanton, in a theory of a scalar (ϕ) and an axion field (H), has been described by Lavrelashvili, Rubakov and Tinyakov [84] and Rubakov and Tinyakov [107]; its analytical continuation is a closed expanding universe born at minimal radius and then undergoing an inflationary phase. Its cosmological relevance is of clear evidence: one has the possibility of describing the self creation of universes which are very similar to ours form the use of wormholes produced at very early epochs. It is also interesting to note the parallelism to our new set of solutions, described in chapter 3, which, on the other hand, describe the birth of closed collapsing baby universes.

The action considered is:

$$S = \int d^4x\sqrt{g}\left(-\frac{R}{2\varkappa} + \frac{1}{2}(\partial_\mu\phi)^2 + V(\phi) + \frac{1}{6}H^2\right) \tag{2.75}$$

($\varkappa = 8\pi G$) with the same essential assumptions as in [34] for the sign choice in front of the strength of H, which is defined as in [34]. Once again, the existence of a conserved (axion) charge is determinant for the existence of the wormhole. An instanton solution is looked for in the Euclidean ansatz:

$$\begin{cases} ds^2 = d\tau^2 + a^2 d\Omega_3^2 \\ \phi = \phi(\tau) \\ H_{ijk} = N\epsilon_{ijk} \end{cases} \tag{2.76}$$

N is a constant and the other H components vanish. The equations of motion are:

$$\begin{cases} \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = V' \\ \dot{a}^2 - 1 = \frac{\varkappa}{3}[a^2(\frac{1}{2}\dot{\phi}^2 - V) - \frac{N^2}{a^4}] \end{cases} \tag{2.77a, b}$$

(a prime denotes a derivative with respect to ϕ) and the boundary conditions are taken of the form:

$$\begin{cases} \phi(\tau \to \infty) \to \phi_\infty & \text{and } \phi(0) = \phi_o \\ a(\tau \to \infty) \to \tau & \text{and } a(0) = a_o \\ \dot{\phi}(0) = 0 & \text{and } \dot{a}(0) = 0 \end{cases} \tag{2.78}$$

The behaviour of a at infinity is that proper to an asymptotically flat wormhole. The conditions at $\tau = 0$ are to ensure the analytical continuation to the Lorentzian sector. At small τ we have $a(\tau) \simeq a_o + \frac{1}{2}\alpha\tau^2 + O(\tau^3)$, with $\alpha = \ddot{a}(0)$; analytic continuation of the instanton to the Lorentzian signature ($\tau = it$) clearly leads to the creation of an expanding universe if $\alpha < 0$. The corresponding Euclidean geometry is shown in fig. (21b).

The Giddings-Strominger instanton is recovered for $\phi = const, V = const$, and $\alpha > 0$ (see fig. (21a)).

The new solution exists for the nontrivial potential of fig. (22a). Eq. (2.77a) describes the classical motion of a particle in a potential $-V$ under the friction force $-3\frac{\dot{a}}{a}\dot{\phi}$ (see fig. (22b)). The asymptotic conditions on the scalar field ($\phi = \phi_\infty$ and $V(\phi_\infty) = 0$) are appropriate for eq. (2.77b) to be consistent with the asymptotic behaviour $a \propto \tau$. The position of ϕ_o is chosen such that (at $\tau = 0$) the "particle" starts moving towards ϕ_∞. At $\tau = 0$, eq. (2.77b) takes the form:

$$1 = \frac{x}{3}\left(a_o^2 V(\phi_o) + \frac{N^2}{a_o^4}\right) \tag{2.79}$$

This admits two positive solutions for a_o, provided that $\frac{2}{N} > V(\phi_o)x^{\frac{3}{2}}$. The largest one ($a_o^1$) corresponds to the initial radius of the expanding universe case. In fact, this obeys $(a_o^1)^2 > \sqrt{x}N$ and, combining the derivative of eq. (2.77b) with eq. (2.77a), one finds:

$$\ddot{a}(0) = -\frac{1}{a_o^1}\left(1 - \frac{xN^2}{(a_o^1)^4}\right) < 0 \tag{2.80}$$

As τ increases from zero, the friction coefficient becomes negative, so the "particle" can gain enough energy to reach the point ϕ_∞. Since the last term in eq. (2.77b) is singular at $a = 0$, the decrease of a must stop at some $\bar{\tau}$ where $\dot{a}$ changes sign. It can be seen, in particular, that the motion proceeds in two stages. At first, ϕ quickly rolls towards ϕ_∞ with a staying approximately at a_o; then, ϕ slowly reaches ϕ_∞ while a evolves as in the Giddings-Strominger instanton. This can be easily seen through the numerical solution provided by [84] for a particular choice of the potential V and which is shown in fig. (23).

The analytic continuation (at $\tau = 0$) of this instanton is the solution to the field equations:

$$\begin{cases} \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = -V' \\ \dot{a}^2 + 1 = \frac{x}{3}[a^2(\frac{1}{2}\dot{\phi}^2 + V) + \frac{N^2}{a^4}] \end{cases} \tag{2.81}$$

with the same initial conditions as in eq. (2.77). Eq. (2.81) clearly describes an expanding universe with the scalar field oscillating in the potential $V(\phi)$ around the false vacuum ϕ_1. These oscillations are damped because of the expansion of the universe (the friction is now positive); therefore, since also the H contribution to the energy-momentum tensor decreases as a^{-4} (see eq. (2.77b)), the universe rapidly enters an inflationary regime dominated by the false vacuum energy $V(\phi_1)$. The false vacuum has a chance to decay in the standard mechanism [12]. After the decay, the new universe can again lead to the creation of other new universes with the aid of the same instanton (see fig. (24)).

2.7 End point solutions

A pure gravity asymptotically flat instanton solution in a Wilsonian effective Lagrangian theory, with all fluctuations and topological features at distances shorter than the energy cutoff E integrated, was found by Polchinski [100]. The Ricci tensor is supposed to have a cutoff $||R^\nu_\mu|| \leq E^2$, where $||..||$ is the magnitude of the maximum eigenvalue. In this way one has a bounded region of functional integration. A wormhole solution exists corresponding to the endpoint condition $||R^\nu_\mu|| = E^2$.

2.8 Quantum solutions

A more general approach in which wormholes are regarded as full quantum solutions of the WdW equation has ben recently proposed, in a minisuperspace context, by Hawking and Page [66] and by Campbell and Garay [8]. The importance of this class of solutions is evident not only for the construction of a more fundamental theory about topological fluctuations ($\Lambda = 0$ model, third quantization etc.), but also to finally provide a mechanism for the black hole "evaporation" suggested by Hawking [58]: classical wormholes are few, appear to exist only for some specific matter content and are all Planck sized. The final goal should be, in fact, to look for a more general class of solutions: we do not think that this task has actually been achieved up to now.

The boundary conditions required for the wave function (Ψ_{wh}) solution of the WdW equation in order to represent an asymptotically flat wormhole are the following:

the 4-geometry is non singular even when the 3-geometry degenerates$\rightarrow$ Ψ_{wh} should be regular or go to zero as a power of the scale factor a.

the 4-geometry is asymptotically Euclidean when the 3-geometry becomes infinite (no gravitational excitations)$\rightarrow$ Ψ_{wh} should be exponentially damped for large a.

Ψ_{wh} is given by a path integral over all asymptotically Euclidean geometries and matter fields taking specific values on a 3-surface S (the mouth of the wormhole). The wormhole ground state is found with the further restriction that matter fields are asymptotically gauge equivalent to zero. Excited states are found introducing sources at infinity for the matter fields; they can be interpreted as saying that there are particles passing through the wormhole.

2.8.1 FLRW

In the ansatz $ds^2 = \sigma^2(N^2dt^2 + a^2d\Omega_3^2)$, the WdW equation for a minimally coupled massless scalar field ϕ is separable and, for $\Psi = c(a)e^{ik\phi}$, is:

$$\frac{d^2c}{da^2} + \frac{1}{a}\frac{dc}{da} + \left(\frac{k^2}{a^2} - a^2\right)c = 0 \tag{2.82}$$

This has the two independent solutions $J_{\pm\frac{ik}{2}}(\frac{ia^2}{2})$. They are eigenstates of the operator $\Pi_\phi = -i\frac{\partial}{\partial\phi}$ with eigenvalue k, and carry a conserved charge $q = i\int_S \phi_{;\mu}d\sigma^\mu = 2\pi^2 k$. This continuous set of solutions oscillates for $0 < a < k^{\frac{1}{2}}$, and can be interpreted as corresponding to classical Lorentzian Friedmann solutions with a scalar flux q, bouncing between a singularity and a sphere of maximum radius. For $a > k^{\frac{1}{2}}$, Ψ decreases like $e^{-\frac{a^2}{2}}$; the lower bound on a and the existence of the scalar flux q are determinant in indicating it as a wormhole solution connecting two asymptotically flat regions.

Another class of solutions (with discrete spectrum), regular at $a = 0$ and damped at large radius can be found if one substitutes $x = a\sinh\phi, y = a\cosh\phi$, for which the WdW equation becomes:

$$\left(\frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial x^2} - y^2 + x^2\right)\Psi = 0 \tag{2.83}$$

whose solutions are just products of harmonic oscillator wave functions with the same energy:

$$\Psi = \Psi_n(x)\Psi_n(y) \tag{2.84}$$

and $\Psi_n(x) = H_n(x)e^{-\frac{x^2}{2}}$ (H_n are Hermite polynomials). In general, it is possible to relate the two kinds of solutions. In fact, the Killing vector for the WdW equation, $\frac{\partial}{\partial\phi} = y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$, can be rewritten, introducing the harmonic creation and annihilation operators $a, a^\dagger$, as:

$$\frac{\partial}{\partial\phi} = a_x a_y - a_x^\dagger a_y^\dagger \tag{2.85}$$

Therefore, if we expand the $\frac{\partial}{\partial\phi}$ eigenstates $|k>$ in terms of harmonic eigenstates $|n>$:

$$|k>=\sum c_n(k)|n> \tag{2.86}$$

we find the recursive relation $ikc_n = (n+1)c_{n+1} - nc_{n-1}$, which can be solved iteratively in terms of hypergeometric functions; this tells that $|k>$ eigenstates can be seen as superpositions of an infinite number of regular harmonic solutions, which therefore satisfy all the asymptotic conditions required for the wormholes. A similar result is found for the case of a conformally invariant field ϕ. It is also shown that one can find explicit solutions, which are regular everywhere and exponentially damped at $a = \infty$, for the case of a free massive minimally coupled scalar in minisuperspace geometry. The method is that of recursive approximations around a semiclassical WKB solution of the WdW equation. This set of solutions should be somehow related to the semiclassical results found in the recent paper [9] described in chapter 3.

2.8.2 Kantowski-Sachs

An analysis very close to the FLRW model of Hawking and Page is found in Campbell and Garay [8] for a minimally coupled scalar field in the ansatz $ds^2 = \sigma^2(N^2d\tau^2 + a^2dr^2 + b^2d\Omega_2^2)$. Here we will state the main results of the work.

The interest in studying such a spacetime is that the interior of a Schwartzschild black hole has the same metric [81].

A more general form for the operator ordering in the WdW equation is considered. Two kinds of wormhole solutions are looked for, with the asymptotic behaviour $R^3 \times S^1$ $(a \to a_o, b \to \tau)$ and $R^2 \times S^2$ $(a \to \tau, b \to b_o)$, respectively described by the asymptotic ground state wave functions, $\Psi \simeq e^{-ab}$ and $\Psi \simeq e^{-\frac{a^2}{4}}$.

An explicit continuous set of solutions is given by:

$$\Psi_{k_o,\theta_o} = a^{ik\cos\theta_o}e^{ik\sin\theta_o\phi}K_{ik}(ab) \tag{2.87}$$

$(k, \theta_o$ arbitrary constants, K is a Bessel function). These are not regular in the origin of the coordinates. Their asymptotic behaviour is appropriate to that of a ground state wormhole only for $b \to \infty$ ($R^3 \times S^1$); they are eigenfunctions of Π_ϕ and $a\Pi_a$. In the latter case, the eigenvalue is zero ($a = a_o$), and for $a_o b > k$ the classical wormhole [74] is reproduced. If the eigenvalue is not zero, wormhole solutions may exist only for a $R^3 \times S^1$ case.

Another set of regular solutions , related to the previous ones by a Fourier transform, is:

$$\Psi_{\lambda_o \theta_o} = \exp\left(-ab\cosh(\cos\theta_o \log\frac{a}{r_o} + \phi\sin\theta_o + \lambda_o)\right) \tag{2.88}$$

λ_o, θ_o, r_o are constants; λ_o labels the excitation of the wormhole state. For $R^3 \times S^1$ solutions, $\lambda_o = 0$ gives a continuous set of degenerate ground states, while $\lambda_o \neq 0$ gives excited states. For $R^2 \times S^2$ solutions, $\lambda_o = \theta_o = 0$ is the only state (ground state) and it corresponds to pure gravity. A discrete spectrum (in λ_o) of (harmonic oscillators) wormholes is found as in [66]. It is easy to see, however, that both this and the previous set of quantum wormhole solutions still appear just an extension of some already known semiclassical solutions, in particular those whose existence is strictly linked with the existence of a conserved charge.

3 A new set of solutions

The main goal of this chapter is to describe a new set of recently found [9] semiclassical wormhole solutions. We have essentially considered asymptotically flat gravitational instantons (AFGIs), relevant for the vacuum to vacuum tunneling in the flat spacetime. The solutions that have been considered so far are usually presented in the form that seems to suggest that wormhole spacetimes and the field theoretical models in which they occur are rather exotic. By contrast, we have demonstrated that wormholes are no more exotic than Friedmann solutions.

The method that we used is very simple: wormholes are analytic continuation of closed expanding solutions in General Relativity (and its simple modifications). First, in section 3.1, I will consider the general case with bulk matter sources, and work out the simple conclusion: wormholes are driven by the matter sources that obey the strong energy condition. Thus, for every solution of that class in classical cosmology there is a wormhole solution. They may be considered more explicitly in the case of an isotropic and homogeneous geometry, and this is shown in section 3.2. Originally introduced solutions of Hawking, and Giddings and Strominger are recovered there, and many similar ones found. In section 3.3, field theoretical models that lead to wormhole solutions are introduced. It came up that, for a fairly general case of a spatially homogeneous minimally coupled complex valued scalar field, this is not possible if the analytic continuation is done in the usual way. Instead, one should perform an asymmetric transition to the Euclidean regime: $N = \pm iN_e$ in the gravitational sector, but $N = \mp iN_e$ in the matter field sector, where N, N_e are lapse functions in the two regimes. This is just the trick that has been introduced by Linde [86], essentially in a complementary case, to obtain the "tunneling" amplitude as the amplitude for the quantum creation of an inflationary universe. In section 3.4 I will explicitly exhibit a number of wormhole solutions driven by a scalar field. Unlike the cases previously considered, all new solutions given here have a nontrivial potential term. Finally, in section 3.5, I will explore the fact that spacetime wormholes in fact have a $S^1 \times S^3$ topology (see also [92]). The periodicity in the Euclidean time may be interpreted as that wormholes have a finite temperature, inversely proportional to their sizes. If correct, this feature may be of major importance for a number of issues in the study of wormholes, some of which I briefly mention. Section 3.6 summarizes our main conclusions and I discuss different prescriptions to ensure the existence of wormhole solutions and the prospects within the quantum gravity.

3.1 The existence of wormholes driven by the bulk matter.

The basic idea is as follows. Consider the general case of an anisotropic and inhomogeneous spacetime with bulk matter sources, characterized by some stress tensor $T_{\mu\nu}$. This is the typical case considered in General Relativity (GR). Let γ_{ab} be the intrinsic geometry on a family of spacelike hypersurfaces, N the lapse function, K_{ab} and K corresponding extrinsic curvature and its trace, σ_{ab} is the shear, and R_3 the scalar curvature for the geometry γ_{ab} [120]. By the usual substitution that changes the signature of the metric, $N = iN_e$ (the sign in the front of i is not important here), one may evaluate these quantities also for the Euclidean signature spacetime. We will distinguish that regime by the subscript e. Then, an asymptotically flat Euclidean spacetime is characterized by the following asymptotic behaviour:

$$ds_e^2 \to dt_e^2 + t_e^2 d\Omega_3^2 \quad , \tag{3.1}$$

or, in more detail,

$$N_e d\tau_e \to dt_e \quad , \tag{3.2a}$$

$$\gamma_{eab} \to t_e^2 \Omega_{ab} \quad , \tag{3.2b}$$

$$K_{eab} \to it_e \Omega_{ab} \quad , \tag{3.2c}$$

$$K_e \to \frac{3i}{t_e} \quad , \tag{3.2d}$$

$$\sigma_{eab} \to 0 \quad , \tag{3.2e}$$

$$R_3 \to R_3(S_3) \sim \frac{1}{t_e^2} \quad . \tag{3.2f}$$

For bulk matter the stress tensor does not change when one changes the signature of spacetime (since it does not include any time derivative). Thus, the Euclidean version of the constraint equation is (see, e.g. [120]),

$$K_e^2 = \frac{3}{2}R_3 + \frac{3}{2}\sigma_e^2 - 24\pi G T_0^0 \quad . \tag{3.3}$$

From eqs. (3.2) and (3.3) one can easily see that for AFGI to exist, $\rho\ (= T_0^0)$ must decay in the asymptotic regime faster than t_e^{-2}. Moreover, the energy conservation law in the asymptotic regime is:

$$\frac{\dot{\rho}}{N_e} = -3K_e\gamma\rho \quad , \qquad \gamma \equiv 1 + p/\rho \quad , \tag{3.4}$$

which admits the well known solution (for $\gamma = const$):

$$\rho = \frac{\rho_o}{a^{3\gamma}} \tag{3.5}$$

Then, one can conclude that asymptotically $\gamma > 2/3$, or $\rho + 3p > 0$. This is the strong energy condition, usually automatically assumed for the bulk matter. Although one needs here for this condition to hold only in the asymptotic regime, normally one expects it to be true everywhere. Thus, for every classical solution of the Einstein equations with the closed spatial geometry and with the bulk matter source that obeys the strong energy condition, its continuation to the Euclidean domain can be shown to represent a wormhole. This is because with these conditions the GR solution has a maximal radius (see the next section). The wormhole solution is just the analytic continuation of that solution above the maximal radius, see fig. (25).

Several comments should be made here. Firstly, it was crucial for this statement that the stress tensor was the same in both the Lorentzian and the Euclidean regimes. This is automatic for the bulk matter, but not in the field theory (see section 3.3). Secondly, the strong energy condition that assures the existence of wormholes is exactly complementary to the condition for the existence of an inflationary phase. In what follows we will see that this complementarity carries on in many places.

Finally, what is the significance of wormhole solutions obtained in this way? Certainly we are not trying to construct the Euclidean theory with the bulk matter driven tunneling solutions. As they stand, these solutions are not yet relevant for the quantum theory. They are however important geometrically, just as any solution in the standard GR. It has been observed in [34, 59] that their wormholes have analytic continuation to the collapsing Lorentzian universes. What is proposed to do is to look at this from the other end, and to construct wormhole geometries from the known Lorentzian signature solutions in GR. In this way we obtain infinitely many wormholes. Since gravity sees the stress tensor regardless of its particular realization, we will continue with this inversion, and try to construct the field theoretical models that have the same wormhole solutions as the bulk matter with the same equation of state. Such models should be relevant for Quantum Gravity (QG). As I will show in section 3.4, the complete construction will require further specification of the analytic continuation. To see all this explicitly, from now on I will work under the restriction to the FRLW geometries.

3.2 FLRW wormholes with the bulk matter.

Consider now the Friedmann constraint equation for a closed FRLW universe (with no cosmological constant) together with the conservation equation, ($\mu^2 \equiv 3/(8\pi G)$),

$$H^2 = \frac{\rho}{\mu^2} - \frac{1}{a^2} \quad , \qquad \dot{\rho} = -3H\gamma\rho \quad . \tag{3.6}$$

Their Euclidean signature versions are,

$$H_e^2 = \frac{1}{a^2} - \frac{\rho}{\mu^2} \quad , \qquad \dot{\rho} = -3H_e\gamma\rho \quad . \tag{3.7}$$

The corresponding line elements are related as,

$$ds^2 = -dt^2 + a^2(t)d\Omega_3^2 \ \rightarrow \ ds_e^2 = dt_e^2 + a^2(t_e)d\Omega_3^2 \quad , \tag{3.8}$$

where $H \equiv \dot{a}(t)/a(t)$, $H_e \equiv \dot{a}(t_e)/a(t_e)$, $\dot{\rho}_e = \frac{d\rho}{dt_e}$ and $d\Omega_3^2$ is the line element on the three-sphere.

One knows from the standard GR that there are many closed universe solutions with some maximum radius a_0, covering the interval $[0, a_0]$. If such solution obeys eq. (3.6), its analytic continuation for $a > a_0$ obeys eq. (3.7). Thus, one can define the AFGI as an analytic continuation of the closed expanding universe with the asymptotics $a^2(t_e) \to t_e^2$ at $t_e \to \pm\infty$. The two solutions together cover $a \in [0, \infty]$, have the same equation of state, and obey eq. 's (3.6) and (3.7) respectively. The imposed asymptotic condition assures that the wormhole joins the two nearly flat sections.

From eq. (3.7), it can be seen immediately that AFGIs exist only for fluids with energy density ρ that decays faster than a^{-2}. This implies $\gamma > 2/3$, or $\rho + 3p > 0$, $a''(t_e) > 0$.

Common solutions in classical cosmology are with $\gamma = const.$, and in this case solutions may be written down explicitly. The energy density decays as in eq. (3.5). Putting this formula inside the Lorentzian constraint equation and using the trick of defining a new time variable as:

$$dt = N d\tau = a^{(4-3\gamma)/2} d\tau \tag{3.9}$$

one finds:

$$\frac{da}{Aa^{2-3\gamma} - 1} = d\tau \tag{3.10}$$

(where I have denoted $\frac{\rho_o}{\mu^2} = A$, for sake of clarity). This can be easily integrated and gives:

$$a(\tau) = \left[A - \left[\left(\frac{2-3\gamma}{2}\right)\tau + \left(A - a_o^{3\gamma-2}\right)^{\frac{1}{2}}\right]^2\right]^{1/(3\gamma-2)} \tag{3.11}$$

(the constant of integration has been put equal to zero, ensuring that the Lorentzian evolution starts at the throat of the wormhole -a_o- at $t = 0$). To fix A, one has to impose that we have a stationary value of the scale factor at $\tau = 0$. Therefore, one finds $A = a_o^{3\gamma-2}$; substituting this result back into eq. (3.11), one finally finds the temporal behaviour of a:

$$a(\tau) = \left[A - \left[\left(\frac{2-3\gamma}{2}\right)\tau\right]^2\right]^{1/(3\gamma-2)} \tag{3.12}$$

Analytically continuing into the Euclidean sector, one can then express the wormhole solution as:

$$ds_e^2 = a^{4-3\gamma} d\tau_e^2 + a^2 d\vec{x}^2 \quad , \qquad a(\tau_e) = \left[a_0^{3\gamma-2} + \left(\frac{2-3\gamma}{2}\right)^2 \tau_e^2\right]^{1/(3\gamma-2)} \quad . \tag{3.13}$$

(with $N^2d\tau^2 \to -N_e^2 d\tau_e^2$) This solution passes through the minimum radius a_0 at $\tau_e = 0$, which is the maximum radius for the corresponding Robertson–Walker universe, see fig. (25). These solutions may also be written in the form that is conformally equivalent to the asymptotically flat metric,

$$ds_e^2 = \Omega^2(r)\left[dr^2 + r^2 d\Omega_3^2\right] \quad . \tag{3.14}$$

Comparing eq. (3.13) and (3.14), one finds the two constraints:

$$\Omega(r)dr = a(\tau_e)^{(4-3\gamma)/2} d\tau_e \tag{3.15a}$$

$$\Omega(r)r = a(\tau_e) \tag{3.15b}$$

Dividing eq. (3.15a) by (3.15b) and then integrating, assuming to start with $r = a_o$ at $\tau_e = 0$, one can easily find the evolution of the wormhole size r:

$$r(t_e) = a_0 \exp\left[\frac{2}{3\gamma - 2}\ Arcsinh\left(\frac{3\gamma-2}{2} a_0^{-(3\gamma-2)/2} \tau_e\right)\right] \quad . \tag{3.16}$$

Putting back this result in eq. (3.15b), the conformal factor is found to be:

$$\Omega(r) = 2^{-2/(3\gamma-2)}\left[1 + \left(\frac{a_0}{r}\right)^{3\gamma-2}\right]^{2/(3\gamma-2)} \quad . \tag{3.17}$$

One has the asymptotic behaviour for r:

$$\begin{cases} r \to \infty & \text{for } \tau_e \to +\infty \\ r \to a_o & \text{for } \tau_e = 0 \\ r \to 0 & \text{for } \tau_e \to -\infty \end{cases} \tag{3.18}$$

The regions with $r > a_0$ and $r < a_0$ are equivalent asymptotically flat spaces, as it may be seen in this form from the invariance of the metric under the transformation $r \to r' = a_0^2/r$. To be sure one has really found a general class of wormhole solutions, one also has to check that the "baby" universe branches off at the minimum radius of the wormhole, i.e. one has to look for the sign of the second derivative of $a(\tau_e)$ at $\tau_e = 0$. It is easy to find:

$$\frac{d^2a(\tau_e)}{d\tau_e^2} = \frac{(3\gamma-2)}{2}\left[1 + \frac{3}{2}(1-\gamma)(3\gamma-2)\tau_e^2 a^{2-3\gamma}\right] a^{3(1-\gamma)} \tag{3.19}$$

As we knew, one can have a wormhole solution only if he considers values of $\gamma > \frac{2}{3}$. This wormhole joins two asymptotically flat regions and creates, as it is easy to see from eq. (3.12), a closed universe branching off at maximum radius and collapsing in a finite time $\tau_o = 2\sqrt{A}/(2-3\gamma)$ (the limiting case $\gamma = \frac{2}{3}$ has the only analytically continuable solution $\dot{a}_e = \dot{a}_L = 0$ for $A = 1$, which clearly does not represent any wormhole). From a cosmological point of view, it is also interesting to consider these solutions as corresponding to a possible mechanism generating the wormholes from an expanding closed FRLW universe. The AFGIs that are analytic continuation of the most common solutions in classical cosmology are:

(i) $p = 0$, $\gamma = 1$, the matter dominated closed Friedmann universe;

$$\begin{aligned} ds_e^2 &= \left(a_0 + \frac{\tau_e^2}{4}\right) d\tau_e^2 + \left(a_0 + \frac{\tau_e^2}{4}\right)^2 d\Omega_3^2 \\ &= \frac{1}{16}\left[1 + \frac{a_0}{r}\right]^4 \left[dr^2 + r^2 d\Omega_3^2\right] \end{aligned} \tag{3.20}$$

(ii) $p = \rho/3$, $\gamma = 4/3$, the radiation dominated closed Friedmann universe;

$$\begin{aligned} ds_e^2 &= d\tau_e^2 + (a_0^2 + \tau_e^2) d\Omega_3^2 \\ &= \frac{1}{4}\left[1 + \frac{a_0^2}{r^2}\right]^2 \left[dr^2 + r^2 d\Omega_3^2\right] \end{aligned} \tag{3.21}$$

This is the wormhole introduced by Hawking [58]. The original motivation might have been different, however, as Hawking does not mention the source term, and he seems to attach some significance to this solution in the context of the contour problem, due to its vanishing scalar curvature. But the metric is the same.

(iii) $p = \rho$, $\gamma = 2$, the stiff matter driven Tolman universe;

$$\begin{aligned} ds_e^2 &= (a_0^4 + 4\tau_e^2)^{-1/2} d\tau_e^2 + (a_0^4 + 4\tau_e^2)^{1/2} d\Omega_3^2 \\ &= \frac{1}{2}\left[1 + \left(\frac{a_0}{r}\right)^4\right] \left[dr^2 + r^2 d\Omega_3^2\right] \end{aligned} \tag{3.22}$$

This is the bulk matter driven Giddings–Strominger instanton [34]. The field theoretical model will be recovered in section 3.4. In the later discussion I will need the actions for the wormhole solutions, and I will work out the gravitational parts here. Since the signs will turn out to be important, I will be very explicit about our notation and conventions.

The gravitational part of the action is,

$$\begin{aligned} S_g &= S_R + S_{btg} \\ &= -\frac{1}{16\pi G}\int_{\mathcal{M}} d^4x\ \sqrt{-g}R + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^3x\ \sqrt{h}K \quad . \end{aligned} \tag{3.23}$$

(where γ is the determinant of the wormhole metric on the 3-sphere) One can calculate the scalar curvature for the Lorentzian FLRW background to be:

$$R = 6\left[\frac{2}{a^2}\frac{\dot{a}^2}{N^2} + \frac{d}{Nd\tau}\left(\frac{\dot{a}}{Na}\right) + \frac{1}{a^2}\right] \tag{3.24}$$

For the extrinsic curvature contribution, one has:

$$K = -\frac{\gamma^{ij}}{2N}\frac{\partial \gamma_{ij}}{\partial \tau} \tag{3.25}$$

with $\gamma_{ij} = a^2\Omega_{ij}$ being the metric on the 3-sphere. This gives:

$$S_R = -2\pi^2\mu^2 \int d\tau \; \left[2\frac{a\dot{a}^2}{N} + a^3\frac{d}{d\tau}\left(\frac{\dot{a}}{Na}\right) + Na\right] \quad , \tag{3.26a}$$

$$S_{btg} = 2\pi^2\mu^2\left[a^3\frac{\dot{a}}{Na}\right]_{|b} \quad , \tag{3.26b}$$

$$S_g = 2\pi^2\mu^2\int d\tau\left[\frac{a\dot{a}}{N} - Na\right] \quad . \tag{3.26c}$$

To pass to the Euclidean regime, i.e., to describe the trajectory for $a > a_0$, one has to do the following: (i) substitute the real parameter (coordinate) τ with the real parameter (coordinate) τ_e, and all derivatives $d/d\tau$ with $d/d\tau_e$; (ii) make the substitution $N = \pm iN_e$, $N_e \in \mathcal{R}$; (iii) define the Euclidean action S_e as $S(\pm iN_e) \equiv iS_e$, so that $\exp[iS]$ leads to $\exp[-S_e]$; (iv) finally, regularize the Euclideanized boundary term for gravity by subtracting from it the same expression evaluated for the flat Euclidean metric, eq. (3.8) with $a = t_e$,

$$\frac{1}{8\pi G}\int d^3x \; \sqrt{h_{ef}}K_{ef} = 2\pi^2\mu^2t_e^2 \quad . \tag{3.27}$$

Thus, one finds for the gravitational Euclidean action:

$$S_e = \pm 2\pi^2\left[\tilde{S}_{eR} + \tilde{S}_{rebtg}\right] \quad . \tag{3.28}$$

where the separate contributions are,

$$\tilde{S}_{eR} = \mu^2\int d\tau_e \; \left[2\frac{a\dot{a}_e^2}{N_e} + a^3\frac{d}{d\tau_e}\left(\frac{\dot{a}_e}{aN_e}\right) - aN_e\right] \quad , \tag{3.29}$$

for the curvature term, and,

$$\tilde{S}_{rebtg} = \mu^2\left[-a^3\frac{\dot{a}}{aN_e}\right]_{|b} + \mu^2t_e^2 \quad , \tag{3.30}$$

for the regularized boundary term. The sign in the front of the second term is, by (iv), the opposite from the sign of the first term. Finally, we have

$$\begin{aligned}\tilde{S}_{eg} &\equiv \tilde{S}_{eR} + \tilde{S}_{rebtg} \\ &= \mu^2 \int d\tau_e \left[-\frac{a\dot{a}_e^2}{N_e} - aN_e \right] + \mu^2 t_e^2 \quad , \end{aligned} \tag{3.31}$$

for the total Euclidean action for gravity (modulo prefactors).

The integrals are to be evaluated between the two boundaries at some radius a, on the two sides of the throat. For wormhole solutions the integrands are even functions, so the integrals may be taken on half-wormhole only, times the factor of 2.

Our solutions, eq. (3.13), may conveniently be parametrized through the hyperbolic angle $\theta \in (-\infty, \infty)$, introduced as,

$$a = a_0[\cosh\theta]^{2/(3\gamma-2)} \quad , \qquad \tau_e = \frac{2}{3\gamma-2} a_0^{(3\gamma-2)/2} \sinh\theta \quad , \tag{3.32}$$

while the flat space time variable is

$$t_e = \int d\tau_e N_e = \frac{2a_0}{3\gamma-2} \int_0^\theta d\tilde{\theta}\ [\cosh\tilde{\theta}]^{2/(3\gamma-2)} \quad . \tag{3.33}$$

After substituting the explicit solutions for a and N_e (eq. (3.9) and (3.13)), one finds,

$$S_{eR} = \pm 2\pi^2 \mu^2 a_0^2 \frac{3\gamma-4}{3\gamma-2} \int_0^\theta d\tilde{\theta}\ [\cosh\tilde{\theta}]^{(8-6\gamma)/(3\gamma-2)} \quad , \tag{3.34a}$$

$$S_{rebtg} = \pm 8\pi^2 \mu^2 \left[\left(\frac{2-3\gamma}{2} \right) \tau_e a^{3(2-\gamma)/2} + t_e^2 \right] \Bigg|_\infty \tag{3.34b}$$

To calculate it in the asymptotic limit $|\tau_e| \to \infty$, one can approximately write eq. (3.18) as:

$$dt_e \simeq \left[\left(\frac{3\gamma-2}{2} \right) \tau_e \right]^{(4-3\gamma)/(3\gamma-2)} d\tau_e \tag{3.35}$$

This can be integrated to give:

$$t_e^2 = \left[\left(\frac{3\gamma-2}{2} \right) \tau_e \right]^{4/(3\gamma-2)} \simeq a_o^2 2^{4/(2-3\gamma)} e^{4\theta/(3\gamma-2)} \left[1 - \frac{2}{3\gamma-2} e^{-2\theta} \right]^2 \tag{3.36}$$

Expanding also the first term on the right hand side of eq. (3.34b) for large τ_e one obtains:

$$\left[a^3 \frac{\dot{a}_e}{aN_e} \right]_{a \gg a_0} = a_0^2\ 2^{4/(2-3\gamma)} e^{4\theta/(3\gamma-2)} \left[1 + \frac{8-6\gamma}{3\gamma-2} e^{-2\theta} \right] \quad , \tag{3.37}$$

so the total regularized boundary term (for both boundaries) is,

$$S_{rebty} = \pm 3\pi^2\mu^2 a_0^2 \left(\frac{\gamma-2}{3\gamma-2}\right) 2^{(9\gamma-10)/(3\gamma-2)} \exp\left[\frac{8-6\gamma}{3\gamma-2}\theta\right]\Bigg|_{\infty} + O\left(\exp\left[\frac{12(1-\gamma)}{3\gamma-2}\theta\right]\right) \tag{3.38}$$

Summing all these terms, the result is (for $a \gg a_o$):

$$\begin{aligned} S_e = &\pm \frac{3\gamma-4}{3\gamma-2} 4\pi^2\mu^2 a_0^2 \int_{-\infty}^{+\infty} d\theta \; [\cosh\theta]^{(8-6\gamma)/(3\gamma-2)} \\ &\pm 3\pi^2\mu^2 a_0^2 \left(\frac{\gamma-2}{3\gamma-2}\right) 2^{(9\gamma-10)/(3\gamma-2)} \exp\left[\frac{8-6\gamma}{3\gamma-2}\theta\right]\Bigg|_{\infty} \end{aligned} \tag{3.39}$$

Note that for the Hawking wormhole ($\gamma = 4/3$), one has the finite action due to the boundary term, $S_e = \mp 3\pi a_0^2/(4G)$. This is supposed to be the full action for this purely gravitational wormhole [58]. For the case $\gamma = 2$, the boundary term for gravity has no contribution, and from the first term we recover the half of the action for the Giddings–Strominger instanton, $S_e = \pm 3\pi^2 a_0^2/(8G)$ [34]. Note that for a given convention those two come out with the opposite signs.

