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Some considerations about conjugate and focal points

Thesis submitted for the degree of
"Magister Philosophiæ"

CANDIDATE
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SUPERVISOR
Prof. Giovanni Vidossich

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Basic results

In this section we introduce the basic concepts. Let $A(t)$ be a continuous $n \times n$ matrix and let a, b be real numbers. Consider the system:

$$(*) \qquad x'' + A(t)x = 0.$$

We say that b is a conjugate point of a with respect to the system $(*)$ if the following problem

$$x'' + A(t)x = 0, \quad x(a) = 0, \quad x(b) = 0$$

has a non trivial solution.
We say that b is a focal point of a with respect to the system $(*)$ if $b > a$ and the following problem

$$y'' + A(t)y = 0, \quad y'(a) = 0, \quad y(b) = 0$$

has a non trivial solution.

Suppose now that $a < b$.
We say that the system $(*)$ is disconjugate on $[a, b]$ if the interval $(a, b]$ contains no conjugate points of a with respect to $(*)$.
We say that the system $(*)$ is disfocal on $[a, b]$ if the interval $(a, b]$ contains no focal points of a with respect to $(*)$.

The following theorems are well known:

Theorem A. Let $A(t) = \{a_{i,j}(t)\}$, $B(t) = \{b_{i,j}(t)\}$ be continuous $n \times n$ matrices; let a, b be real numbers, $a < b$ and suppose that $0 \leq a_{i,j}(t) \leq b_{i,j}(t)$ for every $t \in [a, b]$ and $1 \leq i, j \leq n$. Consider the systems:

$$(1) \qquad x'' + A(t)x = 0$$

$$(2) \qquad y'' + B(t)y = 0$$

and suppose further that b is a conjugate point of a with respect to (1). Then there exists a real number c such that $a < c \leq b$ and c is a conjugate point of a with respect to (2) .

For the proof see [S].

Theorem B. Let $A(t) = \{a_{i,j}(t)\}$, $B(t) = \{b_{i,j}(t)\}$ be continuous $n \times n$ matrices; let a, b be real numbers, $a < b$ and suppose that $0 \leq a_{i,j}(t) \leq b_{i,j}(t)$ for every $t \in [a, b]$ and $1 \leq i, j \leq n$. Consider the systems:

$$(1) \qquad x'' + A(t)x = 0$$

$$(2) \qquad y'' + B(t)y = 0$$

and suppose further that b is a focal point of a with respect to (1). Then there exists a real number c such that $a < c \leq b$ and c is a focal point of a with respect to (2).

For the proof see [S].

Let E be a Banach space with the cone K. The linear operator B is called positive if it trasforms the cone K into itself.

Theorem C. Let B be a linear positive operator; assume that λ_0 is the least upper bound of the absolute magnitudes of the characteristic values of the operator B; consider the following equation in E :

$$(*) \qquad \lambda\phi = B(\phi) + F$$

where $\lambda \in R$ and $F \in E$. If $F \in K$ and $\lambda > \lambda_0$, then the equation $(*)$ has a unique solution in the cone K.

For the proof see [K].

Let $A = \{a_{i,j}\}$ be a $n \times n$ matrix and let H_A be the homomorphism generated by A in the usual way: for $x = Col(x_1 \dots x_n)$ let $H_A x = Col(y_1 \dots y_n)$ where $y_i = \sum_{j=1}^{n} a_{i,j} x_j$. With an abuse of notation we identify the matrix A with the homomorphism H_A and write A in place of H_A. So, for example, the expression as $KerA$, $RangeA$, the restriction of A etc. mean $KerH_A$, $RangeH_A$, the restriction of H_A etc.

