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Multiple Crossover among an Infinite Series of Critical Points in 2-D Quantum Field Theory

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Chapter 1

Renormalization Group in Two Dimensions

1.1 General Ideas

The fundamental objects in quantum field theory are the correlation functions of local fields A_k:

$$< A_1(x_1)A_2(x_2)\ldots A_N(x_N) > \quad . \tag{1.1.1}$$

In the Lagrangian formulation these quantities are given (up to a normalization) by the functional integrals

$$\int \mathcal{D}\phi\, A_1(x_1)\ldots A_N(x_N)\, \exp\{-H[\phi]\} \, , \tag{1.1.2}$$

where ϕ stays for some set of *fundamental fields* such that

$$A_k(x) \;=\; A_k(\phi(x), \partial_\mu\phi(x), \partial_\mu\partial_\nu\phi(x), \ldots) \, , \tag{1.1.3}$$

and $H[\phi]$ is the (Euclidean) action (Hamiltonian in statistical field theory), which in d dimensions is the integral of a local density $\mathcal{H}$:

$$H[\phi] \;=\; \int d^d x\, \mathcal{H}(\phi(x), \partial_\mu\phi(x)) \quad . \tag{1.1.4}$$

The correlators (1.1.1), and then the action H, will in general depend on a set of parameters $g_1, \ldots, g_n$ which are the coupling constants of the theory. As is well known [1], a renormalization procedure is needed in an interacting theory in order to cancel the ultraviolet divergences present in the *naive* formulation of the theory. Cancellation of infinities leaves undetermined finite parts which have to be fixed through the renormalization conditions, namely by assigning the values that some quantities should acquire when computed at a certain scale μ. This dependence on the arbitrary scale parameter μ, which is an unavoidable consequence of renormalization, can be consistently taken into account introducing *running coupling constants* $g_i(\mu)$. Obviously, the physical predictions of the theory must be independent on the choice of μ and this fundamental requirement leads to the equations of the renormalization group (RG) which determine the dependence on μ of the running coupling constants. We will return in more detail on these formal developments in sec. 1.5. For the moment we are interested in introducing a series of basic ideas and definitions concerning the RG which will be widely used in the following.

A very useful and intuitive geometrical picture arises if we associate to an effective action H a point in the *interaction space* Q with coordinates $g_1, \ldots, g_n$ [2, 3]. Generally speaking, Q is an infinite dimensional space. Nevertheless, it is assumed that one can deal with it as with a finite dimensional manifold since only finite dimensional subspaces turn out to be essential. In light of previous considerations, a trajectory in Q parameterized by the scale parameter μ will correspond to a field theory: different points on the trajectory correspond to the effective actions which describe physical systems differing only by a scale transformation; the passage from an effective action to another is realized by a so called RG transformation:

$$\{g_1(\mu), \ldots, g_n(\mu)\} \rightarrow \{g_1(\mu'), \ldots, g_n(\mu')\} \tag{1.1.5}$$

Our convention is that μ is a mass parameter, so that, if $\mu = \mu_0\, e^{-t}$ with μ_0 some fixed mass, the value $t = +\infty$ $(-\infty)$ corresponds to the infrared (ultraviolet) limit. Once the initial conditions $g_i(t_0)$ are given, the evolution of the trajectory in the interaction space Q is completely determined by the RG equations.

Clearly a very special role in the theory is played by the fixed points of the RG, namely by the points of Q which are invariant under the RG transformations:

$$g_i^*(t) \rightarrow g_i^*(t') = g_i^*(t) \; , \quad i = 1, \ldots, n \quad . \tag{1.1.6}$$

This means that such a point represents itself a whole trajectory and that the corresponding quantum field theory is invariant under scale transformations, namely it is a massless theory with an infinite interaction range ξ^*.

The general solution of the RG equations, which amounts to a complete *topological* characterization of the interaction space Q, is a formidable task which remains so far unsolved. The best we can do in the general case is to use perturbation theory in order to study the neighbourhood of fixed points. This gives rise to a series of useful concepts that we now briefly present. To this aim let's write the effective Lagrangian density in the form

$$\mathcal{H}(x) = \sum_{i=1}^{n} g_i \Phi_i(x) \quad . \tag{1.1.7}$$

The operators Φ_i so introduced are said to be conjugated to the coupling constants g_i. Let's now consider in the interaction space Q a point P whose coordinates $g = \{g_1, \ldots, g_n\}$ differ very slightly from those of a fixed point $P^* = g^*$. In other words

$$g_i = g_i^* + \bar{\eta}_i \; , \tag{1.1.8}$$

*In the framework of critical phenomena ξ si called correlation length and the zero mass condition is substituted by that of criticality; in the following we will use interchangeably the two terminologies.

where the $\bar{\eta}_i$'s are infinitesimal constants.

Under a RG transformation corresponding to the change of scale $\mu \longrightarrow \mu/s$ the point P will be mapped in a point P_s with coordinates

$$g_i(s) = g_i^* + \eta_i(s, \bar{\eta}_i) \ , \tag{1.1.9}$$

Since the $\bar{\eta}_i$'s are infinitesimal we can linearize the previous relation by taking the first order expansion in $\bar{\eta}_i$:

$$\eta_i(s) \simeq A_{ij}(s)\bar{\eta}_j \ . \tag{1.1.10}$$

After diagonalization this relation reads

$$\eta_i'(s) \simeq f(s)\bar{\eta}_i' \ . \tag{1.1.11}$$

The requirement that the result of two RG transformations parameterized by s_1 and s_2 should correspond to a unique RG transformation parameterized by $s_1 s_2$ forces $f(s)$ to be simply a power of s:

$$\eta_i'(s) \simeq s^{y_i}\bar{\eta}_i' \ . \tag{1.1.12}$$

From this result the transformation law of the fields Φ_i' conjugated to the coupling constants g_i' under the scale transformation $x \longrightarrow sx$ can be immediately read off noting that the Lagrangian density eq.(1.1.7) should acquire a factor s^d coming from the contraction of the volume when measured in the new units. This implies

$$\Phi_i'(x) \longrightarrow s^{d_i}\Phi_i'(sx) \ . \tag{1.1.13}$$

The Φ_i' are called *scaling operators* and (recall eq.(1.1.7) and eq.(1.1.12))

$$d_i = d - y_i \tag{1.1.14}$$

is their *anomalous scale dimension*; this is in general different from the canonical scale dimension dictated by dimensional analysis. The spectrum of anomalous dimensions characterizes a fixed point and its determination represents the central problem in the framework of critical phenomena since it amounts to the determination of critical exponents. We stress that the identification of scaling operators is essentially a local procedure since it holds only in a neighbourhood of the fixed point under consideration. With this remark in mind we can proceed to the classification of scaling operators according to their anomalous dimensions. We distinguish three cases:

1. $d_i < d$: from eqs.(1.1.14) and (1.1.12) we see that $\eta_i'(s)$ increases with s and the point P_s moves away from the fixed point P^*; the operator Φ_i' is said to be *relevant.*

2. $d_i = d$: $\eta_i'(s)$ remains constant varying s; Φ_i' is said to be *marginal.*

3. $d_i > d$: the situation is reversed with respect to case 1 and $\eta_i'(s)$ vanishes in the infrared limit; Φ_i' is said to be *irrelevant.*

The basis of scaling operators defines a system of local axes in the neighbourhood of a fixed point P^* (see fig. 1). Since our convention is to follow the evolution of RG trajectories as s increases (to this refer the arrows in fig. 1), we will call centrifugal (centripetal) the axes corresponding to relevant (irrelevant) operators. The subspace W_{IR} (W_{UV}) spanned by the centripetal (centrifugal) axes represents the IR (UV) *catchment area* of the fixed point P^* in the sense that any trajectory intersecting W_{IR} (W_{UV}) will converge to P^* in the IR (UV) limit. This gives a very suggestive picture of the property of *universality*, namely of the phenomenon observed in statistical mechanics for which systems with apparently different interactions actually exhibit the same critical behaviour.

W_{IR} is contained in the *critical surface* S_∞ which we define as the submanifold of Q formed by all points corresponding to an infinite correlation length ξ (in particular S_∞ contains all fixed points). To prove this statement we simply observe that in a RG transformation related to the change of scale $\mu \to \mu/s$ the correlation length undergoes the transformation $\xi \to \xi/s$; but, since $P \in W_{IR}$ converges to P^* in the limit $s \to \infty$ and $P^* \in S_\infty$, then it should be $P \in S_\infty$.

It may happen that a trajectory passes close to a fixed point before finally converging to another. If this is the case the theory exhibits at different scales the behaviour characteristic of the two fixed points originating a so called *crossover* phenomenon.

1.2 Operator Algebra and Conformal Field Theories

We have seen in previous section that fixed points play a central role in our study of RG. As a consequence, the investigation of the field theory solutions corresponding to fixed points is of fundamental importance. By definition such solutions should be invariant at all scales under the scale transformation

$$x^\mu \to s x^\mu \quad . \tag{1.2.1}$$

On the other hand it is known that scale invariance of a local, homogeneous and isotropic field theory implies the invariance under a larger transformation group, the conformal group, which preserves the angles of two arbitrary vectors at any given point but may change their lengths. In other words, a conformal transformation

$$x^\mu \to y^\mu(x) \ , \tag{1.2.2}$$

is characterized by the fact that it changes the metric multiplicatively:

$$dy^\mu dy_\mu = \rho(x) dx^\mu dx_\mu \quad . \tag{1.2.3}$$

Thus we see that the important problem of the classification of fixed points of RG is equivalent to the construction of all conformal invariant solutions of field theories. This problem appears very hard to face in the framework of Lagrangian field theory.

In alternative, Polyakov [4] proposed a *bootstrap* approach directly for the correlation functions based on the hypothesis of the *algebra of local fields.* According to this hypothesis [5, 6, 7], in the field theory there is an infinite basis of local operators $A_j(x)$ such that any local operator of the theory can be decomposed as

$$A(x) = \sum_i \mu_i A_i(x) \quad . \tag{1.2.4}$$

Moreover, the space $\mathcal{A}$ of local operators of the theory form a bilocal algebra with respect to the operator product expansion (OPE)

$$A_i(x)A_j(0) = \sum_k C_{ij}^k(x)A_k(0) \ , \tag{1.2.5}$$

where the functions $C_{ij}^k(x)$ clearly contain all the informations about the field theory. Relations like eqs.(1.2.4), (1.2.5) and similar are to be understood as relations between correlation functions.

In the *scaling region* around a fixed point we can suppose to identify the set of scaling fields $\Phi_i'(x)$ (see previous section) as a basis for $\mathcal{A}$. Consequently, the transformation law eq.(1.1.13) can be used to fix the form of the structure functions

$$\Phi_i'(x)\Phi_j'(0) = \sum_k \frac{C_{ij}^k}{x^{d_i+d_j-d_k}}\Phi_k'(0) \ , \tag{1.2.6}$$

where the C_{ij}^k's are now constants.

In Lagrangian theory, $\mathcal{A}$ includes, apart from the "fundamental" fields ϕ and their derivatives, any composite field of the type $: \phi^n :, : \partial_\mu\phi\phi^n :, \ldots$. In the present approach, the basic dynamical equations are provided by the requirement of the associativity of the algebra (1.2.5), which amounts to the condition of crossing symmetry of the correlation functions. Combining this condition with the requirement of conformal invariance of the operator algebra (1.2.5), Polyakov obtained a system of *bootstrap* equations for the anomalous dimensions and the *structure constants* C_{ij}^k. While the general solution of these equations remains an open problem, in many cases they can be exactly solved in the two-dimensional case.

In order to understand the reason for this, let's consider the infinitesimal version of eq.(1.2.2):

$$y^\mu(x) = x^\mu + \varepsilon^\mu(x) \quad . \tag{1.2.7}$$

It is easily seen that, starting from a flat matric $\delta_{\mu\nu}$, eq.(1.2.3) constrains $\varepsilon^\mu(x)$ to be solution of the equation

$$\partial_\mu\varepsilon_\nu(x) + \partial_\nu\varepsilon_\mu(x) = \frac{2}{d}\partial_\lambda\varepsilon^\lambda(x)\delta_{\mu\nu} \ , \tag{1.2.8}$$

where d stays for the number of dimensions. While for $d > 2$ eq.(1.2.8) allows $\varepsilon^\mu(x)$ to be at most quadratic in x, for $d = 2$ it reduces precisely to the Cauchy-Riemann equations:

$$\begin{aligned} \partial_1\varepsilon_1(x) &= \partial_2\varepsilon_2(x) \ ; \\ \partial_1\varepsilon_2(x) &= -\partial_2\varepsilon_1(x) \quad . \end{aligned} \tag{1.2.9}$$

Thus we arrive at the crucial result that for $d = 2$ the transformation (1.2.7) is conformal for any holomorphic (antiholomorphic) function $\varepsilon(z) \equiv \varepsilon^1(z) + i\varepsilon^2(z)$ $(\bar{\varepsilon}(\bar{z}) \equiv \varepsilon^1(\bar{z}) - i\varepsilon^2(\bar{z}))$, with $z, \bar{z}$ defined to be the complex coordinates

$$
\begin{aligned}
z &= x^1 + ix^2 \; ; \\
\bar{z} &= x^1 - ix^2 \quad .
\end{aligned}
\tag{1.2.10}
$$

Then we see that the conformal group of two-dimensional space is an infinite dimensional group and it is precisely this infinite-dimensional symmetry which allows us to advance in the research of conformal field theory for $d = 2$ much further than in higher dimensional cases [8].

For the systematic study of this symmetry, it is convenient to consider z and $\bar{z}$ (which are related by complex conjugation in the Euclidean space $\mathcal{R}^2$) as independent complex variables in the complex space $\mathcal{C}^2$ of which $\mathcal{R}^2$ is some real section; correspondingly, the correlation functions of the theory are analytically continued in some domain in $\mathcal{C}^2$. As a consequence, $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$ become independent functions and the conformal group splits in the direct product $\Gamma \times \bar{\Gamma}$ of *right* and *left* groups of analytic substitutions of the variables z and $\bar{z}$.