3.3 The existence of FLRW wormholes driven by a scalar field.

One can consider the simplest, but physically sufficiently general and important case of a spatially homogeneous, complex valued, minimally coupled scalar field ϕ.

Using the line element with $dt = N d\tau$, the Lorentzian matter field action will be taken as:

$$S_M = 2\pi^2 \int d\tau \; a^3 \left[-\frac{|\dot\phi|^2}{2N} + NV(\phi)\right] \quad , \tag{3.40}$$

where $\dot\phi \equiv d\phi/d\tau$. The energy density and the pressure can be calculated comparing the energy-momentum tensor $T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{\mu\nu}}$ for the Lagrangian density $\mathcal{L}$ of the theory and that for a perfect fluid $T_{\mu\nu} = (p+\rho)U_\mu U_\nu + p g_{\mu\nu}$, and are:

$$\rho = \frac{|\dot\phi|^2}{2N^2} + V(\phi) \equiv T + V \quad , \tag{3.41a}$$

$$p = \frac{|\dot\phi|^2}{2N^2} - V(\phi) \equiv T - V \quad . \tag{3.41b}$$

The relevant equations are now the constraint equation:

$$H^2 = \frac{1}{\mu^2}(T+V) - \frac{1}{a^2} \quad , \qquad H \equiv \frac{1}{a}\frac{da}{N d\tau} \quad ; \tag{3.42}$$

the Raychaudhuri equation:

$$\frac{1}{a}\frac{d^2a}{dt^2} = \frac{\dot{H}}{N} + H^2 = -\frac{1}{2\mu^2}(\rho + 3p) = -\frac{1}{\mu^2}[2T - V] \quad ; \tag{3.43}$$

the energy conservation equation:

$$\frac{\dot{\rho}}{N} = -3H\gamma\rho \quad , \qquad \gamma = \frac{2T}{T+V} \quad ; \tag{3.44}$$

and, finally, the scalar field equation:

$$a^{-3}\frac{d}{Nd\tau}\left(a^3\frac{d\phi}{Nd\tau}\right) + V'(\phi) = 0 \quad . \tag{3.45}$$

(where a dot means the derivative with respect to the time variable τ and a prime that with respect to ϕ^*). Let us now check when Lorentzian signature solution has maximal radius. From eq. (3.43) it follows that trajectory is convex for $\rho + 3p > 0$ or, equivalently $2T > V$, which implies $\gamma > \frac{2}{3}$. Then, from eq. (3.44), it follows that ρ decays faster than a^{-2}. Thus, for every solution that has finite ρ_i at some finite a_i there is a maximum radius a_0 ($> a_i$), such that $H^2(a_0) = 0$.

One would now like to find analytic continuation of these solutions to domain $a > a_0$. Continued solutions are expected to be described by the Euclidean version of eq.'s (3.42-3.45), obtained in the standard way: setting $N = \pm iN_e$, $N_e \in \mathcal{R}$, one has,

$$S_{em} = 2\pi^2 \int d\tau_e \; a^3 \left[\frac{|\dot{\phi}_e|^2}{2N_e} + N_e \; V\right] \quad , \tag{3.46}$$

$$\rho_e = -\frac{|\dot{\phi}_e|^2}{2N_e^2} + V(\phi) \equiv -T_e + V \quad , \tag{3.47a}$$

$$p = -\frac{|\dot{\phi}_e|^2}{2N_e^2} - V(\phi) \equiv -(T_e + V) \quad , \tag{3.47b}$$

where $\dot{\phi}_e \equiv d\phi/d\tau_e$. The sign in the front of i is normally chosen in such a way that the resulting Euclidean action for a scalar field is positive definite. By varying the action, the corresponding equations are then,

$$H_e^2 = \frac{1}{\mu^2}(T_e - V) + \frac{1}{a^2} \quad , \qquad H_e \equiv \frac{1}{a}\frac{da}{N_e d\tau_e} \quad ; \tag{3.48a}$$

$$\frac{1}{a}\frac{d^2a}{dt_e^2} = \frac{\dot{H}_e}{N_e} + H_e^2 = +\frac{1}{2\mu^2}(\rho_e + 3p_e) = -\frac{1}{\mu^2}[2T_e + V] \quad ; \tag{3.48b}$$

$$\frac{\dot{\rho}_e}{N_e} = -3H\gamma_e\rho_e \quad , \qquad \gamma_e = \frac{2T_e}{T_e - V} \quad ; \tag{3.48c}$$

$$a^{-3}\frac{d}{N_e d\tau_e}\left(a^3\frac{d\phi}{N_e d\tau_e}\right) - V'(\phi) = 0 \quad . \tag{3.48d}$$

These equations are the same as what one would get by the substitution $N = \pm iN_e$ directly in equations (3.42-3.45).

One can see immediately that there is a problem with these equations if they are to describe wormhole solutions. These solutions are expected to be concave, while from (3.48b) it follows that solution is concave only if $\rho_e + 3p_e < 0$, or $V < -2T_e < 0$. This is in conflict with Eq. (3.48a), which says that the left hand side may vanish only if $V_0 > 0$ at the matching point. To see various possibilities one can introduce $T_a \equiv (1/2)|d\phi/da|^2$, so that $T = a^2H^2T_a$, and, $T_e = a^2H_e^2T_a$. From (3.42) we find,

$$H^2 = \frac{V - \mu^2/a^2}{\mu^2 - a^2T_a} \quad , \tag{3.49}$$

and from (3.48a),

$$H_e^2 = \frac{\mu^2/a^2 - V}{\mu^2 - a^2T_a} \quad . \tag{3.50}$$

At the matching point, the condition $H^2(a_0) = 0$ implies either,

(i) $a_0^2T_{a_0} \sim finite \neq \mu^2$, with $V_0 = \mu^2/a_0^2$, or,

(ii) $a_0^2T_{a_0} \sim divergent$, with $V_0 < \mu^2/a_0^2$.

On the other side, $H_e^2(a_0) = 0$ implies either,

(iii) $a_0^2T_{a_0} \sim finite \neq \mu^2$, with $V_0 = \mu^2/a_0^2$, same as (i), or,

(iv) $a_0^2T_{a_0} \sim divergent$, with $V_0 > \mu^2/a_0^2$.

The analytic continuation is possible whenever these conditions combine consistently.

Suppose first that (i) and (iii) are true. Then $T(a_0) = a_0^2H^2(a_0)T_{a_0} = 0$, and, from eqs. (3.41) or (3.47), $(\rho + 3p)_0 = -2V_0 < 0$. This violates the starting condition for the existence of the maximal radius for the Lorentzian trajectory. Thus, the continuation to wormhole solution is not possible, but, if one imagines to invert the temporal sequence of the two trajectories (i.e., first the Euclidean one and then the Lorentzian one), this case can describe quantum creation of an expanding inflationary universe, a Lorentzian de Sitter space, at a_o.

Suppose now that (ii) is true. It is clearly incompatible with (iii), and to agree with (iv) one needs to have a potential with a finite discontinuity at the matching point. This case is unattractive, as a_0 is determined by the initial data, independent of the shape of $V(\phi)$, and the position of the possible discontinuity. In particular, it does not allow for the solutions in the $V = 0$ case, which corresponds to the Giddings–Strominger wormhole, (see next section).

Thus, one concludes that the standard analytic continuation for the scalar field driven solutions is possible between the Euclidean solutions and the inflationary solutions; and that it is not possible between the expanding Lorentzian solutions and wormhole solutions, unless the potential has a finite discontinuity at the right place.

The idea is, therefore, to redefine the procedure of the analytic continuation so that it allows for the wormhole solutions driven by scalar fields. One can do it as follows:

(i) analytic continuation is an extension of a Lorentzian signature solution from the finite range $(0, a_0)$, to the Euclidean signature trajectories at $a > a_0$;

(ii) lapse functions in the two regions are related as $N^2 \to -N_e^2$, but the actual transition from one region to another is to be accomplished by the substitution

$$N = \pm iN_e \quad , \tag{3.51}$$

in the gravitational part of the Lorentzian action, and,

$$N = \mp iN_e \quad , \tag{3.52}$$

in the scalar field part of the Lorentzian action.

(iii) all expressions in the Euclidean sector should follow by operating on the Euclidean action obtained in the manner above, not by the continuation of the corresponding equations. Coordinates t, t_e, etc., are all real.

This asymmetric rotation to the Euclidean metric has been introduced by Linde [86] to obtain the "tunneling" wave function as the initial state for an inflationary universe [105, 117, 124]. As in [86], one is tempted here to choose the sign in front of i adapted to the convention in the Euclidean phase in such a way that the total Euclidean action is positive definite. I will briefly discuss this and other prescriptions concluding the chapter.

Now let us list the consequences of these rules for our case. The Euclidean action becomes, from eq. (3.31) and (3.40):

$$S_e = \pm 2\pi^2 \int d\tau_e \left[\mu^2 \left(-\frac{a\dot{a}_e^2}{N_e} - N_e a \right) - a^3 \frac{|\dot{\phi}_e|^2}{2N_e} - a^3 N_e V(\phi) \right] \pm 2\pi^2 \mu^2 t_e^2 \tag{3.53}$$

One can see that the relative sign between the gravitational and matter parts has been changed. The energy density ρ_e and the pressure p_e come out as in eq. (3.47). The field equation (3.48d) remains the same, and so does the conservation equation (3.48c). Parameter γ_e is unchanged. Thus, the change occurs only in desired places: in the action, in the constraint equation which becomes,

$$H_e^2 = \frac{1}{\mu^2}(-T_e + V) + \frac{1}{a^2} \quad , \tag{3.54}$$

and in the Raychaudhuri equation which is now,

$$\frac{1}{a}\frac{d^2 a}{dt_e^2} = \frac{\dot{H}_e}{N_e} + H_e^2 = \frac{1}{2\mu^2}(\rho_e + 3p_e) = \frac{1}{\mu^2}[2T_e + V] \quad . \tag{3.55}$$

Now it is manifest that wormhole solutions are possible, as

$$H_e^2 = \frac{V + \mu^2/a^2}{\mu^2 + a^2 T_a} \quad , \tag{3.56}$$

allows for the matching when $a_0^2 T_{a_0} \sim divergent$, $T(a_0) = 0$, $a_0 \sim finite$, all consistent with (ii). (In contrast, now the complementary possibility does not describe a creation of an inflationary universe.) The Euclidean solution is concave, providing $V > -2T_e$, which is consistent with the existence of a minimal radius with $H_e^2 = 0$. Parameters γ and γ_e need not to be the same, and both are constrained. In order to have a maximum radius for the Lorentzian solution one needs (from (3.41) and (3.43)),

$$\lim_{a \to a_0} \gamma \equiv \gamma_0 > 2/3 \quad , \tag{3.57}$$

while, in order for the Euclidean solution to be asymptotically flat, one needs (from (3.48c), (3.5) and (3.54)),

$$\lim_{a \to \infty} \gamma_e \equiv \gamma_{e\infty} > 2/3 \quad . \tag{3.58}$$

Demanding vanishing expansion rates at the throat one finds,

$$T_0 - T_{e0} + 2V_0 = 0 \quad , \tag{3.59}$$

and using the relations,

$$T = \frac{\gamma}{2-\gamma} V \quad , \qquad T_e = \frac{\gamma_e}{\gamma_e - 2} V \quad , \tag{3.60}$$

the following matching condition for the gamma parameters can be derived,

$$\gamma_0 + \gamma_{e0} = 4 \quad . \tag{3.61}$$

In the case of $\gamma = const.$ solutions driven by a scalar field, these conditions restrict parameters γ, γ_e to be in the range $[2/3, 10/3]$. Therefore, in general, the equation of state has to change under the analytic continuation, the only exception being the Giddings–Strominger case ($\gamma = \gamma_e = 2$).

3.4 Explicit solutions for scalar field driven wormholes.

Let us now show how scalar field driven solutions may be actually built. For this, let us use a trick, introduced in another context by Ellis and Madsen [26]. Combining the constraint equation (3.54) and the Raychaudhuri equation (3.55), one may express the kinetic and potential term for the scalar field through the geometry:

$$T_e = \frac{\mu^2}{3}\left[\frac{\dot{H}_e}{N_e} + \frac{1}{a^2}\right] \quad , \tag{3.62a}$$

$$V(\phi) = \frac{\mu^2}{3}\left[\frac{\dot{H}_e}{N_e} + 3H_e^2 - \frac{2}{a^2}\right] \quad . \tag{3.62b}$$

As it can easily be seen, the asymptotic behaviour of the wormhole solution requires (since: $H_e, \dot{H}_e \to 0$, as $a \to \tau \to \infty$):

$$\begin{cases} |\dot{\phi}_e^2| \to 0 \\ V(\phi_e) \to 0 \end{cases} \quad \text{for } a \to \infty \quad . \tag{3.63}$$

Given a wormhole solution, in principle one may determine both the trajectory of ϕ and the potential . This can be carried on with no difficulty for the $\gamma = const.$ solutions. Using the known forms for N_e and a (eq. (3.9, 3.13)), from eq. (3.54, 3.62) one can find,

$$|\dot{\phi}_e| = \mu\sqrt{\gamma_e a_0^{3\gamma_e - 2}} a^{2-3\gamma_e} \quad . \tag{3.64}$$

This equation can be easily integrated using the explicit expression for $a(\tau_e)$ (eq. (3.13)), to obtain,

$$\phi_e(\tau_e) = \frac{2}{(3\gamma_e - 2)}\mu\sqrt{\gamma_e}\arctan[(3\gamma_e - 2)\tau_e/2a_0^{(3\gamma_e-2)/2}] \quad . \tag{3.65}$$

On the other hand, from the relation (3.60) and eliminating the time dependence with the aid of eq. (3.64), it is also possible to obtain the explicit expression for the potential V as a function of the scalar field ϕ,

$$V(\phi_e) = \frac{\mu^2(\gamma_e - 2)}{2a_0^2}\left|\cos\left[\frac{(3\gamma_e - 2)}{2\mu\sqrt{\gamma_e}}\phi_e\right]\right|^{6\gamma_e/(3\gamma_e-2)} \quad . \tag{3.66}$$

At the throat $\phi = 0$, and the potential has a maximum. Both V and τ_e are periodic functions of ϕ_e; the importance of this peculiar ϕ dependence will be discussed in the next section. The corresponding solutions for the analytically continued FLRW closed universe are essentially described by the same explicit structure for V and ϕ, the only changes being in the value of γ and in the substitution of trigonometric functions by their hyperbolic analogues.

It is also easy to explicitly calculate the full action for this class of scalar field wormholes. Its general form is:

$$S_e = S_{eR} + S_{rebtg} + S_{em} + S_{ebtm} \tag{3.67}$$

The first two terms are as before (eq. (3.31) and (3.39)), with the substitution $\gamma \to \gamma_e$, the third may be read off eq. (3.53) and the last one is a possible boundary term for the scalar field. It is zero if the field is fixed on the boundary, and it is:

$$S_{ebtm} = \pm\pi^2\frac{\dot{\phi}_e\phi_e^* a^3}{N_e} \quad , \tag{3.68}$$

if the momentum $\Pi_{\phi^*} \equiv -\pi^2\frac{a^3\dot{\phi}_e}{N_e}$ is fixed on the boundary.

The matter contribution to the Euclidean action may be calculated from eq. (3.53) using the explicit expressions for $\dot{\phi}_e, V(\phi_e), a$ and N (see eqs. (3.46), (3.60), (3.13)), and is found:

$$S_{em} = \pm 4\pi^2\mu^2\frac{(1-\gamma_e)}{(3\gamma_e - 2)}a_0^2\int_{-\infty}^{+\infty} d\theta[\cosh\theta]^{2(4-3\gamma_e)/(3\gamma_e-2)} \tag{3.69}$$

(where θ has been introduced as before, eq. (3.32)). In an analogous way, one can also find the boundary term for the fixed momentum condition. Since

$\phi(a_o) = 0$ (eq. (3.65)), the contribution may come only from the two upper boundaries:

$$S_{ebtm} = \frac{4\pi^3\mu^2}{(3\gamma_e - 2)} a_0^{(3\gamma_e-2)/2} a^{3(2-\gamma_e)/2}\Big|_{\infty} \tag{3.70}$$

The total Euclidean action is then:

$$\begin{aligned} S_e = &\pm 2\pi^2\mu^2 \frac{(\gamma_e - 2)}{(3\gamma_e - 2)} a_0^2 \int_{-\infty}^{+\infty} d\theta [\cosh\theta]^{(8-6\gamma_e)/(3\gamma_e-2)} \\ &\pm \frac{\pi^2\mu^2 2^{\frac{9\gamma_e-10}{3\gamma_e-2}}}{(3\gamma_e - 2)} a_0^2 \left[3(\gamma_e - 2)\exp\left(\frac{2(4-3\gamma_e)}{3\gamma_e - 2}\theta\right) + \pi\gamma_e \exp\left(\frac{3(2-\gamma_e)}{3\gamma_e - 2}\theta\right)\right]\Big|_{\infty} \end{aligned} \tag{3.71}$$

One can also check that the introduction of a boundary term for the scalar field does not change any equation of motion (such as (3.54) (3.48d)), provided one makes variations of the action at Π_ϕ and N fixed on the boundary. These expressions have some interesting consequences. Consider first the case $V = 0$, $\gamma = \gamma_e = 2$, the Giddings–Strominger wormhole. Only the matter field boundary term may contribute, and one obtains

$$S_e[2] = 0 \quad , \tag{3.72}$$

if the scalar field is fixed on the boundary, and

$$S_e[2] = \pm S_{ebtm}[2] = \pm \frac{3\pi^2}{4G} a_0^2 \quad , \tag{3.73}$$

if the momentum of a scalar field is held fixed on the boundary. One has a positive action, and the expected form of the semiclassical amplitude, $\Psi \sim \exp[-a_0^2/G]$, for the sign choice that makes the starting (total) action (3.53) apparently manifestly unbounded from below. The finite value of the action is entirely due to the boundary term for a scalar field. Similarly, for $\gamma_e = 4/3$, and neglecting S_{em}, one recovers the action for the Hawking wormhole, $S_e[4/3] = \mp 3\pi a_0^2/(4G)$.

In general, when one deals with wormholes that are solutions to analytically continued classical equations, the naive semiclassical amplitude is finite for $2 < \gamma_e < 10/3$. However, the sign is as in the Giddings–Strominger case: wormholes have amplitudes that exponentially damp large values for a_0 only when the starting Euclidean action is unbounded from below. I will come back to this in the final section.

Another remark can be made, on the other hand, on a previous work by Jungman and Wald [73], which demonstrated a theorem about the conditions for the existence of some kinds of asymptotically flat euclidean instanton solutions (see section 2.2). In particular, they showed that the matter equation alone should suffice to rule out solutions for scalar fields satisfying the condition $\phi \frac{\partial V}{\partial \phi} > 0$, such as a free massless minimally coupled Klein-Gordon field. However, both these results hold only for a particular assumption about the asymptotic behaviour of the matter fields which ensures the vanishing of the matter boundary term in the action, but which is not satisfied here. For the solutions presented here, the form of the potential is fixed a posteriori by the Euclidean equation of motion for ϕ and the equation of state.

3.5 The finite temperature of wormholes.

One striking feature is apparent in the solutions introduced above: in all of them the scalar field is restricted to a finite range, or rather, the wormhole is travelled from one end to another while the scalar field evolves for a finite amount. It is straightforward to check that this is a general feature of all scalar field driven wormhole solutions. In fact, expression (3.64) may be integrated to give:

$$|\phi - \phi_0| = \int_{\tau_{e0}}^{\tau_e} d\tau_e \, \sqrt{\frac{2}{3}\mu} \left(\dot{H}_e + \frac{1}{a^2} \right)^{1/2} \quad . \tag{3.74}$$

The integrand is never negative, from the positive definiteness of T_e. It vanishes in the asymptotic regime in order for wormhole to join the asymptotically flat background. Now one only has to check that it decays sufficiently fast for the integral to be finite and not divergent. For this one can keep next to the leading order in the asymptotic behaviour of the metric,

$$a \sim t_e + \frac{B}{a^p} \quad , \qquad p > 0 \quad , \quad B = const. \quad ; \tag{3.75}$$

from which it is easy to obtain:

$$\dot{H}_e \sim -\frac{1}{a^2} + \frac{p(p+3)B}{a^{p+3}} \quad . \tag{3.76}$$

Thus, ϕ changes for a finite amount whenever $(p+3)/2 > 1$, which is automatically satisfied.

Now, the kinetic term is invariant under any constant shift in ϕ, while there are no other constraints on the potential term apart from Eq. (3.62b). Given a wormhole solution, this equation determines the potential in the range $(\phi_0, \phi_0 + P)$, where P is the finite, maximal value of the integral (3.74). Thus, one may now take the same solution and define the potential on the interval $(\phi_0 + P, \phi_0 + 2P)$, etc..

One way to understand this is to say that the field ϕ may always be interpreted as a phase. Indeed, this is the explicit realization of the wormhole driving field in some models [16]. But one is not forced to think that way only. Some of our potentials may be naturally realized in some theory where ϕ has a completely different physical or geometrical interpretation, (e.g. [92]). On the other hand, wormhole geometry is completely determined by the evolution of a field ϕ through the classical equations. In particular, one may always choose a gauge such that ϕ plays a role of the Euclidean time coordinate (suitably rescaled for the proper dimension). The wormhole geometry will appear periodic in the time coordinate. (A better way to say it is that due to wormholes the ground state in quantum gravity has a periodic structure.) In analogy with other gravitational and non–gravitational systems one may interpret this as the finite temperature of wormhole spacetime. Since there is only one parameter that characterizes the wormhole, its size, one expects for wormhole temperature to be inversely proportional to the size of the wormhole.

All this may be explicitly checked for the family of the explicit solutions with $\gamma = const.$, (see also [92]). For this it is useful to use the line element r, eq. (3.15, 3.16). From eq. (3.16) and (3.64) one can write:

$$d\phi_e = \mu\sqrt{\gamma_e} a_0^{(3\gamma_e-2)/2} a^{2-3\gamma_e} d\tau_e = \mu\sqrt{\gamma_e} a_0^{(3\gamma_e-2)/2} \Omega^{1-3\gamma_e/2} r^{-3\gamma_e/2} dr \quad . \tag{3.77}$$

Introducing the auxiliary mass parameter:

$$M \equiv \frac{4\sqrt{\gamma_e}\mu}{3\gamma_e - 2} \quad , \tag{3.78}$$

and integrating eq. (3.77), the solution may be written as,

$$r = a_0 \left| \tan\left(\frac{\phi - \phi_0}{M}\right)\right|^{2/(3\gamma_e-2)} \quad . \tag{3.79}$$

Therefore, the line element becomes (using eqs. (3.14), (3.15) and (3.79)):

$$ds_e^2 = \left(\frac{a_0}{\sqrt{\gamma_e}\mu}\right)^2 \left|\sin\left(2\frac{\phi-\phi_0}{M}\right)\right|^{-6\gamma_e/(3\gamma_e-2)} d\phi^2 + a_0^2 \left|\sin\left(2\frac{\phi-\phi_0}{M}\right)\right|^{-4/(3\gamma_e-2)} d\Omega_3^2 \ . \quad (3.80)$$

The periodicity is manifest now. The time coordinate with the proper dimension is $z \equiv a_0\phi/(\sqrt{\gamma_e}\mu)$, and the period is $\Delta z = 2\pi a_0/(3\gamma_e - 2)$. Thus, one may conclude that a scalar field driven $\gamma_e = const.$ wormhole has the temperature,

$$T_w = \frac{3\gamma_e - 2}{2\pi a_0} \ . \quad (3.81)$$

(where I have set $K_B = \hbar = c = 1$ for simplicity). Of course, this interpretation needs to be supported by further, more detailed considerations. If confirmed, thermal properties of wormholes should be interesting on at least two counts. One is Hawking's idea about the role of wormholes in the evaporation of black holes. Trapped particles, that ultimately reduce the mass of the black hole to zero, are supposed to travel through the wormhole away from our universe. It would be very interesting to further develop thermodynamics of wormholes and to see is there any direct connection to the thermal properties of black holes and some gain in the understanding of the evaporation process. The second area of interest could be in explicit computations of the effective action on the wormhole background, in order to examine in some details how coupling constants become statistical parameters. One expects that the thermal nature of the background should be of some help to carry on and understand such calculations (e.g., in the determination of Green functions, which become subject to periodic conditions, etc..). Finally, possible thermal nature of the ground state may play an important role in the context of a general discussion about the quantum coherence or decoherence in quantum gravity.

3.6 Discussion and conclusions.

There are three main results in this work: an understanding of wormholes as analytically continued closed expanding universes, and resulting construction of wormhole solutions; definition of the analytic continuation through the Linde rotation in order to ensure the existence of scalar field driven wormholes; and the observation about the thermal properties of wormholes. There is nothing more to say about the last topic, and I will now discuss the other two.

There is certainly nothing controversial about the first of our main results. There are just about infinitely many solutions in GR, with the closed spatial geometry and with the bulk matter source that obeys the strong energy condition. They could be homogeneous or inhomogeneous, isotropic or anisotropic. Each of them has a maximal radius (volume). For each of them the analytic continuation above the maximal radius is a spacetime wormhole. Thus, there are infinitely many exact wormhole solutions that can be obtained in this way. Furthermore, at least in the FLRW case, one may use the trick from [26] to construct the field theoretical models that may drive such wormholes. In fact, since the equations of GR are deterministic, one would claim that appropriate field theoretical models exist also in the cases of anisotropic and inhomogeneous wormhole solutions, even if they cannot be explicitly constructed. Similarly, instead of using the scalar field, one may represent the stress tensor through some higher spin classical field. Again, in principle there exists a configuration which drives a given wormhole. Wormholes driven by an antisymmetric tensor field, or by the electromagnetic field should be found or recovered in this way. And using the powerful techniques of conformal transformation these results may be extended to the induced gravity, higher derivative gravity or non–minimally coupled scalar fields. A number of wormhole solutions in such theories has been constructed by Coule and Maeda [17]. In any case in the asymptotic regime the FLRW line element may be used, and everything can be done explicitly.

The existence of all these solutions is interesting not only from the point of view of GR, but also from the point of view of the fundamental particle theory. In such theory the Lagrangian is considered fixed, so only some of all the possible solutions may be important. But as our simple construction shows, wormholes are rather ordinary objects. There is no need to relay on a very particular field content or special values for the coupling constants in order to have wormhole solutions. Their existence is solely based on the two properties of the theory: the equation of state that obeys the strong energy condition, and the analytical continuation to which we now turn.

The real issue is what defines the Euclidean theory. In all works except [86] this is done through the standard rotation $N \to \pm iN_e$. The main motivation in [86] was to obtain a positive definite Euclidean action (for that case), and the "tunneling" wave function was the (desired) result. In this case, the motivation is to obtain wormhole solutions. Since gravity sees the energy density regardless of what is its source, if the bulk matter model shows that a particular wormhole exists, the scalar field model with the same equation of state somehow has to reproduce it. This should determine what is the analytical continuation.

A good motivation for the Euclidean equations that have wormhole solutions is an issue that is recognized by many authors. The most convincing argument so far is probably the one in [16, 85], but it relays on the existence of the conservation law for the wormhole charge. As a consequence, the crucial role for the existence of wormholes is played by the phase of a complex scalar field and its boundary conditions. The conserved charge is just the momentum of the phase which appears in the action as a massless minimally coupled scalar field. In contrast, there is no conserved charge "stabilyzing" our solutions, and, in general, our wormhole driving field is not a phase. Our method covers the case of a self–interacting wormhole driving fields, and we have obtained a number of *exact* solutions in these cases. Our solutions cannot be obtained using methods of [16, 85] as they stand, but their solutions may be obtained by applying our method to the models they are considering.

Classically, our construction is a rather innocent thing to do, as through continuation one may always define the theory whose solutions are continuation of solutions in GR. But one may not have such a freedom in quantum theory, since our analytical continuation defines the Euclidean path integral which is different from the usual one. In fact, one may argue that eq. (3.51, 3.52) make no sense, as there is a single integral over the lapse function in the path integral and the rotation has to be done one way or another. However, as yet there is no treatment of the conformal rotation in a non–trivial model involving both gravity and matter fields. The existence of the conformal rotation that leads to the well defined Euclidean path integral has been explicitly shown only for the linearized gravity [50, 108], where one may work with the true physical degrees of freedom, and for the pure gravity in the minisuperspace [44, 45, 109], in which case there are no clashing signs in the action. It still remains to be seen is it possible to define the Euclidean Quantum Gravity with the nontrivial matter fields by a straightforward contour integration.

Our continuation rule makes a difference between the matter and gravity, and we find some comfort in the fact that the quantum theory certainly makes such difference. More specifically however, the fact that we are starting from the different path integral than the usual one, may have its natural explanation in the different boundary conditions. Here we are assuming the existence of the flat spacetime as the ground state, while in quantum cosmology with the standard path integral one assumes that there is nothing. At the classical level this is supported by the observation in section 3.4 that the usual analytic continuation allows for the continuation from the expanding Euclidean phase, (with the maximal radius), to the inflationary universe, (expanding from the minimal radius), but does not allow for the continuation to the wormhole solutions, while the asymmetric rotation allows for wormhole solutions (with the minimal radius), but not for the quantum creation of an expanding universe (ref. [86] makes specific claim only about the quantum amplitude, not about the classical equation, so the logic there may survive if ours fails, and vice versa.) Both prescriptions should be implemented together in the combinatorial argument for the quantum amplitudes in the presence of wormholes [14, 78]: the asymmetric rotation for wormholes, that ensures their existence, and the standard rotation for the large universes, resulting in the usual Hartle–Hawking amplitude [49], and the subsequent double exponential.

Clarification of these issues, and the final judgement of our method within the quantum theory should come from the explicit computations of wave functions for wormholes. Naively, knowing the action, one may write down what is the semiclassical form of these wave functions, as I have shown in $\gamma = const.$ case. Depending on boundary conditions one may be interested in amplitudes for a given value of the scalar field or for the given value of the momentum. The two are related by the Laplace transformation.

As I have shown the amplitude either decreases or grows with the size of the wormhole. This is quite general, and it happens because the overall sign in our analytical continuation has not been fixed. Here one may follow the logic of [86] and choose the sign that makes the total action for every history manifestly positive definite. This results in the amplitude that grows with the size of the wormhole, contrary to assertions of other authors, and contrary to our expectations. This looks like the complementarity between the "tunneling" and the Hartle–Hawking wave functions in quantum cosmology, so that one may think that our procedure defines one of them, probably the wrong one. There are however good reasons why one might prefer the other choice, the one that makes the total action manifestly negative definite. In that one differs from other authors only because one does not keep matter action positive definite. But this is not very relevant. To make quantum amplitudes well defined one has to specify the contour of integration that makes the integrand well behaved. In any case this would involve distortion of the original contour for both the gravity and the matter fields, and it matters little was it all the action that was negative definite or just the part of it. In fact, just for that reason, and because of the common sign in our action, one would expect such contour to be easier to find. In other words, as manifest from (3.53) speaking about the positive or negative definite actions is impossible without fixing the contour of integration for the lapse function. As I mentioned, as yet there has been no success in explicitly applying some of the general procedures for finding the proper contour [33, 42] to the case when the nontrivial matter fields are present. It is known however that there are possible problems [68]. One may have some reason to hope that things will be simpler when the *total* action is negative definite.

Finally, I mention that the (starting) negative definite action is necessary if one wants to describe the region $a < a_0$. The path integral with the everywhere positive integrand is always real and positive, and cannot reproduce the oscillatory wave function that characterizes the Lorentzian regime below the size a_0. As in quantum cosmology, this may be achieved with the complex contour of integration, for which one has no motivation unless the action is negative definite.

To summarize: our procedure generates many exact, classical wormhole solutions. At the quantum level it leads to the path integral different from the usual one. This should be tied to the assumption about the existence of the flat spacetime. In our case the total Euclidean action is negative definite. As in the standard case, *after this* one should independently specify the (complex) contour of integration along which the path integral is well defined. This should reproduce the expected exponential behavior of the wave function, $\sim \exp[-a_0^2/G]$ in the Euclidean regime and the oscillatory wave function in the Lorentzian regime. Thus, boundary conditions and the choice of the contour of integration should justify our rules of analytical continuation for the classical solutions. This will need further investigation.

4 Do wormholes fix the low energy couplings?

Certainly, one of the fundamental motivations which incited many physicists to excitingly address the study of the topological features of quantum gravity resides in the idea, first mentioned by Hawking [57] and later explicitly developed by Coleman [14], that wormholes may have an important role in fixing the couplings of the physical interactions at low energies.

In particular, the summation over wormholes in the EPI for gravity seemed to turn all the coupling parameters into dynamical variables, sampled from a probability distribution. A saddle point analysis, then, resulted in a distribution with a sharp peak at the cosmological constant (Λ) set equal to zero, which appeared to solve a critical problem of both cosmology and particle physics.

In this chapter I will first proceed historically, schematically describing the feature of the cosmological constant problem (essentially following Weinberg [121]), and then introducing the main results about the wormhole theory, its developments and implementations in the goal of fixing other couplings, such as the gravitational constant and the masses of the particles etc..

4.1 The cosmological constant problem

When Einstein first tried to apply GR to cosmology [24], not yet aware of the Hubble expansion law of the Universe, he did look for a static model. However, this assumption was not compatible with his original equations, and he was obliged to introduce an extra free parameter Λ, the cosmological constant. This enters the Einstein-Hilbert action functional as:

$$I = \int d^4x\sqrt{-g}\left(-\frac{R}{16\pi G} - \Lambda + \mathcal{L}_{matter}\right) \tag{4.1}$$

and gives the equation of motion:

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G(T_{\mu\nu} + \Lambda g_{\mu\nu}) \tag{4.2}$$

where $T_{\mu\nu}$ is the stress tensor, $R_{\mu\nu}$ the Ricci curvature tensor, G the gravitational constant, $g_{\mu\nu}$ the 4-metric and g its determinant. This also admits static solutions, even if they are unstable, as firstly shown by [23]. Even if the cosmological constant was no longer necessary after Hubble's fundamental discovery, and even if after it was rejected by Einstein himself, it was not easy to drop the Λ term, because anything contributing to the energy density of the vacuum acts as a cosmological constant [123].

From Lorentz invariance, the vacuum expectation formula for $T_{\mu\nu}$ of the vacuum should be, in fact:

$$< T_{\mu\nu} >= - < \rho > g_{\mu\nu} \tag{4.3}$$

where $< \rho >$ is the vacuum mass density. Putting this into eq. (4.2), it has the same effect as introducing an effective cosmological constant or vacuum energy:

$$\Lambda_{eff} = \Lambda + < \rho >= \rho_V \tag{4.4}$$

Assuming the hypothesis of homogeneity and isotropy of the Universe (supported, e.g., by the observation of the cosmic microwave background and the galaxy spatial correlation function), the time-time component of the Einstein equation now becomes:

$$\left(\frac{\dot{a}}{a}\right)^2 = -\frac{k}{a^2} + \frac{8\pi G}{3}(\rho + \Lambda) \tag{4.5}$$

The present expansion rate is estimated as:

$$\left.\frac{\dot{a}}{a}\right|_o \doteq H_o \simeq (50 - 100) km sec^{-1} Mpc^{-1} \tag{4.6}$$

It also appears that we do not see gross effects of the spatial curvature, so:

$$\frac{k}{a_o^2} \leq H_o^2 \tag{4.7}$$

and the total mass density ρ is not much different than its critical value, giving:

$$|\rho - < \rho > | < \rho \leq \frac{3H_o^2}{8\pi G} \tag{4.8}$$

Therefore, from eq. (4.4), one obtains the upper bound:

$$|\Lambda_{eff}| \leq \frac{H_o^2}{8\pi G} \simeq 10^{-47} GeV^4 \tag{4.9}$$

However, this immediately appears to be in conflict with the usual expectations of QFT. For example, if one considers a free massive (m) scalar field (ϕ) theory with Lagrangian :

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \tag{4.10}$$

it is easy to find out that the zero point energy summed over all normal modes, with a wave number cutoff $M_p \simeq 10^{19} GeV$ introduced (if one believes GR up to the Planck scale), is:

$$<\rho>= \int_0^{M_p} dk \frac{4\pi k^2}{(2\pi)^3}\sqrt{k^2+m^2} \simeq \frac{M_p^4}{16\pi^2} \simeq 10^{74} GeV^4 \tag{4.11}$$

Therefore, the two terms in eq. (4.4) must cancel each other to better than 121 significant places to satisfy the observational bound of eq. (4.9)! Moreover, one has also to add infinite corrections to $<\rho>$ coming from the field interactions, and the bare constant Λ would have to be fine tuned at each order of the perturbation theory. This is the well known problem of fine tuning for Λ: it seems a miracle that microphysics should be fine tuned so precisely throughout all the phase transitions in the cosmological history, so that now the Universe can be big and flat and the macroscopical $\Lambda_{eff} \simeq 0$. Symmetries are of no help, since they are all expected to be broken at low energy.