§1: A Theorem about the set of conjugate points

In this section we study some properties of conjugate points in the nonselfadjoint case. Let $A(t)$ be a continuous $n \times n$ matrix, $a \in R$; consider the following matrix Cauchy's problems:

$$(a) \qquad X'' + A(t)X = 0_n, \quad X(a) = 0_n, \quad X'(a) = I_n$$

$$(b) \qquad Y'' + A(t)Y = I_n, \quad Y(a) = I_n, \quad Y'(a) = 0_n.$$

Let $W_0(A,a,t)$ be, by definition, the solution of (a) and $W_1(A,a,t)$ be the solution of (b). We know that b is a conjugate point of a with respect to the system $x'' + A(t)x = 0$ if and only if $det W_0(A,a,b) = 0$. If this is the case then the multiplicity of b is, by definition, equal to $dim Ker W_0(A,a,b)$.

Proposition. For the matrices just introduced the following equations hold:

$$(1) \qquad W_0'^T(A^T,a,t)W_0(A,a,t) - W_0^T(A^T,a,t)W_0'(A,a,t) = 0_n$$
$$(2) \qquad W_1^T(A^T,a,t)W_0'(A,a,t) - W_1'^T(A^T,a,t)W_0(A,a,t) = I_n$$
$$(3) \qquad W_0(A,a,t)W_1^T(A^T,a,t) - W_1(A,a,t)W_0^T(A^T,a,t) = 0_n$$
$$(4) \qquad W_0'(A,a,t)W_1^T(A^T,a,t) - W_1'(A,a,t)W_0^T(A^T,a,t) = I_n$$

Proof. For simplicity we suppress the variable a and write, for example, $W_0(A,t)$ in place of $W_0(A,a,t)$. Consider the following matrix Cauchy's problem:

$$(m) \qquad Z' = \begin{pmatrix} 0_n & I_n \\ -A(t) & 0_n \end{pmatrix} Z, \quad Z(a) = I_{2n}$$

and let $W(t)$ be the solution of (m). An easy calculation shows that

$$W(t) = \begin{pmatrix} W_1(A,t) & W_0(A,t) \\ W_1'(A,t) & W_0'(A,t) \end{pmatrix}.$$

Consider now the following matrix Cauchy's problem:

$$(q) \qquad Y' = -Y \begin{pmatrix} 0_n & I_n \\ -A(t) & 0_n \end{pmatrix}, \quad Y(a) = I_{2n}$$

and let $V(t)$ be the solution of (q). An easy calculation shows that

$$V(t) = \begin{pmatrix} W_0'^T(A^T,t) & -W_0^T(A^T,t) \\ -W_1'^T(A^T,t) & W_1^T(A^T,t) \end{pmatrix}.$$

It is easy to prove that the derivative of $V(t)W(t)$ is zero everywhere (Proof :$(V(t)W(t))' = V'(t)W(t) + V(t)W'(t) = -V(t)\begin{pmatrix} 0_n & I_n \\ -A(t) & 0_n \end{pmatrix} W(t) + V(t)\begin{pmatrix} 0_n & I_n \\ -A(t) & 0_n \end{pmatrix} W(t) = 0_n$). From this it follows that $V(t)W(t) \equiv V(a)W(a) = I_{2n}$ and so $W(t)V(t) \equiv I_{2n}$. Therefore

$$\begin{pmatrix} W_0'^T(A^T,t) & -W_0^T(A^T,t) \\ -W_1'^T(A^T,t) & W_1^T(A^T,t) \end{pmatrix} \begin{pmatrix} W_1(A,t) & W_0(A,t) \\ W_1'(A,t) & W_0'(A,t) \end{pmatrix} = I_{2n}$$

leads us to the first two equation and

$$\begin{pmatrix} W_1(A,t) & W_0(A,t) \\ W_1'(A,t) & W_0'(A,t) \end{pmatrix} \begin{pmatrix} W_0'^T(A^T,t) & -W_0^T(A^T,t) \\ -W_1'^T(A^T,t) & W_1^T(A^T,t) \end{pmatrix} = I_{2n}$$

leads us to the last two equations.

Theorem 1. Let $A(t)$ be a continuous $n \times n$ matrix. Consider the systems:

(a) $$x'' + A(t)x = 0$$

(b) $$y'' + A^T(t)y = 0.$$

Let a be a real number. Then b is a conjugate point of a with respect to (a) if and only if b is a conjugate point of a with respect to (b). Moreover both multiplicities coincide.