In any homogeneous and isotropic field theory there is a local symmetric energy-momentum tensor $T_{\mu\nu}(x)$ contained in the algebra of local operators $\mathcal{A}$ which satisfyies the conservation equations

$$
\partial_\mu T^{\mu\nu}(x) = 0 \quad . \tag{1.2.11}
$$

From the Lagrangian point of view, $T_{\mu\nu}(x)$ describes a variation of the action H under the arbitrary infinitesimal coordinate transformation eq.(1.2.7) (from now on we consider d=2)

$$
\delta_\varepsilon H = 2\int d^2x\, \partial_\mu \varepsilon_\nu(x) T^{\mu\nu}(x) \quad . \tag{1.2.12}
$$

In particular, for the dilatation eq.(1.2.1) (with $s = 1 + \varepsilon$) we have $\varepsilon_\nu(x) = \varepsilon x_\nu$ and eq.(1.2.12) specializes to

$$
\delta_\varepsilon H = 2\varepsilon \int d^2x\, \Theta(x) \; , \tag{1.2.13}
$$

where $\Theta = T^\mu_\mu$. Hence scale invariance constrains the energy-momentum tensor of field theories corresponding to fixed points to be traceless.

Coming back to the general transformations, eq.(1.2.12) can be used in the functional integrals (1.1.2) to compute the variation of correlation functions. We obtain

$$
\begin{aligned}
&\sum_1^N < A_1(x_1) \ldots \delta_\varepsilon A_i(x_i) \ldots A_N(x_N) > \\
&= 2\int d^2 \partial_\mu \varepsilon_\nu(x) < T^{\mu\nu}(x) A_1(x_1) \ldots A_N(x_N) > \; ,
\end{aligned}
\tag{1.2.14}
$$

where $\delta_\varepsilon A(x)$ represents the variation of the field $A(x)$ under the transformation eq.(1.2.7). In the non-Lagrangian approach based on the algebra of local fields this equation is postulated and should be considered as defining the action of the operator δ_ε on $\mathcal{A}$. Due to eq.(1.2.11) we can write the integrand in last equation in divergence form apart from the points $x = x_i$ where the correlator in right-hand side is singular. Calling S_{x_i} an arbitrary small domain in $\mathcal{R}^2$ containing the point x_i we can write the integral in eq.(1.2.14) as

$$\int_{\mathcal{R}^2 - \bigcup_{i=1}^{N} S_{x_i}} d^2x\, \partial_\mu < \varepsilon_\nu(x) T^{\mu\nu}(x) A_1(x_1) \dots A_N(x_N) >$$

$$+ \sum_{i=1}^{N} \int_{S_{x_i}} d^2x\, \partial_\mu \varepsilon_\nu(x) < T^{\mu\nu}(x) A_1(x_1) \dots A_N(x_N) > , \qquad (1.2.15)$$

so that we conclude

$$\frac{1}{2}\delta_\varepsilon A(x) = \oint_{C_x} dy^\lambda\, \varepsilon_{\mu\lambda} \varepsilon_\nu(y) T^{\mu\nu}(y) A(x) + \int_{S_x} d^2y\, \partial_\mu \varepsilon_\nu(y) T^{\mu\nu}(y) A(x) , \qquad (1.2.16)$$

where C_x is the boundary of S_x and $\varepsilon_{\mu\nu}$ an antisymmetric tensor ($\varepsilon_{12} = 1$).

It is convenient to introduce the spin operator S and the dilatation operator D acting on $\mathcal{A}$ in order to express the field variation corresponding to rotations and translations translations $\varepsilon^\mu(x) = \varepsilon^\mu + \omega^{\mu\nu} x_\nu$ ($\omega^{\mu\nu} = -\omega^{\nu\mu}$)

$$\delta_\varepsilon A(x) = \varepsilon^\mu \partial_\mu A(x) + \omega^{\mu\nu}(x_\mu \partial_\nu + \varepsilon_{\mu\nu} S) A(x) \qquad (1.2.17)$$

and uniform dilatations $\varepsilon^\mu = \varepsilon x^\mu$

$$\delta_\varepsilon A(x) = \varepsilon(x^\mu \partial_\mu + D) A(x) \ . \qquad (1.2.18)$$

Let's restrict our attention to a field theory related to a fixed point of the RG in order to see how conformal symmetry allows a characterization of the space of local fields $\mathcal{A}$. If we define

$$T = T^{11} - T^{22} + 2iT^{12} \ ;$$
$$\bar{T} = T^{11} - T^{22} - 2iT^{12} \ , \qquad (1.2.19)$$

due to the condition $\Theta = 0$, equations (1.2.11) become

$$\partial_{\bar{z}} T = 0 \ ; \qquad \partial_z \bar{T} = 0 \ , \qquad (1.2.20)$$

showing that $T = T(z)$ ($\bar{T} = \bar{T}(\bar{z})$) is a holomorphic (antiholomorphic) function. Since under a conformal transformation the second term in eq.(1.2.16) drops out due to eq.(1.2.8) and to $\Theta = 0$, we obtain

$$\delta_\varepsilon A(z, \bar{z}) = \oint_{C_z} \frac{dw}{2\pi i} \varepsilon(w) T(w) A(z, \bar{z}) \ . \qquad (1.2.21)$$

This equation and the similar one for $\delta_{\bar{\varepsilon}} A(z, \bar{z})$ involving $\bar{T}(\bar{z})$ show that T and $\bar{T}$ can be considered as the generators of *right* and *left* conformal transformations respectively. A way to formalize this fact is to expand $T(z)$ and $\bar{T}(\bar{z})$ in Laurent series

$$T(z) = \sum_{n=-\infty}^{+\infty} z^{-n-2} L_n \ ;$$
$$\bar{T}(\bar{z}) = \sum_{n=-\infty}^{\infty} \bar{z}^{-n-2} \bar{L}_n \ , \qquad (1.2.22)$$

with the L_n's and $\bar{L}_n$'s acting as operators on $\mathcal{A}$ and representing the infinite generators of the two-dimensional conformal group $\Gamma \times \bar{\Gamma}$. By using these definitions in eq.(1.2.21) and composing with eqs.(1.2.17) and (1.2.18), the following results are easily obtained

$$L_{-1}A(z,\bar{z}) = \partial_z A(z,\bar{z}) \ , \qquad \bar{L}_{-1}A(z,\bar{z}) = \partial_{\bar{z}} A(z,\bar{z}) \ ; \tag{1.2.23}$$

$$L_0 - \bar{L}_0 = S \ , \qquad L_0 + \bar{L}_0 = D \ . \tag{1.2.24}$$

It is convenient to choose in $\mathcal{A}$ a basis of eigenvectors A_i of the operators L_0 and $\bar{L}_0$:

$$L_0 A_i = \Delta_i A_i \ ; \qquad \bar{L}_0 A_i = \bar{\Delta}_i A_i \ . \tag{1.2.25}$$

Here the real numbers Δ_i and $\bar{\Delta}_i$ are called *right* and *left* dimensions (or conformal weights) of the field A_i; in light of eq.(1.2.24) the quantities $s_i = \Delta_i - \bar{\Delta}_i$ and $d_i = \Delta_i + \bar{\Delta}_i$ represent the spin and the anomalous scale dimension of A_i.

On general grounds it can be shown that the field $T(z)$ transforms under a conformal transformation as

$$\begin{aligned} \delta_\varepsilon T(z) &= \varepsilon(z)T'(z) + 2\varepsilon'(z)T(z) + \frac{c}{12}\varepsilon'''(z) \ ; \\ \delta_{\bar{\varepsilon}} T(z) &= 0 \ , \end{aligned} \tag{1.2.26}$$

where primes indicates differentiation with respect to z. The parameter c, called *central charge*, is an important quantity which characterizes the conformal field theories (CFT). The transformation law of $T(z)$ under a finite conformal transformation $z \rightarrow f(z)$ reads

$$T(z) \rightarrow (f')^2 T(f(z)) + \frac{c}{12} S(f,z) \ , \tag{1.2.27}$$

where the quantity

$$S(f,z) = \frac{f'f''' - \frac{3}{2}(f'')^2}{(f')^2} \tag{1.2.28}$$

is known as the Schwartzian derivative. It can be shown that $S(f,z) = 0$ only for the three-parameter subgroup of the conformal group corresponding to

$$f(z) = \frac{az+b}{cz+d} \ , \qquad ad - bc = 1 \ . \tag{1.2.29}$$

These are the only conformal transformations defined and invertible in the whole complex plane.

The expression eq.(1.2.26) is equivalent, through eq.(1.2.21), to the following OPE:

$$T(z)T(0) = \frac{c}{2z^4} + \frac{2}{z^2}T(0) + \frac{1}{z}T'(0) + regular\ \ terms \ . \tag{1.2.30}$$

Equating (1.2.26) to (1.2.21) with $A(z,\bar{z}) = T(z)$ and using eq.(1.2.22) one can find the following commutation relations obeyed by the generators L_n:

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \ . \tag{1.2.31}$$

This is called *Virasoro algebra* and we denote it by $\mathcal{L}$. Recalling that an analogous algebra $\bar{\mathcal{L}}$ is satisfied by the left generators $\bar{L}_n$, we conclude that the conformal group in two dimensions is generated by $\mathcal{L} \oplus \bar{\mathcal{L}}$.

The representation theory of Virasoro algebra can be used to study the structure of the space $\mathcal{A}$ of fields of the conformal theory. It is immediately seen from eq.(1.2.31) that the operator L_n map the space $\mathcal{U}_\Delta \subset \mathcal{A}$ of eigenvectors of L_0 with right conformal weight Δ into $\mathcal{U}_{\Delta-n}$. By the requirement that the spectra of anomalous dimensions $d = \Delta + \bar{\Delta}$ is bounded from below, there exist lowest weights representations (LWR) whose lowest weight vectors Φ_j satisfy the equations

$$\begin{aligned} L_0\Phi_j &= \Delta_j\Phi_j , \qquad \bar{L}_0\Phi_j = \bar{\Delta}_j\Phi_j ; \\ L_n\Phi_j &= \bar{L}_n\Phi_j = 0 \qquad \text{for } n > 0 . \end{aligned} \tag{1.2.32}$$

The operators $\Phi_j(z,\bar{z})$ are called *primary fields.* Using eq.(1.2.21) their variation under an infinitesimal "right transformation" is simply given by

$$\delta_\varepsilon\Phi_j(z,\bar{z}) = \varepsilon(z)\partial_z\Phi_j(z,\bar{z}) + \Delta_j\varepsilon'(z)\Phi_j(z,\bar{z}) . \tag{1.2.33}$$

Specializing this variation to the case of a dilatation and taking into account also the left part, one can immediately see that it coincides with the infinitesimal form of the transformation eq.(1.1.13). Thus we conclude that the primary fields of a conformal field theory can be identified with the scaling fields of the corresponding fixed point.

The other vectors of the LWR are obtained by applying to the primary field the step operators $L_n, \bar{L}_n$ with negative n and are called *descendant fields*; the LWR associated to the primary field Φ_j is called *conformal family* and will be denoted with $[\Phi_j]$. Due to the decomposition of the Virasoro algebra in right and left parts, a conformal family can be written in the direct product form

$$[\Phi_j] = [\Delta_j] \times [\bar{\Delta}_j] , \tag{1.2.34}$$

where $[\Delta_j]$ and $[\bar{\Delta}_j]$ are LWR of $\mathcal{L}$ and $\bar{\mathcal{L}}$ respectively; $[\Delta_j]$, for example, is spanned by all vectors of the form

$$L_{-n_1} \dots L_{-n_N}\Phi_j , \qquad 1 \le n_1 \le \dots \le n_N . \tag{1.2.35}$$

The quantity $L = \sum_{i=1}^{N} n_i$ is called the *level* of the vector (1.2.35). The space $\mathcal{A}$ of local fields of a CFT contains some (in general infinitely many) primary fields and can thus be written as direct sum of conformal families:

$$\mathcal{A} = \bigoplus_j [\Phi_j] . \tag{1.2.36}$$

1.3 Minimal Models

While in the general case these informations are not still sufficient to solve the bootstrap equations and, consequently, to determine the anomalous dimensions and the structure constants of the theory, this program can be successfully completed for the so called *minimal models* [8]. These are obtained by the condition that the space $\mathcal{A}$ contains reducible representations of the Virasoro algebra, namely representations having invariant subspaces. To be concrete, suppose that the subspace $\mathcal{V}_\Delta$ spanned by the vectors (1.2.35) contains at level L a vector $\chi_{\Delta+L}$ (null-vector) satisfying the equations

$$\begin{aligned} L_n\chi_{\Delta+L} &= 0 \qquad \text{for } n>0 \;, \\ L_0\chi_{\Delta+L} &= (\Delta+L)\chi_{\Delta+L} \;. \end{aligned} \tag{1.3.1}$$

In this case the subspace $\mathcal{V}_{\Delta+L} \subset \mathcal{V}_\Delta$ generated by applying the operators L_n with $n > 0$ to $\chi_{\Delta+L}$ is invariant under the action of the right Virasoro algebra $\mathcal{L}$. In order to get an irreducible representation we consider the factor space $[\Delta] = \mathcal{V}_\Delta/\mathcal{V}_{\Delta+L}$, i.e. we set

$$\chi_{\Delta+L} = 0 \tag{1.3.2}$$

The irreducible representation $[\Delta]$ obtained in this way is said to be *degenerate*. The requirement that null-vectors exist, supplemented by the requirement that anomalous dimensions are real and bounded from below, select the following values of the central charge for which physically sensible degenerate representations are possible:

$$c(p,q) = 1 - 6\frac{(p-q)^2}{pq} \;, \tag{1.3.3}$$

where p and q are mutually prime natural numbers. It turns out that the operator algebra of each minimal model $\mathcal{M}(p,q)$ closes on a finite number of primary fields $\Phi_{(n,m)}$ $(1 \le n \le p-1, 1 \le m \le q-1)$ whose conformal dimensions are given by the Kac's formula [9, 10]

$$\Delta_{(n,m)} = \frac{(qn-pm)^2-(q-p)^2}{4pq} \;. \tag{1.3.4}$$

The degenerate representation with lowest-weight vector $\Phi_{(n,m)}$ turns out to contain a null vector at level nm. Since $\Delta_{(p-n,q-m)} = \Delta_{(n,m)}$, we conclude that $\Phi_{(p-n,q-m)} = \Phi_{(n,m)}$. The field $\Phi_{(1,1)} = \Phi_{(p-1,q-1)}$ has conformal dimensions $\Delta = \bar{\Delta} = 0$ and is identified with the identity operator I of the theory. The structure of the operator algebra is described by the *fusion rules*

$$\Phi_{(n_1,m_1)}\Phi_{(n_2,m_2)} = \sum_{k=|n_1-n_2|+1}^{n_1+n_2-1} \; \sum_{l=|m_1-m_2|+1}^{m_1+m_2-1} [\Phi_{(k,l)}] \;, \tag{1.3.5}$$

where the variable k (l) runs over the even integers, provided n_1+n_2 (m_1+m_2) is odd and vice versa.