4.2 The theory of the cosmological constant

A way out of the problem of the cosmological constant within the formalism of quantum gravity (QG) was suggested in a paper by Hawking [57], who studied the saddle point approximation (dominated by large 4-spheres) in EPI gravity where Λ is a positive effective dynamical variable. He showed (explicitly using an ad hoc 3-form field) that the probability of a given configuration is exponentially peaked at $\Lambda = 0$, like $\exp\left(\frac{3\pi}{G\Lambda}\right)$.

A similar method was used by Baum [5] to recover the same peak at $\Lambda = 0$, but he did not mention the role of topological fluctuations and studied a minimally coupled scalar field to make Λ dynamical.

However, the first detailed study and theory about the effects of wormholes on gravity, Λ and the other coupling constants is due to Coleman [14]. The analysis is essentially semiclassical and, assuming that the E.P.I. is correctly represented by the H.H. wave function, it shows that our universe should be selected out from a probability distribution which is peaked at $\Lambda = 0$. The argument is constructed, however, on a certain number of hypothesis which appear, if not debatable, at least unclear (see chapter 5). On the other hand, Coleman himself quoted [14]: " This is doubly a house built on sand...I think I have threaded my way through it safely but it is always possible that unknown to myself I am up to my neck in quicksand and sinking fast."

After Coleman's paper was published, parallelly to a literature tackling the study of a related topic (the 3^{rd} quantization models-see chapter 8), a number of works appeared trying to derive his main results in a more general and transparent way, possibly dropping the semiclassical assumptions. One of the most interesting is that by Banks, Klebanov and Susskind [78].

The main assumptions of the analysis are the same as in [14]. The E.P.I. for Q.G. is assumed; one integrates over all compact topologies, focusing on large spherical universes, eventually connected by tiny wormholes; they use the "dilute approximation" (wormhole average space-time separation much greater than their size, $L_w \sim M_p^{-1}$); wormholes have no characteristic length, and those much thicker than M_p^{-1} are neglected. Moreover, one neglects the possibility that a wormhole can divide into two or more and assumes no interactions among them, but only with the low energy fields in the large regions.

At first one can focus on a single universe with no wormholes. For the expectation value of some local gauge invariant observable M in a single universe, one can tentatively assume (this is in fact a crucial assumption, as I will explain in chapter 8):

$$< M >_\lambda = \frac{\int dg e^{-I(g,\lambda)} M}{\int dg e^{-I(g,\lambda)}} \tag{4.12}$$

where all the couplings are collectively indicated by λ, and g represents the metric and the other local fields. I is the action functional.

If one lets $\phi_i(x)$ be a basis for local operators at an event x, in an effective theory at $L \gg L_w$ the effect of a wormhole connecting two points x, x' of a single universe can be assumed to be represented by the insertion of the bilocal action:

$$\sum_{ij} C_{ij}\phi_i(x)\phi_j(x') \tag{4.13}$$

in the integrand of the path integral (e.g., the numerator of eq. (4.12)), where $C_{ij} \sim e^{-S_w}$ is the amplitude for a wormhole insertion, and S_w is the wormhole action. C_{ij} does not depend on spacetime, at least when the two points x and x' are far enough: this is because of the assumption that wormholes short circuit spacetime and have no characteristic length. The reason why one can effectively use expression (4.13) will be explained, at least for some particular cases, in section 6.1. Thus, the numerator of eq. (4.13) would be replaced by:

$$\int dg\ Me^{-I(g,\lambda)}\frac{1}{2}\int dxdx'\sum_{ij} C_{ij}\phi_i(x)\phi_j(x') \tag{4.14}$$

This process can be represented by a picture in which a line connects x and x' (see fig. (26)). The integrations on x and x' are to take into account all possible locations of the wormhole ends on spacetime.

Now consider the sum over any number of wormholes, as in fig. (27). If the wormholes are considered independent of one another (because of the dilute approximation), it is easy to see that a N wormhole contribution factorizes in the integrand of eq. (4.14) as:

$$\frac{\left(\frac{1}{2}\int dxdx'\sum_{ij} C_{ij}\phi_i(x)\phi_j(x')\right)^N}{N!} \tag{4.15}$$

where the $N!$ compensates for overcounting identical wormholes. For an arbitrary wormhole configuration one has to sum over N, and eq. (4.15) exponentiates giving:

$$e^{\frac{1}{2}\int dxdx'\sum_{ij} C_{ij}\phi_i(x)\phi_j(x')} \tag{4.16}$$

One can make use of the identity:

$$e^{\frac{1}{2}C_{ij}V_iV_j} \simeq \int \Pi_k d\alpha_k e^{-\frac{1}{2}D_{ij}\alpha_i\alpha_j}e^{-\alpha_l V_l} \tag{4.17}$$

(where the α are arbitrary parameters and $D_{ij} = C_{ij}^{-1}$ is the inverse of C_{ij}) and obtain for the matrix element:

$$\int \Pi_k d\alpha_k e^{-\frac{1}{2}D_{ij}\alpha_i\alpha_j}\int dg\ Me^{-I(g,\lambda)}e^{-\alpha_l\int dx\phi_l(x)} \tag{4.18}$$

If the λ_i are the coefficients of $\int dx\phi_i$ in the Lagrangian, one can write eq. (4.18) as:

$$\int \Pi_k d\alpha_k e^{-\frac{1}{2}D_{ij}\alpha_i\alpha_j} \int dg M e^{-I(g,\lambda+\alpha)} \tag{4.19}$$

where

$$I(g, \lambda + \alpha) = (\lambda_i + \alpha_i) \int dx\phi_i + I'(g) \tag{4.20}$$

Similarly, one can consider to take into account processes involving additional closed universes (see fig. (28)). Each additional universe is expected to give a factor $\int dg e^{-I(g,\lambda+\alpha)}$ in the α integrand (but note that the α's of different universes need not to be the same). Summing over any possible number of such universes (which have to be considered the same and independent of each other), once again turns into an exponential giving:

$$< M >= \frac{1}{N} \int d\alpha e^{-\frac{1}{2}D_{ij}\alpha_i\alpha_j} \int dg M e^{-I(g,\lambda+\alpha)} \exp\left(\int dg' \exp(-I(g', \lambda + \alpha))\right) \tag{4.21}$$

where N is a normalization factor. Comparing this equation with eq. (4.12), one can write:

$$< M >= \int d\alpha \rho(\alpha) < M >_{\lambda+\alpha} \tag{4.22}$$

where :

$$\rho(\alpha) = \frac{1}{N} e^{-\frac{1}{2}D_{ij}\alpha_i\alpha_j} X e^X, \qquad X = \int dg e^{-I(g,\lambda+\alpha)} \tag{4.23}$$

The functional integral in eq. (4.23) is intended over geometries with no wormholes. Eq. (4.22) and (4.23) say that any expectation value computed in our universe is a weighted average over expectation values in universes without wormholes and couplings $\lambda + \alpha$. This is the formula for an ensemble of worlds with a statistical distribution of coupling constants. An observer in one of the members of the ensemble would have no way to deduce the existence of the others. One has "superselection" sectors labelled by the α's, not communicating through any local physics, with α dependent coupling constants. The number of independent α's is in principle the same as that of the gauge invariant local operators, i.e. infinite. Because the integration variables are not functions of the position, the effects of wormholes is to equalize the couplings in all the regions of spacetime.

The next task is to compute the probability distribution ρ . A simple way to do this is to consider only large smooth spherical topologies and to assume that the leading approximation to X is the contribution from the classical stationary point associated with Euclidean de Sitter space. I will discuss these critical assumptions in the next chapter. One can calculate the effective action for gravity at a scale $L \gg L_w$, integrating over all fluctuations including matter and gauge fields. The result can be then expanded in powers of the curvature tensor and its derivatives:

$$S_{eff} = \int \sqrt{g} \left(\Lambda - \frac{1}{16\pi G} R + a R_{abcd} R^{abcd} + b R_{ab} R^{ab} + c R^2 + ... \right) \qquad (4.24)$$

Loop corrections are small (suppressed by powers of $\frac{L_w}{L}$) and the massive fields (heavier than L^{-1}) have been integrated out and light fields set equal to values minimizing Γ. λ, G, a etc. are the fully renormalized couplings , including all effects of all interactions; they all depend on the shifted fundamental parameters $\lambda+\alpha$, due to the integration of wormholes and loop fluctuations. Approximating S_{eff} by Einstein gravity in a small curvature limit (i.e., initially neglecting a, b, c), the variational equation derived from eq. (4.24) is:

$$R_{\mu\nu} = 8\pi G \Lambda g_{\mu\nu} \qquad (4.25)$$

For $\Lambda > 0$, one has a space of maximum volume, the 4-sphere whose radius becomes large as $\Lambda \rightarrow 0$, while for $\Lambda < 0$ there is no known maximum volume. Therefore, restricting to large 4-spheres of radius r, one can write:

$$R_{abcd} = \frac{1}{r^2}(g_{ac}g_{bd} - g_{ad}g_{bc}) \qquad (4.26)$$

Substituting this back in eq. (4.24) one finds:

$$S_{eff} = \frac{8}{3}\pi^2 \left(\Lambda r^4 - \frac{3}{4\pi G} r^2 + A_1 + \frac{A_2}{r^2} + .. \right) \qquad (4.27)$$

The stationary point of S_{eff} can easily be seen to occur (for large r or small Λ) at $r \simeq \sqrt{\frac{3}{8\pi G(\alpha)\Lambda(\alpha)}}$. This gives

$$S_{eff} \simeq -\frac{3}{8G^2\Lambda} \qquad (4.28)$$

Then, from eq. (4.23) one has :

$$\ln \rho = \begin{cases} \propto e^{3/8G^2(\alpha)\Lambda(\alpha)} & \Lambda \rightarrow 0^+ \\ \rightarrow 0 & \Lambda \rightarrow 0^- \end{cases} \qquad (4.29)$$

This shows the fundamental infrared divergence as $\Lambda \to 0$, or better a peak at $G^2(\alpha)\Lambda(\alpha) = 0$.

To properly normalize the probability distribution, one can introduce a volume cutoff, restricting the E.P.I. to 4-spheres with radius less than r_{max}. In this case, the minimum of S_{eff} (for $\Lambda > 0$) occurs at $r^2 = \frac{3}{8\pi G}$ for $\Lambda \geq \frac{3}{8\pi G r_{max}^2}$, and at $r^2 = r_{max}^2$ for $\Lambda \leq \frac{3}{8\pi G r_{max}^2}$ (see fig. (29)). Therefore, the stationary action is:

$$S_{eff}(r) \simeq \begin{cases} -\frac{3}{8G^2\Lambda} & \Lambda \geq \frac{3}{8\pi G r_{max}^2} \\ -\frac{3}{8G^2\Lambda}\left[2(\frac{8\pi G\Lambda r_{max}^2}{3}) - (\frac{8\pi G\Lambda r_{max}^2}{3})^2\right] & 0 < \Lambda \leq \frac{3}{8\pi G r_{max}^2} \end{cases} \tag{4.30}$$

If one chooses $r = L$, not only fluctuations at wavelengths less than r are absorbed in the renormalized Γ, but also those of larger wavelengths are absent because the volume acts as an infrared cutoff. The typical value in the distribution is $\Lambda \propto \frac{1}{r_{max}^2 G}$. Normalizing in α and removing the cutoff ($r_{\max} \to \infty$), $\rho(\alpha)$ becomes highly concentrated on that submanifold of α-space (if it exists) on which $\Lambda(\alpha) = 0$.

Essentially, the solution to the cosmological constant problem works like this: wormholes say that, on extremely small scales, our universe is in contact with other universes governed by the "same" physics; even if one imagined to live in the inflationary epoch (!), with our universe small and hot, the other universes would be large and cool, and see $\Lambda = 0$: prearrangement is turned in precognition!

4.3 Comments on the Coleman approach

One of the points in which Coleman's original work differs from that by [78] is that Coleman treats wormholes semiclassically, i.e. he assumes that the only wormhole configurations contributing to the E.P.I. are stationary points of the Euclidean equation of motion, and in considering small fluctuations about them. Moreover, it is assumed that the size of dominant wormholes is greater than the Planck scale and their action is a large multiple of $\hbar$. The whole argument of [78], on the other hand, seems to depend very little on the details of physics near the Planck scale and, in particular, on the existence of classical wormholes of Planck size. The idea is that, when the size a of the wormhole approaches the Planck scale, one does not know how the action varies. In fact, one does not even know if the notions of metric and manifold correctly define a wormhole. The only important assumption is that the integral over a makes sense and gives rise to an effective description in terms of bilocal operators. Moreover, in Einstein gravity, the contribution of a wormhole with positive action is maximized at the end point $a = 0$, not at the saddle point associated with the classical solution.

Let us now briefly discuss the main points in Coleman's theory, both for a historical reason and because they will also be useful for a better understanding of chapter 8. One neglects the possibility that a wormhole can divide into two or more and assumes no interactions among them (i.e., they are far enough form each other), but only with the low energy fields in the large regions.

In general, "baby universes" (the midpoint spatial sections of the wormholes, in semiclassical approximation) are distinguished by their size, the state of internal excitation, their charge etc., and are labelled, for simplicity, with a discrete index i. Annihilation (a_i) and creation ($a_i^\dagger$) operators are introduced for each type of "baby", satisfying the usual bosonic commutation rule:

$$[a_i, a_j^\dagger] = \delta_{ij} \tag{4.31}$$

"Babies" are closed universes, therefore carry zero energy, linear and angular momentum, and zero values of any conserved gauge local charge; they may, however, carry ungauged charges, i.e. violate global conservation laws. The action of each wormhole is defined as $2S_i$, and the amplitude for inserting the end of a semiwormhole in the background is $K_i\sqrt{g}d^4x$, with K_i some function of the background fields in the manifold.

Coleman imagines to start with a local field theory at the Planck scale L_p, and to introduce wormholes at a larger scale L_w. At a scale $L > L_w$, one can then build a new local effective theory where shorter wavelength loop fluctuations and wormholes have been integrated out, leaving only topologically trivial manifolds. The E.P.I. for one large universe stripped of "handles" like that in fig. (30) is calculated as (see [13]):

$$\int dG\ e^{-S(G)} = \int dg\ \ < n'_1 n'_2..|e^{-S_{eff}}|n_1 n_2.. > \tag{4.32}$$

where G denotes all possible 4-geometries while g only those without wormholes. The $|n_i >$'s represent the baby-number eigenstates which have to be regarded as a complete unknown to any observer in our universe (see section 6.2), and are presumably determined by Planck scale physics, while

$$S_{eff} = \int_M d^4x\sqrt{g}(\mathcal{L}_o + \sum_i (a_i^\dagger + a_i)e^{-S_i}K_i) \tag{4.33}$$

($\mathcal{L}_o$ is the renormalized Lagrangian). One can now introduce a new set of operators:

$$A_i = a_i + a_i^\dagger \tag{4.34}$$

They are Hermitian and mutually commuting operators, i.e. they can be simultaneously diagonalized by eigenstates $|\alpha_i >$ with eigenvalues α_i.

$$A_i|\alpha_j >= \alpha_i|\alpha_j >, \qquad < \alpha'_j|\alpha_k >= \Pi_i\delta(\alpha_i - \alpha'_i) \tag{4.35}$$

The parameters α do not depend on spacetime (i.e. $\partial_\mu A_i = 0$). The new eigenstates are related to the $|n >$ basis through:

$$< \alpha|n >= \Pi_i\psi_{n_i}\left(\frac{\alpha_i}{\sqrt{2}}\right) \tag{4.36}$$

where $\psi_n(\alpha)$ is the n^{th} energy eigenfunction of the harmonic oscillator $H = (p^2 + q^2)/2$. The $|\alpha >$ states have the property:

$$\begin{aligned} < \alpha'_1...|e^{-S_{eff}}|\alpha_1... > &= e^{-S_{eff}(\alpha)}\Pi\delta(\alpha'_i - \alpha_i) \\ &= e^{-\int_M d^4x\sqrt{g}(\mathcal{L}_o + \sum \alpha_i e^{-S_i}K_i)}\Pi\delta(\alpha'_i - \alpha_i) \end{aligned} \tag{4.37}$$

At this level, from the point of view of an observer in our universe who cannot detect baby universes, the α's appear just as arbitrary parameters. The effect of wormholes is to add local interactions to the bare Lagrange density, one for each kind of wormhole (which, a priori, is an infinite number). As a result, one will observe the fundamental coupling constants shifted as

$$\lambda_{eff} = \lambda_i + \alpha_i e^{-S_i} \tag{4.38}$$

This is essentially the same result as that in [78].

Now, if one allows for an arbitrary number of universes, assuming the H.H. prescription for the definition of the E.P.I., in the E.P.I. one has to sum over all closed connected 4-manifolds, including those that become disconnected when all the wormholes are removed (the usual typical geometry is that in fig. (31)). Each connected manifold can be factorized into a connected component on which a given local observable O can be measured, and a remainder on which O does not act. Since the sum over all closed 4-D geometries can be written as the exponential of a sum over all connected closed 4-geometries [71], the expression for the E.P.I. is shown to become:

$$Z = N \int d\alpha P(\alpha) dg' e^{-S_{eff}(g',\alpha)} \exp\left(\int dg \ e^{-S_{eff}(g,\alpha)}\right) \tag{4.39}$$

where the integrations over g and g' denote sums over connected closed 4-geometries without wormholes and N is a normalization factor. $P(\alpha)$ comes from a projection of the initial and final $|n>$ states on the $|\alpha>$ ones, and is essentially the square of the product of harmonic oscillator wave functions for each type of wormhole. The dominant term in the probability distribution appearing in eq. (4.39) is, as before, the double exponential term. The semi-classical interpretation of the distribution may be described in this way. For an observer doing experiments at scales greater than the wormhole one, the A's eigenstates are simultaneous eigenstates for all the coupling constants, which are then renormalized by the α's. Which A-eigenspace we are in determines the coefficients of all the various operators in the effective Lagrangian in eq. (4.37). Measuring one coupling constant, one will find a spread of values described by $P(\alpha)$; our universe is chosen at random from an ensemble of possible universes, each with different values of the α's and, hence, of the constants. Once measured, or selected a theory in which the couplings have certain values, successive measurements will then always give the same values (as in the collapse of the wave function in ordinary quantum mechanics). This is reminiscent of a sort of statistical [13] or quantum [64, 102] indeterminacy in the distribution of the physical coupling constants. In other words, the interaction terms in the effective Lagrangian may not have definite values that can be predicted with certainty by some ultimate theory [64]. The procedure to calculate the peak of the probability distribution is then essentially the same as in [78].

4.4 Fixing G and the other constants of nature

The interesting and fundamental idea that wormholes might fix most, if not all, the constants of nature which appear in an effective Lagrangian theory was first introduced by Coleman [14], who gave it the name of the "big fix". Later on, a lot of papers [11, 15, 31, 38, 39, 78, 98, 102, 103] tackled the problem, in the context of different and more or less complicated models for gravity coupled to matter fields, but no unique, certain and, sometimes, reasonable physical results have been found so far.

In his original paper, Coleman defines a nonlinear change of variables in the α space, $\alpha_o \doteq \frac{8}{3}G^2\Lambda$, while the other α 's are denoted by $\hat{\alpha}$; in this way, the stationary value of S_{eff}, including higher order curvature operators, is expected to have an expansion of the kind (see also [60]):

$$S_{eff} = -\frac{1}{\alpha_o} + S_o(\hat{\alpha}) + \alpha_o S_1(\hat{\alpha}) \tag{4.40}$$

One can compute:

$$\ln \frac{\rho(\alpha_o, \hat{\alpha})}{\rho(\alpha_o, \hat{\alpha}')} = e^{1/\alpha_o} \left(e^{-S_o(\hat{\alpha})+..} - e^{-S_o(\hat{\alpha}')+..} \right) \tag{4.41}$$

which is the same as:

$$\frac{\delta\rho(\alpha)}{\rho(\alpha)} = e^{-S_{eff}} \delta S_{eff} \tag{4.42}$$

These equations say that, for $\Lambda \to 0^+$, a small correction to S_{eff} will have a big effect on the probability, which then would be concentrated also on the submanifold where S_o is minimum (if it has a minimum for finite α!); similarly, $\rho(\alpha)$ would be concentrated at the minimum of the minimum of the higher order coefficients in the S_{eff} expansion. This would lead to an infinite number of conditions on the α's, and it is hoped that these can cause $\rho(\alpha)$ to collapse at a single fixed value of all the α's and, as a consequence, of the couplings of nature which should all depend on the α's (see eq. (4.38)). This is the "big fix".

Obviously, to find out such condition on the α's (i.e., minimize S_o, S_1 etc..) is another problem. For example, the addition of a term like the Euler density, which is assumed to have no effects in the low energy effective theory, might require a detailed knowledge of the wormhole physics.

Another mechanism for fixing the effective couplings has been proposed by Preskill [102]. The main idea derives from the observation that, if the dominant term in S_{eff} is $-\frac{3}{8G^2\Lambda}$, the probability distribution of eq. (4.29), should be peaked at $G^2\Lambda = 0$, i.e. not only at $\Lambda(\alpha) = 0$, but also at $G(\alpha) = 0$. However, since one knows that $G(\alpha) \neq 0$, because one observes gravity, there should be a peak at some minimum value of $G(\alpha)$ on the surface $\Lambda(\alpha) = 0$. One might hope that this minimum would occur at an isolated point in α space, where all the α's are fixed, and therefore determine all the constants of nature, since all contribute to G through renormalization effects (everything couples to gravity). The only problem would be to compute the exact dependence of G and of the constants on the α's.

To go beyond the "dilute approximation" and consider instanton interactions is essential in ensuring that G has a non zero minimum. An easy way to see this is to note that $G = 0$ is a fixed point under renormalization (no renormalizations in G if gravity is free); since $G_{obs}(\alpha) \geq G_{min} > 0$, one has to impose that the bare coupling renormalized by wormholes at L_w satisfies: $G_o(\alpha) \geq G_{o,min} > 0$. Clearly, if the effective action is linear in α (dilute approximation), $G_o^{-1}(\alpha)$ is also linear in α, and G_o might vanish for an infinite combination of the α's: this is in contrast with the existence of the lower bound on G_o, and the only way out is to assume that the dilute approximation is not valid.

Another motivation comes from the fact that, in the effective theory with the cutoff at $L_w = M_w^{-1}$, one expects Λ to be renormalized (on dimensional grounds) by $\sim M_w^4$; therefore, to avoid an unnatural fine tuning of the bare Λ_o to cancel this renormalization, the linear α dependent shift should be of the order M_w^4. But since the instanton density is just of the order $\alpha \sim M_w^4$ (see section 6.3), this implies a mean instanton separation to be L_w, which clearly violates the dilute approximation.

A toy model shows that, with instanton interactions taken into account,

$G(\alpha)$ may effectively be a bounded function of α. One can divide the Euclidean spacetime into identical cells of volume L_w^4, and describe the configuration of the instanton gas by giving each cell i the number n_i of instantons that occupy that cell (see fig. (32)). Only one type of instanton and of α are assumed. Instantons in distinct cells are assumed not to interact, while those in the same cell do interact. If, at first, one considers the instantons as non interacting identical particles, the effective action generated by the sum over arbitrary configurations for the instanton gas is given by:

$$e^{-\delta S(\alpha)} = \Pi_i \sum_{n_i=0}^{\infty} \frac{1}{n_i!}(\alpha O_i)^{n_i} = e^{(\alpha \sum_i O_i)} \tag{4.43}$$

where O_i expresses the dependence of the instanton on the background geometry. This is the same as in the dilute approximation. To simulate interactions, one can reproduce the sum over n by $\sum_{n=0} \frac{a_n}{n!}(\alpha O_i)^n$: $a_n > 1$ (< 1) represents an attractive (repulsive) short distance range n-body interaction. The effective action becomes now:

$$\delta S(\alpha) = -\sum_i \ln f(\alpha O_i), \qquad f(z) = \sum \frac{a_n z^n}{n!} \tag{4.44}$$

If one expands O in powers of derivatives of the background metric (i.e., we are in a smooth manifold) as:

$$O = c_o + c_1 R + \ldots \tag{4.45}$$

then one has (the sum over cells is replaced by a volume integral):

$$\delta S(\alpha) = \int d^4x \sqrt{g} L_w^{-4} \left(-\ln f(c_o\alpha) - \frac{\alpha f'(c_o\alpha)}{f(c_o\alpha)} c_1 R + \ldots \right) \tag{4.46}$$

Therefore, the shift in the gravitational constant is:

$$\delta \left(\frac{1}{16\pi G_o} \right) \simeq \frac{\alpha f'(\alpha)}{f(\alpha) L_w^4} \tag{4.47}$$

which can be bounded for appropriate choices of the a_n's.

For the whole theory to work, now, the assumption on the bare G_o ($\geq G_{o,min}$) must guarantee that also the renormalized (physical) $G \geq G_{min}$. Moreover, one has also to find some suitable finite bounds for the renormalized couplings of the terms in the action (4.24) involving higher derivatives of the metric (which could, otherwise, cancel the infrared peak in S_{eff}, and might be inconsistent with the Einstein theory of gravity at the classical level), even if the bare couplings at the wormhole scale are in principle allowed to vary without restrictions (as a function of the α's).

To actually determine the α's, one has to compute $G(\alpha)$ in an effective theory with an ultraviolet cutoff M much smaller than M_w. If M descends below the lightest massive particle in the original bare theory at scale M_w, the only remaining degrees of freedom are that of massless fields, such as the graviton and the photon: $M \to 0$ is called the "continuum" limit of the bare theory. In this limit, even if the photon may couple to the graviton, the self interactions of matter have been integrated out, and the only surviving interaction is gravity. Also Λ should go to zero, according to the results of sections 4.2 and 4.3. The assumption is now that Einstein gravity has no nontrivial continuum limit (other than a free theory). This can be stated with more precision saying that, for any choice of the bare theory with cutoff M_o, all dimensionless renormalized couplings λ^a_m approach zero as $M \to 0$, or:

$$\lambda^a_M \leq \lambda^a_{MAX} \left(\frac{M_o}{M}\right) \to 0 \tag{4.48}$$

The fact that renormalized couplings are bounded even if the bare ones are not is a common phenomenon in renormalization theories, and can be described in terms of strong charge screening effects. This assumption is also essential to justify neglecting (at least perturbatively) higher derivative terms in the action. This renormalization group argument seems just to provide an upper bound on G, but not to avoid $G \to 0$, which obviously is not a good result. At this point, Preskill guesses that the same screening effects mentioned above might prevent infinite field renormalizations on G_o, which would give $G \sim G_o$: if one has a minimum $G_o \geq M_o^{-2}$ (on dimensional grounds), the scheme might work. This discussion appears at the moment rather speculative.

An explicit calculation is done for the case of the stationary effective action (eq. (4.24)), where the expansion in M has become an expansion in r^{-1} (r is the radius of the classical solution).

From a perturbative analysis and power counting of Feynman diagrams, it turns out that the renormalization of G is dominated by short distance quantum fluctuations sensitive to the details of Planck scale physics.

At 1-loop it is found that a diagram like that in fig. (33) gives a renormalization:

$$\delta\left(\frac{1}{16\pi G}\right) = -\frac{1}{192\pi^2}m^2 \ln\left(\frac{M_o^2}{m^2}\right) + .. \qquad (4.49)$$

where m is the (physical) mass of a free minimally coupled fermion or scalar. The criterion of minimizing G, therefore, would favour lighter matter particles, but one should also take into account the contributions from power divergent renormalizations to G^{-1}. These depend on whether the light particles have adjustable bare masses (m_o), or if they acquire their masses from interactions. In the first case, in fact, maximizing G^{-1} with respect to m_o one finds [102] the condition:

$$\ln\left(\frac{M_o^2}{m^2}\right) \simeq 1 \qquad (4.50)$$

Therefore, the maximum of G^{-1} will be destabilized if $m \ll M_o \sim M_w$. This instability can be avoided only if the physical mass of the particle vanishes exactly. To be compatible with the known features of particle physics, this suggests that the light mass scales are to be determined dynamically.

A slightly different approach to the Newton constant problem has been described by Grinstein [38], for the case of charged wormholes.

Using the explicit form of the vertex found for a charged scalar field ($\phi \simeq \rho e^{i\theta}$) wormhole (see section 6.1), one can expect additional terms in the effective Lagrangian of the kind $cM_w\phi R$ (c constant, R is the scalar curvature and it is assumed a charge $Q = 1$). In the broken symmetry case, this renormalizes the bare G_o like $\Delta(\frac{1}{G_o}) = c\alpha v M_w$, where v is the vacuum expectation value of $|\phi|$; this is distressing since, naively, for $|\alpha| \to \infty$ and $-\frac{\pi}{2} - arg(c) < arg(\alpha) < \frac{\pi}{2} - arg(c)$, one obtains $G_o \to 0$. The idea is that also v will in general depend on α, and the phases of ϕ and α may be correlated in such a way to decrease $\frac{1}{G_o}$.

One may study a simple toy model with an effective potential for ϕ at the wormhole scale:

$$V_{eff}(\phi) = \frac{1}{2}\lambda(|\phi|^2 - v^2)^2 + bM_w^3\alpha\phi + h.c. \qquad (4.51)$$

with the phase of α absorbed in ϕ such that α is real. Minimization of V with respect to θ gives the conditions:

$$\begin{cases} \alpha|b|\rho\sin(\theta + arg(b)) = 0 \\ \alpha|b|\rho\cos(\theta + arg(b)) < 0 \end{cases} \tag{4.52}$$

or, $\theta_o = \pi - arg(b)$ at $|\phi| = v$. Now, one has:

$$\Delta\left(\frac{1}{G_o}\right) = -\alpha|c|M_w\rho_o\cos(arg(c) - arg(b)) \tag{4.53}$$

which clearly is non positive (and decreases G_o^{-1}) for $\cos(arg(c) - arg(b)) \geq 0$. This might solve the problem. Another example is to consider the effect of a real scalar particle of mass m. Taking into account quantum fluctuations, one has a 1-loop renormalization of G (at the scale μ)[39]:

$$\Delta\left(\frac{1}{16\pi G}\right) = -\frac{1}{8\pi^2}\left(\xi - \frac{1}{6}\right)m^2\ln\left(\frac{m^2}{\mu^2}\right) + \frac{19}{24}m^2 \tag{4.54}$$

where ξ is the coupling of the scalar to R ($\xi = \frac{1}{6}$ for conformal coupling). Both ξ and m depend on α. If the scalar is the same charged scalar as before, then $\xi(\alpha) = 0$ and $m^2(\alpha) = 2\lambda(3\rho_o^2(\alpha) - v^2)$, at tree level. From eq. (4.54) one has $\Delta\left(\frac{1}{G}\right) \to \infty$ and, therefore, $G \to 0$ (G_o is assumed finite), if $\rho_o^2(\alpha) \to \infty$. To avoid a trivial free gravity, therefore, one must check not to upset the condition $G_o \neq 0$ or, in other words, one has to check that $G_o m^2 \ln m^2$ does not diverge as $\rho_o^2(\alpha) \to \infty$.

For $Q = 1$ and large α, $\rho_o \sim \alpha^{\frac{1}{3}}$ (from the minimum of eq. (4.51)) and $G_o \sim (\alpha\rho_o)^{-1} \sim \alpha^{-\frac{4}{3}}$, so that $G_o m^2 \ln m^2 \sim \alpha^{-\frac{2}{3}}\ln\alpha \to 0$ and the theory is safe ($G \not\to 0$).

For general Q, the leading term in V_{eff} is $\sim \alpha_Q\rho^Q$ (see section 6.1), $\rho_o \sim \alpha_Q^{\frac{1}{4-Q}}$ (from the minimum of eq. (4.51)) and $G_o \sim \alpha_Q^{\frac{4}{4-Q}}$, so that $G_o\rho_o^2 \sim \alpha_Q^{-\frac{2}{4-Q}}$: for $Q < 4$ the theory is safe again. For larger wormholes ($Q > 4$) one hopes in the existence of some mechanism for their suppression (see section 5.4).

It is interesting to note that the condition $\Lambda = 0$ is (luckily) incompatible with $G = 0$, since the former is achieved for finite α, while the latter may occur in this model only for $\alpha \to \infty$.

On the other hand, a simple argument seems to somehow fix the value of the mass for a free fermion field (eq. (4.54) with $\xi = 0$). If the method is to maximize M_p, then eq. (4.54) naively says that $m^2 \to \infty$ is favoured. However, this possibility is clearly inconsistent with an effective theory for long wavelength fluctuations. This suggests a "bootstrap" condition for $m^2(\alpha)$: to consistently consider the case $m^2(\alpha) \geq M_w^2$, one has to integrate the field before the wormholes, presumably implying a renormalization giving $m \sim M$. Therefore, unless protected by a symmetry (unbroken by wormholes), the particle masses should be driven to the wormhole scale, giving finite renormalizations to G^{-1}. Once again, to be consistent with experimental measures, dynamical generation is required.

Klebanov, Susskind and Banks [78] tried to determine some limits on the pion mass. They considered the action (4.24), minimized it with respect to r and found:

$$S_{eff} = -\frac{3}{8G^2\Lambda} + \frac{8\pi^2}{3}A_1 + O(G\Lambda) \tag{4.55}$$

They assume that the maximization of $\rho(\alpha)$ (eq. (4.23)) is achieved by the conditions $G^2\Lambda = 0$ and A_1 to be at its minimum. Since A_1 is dimensionless, it will be generally logarithmically divergent in the ultraviolet, and it will depend on the short distance physics. For the Lagrangian of a free minimally coupled pion π with coupling f_π and mass m_π:

$$\mathcal{L} = \frac{1}{2}[(\nabla\pi)^2 + m_\pi^2\pi^2] \tag{4.56}$$

it is found that [39]:

$$A_1 = +\frac{29}{480\pi^2}\ln\frac{m_\pi^2}{f_\pi^2} + O(m_\pi^2) \tag{4.57}$$

which is minimized for $m_\pi = 0$!

A suggested possible way out of this unphysical result is to abandon the dilute approximation for the wormholes and introduce a nonlinear dependence on α in the couplings. The idea is that in α space the surfaces $\Lambda(\alpha) = 0$ and $m_\pi(\alpha) = 0$ have no particular reasons to intersect (see fig. (34)). For instance, wormholes shifting m_π and Λ carry different quantum numbers: the former has a chiral charge, the latter is a chiral singlet.