Proof. It is sufficient to prove that for every $t \in R$ we have that

$$dim Ker W_o(A,a,t) = dim Ker W_0(A^T,a,t).$$

We shall first prove the following equality:

$(*)$ $$W_0'(A,a,t) Ker W_0(A,a,t) = Ker W_0^T(A^T,a,t) \quad \forall t \in R.$$

Let $h \in W_0'(A,a,t)KerW_0(A,a,t)$. Then there exists $\eta \in KerW_0(A,a,t)$ such that $h = W_0'(A,a,t)\eta$. If we multiply equation (1) of the Proposition from the right by η then we obtain that $W_0^T(A^T,a,t)h = 0$ because $W_0(A,a,t)\eta = 0$ by hypothesis. From this it follows that $h \in KerW_0^T(A^T,a,t)$. Let $k \in KerW_0^T(A^T,a,t)$. If we multiply equation (4) of the Proposition from the right by k then we obtain that $k = W_0'(A,a,t)W_1^T(A^T,a,t)k$ because $W_0^T(A^T,a,t)k = 0$ by hypothesis. We must check that $W_1^T(A^T,a,t)k \in KerW_0(A,a,t)$. By hypothesis $k \in KerW_0^T(A^T,a,t)$ and so, if we multiply equation (3) of the Proposition from the right by k we obtain that $W_0(A,a,t)W_1^T(A^T,a,t)k = 0$. The equality $(*)$ is therefore proved.

Next we shall prove that the restriction of $W_0'(A,a,t)$ on $KerW_0(A,a,t)$ is injective. Let $\eta_1, \eta_2 \in KerW_0(A,a,t)$ and suppose $W_0'(A,a,t)\eta_1 = W_0'(A,a,t)\eta_2$. From this it follows that

$$\eta_1 - \eta_2 \in KerW_0'(A,a,t)$$

and, obviously, $\eta_1 - \eta_2 \in KerW_0(A,a,t)$. Now, if we multiply equation (2) of the Proposition from the right by $\eta_1 - \eta_2$ then we obtain that $0 = \eta_1 - \eta_2$ and so $\eta_1 = \eta_2$.

From this it follows that

$$dimW_0'(A,a,t)KerW_0(A,a,t) = dimKerW_0(A,a,t)$$

and so, from $(*)$ it follows that

$$dimKerW_0(A,a,t) = dimKerW_0^T(A^T,a,t) = dimKerW_0(A^T,a,t).$$

Theorem 2. Let $A(t)$ be a continuous $n \times n$ matrix. Consider the system:

$$x'' + A(t)x = 0. \tag{1}$$

Let a be a real number. Put, by definition, C_a = the set of conjugate points of a with respect to (1) that have multiplicity strictly greater than $n/2$. Then C_a is discrete.

Proof. We note, first of all, that C_a is closed in virtue of the lower-semicontinuity of the function $t \longrightarrow RankW_0(A,a,t)$. Suppose, by contradiction, that b is an accumulation point of C_a: there exists a sequence (s_i) in C_a such that $\lim_{i\to\infty} s_i = b$. W.L.G. we may assume that $dimKerW_0(A,a,s_i) = m > n/2$ for every $i = 1,2,3,\ldots$, where m is a constant. Put, by definition, $K_i = KerW_0(A,a,s_i)$ and $L_i = KerW_0(A^T,a,s_i)$. From the proof of Theorem 1 we know that

$$W_0'(A,a,s_i)K_i = KerW_0^T(A^T,a,s_i)$$

and the restriction of $W_0'(A,a,s_i)$ on K_i is injective for $i = 1,2,3,\ldots$. If we replace A with A^T in the previous formula then we obtain that

$$W_0'(A^T,a,s_i)L_i = KerW_0^T(A,a,s_i)$$

and the restriction of $W_0'(A^T,a,s_i)$ on L_i is injective for $i = 1,2,3,\ldots$. From Theorem 1 $dimL_i = dimK_i = m$ and so $dimW_0'(A^T,a,s_i)L_i = dimW_0'(A,a,s_i)K_i = m$. By hypothesis $m > n/2$. So