For minimal models the structure constants of the operator algebra (1.2.5), and then the correlation functions, can be determined by exploiting differential equations (null-vector equations) for correlation functions obtained using eq.(1.3.2) and OPE. This completes the bootstrap program.

An important class of minimal models can be selected by imposing the unitarity condition, namely the requirement that the space of states of the field theory is positive definite. In [11] it was shown that for $c < 1$ this constraint can be satisfied only for the discrete set of values of the central charge given by eq.(1.3.3) with $q = p + 1$. In this case eqs.(1.3.3) and (1.3.4) specialize to

$$c_p = 1 - \frac{6}{p(p+1)} \tag{1.3.6}$$

and

$$\Delta_{(n,m)} = \frac{((p+1)n - pm)^2 - 1}{4p(p+1)} \tag{1.3.7}$$

It can be easily verified from eq.(1.3.4) that all the fields contained in such theories have non negative anomalous dimensions.

Minimal models describe the critical behaviour of many statistical mechanical systems. For given values of p and q, different universality classes (namely different critical behaviours) correspond to different subset of the allowed operators. A complete classification of these subsets can be obtained by requiring the *modular invariance* of the partition function on a torus [12, 13]. We just sketch the basic idea of this approach. Consider the conformal mapping

$$w = u + iv = \frac{l}{2\pi} \ln z \ , \tag{1.3.8}$$

which maps the whole z-plane onto an infinite strip of width l with periodic boundary conditions (a cylinder). Using the transformation law eq.(1.2.27) one obtains the following expression for the quantum hamiltonian of the system on the cylinder (the euclidean time direction is chosen along the v axis):

$$\hat{H} = \frac{1}{2\pi} \int_0^l dv\, T_{vv} = \frac{2\pi}{l} \left(L_0 + \bar{L}_0 - \frac{c}{12} \right) \ . \tag{1.3.9}$$

This equation relates the eigenvalues E_k of the hamiltonian $\hat{H}$ to the scaling dimensions $d_k = \Delta_k + \bar{\Delta}_k$ of the operators of the CFT (recall that d_k are the eigenvalues of the generator $D = L_0 + \bar{L}_0$ of dilatations in the plane). Then the partition function for a rectangle of sides l and l' with periodic boundary conditions on both pairs of opposite sides (a torus) is

$$Z(l, l') = \mathrm{Tr} e^{-l'\hat{H}} = e^{\pi c \delta/6} \sum_k e^{-2\pi d_k \delta} \ , \tag{1.3.10}$$

where $\delta = l'/l$. For a minimal model we know that the scaling dimensions d_k of a descendent operator at level $(L, \bar{L})$ in the conformal family of a primary field of dimensions $(\Delta_{(n,m)}, \Delta_{(\bar{n},\bar{m})})$ is $\Delta_{(n,m)} + \Delta_{(\bar{n},\bar{m})} + L + \bar{L}$. Then the sum in eq.(1.3.10) can be rewritten as

$$\sum_{n,m;\bar{n},\bar{m}} \mathcal{N}(n,m;\bar{n},\bar{m}) d_{n,m}(L) d_{\bar{n},\bar{m}}(\bar{L}) \exp(-2\pi\delta(\Delta_{(n,m)} + \Delta_{(\bar{n},\bar{m})} + L + \bar{L})) \ , \tag{1.3.11}$$

where the factors d denote the degeneracy at the appropriate level in the conformal family and the integers $\mathcal{N}$ define the operator content of the theory. Introducing the Virasoro characters

$$\chi_{n,m}(\delta) = q^{-c/24}\mathrm{Tr}q^{L_0} \ , \tag{1.3.12}$$

$$q \equiv e^{-2\pi\delta} \ , \tag{1.3.13}$$

we rewrite the partition function (1.3.10) in the form

$$Z(\delta) = \sum_{n,m;\bar{n},\bar{m}} \mathcal{N}(n,m;\bar{n},\bar{m})\chi_{n,m}(\delta)\chi_{\bar{n},\bar{m}}(\delta) \ . \tag{1.3.14}$$

Now we impose the fundamental condition

$$Z(\delta) = Z(1/\delta) \tag{1.3.15}$$

reflecting the fact that the partition function on a torus can be calculated in two different ways corresponding to the two possible choices for the time axis. More in general one can require the invariance of Z under the whole modular group generated by $\delta \to 1/\delta$ and $\delta \to \delta + i$. This gives rise to a system of linear Diophantine equations for the numbers $\mathcal{N}$ which permits a complete classification of modular invariant partition functions for minimal conformal theories [13]. Such classification is known as ADE classification since all the solutions are labelled by the simply laced Lie algebras.

Let us restrict now our attention to the unitary minimal models with central charges given by eq.(1.3.6). For each p there is always the solution $\mathcal{N}(n,m;\bar{n},\bar{m}) = \delta_{n,\bar{n}}\delta_{m,\bar{m}}$ where each allowed scalar operator is present just once and no others appear. These diagonal solutions correspond to the so called $A-A$ series (or principal series) in the ADE classification and were identified with the universality classes of critical ($p=3$), tricritical ($p=4$), tetracritical ($p=5$), etc. points in Ising models. In additions, for $p=5$, and $p=6$ there exist two further (non diagonal) solutions identified with the critical and tricritical three-state Potts universality classes respectively. For $p \leq 6$ these are the only solutions of the modular invariant constraint.

An interesting mapping can be established between the unitary model of the principal series (we will denote them $\mathcal{M}_p$) and the lagrangian theories of a scalar field ϕ subject to an even (i.e. invariant under the replacement $\phi \to -\phi$) polynomial interaction [14]. Indeed in the Landau-Ginzburg classification a $(p-1)$-critical point is described by the effective action

$$H = \int d^2x \left[\frac{1}{2}\partial_\mu\phi\partial_\mu\phi + \sum_{k=1}^{p-1} g_k\phi^{2k} \right] \ , \tag{1.3.16}$$

with $g_1 = g_2 = \ldots = g_{p-2} = 0$ and $g_{p-1} > 0$. Let's define the composite operator $:\phi^2:$ by the OPE

$$\phi(x)\phi(0) - <\phi(x)\phi(0)> = |x|^{d_2-2d_1} :\phi^2:(0) + \ldots \ , \tag{1.3.17}$$

where d_2 and d_1 are the anomalous dimensions of $:\phi^2:$ and ϕ respectively and only the most singular term for $x \to 0$ was written in the right-hand side. The other composite fields $:\phi^{k+1}:$ can be defined recursively through the OPE $\phi(x):\phi^k:(0)$

$$:\phi^{k+1}:(0) \equiv \lim_{x\to 0}|x|^{d_1+d_k-d_{k+1}}\Big\{\phi(x):\phi^k:(0) - \sum_{q=1}^{[(k+1)/2]} A_q|x|^{d_{k+1-2q}-d_1-d_k}:\phi^{k+1-2q}:(0)\Big\} \quad . \qquad (1.3.18)$$

In this relation the coefficients A_q are chosen in such a way that the limit is finite; an average over the directions of the vector x^μ is understood to ensure the absence of vector terms. The fields $:\phi^{2p-3}:$ constructed in this way should satisfy the operator equation of motion

$$g:\phi^{2p-3}: = \partial_\mu\partial_\mu\phi \quad . \qquad (1.3.19)$$

Consider now the operator algebra of the model $\mathcal{M}_p$. From eq.(1.3.5) we obtain

$$\Phi_{(2,2)}\Phi_{(n,m)} = \sum_{k,l=\pm 1} C^{(k,l)}_{n,m}[\Phi_{(n+k,m+l)}] \quad . \qquad (1.3.20)$$

We are not interested in the precise values of the structure constants $C^{(k,l)}_{n,m}$; for us it is important to know only that

$$C^{(-,-)}_{1,m} = C^{(-,+)}_{1,m} = C^{(+,-)}_{p-1,m} = C^{(+,+)}_{p-1,m} = C^{(-,-)}_{n,1} = C^{(+,-)}_{n,1} = 0 \quad . \qquad (1.3.21)$$

If we use the notation $\Phi_{(2,2)} = \phi$ and define *composite fields* $:\phi^k:$ using the rules (1.3.18), we can verify using eqs.(1.3.20), (1.3.21) and (1.3.7) that

$$\begin{aligned} :\phi^k: &\sim \Phi_{(k+1,k+1)} && \text{for } k=0,1,\ldots,p-2\ ; \\ :\phi^k: &\sim \Phi_{(k-p+3,k-p+2)} && \text{for } k=p-1,\ldots,2p-4\ . \end{aligned} \qquad (1.3.22)$$

The OPE $\phi:\phi^{2p-4}:$ is given by

$$\Phi_{(2,2)}\Phi_{(p-1,p-2)} = [\Phi_{(p-2,p-3)}] + [\Phi_{(2,2)}] \qquad (1.3.23)$$

(recall that $\Phi_{(n,m)} = \Phi_{(p-n,p+1-m)}$). According to eq.(1.3.18) $:\phi^{2p-3}:$ is obtained subtracting from eq.(1.3.23) the most singular contributions due to the primary fields $\Phi_{(2,2)} = \phi$ and $\Phi_{(p-2,p-3)} = \phi^{2p-5}$:. Therefore $:\phi^{2p-3}:$ will receive a contribution from the field $\partial_\mu\partial_\mu\phi$ which is the most singular scalar representative of the conformal family $[\phi]$ after the field ϕ itself. Thus we see that the field $\phi = \Phi_{(2,2)}$ of the model $\mathcal{M}_p$ formally satisfies the operator equation of motion (1.3.19). In this sense we say that the $(p-1)$-critical behaviour of theory (1.3.16) is described by the unitary minimal model $\mathcal{M}_p$ and identify the composite operators $:\phi^k:$ with conformal primary fields according to (1.3.22).

It was shown in ref. [15] that the models $\mathcal{M}_p$ describe the critical points of the exactly solvable "RSOS models" [16], which apparently have the same physical origin as the $(p-1)$-critical points described by (1.3.16). This leads to conclude that a critical theory $\mathcal{M}_p$ is not a specific feature of some exactly solvable model, but describe the general $(p-1)$-critical behaviour of two-dimensional systems with scalar order parameter ϕ and symmetry $\mathbf{Z}_2$ $(\phi \to -\phi)$.

1.4 Coset Construction

A more general CFT can be constructed by means of algebraic methods (see ref. [17] and references therein). Consider the affine Kac-Moody algebra $\hat{\mathcal{G}}$ defined by the commutation rules

$$[J^a_m, J^b_n] = i f^{abc} J^c_{m+n} + \tilde{k} m \delta^{ab} \delta_{m+n,0} , \tag{1.4.1}$$

where f^{abc} are the structure constants of some Lie algebra $\mathcal{G}$ associated to a compact Lie group G of dimension $\dim G \equiv |G|$, $\tilde{k}$ is the so-called central extension and $n, m \in \mathbf{Z}$. If we introduce the *currents* $J^a(z)$ in terms of the mode expansion

$$J^a(z) = \sum_{n \in \mathbf{Z}} J^a_n z^{-n-1} , \tag{1.4.2}$$

eq.(1.4.1) is equivalent to the OPE

$$J^a(z) J^b(0) = \frac{\tilde{k}}{z^2} \delta^{ab} + i \frac{f_{abc}}{z} J^c(0) + \ldots \quad . \tag{1.4.3}$$

The representation theory of affine algebras shares many features with that of the Virasoro algebra. In particular, there exist primary fields $\phi(z)$ having the OPE

$$J^a(z) \phi(0) = \frac{t^a}{z} \phi(0) + \ldots \tag{1.4.4}$$

and corresponding to highest weight vectors of some highest weight representation . In the following we will restrict our attention to unitary representations. Unitarity is implemented as the condition of hermiticity of the generators: $J^{a\dagger}(z) = J^a(z)$. It can be shown that this implies $J^{a\dagger}_n = J^a_{-n}$.

Virasoro generators L_n can be constructed in terms of the modes J^a_n. Indeed one can verify that the operators

$$L_n(z) = \frac{1}{2\tilde{k} + C_A} \sum_{a=1}^{|G|} \sum_{m=-\infty}^{+\infty} : J^a_{m+n} J^a_{-m} : \tag{1.4.5}$$

satisfy the Virasoro algebra, the quadratic Casimir C_A of the adjoint representation of $\mathcal{G}$ being defined by $f^{acd} f^{bcd} = C_A \delta^{ab}$. Eq.(1.4.5) automatically leads to the introduction of the so-called Sugawara form of the stress-energy tensor

$$T(z) = \frac{1}{2\tilde{k} + C_A} \sum_{a=1}^{|G|} : J^a(z) J^a(z) : \quad . \tag{1.4.6}$$

T(z) satisfies the canonical OPE eq.(1.2.30) with central charge

$$c_G = \frac{\tilde{k}|G|}{\tilde{k} + C_A/2} \quad . \tag{1.4.7}$$

From eq.(1.4.5) one obtains

$$[L_m, J^a_n] = -nJ^a_{m+n} \ , \tag{1.4.8}$$

which in turn implies the OPE

$$T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} + \dots \tag{1.4.9}$$

showing that $J^a(z)$ is a conformal field of dimensions $(1,0)$.