A fundamental and detailed analysis of the model for an effective action with a scalar field interacting with itself and coupled to a high derivative gravity is due to Grinstein and Wise [39]. They considered the action, neglecting terms of order M_p^2:

$$S_{eff}(\alpha) = \int d^4x\sqrt{g}\left(\frac{1}{2}(\partial\phi)^2 + V(\phi) - U(\phi)R + \beta R_{abcd}R^{abcd} + \gamma R_{ab}R^{ab} + \delta R^2\right) \tag{4.58}$$

On S^4, using $R_{abcd} = (g_{ac}g_{bd} - g_{ad}g_{bc})$, $\sqrt{g}d^4x = r^4 d\Omega_5, \Omega_5 = \frac{8\pi^2}{3}$, this becomes:

$$S_{eff}(\alpha) = \int d\Omega_5 r^4 \left[-\frac{1}{2r^2}\phi\Box\phi + V(\phi) - 12\frac{U(\phi)}{r^2} + \frac{\hat{c}}{r^4} \right] \tag{4.59}$$

To obtain the effective action for gravity one has to integrate out ϕ, which is equivalent to expand S_{eff} to quadratic order about the stationary point $\bar{\phi}$, assumed to be constant on the S^4 sphere. Using dimensional regularization at 1-loop, this gives:

$$\Gamma_\alpha(r) = \mu^{n-4}\Omega_{n+1}r^n[V(\bar{\phi}) - n(n-1)U(\bar{\phi})/r^2 + \hat{c}/r^4 + V_{1-loop}(\bar{\phi})] \tag{4.60}$$

where μ is the subtraction point, $n = 4 - \epsilon$ and V_{1-loop} can be expanded in eigenfunctions on the sphere. The mass of the scalar is $m^2 = V''(\bar{\phi})$, and the assumed criterion for minimizing Γ is to minimize $\hat{c}$:

$$\delta\hat{c} = \frac{1}{64\pi^2}\left[(12U'' + 2) - \frac{2}{15}\right] \ln\left(\frac{V''}{\mu^2}\right) \tag{4.61}$$

All the couplings $(m^2, \hat{c}, U'')$ are assumed to be bounded functions of α depending on the wormhole physics. A detailed study of the renormalization group equation for this model (which I will not discuss here) shows the following conclusion: if the minimization of G does not fix all the α's, then, in the region of α space with $m < M_p$, $\rho(\alpha)$ is peaked for m^2 very small, i.e. one may have naturally small scalars.

A similar but simpler model has also been considered by Myers and Periwal [98], who reached, however, a different conclusion. The effective action is essentially the same as that of eq. (4.56). They assumed that wormholes convert $V \to V_\alpha$ ($U \to U_\alpha$), but that derivative terms are unaffected. Assuming that the saddle point in eq. (4.58) occurs for constant ϕ, the ϕ equation of motion is:

$$V'_\alpha(\phi) - \frac{12}{r^2}U'_\alpha(\phi) = 0 \tag{4.62}$$

where r is the radius of a S^4 sphere, $r^2 = 6U_\alpha(\phi)/V_\alpha(\phi)$ (as it can be shown from the Einstein equations). Therefore, from eq. (4.62), one gets:

$$\left(\frac{V_\alpha(\phi)}{U_\alpha^2(\phi)}\right)' = 0 \tag{4.63}$$

whose solution is ϕ_c. The action at the saddle point then becomes:

$$S = -48\pi^2\frac{U_\alpha^2(\phi)}{V_\alpha(\phi)} + c + .. \tag{4.64}$$

where c is the constant contribution from R^2 terms.

If one denotes ϕ_o as a zero of the potential, $V_\alpha(\phi_o) = 0$, it can be seen that this provides the dominant saddle points, even if, in general, $\phi_c \neq \phi_o$. In other words, the important configurations do not necessarily come from solutions of the equation of motion (see [77, 78]). Now, to maximize $\rho(\alpha)$, the criterion is to look first at the zeros of $V_\alpha(\phi)$ in eq. (4.64), then to compare $U_\alpha^2(\phi)/V_\alpha(\phi)$ for different zeros and, finally, to minimize c. It is easy to understand that the α values are determined by the requirement, if V_α is a polynomial in ϕ, that its zeros are degenerate (provided $U_\alpha \neq 0$ for the same α) and that V is a polynomial of as high a degree as possible (with a high order zero). In particular, this analysis shows that, if $\Lambda = 0$, the mass of the scalar field should vanish exactly, independently of calculating loop corrections to V and overwhelming any effect due to the c term.

More recently, Gavilanes and Perez-Mercader [31] computed the 1-loop effective action in the dominant, zero cosmological constant, spherical spacetime (S^4), where there are scalar (m_s), fermionic (m_f) and vector (m_v) degrees of freedom, taking into account corrections up to the 3^{rd} power in the curvature. The stationary action, extremized with respect to the radius of S^4 is:

$$\Gamma = -\frac{3}{8G^2\Lambda} - \frac{64\pi^3}{9}G\Lambda B_3 + O(G^2\Lambda^2) \tag{4.65}$$

A detailed study of the coefficient B_3 (dependent on the mass of the particles) for the cases where only one of the particle fluctuations is picked up shows that it is not possible to fix the values of their masses but, rather, only constrain them on certain ranges. These constraints become less rigid when a richer particle variety is included. For example, for a theory containing scalars, fermions and gauge degrees of freedom, B_3 is positive and driven to infinity, and the phenomenological ratio $m_s^2 \gg m_f^2$ is admitted provided $m_v^2 \to 0$.

Another interesting issue has been tackled by Preskill, Trivedi and Wise [103] and by Choi and Holman [11], who used wormhole effects to calculate the dependence of M_p on the θ angle of QCD and to show that it should be fixed at π, which is in apparent contradiction with recent experimental results based on chiral perturbation theory calculations (which give $\theta = 0$). This will probably be a big stimulation for testing the validity of the wormhole theory.

As it is easy to see from the analysis of all these works, we are at the moment far away from a well defined and unique theory about the "big fix", and the actual results appear still rather controversial if not, sometimes, in contradiction with well known experimental facts. This is one of the tasks of the future work for a deeper understanding of the wormhole theory.

5 Problems

5.1 Difficulties of the theory

The whole scheme just presented about the wormhole theory and its claimed effects for the elegant solution of the cosmological constant problem, for the idea of the possible determination of the coupling constants of nature and for the introduction of an extra degree of indeterminacy in the laws of physics and in quantum gravity may, however, be seriously threatened by a lot of unclear or ill defined issues, some of which I will begin to enumerate here (see also section 1.9).

First of all, all the models presented so far are based on a not yet existent well defined formulation of gravity in terms of the EPI. This is related to the problem of finding a contour of integration for which the EPI is made convergent, as I will explain in section 5.2.

Both [14] and [78] appear to restrict their attention (essentially for simplifying the calculation which gives $\Lambda = 0$) only to the stationary points in S_{eff} which are large smooth geometries. But are they the dominant ones? What about more complicated topologies or, at least, the simple case of spheres of different sizes (the argument of [78], which I will discuss in section 7.1, does not appear to be so convincing)? This question is, in fact, strictly related to the problem of finding a contour, even in semiclassical approximation, passing through all these stationary points. Moreover, we do not yet know whether wormholes really have a characteristic size (which seems true, at least for the classes of known solutions), and one has not considered the possibility of having other exotic topological features near the Planck scale. How to deal with little or even large inhomogeneities and anisotropies of the metric? These might be important particularly in the first stages of the cosmic evolution. Preskill [102] takes the perspective that the EPI may also receive important contributions by strong renormalizations due to quantum fluctuations at scales just below

M_p. Then, is there a clear distinction between small wormholes and large universes (which seems essential in the construction of an effective theory "a la Coleman")? Another important point is the following. In the discussion of the Λ theory, one appears to confuse properties of a single universe theory and possible effects coming from universes interactions. This is quite evident, for instance, in Coleman [14], where the HH wave function is used without taking into account possible modifications of the WdW equation due to the presence of wormholes. One has also to properly take into account the interactions among the wormholes themselves (a model is presented in section 6.3). In this sense, the approach to a 3^{rd} quantized theory (as described in chapter 8) appears essential. Then, what about the eventual presence of macroscopically or even cosmically large wormholes (which would violate locality)? These large wormholes might be a non secondary effect in the $\Lambda = 0$ theory. This will be the argument of section 5.4. One important mathematical task is, moreover, to explicitly demonstrate that the α's maximizing $\rho(\alpha)$ at some fixed values of the couplings really exist, and to be able to properly normalize the α measure in the PI. Strictly connected to this is the problem of the regulator for the infrared divergence at $\Lambda = 0$ of the probability measure. This will be discussed in details in section 5.3.

Another fundamental and debatable assumption for the whole Coleman's mechanism is that about the definition of probabilities and transition amplitudes in the EPI formalism. In section 4.2 I superficially played with expressions like eq. (4.12) and others, but I did not stress the problem of answering the following questions. How actually determine the "right" wave function of the multiuniverse theory (Coleman's prescription picks out one, but obviously it is not necessarily the only choice)? Given the wave function, what are the observables? In particular, one should be interested not in a "meta-observer" magically able to couple to all the universes, but rather in the probability that our own universe has some given properties, i.e. in looking for single universe observables. Moreover, in general, path integrals represent transition amplitudes: only if the Hamiltonian of the theory is "time" independent (here, obviously, there is the problem of defining such a time-see for instance section

8.2.1), the initial and final (ground) states are the same, and these amplitudes become (ground state) expectation values. The problem is to realize whether Coleman's theory effectively describes a probability distribution ($\rho(\alpha)$), i.e. if its initial and final states may actually both be a sort of ground state for the theory. Is flat space (as suggested by the use of large smooth geometries in the computations of [14] and [78]) the ground state? If so, the problem might appear to be essentially solved. Ref. [27] does not agree on this.

5.2 The contour problem in the EPI

The Einstein-Hilbert action for gravity is unbounded from below, and the EPI quantization of gravity is, at present, only a formal technique. The problem of defining a good contour for the EPI (together with the need of a consistent regularization scheme, the fixing of the "right" boundary conditions and of a proper measure) is certainly one of the most important and not yet solved issues in quantum gravity. Gibbons, Hawking and Perry [33] first observed that the negativity of the Euclidean action for gravity was caused by the rapidly oscillating conformal part of the Euclidean 4-metric. The idea was to split the EPI into a sum over conformal equivalence classes and a sum over conformal factors within each class, and the integral was claimed to be made convergent if the integration over conformal factors was rotated to lie parallel to the imaginary axis. One of the main objections to this method is that it is not valid when the metric is coupled to non conformally invariant matter and that it is very difficult to implement in more concrete examples.

An alternative proposal was made by Hartle [47], who suggested that the EPI should be calculated along the "steepest descent path" in the space of complex 4-geometries. In this approach, one does not link himself to a peculiar signature of the spacetime (Euclidean or Lorentzian), but rather regards the metric as complex and the integration taken along the contour along which the real part of the action increases most rapidly. Halliwell and Louko [44, 45] started a program in which they applied this idea to a simple de Sitter minisuperspace model (a FLRW metric with cosmological constant), which is the simplest nontrivial exactly soluble model. They determined all possible contours yielding a convergent PI and solutions of the WdW equation, and found, for instance, that the HH boundary condition proposal does not fix the contour uniquely. The problem of fixing the "right" boundary conditions is a strictly related problem, which may be used as a sort of an extra degree of freedom in order to build up different kinds of topologies (see the discussion in chapter 3). One of the interesting consequences of a complex contour is also that there can be saddle points in the PI which have neither Euclidean nor Lorentzian signature. The next and truly fundamental task is to implement all these models for more relevant cases, including matter and, possibly, anisotropies in the metric. Other papers such as that by Hartle and Schleich [50] seem to avoid the contour problem studying a linearized gravity model. This is an interesting challenge for the future.

The contour problem directly clashes with the wormhole theory in this sense. It appears, in particular, that Coleman's mechanism for the vanishing of Λ heavily relies on the apparent instability with respect to nucleating an arbitrarily large number of Euclidean 4-spheres, each contributing $\exp(-\frac{2}{3\lambda})$ to the action ($\lambda \doteq \frac{16\Lambda G^2}{9}$, see [27]). Neither the mathematical prescriptions for eliminating these instabilities seem to work univocally. For example, the EPI for spherical conformally flat geometries, with metric $g_{ij} = \phi^2\delta_{ij}$, includes the functional integral (for the Einstein gravity, eq. (4.1), with $\mathcal{L}_M = 0$):

$$\int [d\phi] \exp\left(\int d^4x \left(\frac{3}{8\pi G}(\partial\phi)^2 - \Lambda\phi^4\right)\right) \tag{5.1}$$

The Gibbons-Hawking-Perry [33] prescription for the rotation of the conformal factor ($\phi \to i\phi$), gives the result:

$$\int [d\phi] \exp\left(-\int d^4x \left(\frac{3}{8\pi G}(\partial\phi)^2 + \Lambda\phi^4\right)\right) \tag{5.2}$$

This corresponds to the stable ϕ^4 theory. It may define a consistent theory of gravity, but may also be expected to eliminate any divergence as $\lambda \to 0$ (which should be, in fact, just the reflection of the unboundedness of gravity, see also [61]). Other prescriptions rotate the contour about the Euclidean saddle points associated with wormhole connected 4-spheres. For instance, studying the modes of fluctuation around these saddle points, Polchinski [101] seemed to reach the conclusion that, in front of the Hawking $\exp(\frac{2}{3\lambda})$ amplitude, there should also be an additional prefactor $(i)^{D-2}$, depending on the dimension D of spacetime. It is clear that, in 4 dimensions, this prefactor would transform the Coleman's double exponential into the disappointing $\exp(-\exp(\frac{2}{3\lambda}))$, not at all peaked at $\lambda = 0$. A recent paper by Mazur and Mottela [88], however, does not confirm this result, but claims that use of the correct measure in the EPI should lead to a completely real 1-loop partition function. The question remains open.

5.3 The infrared regulator and the measure problem

A convergence problem in the definition of the EPI with the extra integration variable α introduced by the wormholes (see section 4.2) has been put in evidence by Unruh [116] and by Hawking [60]. Since the bilocal action readable from eq. (4.16) is negative definite, the path integral does not converge: if one calculates the wormhole vertex for a conformally or minimally coupled scalar field (see section 6.1), the action (4.16) becomes $\propto (\int d^4x\phi^4)^2$, which gives ϕ an effective potential unbounded from below, and the functional integration over ϕ diverges. In particular, if there is a direct wormhole contribution to Λ_{eff}, the EPI would be:

$$Z = \int [dg][d\phi]\, e^{-I} \tag{5.3}$$

with

$$I = \int (L\ dx) - CV^2 \tag{5.4}$$

where L is the bare Lagrangian, V is the volume of spacetime and C is a constant. If C is negative, the integral over α would diverge, but if positive, the EPI would diverge. Rotation of the contours for the conformal integration would change nothing, since V would remain real and positive. One can also see that the introduction of the α parameters does not commute with taking the saddle point in the integral. In fact, the bilocal integral over the wormhole propagator has the same effect in the EPI as a cosmological constant $\lambda_1 = -2CV$ (eq. (5.4)).

If $C < 0$, this leads to a saddle point 4-sphere metric of Planck size. The EPI is finite (but one also needs a large bare λ_o to balance λ_1 and to avoid infinite suppression). On the other hand, if one introduces a parameter α as in section 4.2, one gets a direct α contribution that may cancel any cosmological term in the original Lagrangian. The saddle point metric becomes a 4-sphere of infinite radius and Z diverges.

Another crucial point is that of the regularization of the α measure (see [60]):

$$\mu(\alpha) = P(\alpha)Z(\alpha) \tag{5.5}$$

where $P(\alpha) = e^{-\frac{1}{2}\alpha^2}$ and $Z(\alpha) = \exp(\exp(-\Gamma(\alpha)))$. For the S^4 saddle point, with $\Gamma = -\frac{3}{8G^2\Lambda}$, the total measure in α space is clearly infinite. A way to correctly define (mathematically) $\mu(\alpha)$ might be the following. One can impose a cutoff to $\mu(\alpha)$ by introducing a function $F(\alpha)$, which is zero on the surface where $-\frac{1}{\Gamma} \simeq G^2\Lambda = 0$, and which is positive for small negative $\frac{1}{\Gamma}$. Then one may cut the region $0 \leq F \leq \epsilon$ out of α space. $\mu(\alpha)$ is therefore finite and gives a well defined probability distribution on the rest of α space. $Z(\alpha)$ is highly peaked near the minimum of Γ on the surface $F = \epsilon$: as $\epsilon \to 0$, the peak would be concentrated at a single point of α space. The problem is that this point will depend on the choice of the cutoff (F), and will give different results. As an example, Coleman's procedure [14] seems to be simply putting $F = \Lambda$, while Preskill's one [102], $F \simeq r^{-2} = G^2\Lambda^2$ (volume cutoff). Another possible candidate is $F = -\frac{1}{\Gamma}$, leading to $\Lambda = 0$ and a $P(\alpha)$ distribution of the other couplings.

Another very interesting analysis has been carried on by Elizalde and Gaztanaga [25]. They included higher derivative gravitational corrections to the stationary point for large S^4 spheres in the effective action (up to terms $\propto \Lambda^2$) and showed the surprising result that the Coleman's peak at $\Lambda = 0$ disappears, being substituted by a random distribution around $\Lambda = 0$. The probability distribution $Z(\alpha)$ can be written:

$$Z(\Lambda(\alpha)) = N \ \exp\left(\exp\left[\frac{3}{8G^2\Lambda} + A + BG\Lambda + C(G\Lambda)^2\right]\right) \tag{5.6}$$

where A, B, C are constants, N is a normalization and all parameters depend on α. Successive quantum corrections may introduce higher powers of Λ. The aim is to normalize $Z(\alpha)$. First, Λ is a dynamical variable [14] and it is assumed that the corresponding integration range in the PI has some physical symmetric cutoff $[\Lambda_c^{-1}, \Lambda_c]$. For notational simplicity, one can define

$$\rho_y(x) = g(y)f(x) = g(y)\exp\left(\exp\left(\frac{a}{x} + \sum_k a_k x^k\right)\right) \tag{5.7}$$

where x stands for Λ, and $g(y)$ is the normalization:

$$g(y) = \left(\int_y^{\frac{1}{y}} f(x)dx\right)^{-1} \tag{5.8}$$

with cutoff $0 < y < 1$. The probability may then be defined as:

$$Z(\alpha) \propto \rho(x) = \lim_{y \to 0} \rho_y(x) \tag{5.9}$$

A simplified toy model is that for $f(x) = \frac{a}{x} + bx$ $(a, b, x > 0)$; then $g(y) = (-2a \ln y + \frac{b}{2y^2} - \frac{by^2}{2})^{-1}$. For $b = 0$, the singularity in $x = 0$ is transported in the normalized $\rho(y) \sim -\frac{1}{2y \ln y} \to +\infty$. But for $b \neq 0$, after normalization the peak completely disappears: $\rho \sim \frac{2ay}{b} \to 0$! For the more general case of eq. (5.7), it is shown numerically a similar behaviour (see fig. (35)): provided at least one of the coefficients a_k is positive, the normalized $Z(\alpha)$ becomes a uniform smooth distribution in Λ.

5.4 The giant wormhole disaster

One of the major and not yet solved issues in the wormhole theory is that about the eventual presence and effects of the so called "giant" wormholes, i.e. wormholes of size $\gg M_p$ or even cosmologically large. As I have already noted, these macroscopic wormholes might be of great use in the context of a mechanism explaining the "evaporation" of black holes as suggested by Hawking [61], but also lead to a catastrophic result if they are free to join into an arbitrary region of spacetime. In the last case, in fact, they might violate the well tested successes of local field theory in describing low energy physics.

In this context, the first explicit computation about the physical relevance of these giant wormholes is due to Fischler and Susskind [28]. They essentially show that the main assumptions leading to the Coleman's wormhole solution to the Λ problem (quantum gravity is described by an EPI dominated by large spherical universes connected by wormholes; the amplitude for a large universe is of the order $\exp\left(\frac{1}{G^2\Lambda}\right)$; the sum over wormholes is dominated by Planck scale wormholes) are mutually inconsistent or give rise to wormholes of every size materializing in the vacuum with the maximum density allowed by kinematics. The starting point in the discussion is to assume an effective theory where all short distance fluctuations and wormholes of scale less than ρ have been integrated out. In this case, the expectation value of an operator in a single universe is given by eq. (4.22, 4.23) of section 4.2. The Euclidean effective action which is explicitly considered is of the second order in the scalar curvature:

$$I(\alpha) = \int [(\Lambda(\rho) + \alpha_1)\sqrt{g} - (k^2(\rho) + \alpha_2)^{-1} R\sqrt{g} + (\gamma(\rho) + \alpha_3) R^2 \sqrt{g}] \tag{5.10}$$

($k \propto M_p^{-1}$) and at saddle points it gives a probability distribution :

$$\int d\alpha_i e^{-D_{ij}\alpha_i\alpha_j} \exp\left[\exp\left(\frac{1}{(\Lambda(\rho)+\alpha_1)(k^2(\rho)+\alpha_2)^2} + (\gamma(\rho)+\alpha_3)\right)\right] \tag{5.11}$$

However, one in principle has to account also for fluctuations on scales larger than ρ. To simplify the model, one may think about discrete well separated scales, say ρ and $\rho' \gg \rho$ (see fig. (36)). Naively, the effects of larger wormholes should be suppressed. In fact, the action of known wormholes is of order $\frac{\rho^2}{k^2}$ (see chapter 2), and the relative amplitude for ρ' and ρ wormholes is:

$$\frac{e^{-\rho'^2/k^2}}{e^{-\rho^2/k^2}} \tag{5.12}$$

However, what is important is the contribution of the wormhole action in the distribution (5.11). There, $D = C^{-1} \simeq \exp\ (action)$ (see section 4.2), and this suppresses α fluctuations away from zero. On the other hand, the other factor in eq. (5.11) is much stronger in driving α_1 to $-\Lambda(\rho)$: so one may expect larger wormholes to be driven by a similar force which cannot be overcome by a large action.

To proceed, one can integrate out the next round of fluctuations. The combined effect of the perturbations and of wormholes is to shift the effective couplings $\lambda_i + \alpha_i$ (see section 4.2) as:

$$\lambda_{eff} = \lambda_i(\rho') + \beta_i = \lambda_i(\rho) + \alpha_i + B_i(\lambda(\rho) + \alpha) + \beta_i \tag{5.13}$$

where B_i, β_i are the renormalizations of perturbations and wormholes. The distribution (5.11) changes into:

$$\int d\alpha_i d\beta_i e^{D_{ij}(\rho)\alpha_i\alpha_j + D^{\alpha}_{ij}(\rho')\beta_i\beta_j} \exp\left[\exp\left(\frac{1}{\Lambda_{eff}(\rho')k^4_{eff}(\rho')} + \gamma_{eff}(\rho')\right)\right] \tag{5.14}$$

$D(\rho')$ depends on α because the effect of a large wormhole is calculated using bare parameters $\lambda + \alpha$. Λ_{eff} is driven to zero and, to maintain perturbation theory reliable, one has to maximize γ_{eff} at a finite value. From perturbation theory it turns out that $\gamma(\rho') \leq k^{-2}(\rho')^2$.

The same bound can be obtained by requiring the density of wormholes of scale ρ' to be less than $(\rho')^{-4}$, i.e. the close packing density. In fact, if one expands the EPI for a single universe with shifted action, $\int e^{(I+\alpha\theta)}$, as:

$$\sum_N C_N \alpha^N \doteq Z(\alpha) = \exp\left(\frac{1}{\Lambda_{eff}k^4_{eff}} + \gamma_{eff}\right) \tag{5.15}$$

one can interpret the N^{th} term as representing N wormholes inserted into the large universe ($1\ \alpha \rightarrow 1\ wormh.$) and define an average N:

$$< N >= \frac{\sum N C_N \alpha^N}{\sum C_N \alpha^N} = \frac{\alpha}{Z}\frac{\partial Z}{\partial \alpha} \tag{5.16}$$

A mean density ν of wormholes may be defined dividing $< N >$ by the volume $(\Lambda_{eff}k^2_{eff})^{-2}$. For small Λ_{eff} this becomes:

$$\nu \simeq -\alpha \frac{\partial \Lambda_{eff}}{\partial \alpha} \tag{5.17}$$

On dimensional grounds, one expects contributions to Λ of order $\frac{k^2}{\rho^6}(\gamma + \alpha_3)$, which give:

$$\nu \simeq \alpha_3 \frac{k^2}{\rho^6} \tag{5.18}$$

and the maximum density ($\sim \rho^{-4}$) is obtained for $\alpha_3 \sim \frac{\rho^2}{k^2}$, consistent with the previous result.

A way out of the problem of large wormholes was first suggested by Preskill [102]. The idea is that interactions between instantons may be responsible for the suppression of large ones. In particular, small instantons should crowd out the large ones. Naively, if the goal is to maximize G^{-1} (see section 4.4), configurations with many large instantons are inefficient, because the large instantons "exclude" from spacetime a region that small ones would like to occupy (see fig. (37)). If one imagines two separated classes of instantons, the small ones with size R_S and the large ones with size R_L, on dimensional grounds the α dependence of the bare G_o^{-1} is expected to be:

$$G_o^{-1} \simeq C_S R_S^{-2} n_S(\alpha_S) + C_L R_L^{-2} n_L(\alpha_L) \tag{5.19}$$

where $n_{L,S}$ is a dimensionless density of large (small) instantons and $C_{L,S}$ numerical constants of $O(1)$. If small instantons cannot sit on top of large ones, G_o^{-1} is maximized with respect to the constraint:

$$n_S + n_L \leq 1 \tag{5.20}$$

which gives $n_S = 1, n_L = 0$, assuming $C_S R_S^{-2} > C_L R_L^{-2}$. The large instantons are completely crowded out by small ones, but apparently at the high cost of a violation of the principle that short distance physics is effectively decoupled from long distance physics.

Polchinski [100] reconsidered the analysis of [102] in a more general context. It is assumed that wormholes have no interactions other than the excluded volume effect. First consider instantons of only one size l. Divide a large 4-volume V_4 into cells of volume l^4, each containing zero or one instanton. The total functional integral is factorized into separate pieces on each cell. The EPI in one cell over all configurations without instantons in the cell is:

$$w_1 = \int_{n.i.} [d\ fields] e^{-S_{cell}} \tag{5.21}$$

while that for one instanton is:

$$x_1 = \int_{1\ i.} [d\ fields] e^{-S_{cell}} \tag{5.22}$$

The probability to find an instanton in a given cell, or the average density of instantons, is:

$$n_1 = \frac{x_1}{x_1 + w_1} \tag{5.23}$$

Defining the partition functions $z_o = (w_1)^{\frac{V_4}{l^4}}$ (without instantons) and $z_1 = (w_1 + x_1)^{\frac{V_4}{l^4}}$ (with 1 instanton), the shift in the cosmological constant is given by :

$$\frac{z_1}{z_o} = e^{-V_4 \delta\lambda_1} = l^{-4} \ln(1 - n_1) \tag{5.24}$$

Allowing for instantons on many scales, one can divide V_4 into cells of side $2^{n-1}l$ (n integer). A cell of side $L_k = 2^{k-1}l$ is a k-cell. Each n-cell may contain an instanton of size L_n. If it does not, it is subdivided into 16 $(n-1)$-cells, each of which may be occupied by a L_{n-1} sized instanton, or further divided into $n-2$ cells and so on (see fig. (38)). One can then define an EPI on a k-cell over all topologies (z_k), over configurations with a L_k sized instanton (x_k) and with instantons of size L_{k-1} or less (w_k). Obviously, $z_k = x_k + w_k$ and $w_k = (z_{k-1})^{16}$. Once again, the effective cosmological constant at scale L_k is given by $\lambda_k = -L_k^{-4} \ln z_k$ and, summing over all scales, it satisfies the renormalization group equation (RGE):

$$\delta\lambda = \sum_j L_j^{-4} \ln(1 - \tilde{n}_j) \tag{5.25}$$

where $\tilde{n}_k = \frac{x_k}{x_k + w_k}$ is the fraction of k-cells occupied by an instanton of size L_k. The Planck mass satisfies a similar RGE. The excluded volume constraint becomes:

$$0 \leq 1 - \sum_k n_k = \Pi_k (1 - \tilde{n}_k) \tag{5.26}$$

This equation, apart from the condition $0 \leq \tilde{n}_k \leq 1$, gives no further restrictions on the $\tilde{n}_k$. Therefore, in the maximization of G_N^{-1}, nothing stops the tendency of $\tilde{n}_k$ to be of order 1 at large L_k: this is the "return of the giant wormholes". Also the claimed violation of the decoupling principle is an illusion due to the nonlocal nature of the n_k. In fact, while the $\tilde{n}_k$ just depend on the effective Lagrangian at scale L_k (and are local quantities), n_k is the probability that a L_k instanton is found at a given point , i.e. $\tilde{n}_k$ times the probability of no larger instantons at that point.

Another "escape from the menace of the giant wormholes" has been looked for by Coleman and Lee [15]. The main assumptions are those of doing systematic semiclassical computations (see section 4.3), considering only stationary points of the EPI, a discrete set of wormhole types carrying a conserved global U(1) charge (q_i) and of size L_i, and restricting to unbroken Abelian symmetry. The method is to impose the constraints of a vanishing (effective) cosmological constant λ and of maximizing the (effective) Planck mass, to lowest order in the α's. Wormhole interactions are considered by imposing additional constraints. Because the ground state of the theory is assumed U(1) invariant and the wormhole induced terms in the effective Lagrangian are charge changing, the corrections to the bare λ_o first arise at the second order in α (see section 6.3):

$$\lambda = \lambda_o - \sum b_i |\alpha_i|^2 e^{-2S_i} L_i^{-4} \tag{5.27}$$

with $2S_i$ the wormhole action and b_i dimensionless constants.

Analogously:

$$M_p^2 = M_{po}^2 + \sum c_i |\alpha_i|^2 e^{-2S_i} L_i^{-2} \tag{5.28}$$

where M_{po} is the bare Planck mass and c_i dimensionless constants. From a QFT theorem, stating that the 2^{nd} order perturbation of the ground state energy (in this case a 3-sphere of volume $\propto \lambda$) is always negative, it easily follows that $b_i \geq 0$. Even if, in principle, the c_i's may be of either sign, since one wants to maximize M_p, they are also chosen to be positive.

It can be easily shown that

$$\nu_i = b_i |\alpha_i|^2 e^{-2S_i} \tag{5.29}$$

may be interpreted as the fractional 4-volume occupied by wormhole ends of type i. In fact, in the case of some large smooth 4-geometry with volume V_4, the λ contribution to the EPI amplitude may be approximated as:

$$e^{-\Gamma} = e^{-\lambda_o V_4} \Pi_i e^{\nu_i L_i^{-4} V_4} \tag{5.30}$$

For each of the exponentials on the right hand side of eq. (5.30), the dominant contribution is that of the order:

$$n_i \sim \nu_i L_i^{-4} V_4 \tag{5.31}$$

Thus the number of factors e^{-S_i} in the dominant term is of order n_i. But this number is just the number of wormhole ends of type i and

$$\nu_i \sim n_i L_i^4 / V_4 \tag{5.32}$$

is just the corresponding dimensionless wormhole density. The principal constraints thus become:

$$\begin{cases} \sum \nu_i L_i^{-4} = L_o^{-4} \\ M_p^2 = M_{po}^2 + \sum \beta_i \nu_i L_i^{-2} \end{cases} \tag{5.33a, b}$$

with $\beta_i \doteq \frac{b_i}{c_i} > 0$, dimensionless, $\hbar$ independent numbers, slowly varying with L_i, which can be put equal to 1 for simplicity. M_p is then renamed M_s and $L_o^{-4} \doteq \lambda_o$. Maximizing M_s with λ fixed has, in general, no solution (M_s is a linear function of the ν's). The first trial is to introduce a constraint simulating the idea that for *each* type of wormhole, there cannot be more 4-volume occupied by wormhole ends than the total 4-volume available for occupation. In equations, for each i, $1 \geq \nu_i \geq 0$. If M_s has a maximum, this will occur at $\nu_i = 1$ for all L_i greater than some lower bound. This is the giant wormhole catastrophe.

A second trial is to impose the constraint that the sum of the volume of *all* the wormhole ends cannot be greater than the total amount of 4 space. In equations, $\sum_i \nu_i \leq 1$. Since M_s^2 is a linear function of the ν_i's, the maximum is on the boundary $\sum_i \nu_i = 1$. Thus the ν_i can be taken to define a probability distribution, in terms of which eq. (5.33a) becomes:

$$< L^{-4} >= L_o^{-4} \tag{5.34}$$

and one has to maximize $M_s^2 - M_{po}^2 =< L^{-2} >$.

Since $< L^{-4} >=< L^{-2} >^2 + < L^{-2} - < L^{-2} >>^2$, this is equivalent to minimizing the variance. If the solution L_o is one of the L_i, the associated ν is equal to 1 and the others vanish. If it is not, the only nonzero ν's are the two L's nearest to L_o. This totally overcomes the giant wormholes catastrophe. Unfortunately, the starting hypothesis (similar to Preskill's one) is an unphysical one, since the presence of large wormholes does not reduce the volume available for small ones. Small wormholes can still attach to large ones, and smaller ones attach to them (fig. (39)).

The working trial is to assume that small wormholes can destabilize large ones (they "bleed" them). Small wormholes induce charge nonconserving interactions. As charge flows into the throat of a large wormhole, it can be diverted into small wormholes, until there is too little charge left to support the large one. The condition for stability is that the mean square charge carried by small wormholes is less than that of the large one:

$$\sum_{L_i < L_1} \nu_i (L_1/L_i)^4 q_i^2 \leq q_1^2 \tag{5.35}$$

where $L_{1,i}$ and $q_{1,i}$ are the sizes and charges of the large (small) wormholes. For the model presented in section 2.5.2, if $mL \ll 1$ (m is the mass of the scalar field), one has $q_i \propto M_p^2 L_i^2$, and eq. (5.35) gives $\sum_{L_i<L_1}(\nu_i) \leq 1$. Assuming there is a largest size with nonzero density, this becomes $\sum \nu_i \leq 2$. If $mL \gg 1$, $q_i \propto \frac{M_p^2 L}{m}$ and the condition on the ν_i is stronger than before, since $\sum_{mLi<1}(\nu_i m^2 L_1^2) + \sum_{1<mL_i<mL_1}(\nu_i \frac{L_1^2}{L_i^2}) < 1$. The fact that large wormholes make extra space for small ones even strengthens this condition. From the previous discussion, it turns out that the giant wormhole problem is apparently solved. Unfortunately, also this solution strongly depends on the existence of a peculiar kind of (charged) wormhole solutions and on a lot of semiclassical approximations, such as the restrictive definition of stationary point wormholes (but there are also end point solutions, see section 2.7).

6 Wormhole features

6.1 The wormhole vertex

One of the main goals in the study of the wormhole theory is to explicitly find out the effective interaction vertices induced by the presence of these topological features in the low energy physical effective Lagrangian (see eq. (4.13)). It is essential to work out a consistent scheme for the α dependent shifts of the coupling constants of nature and their subsequent theoretical determination, where it is implicitly expected that each wormhole type should correspond to the addition of a given local operator in the low energy action.

A simple way to understand the effect of wormholes is the following. One of the basic ideas about wormholes is that they can work as a sort of "tunnels" in spacetime, through which particles can reach far regions in an efficient way. Since one has to integrate over all possible positions for a wormhole to join into spacetime (or, more simply, since the "babies" are closed universes), energy will be conserved at each junction point. This means, for example, that an electron-positron pair could not just fall into a wormhole and disappear leaving nothing. However, a single electron could go into a wormhole which would in its turn emit the antiparticle to the positron, that is, another electron. It is just this interaction with the wormhole that will appear to shift the electron mass (see below). Similarly, one can think about wormholes containing 4 fermions etc. (see fig. (40)).