$$(W_0'(A^T,a,s_i)L_i) \cap (W_0'(A,a,s_i)K_i) \neq \{0\}$$

for $i = 1, 2, 3, \ldots$. For every i we can therefore select from this intersection an element ρ_i with norm equal to 1. Now, (ρ_i) is a sequence in the unit sphere of R^n, we can therefore find a convergent subsequence (ρ_{i_r}). Put $\rho = \lim_{r\to\infty} \rho_{i_r}$. Obviously $\|\rho\| = 1$. From the definition of ρ_i there exists $\eta_i \in K_i$ and $\eta_i^* \in L_i$ such that $\rho_i = W_0'(A, a, s_i)\eta_i$ and $\rho_i = W_0'(A^T, a, s_i)\eta_i^*$. We may assume that the sequences (η_{i_r}) and $(\eta_{i_r}^*)$ are convergent. Infact: to see, for example, that the sequence (η_i) is bounded we proceed as follows: put $R_i =$ the restriction of $W_0'(A, a, s_i)$ on K_i. We know that R_i is injective and so $\eta_i = R_i^{-1}\rho_i$. From this it follows that $\|\eta_i\| \leq \|R_i\|^{-1}$. Therefore it is sufficient to show that there exists an $\epsilon > 0$ such that $\|R_i\| > \epsilon$ for every i. Assume, by contradiction, that there exist $k_i \in K_i$, $\|k_i\| = 1$ and $\lim_{i\to\infty} \|R_i k_i\| = 0$. W.L.G. we may assume that $\lim_{i\to\infty} k_i = k$. Then $W_0'(A, a, b)k = \lim_{i\to\infty} W_0'(A, a, s_i)k_i = \lim_{i\to\infty} R_i k_i = 0$ and $W_0(A, a, b)k = \lim_{i\to\infty} W_0(A, a, s_i)k_i = 0$. So $\|k\| = 1$, $k \in Ker W_0'(A, a, b)$ and $k \in Ker W_0(A, a, b)$. But this is impossible.

Put $\eta = \lim_{r\to\infty} \eta_{i_r}$, $\eta^* = \lim_{r\to\infty} \eta_{i_r}^*$. From

$$W_0(A, a, s_{i_r})\eta_{i_r} = 0 \quad \forall r \in N$$

and

$$W_0(A^T, a, s_{i_r})\eta_{i_r}^* = 0 \quad \forall r \in N$$

it follows that

$$W_0(A, a, b)\eta = 0$$

and

$$W_0(A^T, a, b)\eta^* = 0.$$

So $\eta \in Ker W_0(A, a, b)$ and $\eta^* \in Ker W_0(A^T, a, b)$. From

$$W_0'(A, a, s_{i_r})\eta_{i_r} = \rho_{i_r} \quad \forall r \in N$$

and

$$W_0'(A^T, a, s_{i_r})\eta_{i_r}^* = \rho_{i_r} \quad \forall r \in N$$

it follows that

(*i*)
$$\rho = W_0'(A, a, b)\eta$$

and

(*ii*)
$$\rho = W_0'(A^T, a, b)\eta^*.$$

From $\eta^* \in Ker W_0(A^T, a, b)$ and from

$$W_0'(A^T, a, b) Ker W_0(A^T, a, b) = Ker W_0^T(A, a, b)$$

it follows, from (*ii*), that $\rho \in Ker W_0^T(A, a, b)$ and so $\rho \in (Range W_0(A, a, b))^\perp$. Now, $\langle \rho, W_0(A, a, s_{i_r})\eta_{i_r} \rangle = 0$ and $\langle \rho, W_0(A, a, b)\eta_{i_r} \rangle = 0$ because $\rho \in (Range W_0(A, a, b))^\perp$. Put, for