The numbers C_A and $\tilde{k}$ in eq.(1.4.7) depend in general on the normalization of the structure constants f^{abc}. Denoting by ψ the highest root of the Lie algebra $\mathcal{G}$, we define the normalization independent quantity $\tilde{h}_G \equiv C_A/\psi^2$, known as the dual Coxeter number. Only for simply-laced algebras (A,D,E series) $\tilde{h}_G$ coincides with the Coxeter number, defined as the number of (non-zero) roots divided by the rank of $\mathcal{G}$. The dual Coxeter numbers for all the compact simple Lie algebras are listed below:

G	$\tilde{h}_G$
$SU(n)\ (n \geq 2)$	n
$SO(n)\ (n \geq 4)$	$n-2$
E_6	12
E_7	18
E_8	30
$Sp(2n)\ (n \geq 1)$	$n+1$
G_2	4
F_4	9

Analogously, we introduce the normalization independent quantity $k \equiv 2\tilde{k}/\psi^2$, known as the *level* of the affine algebra $\tilde{\mathcal{G}}$. It can be shown that k is quantized as an integer in a highest weight representation. In terms of the integers k and $\tilde{h}_G$ the formula (1.4.7) for the central charge reads

$$c_G = \frac{k|G|}{k+\tilde{h}_G} \quad . \tag{1.4.10}$$

For any group G the central charge (1.1.4) satisfies the inequality

$$\text{rank}G \leq C_G \leq |G| \quad . \tag{1.4.11}$$

Thus we see that the Sugawara stress-energy tensor (1.4.6) does not allow to make contact with conformal theories having $c < 1$. This limitation can be circumvented through the so-called *coset construction.*

Denote by J^a_G the G currents and, among them, denote by J^i_H the currents corresponding to a subgroup $H \subset G$ ($i = 1, \dots, |H| \equiv \text{dim}H$). We can construct the two Sugawara stress-energy tensors (in the following the normalization of structure constants is fixed by $\psi^2 = 2$)

$$T_G(z) = \frac{1/2}{k_G + \tilde{h}_G} \sum_{a=1}^{|G|} : J^a_G(z) J^a_G(z) : \ , \tag{1.4.12}$$

$$T_H(z) = \frac{1/2}{k_H + \tilde{h}_H} \sum_{i=1}^{|H|} : J_H^i(z) J_H^i(z) : , \tag{1.4.13}$$

and, correspondingly, two sets of Virasoro generators L_n^G and L_n^H. One can verify that the operators $L_n^{G/H} \equiv L_n^G - L_n^H$ commute with the L_n^G's and satisfy the Virasoro algebra with central charge

$$c_{G/H} = c_G - c_H = \frac{k_G |G|}{k_G + \tilde{h}_G} - \frac{k_H |H|}{k_H + \tilde{h}_H} \quad . \tag{1.4.14}$$

This realizes a decomposition of the Virasoro algebra generated by

$$T_G = (T_G - T_H) + T_H \equiv T_{G/H} + T_H \tag{1.4.15}$$

into two mutually commuting Virasoro subalgebras generated by $T_{G/H}$ and T_H. From eq.(1.4.14) a central charge less than the rank of G can now be obtained.

As an example, consider the case of coset spaces of the form $G' \times G''/H$, where H is the diagonal subgroup generated by $J^a = J^a_{(1)} + J^a_{(2)}$, $J^a_{(1)}$ and $J_{(2)}$ being the generators of G' and G'' respectively. From eq.(1.4.3) we obtain

$$J^a(z) J^b(w) = J^a_{(1)}(z) J^b_{(1)}(w) + J^a_{(2)}(z) J^b_{(2)}(w) = \frac{k_1 + k_2}{(z-w)^2} \delta_{ab} + \ldots \tag{1.4.16}$$

so that the level of H is determined to be $k = k_1 + k_2$.

In the case $G/H = SU(2)_k \times SU(2)_1 / SU(2)_{k+1}$, with the indices denoting the levels, eq.(1.4.14) gives

$$c_{G/H} = \frac{3k}{k+2} + 1 - \frac{3(k+1)}{(k+1)+2} = 1 - \frac{6}{(k+2)(k+3)} \tag{1.4.17}$$

and we recognize the values of the $c < 1$ unitary series eq.(1.3.6) with $p = k + 2 = 3, 4, \ldots$.

For $G/H = SU(2)_k \times SU(2)_2 / SU(2)_{k+2}$ we find

$$c_{G/H} = \frac{3k}{k+2} + \frac{3}{2} - \frac{3(k+2)}{(k+2)+2} = \frac{3}{2}\left(1 - \frac{8}{(k+2)(k+4)}\right) , \tag{1.4.18}$$

corresponding to the central charge values for the unitary series of $N = 1$ superconformal models [18].

1.5 c-Theorem

We have seen in previous sections that to each fixed point of the RG can be associated a CFT identified by the value of the central charge c. A.B. Zamolodchikov obtained in ref. [19] an important result, known as c-theorem, which gives the possibility to interpret the ordering of CFT solutions according to the value of their central charge.

Let's consider an unitary, homogeneous and isotropic field theory in two dimensions. In the spirit of renormalization theory, we have to require that the Lagrangian density $\mathcal{H}$ of the theory does not globally depend on the arbitrary scale parameter μ introduced by the regularization procedure, Taking into account the dependence of $\mathcal{H}$ on the running coupling constants $g_i(\mu)$, $i = 1, \ldots, n$ we have to impose

$$\frac{d}{d\mu}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial \mu} + \frac{\partial g_i}{\partial \mu}\frac{\partial \mathcal{H}}{\partial g_i} = 0 \qquad (1.5.1)$$

(in this section sum over repeated indices is understood). If we now parameterize $\mu = \mu_0 e^{-t}$, introduce the *beta functions* by means of

$$\beta_i(g) = \frac{\partial g_i}{\partial t} , \qquad (1.5.2)$$

and the spinless fields $\Phi_i(x)$ conjugated to g_i through

$$\Phi_i = \frac{\partial \mathcal{H}}{\partial g_i} , \qquad (1.5.3)$$

we can rewrite eq.(1.5.1) in the form

$$\frac{\partial \mathcal{H}}{\partial t} = -\beta_i \Phi_i \quad . \qquad (1.5.4)$$

On the other hand the same variation of the Lagrangian density under an infinitesimal dilatation can be obtained from eq.(1.2.13). Therefore, comparing with eq.(1.5.4), we get the relation

$$\Theta(x) = -\beta_i(g)\Phi_i(x) \quad . \qquad (1.5.5)$$

We note that in correspondence of fixed points, where the theory becomes scale invariant and $\Theta = 0$, all the coefficients of expansion (1.5.5) should vanish as expected from the definition eq.(1.5.2).

The variation of correlation function (1.1.1) with coupling constants can be written, using eqs.(1.1.2) and (1.5.3) as

$$\begin{aligned} \frac{\partial}{\partial g_i} < A_1(x_1)\ldots A_N(x_N) > \;&=\; \sum_{a=1}^{N} < A_1(x_1)\ldots \frac{\partial}{\partial g_i} A_a(x_a)\ldots A_N(x_N) > \\ &\quad - \int d^2y \; < A_1(x_1)\ldots A_N(x_N)\Phi_i(y) > \; . \qquad (1.5.6) \end{aligned}$$

On the other hand, the scale transformation eq.(1.2.18) operated on all the fields contained in the correlator (1.1.1) results simply in a change of integration variables in the functional integral (1.1.2) so that we write the equation as

$$\sum_{a=1}^{N} < (x_a^\mu \partial_\mu^a + D_a) A_1(x_1)\ldots A_N(x_N) > + \int d^2y < A_1(x_1)\ldots A_N(x_N)\Theta(y) >= 0 \; , \qquad (1.5.7)$$

where D_a denotes the dilatation operator D applying to the field $A_a(x_a)$ and we made use of eq.(1.2.13). Combining now eqs.(1.5.5), (1.5.6) and (1.5.7) we obtain the equations of RG in the Callan-Simanzik form

$$\left\{ \sum_{a=1}^{N} (x_a^\mu \partial_\mu^a + \Gamma_a(g)) - \beta_i(g) \frac{\partial}{\partial g_i} \right\} < A_1(x_1) \ldots A_N(x_N) >= 0 \ , \tag{1.5.8}$$

where the operator

$$\Gamma(g) = D + \beta_i(g) \frac{\partial}{\partial g_i} \tag{1.5.9}$$

is the so called matrix of anomalous dimensions. By verifying the compatibility of the equations (1.5.6) and (1.5.7) it is possible to show that the space $\Phi \subset \mathcal{A}$ spanned by the fields Φ_i is invariant under the action of Γ:

$$\Gamma \Phi_j(x) = \left(2\delta_{ij} - \frac{\partial \beta_i}{\partial g_i} \right) \Phi_i(x) \quad . \tag{1.5.10}$$

This relation, together with eq.(1.5.5), allows to obtain the equation

$$\Gamma \Theta = 2\Theta \ , \tag{1.5.11}$$

showing the coincidence of the anomalous dimension of Θ with his canonical dimension, i.e. the absence of renormalization for the components of the energy-momentum tensor.

Let's consider now the correlators of the components of $T_{\mu\nu}$ (we pass to complex coordinates according to eqs.(1.2.10) and (1.2.19)). By euclidean invariance and the absence of renormalization of the energy-momentum tensor, they can be parameterized as follows:

$$\begin{aligned} < T(z,\bar{z})T(0,0) > &= \frac{F(\tau)}{z^4} \ ; \\ < T(z,\bar{z})\Theta(0,0) > &= \frac{H(\tau)}{z^3 \bar{z}} \ ; \\ < \Theta(z,\bar{z})\Theta(0,0) > &= \frac{G(\tau)}{(z\bar{z})^2} \ , \end{aligned} \tag{1.5.12}$$

where $\tau = ln(z\bar{z})$. In the present notation equations (1.2.11) are rewritten in the form

$$\partial_{\bar{z}} T + \partial_z \Theta = 0 \ ; \qquad \partial_z \bar{T} + \partial_{\bar{z}} \Theta = 0 \quad . \tag{1.5.13}$$

These equations imply the following relations for the functions F, G and H:

$$\dot{F} = 3H - \dot{H} \ ; \qquad \dot{H} - H = 2G - \dot{G} \ , \tag{1.5.14}$$

where the dot means differentiation with respect to τ. If we now define the function

$$C = 2F - 4H - 6G \ , \tag{1.5.15}$$

relations (1.5.14) immediately give

$$\dot{C} = -12G \quad . \tag{1.5.16}$$

At this point we can use this relation, the RG equations (1.5.8) and the absence of renormalization for $T^{\mu\nu}$ to obtain

$$\beta_i(g)\frac{\partial}{\partial g_i}C(\tau,g) = -24G(\tau,g) \quad . \tag{1.5.17}$$

By fixing $\tau = 0$ and defining

$$\begin{aligned} G_{ij}(\tau,g) &= (z\bar{z})^2 < \Phi_i(z,\bar{z})\Phi_j(0,0) > ; \\ G_{ij}(g) &= G_{ij}(0,g) , \end{aligned} \tag{1.5.18}$$

eq.(1.5.17) becomes

$$\beta_i(g)\frac{\partial}{\partial g_i}C(g) = -24G_{ij}(g)\beta_i(g)\beta_j(g) , \tag{1.5.19}$$

(where eq.(1.5.5) was used).

So far we did not make use of the initial assumption about unitarity of the theory but now we observe that this condition implies the matrix G_{ij} to be positive definite (note that in force of this requirement the quantity $ds^2 = G_{ij}dg_i dg_j$ can be regarded as the metric in the interaction space Q). As a consequence, from eqs.(1.5.2) and (1.5.19) we obtain the important inequality

$$\frac{d}{dt}C(g) \leq 0 , \tag{1.5.20}$$

showing that the function C is monotonically decreasing along the RG trajectories. In particular this excludes the existence of limit cycles in the RG flow.

In correspondence of a fixed point of the RG identified by coordinates g^* in Q the theory becomes conformal invariant and $\Theta = 0$ so that eq.(1.5.15) reduces to

$$C(g^*) = 2z^4 < T(z)T(0) > \quad . \tag{1.5.21}$$

The two-point function in the right-hand side is immediately obtained from eq.(1.2.30) observing that $< T >= 0$ in an infinite system, so that

$$< T(z)T(0) >= \frac{c}{2z^4} , \tag{1.5.22}$$

where c is the central charge of the conformal theory. Thus we can conclude that the function C introduced in eq.(1.5.15) is stationary in correspondence of fixed points of the RG where it equals the central charge of the corresponding conformal field theory:

$$C(g^*) = c \quad . \tag{1.5.23}$$

1.6 Perturbation Theory around Fixed Points

The exact knowledge of the field-theory solutions corresponding to a fixed point of the RG can be used as the starting point for the study of the properties of RG around the fixed point itself. Indeed we can think to deform a CFT by adding to it a small perturbation and to study perturbatively the field theory obtained in this way, with action

$$H = H_{CFT} + H_{Pert} \quad . \tag{1.6.1}$$

However, this approximation is useful only if the RG exhibits a nontrivial topological behaviour in a sufficiently small neighborhood of the fixed point, namely if it has other fixed points in this region, since otherwise we would have to sum up at all orders the perturbative series to get any exact topological information. In order to determine the conditions under which such nontrivial behaviour can arise, let's consider a fixed point with coordinates g^* in Q and choose it as the origin of the coordinate system, namely $g^* = 0$. We now introduce the conjugated fields Φ_i according with eq.(1.5.3) and orient the coordinate system in Q in such a way that the fields $\Phi_i^0 \equiv \Phi_i|_{g=0}$ have well defined conformal dimensions $\Delta_i = \bar{\Delta}_i$ so that the matrix of anomalous dimension is diagonal in the origin:

$$\Gamma(0)\Phi_i^0 = \Delta_i \Phi_i^0 \quad . \tag{1.6.2}$$

Then using eq.(1.5.10) we obtain

$$\beta_i(g) = 2\varepsilon_i g_i + O(g^2) \ , \tag{1.6.3}$$

where $\varepsilon_j \equiv 1 - \Delta_j$. Therefore we conclude that if $|\varepsilon_i| \sim \varepsilon \ll 1$ there is the possibility to find a new fixed point for $g_i \sim \varepsilon$, namely inside the region for which a perturbative study at finite order is allowed.

In the following we will restrict our attention to the cases in which a CFT is perturbed only along one direction in the interaction space Q, namely we will consider *single-charge* perturbing actions of the form

$$H_{Pert} = g \int d^2x \, \Phi(x) \quad . \tag{1.6.4}$$

This restriction is consistent only if the perturbing field Φ is the only relevant field in the operator algebra $\mathcal{A}$ or, at least, in some subalgebra of $\mathcal{A}$. This condition ensures that under repeated fusions Φ closes on itself (modulo irrelevant operators) and the identity, so that renormalization of ultraviolet divergences in the perturbation theory based on eq.(1.6.4) does not require additional counterterms.