The first works studying the subject in more details are due to Hawking [59, 60, 62]. In these papers it is assumed to work in the ansatz of conformally flat wormhole solutions (see section 2.4), and the effect of branching off of little closed universes is considered on ordinary, non gravitational fields (particles) in the asymptotically flat regions at energies low compared to the Planck mass. Wormhole interactions are neglected. The effects of non conformally flat wormholes are expected to be similar (but for the inclusion of gravitons). The matter fields propagating down the wormhole are taken conformally invariant, with the effect of masses eventually included as a perturbation. To calculate the effective interaction caused by a wormhole, one needs a full quantum treatment. One introduces a three surface, S, which is a cross section of the wormhole, and describes the quantum state of the wormhole by a wave function Ψ, obeying the WdW equation and appropriate asymptotic boundary conditions, as I have described in section 2.8. The idea is to consider perturbations about minisuperspace models, where the metric is taken as :

$$ds^2 = N^2 d\tau^2 + a^2(\Omega_{ij} + \epsilon_{ij})dx^i dx^j \tag{6.1}$$

where Ω_{ij} is the round metric on S^3 and ϵ_{ij} a small perturbation which can be expanded in terms of hyperspherical harmonics on S (see [46]). A similar expansion is done for the case of a scalar matter field, which is written explicitly:

$$\phi_o = a^{-1} \sum_n f_n Q_n \tag{6.2}$$

with f_n the coefficients of the scalar harmonics Q_n. Ψ is then a function of the scale factor a and of the coefficients of the harmonics. Expanding the WdW equation to all orders in a and to the second order in the other coefficients, one finds that the conformal scalar coefficients decouple and part of the WdW operator can be written:

$$-\frac{\partial^2}{\partial f_n^2} + (n^2+1) f_n^2 \tag{6.3}$$

acting on the wave function:

$$\Psi_{n,m}(f_n) = \left[\frac{\sqrt{n^2+1}}{\pi 2^{2m}(m!)^2}\right]^{1/4} e^{-f_n^2\sqrt{n^2+1}/2} H_m((n^2+1)^{1/4} f_n) \tag{6.4}$$

which can be interpreted as corresponding to a closed universe containing m scalar particles in the n^{th} harmonic mode. The case of higher spin particles is similar, but the lowest mode has $n \neq 0$. While the other coefficients can be made zero by a suitable diffeomorphism, the scale factor a appears in the WdW equation as the operator:

$$\frac{\partial^2}{\partial a^2} - a^2 \tag{6.5}$$

Therefore, Ψ essentially results as a product of two harmonic oscillator wave functions ($\Psi = \psi(a)\psi(f)$) in which the positive energy of the matter modes is balanced by the negative gravitational potential energy. This is another way to say that the total energy of a wormhole is zero. To estimate the effect on low energy physics of a small wormhole joining into an event x_o of spacetime, one can also calculate the n-point Green function between a wormhole and an asymptotically Euclidean space:

$$< \Psi|\phi(y_1)...\phi(y_r)|0 >= \int dh\ d\phi_o \Psi[h_{ij}, \phi_o] \int dg\ d\phi\phi(y_1)...\phi(y_r)e^{-I(g,\phi)} \tag{6.6}$$

with C the class of asymptotically flat metrics at infinity and with induced metric h_{ij} on S, and the scalar field is zero at infinity and ϕ_o on S. For a wormhole in the $n = 0$ homogeneous scalar mode, the integral over Ψ essentially becomes:

$$\int dadf_o \psi_E(a)\psi_{om}(f_o) \tag{6.7}$$

(E is the level number of the radial oscillator). The saddle point for the path integral is given by flat space outside a 3-sphere of radius a centered on x_o and:

$$\phi = \frac{af_o}{(x - x_o)^2} \tag{6.8}$$

and has action $\frac{a_o^2+f_o^2}{2}$ (59). Therefore, one will have an integral over f_o:

$$\int df_o f_o^r e^{-f_o^2} H_m(f_o) \tag{6.9}$$

which is interesting only for $r = m$ (for $m > r$ it is zero, for $m < r$ one can have particles not going into the wormhole), and an integral over a:

$$\int daa^m e^{-a^2} H_E(a)\Delta(a) \tag{6.10}$$

which is $\sim O(1)$ for small m ($\Delta(a)$ is the determinant of the saddle point fluctuations). In conclusion, the matrix element is of the form:

$$D(m)\Pi \frac{1}{(y_i - x_o)^2} \tag{6.11}$$

($D(m) \sim O(1)$), which has then to be integrated over the possible positions of the wormhole (with a measure of the form $M_p d^4 x_o$) and over an orthogonal matrix O which specifies its orientation with respect to the y_i. The result (6.11) is the same as that obtained in flat space with an effective interaction of the form (on dimensional grounds):

$$F(m)M_p^{4-m}\phi^m(c_{om} + c_{om}^\dagger) \tag{6.12}$$

where $F(m) \sim O(1)$ and $c_{o,m}$ ($c^{\dagger}_{o,m}$) are the annihilation (creation) operators for a closed universe containing m scalar particles in the $n = 0$ mode (see [60]). These interactions appear alarmingly large. The $m = 1$ case would be a disaster: a single scalar would get a position independent propagator, because it could go into a wormhole whose other end is at a great distance in the asymptotically flat region. If the scalar is coupled to a Yang-Mills field, however, this interaction is exactly zero. The reason is that a closed universe must also be a singlet under the Poincare' group and any Yang-Mills gauge group, i.e. it will carry zero local gauge charges. A closed universe with a single scalar is not a Yang-Mills singlet. For incidence, it is interesting to note that a closed universe may, however, carry nonzero charges conserved by a global symmetry-such as electron number minus muon number: violation of global conservation laws in our universe might be an evidence of the existence of wormholes. The $m = 2$ case for a scalar particle and its antiparticle will be relevant because a particle-antiparticle state contains a Yang-Mills singlet. However, this again would be a disaster. With two vertices like (6.12), in fact, one could make a closed loop with a closed universe (propagator $\delta^4(p)$) and a scalar particle (propagator p^{-2}): this is infrared divergent with the momentum p. The divergence could be cutoff by giving the scalar a mass, but the effective mass would be of order M_p! The presence of closed universes would also cause the phase of the particle state to become undefined. For 4 scalar particles, the interaction coupling would be of order 1, which would cause a significant loss of quantum coherence in collisions of truly elementary particles (the Higgs fields?). Unfortunately, this interaction corresponds to the emission of a zero energy particle, which is not detected. To reduce the effects of these $n = 0$ interactions one might hope, for instance, in fortunate (SUSY ?) cancellations or in the idea that any scalar is actually a bound state of higher spin particles.

Similarly, one can calculate vertices for higher excited particles and higher particle numbers. However, analysis of the ϕ saddle point shows that the only nonnegligible one should be that for a universe containing 2 particles in the $n = 1$ mode, with explicit structure like:

$$\nabla\phi\nabla\phi(c_{12} + c^{\dagger}_{12}) \tag{6.13}$$

For wormholes containing higher spin particles, averaging over O of similar amplitude elements restricts the effective interactions to be Lorentz invariant, or to contain an even number of fermion fields (en passant, this is another motivation for the wormholes to be treated as bosons-see also chapter 8). For instance, for spin $\frac{1}{2}$ particles, one may have the effective interaction (see Lyons [87]):

$$M_p^{4-(3m/2)}\psi^m d_m + h.c. \tag{6.14}$$

where ψ^m is some Lorentz invariant combination of m spinor fields and d_m is the annihilation operator for "babies". This is suppressed for four or more particles. The only place where it might be observed is in the decay of a barion, for instance in a wormhole carrying three quarks and a neutrino or a positron. Unfortunately, the lifetime would be of the order of $10^{50}yr$, too long to be detected.

For spin 1 gauge particles ($F_{\mu\nu}$ field), the effective interaction is expected to be of the kind (see Dowker [21]):

$$M_p^{4-2m}[(F_{\mu\nu})^m(g_m + g_m^\dagger)] \tag{6.15}$$

From these results one might expect that wormholes containing gravitons would give effective interactions of the form "curvature to the n^{th} power". This has been confirmed by calculations by Dowker and Laflamme [22]. Another explicit computation of the wormhole induced vertex for the theory of a charged scalar field ϕ can be found in Coleman and Lee [16], Abbott and Wise [1] and in Grinstein [38]. As shown in section 2.5.2, the total action for the insertion of a charged wormhole in the flat background is given by:

$$S = -q \ln \frac{f}{\sqrt{3}\pi M_p} \tag{6.16}$$

where $\phi = \frac{f}{\sqrt{2}}e^{-i\theta}$ and q is the (quantized) charge carried by the wormhole. Therefore, to leading (no-loop) order in $\hbar$, this corresponds to the insertion of an interaction in the effective Lagrangian of the kind:

$$\mathcal{L}_q = \left(\frac{\psi}{\sqrt{3}\pi M_p}\right)^{\frac{q}{\hbar}} \tag{6.17}$$

(the factor $e^{\frac{iq\theta}{\hbar}}$ comes from the boundary piece of the action, see section 2.5.2, eq. (2.13)). Similarly, in [1] it is shown that the effect of a single wormhole of charge n is equivalent to the insertion, in the transition amplitude Z between states of definite charge (n), of the operator:

$$g_n \int d^4y \phi^n(y) \tag{6.18}$$

where the coefficients g_n essentially depends on the size and the charge of the wormhole. Moreover, as one approaches the asymptotically flat region, the g_n's are finite, indicating that U(1) violating processes occur even if the wormhole action is divergent (see section 2.5.2).

5.3 Loss of coherence in quantum gravity

Historically, the possibility that topological fluctuations might lead to some rather peculiar effects in particle physics, such as the loss of quantum coherence, an extra degree of uncertainty in QG over that normally associated with QFT, was first debated by Dyson [20], Hawking [54], Gross [40], Banks, Peskin and Susskind [4]. A more definite discussion of the problem has been recently presented for the case of wormholes by Hawking and Laflamme [64], Lavrelashvili, Rubakov and Tinyakov [82, 83], Unruh [116], Laflamme [80], Giddings and Strominger [35] and Coleman [13]. The conclusion of the first set of papers [64, 80, 82, 83, 116] is that quantum coherence is indeed lost. This is not, however, the same for [13] and [35], which essentially agree on the main results of their calculations. Let us first discuss the results of [82, 83]. If one considers the process (see fig. (41)) in which a closed universe Σ_3 splits off from a large universe Σ_1, so that the final 3-D manifold is disconnected, the initial state of a particle (e.g., an electron) of momentum p ($\Psi_{in} = |e^-, p>$) will be converted into the state:

$$|\Psi_{out}>= aM_p^4|\Psi_{in}> \otimes(|0> +\epsilon_p|e^-e^+>) \tag{6.19}$$

where aM_p^4 is the amplitude for the topological transition and ϵ_p that for the production of the e^+e^- pair. Calculating the final density matrix, one can then show that an initial superposition $|e^-, p_1> + |e^-, p_2>$ looses coherence at a rate $aM_p^4|\epsilon_{p_1} - \epsilon_{p_2}|$. For particles of spin s and mass m, this becomes:

$$\left|\frac{1}{\omega_{p_1}} - \frac{1}{\omega_{p_2}}\right| a \begin{cases} M_p^2 & s = 0 \\ M_p m & s = \frac{1}{2} \\ m^2 & s = 1 \end{cases} \tag{6.20}$$

where $\omega_p = \sqrt{p^2 + m^2}$. Since, for instance, a loss of quantum coherence would lead to the disappearance of the K_o meson oscillations, one can compare the prediction of eq. (6.20) with the oscillation time and find the limit $a \sim 10^{-17}$, contrary to a naively expected $a \sim 1$.

The results of ref. [13] and [35] are quite different. Giddings and Strominger [35], for instance, consider the explicit case of an instanton for gravity coupled to an axion (see section 2.5.1). The expected effective Hamiltonian produced by these instantons is of the form:

$$H_q = \gamma(a_+^\dagger + a_-^\dagger + a_+ + a_-) = \gamma(C + C^\dagger) \tag{6.21}$$

where $\pm q$ is the charge of the "baby", with creation (annihilation) operators $a_\pm^\dagger(a_\pm)$, γ is a coupling constant (essentially the same as in section 4.3, eq. (4.33)), and $C \doteq a_+^\dagger + a_-$ $([C, C^\dagger] = 0)$.

The instantons absorb or emit axions, therefore any particle interacting with axions will interact with the "babies". As a toy model, one can assume a particle which can exist in two states, $|A_{1,2}>$, and take an interaction Hamiltonian:

$$H_I = \frac{\beta r}{i}(C - C^\dagger) \tag{6.22}$$

where $r|A_{1,2}> = \mp i|A_{2,1}>$ and β is a coupling constant. The states $|A_{1,2}>$ might be the kaon states $|K_o>, |\bar{K}_o>$. It can be shown that the behaviour of the A system strongly depends on the choice of the baby wave function. One obvious possibility is that the wave function diagonalizes H_I (i.e., C and $C^\dagger$), with the state:

$$|\alpha> = \frac{1}{\sqrt{\pi}} e^{\alpha\bar{\alpha}/2} e^{-(\alpha - a_+^\dagger)(\bar{\alpha} - a_-^\dagger)}|0> \tag{6.23}$$

($|0>$ is the no-baby state) satisfying:

$$C|\alpha> = \alpha|\alpha> \qquad C^\dagger|\alpha> = \bar{\alpha}|\alpha> \tag{6.24}$$

and for which one has:

$$H_I = 2\beta y r \tag{6.25}$$

with $\alpha = x + iy$. This Hamiltonian, describing a constant rotation of $|A_1>$ into $|A_2>$, clearly does not describe any loss of coherence.

A second possibility is that to assume no "babies" at time $t = 0$ (the HH boundary condition). Then the universe is evolved to present time T by H_q, giving the state:

$$|T> e^{i\gamma T(C+C^\dagger)}|0> \tag{6.26}$$

which can be expressed in terms of α states by use of eq. (6.23). If one now creates a particle in the state $|A_1>$, and measures the evolved composed system after a time τ, one finds:

$$|\psi>= [\cos i\beta\tau(C - C^\dagger)|A_1> + \sin i\beta\tau(C - C^\dagger)|A_2>]|T + \tau> \tag{6.27}$$

Since the state of "babies" is not measurable, one is obliged to trace over the possible states of these universes, obtaining the density matrix:

$$\rho = Tr|\psi>< \psi| = \frac{1}{2}(1 + e^{-4\beta^2\tau^2})|A_1>< A_1| + \frac{1}{2}(1 + e^{-4\beta^2\tau^2})|A_2>< A_2| \tag{6.28}$$

which, for $T \to \infty$, tends to $\frac{1}{2}(|A_1>< A_1| + |A_2>< A_2|)$. An initial pure state has evolved into a mixed state.

However, this may not necessarily imply a loss of coherence. One, in fact, may consider doing a sequence of experiments on the A system, the time between consecutive experiments being τ. After N measurements, if one has found $|A_1>$ n times and $|A_2>$ m times, the state of the system can be expressed as [35]:

$$\int d^2\alpha[\cos^n(2\beta\tau y)\sin^m(2\beta\tau y)]e^{2i\gamma(T+N\tau)x-(\alpha\bar{\alpha})/2} \tag{6.29}$$

As $N = n + m$ gets large, this is sharply peaked around:

$$\tan^2 2\beta\tau y = \frac{m}{n} \tag{6.30}$$

Now, by varying τ and measuring different linear combinations of $|A_{1,2}>$, one can collapse the wave function to one with support on a fixed y. However, no amount of measurements of the A system will be able to fix x, i.e. the wave function has collapsed into a state where quantum coherence is not observable.

Another simple way to see this (see also [13]) is to calculate the entropy

$$S = -Tr(\rho \ln \rho) \tag{6.31}$$

as a function of the number of experiments n. If the "baby" is in a state:

$$|b>= \int d^2\alpha f(\alpha)|\alpha> \tag{6.32}$$

after a time τ, the initial state $|A_1>$ is evolved by H_I to the state:

$$\cos(2\beta y\tau)|A_1> - \sin(2\beta y\tau)|A_2> \tag{6.33}$$

Tracing over the $|b>$ baby universe states the r^{th} power of the density matrix ρ_n after n identical experiments, one obtains:

$$\begin{aligned}\sigma_r(n) &\doteq Tr(\rho_n^r) \\ &= \int d^2\alpha_1 ... d^2\alpha_r |f(\alpha_1)|^2 ... |f(\alpha_r)|^2 \cos^n(2\beta\tau(y_1 - y_2)) ... \cos^n(2\beta\tau(y_r - y_1))\end{aligned} \tag{6.34}$$

Each $[\cos(2\beta\tau(y_i - y_{i+1}))]^n$ may be approximated by a sum of Gaussians of widths $\sim \frac{1}{\sqrt{n}}$, thus:

$$\sigma_r \simeq \left(\frac{1}{\sqrt{n}}\right)^{r-1} F(r) \tag{6.35}$$

with $F(r)$ a function such that $F(1) = 1$. The entropy is:

$$S(n) = -\frac{d}{dr}\sigma_r(n)|_{r=1} = \frac{1}{2}\ln n \tag{6.36}$$

which, for large n, gives $\frac{dS}{dn} \simeq \frac{1}{2n}$: the loss of coherence becomes negligible after a large number of experiments.

In terms of the "manyworlds" interpretation of section 4.3: if one started in a superposition of the $|\alpha>$ states, after a large number of experiments the observer will have evolved into a branch which is a y eigenstate, and there he will see ordinary quantum mechanical evolution governed by $H(y)$.

Hawking and Laflamme's [64] interpretation appears a bit different. For them, one should expect loss of quantum coherence in any theory of quantum gravity that allows non simply connected manifolds. The reason is that one does not know how many "handles" the spacetime manifold may have. This means that one will never be sure to have measured all of the final state. There may also be little closed universes whose presence one cannot detect (they have zero energy), and whose state one cannot measure. If one measures only part of the final state, he has to sum over all the possibilities for the part that he does not observe. This reduces the pure quantum state to a mixed state and one looses quantum coherence.

What Hawking strengthens, differently from [13] and [35], is the following. It is evident that, only if one carefully tunes the baby state to be a coherent state $|\alpha>$ of the C operators, the effective interactions would leave it in the same state, with no loss of quantum coherence. If the initial state were not an α state, the final state will be mixed: for Coleman, this is just a reflection of our lack of knowledge of the initial state, not of the loss of quantum coherence (this strong dependence on the choice of the initial state was also pointed out in the context of an interesting 3^{rd} quantization toy model by [84]). The critic is that it is really metaphysical to argue whether wormholes lead to one or the other consequence. The important point is that there is an extra degree of uncertainty over and above that expected in ordinary flat QFT.

Moreover, even if one assumes an initial no baby state $|0>$, after a scattering experiment one will not be able to predict the strength of the effective interactions, which will be fixed only after the collapse of the probability distribution (see section 4.3). There will be, in particular, an infinite sequence of higher and higher order effective interactions, corresponding to "babies" containing more and more particles. One will never measure more than a finite number of them. Thus, even if the underlying fundamental theory of gravity may be finite and precisely defined with no arbitrary coupling constants, the effective theory will be non renormalizable, with an infinite number of undetermined couplings.

6.3 Nonlocality, causal properties and wormhole interactions

The existence of wormholes might lead to other interesting and exotic effects, like nonlocality and acausality, wormhole correlations and interactions, which have been studied and debated, at the moment without reaching a unique and accepted interpretation, by a lot of physicists. Here, I will not give a comprehensive and complete description of all these works, but just introduce some basic ideas. A more complete literature can be reconstructed from the papers here cited.

Coleman [13] assumes locality in QFT as an axiom. The nonlocal term in the action obtained after integration of the wormholes is resolved introducing annihilation and creation operators for universes and interpreting this term as the expectation value of a shift operator in the ground state for these operators.

An interesting interpretation of the problem of nonlocality and how it arises in the wormhole physics is given by Myers and Periwal [98]. For a free scalar field ϕ of mass m, with action:

$$S = \int \frac{1}{2}\phi(-\partial^2 + m^2)\phi + a\phi + b \tag{6.37}$$

the wormhole may be expected to introduce at least a linear correction to ϕ of the kind $S_w = \int \alpha g\phi + \beta h$. The vanishing of the total vacuum energy fixes β at some β_o, which amounts in having a probability distribution (eq. (5.5)) of the kind:

$$d\mu \simeq d\alpha e^{-(\alpha^2+\beta_o^2)/2} \tag{6.38}$$

The saddle point solution to the EPI for a fixed α is given by $\phi_c = -\frac{(a+g\alpha)}{m^2}$, which may be interpreted to say that the "probable" two-point function for the scalar is:

$$< \phi(x)\phi(y) > = < \phi(x)\phi(y) >_{m^2} + \frac{a^2 + 2agJ_1 + g^2 J_2}{m^4} \tag{6.39}$$

where $J_i = \frac{\int d\mu\alpha^i}{\int d\mu}$ and $< \phi\phi >_{m^2}$ is the usual free propagator. This probable propagator is rather badly nonlocal (it depends on the integration over the α's, which carry the information about the sum over wormholes connecting different regions of spacetime). However, any propagator used in a calculation of physical quantities should be a propagator for a specific value of α, which should be fixed after the first measurement has collapsed the probability distribution $d\mu(\alpha)$ at its peak. In this sense, nonlocality might no longer be a problem.

The inclusion of possible interactions among wormholes has been investigated by Preskill [102] and others. Just for convenience, one can distinguish (but this is not a definition !) between an instanton as the region of spacetime where a wormhole hooks on, and a wormhole as the connection between two instantons. Wormhole interactions are described by vertices where a branching of wormholes occurs (see fig. (42)), and they introduce non Gaussian corrections to the α probability distribution (see, for instance, section 8, eq. (8.15)). More interesting are the instanton interactions, which occur when two approach closely to one another (beyond the dilute approximation), and which induce (assuming they are of short range, see [41]) non linear corrections in the effective action Γ_α. One can therefore introduce a kind of "cluster" expansion defining the concept of "density" of instantons as:

$$\frac{N}{V} \simeq \alpha D^{1/2} \tag{6.40}$$

This comes from considering the dominant term in the Taylor expansion of the first exponential of eq. (4.23), which can be rescaled as

$$e^{\alpha D^{1/2} V} = \sum \left(\frac{\alpha D^{\frac{1}{2}} V}{N!} \right)^N \tag{6.41}$$

where V is the volume of spacetime. The term in eq. (6.41) of order V^N may be interpreted as the contribution at fixed α due to a N-instanton configuration and, since the sum over N is dominated by $\alpha D^{\frac{1}{2}} V \simeq N$, one can in this sense think of eq. (6.40) as a density. The instanton density is the expansion parameter of the cluster expansion, and the dilute approximation is valid for $\alpha D^{1/2} \ll 1$, or α small for a generic D. In particular, the probability distribution (4.23) is not correct at large α. It can then be shown [102] that the effect of (connected) n-body instanton interactions is to induce α^n shifts in the effective action:

$$\begin{cases} \delta\Gamma^n(\alpha) = -\frac{1}{n!} \sum_{a_1..a_n} \alpha_{a_1}...\alpha_{a_n} \lambda^{(n)}_{a_1..a_n} \\ \lambda^{(n)}_{a_1..a_n} = \int d^4x_1 \sqrt{g_1}...d^4x_n \sqrt{g_n} R^{(n)}_{a_1...a_n}(x_1..x_n) \end{cases} \tag{6.42a, b}$$

where R is a translational invariant operatorial valued function dropping rapidly as $|x_i - x_j|$ becomes large. If the interactions are of short range, one can formally perform $n-1$ integrals in eq. (6.42b) and find:

$$\Gamma(\alpha) = \Gamma^o(\alpha) + \sum_{n=1}^{\infty} \Gamma^n(\alpha) \tag{6.43}$$

as a sum of local operators, each with a coefficient expanded as a power series in α. For example, one can write for the cosmological constant:

$$\Lambda = \Lambda_o + \sum_n \frac{1}{n!} \sum_{a_1..a_n} \alpha_{a_1}...\alpha_{a_n} \Lambda^{(n)}_{a_1..a_n} \tag{6.44}$$

The linear term $\sum_a \Lambda_a^1 \alpha_a$ is the density of instantons.

An explicit calculation for the effect of such interaction terms on the cosmological constant is due to Abbott and Wise [1]. Charge conservation requires that in the vacuum wormholes appear as $\pm n$ charged pairs. The contribution of a charge n wormhole-antiwormhole system to the vacuum-vacuum transition is :

$$< 0| \int d^4x \alpha_n g_n \phi^n(x) \int d^4y \alpha_n^* g_n \phi^{*n}(y)|0 > \tag{6.45}$$

Connecting the two insertions with n scalar propagators ($D_F(t)$) and summing over multiple uncorrelated pairs and n, one finds a shift for the bare cosmological constant Λ_o which is:

$$\Lambda = \Lambda_o - \sum_n |\alpha_n|^2 g_n^2 n! 2\pi^2 \int_o^\infty dt t^3 (D_F(t))^n \tag{6.46}$$

where the integral is dominated by $t < m^{-1}$, $D_F \sim t^{-2}$ (t is defined as in section 2.5.2). The quantity $-n \ln D_F(t)$ plays the role of an interacting potential between a wormhole and an antiwormhole. At long distances, $D_F(t) \sim e^{-mt}$, and a linear potential prevents the pair from separating. At shorter distances, a logarithmic potential confines the pair to a separation $\sim t_n$. $n = 1$ pairs can however separate out to distances $\sim m^{-1}$.

Coleman and Lee [16], similarly studying the charged wormholes of action (2.33), came to the conclusion that the interaction between wormhole ends falls off exponentially with distance. The reason of the absence of long range interactions is explained by the simple fact that outside a spherical instanton space is flat. A computation on the effects of the wormhole interactions on the wormhole effective vertex and nonlocality can also be found in [41].

I would like to conclude this section by just mentioning an interesting paper by Visser [118], translating the study of the causal properties of the wormholes in a Lorentzian spacetime context. There, it is stated that "transient" wormholes (where a parent universe emits a virtual "baby" and subsequently reabsorbs it, see fig. (43)) cannot exist in bounded regions of any stable causal spacetime. Since localized topology changes are forbidden, to avoid acausality one may think of a process like that in fig. (44, 45), where a "bubble" blows out and starts growing while its connection to the parent shrinks: an "umbilical cord" at all times connects the "parent" and the "baby". Its presence is a necessary but not sufficient condition for preventing causality violations. Physical probes will not be able to pass through the umbilical cord of size R unless they have an energy greater than $\frac{\hbar c}{R}$. The physical topology becomes an energy dependent concept. At low energy, the cord will behave like an elementary particle (see Wheeler's idea [122]). A third kind of wormhole is the "permanent" wormhole, which is an intra (inter)-universe connection formed "ab initio". This may allow the possibility of permanently trapping some magnetic flux lines and introduce, say, Planck mass magnetic monopoles. To make permanent wormholes compatible with stable causality one has to enforce the condition that their end points connect regions which are simultaneous in terms of some peculiar cosmic time function. Other classical or semiclassical investigations on Minkowski signature wormholes can be found in Morris and Thorne [93] and Morris, Thorne and Yurtsever [94]. Another result is that violations of the weak energy condition occur at the throat of such wormholes. On the other hand, it is experimentally known that quantum mechanics violates all the energy hypothesis (remember, e.g., the Casimir effect).

Implementation to a quantum mechanical interpretation of Lorentzian wormholes has been recently considered by Visser [119]. Solving the WdW equation for Einstein gravity in a minisuperspace frame, the wave function for a simple class of wormholes is explicitly found and it is shown that the wormhole becomes quantum mechanically stabilized by an average radius of the order of the Planck length.

7 Wormholes and cosmology

The most serious attempts to really trying to reconcile the wormhole theory with a reasonable cosmology can be found, for example, in Lee [85], Lavrelashvili, Rubakov and Tinyakov [84], Rubakov and Tinyakov [107] and Midorikawa [91]. These works have already been considered in chapter 2, and will not be addressed any longer. The idea of this chapter is to describe some other interesting aspects such as that of the inclusion of different sized universes (possibly including some similar to ours!) in the wormhole theory of the cosmological constant, and that of a possible scenario for the dynamical evolution of the same cosmological constant driven by a conformally coupled scalar field.

7.1 Cold and hot universes

An interesting analysis is that by Klebanov, Susskind and Banks [78].

The idea is to consider a theory where wormholes connect large spherical universes also of different size, i.e. with different Λ and temperature. They first consider a minisuperspace model with a scalar field ϕ and without wormholes, described by the Euclidean action:

$$I = \int dt \left[-\frac{1}{2}a\dot{a}^2 - \frac{1}{2}a + a^3\left(\frac{1}{2}\dot{\phi}^2 + V(\phi)\right)\right] \tag{7.1}$$

and with equations of motion:

$$\begin{cases} \ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} = \frac{dV}{d\phi} \\ (\frac{\dot{a}}{a})^2 = \frac{1}{a^2} + \dot{\phi}^2 - 2V \end{cases} \tag{7.2}$$

For the potential shown in fig. (46), it is easy to distinguish two possible saddle points, corresponding to trajectories starting at $a = 0$ with $\dot{\phi} = 0$ and $\phi = \phi_a$ or $\phi = \phi_b$ ($\phi_b > \phi_a$). The explicit solutions are (with $h^2_{a,b} \doteq 2V(\phi_{a,b})$):

$$\begin{cases} \phi = \phi_{a,b} = constant \\ a = \frac{1}{h_{a,b}}\sin(h_{a,b}t) \end{cases} \tag{7.3}$$

A "warm" universe can be identified with the Euclidean trajectory emerging in the classically allowed region of a near ϕ_b. The other Euclidean solution is associated to a "cold" universe. To extrapolate to large values of a, one has to analytically continue to the Minkowskian equations of motion. Eventually, in the classically allowed region $a^2 > (2V)^{-1}$, tunnelling to the lower well takes place (accompanied by heat production). One then assumes that the tunneling between the wells of V is much slower than that for a from $a = 0$ to the classically allowed region. A wormhole can be thought as a reflection off $a = 0$. There are three types of wormholes: those representing diagonal transitions $\phi_{a,b} \to \phi_{a,b}$, and those exchanging $\phi_{a,b} \leftrightarrow \phi_{a,b}$. The reason appreciable transitions exist is that the ϕ motion at $a = 0$ costs no action, so the EPI is effectively saturated by a series of small and large universes (see fig. (47)). The Coleman's probability $Z(\alpha)$ will be now of the kind:

$$\rho \sim \exp\left(F_a \exp\left(\frac{3}{8G^2\Lambda_a}\right) + F_b \exp\left(\frac{3}{8G^2\Lambda_b}\right)\right) \tag{7.4}$$

with $\Lambda_{a,b} = \frac{9V(\phi_{a,b})}{8G^2}$ and $F_{a,b}$ constants, both dependent on the wormhole shifted couplings of the theory $(\lambda + \alpha)$, and in principle they should be calculable from the PI. One peculiar trial is to assume the dependence:

$$F_{a,b} = F_{a,b}(\Lambda_a, \Lambda_b) \tag{7.5}$$

If, moreover, one assumes the further condition:

$$F_a(0,0) + F_b(0,0) \leq F_a(0,x) \tag{7.6}$$

where $F_a(0,x)$ is the maximum of $F_a(0,\Lambda_b)$, then ρ is infinitely peaked at $\Lambda_a = 0$ & $\Lambda_b = x$. This means that one still has a number of warm universes. In fact, the mean number of universes can be calculated as:

$$N_{a,b} \simeq \exp\left(\frac{3}{8G^2\Lambda_{a,b}}\right) \tag{7.7}$$

Then, while the cold universes $(\Lambda_a = 0)$ are infinite, the warm ones are finite: the cosmological constant may be driven to zero also in these ones by contact with the cold ones. The main objections to this model are that it does not clearly justify the peculiar assumptions on the F amplitudes and, moreover, it seems far away from explaining a honest inflationary scenario for our universe. It may be considered, however, a good stimulation for more detailed works.

7.2 A time dependent model for Λ

Another important issue which is not clarified by the Coleman's mechanism is connected with the possibility that the effective cosmological constant might have been different from zero in an earlier cosmological epoch, i.e. during an inflationary phase. The problem of the (temporal) dynamical evolution of the gravitational and cosmological constants in the frame of the wormhole model has been recently tackled in an interesting paper by Fukuyama and Morikawa [29]. The main lines of the model are the following. The field variables are divided into "slow" ones (determining the dynamics of Λ) and "fast" ones (contributing to the constant renormalization of Λ), relative to the gravitational timescale τ_g. The Lorentzian action of the model is:

$$S = S_g - \int d^4x\sqrt{-g}\left(V(\phi) + \frac{1}{2}\xi R\phi^2\right) + \frac{1}{2}\int d^4x\sqrt{-g}(\partial\phi)^2 \tag{7.8}$$

where ϕ is a homogeneous field coupled to gravity through the parameter ξ and S_g is the usual action for gravity (see eq. (4.1), with $\mathcal{L}_M = 0$). ϕ is assumed to be a "slow" field, therefore the first two terms in the action ($\doteq S_g(\phi)$) dominate the last kinetic term ($\doteq K(\phi)$). The effective constants in $S_g(\phi)$ become now (a suffix $_o$ means the renormalized value without wormholes):

$$\Lambda = \frac{G}{G_o}\Lambda_o + 8\pi G V(\phi), \qquad G^{-1} = G_o^{-1} - 8\pi\xi\phi^2 \tag{7.9}$$

The idea is to adiabatically integrate the metric g to get a low frequency effective action S_{eff} for the "slow" variable ϕ:

$$e^{iS_{eff}(\phi)} = \int dg e^{iS} \tag{7.10}$$

To pick up only the low frequency dynamics of ϕ, one can cut up the time coordinate into pieces so that the $i-th$ piece is given by $(t_i - T - i,\ t_i + T_i)$ $(\ni t = t_i + \tau)$

$$\begin{aligned}&\int [dg] \exp\left(i\int dt \int d^3x\ \mathcal{L}(g(t), \phi(t))\right) = \\ &= \int \Pi_i \Pi_\tau dg(t_i, \tau)\ \exp\left(i\sum_i \int_{-T_i}^{T_i} d\tau \int d^3x \mathcal{L}(g(t_i, \tau), \phi(t_i, \tau))\right)\end{aligned} \tag{7.11}$$

where $2T_i$ is chosen to be smaller than the time scale of ϕ around t_i. $\mathcal{L}$ is the Lagrangian corresponding to S, and for the moment the K contribution may be ignored. The τ dependence of ϕ may be neglected. One may now Wick rotate the variable τ taking t_i as a center and obtaining:

$$\Pi_i \int [dg] \ \exp\left(-\int_{T_i}^{T_i} d\tau \int d^3x \mathcal{L}_g^e(g(\tau), \phi(t_i))\right) \tag{7.12}$$

where $\mathcal{L}_g^e$ corresponds to $S_g(\phi)$. Further calculation is essentially the same as in [78], giving as a result an EPI approximated by:

$$\sum_{topologies} e^{\Gamma_g^e(\bar{g}, \phi(t_i))} \tag{7.13}$$

where $\bar{g}$ is the background metric extremizing Γ. The single exponential is due to the fact that one, a priori, does not know the wormhole timescale, and so cannot sum over "disconnected" universes. As usual, Γ is stationary for a 4-sphere of radius $r = \sqrt{\frac{3}{\Lambda}}$ (analytically continued to a de Sitter space with scale factor $a = \cosh\sqrt{\frac{\Lambda}{3}}t$), and one can identify $T_i = \frac{\pi}{2}r$. At each t_i there is an instantaneous Euclidean S^4 universe which turns into a $S^3 \otimes R$ Lorentzian one. Continuing back to the Lorentzian signature, the kinetic term is then:

$$K(\phi) = \pi^2 \int dt \left(\frac{3}{\Lambda}\right)^{3/2} a^3(t)\dot{\phi}^2 \tag{7.14}$$

and, therefore, replacing for the slow field ϕ, $\sum_i T_i$ by $\int dt$:

$$S_{eff} = \int dt \left[\pi^2 \left(\frac{3}{\Lambda}\right)^{3/2} \cosh^3\left(\sqrt{\frac{\Lambda}{3}}t\right)\dot{\phi}^2 + \frac{1}{G}\sqrt{\frac{3}{\Lambda}}\right] \tag{7.15}$$

The ϕ equation of motion, assuming $\frac{\partial \Lambda}{\partial \phi} = 0$ (ϕ is a slow field) and $\sqrt{\Lambda}t \gg 1$, can be written as:

$$e^{\sqrt{3\Lambda}t}\dot{\phi} = -\frac{\partial}{\partial \phi}W(\phi) \tag{7.16}$$

where $W(\phi)$ is an effective potential:

$$W(\phi) = \frac{16\sqrt{3}}{9\pi} \int d\phi \left(6\xi\phi\sqrt{\Lambda} + \frac{1}{\sqrt{\Lambda}}\frac{\partial V}{\partial \phi}\right) \tag{7.17}$$

If one puts $\xi = 0$ (minimally coupled field), $W \propto \frac{\sqrt{\Lambda}}{G}$; which has a minimum at $\Lambda = 0$ and finite G. Even if $\xi \neq 0$, but sufficiently small, the minimum is not expected to change: ϕ gradually rolls down towards it and the fully renormalized Λ dynamically decreases.