$$c = \min\{s_{i_r}, b\} \leq t \leq \max\{s_{i_r}, b\} = d,$$

$$F_r(t) = \langle \rho, W_0(A,a,t)\eta_{i_r} \rangle.$$

Then $F_r(t)$ is a real valued, differentiable function, defined on the interval $[c,d]$ and such that $F_r(c) = F_r(d) = 0$. From Rolle's Theorem there exists $c \le t_r \le d$ such that $F_r'(t_r) = 0$ i.e. $\langle \rho, W_0'(A,a,t_r)\eta_{i_r} \rangle = 0$. Obviously, $\lim_{i\to\infty} t_i = b$ and so, passing to the limit in the last equation, we obtain that $\langle \rho, W_0'(A,a,b)\eta \rangle = 0$. From (i) we have that $\langle \rho, \rho \rangle = 0$ and so $\rho = 0$. But this contradicts the fact that $\|\rho\| = 1$.

§2: Existence of positive solution in the disconjugate case

In this section we prove the following theorem:

Theorem 1. Let $A(t) = \{a_{i,j}(t)\}$ be a continuous $n \times n$ matrix and let $f(t) = Col(f_1(t), \ldots, f_n(t))$ be a continuous function defined on R with values in R^n. Consider the system:

$$x'' + A(t)x = f(t). \tag{1}$$

Let a, b be real numbers, $a < b$. Suppose that the system

$$y'' + A(t)y = 0 \tag{2}$$

is disconjugate on $[a,b]$; suppose further that $a_{i,j}(t) \ge 0$ for $t \in [a,b]$, $1 \le i,j \le n$ and $f_i(t) \le 0$ for $t \in [a,b]$, $1 \le i \le n$. Then there exists an unique solution $z(t) = Col(z_1(t), \ldots, z_n(t))$ of (1) such that $z(a) = z(b) = 0$ and moreover $z_i(t) \ge 0$ for $t \in [a,b]$ and $1 \le i \le n$.

Proof. Let $A[a,b]$ be the usual space of admissible function: $A[a,b] = \{f : [a,b] \longrightarrow R^n | f$ is absolutely continuous, $f(a) = f(b) = 0$ and $\|f'\| \in L^2[a,b]\}$. Let

$$G(s,t) = \begin{cases} \frac{(b-t)(s-a)}{b-a} & for \quad a \le s \le t \le b \\ \frac{(b-s)(t-a)}{b-a} & for \quad a \le t \le s \le b \end{cases}$$

be the usual Green's function. Define on $A[a,b]$ the following operator: for $\phi(t) \in A[a,b]$ put

$$B(\phi)(t) = \int_a^b G(s,t)A(s)\phi(s)ds.$$

We know that B is a linear, continuous and compact operator that transforms the space $A[a,b]$ into itself; moreover, $x(t)$ is a solution of

$$x'' + A(t)x = 0, \quad x(a) = x(b) = 0$$

if and only if $x(t)$ is a fixed point of B. From the hypothesis of the Theorem it follows that B is a positive operator with respect to the cone of all functions $x(t) \in A[a,b]$ with non negative components. Let $C(B)$ be the set of characteristic values of the operator B. We know that $0 \in C(B)$ and so $C(B)$ is not empty.

Claim $\sup C(B) < 1$.

Infact: suppose first that there exists $r \in C(B)$ such that $r \geq 1$. There exists a non trivial $\phi \in A[a,b]$ such that

$$r\phi = B(\phi) = \int_a^b G(s,t)A(s)\phi(s)ds$$

and so

$$(*) \qquad \phi = \int_a^b G(s,t)r^{-1}A(s)\phi(s)ds.$$

Put $b_{i,j}(t) = r^{-1}a_{i,j}(t)$, $B(t) = \{b_{i,j}(t)\}$. The equation $(*)$ shows that ϕ is a non trivial solution of the following problem:

$$y'' + B(t)y = 0, \quad y(a) = y(b) = 0.$$

From $r \geq 1$ it follows that $b_{i,j}(t) \leq a_{i,j}(t)$ for $t \in [a,b]$ and $1 \leq i,j \leq n$. From Theorem A there exists a conjugate point c of a with respect to (2) such that $a < c \leq b$, but this contradicts the disconjugacy of (2) on $[a,b]$.Therefore every element of $C(B)$ is strictly less than 1. Suppose now that $\sup C(B) = 1$. There exists a sequence (r_i) in $C(B)$ such that $\lim_{i\to\infty} r_i = 1$. We can suppose that $r_i > 0$ for every $i = 1,2,3,\ldots$. Put $B_i(t) = r_i^{-1}A(t)$. Then $B_i(t) \to A(t)$ for $i \to \infty$ and the problems:

$$z'' + B_i(t)z = 0, \quad z(a) = z(b) = 0$$

for $i = 1,2,3,\ldots$ have all a nontrivial solution. From this it follows that $det W_0(B_i,a,b) = 0 \quad \forall i \in N$. From the Continuous Dependence Theorem it follows that $det W_0(A,a,b) = \lim_{i\to\infty} det W_0(B_i,a,b) = 0$. But this contradicts the disconjugacy of (2) on $[a,b]$. The claim is therefore proved.

Put now, by definition,

$$F(t) = -\int_a^b G(s,t)f(s)ds,$$

$$F(t) = Col(F_1(t),\ldots,F_n(t)).$$

Obviously $F(t) \in A[a,b]$. From the hypotesis of the Theorem it follows that $F_i(t) \geq 0$ for $t \in [a,b]$ and $i = 1,2,\ldots,n$. Consider now the equation

$$(+) \qquad \phi = B(\phi) + F.$$

From Theorem C the equation $(+)$ has a unique solution ϕ with non negative components. But this function solves also the problem (1).

Theorem 2. Let $A(t) = \{a_{i,j}(t)\}$ be a continuous $n \times n$ matrix and let $f(t) = Col(f_1(t), \ldots, f_n(t))$ be a continuous function defined on R with values in R^n. Consider the system:

$$x'' + A(t)x = f(t). \tag{1}$$

Let a, b be real numbers, $a < b$. Suppose that the system

$$y'' + A(t)y = 0 \tag{2}$$

is disconjugate on $[a, b]$. Suppose further that there exists two sets I and J such that $I \cup J = \{1, \ldots, n\}$, $I \cap J = \Phi$ and $a_{i,j}(t) \geq 0$ for $t \in [a, b]$ if $(i, j) \in I \times J \cup J \times I$, $f_i(t) \leq 0$ for $t \in [a, b]$ if $i \in I$ and $f_i(t) \geq 0$ for $t \in [a, b]$ if $i \in J$. Then there exists a unique solution $z(t) = Col(z_1(t), \ldots, z_n(t))$ of (1) such that $z(a) = z(b) = 0$ and moreover $z_i(t) \leq 0$ for $t \in [a, b]$ if $i \in J$ and $z_i(t) \geq 0$ for $t \in [a, b]$ if $i \in I$.

Proof. Put, by definition, for $i = 1, 2, \ldots, n$

$$f_i = \begin{cases} 1 & if \quad i \in I \\ -1 & if \quad i \in J, \end{cases}$$

$$T = diag(f_1, \ldots, f_n).$$

Then $T^T = T = T^{-1}$. Let $B(t) = TA(t)T$, $B(t) = \{b_{i,j}(t)\}$. Then

$$b_{i,j}(t) = f_i f_j a_{i,j}(t) = \begin{cases} a_{i,j}(t) & if \quad (i,j) \in I \times I \cup J \times J \\ -a_{i,j}(t) & if \quad (i,j) \in I \times J \cup J \times I \end{cases} = |a_{i,j}(t)|.$$