Let's now apply these general ideas to the specific case in which the fixed point g_0^* corresponds to a minimal unitary model $\mathcal{M}_p$ with $p \gg 1$ [19]. It is easy to see from eq.(1.3.4) that the only relevant fields in these theories with conformal dimension close to 1 belong to the series $\Phi_{(n,n+2)}$, with $n \ll p$. In particular the field $\Phi_{(1,3)}$, with dimension

$$\Delta_{(1,3)} = 1 - \varepsilon \ ; \qquad \varepsilon = \frac{2}{p+1} \ , \tag{1.6.5}$$

is the only relevant field in the subalgebra $\mathcal{A}_1 = \bigoplus_n [\Phi_{(1,2n+1)}] \subset \mathcal{A}$, as can be verified by eq.(1.3.5). Thus we can take eq.(1.6.4) with $\Phi^0 = \Phi_{1,3}$ as perturbation of the conformal model $\mathcal{M}_p$. We begin by evaluating the second order correction in eq.(1.6.3) to verify the existence of another fixed point g_1^* in the perturbative region. To this aim we compute the two-point function

$$\begin{aligned} < \Phi(x)\Phi(0) > &= < \Phi(x)\Phi(0) > |_{g=0} + g\frac{\partial}{\partial g} < \Phi(x)\Phi(0) >|_{g=0} + O(g^2) \\ &= < \Phi^0(x)\Phi^0(0) > +g\left(< \frac{\partial}{\partial g}\Phi^0(x)\Phi^0(0) > + < \Phi^0(x)\frac{\partial}{\partial g}\Phi^0(0) >\right) \\ &\quad -g\int d^2y \, < \Phi^0(x)\Phi^0(0)\Phi^0(y) > +O(g^2) \ . \end{aligned} \tag{1.6.6}$$

For $g = 0$ the form of two-point and three-point functions is fixed by conformal invariance:

$$\begin{aligned} < \Phi^0(x)\Phi^0(0) > &= |x|^{-4\Delta} \ ; \\ < \Phi^0(x)\Phi^0(0)\Phi^0(y) > &= C_{(1,3)(1,3)}^{(1,3)}(|x||x-y||y|)^{-2\Delta} \ , \end{aligned} \tag{1.6.7}$$

where $\Delta = \Delta_{(1,3)}$ and the structure constant $C_{(1,3)(1,3)}^{(1,3)}$ is equal to $\frac{4}{\sqrt{3}}$ in the large p limit we are considering [21]. With a suitable choice of the coordinate system in Q one can obtain the following condition on the metric:

$$G(g) =< \Phi(x)\Phi(0) > |_{x^2=1} = 1 + O(g^2) \ , \tag{1.6.8}$$

so that

$$\frac{\partial}{\partial g} < \Phi(x)\Phi(0) >|_{g=0,x^2=1} = 0 \ . \tag{1.6.9}$$

Moreover we know that

$$< \frac{\partial}{\partial g}\Phi^0(x)\Phi^0(0) > \sim < \Phi^0(x)\frac{\partial}{\partial g}\Phi^0(0) > \sim |x|^{-4\Delta} \ . \tag{1.6.10}$$

Using all these information we get

$$\begin{aligned} < \Phi(x)\Phi(0) > &\simeq |x|^{-4\Delta}\left(1 - g\frac{16\pi}{(1-\Delta)\sqrt{3}}(|x|^{2(1-\Delta)} - 1)\right) \\ &\simeq |x|^{-4(\Delta + \frac{8\pi}{\sqrt{3}}g)} \ . \end{aligned} \tag{1.6.11}$$

From this relation we can read off the anomalous dimension of the perturbing field Φ:

$$\Delta(g) = \Delta + \frac{8}{\sqrt{3}}\pi g + O(g^2) \ . \tag{1.6.12}$$

We can now use eq.(1.5.10) to obtain

$$\beta(g) = 2g\left(\varepsilon - \frac{4}{\sqrt{3}}\pi g\right) + O(g^3) \ , \tag{1.6.13}$$

showing that, for $g > 0$, an infrared fixed point is present with coordinates

$$g_1^* = \frac{\sqrt{3}}{4\pi}\varepsilon + O(\varepsilon^2) \quad . \tag{1.6.14}$$

The models $\mathcal{M}_p$ are the only unitary conformal theories with $c < 1$. Since unitarity cannot be destroyed perturbatively, we can use the statement of previous section concerning the decay of the C-function to predict that the new fixed point corresponds to a model $\mathcal{M}_r$ with $r < p$. In order to identify the precise value of r, we use eq.(1.5.19) to compute the difference between the central charges at g_0^* and g_1^*:

$$c(g_0^*) - c(g_1^*) = -24 \int_{g_1^*}^{g_0^*} \beta(g) dg = \frac{3}{2}\varepsilon^3 \quad . \tag{1.6.15}$$

Using eq.(1.3.6) we verify that $r = p - 1$. Therefore we conclude that the field theory constructed in this section interpolates between the minimal unitary models $\mathcal{M}_p$ and $\mathcal{M}_{p-1}$. In particular

$$\Delta(g_1^*) = 1 + \varepsilon \tag{1.6.16}$$

and this value corresponds to the conformal dimension of the irrelevant operator $\Phi_{(3,1)}$ in the model $\mathcal{M}_{p-1}$. Thus we see that the perturbing field Φ, identified as $\Phi^p_{(1,3)}$ at g_0^*, renormalizes to $\Phi^{p-1}_{(3,1)}$ at g_1^*.

This flow from $\mathcal{M}_p$ to $\mathcal{M}_{p-1}$ has an immediate interpretation in the Landau-Ginzburg description discussed in sec. 1.3. Indeed, according to (1.3.22), the field $\Phi_{(1,3)}$ corresponds to the composite field $:\phi^{2p-4}:$ so that perturbing the model $\mathcal{M}_p$ by the field $\Phi_{(1,3)}$ amounts to lower from $p-1$ to $p-2$ the degree of criticality of action (1.3.16). Detailed analysis [19, 20] allows to follow explicitly the change of position in the conformal grid of the operators $:\phi^k:$ as $p \to (p-1)$: all operators $\Phi_{(n,n)}$ with $n = 1, 2, \ldots, p-2$ stay at the same position while the other operators move according to $\Phi_{(m,n)} \to \Phi_{(m+1,n+1)}$ for $m - n = 1$ and $\Phi_{(p-1,p-1)} \to \Phi_{(2,1)}$.

Chapter 2

Integrable Deformations of Conformal Field Theories

2.1 Integrals of Motion

We presented in sec. 1.6 an example of a field theory with both ultraviolet and infrared asymptotic limits described by CFTs (the unitary minimal models $\mathcal{M}_p$ and $\mathcal{M}_{p-1}$ respectively). Since the associated RG trajectory interpolates between two fixed points, it necessarily lies on the critical surface S_∞ and corresponds to a massless theory (see sec. 1.1). But in general the RG trajectory obtained perturbing a CFT by a relevant operator goes out from the critical surface, so that the corresponding field theory develops a finite correlation length, namely contains only massive particles.

It is known that a massive theory is equivalent to the relativistic scattering theory and so it is completely specified by the S-matrix, subjected to the usual requirements of analyticity, unitarity and crossing symmetry. The S-matrix gives explicit informations about the infrared properties of the theory, while the CFT data (i.e. the particular ultraviolet fixed point and the perturbing operator) are encoded in the S-matrix in non-trivial way and have to be derived from it by some technique.

The study of the link between the *CFT data* and the *S-matrix data* can be successfully performed for a particular class of perturbations, the so-called *integrable deformations* [22], and represents an important progress in our understanding of the structure of two-dimensional quantum field theories. We begin our discussion by considering the problem of integrals of motion for two dimensions.

The existence of a conserved current in a two-dimensional field theory is expressed by the continuity equation

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1} \ , \tag{2.1.1}$$

where T_{s+1} and Θ_{s-1} are some fields of spin $s+1$ and $s-1$ respectively. The integral

$$P_s = \oint [T_{s+1} dz + \Theta_{s-1} d\bar{z}] \tag{2.1.2}$$

is invariant under contour deformations and represents an integral of motion of spin s for the field theory. If M is the Euclidean rotation generator, the following commutation relation holds:

$$[M, P_s] = sP_s \quad . \tag{2.1.3}$$

A conformal invariant theory in two dimensions should possess an infinite number of conserved currents. They are easily identified considering the subspace Λ of the conformal family of the identity operator I formed by all the fields with left conformal dimension equal to zero. Indeed all the operators $T_s \in \Lambda$ (the spin s coincides with the right dimension) are holomorphic and satisfy the equation

$$\partial_{\bar{z}} T_s = 0 \ , \tag{2.1.4}$$

which is a particular case of eq.(2.1.1). In particular, $T_2 = L_{-2}I$ is simply the holomorphic part T of the energy-momentum tensor.

Consider now the theory obtained perturbing a CFT by a relevant scalar operator Φ of anomalous dimension $d = 2\Delta$ (as already discussed in sec. 1.6, the field Φ is taken to be the most relevant one in the space $\mathcal{A}$ or in some subalgebra $\mathcal{A} \subset \mathcal{A}_1$). The action can be written as

$$H = H_{CFT} + g \int dx^2 \, \Phi(x) \ , \tag{2.1.5}$$

with the coupling constant g carrying anomalous dimension $2(1-\Delta)$. This can be expressed ascribing to g right and left dimensions $(1-\Delta, 1-\Delta)$. We want to study the possibility that some conserved current T_s of the CFT (apart from the case $s = 2$) remains conserved in the perturbed theory [22]. In the following we assume that the structure of the space of operators in the perturbed theory remains unchanged with respect to the space $\mathcal{A}$ of the CFT. Therefore we will keep also the same notations.

Clearly, equation (2.1.4) will be in general modified in the perturbed theory; we rewrite it in the form

$$\partial_{\bar{z}} T_s = gR^1_{s-1} + \ldots + g^n R^n_{s-1} + \ldots \ , \tag{2.1.6}$$

where R^n_{s-1} are some local fields belonging to $\mathcal{A}$ (or $\mathcal{A}_1$. Since the dimensions of T_s are $(s, 0)$, the dimensions of each term on the right-hand side should be $(s, 1)$. This implies that the dimensions of R^n_{s-1} are $(s - n\varepsilon, 1 - n\varepsilon)$, with $\varepsilon = 1 - \Delta$ ($\varepsilon > 0$ since Φ is a relevant operator), so that they become negative for sufficiently high n. Suppose now we restrict our attention to a unitary theory. Since such a theory does not contain fields with negative dimensions we conclude that the perturbative series in eq.(2.1.6) should be finite. In fact, the term $g^n R^n_{s-1}$ with $n > 1$ can be different from zero only if the condition

$$1 - n\varepsilon = \Delta_r \tag{2.1.7}$$

is satisfied for some relevant dimension $\Delta_r < \Delta$, since otherwise one cannot find a field with appropriate dimension in $\mathcal{A}$. But we have chosen Φ to be the most relevant field in $\mathcal{A}$ (or in $\mathcal{A}_1$) so that it must be $\Delta_r = 0$ and the relation (2.1.7) can be verified only if ε is an inverse integer, namely

$$\Delta = 1 - \frac{1}{N} \ , \qquad N > 1 \quad . \tag{2.1.8}$$

We assume this is not the case and write

$$\partial_{\bar z} T_s = g R^1_{s-1} \quad . \tag{2.1.9}$$

It was shown in ref. [22] that there is a set of linear operators D_n with the following properties:

$$\begin{aligned} \partial_{\bar z} &= g D_0 \ , \\ [L_n, D_m] &= -((1-\Delta)(n+1)+m) D_{n+m} \ , \\ D_{-n-1} I &= \frac{1}{n!} L^n_{-1} \Phi(z,\bar z) \quad . \end{aligned} \tag{2.1.10}$$

This allows the direct computation of the right-hand side of equation (2.1.9). For example, for $T_2 = T$ we have

$$\partial_{\bar z} T = g D_0 L_{-2} I = g(\Delta-1) D_{-2} I = g(\Delta - 1) L_{-1} \Phi \ , \tag{2.1.11}$$

But, since $L_{-1} = \partial_z$ (see eq.(1.2.23)), this result can be rewritten in the usual form

$$\partial_{\bar z} T + \partial_z \Theta = 0 \ , \tag{2.1.12}$$

where

$$\Theta = g(1-\Delta)\Phi \tag{2.1.13}$$

expresses the trace of the energy-momentum tensor in terms of the perturbing field. Note that, although eq.(2.1.13) vanishes for $\Delta = 1$, this does not means that Θ really vanishes in the case of marginal perturbation since the assumption $\Delta < 1$ was crucial in the reasoning leading to the truncation to first order of the perturbative series in eq.(2.1.6).

As a second example, consider the field $T_4 = L^2_{-2} I$. We find

$$\begin{aligned} \partial_{\bar z} T_4 &= g D_0 L_{-2} L_{-2} I \\ &= g(\Delta-1)(D_{-2}L_{-2} + L_{-2} D_{-2}) I \\ &= g(\Delta-1)\left(2L_{-2}L_{-1} + \frac{\Delta-3}{6} L^3_{-1}\right)\Phi \ , \end{aligned} \tag{2.1.14}$$

but since the right-hand side is not a total z-derivative, this does not imply the conservation of the quantity under consideration.