Here, the friction coefficient in front of $\dot{\phi}$ in eq. (7.16) may play an important role. If the duration of the roll-down is too large, the evolution of ϕ may stop before it reaches the minimum. However, since τ_g has become larger, one may expect another field with the next longer timescale to supersede ϕ in controlling the dynamics of Λ. If the duration is, on the other hand, too short, friction cannot work and τ_ϕ decreases to zero: the adiabatic approximation breaks down when $\tau_g \simeq \tau_\phi$ and ϕ contributes to the renormalization of Λ. In the ideal case that there is always a field with the appropriate timescale (much greater than τ_g), Λ should reduce to zero in a finite time (G remaining finite). The interesting side of the argument is that it does not rely upon any (unclear) probabilistic interpretation of the Euclidean partition function. It would be worth exploring the case of analogue models for different matter contents and to implement it with the description of a possible inflationary scenario.

8 Towards a 3^{rd} quantized theory for the universe

The idea that topological fluctuations may have a dominant role in quantum gravity and the study about the initial quantum state of the universe was a strong support to the rise and development of an ambitious and challenging area in theoretical physics: the search for a 3^{rd} quantization theory for the universe [2, 3, 7, 27, 36, 69, 72, 77, 79, 84, 89, 90, 95, 106, 111, 114]. In these theories, the "universe" becomes in fact a "multiuniverse", where disconnected pieces can be created or annihilated by appropriate operators, and there can be interactions between them as well (read: 3^{rd} quantized topological couplings, vertices, propagators etc.) The suggestion by Coleman [14] about wormholes and the cosmological constant, and the necessity of its inclusion in a more fundamental framework, was then accompanied by a series of papers arguing about the behaviour and the values of the physical couplings, the correct meaning of the concepts of probabilities, expectation values of observables and their relation to the standard cosmological picture. A lot of papers assumed as a starting point that quantum gravity is formulated in the usual Euclidean context (see section 8.1). On the other hand, a set of more recent works, aware of the necessity to overcome at least some of the major difficulties inherent in these theories (see chapter 5), investigated the main features and consequences of a Lorentzian approach to the problem (see section 8.2).

8.1 The Euclidean approach

Historically, the first ideas on the possibility of a third quantization can be found in [2, 7, 72, 79, 95, 111], and in the references quoted therein. Among the first important works on third quantization and the cosmological constant one finds Rubakov's paper [106]. There, third quantization is studied in a minisuperspace model, with a matter field treated as a small perturbation. The WdW wave function of the universe is raised to the rank of an operator, expanded in terms of annihilation and creation operators for small Friedmann or large de

Sitter classical solutions. These operators obey the usual bosonic commutation rules. Universes are treated as bosons. Topological changes and interactions, associated with non linear terms in the 3^{rd} quantized action, are neglected. There exist no states containing neither Friedmann nor de Sitter universes: de Sitter states and Friedmann states are related by a Bogoliubov transformation. Rather, it is argued that, if there are no universes at small values of the scale factor, then the number of large size universes is proportional to $\exp(\frac{3}{8G^2\Lambda})$. This is close to the earlier suggestion by Vilenkin [117] for the creation of universes from nothing, even if there the tunneling interpretation of the wave function leads to an exponential suppression. This scheme gives a natural interpretation of Hawking's [57] proposed solution to the Λ problem (Coleman's double peak might emerge if accounting also for topological changes). The main difficulties of the theory are that the most probable universe (with $\Lambda = 0$) seems to be empty and cold, and the evidence about the strong dependence of the calculation on the initial state.

A good description of the problem of how to 3^{rd} quantize a minisuperspace model for a minimally coupled homogeneous scalar field (without including universe interactions, but taking into account details such as the operator ordering problem), with a more explicit asymptotic expression for large values of the scale factor of the WdW operatorial valued wave function, can be found in Hosoya and Morikawa [69].

A more fundamental and general approach to the issue of 3^{rd} quantization in an Euclidean context, also addressed to the description of a possible solution to the cosmological constant problem, is due to Giddings and Strominger [36]. To give an idea of the main concepts of these theoretical models, I will essentially follow the review by Strominger [114].

8.1.1 A 0+1 dimensional model

An ordinary quantum mechanical system is described by a set of initial data with laws governing its time evolution. Unfortunately, no clear definitions of time exist at the first stages of the evolution of the universe; moreover, each universe has its own time. Therefore, the description of the universe cannot be simply given in such a way. The task is to find a different consistent interpretation for the many universe system.

As a toy model, one can first consider a QFT in universes with zero space and one time dimension, and then generalize it to a 3+1 case. If topology change is allowed, space is described as a set of points, no longer equivalent to a quantum mechanics model but to a 3^{rd} quantized many particle QFT, where the 3^{rd} quantized equation of motion is an equation for the effective couplings in the 1-dimensional universe. Actually, there are some differences between the toy model and a more realistic 4-dimensional model: it has been shown that, in 4-D, given any two 3-manifolds, there always exists a smooth interpolating 4-manifold whose boundary is the two 3-manifolds. This does not happen in 1-D, where topology change has to be introduced in a rather artificial way. However, let us consider a 1-D universe described by the 2^{nd} quantized action:

$$S = \int_{\tau_i}^{\tau_f} d\tau (e^{-1}\dot{X}^{\mu}\dot{X}^{\nu}g_{\mu\nu} - em^2) \tag{8.1}$$

The second quantized fields are the einbein e and the D "matter" fields X^{μ} ($\mu = 1....D$). The 1-D line element is $ds^2 = -e^2 d\tau^2$ and m^2 is the 1-D cosmological constant. S is invariant under local diffeomorphisms of the world line ($\tau \to \tau'; e \to e' = e\frac{d\tau}{d\tau'}$), and to define a quantum theory one has to fix a gauge. For the synchronous gauge, $e(\tau) = N = constant > 0$. There is one gauge symmetry left, corresponding to the translations $\tau \to \tau + constant$, generated by the Hamiltonian:

$$H = g_{\mu\nu}P^{\mu}P^{\nu} + m^2 \tag{8.2}$$

($P_{\nu} = \frac{1}{N}\dot{X}^{\mu}g_{\mu\nu}$ is the momentum conjugate to X_{μ}). Quantization of the theory gives $P^{\mu} = -i\frac{\partial}{\partial X^{\mu}}$ and, for the translational symmetry, the constraint (WdW) equation:

$$H\Phi(X^{\mu}) = 0 \tag{8.3}$$

Φ is the 2^{nd} quantized wave function of the universe in 1-D, giving the probability amplitude for observing a field configuration X^{μ} in the universe. The 3^{rd}

quantization can be achieved writing the action:

$$\mathcal{S} = \frac{1}{2}\int d^D X\sqrt{-g}\Phi H\Phi = \frac{1}{2}\int d^D X\sqrt{-g}\left[(\nabla\Phi)^2 + m^2\Phi^2\right] \quad (8.4)$$

Variation of $\mathcal{S}$ with respect to Φ gives the WdW equation. To allow for the possibility of topology change, one can describe the system in terms of an EPI sum over 1-geometries. Choosing the coordinate system in the synchronous gauge to run from $\tau_i = 0$ to $\tau_f = 1$, this is defined as:

$$G_e(X_i^\mu, X_f^\mu) = \int_{X_i^\mu}^{X_f^\mu} dX^\mu(\tau)\int_0^\infty dN e^{-S_e} \quad (8.5)$$

with:

$$S_e = \int_0^1 \left(\frac{1}{N}\dot{X}^\mu \dot{X}_\mu + Nm^2\right) \quad (8.6)$$

It is then easy to see that G_e is, in fact, a Green function for H:

$$HG_e = \delta^D(X_i^\mu - X_f^{mu}) \quad (8.7)$$

Its third quantized formula is (including normalization to divide out disconnected vacuum diagrams):

$$G_e(X_i^\mu, X_f^\mu) = \int d\Phi\Phi(X_i^\mu)\Phi(X_f^\mu)e^{-\mathcal{S}_e(\Phi)} \quad (8.8)$$

where $\mathcal{S}_e$ is the Euclidean version of eq. (8.4). Topology change is then introduced by allowing the basic process of fig. (48), with weighting factor λ, described by a 3^{rd} quantized action:

$$\mathcal{S}_e = \frac{1}{2}\int d^D X^\mu\sqrt{g}\left((\nabla\Phi)^2 + m^2\Phi^2 + \frac{\lambda}{3}\Phi^3\right) \quad (89)$$

As in ordinary QFT, it can be shown that the n-point Green function is:

$$G_e(X_1^\mu, X_2^\mu, ..X_n^\mu) = \int d\Phi e^{-\mathcal{S}_e}\Phi(X_1^\mu)\Phi(X_2^\mu)...\Phi(X_n^\mu) \quad (8.10)$$

To describe a theory with topology change, one has clearly to enlarge the Hilbert space, since the "physical states" now may have support on configurations with all possible numbers of universes (3^{rd} quantized Hilbert space). One can then postulate that the many universe system is described by a Shröedinger state $\Psi(\Phi(X^i), X^o)$ obeying (at least formally):

$$\mathcal{H}|\Psi> = i\frac{\partial}{\partial X^o}|\Psi> \quad (8.11)$$

where X^o is a 2^{nd} quantized field operator playing the role of a 3^{rd} quantized time coordinate and $\mathcal{H}$ is the Hamiltonian of the 3^{rd} quantized action (8.4). Now, if one introduces the universe number operator N, diagonalized as:

$$N|n>= n|n> \tag{8.12}$$

the state $|\Psi>$ can be decomposed into:

$$|\Psi>= \sum_n \Psi_n(X_o)|n> \tag{8.13}$$

where $\Psi_n(X^o)$ represents the probability amplitude for n universes at time X^o (or with field values X^o). More generally, it will be possible to decompose $|\Psi>$ as a coherent sum over eigenstates of some other complete set of observables (i.e. Φ), and the answer to a question such as "How many universes are there?" is sensible since one has defined the auxiliary time variable X^o.

The next step is to describe the so called "single universe" approximation, which is fundamental to understand the effects of the theory for an observer living in our own universe. The idea is to consider two separated classes of universes, small ($\sim$ Planck scale) baby universes and large ($\sim$ Hubble scale) parent universes. To mimic this situation in 1-D, one can write the actions for a "parent" and a "baby" (S_p, S_b) as:

$$S_{p,b} = \int d\tau \left(\frac{\dot{X}^2}{N} + N m_{p,b}^2\right) \tag{8.14}$$

where $m_p \neq m_b$ and, for simplicity, D=1.

The allowed topology changing interactions are assumed to be: "parent-parent-baby" and "baby-baby-baby" (see fig. (49)). The corresponding third quantized action is:

$$S_c(\Phi) = \frac{1}{2g^2}\int dX \left((\nabla\Phi_p)^2 + m_p^2\Phi_p^2 + (\nabla\Phi_b)^2 + m_b^2\Phi_b^2 + k\Phi_p^2\Phi_b + \frac{\lambda}{3}\Phi_b^3\right) \tag{8.15}$$

with $\Phi_{b,p}$ acting as annihilating and creating operators for "babies" and "parents"; g^2 is a scaled out 3^{rd} quantization coupling. In the limit of very large m_p, pair production of parent universes is suppressed and, since the couplings preserve parent universe number modulo 2, one can assume to restrict to the case of a single "parent" propagating in a "plasma" of "babies" (see fig. (50)). To describe the "parent" dynamics by an ordinary 2^{nd} quantized effective action, one can use the "hybrid" description in which the "babies" are treated in a 3^{rd} quantized manner, the "parents" in a 2^{nd} quantized manner and "parent-baby" interactions by means of a mixed interaction action S_I:

$$S_I = \int d\tau N \sum_i \mathcal{L}_i(\tau)\Phi_b^i \tag{8.16}$$

where $\mathcal{L}_i(\Phi)$ are local 2^{nd} quantized operators on the parent universe and Φ_B^i the modes of the 3^{rd} quantized baby field operator (i.e., in the Coleman's theory, the annihilation or creation of a "baby" in the mode Φ_B^i is accompanied by an operator insertion $\mathcal{L}_i(\Phi)$). The 3^{rd} quantized functional integral for the parent propagator in a babies' background may be expressed by a 2^{nd} quantized path integral:

$$G(X_i, X_f) = -g^2 \int_{X_i}^{X_f} dX(\tau) \int_0^\infty dN < \Psi_B| e^{-(S_p+S_I)} |\Psi_B > \tag{8.17}$$

where $|\Psi_B>$ is the Shröedinger baby state and one can write:

$$S_I = k \int_0^1 d\tau \Phi_B X(\tau) \tag{8.18}$$

(equivalent to eq. (8.16) if one introduces the Fourier transform $\tilde{\Phi}_B(k)$ and replaces the discrete index i by the continuous one k).

The semiclassical limit of the 3^{rd} quantized theory is obtained as $g^2 \to 0$. In this limit the field operators all commute and it is possible to diagonalize Φ_B in terms of 3^{rd} quantized baby orthonormal eigenstates $|\alpha(X)>$:

$$\Phi_B(X)|\alpha(X)>= \alpha(X)|\alpha(X)> \tag{8.19}$$

The eigenvalues $\alpha(X)$, in the absence of "parent" sources, obey the dynamical constraint (from variations of eq. (8.15) with respect to Φ_B, and replacing Φ_B by its eigenvalue):

$$(-\nabla^2 + m_b^2)\alpha(X) + \frac{1}{2}\lambda\alpha^2(X) = 0 \tag{8.20}$$

In the parent propagator (eq. (8.17)), the expectation value between the baby state gives rise to a second quantized effective action for a 1-D universe:

$$S_p + S_I(\alpha) = \int d\tau \left(\frac{1}{N}\dot{X}^2 + Nm^2 + Nk\alpha(X)\right) \tag{8.21}$$

The effect of these "babies" is thus summarized by the addition of an ordinary potential $\alpha(X)$ into the field theory (see [36]). It is this term, subject to the constraint (8.20), that will appear to be essential for the vanishing of Λ and the "big fix" proposal.

8.1.2 The third quantized uncertainty principle

Already at this level, it is very nice and easy to discuss some interesting and typical features introduced by the presence of baby universes.

In general, there is no obvious reason to suppose that the "babies" are in an $|\alpha>$ eigenstate (see, e.g. [64]), but one may describe their state as:

$$|\Psi_B>=\beta|\alpha(X)>+\beta'|\alpha'(X)> \tag{8.22}$$

($|\beta|^2+|\beta'|^2=1$). Supposing to have an ideal clock in the parent universe, one can calculate the correlation function of n-field operators at times $\tau_1 ... \tau_n$:

$$< X(\tau_1)...X(\tau_n)>_{\alpha,\alpha'}=<\alpha\ \alpha'|\int dX(\tau)e^{-(S_p+S_I)}X(\tau_1)...X(\tau_n)|\alpha\ \alpha'> \tag{8.23}$$

Since the α states are orthogonal, this separates into two pieces:

$$\begin{aligned}< X(\tau_1)...X(\tau_n)>_{\alpha,\alpha'} &= |\beta|^2\int dX(\tau)e^{-(S_p+S_I(\alpha))}X(\tau_1)...X(\tau_n)\\ &+|\beta'|^2\int dX(\tau)e^{-(S_p+S_I(\alpha'))}X(\tau_1)...X(\tau_n)\end{aligned} \tag{8.24}$$

The two pieces represent ordinary correlation functions but in universes with different couplings α and α'. This explicitly explains the claim by [14] according to which two observers measuring two different values, α and α', cannot communicate to each other.

All this leads to the existence of a "3^{rd} quantized uncertainty principle". Initially, the coupling constants of the universe are not defined, but depend on a probability distribution. However, any measurement of the α's collapses the 3^{rd} quantized wave function into the states $|\alpha>$ ($|\alpha'>$) with probability $|\beta|^2$ ($|\beta'|^2$). All future measurements are then consistent with some definite α (or coupling constant). When $g^2\to 0$, after fixing the values of $\alpha(X)$ and $\frac{\partial\alpha(X)}{\partial X}$ at a given X, eq. (8.20) uniquely determines $\alpha(X)$ for each other X. But when $g^2 \neq 0$, the baby state is subject to quantum fluctuations, and one has to speak about conditional probability amplitudes for the results of the measurements. For example, if one measures $\alpha_{1,2} \doteq \alpha(X_{1,2})$, one can ask for the normalized amplitude $A(\alpha_3) = c \int D\Phi(X) e^{-S_B(\Phi)}$ for $X_1 < X_3 < X_2$ (S_B is the third quantized baby universe action). If X runs over an infinite range, $A(\alpha_3)$ is zero for all α_3. Even if X has a finite range, there will be difficulties in measuring the first derivative of α at X_3: this corresponds to the well known fact that the momentum spread of a quantum mechanical particle is very large immediately after a precise measure of its position. What is "uncertain" here is the prediction for the relation among the coupling constants.

8.1.3 The 3+1 model

The generalization to a 4-D model is not much complicated. Correspondingly to the 0+1 toy model, the new arena for the quantum dynamics of the multiuniverse theory is superspace. The 3^{rd} quantized field operators act on the "void", the 3^{rd} quantized state with no universes, and create 2^{nd} quantized states in the field theory of a single universe. In the absence of interactions, these operators obey the WdW equation; interactions generalize it to a non linear form, which is seen as a dynamical equation for spacetime couplings. The main differences arise due to the following facts. The 4-D theory is a gauge theory, and one has to take into account the 3^{rd} quantized gauge symmetries when constructing the action. This can be accomplished, e.g., by the BRST technique, which essentially implies the introduction of a BRST charge Q annihilating the physical state Φ and leading to a more general structure of the WdW equation [108]. However, this problem, along with the other of non renormalizability, does not arise in the semiclassical approximation made when introducing baby universes.

One can then, as before, make a sharp distinction between the baby universe scale, μ, and the parent universe scale, $\frac{\Lambda^{\frac{1}{2}}}{M_p}$. The 3^{rd} quantized theory will have an expansion in the small parameter $\frac{\Lambda}{\mu^2 M_r^2}$. Since the joining or splitting of a macroscopic universe from our own would lead to strong effects which have not been observed [83, 102], one has to assume that these processes are suppressed. As I have shown in chapter 2, in the semiclassical approximation for theories governed by the Einstein action at long distances, this is automatically the case. The validity of the instanton approximation can also be seen to correspond to the condition $\frac{\mu}{M_r} \ll 1$ (see [114]). To leading order in this approximation one only considers wormholes connecting "parents" to themselves or to one another. Bifurcations of "parents" are exponentially suppressed by $\frac{M_r^4}{\Lambda}$; bifurcation of "babies" is negligible with respect to the interaction of three "babies" coupled via a "parent", since this latter is enhanced by a space factor of the cube of the parent universe. Essentially, wormholes represent the propagators of a QFT where the vertices are the parent universes, whose 2^{nd} quantized state is in its turn determined by the baby universe state.

The "parent-baby" interactions are studied, once again, in the "hybrid" representation. The 3^{rd} quantized baby field operator Φ (functional of the 3-metric ^3g) is decomposed in Φ_i modes as:

$$\Phi(^3g) = \sum \Phi_i f_i(^3g) \tag{8.25}$$

with f_i a set of orthonormal functions on the space of S^3 metrics. The nucleation of a "baby" of mode i at the event x is equivalent to the insertion of a set of local operators $\mathcal{L}_i(x)$. The G_{ij} propagator from a "baby" of type i to one of type j is defined as:

$$G_{ij} = \int d^3g \int d^3g' f_i(^3g) f_j(^3g') G_e(^3g, ^3g') \tag{8.26}$$

The 2^{nd} quantized sum over 4-geometries is reproduced by the functional $Z = \int D\Phi e^{-S(\Phi)}$ where the action is (for a derivation, see [36] and [78]):

$$S(\Phi_i) = e^{2\gamma} \int_{S^3} d^3g_s \left[\frac{1}{2}\Phi_i G_{ij}^{-1} \Phi_j - \int_{S^4} d^4g_l e^{(\Phi_i - \lambda_i)\int d^4x \sqrt{g}\mathcal{L}_i(x)} \right] \tag{8.27}$$

γ is a topological coefficient (counting the number of closed loops of universes), $D^3g_s andD^4g_l$ respectively represent an integration over small ("babies") and large ("parents") 3 and 4-geometries (plus eventual matter fields), and λ_i are the fundamental coupling constants. $e^{-2\gamma}$ plays the role of a 3^{rd} quantized Planck's constant. The effective parent universe couplings $(\lambda_i - \Phi_i)$ are clearly shifted due to the "babies" (cf. with [14]). Without going through all the details, it is easy to qualitatively see how eq. (8.27) can reproduce the sum over 4-geometries. For instance, the integral over large 4-geometries is due to the fact that for each "parent" configuration there is one vertex. The use of G^{-1} as a kinetic operator then gives the desired wormhole propagator (see fig. (51)).

8.1.4 Couplings determined as dynamical variables

Assuming, as before, the existence of a Hilbert space representation for the 3^{rd} quantized field theory, with Φ_i acting as operators and the baby state described by a state $|\Psi>$, measurements of 2^{nd} quantized operators $O_i(X_i)$ on a single parent universe can be represented as:

$$< O_1(X_1)...O_n(X_n) >_\Psi = < \Psi| \int_{S^4} d^4 g_I e^{-(\lambda_i - \Phi_i)\int d^4x\sqrt{g}\mathcal{L}_i(x)} O_1(X_1)...O_n(X_n)|\Psi > \qquad (8.28)$$

In the semiclassical limit $e^{-2\gamma} \to 0$, the operators Φ all commute and they can be diagonalized by the coherent states $|\alpha_i>$ (cf. with section 4.3):

$$\begin{cases} \Phi_i|\alpha_j> = \alpha_i|\alpha_j> \\ <\alpha_i|\alpha_i'> = \Pi\delta(\alpha_i - \alpha_i') \end{cases} \qquad (8.29)$$

Then, the expectation value (8.28) becomes:

$$< O_1(X_1)...O_n(X_n) >_\alpha = \int_{S^4} d^4 g_I e^{-(\lambda_i - \alpha_i)\int d^4x\sqrt{g}\mathcal{L}_i(x)} O_1(X_1)...O_n(X_n) \qquad (8.30)$$

Since, in general, there is at least one kind of "baby" for each local operator in $\mathcal{L}_i(X)$, this formula shows that all couplings are shifted by the α's eigenvalues. The Φ_i equation of motion becomes a dynamical constraint for the 2^{nd} quantized couplings and, in the classical limit, it is:

$$G_{ij}^{-1}\alpha_j - \int_{S^4} d^4 g_I e^{-(\lambda_j - \alpha_j)\int d^4x\sqrt{g}\mathcal{L}_j(x)} \int d^4x\sqrt{g}\mathcal{L}_i(x) = 0 \qquad (8.31)$$

In the baby ground state $|0>$, eq. (8.30) for a single operator O becomes:

$$< O(X) > = \sum_{topologies} \int d^4 g_I e^{-\lambda_j \int d^4x\sqrt{g}\mathcal{L}_j(x)} O(X) \qquad (8.32)$$

which exactly reproduces Coleman's formula (4.22). In general, as for the 0+1 model, the baby universe state is not an $|\alpha>$ eigenstate, rather one can use the expansion:

$$|\Psi>= \Pi_i \int_{\infty}^{\infty} d\alpha_i f(\alpha_i)|\alpha_i> \qquad (8.33)$$

where $f(\alpha_i)f^*(\alpha_i)$ is the probability amplitude that the universe is governed by the couplings α_i. Exact predictions can be made only if $|\Psi>$ is peaked on a subspace of the α_i. The Coleman's singular peak is strong enough to fix the effective Λ to zero. Once again, one finds a superselection rule dividing the α sectors and the existence, out of the semiclassical limit, of a 3^{rd} quantized uncertainty principle, saying that the relations among the spacetime couplings cannot be fixed to arbitrary accuracy.

8.2 The Googolplexus: i.e. a Minkowskian approach to quantum gravity

As already stressed more than once, there are neither few nor unimportant unsolved troubles which may seriously undermine the whole Euclidean formulation of quantum gravity including topology change, both in the Coleman's theory and in the 3^{rd} quantization models. Some of the major difficulties have been described in chapter 5 and will not be repeated here. Because of these difficulties, Fishler, Klebanov, Polchinski and Susskind [27] started a new program, consisting in the construction of the Minkowskian version of a 3^{rd} quantized theory for interacting universes. This is the argument of the next section, where I will essentially follow the nice review by Klebanov [76].

8.2.1 Third quantization in minisuperspace

The new theory is based on a Hilbert space analysis of topology change in a minisuperspace model of quantum gravity derived from the Minkowskian path integral (MPI). Since one of the main results is an average number of large universes of $O\left(\exp(\frac{2}{3\lambda})\right)$ which, using the observational bound on $\lambda \leq 10^{-120}$, becomes $\geq 10^{10^{120}}$, the theory has been given the name of "Googolplexus" (googolplex=largest finite integer with a special name ($10^{10^{100}}$); plexus=network).

The WdW equation in minisuperspace is derived from the metric:

$$ds^2 = \sigma^2(-N^2(\tau)d\tau^2 + a^2(\tau)d\Omega_3^2) \tag{8.34}$$

($\sigma^2 = \frac{2G}{3\pi}$) and the action for gravity minimally coupled to a set of homogeneous real scalar fields ϕ_i is:

$$I = \frac{1}{2}\int_0^T d\tau N \left(-\frac{a\dot{a}^2}{N^2} + a + a^3\left(\frac{\dot{\phi}_i^2}{N^2} - \lambda - V(\phi_i)\right)\right) \tag{8.35}$$

The classical constraint equation (varying I with respect to N) is:

$$-\frac{\pi_a^2}{a} - a + a^3(\lambda + V(\phi_i)) + a^{-3}\pi_i^2 = 0 \tag{8.36}$$

where the momenta conjugate to a and ϕ_i are $\Pi_a = -\frac{a\dot{a}}{N}$ and $\Pi_i = \frac{a^3\dot{\phi}_i}{N}$.

Quantization is carried out taking $\Pi_{a,i} \to -i\frac{\delta}{\delta a,(\phi_i)}$ and annihilating the constraint operator on the wave function of the universe $\Phi(a, \phi_i)$:

$$\left[a^{-p}\frac{\partial}{\partial a}a^p\frac{\partial}{\partial a} - a^{-2}\frac{\partial^2}{\partial\phi_i^2} - a^2 + a^4(\lambda + V(\phi_i))\right]\Phi(a, \phi_i) = 0 \tag{8.37}$$

where p expresses part of the operator ordering ambiguities. This resembles a Klein-Gordon equation for gravity. Indeed, here the scale factor a seems to play the role of "time" [19] and ϕ_i that of "position". Moreover, there is a time dependent "mass"

$$m^2 = -a^2 + a^4(\lambda + V(\phi_i)) \tag{8.38}$$

The standard one particle interpretation of this equation is not appropriate (as is well known: for instance one can have pair production, the probability density is not positive definite etc..), and it is better to think of Φ as a quantum field in a theory of (interacting) universes on a Hilbert space (the Googolplexus) with coordinates a and ϕ_i.

Third quantization of the WdW equation is equivalent to the 2^{nd} quantization of the Klein-Gordon equation. The Hilbert space consists of all functionals of Φ at a fixed time a_o, $|\Psi> \sim \Psi(\Phi(a_o, \phi_i))$. $\Phi(a_o, \phi_i)$ is a complete set of commuting operators, and the WdW equation, interpreted as a Heisenberg field equation, determines the field at all other times in terms of these.

The first step is to consider a pure gravity theory ($\phi_i = 0$) without topology changes (no wormholes). Adopting the operator ordering choice given by $p = -2$ and defining $a^3 = V$, the WdW equation can be written:

$$\left(\frac{\partial^2}{\partial V^2} - \frac{1}{9V^{2/3}} + \frac{\lambda}{9}\right)\Phi(V) = 0 \tag{8.39}$$

which clearly resembles, if one thinks of V as a time and of $\Phi(V)$ as a real coordinate, the classical solution for a Harmonic oscillator with a time dependent spring constant $\omega^2 \doteq \frac{1}{9}(\lambda - V^{-\frac{2}{3}})$. The oscillator is upside-down for $V < \lambda^{-\frac{3}{2}}$, and an ordinary one with frequency $\omega_o = \frac{\sqrt{\lambda}}{3}$ as $V \to \infty$. Changing notation and replacing $\Phi(V)$ with $X(t)$, the third quantized Hamiltonian, including a delta function source at $t = 0$, is:

$$H = \frac{1}{2}P^2 + \frac{1}{2}\omega^2(t)X^2 + JX\delta(t), \qquad (P = \dot{X}) \tag{8.40}$$

$X(t)$ creates or annihilates whole universes (particles) with spatial volume t, and can be expanded as:

$$X(t) = f(t)a + f^*(t)a^\dagger \tag{8.41}$$

with a ($a^\dagger$) the universe annihilation (creation) operators, and f (f^*) the incoming (outgoing) solutions of the WdW equation. Its conjugate momentum is:

$$P(t) = \dot{f}(t)a + \dot{f}^*(t)a^\dagger \tag{8.42}$$

The canonical quantum commutation rules for X and P and for a and $a^\dagger$ are satisfied if one defines the Wronskian: $W(f, f^*) \doteq f\dot{f}^* - \dot{f}f^* = -i$. This fixes the normalization of f in a WKB solution of eq. (8.39) (assuming adiabatic variations of ω^2):

$$f(t \to \infty) = (2\omega_o)^{-1/2}e^{-i(\omega_o t + \delta)} \tag{8.43}$$

The phase δ is fixed by requiring that $\dot{f}_1(0) = 0$, where $f = f_1 + if_2$. The Hamiltonian (8.40) looses its time dependence as $t \to \infty$:

$$H(t \to \infty) = \frac{1}{2}\omega_o\{a, a^\dagger\} \tag{8.44}$$

and the ground state $|OUT>$ at late times is just the oscillator vacuum:

$$a|OUT> = 0 \tag{8.45}$$

Therefore, a and $a^\dagger$ annihilate and create the "out" universes.

The Shröedinger wave function of the Googolplexus, $\Psi(X,t)$, should be determined near $t = 0$ (since a negative time t is physically meaningless) by a smooth match on to short (Planck) distance physics. This physics is expected to be almost completely independent of λ, and therefore the boundary condition is expected to have no strong dependences on λ. Assuming a reflection symmetry under $t \to -t$, the initial wave function should behave like:

$$\begin{cases} \dot{X}(t=0)|I>= P(t=0)|I>= 0 & \text{(Heisenberg)} \\ \Psi_I(X,0) = constant & \text{(Shröedinger)} \end{cases} \tag{8.46a, b}$$

One can choose as a general form of the wave function consistent with these conditions a Gaussian of width w:

$$\Psi_w(X, t=0) = \frac{1}{(\pi w^2)^{1/4}} \exp\left(-\frac{X^2}{2w^2}\right) \tag{8.47}$$

To compare with Coleman's theory, one has to calculate the MPI transition amplitude between the Googolplexus wave function (the $|IN>$ state) and the $|OUT>$ vacuum. One can see that this is:

$$Z = \int dX(t) e^{i\int_0^\infty L(J)dt} =< OUT|IN > e^{-\frac{1}{2}J^2 G(0,0)} \tag{8.48}$$

where L is the Lagrangian corresponding to the Hamiltonian (8.40) and:

$$G(t_1,t_2) = \frac{< OUT|T(X(t_1), X(t_2))|IN >}{< OUT|IN >} \tag{8.49}$$

is the two point Green function with the Dyson temporal ordering operator T [71]. The result of eq. (8.48) can be easily checked a posteriori by using the standard definition of G:

$$G(0,0) = (-i)^2 \frac{1}{Z}\frac{\delta^2 Z}{\delta J^2}\bigg|_{J=0} \tag{50}$$

and that, using eq. (8.48) in eq. (8.50), one gets an identity. The source J has been treated as a perturbation, and Z is given by the sum of the graphs shown in fig. (52). Each line represents a universe created by the source at zero radius, propagated , shrunk back to zero radius and then annihilated by the source: in other words, it represents a sum over minisuperspace spherical geometries. Specializing to the case $|IN>=|I>$, one has (from eqs. (8.41, 8.46a, 8.49)):

$$G(t_1 < t_2) = 2f_1(t_1)f(t_2) \tag{8.51}$$

The general behaviour of $f(t)$ can be seen from eq. (8.39). Under the boundary conditions specified at $t = 0$, f exponentially increases until $t = \lambda^{-\frac{3}{2}}$, from where it will start oscillating with amplitude fixed by eq. (8.43). A detailed analysis of eq. (8.39), with WKB techniques (see also [27]), shows that $f_1(0) \sim \exp(-\frac{1}{3\lambda})$ and $f_2(0) \sim \exp(\frac{1}{3\lambda})$. As a result, from eq. (8.51):

$$G(0,0) \sim O(\exp(-2/3\lambda)) + iO(1) \tag{8.52}$$

This is in clear disagreement with Coleman's prediction [14].

However, it can also be shown that (see [78]) this is the same result one would have obtained by formally summing over saddle points in the minisuperspace EPI. $G(0,0)$ may be approximated as the EPI over all trajectories a with $a(0) = a(T) = 0$ (including multiple bounces off $a = 0$), with T ranging from zero to infinity. Due to the unboundedness of the Euclidean gravity, one "prescription" might be to sum over Euclidean stationary paths, given by $a = \frac{1}{\sqrt{\lambda}} \sin \sqrt{\lambda}\tau$, with duration $T = \frac{n\pi}{\sqrt{\lambda}}$. These paths are simply linear chains of n 4-spheres glued at their poles by zero size wormholes (see fig. (53)). Since the action of each 4-sphere is $I_e = -\frac{2}{3\lambda}$ (see [78]), one has:

$$G(0,0) \simeq \sum_{n=1} \mu^{n-1} e^{2n/3\lambda} \simeq -\frac{1}{\mu} + O(\exp(-2/3\lambda)) \tag{8.53}$$

The weighting factor μ for bounces at $a = 0$ depends on the boundary conditions at $a = 0$; semiclassically it is $\mu \sim e^{-S_w}$, with S_w the Euclidean wormhole action (weakly dependent on λ).

This theory is, however, inconsistent: when one will actually and formally account for wormholes, he will overcount those connecting north to south poles in these chain configurations. To turn off these unwanted wormholes (i.e., effectively setting $\mu = 0$), one needs a careful tuning of either the boundary conditions or the spring constant ω^2 near $t = 0$. For instance, if one makes ω^2 sufficiently attractive there (i.e., if the WdW equation has a zero energy bound state when $\lambda = 0$, maintaining the boundary condition $\dot{f}_1(0) = 0$), one can fine tune f_1 such to be exponentially decreasing with t and $f_1(0) \sim \exp(\frac{1}{\lambda})$, and then $G(0,0) \sim \exp(\frac{2}{\lambda})$, as wanted.

A simpler way to look at this fine tuning prescription is the following. Using $a|OUT> = 0, a = iW(f^*, X)$ and projecting in the position representation, the

Shröedinger wave function of the out vacuum can be found:

$$\Psi_{OUT}(X,t) = \left[\frac{Re(\alpha(t))}{\pi}\right]^{1/4} e^{-\frac{1}{2}\alpha(t)X^2} \tag{8.54}$$

with $\alpha(t) \doteq -i\frac{\dot{f}^*(t)}{f^*(t)}$.