So all the entries of $B(t)$ are non negative. An easy calculation shows that

$$W_0(B, a, t) = W_0(TAT, a, t) = TW_0(A, a, t)T.$$

So $detW_0(B, a, t) = detW_0(A, a, t) \neq 0$ for $a < t \leq b$. From this it follows that the system

$$z'' + B(t)z = 0 \tag{3}$$

is diconjugate on $[a, b]$. Put now $f^*(t) = Tf(t)$, $f^*(t) = Col(f_1^*(t), \ldots, f_n^*)$. Then

$$f_i^*(t) = f_i f_i(t) = \begin{cases} -f_i(t) & if \quad i \in J \\ f_i(t) & if \quad i \in I \end{cases} = -|f_i(t)|$$

for every $t \in [a,b]$. Then the system

$$z'' + B(t)z = f^*(t) \tag{4}$$

satisfies the condition of Theorem 1 and so there exists a solution

$$z(t) = Col(z_1(t), \dots, z_n(t))$$

of (4) such that $z(a) = z(b) = 0$, $z_i(t) \geq 0$ for $t \in [a,b]$ and $i = 1, 2, \dots, n$. If we multiply the equation (4) from the left by T then we obtain:

$$(Tz)'' + A(t)(Tz) = f(t).$$

So Tz satisfies (1), $Tz(a) = Tz(b) = 0$ and an easy calculation shows that if $Tz(t) = Col(g_1(t), \dots, g_n(t))$ then $g_i(t) \leq 0$ for $t \in [a,b]$ if $i \in J$ and $g_i(t) \geq 0$ for $t \in [a,b]$ if $i \in I$.

§3: Existence of positive solution in the disfocal case.

In this section we prove the following Theorem:

Theorem 1. Let $A(t) = \{a_{i,j}(t)\}$ be a continuous $n \times n$ matrix and let $f(t) = Col(f_1(t), \dots, f_n(t))$ be a continuous function defined on R with values in R^n. Consider the system:

$$x'' + A(t)x = f(t). \tag{1}$$

Let a, b be real numbers, $a < b$. Suppose that the system

$$y'' + A(t)y = 0 \tag{2}$$

is disfocal on $[a,b]$. Suppose further that $a_{i,j}(t) \geq 0$ for $t \in [a,b]$, $1 \leq i,j \leq n$ and $f_i(t) \leq 0$ for $t \in [a,b]$, $1 \leq i \leq n$. Then there exists an unique solution $z(t) = Col(z_1(t), \dots, z_n(t))$ of (1) such that $z'(a) = z(b) = 0$ and moreover $z_i(t) \geq 0$ for $t \in [a,b]$ and $1 \leq n$.

Proof. The proof is essentialy the same as the proof of the Theorem 1 of §2. Put, by definition

$$G^*(s,t) = \begin{cases} b-s & for \quad a \leq t \leq s \leq b \\ b-t & for \quad a \leq s \leq t \leq b \end{cases}$$

and define on

$$A(a,b) \stackrel{\text{def}}{=} \{f : [a,b] \longrightarrow R^n | f \quad \text{is} \quad \text{a.c.},$$

$$f(b) = 0, \|f'\| \in L^2[a,b]\}$$

the following operator: for $\phi \in A(a,b)$ put

$$B^*(\phi)(t) = \int_a^b G^*(s,t)A(s)\phi(s)ds.$$

Then B^* is a linear, continuous, compact and positive operator. Moreover $x(t)$ is a solution of

$$x'' + A(t)x = 0, \quad x'(a) = x(b) = 0$$

if and only if $x(t)$ is a fixed point of B^*. Let $C(B^*)$ be the set of characteristic values of the operator B^*. We can show that $\sup C(B^*) < 1$: we proceed as in the proof of the Theorem 1 of §2. It is sufficient to use the Theorem B in place of the Theorem A and replace the matrix $W_0(A,a,t)$ with the matrix $W_1(A,a,t)$. Put next

$$F^* = -\int_a^b G^*(s,t)f(s)ds,$$

$$F^*(t) = Col(F_1^*(t), \dots, F_n^*(t)).$$

Obviously $F^*(t) \in A(a,b)$ and from the hypothesis of the Theorem it follows that $F_i(t) \geq 0$ for $t \in [a,b]$ and $i = 1, 2, \dots, n$. Consider the equation:

$$\phi = B^*(\phi) + F^*.$$

The conclusion is now the same as the conclusion of the Theorem 1 of §2.

References.

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