A general argument for establishing the existence of non-trivial integrals of motion in a perturbed theory was developed in ref. [22] along the following lines. As shown by the dimensional counting explained above and confirmed by previous examples, the field R^1_{s-1} belongs to the subspace Φ_{s-1} of the conformal family $[\Phi]$ containing all the fields of dimensions $(\Delta + s - 1, \Delta)$. Analogously we define Λ_s to be the subspace of Λ containing the fields of dimensions $(s, 0)$:

$$\Lambda = \bigoplus_{s=0}^{\infty} \Lambda_s \quad . \tag{2.1.15}$$

Let us introduce also the factor spaces

$$\hat{\Lambda}_s = \Lambda_s / L_{-1}\Lambda_{s-1} \tag{2.1.16}$$

and

$$\hat{\Phi}_s = \Phi / L_{-1}\Phi_{s-1} \quad . \tag{2.1.17}$$

The symbol $\partial_{\bar{z}}$ in eq.(2.1.9) can be considered as a linear operator acting in the following way:

$$\partial_{\bar{z}} \ : \quad \hat{\Lambda}_s \rightarrow \hat{\Phi}_{s-1} \tag{2.1.18}$$

(in fact, the action of $\partial_{\bar{z}}$ extends to the whole space Λ and the restriction to $\hat{\Lambda}$ is only for later convenience). Let Π_s be the projector $\Phi_s \rightarrow \hat{\Phi}_s$ and B_s the operator

$$B_s = \Pi_s \partial_{\bar{z}} \ : \quad \hat{\Lambda}_s \rightarrow \hat{\Phi}_{s-1} \quad . \tag{2.1.19}$$

Then, by definition, every field $T_{s+1} \in \Lambda_{s+1}$, satisfying $B_{s+1}T_{s+1} = 0$ has the property

$$\partial_{\bar{z}} T_{s+1} = \partial_z \Theta_{s-1} \ , \tag{2.1.20}$$

where Θ_{s-1} is some local field belonging to Φ_{s-1}. So we conclude that an integral of motion of spin s in present in the perturbed theory every time the operator B_{s+1} has a non-vanishing kernel. For a general field Φ taken as the perturbation, $ker\, B_2 = T$ and $ker\, B_s = 0$ for $s \neq 2$.

The presence of non-trivial integrals of motion can be proven if the perturbing field is chosen to be one of the degenerate fields $\Phi_{(n,m)}$ contained in the unitary minimal models $\mathcal{M}_p$ discussed in sec. 1.3. Our choice restricts to the three fields $\Phi_{(1,3)}$, $\Phi_{(1,2)}$ and $\Phi_{(2,1)}$ since these are the only fields among the $\Phi_{(n,m)}$'s with the property to be the most relevant fields in some appropriate subalgebra. As an example, take $\Phi = \Phi_{(1,3)}$. This field satisfies the third-level null-vector equation [8]

$$\left(L_{-3} - \frac{2}{\Delta + 2} L_{-1}L_{-2} + \frac{1}{(\Delta+1)(\Delta+2)} L_{-1}^3 \right) \Phi = 0 \ , \tag{2.1.21}$$

where $\Delta = \Delta_{(1,3)}$ can be computed by the formula (1.3.7). Using this equation we can rewrite eq.(2.1.14) in the form

$$\partial_{\bar{z}} T_4 = \partial_z \Theta_2 \ , \tag{2.1.22}$$

with

$$\Theta_2 = \frac{g(\Delta - 1)}{\Delta + 2} \left[2\Delta L_{-2} + \frac{(\Delta-2)(\Delta-1)(\Delta+3)}{6(\Delta+1)} L_{-1}^2 \right] \Phi \quad . \tag{2.1.23}$$

In the general case, since B_s is a linear operator mapping $\hat{\Lambda}_s$ into $\hat{\Phi}_{s-1}$, $ker\, B_{s+1} \neq 0$ if $dim(\hat{\Lambda}_{s+1}) > dim(\hat{\Phi}_s)$. In the example we just proposed, the nonzero kernel of B_4 comes simply from the fact that $dim(\hat{\Phi}_3) = 0$ due to eq.(2.1.21). The dimensions of the spaces $\hat{\Lambda}_s$ and $\hat{\Phi}_s$ can be computed by using the generating functions

$$\sum_{s=0}^{\infty} q^s dim(\hat{\Lambda}_s) = (1-q) q^{c/24} \chi_{(1,1)}(q) + q \tag{2.1.24}$$

and

$$q^{\Delta_{(n,m)}}\sum_{s=0}^{\infty} q^s dim(\hat{\Phi}_s) = (1-q)q^{c/24}\chi_{(n,m)}(q) \ , \tag{2.1.25}$$

where $\chi_{(1,1)}(q)$ and $\chi_{(n,m)}(q)$ are the Virasoro character functions of the identity operator $\Phi_{(1,1)}$ and the perturbing operator $\Phi_{(n,m)}$. Such functions are defined in the following way:

$$\chi_{(r,s)}(q) = q^{-c/24}P(q)\sum_{k=-\infty}^{+\infty}\left(q^{\Delta_{2pk+r,s}} - q^{\Delta_{2pk+r,-s}}\right) \tag{2.1.26}$$

where

$$P(q) = \prod_{n=1}^{\infty}(1-q^n)^{-1} \ , \tag{2.1.27}$$

and p refers to the model $\mathcal{M}_p$. Using eqs.(2.1.24) and (2.1.25) we can establish for which values of s the condition $dim(\hat{\Lambda}_{s+1}) > dim(\hat{\Phi}_s)$ is satisfied. This *counting argument* permits to show [22] that the theory obtained perturbing a unitary minimal model $\mathcal{M}_p$ by the operator $\Phi_{(n,m)}$ (whose action we denote by $H_p^{(n,m)}$) contains integrals of motion P_s with the following spins:

$$\begin{aligned} s &= 1,3,5,7 && \text{for } H_p^{(1,3)} \\ s &= 1,5,7,11 && \text{for } H_p^{(1,2)}, H_p^{(2,1)} \ ; \ p \geq 5 \ . \end{aligned} \tag{2.1.28}$$

The counting argument turns out to be not valid for values of s larger than those contained in (2.1.28). Nevertheless, it is generally believed these first few conserved charges are only the first representatives of an infinite series. This amounts to assume the complete *integrability* of the theories proposed in this section and gives rise to a series of important developments. The theories $H_p^{(1,3)}$ with $g < 0$, $H_p^{(1,2)}$ and $H_p^{(2,1)}$ are expected to develop a finite correlation length $\xi \sim |g|^{-1/\epsilon}$ (we showed in sec. 1.6 that $H_p^{(1,3)}$ remains on the critical surface for $g > 0$). Therefore they will contain only massive particles and will be characterized by an S-matrix.

2.2 Factorized S-matrix

While in four-dimensional space-time the existence of non-trivial integrals of motion is known to force the S-matrix to coincide with identity [23], in two dimensions this circumstance leads to particularly simple (but non-trivial) scattering theories exhibiting many interesting properties [24].

Let's consider a relativistic scattering theory containing N types of particles with masses m_a, $a = 1, \ldots, N$. We will denote $a(p)$ the particle a with two-momentum p_μ satisfying the mass-shell constraint $p_\mu p^\mu = m_a^2$. We also define the light-cone components of p^μ

$$p = p^0 + p^1 \ ; \qquad \bar{p} = p^0 - p^1 \ . \tag{2.2.1}$$

The states

$$|a_1(p_1)a_2(p_2)\ldots a_n(p_n) >_{in(out)} \tag{2.2.2}$$

form the basis of the asymptotic in (out)-states which is assumed to be complete in a local field theory. The S-matrix is the operator connecting these two sets of asymptotic states. Let's assume that the theory possesses an infinite number of integrals of motion P_s of type (2.1.2) with different spin s, whose action on the asymptotic states (2.2.2) is given by

$$P_s|a_1(p_1)\dots a_n(p_n)>_{in(out)}=\sum_{i=1}^{n}\omega_s^{a_i}(p_i)|a_1(p_1)\dots a_n(p_n)>_{in(out)} \quad . \tag{2.2.3}$$

The additivity of the contributions of the largely separated particles contained in asymptotic states follows from the locality of the conservation law expressed by the fact that P_s is the integral of a local density. Due to eq.(2.1.3), the eigenvalues $\omega_s^a(p)$ have the general form

$$\omega_s^a(p)=k_s^a p^s \ , \tag{2.2.4}$$

where p is defined in eq.(2.2.1) and k_s^a are constants.

If we now consider a generic scattering process in which the state $|a_1(p_1)\dots a_n(p_n)>_{in}$ evolves into the final state $|b_1(q_1)\dots b_m(q_m)>_{out}$, conservation of charges P_s implies

$$\sum_{i=1}^{n}\omega_s^{a_i}(p_i)=\sum_{j=1}^{m}\omega_s^{b_j}(q_j) \quad . \tag{2.2.5}$$

Since we required the existence of infinite charges with different spin, (2.2.5) is a system of infinite equations for a finite number of unknowns which will be satisfied in general only if $n=m$ and the set of initial two-momenta $\{p_1,\dots,p_n\}$ equals the set of final two-momenta $\{q_1,\dots,q_m\}$. In other words, the theory in question admits only elastic scattering processes (there is not particle production) in which each initial two-momentum is individually conserved (this automatically implies the conservation of the number of particles with a given mass). All this does not mean that the scattering is trivial since, for instance, if the initial state contains particles with the same mass, they can exchange momenta in the final state, or be replaced by other particles with the same mass*.

Another important property enjoyed by the scattering theories we are discussing is the complete factorization of multiparticle S-matrix into a product of two-particle S-matrices. Indeed, if we consider as an example the collision process of three particles with spatial momenta $k_1>k_2>k_3$, there are the three possible space-time diagrams depicted in fig. 2. Let's now apply to the initial state the operator $\exp(iaP_s)$, with $s>1$ and a some real parameter. Since the charge is locally conserved and the particles are initially widely separated, the exponential operator will act separately on the wave packet of the i-th particle which we write in the form

$$|\psi_i(x)>=|\int_{-\infty}^{+\infty}dk\ \exp[ik(x-x_i)]\,f(k)> \ , \tag{2.2.6}$$

*Actually, we have to say that it can be shown [25] that to reach the same conclusions it is enough to require the existence of one non-trivial ($s>1$) conserved charge plus analyticity.

where $f(k)$ is any reasonable function peaked at $k = k_i$. We have

$$\begin{aligned} |\psi'_i(x)> &\equiv \exp(iaP_s)|\psi_i(x)> \\ &= |\int_{-\infty}^{+\infty} dk \, \exp[ia\omega_s(k)] \exp[ik(x - x_i)] \, f(k) > \quad . \end{aligned} \quad (2.2.7)$$

By stationary phase approximation, we see that $\psi_i(x)$ is peaked at x_i, while ψ'_i is peaked around the value

$$x = x_i - a \left[\frac{d}{dk} \omega_s(k) \right]_{k=k_i} \quad . \quad (2.2.8)$$

Then, recalling eq.(2.2.4), we conclude that the exponential operator with $s > 1$ displaces the "centre of mass" of the i-th wave packet by an amount depending on the mean momentum k_i, so that taking the parameter a sufficiently large we can alter the relative position between any two particles with different momenta [26]. As a consequence, we can pass in fig. 2 from a diagram to another applying $\exp[iaP_s]$ to the initial state. But, since this operator commutes with the S-matrix, we conclude that the amplitudes for the three possible sequences of collisions in fig. 2 are equal and factorize in the product of three two-body S-matrices. With obvious notations

$$S(123) = S(23)S(13)S(12) = S(12)S(13)S(23) \quad . \quad (2.2.9)$$

These cubic relations, also known as *star-triangle* or *Yang-Baxter* equations, are trivially satisfied if the $S(ij)$ are ordinary functions, but impose on the contrary serious constraints to the two-body S-matrix in the case the spectrum of the theory contains particles with the same mass differing for quantum numbers related to some internal symmetry.

Generalizing the displacement argument just illustrated one can see that any N-particle S-matrix element can be written in factorized form, in such a way that the problem of obtaining the complete S-matrix for a two-dimensional scattering theory possessing non-trivial integrals of motion reduces to the determination of the two-body S-matrix. Exploiting relations (2.2.9) and the usual requirements of unitarity, analyticity and crossing symmetry, this task has been successfully completed for several models.

2.3 Purely Elastic Scattering Theories

The integrable deformations of CFTs discussed in sec. 2.1 have led to diagonal factorizable S-matrix theories, namely to theories in which the final state contains exactly the same particles, with the same momenta and the same quantum numbers, than the initial state [27, 22, 28, 29, 30, 31, 32]. As a consequence, the S-matrix element S_{ab} describing the scattering of particles a and b reduces simply to a pure phase $e^{i\delta_{ab}}$ expressing the time delay (or advance) with respect to the free case and the cubic identities eq.(2.2.9) are trivially satisfied. To these diagonal, or *purely elastic*, scattering theories we will restrict our attention in the following.

We define the two-particle S-matrix element in terms of the relation

$$|a(\theta_a)b(\theta_b)>_{out}= S_{ab}(\theta_a,\theta_b)|a(\theta_a)b(\theta_b)>_{in} \ , \tag{2.3.1}$$

where we introduced the convenient parametrization of two-momenta in terms of the *rapidity* variable θ:

$$p^0 = m\cosh\theta \ ; \qquad p^1 = m\sinh\theta \ . \tag{2.3.2}$$

In terms of the Mandelstam variable $s = (p_a + p_b)^2$, $S_{ab}(s)$ has two cuts along the real axis of the complex s-plane for $s \geq (m_a + m_b)^2$ and $s \leq (m_a - m_b)^2$. These cuts, required by two-particle unitarity, are of purely kinematical origin, corresponding to the opening of the *direct* channel ($ab \to ab$) and of the *crossed* channels ($a\bar{b} \to a\bar{b}$, $\bar{a}b \to \bar{a}b$) respectively. They are the only cuts in the s-plane, due to the absence of particle production. They can be eliminated considering S_{ab} as a function of the relative rapidity $\theta_{ab} = \theta_a - \theta_b$ and using the relation

$$s(\theta_{ab}) = m_a^2 + m_b^2 + 2m_a m_b\cosh\theta_{ab} \tag{2.3.3}$$

to map the s-plane onto the θ-plane as depicted in fig. 3. The *first sheet* of the s-plane is mapped into the *physical strip* $0 < Im\,\theta_{ab} < \pi$ while the second Riemann sheet is mapped into the strip $-\pi < Im\,\theta_{ab} < 0$ (this structure of the θ-plane is obviously $2\pi i$-periodic). Since to poles in the S-plane correspond poles in the θ-plane, we will treat $S_{ab}(\theta)$ as a meromorphic function of θ. The assumption of real analyticity in the s-plane, $S_{ab}(s) = S^*_{ab}(s^*)$, translates into $S_{ab}(\theta) = S^*_{ab}(-\theta^*)$, so that the unitarity condition $SS^\dagger = 1$ reduces, for real values of θ, to

$$S_{ab}(\theta)S_{ab}(-\theta) = 1 \ , \tag{2.3.4}$$

and can be extended to the entire θ-plane by analytic continuation.