Without fine tuning, $Re[\alpha(0)] \simeq O(\exp(-\frac{2}{3\lambda}))$ and $Im[\alpha(0)] \simeq O(1)$: $\Psi_{out}(X,0)$ is an oscillating wave function with a broad envelope. With fine tuning, $Im[\alpha(0)] \simeq O(\exp(-\frac{2}{3\lambda}))$, and the oscillations are eliminated. In the limit $\lambda \to 0$ (where one can clearly separate, since 4-spheres are very large, between deformed "bubbles" and "bubbles" connected by wormholes), $\Psi_{out}(X,0)$ becomes flat and, therefore, indistinguishable from $\Psi_i(X,0)$ (see eq. (8.46b)). This is equivalent to choosing the boundary condition:

$$\begin{cases} \Psi_{IN}(X,0) = \Psi_{OUT}(X,0) & \text{(Shröedinger)} \\ |IN> = |OUT> & \text{(Heisenberg)} \end{cases} \tag{8.55}$$

i.e. the Googolplexus state evolves into the ground state at late times, with no de Sitter universes. For this choice and from eq. (8.41, 8.49), it is easy to show that:

$$G(0,0) = f(0)f^*(0) \sim \exp(2/3\lambda) \tag{8.56}$$

as before.

This result is, however, still in gross contrast with Coleman's one. The path integral is, in fact:

$$Z \simeq \exp\left(-\frac{1}{2}J^2 e^{2/3\lambda}\right) \tag{8.57}$$

which is still a double exponential, but enormously suppressed as $\lambda \to 0$. The origin of the suppression is that the average number N of outgoing universes produced by the source J diverges as $\lambda \to 0$:

$$N \doteq < OUT|e^{iJX(0)}a^\dagger a e^{-iJX(0)}|OUT> = J^2 f(0)f^*(0) \sim J^2 e^{2/3\lambda} \tag{8.58}$$

Eqs. (8.57) and (8.58) are in fact in agreement with a Poisson distribution of large universes, where Z is the amplitude to produce no universes in the final state, since:

$$|Z|^2 = e^{-N} \tag{8.59}$$

8.2.2 Allowing for topology changes

Now one can consider topology changes. To do so, one can introduce in the action (see [27]) a dynamical variable α, representing the field for Planckian size wormholes (baby universes), and obtain:

$$S = \frac{1}{2}\alpha^2 + \int_0^\infty dt \left(\frac{1}{2}\dot{X}^2 + \frac{1}{18}\left[\frac{1}{t^{2/3}} - (\lambda + g\alpha)\right] X^2 - JX\delta(t)\right) \qquad (8.60)$$

The Feynman graphs resulting from this nonlocal action correspond to space-times in which the spheres described by X are connected by wormholes (the α propagator). To keep a Hamiltonian interpretation, one needs making the action local in time formally promoting α into a function of t and introducing a Lagrange multiplier $\beta(t)$:

$$S = \frac{1}{2}\alpha^2 + \int_0^\infty dt \left(\frac{1}{2}\dot{X}^2 + \frac{1}{18}\left[\frac{1}{t^{2/3}} - (\lambda + g\alpha(t))\right] X^2 + \beta(t)\dot{\alpha}(t) - JX\delta(t)\right) \qquad (8.61)$$

Since α and β appear as a pair of conjugate variables, the state of the Googolplexus ($|\Psi >$) may be chosen to depend on X and α. The first term in eq. (8.61) is arbitrary, because of the constraint $\dot{\alpha} = 0$, and it can be absorbed in $|\Psi >$. Since $\dot{\alpha} = 0$, the Hilbert space breaks up into sectors labelled by an α independent of time. Each sector has now an effective cosmological constant $\lambda_{eff} = \lambda + g\alpha$. A generic Googolplexus state can be expanded in a complete set of states ($|\alpha >$) with definite α:

$$|\Psi >= \int d\alpha e^{i\alpha^2/2} F_\Psi(\alpha)|\Psi; \alpha > |\alpha > \qquad (8.62)$$

where $F_\Psi(\alpha)$ is the weighting function for different α sectors, determined by the boundary conditions at $t = 0$, smoothly dependent on λ_{eff}. The $|OUT >$ state is:

$$|OUT >= \int d\alpha F_{OUT}(\alpha)|OUT; \alpha > |\alpha > \qquad (8.63)$$

and the MPI (for $|\Psi >= |IN >$) is:

$$\int d\alpha e^{i\alpha^2/2} F^*_{OUT}(\alpha) F_{IN}(\alpha) < OUT; \alpha|IN; \alpha > \qquad (8.64)$$

where I have already calculated $Z(\lambda + g\alpha) \doteq < OUT; \alpha|IN; \alpha >$ to be exponentially suppressed as $\lambda_{eff} \to 0$.

8.2.3 The probability interpretation

One of Coleman's implicit assumptions for his theory of the cosmological constant is that the EPI with operator insertions computes expectation values in quantum gravity. This is not the correct point of view in the Googolplexus scheme. The main problem (as already announced in section 8.2.1) is that the 3^{rd} quantized Hamiltonian (8.40) is explicitly time dependent. In this theory, the MPI with operator insertion Θ_i is a transition amplitude of the kind:

$$< OUT|\Theta_i|IN > \tag{8.65}$$

while for the expectation value (for instance in the $|IN >$ state) one should have:

$$< IN|\Theta_i|IN > \tag{8.66}$$

For instance, the probability distribution for the effective cosmological constant should be calculated as [27]:

$$P(\lambda_{eff}) =< IN|\delta(\lambda + g\alpha - \lambda_{eff})|IN >= g\left|F_{IN}\left(\frac{\lambda_{eff}-\lambda}{g}\right)\right|^2 \tag{8.67}$$

Therefore, the a priori probability for λ_{eff} is a smooth function entirely depending on the Planck scale physics: it has no sharp peaks at $\lambda_{eff} = 0$. Coleman's precognition is absent in the Googolplexus. The average number of large de Sitter universes is the same as in eq. (8.58), with λ substituted by λ_{eff}, and $|OUT >= e^{iJX(0)}|IN >$: it is exponentially divergent at $\lambda_{eff} = 0$. For incidence, the MPI computes expectation values if $|IN >= |OUT >$, corresponding to choose $J = 0$. From eq. (8.58) it is easy to see that no sharp peaks exist at $\lambda_{eff} = 0$ also in this case. A similar result holds in a more general theory where the α fields are substituted by a X^3 interaction term in the action (8.60), leading to a smooth $P(\lambda_{eff})$ and a large number of de Sitter outgoing universes (see [27]).

All this appears rather disappointing. On the other hand, the a priori probabilities investigated so far are reasonable for ideal "meta-observers" who can couple to the whole Googolplexus. It would be interesting to define a kind of inclusive probabilities, sensitive to an observer confined to our own universe. This evokes a sort of anthropic principle: probabilities should be weighted by the number of universes which resemble our own. If one considers large de

Sitter universes, according to eq. (8.58), this number is exponentially peaked at $\lambda_{eff} = 0$. This should also hold for a more general case, like that for the Gaussian boundary condition (8.47), for which one has (at late times):

$$N(w) =< w|a^{\dagger}a|w >= \frac{1}{2w^2}|f(0)|^2 + \frac{w^2}{2}|\dot{f}(0)|^2 - \frac{1}{2} \quad (8.68)$$

which is exponentially peaked at $\lambda_{eff} = 0$, even without fine tuning at $t = 0$ (see also [27]). The qualitative reason of this behaviour is due to the fact that, from eq. (8.39), when $t < \lambda^{-\frac{3}{2}}$ the wave packet spreads, such that at late times the wave function is in a highly excited state containing a large number of quanta (universes).

To see if this may work in a more realistic frame, one has to include the matter degrees of freedom in the action. As a simple starting model, one can consider a periodic scalar field ϕ (an angle ranging in $(0, 2\pi)$), and expand the WdW field as:

$$X(t,\phi) = \sum_k e^{ik\phi} f_k(t) a_k + h.c. \quad (8.69)$$

Insertion in eq. (8.39) says that f_k is a solution of:

$$\left(\frac{\partial^2}{\partial t^2} + \frac{k^2}{9t^2} - \frac{1}{9t^{2/3}} + \frac{\lambda}{9}\right) f_k(t) = 0 \quad (8.70)$$

For $|k| \ll \frac{2}{3\sqrt{3}\lambda}$, this equation has two classical solutions: a FLRW universe, oscillating between a singularity at $t = 0$ and a maximum volume $\sim |k|^{\frac{3}{2}}$, and a de Sitter universe of minimum volume $\sim \lambda^{-\frac{3}{2}}$. For $|k| > \frac{2}{3\sqrt{3}\lambda}$ the barrier separating the two regions disappears, and one has a universe expanding for ever.

One can assume that a universe similar to ours is described by a FLRW solution with a small curvature:

$$1 \ll |k| \ll \frac{2}{3\sqrt{3}\lambda} \quad (8.71)$$

The inclusive probabilities are weighted by the number (N) of universes of this kind. De Sitter universes with volume $\gg \lambda^{-\frac{3}{2}}$, however, are not to be counted, since they virtually contain no heat (the scalar field energy is negligible for $t > \lambda^{-\frac{3}{2}}$). This appears to destroy the previous argument. Moreover, imposing boundary conditions at early times does not lead to any sharp dependence of N on λ_{eff}.

The other interesting possibility is to impose boundary conditions at late times, as in eq. (8.55). If condition (8.71) is satisfied, there is a thick barrier separating the two classical solutions of eq. (8.70), and the wave function tending to the ground state Gaussian at late times is a wave packet of width $\sim \exp(\frac{2}{3\lambda})$ at early times. Such a state will contain $O\left(\exp(\frac{2}{3\lambda})\right)$ FLRW universes and might solve the cosmological constant problem.

I would like to stress, however, the following consideration. Unfortunately, the model presented in [27, 77] may appear not to be physical: in fact, it suffers from the problem of not considering the possibility that our universe has undergone an inflationary phase. Whether it may be considered interesting will surely need further analysis.

9 Conclusions and prospects

Let us summarize the main aspects of the wormhole theory we have dealt with. I have first tried to give (chapter 2) an overview as comprehensive as possible of all known wormhole solutions in the recent literature. We saw that, for semiclassical gravitational instantons to exist, one has apparently to impose specific conditions, such as the presence of antisymmetric stress-energy tensors in the action functional. This condition appears to be satisfied by most matter fields (axionic, electromagnetic, Yang-Mills, minimally coupled charged scalar, conformal scalar, nonlinear gravity etc.). Almost all the instanton solutions that have been presented so far share the crucial property of the existence of some conserved charge, which is also used, for example, to impose particular boundary conditions. Little attention is, in general, paid to the properties and problems of the analytic continuation of the instanton solutions to the Lorentzian signature regime. A first trial to generalize these wormhole instantons at a quantum level has been tackled in some simple cases.

In chapter 3 I showed that wormhole solutions may actually be no more exotic than Friedmann solutions are for GR. In fact, we demonstrated that, for every classical solution in standard cosmology with the closed spatial geometry and with stress tensor sources obeying the strong energy condition, there is a wormhole solution which is the analytic continuation of a closed expanding universe. Some of these solutions reproduce the main already known ones. In particular, we constructed explicit wormhole solutions driven by a complex scalar field with a nontrivial potential term, and showed that for these to exist it is necessary to analytically continue to the Euclidean regime by an asymmetric rotation of the lapse function in the matter and gravitational part of the action. Our choice is connected to the assumption that flat spacetime is the ground state of QG. No conserved charges stabilyze our solutions. Our analytic continuation defines an EPI which is different from the usual one and, to keep a quantum amplitude which decreases with the wormhole size, its overall sign has to be chosen such that the total action is made negative definite. Finally, we found an evidence for the finite temperature of wormholes, which might be related to the Hawking's idea about black hole evaporation.

In chapter 4, then, I tried to discuss in some details the original proposal that wormholes might fix the coupling constants of nature at definite values (in particular, $\Lambda = 0$). Integrating out wormhole fluctuations gives an effective theory for gravity and matter fields where the couplings become dynamical variables governed by a probability distribution. A saddle point analysis of the action functional in the EPI around large smooth geometries shows that this distribution should be exponentially peaked at $\Lambda = 0$. The attempts to also fix the other couplings of nature appears, at the moment, still far from achieving definite and clear results. In some models, the condition $G > 0$ seems to require a dynamical generation of the particle masses; moreover the pion mass and the θ angle of QCD might be set to zero. Other models allow for naturally small particle masses.

A lot of other difficulties threaten the wormhole theory (chapter 5). As we already said, at present there is no well defined formulation of QG in terms of the EPI. The action for gravity is unbounded from below, and one has to find a contour of integration for which the EPI is made convergent. The proposal [33] for the rotation of the conformal degrees of freedom of the metric has not yet been implemented for more general and complicated cases. Some [50] tried to do that working with the linear gravity. Others [44] used the "steepest descent" methods in the space of 4-geometries with complex signature. Moreover, the choice of the contour for the EPI might be determinant in turning the Coleman's peak at $\Lambda = 0$ into a disappointing broad distribution. No clear justifications have been given for the use of smooth geometries and the distinction between large universes and small wormholes in the derivation of Coleman's theory. The problems of finding a regulator for the infrared divergence of the probability measure, and that of suppressing the amplitudes of "giant" wormholes have not convincingly been solved.

Other aspects of the wormhole theory have been investigated in chapter 6. Some interesting calculations have been done to explicitly construct the effective interaction vertices induced in the low energy Lagrangian by wormholes, and to study their possible consequences on particle physics. The effects of the presence of "unobservable" baby universes outside our universe have also been studied. I would tendentiously agree with Hawking's point of view, which is that of considering "metaphysics" to argue whether wormholes lead either to a lack of knowledge of the initial state of the universe or to the loss of quantum coherence. Wormholes, however, seem to induce an extra degree of uncertainty in QG besides that of ordinary flat QFT. Other features, such as wormhole interactions etc., still appear to need further and deeper understanding.

Some initial and partial efforts to more precisely connect the wormhole theory with the possible scenarios of cosmology can be found in the recent literature (see chapter 7).

Finally, in chapter 8, we tried to introduce to the problems of the 3^{rd} quantization theories. There, the idea is to consider a "multiuniverse" system as a QFT on superspace, where 3^{rd} quantized operators create 2^{nd} quantized states in the field theory of a single universe. The 3^{rd} quantized state is a 2^{nd} quantized WdW wave function. Because of universes' interactions, we showed that the field equation in the 3^{rd} quantized theory becomes nonlinear, and it represents a dynamical equation for the 2^{nd} quantized couplings. These couplings are subject to a 3^{rd} quantized uncertainty principle. The Euclidean version of these theories essentially appears to agree with the main predictions of Coleman's model about the cosmological constant. Contrarily, the Lorentzian version, introduced to overcome some of the difficulties of the Euclidean theory and which fixes the attention on the problem of correctly defining the concepts of probabilities in the PI, seems to reach the conclusion that no peaks at $\Lambda = 0$ should exist.

The excited atmosphere that pervaded the community of high energy physicists, relativists and cosmologists after the appearence of the first works on the wormhole theory and their claimed effects on Λ etc., at present has partly disappeared. Not much progress has been done to solve at least some of the problems in the original Coleman's theory for the cosmological constant. Coleman himself, at the school held in Jerusalem last December, seemed very sceptic about the possibility of overcoming these difficulties in the very next future. Also the different proposals for a 3^{rd} quantization theory have not at the moment been implemented with precise calculations in a more realistic cosmological context and still appear at a rather initial stage and not to agree on their predictions. Apparently only Hawking and a few others [8, 66] have started the program of finding full quantum wormhole solutions. The only area where the number of publications seems to continuously increase is that of new specific semiclassical wormhole solutions.

Our new set of solutions might appear, if confirmed, a more general one. To check whether our rule for the analytic continuation is really justified, and the scalar field driven wormholes are relevant or not, one could take, for instance, the following perspective. First of all, it should be very interesting to look for an explicit computation of the wave function solution to the WdW equation with a generic potential term (see chapter 3, eq. (3.62b)). Naively, one could compare it (at least in some asymptotic limit) with the semiclassical form of the wave function that one can construct using the action for our wormhole solutions (eq. (3.71)). Depending on the boundary conditions, one may be interested in amplitudes for a given value of the scalar field or of its momentum. The two should be related by a Laplace transform. This should also definitely tell us about the right choice of the global sign in front of the action. Then, one might try to applicate methods similar to those used by [50]. There, the existence of a conformal rotation leading to a well defined EPI was explicitly shown for the linearized gravity, working with the true degrees of freedom. It would be nice to see is it possible to recover the WdW wave function for our nontrivial matter content (found before) by a straightforward contour integration. A similar work could be done using the "steepest descent" methods of [44]. As in the standard analytic continuation scheme, one should independently specify the (complex) contour of integration which reproduces the behaviour of our wave function in the Euclidean space, and an oscillatory wave function in the Lorentzian space. Boundary conditions and the choice of the contour should justify our rules for the analytic continuation.

As already said in chapter 7 and 8, other interesting and nice results will probably come from a more serious and quantitative analysis of the wormhole theory in the context of a "honest" cosmological scenario, where one takes into account universes which have undergone an inflationary period and resemble our own. Despite their present problems, also the development of the 3^{rd} quantization theories will probably show to be determinant for the understanding of the role of wormholes in QG.

As we saw, at present, the "wormhole world" still appears with a lot of unclear and unsolved isuues, but surely it is worth being explored in more details.

Aknowledgements

I am greatly indebted to my Supervisor, Prof. D.W. Sciama, who suggested to me this subject, helped me to enter it, and encouraged me during the learning period. He also supported my participation to the schools of Edinburgh and Jerusalem.

I am particularly grateful to Dr. M. Mij'ic, who patiently helped me to better understand the various and obscure aspects of the wormhole theory and quantum cosmology with lots of enlightening discussions, who greatly stimulated me also in the difficult moments during the preparation of our paper on wormholes, and for his very useful comments to improve this thesis.

I would also like to thank Prof. A. Treves and Prof. M. Abramowicz for having understood the real line of my interests and for having encouraged me on this new road.

Finally, I have to say thanks to Dr. Laflamme R. for his useful comments on the paper on wormholes, to Dr. Y. Anini and P. Soriani for helpful discussions, and to M. Orlandini, A. Romeo and J. Acosta that were precious in approaching me to a better use of the TEX and LATEX facilities.

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Figure captions

Fig. 1. The amplitude or wavefunction $\Psi[h_{ij}\phi]$ is given by integrating over all 4-geometries and matter field configurations belonging to the class C, which are bounded by the 3-surface S and which agree with the given 3-metric h_{ij} and matter field configuration ϕ on S.

Fig. 2. On the trousers topology, the direction of time cannot be chosen smoothly. There must be a singularity at least at one point. Arrows show a particular choice of a time-like vector which is ill defined at point A.

Fig. 3. The process of creation and absorption of a baby universe is called a wormhole.

Fig. 4. The birth of a baby universe.

Fig. 5. The particles of a black hole go off into a little closed universe which branches off from our region of spacetime.

Fig. 6. A truncated wormhole representing a topology change from R^3 to $R^3 \oplus S^3$.

Fig. 7. A wormhole with two asymptotically flat Euclidean regions. Two dimensions are suppressed; each circle around the throat represents a 3-sphere.

Fig. 8. A close view of a wormhole.

Fig. 9. An instanton describing tunneling from a topologically R^3 initial geometry Σ_i to a topologically $R^3 \oplus S^3$ final geometry Σ_f.

Fig. 10. Wormhole charge, q, as a function of wormhole size, L. The dotted curves show the large-wormhole and small-wormhole limits.

Fig. 11. The maximally extended solution with $\Lambda = 0$ has two asymptotically flat Euclidean regions. A single r coordinate covers only one-half of the space, while the u coordinate extends over the entire instanton. (D-2) dimensions are suppressed, so that each circle around the throat represents a (D-1) sphere.

Fig. 12. The instanton with $\Lambda > 0$ covered by a single r coordinate patch. Near r_{max}, Λ dominates the curvature, while at r_{min} the axion is the chief source of curvature. (D-2) dimensions are suppressed.

Fig. 13. Examples of instantons constructed by sewing together a numer of copies of the instanton of fig. (12). Such instantons may describe tunneling between a) de Sitter space and a baby universe, b) two baby universes, or c) two de Sitter spaces.

Fig. 14. a) The potential $V(H)$ of the analog mechanical model. The energy E determines the period of the electric and magnetic field oscillations. b) The potential $V(\sqrt{2\lambda}a)$ of the analog mechanical model. The scale factor $a(\tau)$ oscillates between $R_- \leq a \leq R_+$.

Fig. 15. The phase diagram of (a) the gauge potential and (b) the scale factor. The solid lines correspond to $E = 0$. The regions bounded by the solid lines are for $E < 0$ and both a and H show oscillatory behaviour.

Fig. 16. The phase plane of the Euclidean solutions with $\lambda > 0$. The separatrix (the bold curve) is the set of solutions for which $K = 0$. The solutions for which $K < 0$ lie outside this separatrix and are all

singular. The solutions for which $0 < K < 1/4\lambda$ lie inside the separatrix. These are the wormhole solutions. They are periodic and non singular and centre around the point $a = (2\lambda)^{-1/2}, a' = 0$.

Fig. 17. The phase plane of Lorentzian solutions with $\lambda > 0$. The separatrices are the solutions for which $K = 1/4\lambda$ and intersect the a axis at the point $a = (2\lambda)^{-1/2}$. The solutions for which $0 < K < 1/4\lambda$ lie between the separatrices and intersect a' at some stage. The solutions lyiong outside the separatrices are those which $K > 1/4\lambda$.

Fig. 18. (a) Two flat spaces connected by a wormhole of radius a_o.
(b) Two de Sitter spaces with the same radius a_{DS} connected by a wormhole with a radius a_o.

Fig. 19. The potential $V(f)$. The parameters are chosen to be: $\alpha = 200, \beta = 500$, $c = 5$ and $h = 4.71584 \times 10^{-7}$.

Fig. 20. Graphs of the radius r (solid line) and the scalar field f (dashed line) in the case of a double period wormhole solution.

Fig. 21. (a) An instanton whose analytic continuation is a contracting small universe. (b) An instanton whose analytic continuation is an expanding small universe.

Fig. 22. (a) The potential for the scalar field ϕ. (b) The particle with coordinate ϕ moves in the potential $-V$.

Fig. 23. A numerical solution to eq. (2.77) for the potential $V = 0.1\phi^2(\phi-1)^2 + 0.002(\phi - b)$.

Fig. 24. After the decay, new universes can again create expanding baby universes by the same instantons.

Fig. 25. Wormhole as analytic continuation of a closed expanding universe The strong energy condition for matter sources ensures both the existence of the maximal radius for the Lorentzian branch,and the asymptotically flat behaviour for the Euclidean one.

Fig. 26. A large universe with one wormhole.

Fig. 27. A large universe with multiple wormholes.

Fig. 28. Large universes connected by wormholes.

Fig. 29. The action S for a 4-sphere of radius R, with $\Lambda > 0$. the minimum S_o is indicated (a) for $\Lambda \geq 3R_{max}^{-2}$ and (b) for $\Lambda \leq 3R_{max}^{-2}$.

Fig. 30. A schematic diagram of the most general 4-geometry connecting an initial state with n_i baby universes to a final state with n_f baby universes. m of the baby universes do not interact with the parent universe, and n of thit are emitted and reabsorbed by the parent universe.

Fig. 31. A number of large Euclidean manifolds connected by wormholes. Baby universes emitted and absorbed by the vacuum are shown as loose ends.

Fig. 32. Euclidean spacetime has been divided into cells; each cell contains a number of instantons that is assumed to be an integer.

Fig. 33. A 1-loop diagram that induces a logharitmic renormalization of G. The solid line represents a massive particle (either a fermion or a scalar). The wavy lines represent gravitons.

Fig. 1

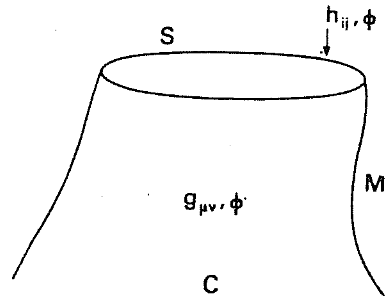


Fig. 2

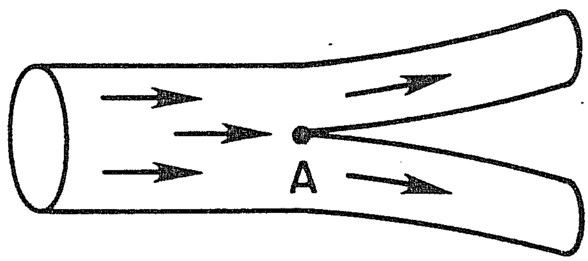


Fig. 3

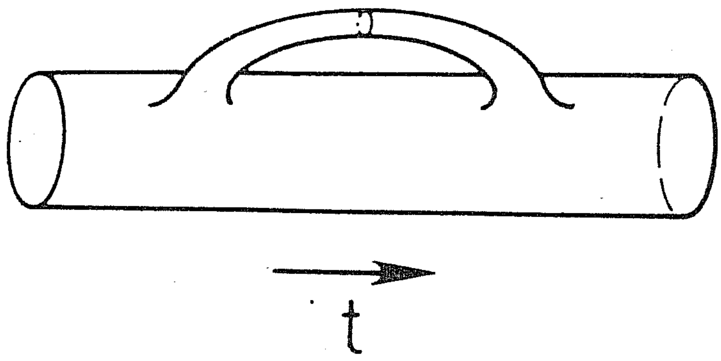


Fig. 4

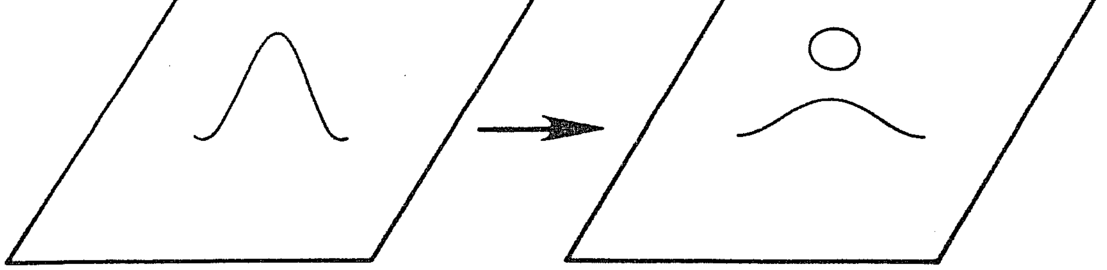
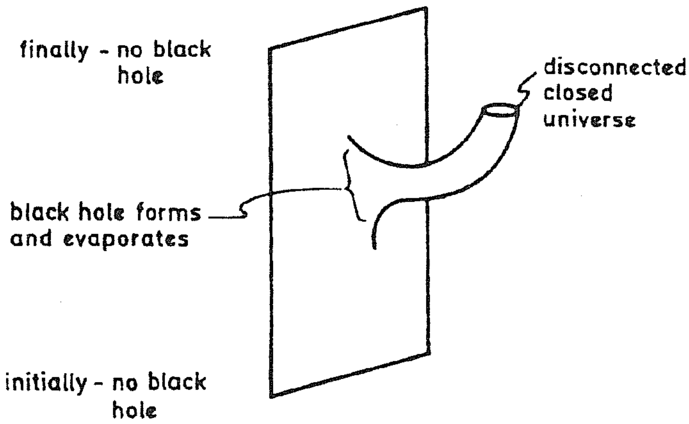


Fig. 5



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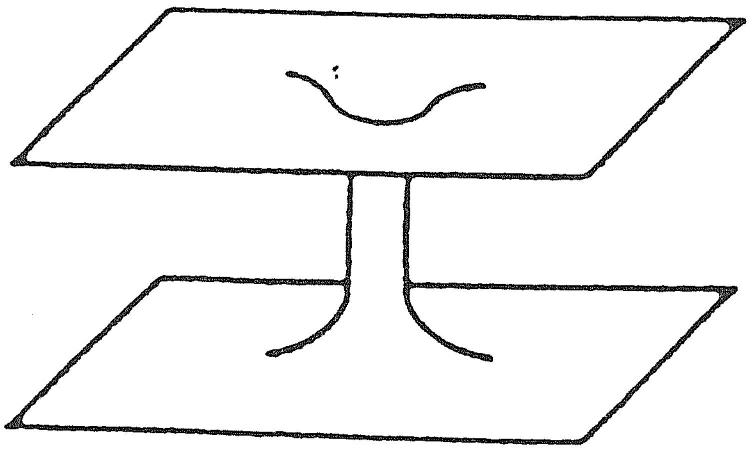


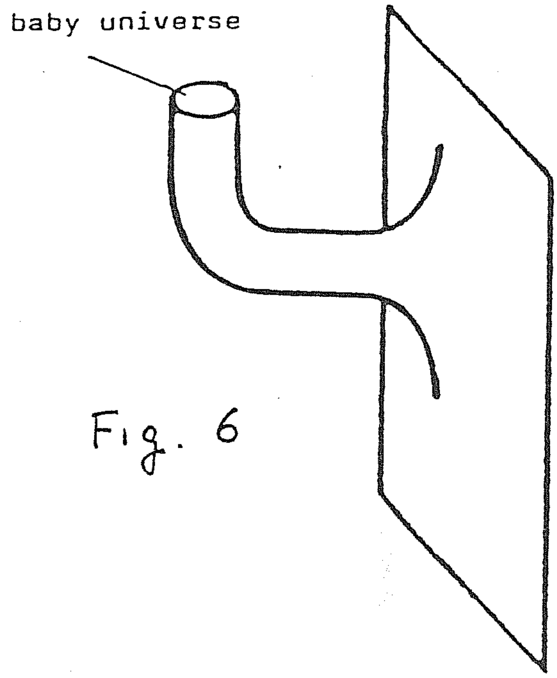
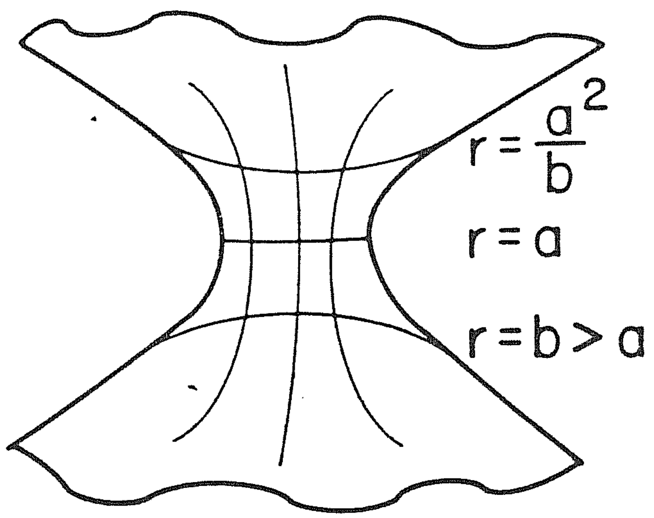


Fig. 6

Fig. 8



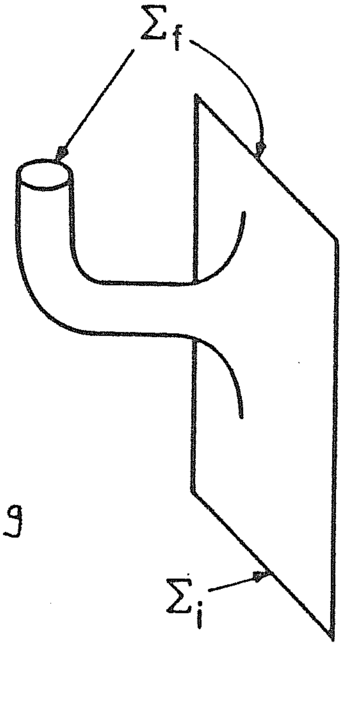
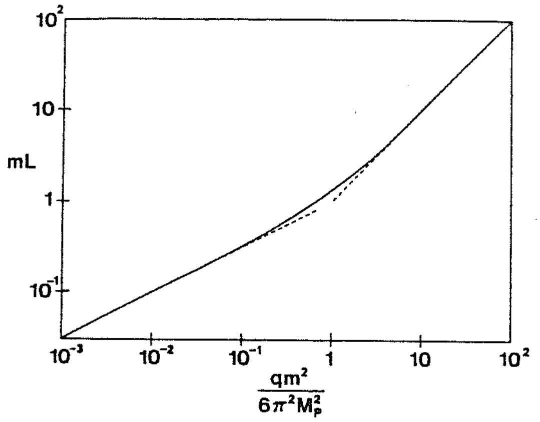


Fig. 9

Fig. 10



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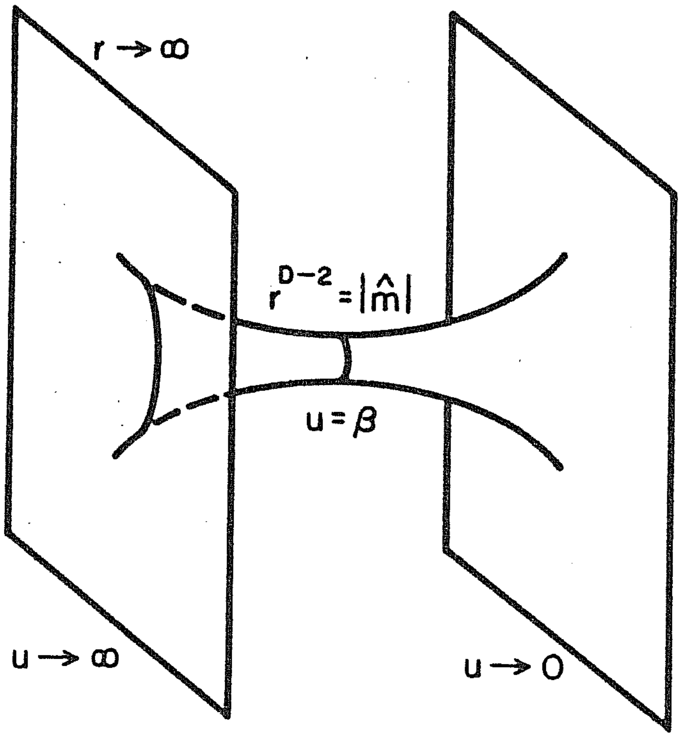

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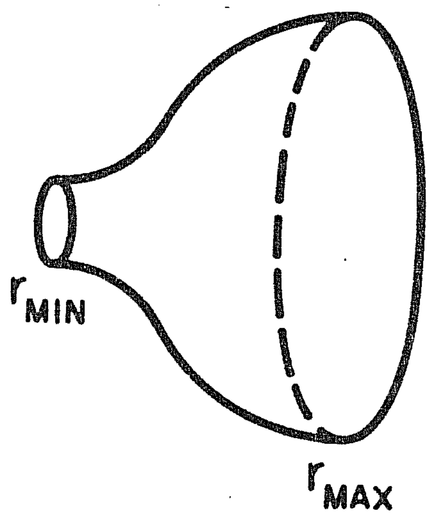

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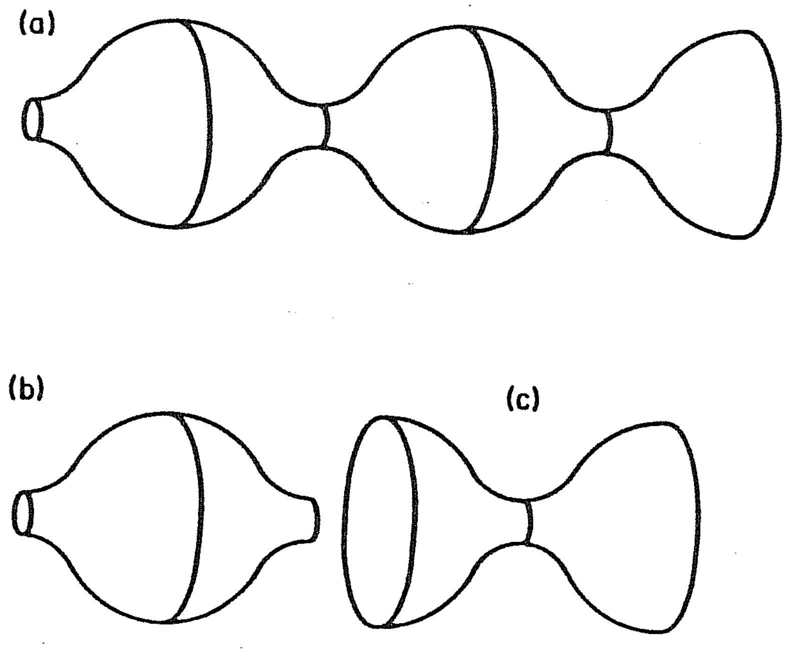