Since $S_{ab}(\theta) = S_{ba}(\theta)$ (we consider parity invariant theories) and $S_{ab}(\theta) = S_{\bar{a}\bar{b}}(\theta)$ due to charge-conjugation symmetry, the crossing symmetry relation $S_{a\bar{b}}(s) = S_{\bar{a}b}(s) = S_{ab}(2m_a^2 + 2m_b^2 - s)$ becomes simply

$$S_{a\bar{b}}(\theta) = S_{\bar{a}b}(\theta) = S_{ab}(i\pi - \theta) \ . \tag{2.3.5}$$

From eqs.(2.3.4) and (2.3.5) we obtain the relation $S_{ab}(\theta + 2\pi i)S_{ab}(-\theta) = 1$ showing that (compare with eq.(2.3.4)) $S_{ab}(\theta)$ should be a $2\pi i$-periodic function of θ.

In a local quantum field theory we expect scattering amplitudes to be polinomially bounded in the momenta (this is a consequence of Wightman axioms [33]). It can be shown [34] that this condition forces any meromorphic , real analytic, $2\pi i$-periodic function $f(\theta)$ satisfying the unitarity condition (2.3.4) to be of the form

$$f(\theta) = \prod_{\alpha\in A} f_\alpha(\theta) \ , \tag{2.3.6}$$

where f_α are the so-called *building blocks*,

$$f_\alpha(\theta) = \frac{\sinh\frac{1}{2}(\theta + i\alpha\pi)}{\sinh\frac{1}{2}(\theta - i\alpha\pi)} \ , \tag{2.3.7}$$

and A is a set of complex numbers invariant under complex conjugation.

We consider theories whose spectrum contains only stable particles to which correspond poles in the interval $(m_a - m_b)^2 < s < (m_a + m_b)^2$ of the real axis in the s-plane. Since this interval is mapped onto the imaginary axis in the θ-plane, we can take every α real and $-1 < \alpha \leq 1$. The building block $f_\alpha(\theta)$ has a simple pole at $\theta = i\alpha\pi$ and a simple zero at $\theta = -i\alpha\pi$, and has the useful properties

$$f_\alpha(\theta) = f_{\alpha+2}(\theta) = f_{-\alpha}(-\theta) \ , \tag{2.3.8}$$

$$f_\alpha(\theta) f_{-\alpha}(\theta) = 1 \ , \tag{2.3.9}$$

$$f_\alpha(i\pi - \theta) = -f_{1-\alpha}(\theta) \tag{2.3.10}$$

$$f_\alpha(\theta - i\pi\beta) f_\alpha(\theta + i\pi\beta) = f_{\alpha-\beta}(\theta) \ , \tag{2.3.11}$$

$$f_0(\theta) = -f_1(\theta) = 1 \tag{2.3.12}$$

If at least one of the particles a and b is its own antiparticle, the crossing invariance relation (2.3.5) reduces to

$$S_{ab}(\theta) = S_{ab}(i\pi - \theta) \ , \tag{2.3.13}$$

implying that, up to a sign, $S_{ab}(\theta)$ must be a product of functions of the form

$$F_\alpha(\theta) = f_\alpha(\theta) f_\alpha(i\pi - \theta) = \frac{\tanh \frac{1}{2}(\theta + i\alpha\pi)}{\tanh \frac{1}{2}(\theta - i\alpha\pi)} \ . \tag{2.3.14}$$

If $S_{ab}(\theta)$ has a simple pole at $\theta = iu_{ab}^c$ in the direct channel, we say that the particle c with mass

$$m_c^2 = s(iu_{ab}^c) = m_a^2 + m_b^2 + 2 m_a m_b \cos u_{ab}^c \tag{2.3.15}$$

is a *bound state* of a and b. The last equation can be geometrically interpreted in terms of a triangle of sides m_a^2, m_b^2 and m_c^2, so that

$$u_{ab}^c + u_{\bar{c}a}^{\bar{b}} + u_{b\bar{c}}^{\bar{a}} = 2\pi \ . \tag{2.3.16}$$

In the vicinity of this singularity, the S-matrix element is given by

$$S_{ab}(\theta) \sim \frac{i g_{abc}^2}{\theta - iu_{ab}^c} \ , \tag{2.3.17}$$

and can be diagrammatically represented as in fig. 4, where the couplings g_{abc} are supposed to be totally symmetric in a, b and c. To this direct channel pole corresponds, through the crossing symmetry relation (2.3.5), a pole at $\theta = i\bar{u}_{ab}^c \equiv i(\pi - u_{ab}^c)$ for the matrix elements $S_{a\bar{b}}(\theta) = S_{\bar{a}b}(\theta)$ of the crossed processes. A direct-channel pole can be distinguished from a crossed-channel one through the sign of the residue, respectively positive and negative in the two cases. For the discussion of this point and many other details, such as higher order poles, we refer the reader to the review article ref. [35] and references therein.

The construction of the S-matrix for factorizable scattering theories in two dimensions can be performed using the so-called *bootstrap principle* [36]. According to this, one uses the scattering amplitudes, supposed to be known, of a small number of *fundamental particles* to obtain the scattering amplitudes of the bound states of these particles through a *fusion procedure*; this procedure is then applied to the bound states of the bound states and stops only if the bootstrap closes on a finite number of particles.

To be concrete, let's return to the case of purely elastic scattering theories discussed in previous section and suppose for simplicity that all the particles of the theory are self conjugated. Consider a particle c appearing as a direct-channel bound state at $\theta_{ab} = iu^c_{ab}$ in the scattering of particles a and b. Now, if we consider the three-particle amplitude

$$S_{abd}(\theta_a, \theta_b, \theta_d) = S_{ab}(\theta_{ab}) S_{ad}(\theta_{ad}) S_{bd}(\theta_{bd}) \tag{2.3.18}$$

for $\theta_{ab} = iu^c_{ab}$, we can use the displacement argument of sec. 2.2 to obtain the relation diagrammatically represented in fig. 5 and corresponding to the bootstrap equation

$$S_{cd}(\theta) = S_{ad}(\theta + iu^b_{ac}) S_{bd}(\theta - i\bar{u}^a_{bc}) \quad . \tag{2.3.19}$$

The fact that for suitable imaginary values of the rapidities particles a and b give rise to the bound state c can be expressed writing

$$|c(\theta)> = \lim_{\varepsilon \to 0} \varepsilon |a(\theta + i\bar{u}^b_{ac} - \varepsilon) b(\theta - i\bar{u}^a_{bc} + \varepsilon)> \quad . \tag{2.3.20}$$

From eq.(2.2.3) follows that, acting on both sides of the last equation with the conserved charge P_s, we obtain [22]

$$k^a_s m^s_a e^{-is\bar{u}^b_{ac}} + k^b_s m^s_b e^{is\bar{u}^a_{bc}} = k^c_s m^s_c \ , \tag{2.3.21}$$

where eqs.(2.2.1), (2.2.4) and (2.3.2) were used. This can be considered as a system of equations for the constants k^a_s which provide significant limitations on the possible values of the spin s. Note that the trivial solution $k^a_s = 0$ for any a and s should be rejected since it would imply the absence of integrals of motion and this is in contradiction with factorization.

As an example consider the case $a = b$ and $k^a_s \neq 0$. Then equation (2.3.21) reduces to

$$\frac{k^c_s}{k^a_s} = 2\cos(s\bar{u}^a_{ac}) \quad . \tag{2.3.22}$$

If $a = b = c$, the above equation has an unique solution

$$\bar{u}^a_{aa} = \frac{\pi}{3} \ , \qquad s = 1, 5 \, (mod. \, 6) \quad . \tag{2.3.23}$$

The non-vanishing of the quantity u^a_{aa} means that the particle a appears as bound state of itself. This is referred to as "ϕ^3-property".

Very interesting applications of the bootstrap principle and of equation (2.3.21) can be found in ref. [22].

2.4 The 3-state Potts Model

As an application of many of the ideas exposed so far we present in this section some results concerning the 3-state Potts Model [37, 27].

Generally speaking the q-state Potts model (see,e.g., [38]) is the lattice model in which the "spin" variable σ_x associated to the generic site x can assume q different values while the nearest neighbour interaction energy takes two possible values, distinguishing the case when the two states are identical (favoured case) from the one when they differ. For $q = 2$ one simply recovers the Ising model. It is clear from the definition that the model has S_q-invariance, S_q being the permutation group of q elements.

In the $q = 3$ case the partition function can be written in the form

$$\begin{aligned} Z(\beta) &= \sum_{\{\sigma\}} \exp\left\{\beta \sum_{(x,x')} \frac{1}{2}(\sigma_x\bar{\sigma}_{x'} + \bar{\sigma}_x\sigma_{x'})\right\} \\ &= \sum_{\phi_x=0,\pm 2\pi/3} \exp\left\{\beta \sum_{(x,x')} \cos(\phi_x - \phi_{x'})\right\} , \end{aligned} \tag{2.4.1}$$

where the sum in the exponent is intended over nearest neighbours and the spin variables $\sigma \equiv \exp(i\phi)$ and $\bar{\sigma} \equiv \exp(-i\phi)$ take as values the cubic roots of identity. Looking at this partition function one immediately realizes that the global symmetry group S_3 can be regarded as the product of the two abelian subgroups $\mathbf{Z}_3$ (with generator $\mathbf{\Omega}$, $\mathbf{\Omega}^3 = E$) and $\mathbf{Z}_2$ (with generator $\mathbf{C}$, $\mathbf{C}^2 = E$) whose action on the spin variables is defined by ($\omega \equiv \exp(2\pi i/3)$)

$$\mathbf{\Omega}\sigma = \omega\sigma , \qquad \mathbf{\Omega}\bar{\sigma} = \omega^{-1}\bar{\sigma} \tag{2.4.2}$$

$$\mathbf{C}\sigma = \bar{\sigma} . \tag{2.4.3}$$

We will call the subgroup $\mathbf{Z}_2$ "charge conjugation".

The partition function of the two-dimensional q-state Potts model obeys a duality relation with a fixed point

$$\beta_c = \frac{2}{3}\ln(1 + \sqrt{q}) \tag{2.4.4}$$

(this is a generalization of the well known duality property of the Ising model). For $q > 4$ there is a first order phase transition while for $q \leq 4$ the transition is continuous [39] and at the corresponding critical point the system can be described by a conformal invariant theory. Some critical exponents for the 3-state Potts model where obtained by Baxter [40] from the assumption that this model is in the same universality class as the hard hexagon model which he solved. These informations permit to compute the conformal dimensions of the operators $\sigma(x)$ and $\bar{\sigma}(x)$, which are the scaling limits of the lattice spin variables, and of the energy operator $\varepsilon(x)$, which results from the scaling limit of the the interaction term $\sigma_x\bar{\sigma}_{x'} + \bar{\sigma}_x\sigma_{x'}$ in the partition function. Such dimensions are $\Delta_\sigma = \Delta_{\bar{\sigma}} = 1/15$ and $\Delta_\varepsilon = 2/5$. Scanning the Kac's tables for the unitary minimal models one finds that these two conformal weights are present in the conformal grid of the model $\mathcal{M}_5$ ($c = 4/5$). The complete conformal grid for this model is shown in table 1.

It was already mentioned in sec. 1.3 that there exist two modular invariant partition functions corresponding to the value $p = 5$. The first one is the diagonal solution which allows only for the presence of scalar primary fields appearing just once. The second partition function belong to the A-D series of the ADE classification and reads

$$\begin{aligned} Z &= |\chi_{1,1} + \chi_{1,4}|^2 + |\chi_{1,2} + \chi_{1,3}|^2 + 2|\chi_{3,3}|^2 + 2|\chi_{3,4}|^2 \\ &= |\chi_0 + \chi_3|^2 + |\chi_{2/5} + \chi_{7/5}|^2 + 2|\chi_{1/15}|^2 + 2|\chi_{2/3}|^2 \end{aligned} \tag{2.4.5}$$

(here we used first the notation $\chi_{n,m}$, then the notation $\chi_{\Delta_{(n,m)}}$). We see that this second solution contains two scalar fields of conformal dimensions $(1/15, 1/15)$ in perfect agreement with the presence of two spin variables σ and $\bar{\sigma}$ in the partition function (2.4.1). Therefore we conclude that eq.(2.4.5) is the correct choice for the partition function of the 3-state Potts model.

Let's consider the other primary operators which, according to eq.(2.4.5) are present in the theory (we will adopt the notation $\Phi_{(\Delta,\bar{\Delta})}$ for the field with conformal dimensions $(\Delta, \bar{\Delta})$). The transformation properties of such operators under the group S_3 were determined in refs. [41, 42]. It turns out that the doublet of fields $\Phi_{(2/3,2/3)}$ represents, as the doublet $\Phi_{(1/15,1/15)}$, a basis for a two-dimensional representation of S_3. The remaining scalar fields $\Phi_{(2/5,2/5)} = \varepsilon$, $\Phi_{(7/5,7/5)}$ and $\Phi_{(3,3)}$ are invariant under S_3 transformations. In conclusion we have the two pairs of fields $V \equiv \Phi_{(7/5,2/5)}$, $\bar{V} \equiv \Phi_{(2/5,7/5)}$ and $W \equiv \Phi_{(3,0)}$, $\bar{W} \equiv \Phi_{(0,3)}$ which are the left and right light-cone components of a spin 1 field and a spin 3 field, respectively; they have zero $\mathbf{Z}_3$ charge but change sign under $\mathbf{C}$ conjugation.

All these properties are in complete agreement with a Landau-Ginzburg description of the 3-state Potts model based on a complex scalar field ϕ with Lagrangian

$$\mathcal{L} = (\partial_\mu \phi)^2 + \lambda(\phi^3 + \phi^{*3}) \quad . \tag{2.4.6}$$

Then the spin density doublets $\Phi_{(1/15,1/15)}$ and $\Phi_{(2/3,2/3)}$ may be identified with the pairs ϕ, ϕ^* and $\phi^*\phi^2, \phi^{*2}\phi$ respectively. In addition the energy-like operators $\Phi_{(2/5,2/5)}$ and $\Phi_{(7/5,7/5)}$ correspond to the S_3-invariant combinations $\phi^*\phi$ and $\phi^3 + \phi^{*3}$. Finally the operators with spins 1 and 3 are associated to $(\phi^*\partial\phi - \phi\partial\phi^*)$ and $(\phi^*\partial^3\phi - \phi\partial^3\phi^*)$ respectively.