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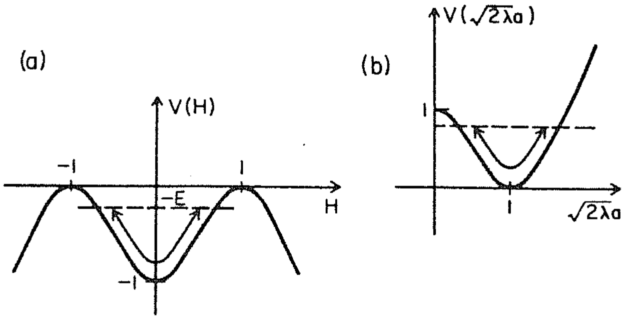

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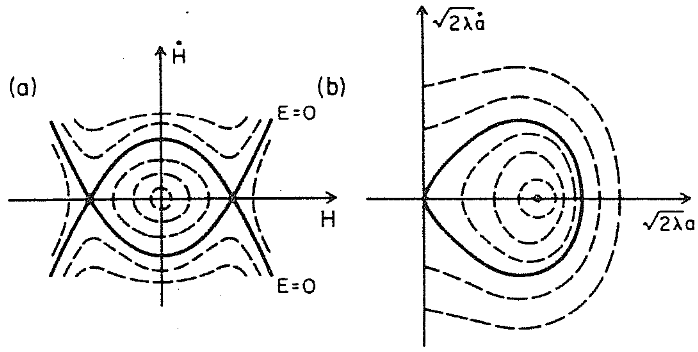

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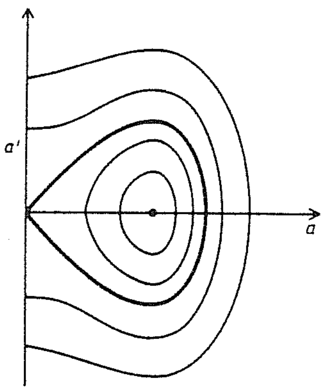

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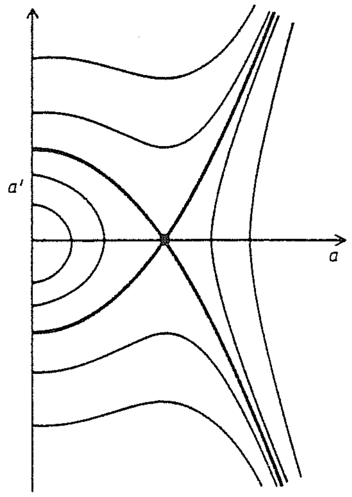

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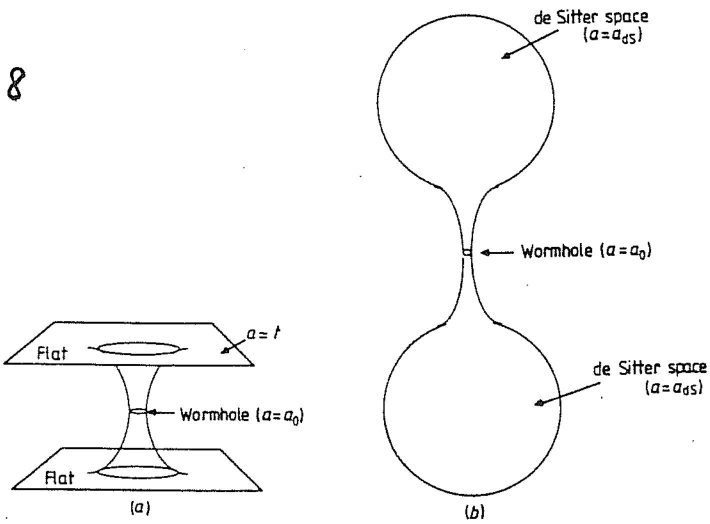

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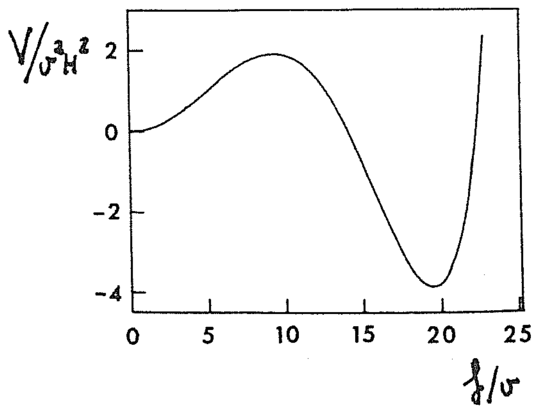


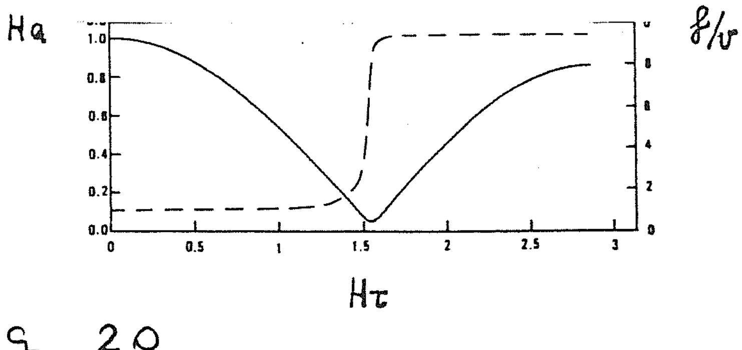
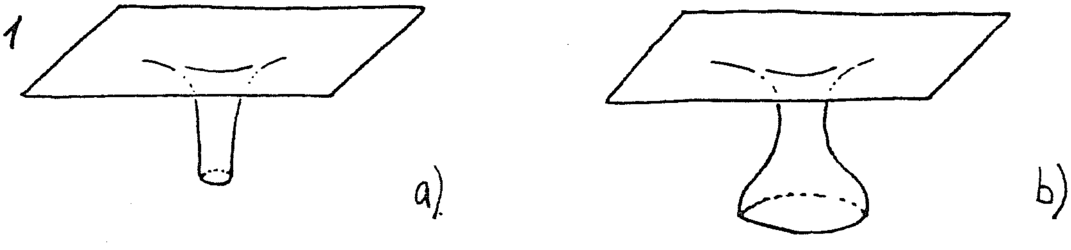


Fig. 20

Fig. 21



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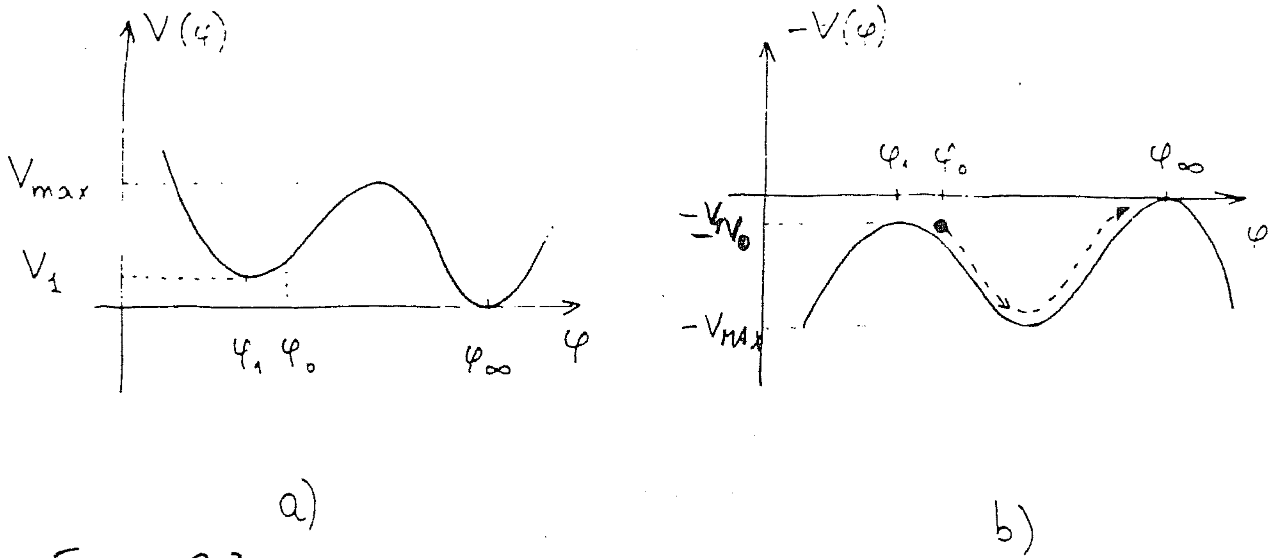

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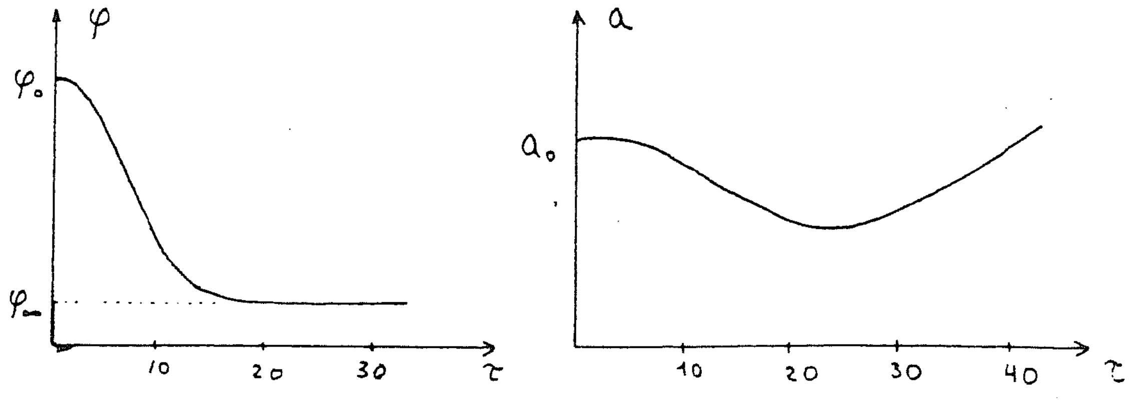

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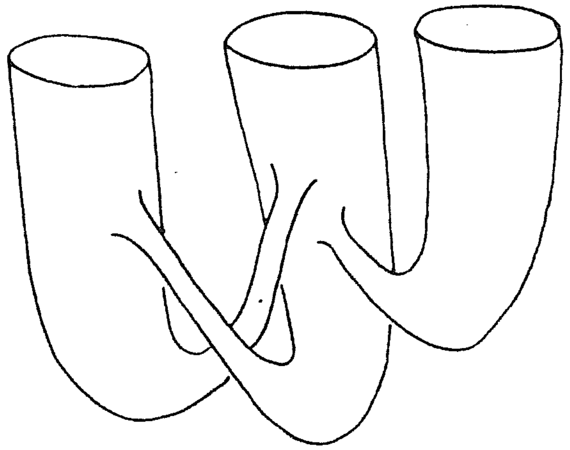

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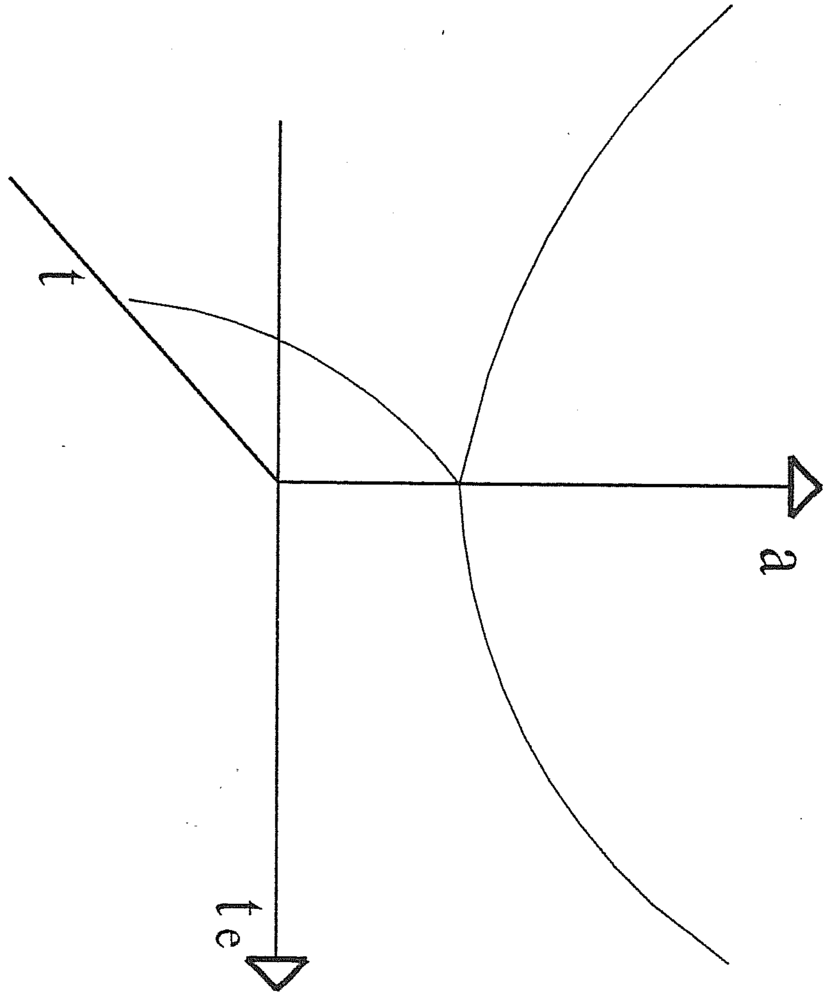

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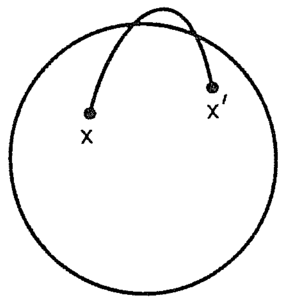

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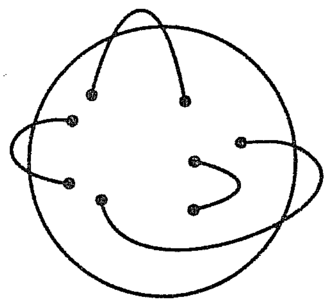

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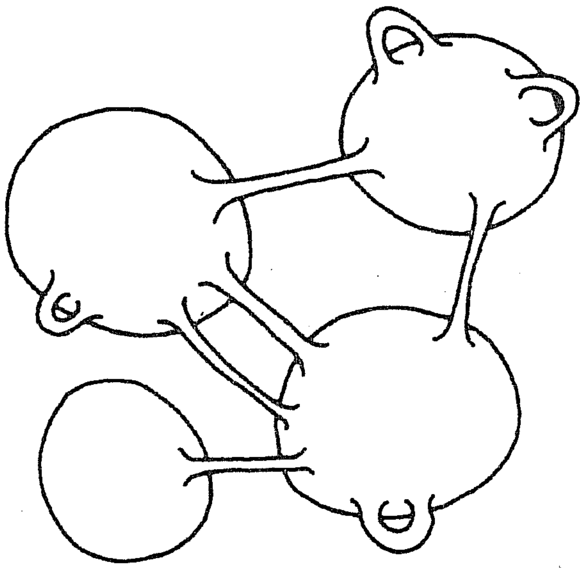

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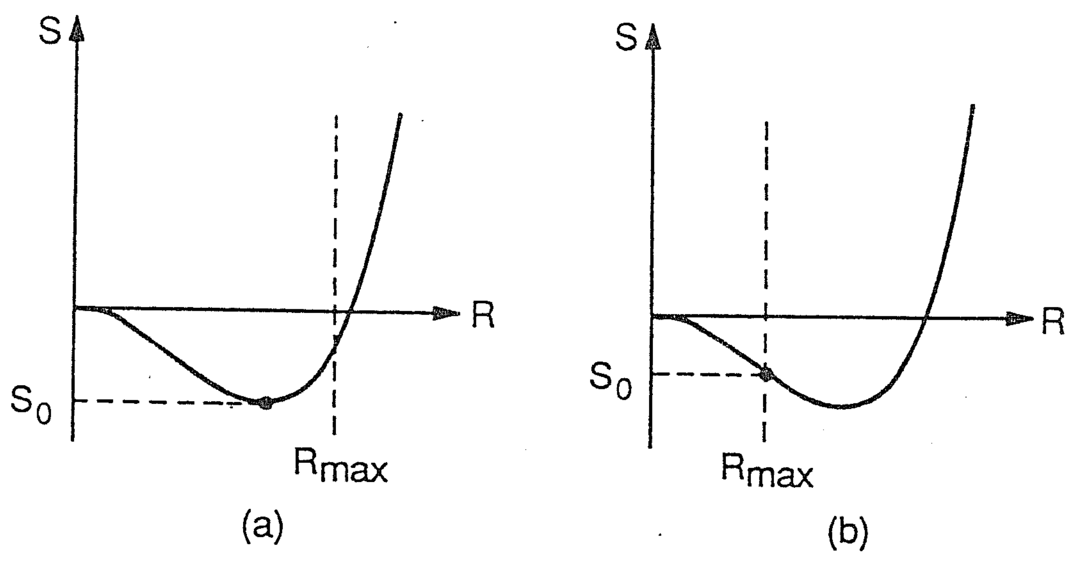

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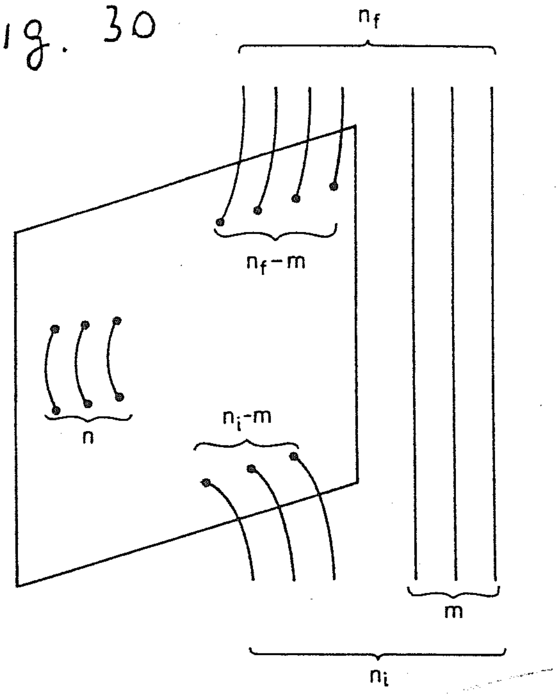

Fig. 31

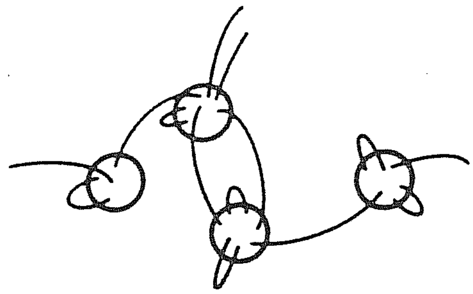

Fig. 32

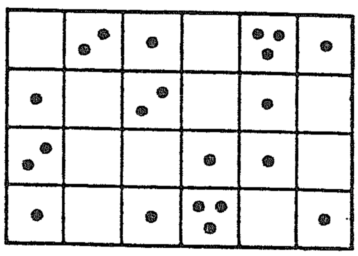

Fig. 33

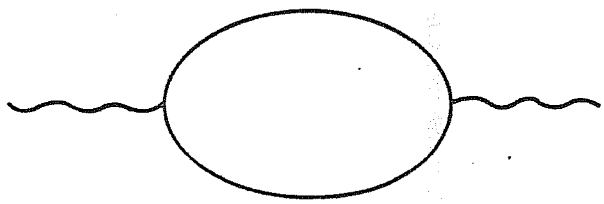

Fig. 34

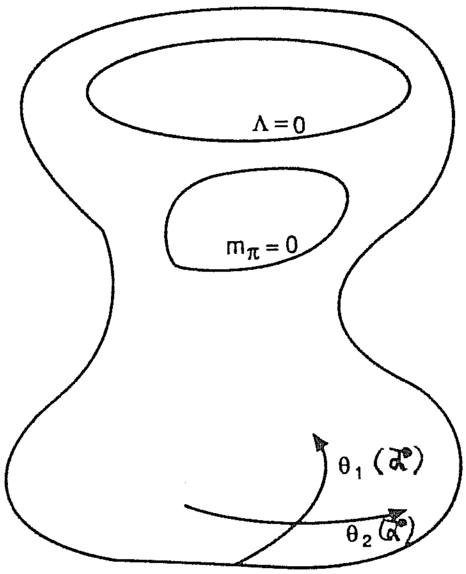

Fig. 35

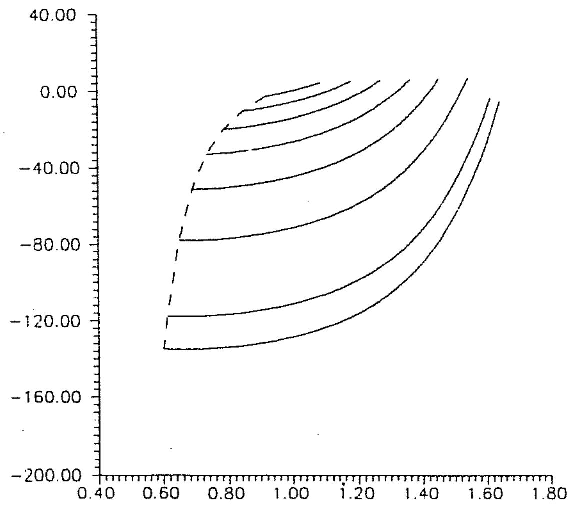

Fig. 36

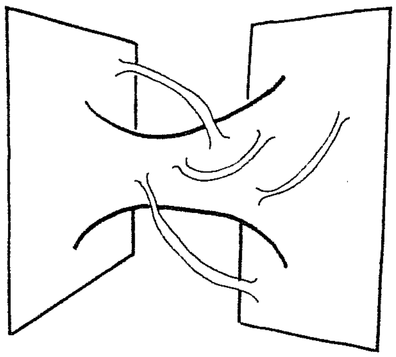

Fig. 37

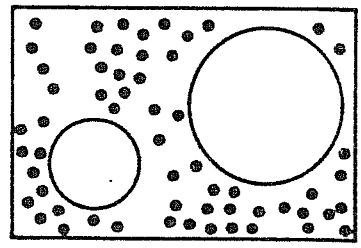

Fig. 38

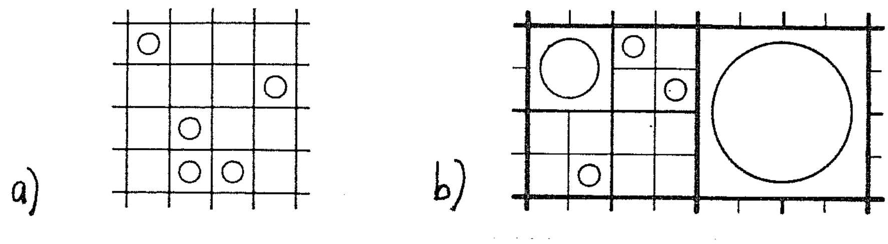

Fig. 39

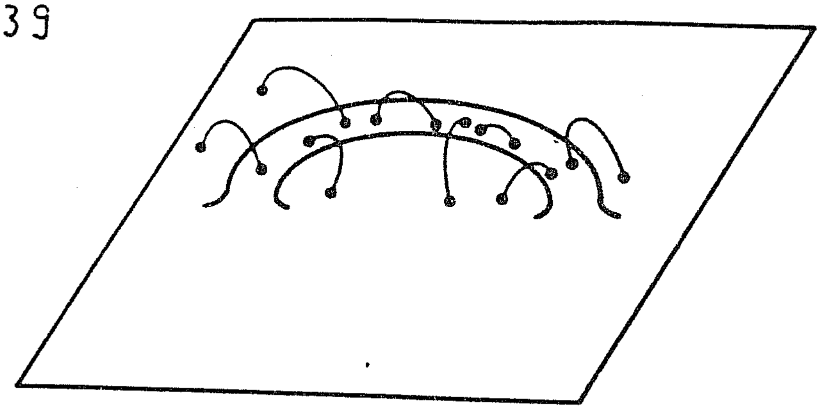

Fig. 40

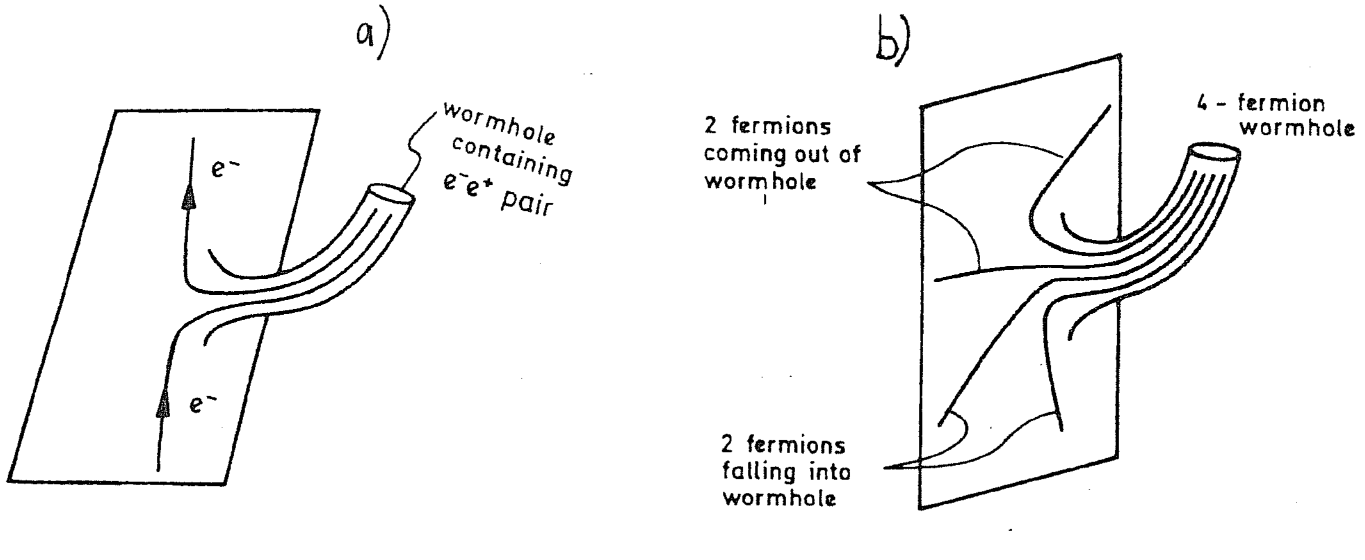

Fig. 41

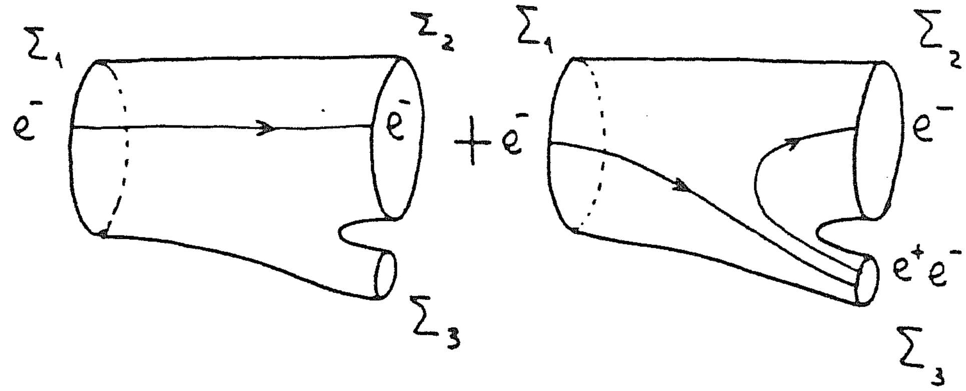

Fig. 42

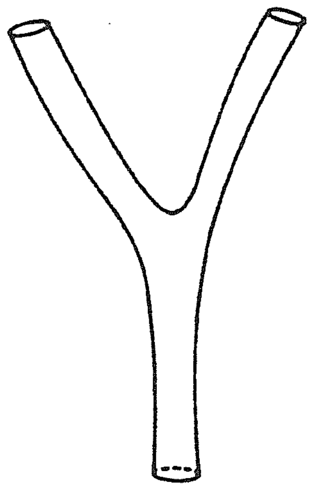

Fig. 43

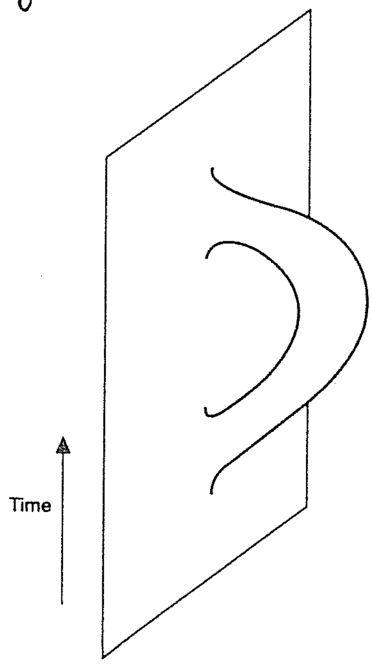

Fig. 44

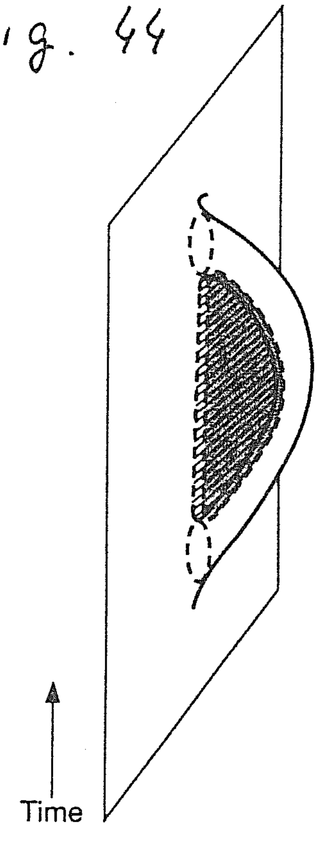

Fig. 45

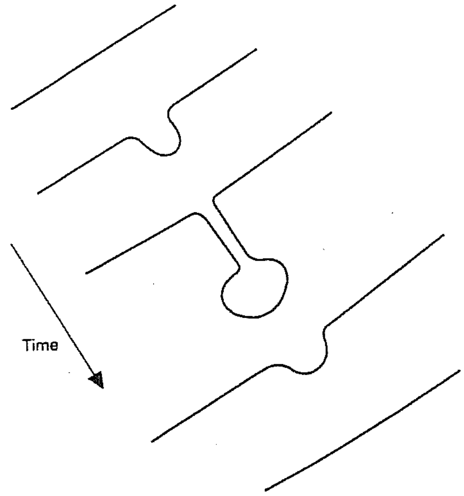

Fig. 46

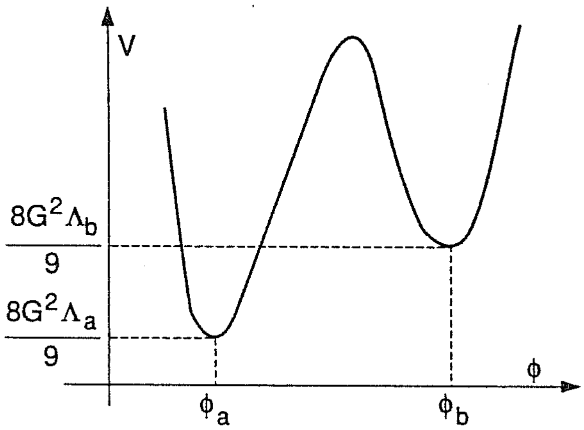

Fig. 47

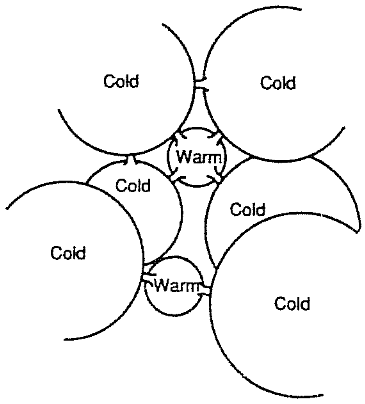

Fig. 48

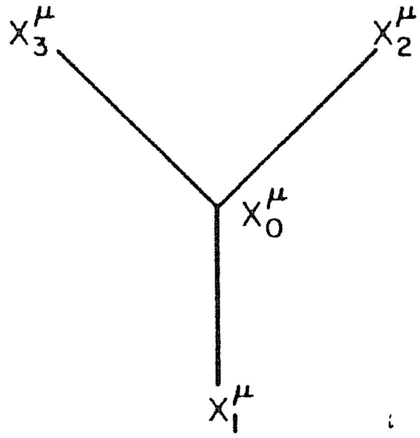

Fig. 49

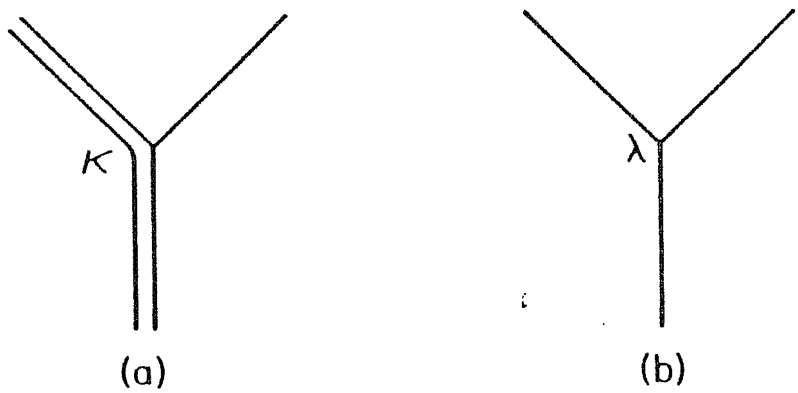

Fig. 50

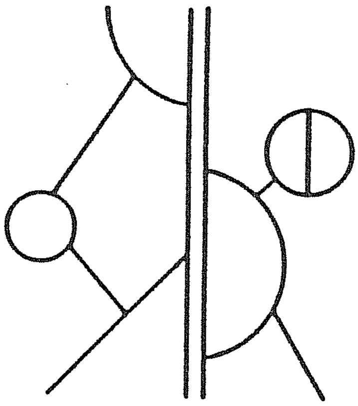

Fig. 51

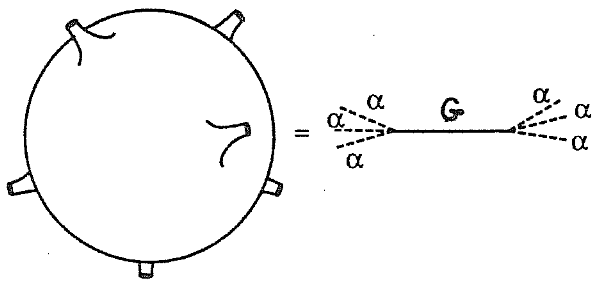

Fig. 52

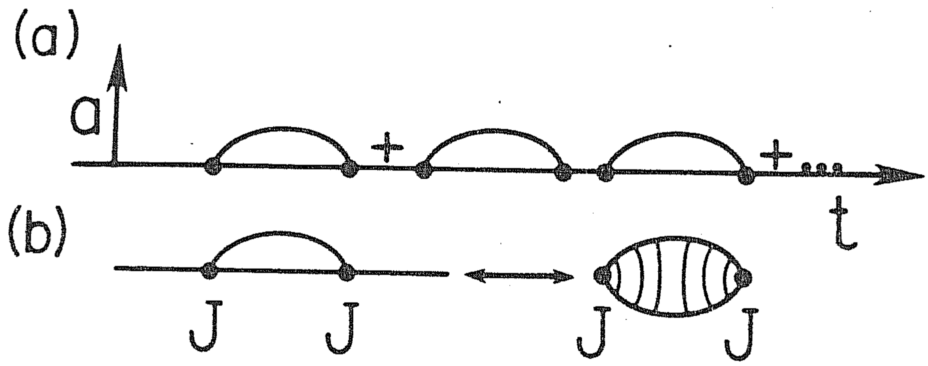

Fig. 53