Only a subset of the anomalous dimensions contained in table 1 appears in the partition function (2.4.5). The remaining anomalous dimensions were related in ref. [42] to "disorder fields" which are non-local with respect to the subalgebra discussed above.

The fields W and $\bar{W}$ are spin 3 conserved currents since they satisfy the equations

$$\partial_{\bar{z}} W = 0 \ , \qquad \partial_z \bar{W} \quad . \tag{2.4.7}$$

showing that $W = W(z)$ and $\bar{W} = \bar{W}(\bar{z})$ (in the following we will consider only the holomorphic component W reminding that the same results hold for $\bar{W}$). An infinite set of generators W_n ($n = 1, \pm 1, \pm 2, \ldots$) for the associated symmetry can be introduced through a mode expansion analogous to that used to define the Virasoro generators L_n from the holomorphic component of the energy-momentum tensor, i.e.

$$W_n = \oint dz\, z^{n+2} W(z) \quad . \tag{2.4.8}$$

These operators form, together with the Virasoro generators, the so-called W-algebra [43].

The 3-state Potts model away from the critical point can be described by perturbing the fixed point Hamiltonian H_0 by the energy density operator $\varepsilon(x)$ which is the only relevant operator in the theory invariant under the group S_3. In other words we consider the Hamiltonian

$$H = H_0 + g \int d^2x\, \varepsilon(x) \ , \tag{2.4.9}$$

where the coupling constant g has conformal dimensions $(3/5, 3/5)$. Since in the notation of sec. 2.1 this Hamiltonian corresponds to $H_5^{(1,2)}$, we already know (see (2.1.28)) that an infinite set of integrals of motion P_s with spin $s = 1, 5, 7, 11, \ldots$ survives in the perturbed theory. These integrals of motion comes from the deformation of conserved currents belonging to the conformal family of the identity and have positive C-parity. On the other hand we have seen that the conformal limit of the theory under consideration contains also infinite conserved currents belonging to the conformal family of W (all the descendents obtained applying the operators L_{-n} to W) which have negative C-parity. In order to see if some of these currents remains conserved off-criticality, let's consider the first of equations (2.4.7). For $g \neq 0$ the right-hand side becomes non-zero and can be represented in the form of a perturbative series in g as in eq.(2.1.6). Using the dimensional counting discussed in sec. 2.1 one immediately sees that the series can contain only the linear term so that we write

$$\partial_{\bar{z}} W = gR \quad . \tag{2.4.10}$$

Here R should be a local field of dimensions $(12/5, 2/5)$ and negative C-parity belonging to the operator algebra of the theory. The only field with these properties is the derivative $\partial_z V$ where the field V was specified before. Thus eq.(2.4.10) can be rewritten in the form

$$\partial_{\bar{z}} W = \partial_z Q \ , \tag{2.4.11}$$

where Q is just $\partial_z V$ up to an appropriate numerical factor. Therefore the operator

$$P_2 = \oint dz\, W + d\bar{z}\, Q \tag{2.4.12}$$

is an integral of motion for the theory having spin 2 and negative C-parity. In ref. [27] integrals of motion with negative C-parity and spin 4 and 8 were also constructed and it was argued that, together with the positive C-parity representatives previously mentioned, they form the first members of an infinite series of conserved charges P_s whose spin s takes all positive values such that $s \neq 0 \pmod 3$.

The existence of these integrals of motion implies the factorizability of the S-matrix associated to the theory defined by the action (2.4.9). Such S-matrix can be constructed associating to the fields σ and $\bar{\sigma}$ the particle and antiparticle a and $\bar{a}$, respectively. They have the same mass m and form a basis of a two-dimensional representation of S_3, i.e.

$$\Omega a = \omega a \ , \qquad \Omega \bar{a} = \omega^{-1} \bar{a} \tag{2.4.13}$$

$$\mathbf{C} a = \bar{a} \tag{2.4.14}$$

Then the most general two-particle S-matrix is given by the relations

$$|a(\theta_1)a(\theta_2) >_{\text{out}} \ = \ u(\theta_{12})|a(\theta_1)a(\theta_2) >_{\text{in}} \ , \tag{2.4.15}$$

$$|a(\theta_1)\bar{a}(\theta_2) >_{\text{out}} \ = \ t(\theta_{12})|a(\theta_1)\bar{a}(\theta_2) >_{\text{in}} + r(\theta_{12})|\bar{a}(\theta_1)a(\theta_2) >_{\text{in}} \ . \tag{2.4.16}$$

The selection rules coming from the conservation of P_2 can be used to show that the reflection amplitude $r(\theta)$ in eq.(2.4.16) must vanish. Indeed the one-particle states $|a(\theta) >$ and $|\bar{a}(\theta) >$ should be eigenstates of P_2 with opposite sign eigenvalues to take in account the fact that P_2 has negative C-parity. Then, according to the results of previous sections, we can write

$$P_2|a(\theta) > \ = \ kme^{2\theta}|a(\theta) > \ , \tag{2.4.17}$$

$$P_2|\bar{a}(\theta) > \ = \ -kme^{2\theta}|\bar{a}(\theta) > \ . \tag{2.4.18}$$

Recalling that P_2 commutes with the S-matrix and using eq.(2.2.3) to compute the result of the action of P_2 on both sides of eq.(2.4.16), one immediately concludes that it should be $r(\theta) = 0$ so that the S-matrix becomes diagonal. The remaining amplitudes $u(\theta)$ and $t(\theta)$ are subject to the constraints coming from unitarity

$$t(\theta)t(-\theta) = 1 \ , \qquad u(\theta) = u(-\theta) = 1 \tag{2.4.19}$$

and crossing symmetry

$$t(\theta) = u(i\pi - \theta) \ . \tag{2.4.20}$$

The minimal $\mathbf{Z}_3$-symmetric solution for these equations was proposed in ref. [44] where the S-matrices for general $\mathbf{Z}_n$-symmetric models were constructed. It reads

$$u(\theta) = \frac{\sinh \frac{1}{2}(\theta + 2\pi i/3)}{\sinh \frac{1}{2}(\theta - 2\pi i/3)} \ . \tag{2.4.21}$$

The pole at $\theta = 2\pi i/3$ in this amplitude is interpreted as an antiparticle $\bar{a}$ appearing as a bound state in aa channel. This is in agreement with the constraints provided by the conservation of P_2. Indeed, in the vicinity of the pole one can write

$$|a(p_1)a(p_2) > \sim |\bar{a}(p_1 + p_2) > \ . \tag{2.4.22}$$

Applying P_2 to both sides of this relation we get the equation

$$mk(e^{2\theta_1} + e^{2\theta_2}) = -mk(e^{\theta_1} + e^{\theta_2})^2 \tag{2.4.23}$$

which is satisfied only for $\theta_1 - \theta_2 = \frac{2\pi i}{3}$.

Chapter 3

Off-Shell Properties from S-matrix

3.1 Form Factors

As we have seen in the previous chapter, the factorization of the scattering characterizing integrable two-dimensional models allows in many cases the construction of the exact S-matrix. An important point in the scheme described so far is represented by the problem of passing from the on-shell data provided by the relativistic scattering theory associated to an integrable model to off-shell quantities, like correlation functions of local operators. This would give, in particular, the possibility of going back from S-matrix data to the conformal theory describing the ultraviolet limit. Unfortunately, apart from a single exception concerning the scaling Ising model [45], there is no known method to compute correlation functions in explicit form. A very important step in this direction was made in a series af papers [46] in which was shown that the factorization of S-matrix, supported by the usual properties of analyticity, crossing symmetry and unitarity, allows for many integrable theories the explicit reconstruction of matrix elements of local operators between asymptotic states. The knowledge of such matrix elements immediately gives the possibility to write down a correlation function as an infinite sum over multiparticle intermediate states. At this point one can, in principle, hope to sum up the series or, more pragmatically, use partial sums if the convergence of the series is fast enough.

At the base of this approach there is the assumption about the existence for the integrable model under consideration of a set of operators $V_a^\dagger(\theta_a)$, $V_a(\theta_a)$ (the index a numerates the particles preset in the theory) of creation and annihilation type, which satisfy the following associative algebra generalizing fermionic and bosonic algebras (we suppose for notational convenience the theory to be purely elastic):

$$\begin{aligned} V_a(\theta_a)V_b(\theta_b) &= S_{ab}(\theta_{ab})V_b(\theta_b)V_a(\theta_a) \,, \\ V_a^\dagger(\theta_a)V_b^\dagger(\theta_b) &= S_{ab}(\theta_{ab})V_b^\dagger(\theta_b)V_a^\dagger(\theta_a) \,, \qquad (3.1.1) \\ V_a(\theta_a)V_b^\dagger(\theta_b) &= S_{ab}(\theta_{ab})V_b^\dagger(\theta_b)V_a(\theta_a) + 2\pi\delta_{ab}\delta(\theta_{ab}) \,. \qquad (3.1.2) \end{aligned}$$

The role played by S-matrix clearly indicates that each commutation of these operators can be interpreted as a scattering process. If we call $L(\varepsilon)$ and T_y the generators of Lorentz transformations and translations respectively, the operators $V_a^\dagger$ and V_a are expected to transform in the following way under the Poincare' group (recall the definition (2.3.2) of the rapidity variable θ):

$$U_L V_a(\theta) U_L^{-1} = V_a(\theta + \varepsilon) \ , \tag{3.1.3}$$

$$U_{T_y} V_a U_{T_y}^{-1} = e^{ip_\mu(\theta) y^\mu} V_a(\theta) \ . \tag{3.1.4}$$

We identify with the vacuum of the theory the state $|0> = |0>_{in(out)}$ annihilated by the operators $V_a(\theta)$:

$$V_a(\theta)|0> = 0 = <0|V_a^\dagger(\theta) \ . \tag{3.1.5}$$

The other states of Hilbert space can then be generated by successive action of the creation operators $V_a^\dagger(\theta)$ on the vacuum:

$$|a_1(\theta_1)\ldots a_n(\theta_n)> \equiv V_{a_1}^\dagger(\theta_1)\ldots V_{a_n}^\dagger(\theta_n)|0> \ . \tag{3.1.6}$$

According to eq.(3.1.2), the normalization of one-particle states is

$$<a(\theta_a)|b(\theta_b)> = 2\pi\delta_{ab}\delta(\theta_{ab}) \ . \tag{3.1.7}$$

It is evident from the commutation rules (3.1.2) that the states generated in this way are not all linearly independent, but linearly independent sets can be selected giving, for example, ordering prescriptions on the rapidities. Then we choose [24] as a basis for the in-states those which are ordered with increasing rapidities

$$\theta_1 > \ldots > \theta_n \ , \tag{3.1.8}$$

and as a basis for the out-states those with decreasing rapidities

$$\theta_1 < \ldots < \theta_n \ . \tag{3.1.9}$$

Let's now consider a hermitean local field $\mathcal{O}(x)$, which for simplicity we will suppose to be scalar, and define the form factors of this operator as

$$F_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n) = <0|\mathcal{O}(0)|a_1(\theta_1)\ldots a_n(\theta_n)>_{in} \tag{3.1.10}$$

(see fig. 6 for a diagrammatic representation). The x-dependence can be easily introduced by the shift $\mathcal{O}(x) = U_{T_x}\mathcal{O}(0)U_{T_x}^{-1}$ and using eq.(3.1.4), while the matrix element between generic asymptotic states can be obtained from eq.(3.1.10) exploiting crossing symmetry. Indeed, by analytic continuation of several θ's up to the line $Im\theta = i\pi$ the corresponding two-momenta are reversed and we can write

$$\begin{aligned} {}_{out}<b_1(\theta'_1)\ldots b_m(\theta'_m)|\mathcal{O}(0)|a_1(\theta_1)\ldots a_n(\theta_n)>_{in} \\ = F_{\bar{b}_1\ldots\bar{b}_m a_1\ldots a_n}(\theta'_1 + i\pi,\ldots,\theta'_m + i\pi,\theta_1,\ldots,\theta_n) \ . \end{aligned} \tag{3.1.11}$$

Two dimensional Lorentz covariance implies, for a generic operator of spin s, the relation

$$F_{a_1\ldots a_n}(\theta_1 + \Lambda,\ldots,\theta_n + \Lambda) = e^{s\Lambda}F_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n) \ , \tag{3.1.12}$$

showing that the form factors (3.1.10) are functions of the differences of rapidities θ_{ij}. We expect they to be analytic in the strip $0 < Im\,\theta_{ij} < 2\pi$ except for simple poles.

The form factors also satisfy a set of equations, known as Watson's equations [47], which in two dimensions assume a particularly simple form. The first one is an immediate consequence of the algebra (3.1.2) and of the definition (3.1.10) and reads

$$F_{a_1\dots a_i a_{i+1}\dots a_n}(\theta_1,\dots,\theta_i,\theta_{i+1},\dots,\theta_n) = S_{a_i a_{i+1}}(\theta_i-\theta_{i+1})F_{a_1\dots a_{i+1}a_i\dots a_n}(\theta_1,\dots,\theta_{i+1},\theta_i,\dots,\theta_n) \quad (3.1.13)$$

The second Watson's equation is obtained by considering the analytic continuation $\theta_1 \to \theta_1 + 2\pi i$ and takes into account the monodromy of the form factor as a function of the rapidities. Indeed, to the proposed substitution does not correspond any variation from the kinematical point of view but the presence of cuts at $\theta_{ij} = 2\pi i$ gives rise to the following relation [46]:

$$\begin{aligned} F_{a_1\dots a_n}(\theta_1+2\pi i,\theta_2,\dots,\theta_n) &= F_{a_2\dots a_n a_1}(\theta_2,\dots,\theta_n,\theta_1) \\ &= \prod_{i=2}^{n} S_{a_i a_1}(\theta_i-\theta_1)F_{a_1\dots a_n}(\theta_1,\dots,\theta_n)\,. \end{aligned} \quad (3.1.14)$$

In the case $n = 2$ Watson's equations specializes to

$$F_{a_1a_2}(\theta) = F_{a_1a_2}(-\theta)S_{a_1a_2}(\theta)\ , \quad (3.1.15)$$

$$F_{a_1a_2}(i\pi-\theta) = F_{a_1a_2}(i\pi+\theta)\ . \quad (3.1.16)$$

The fundamental informations which allow in many cases the explicit reconstruction of form factors came from their pole structure in the strip $0 < Im\,\theta_i < 2\pi$ and from the recursive equations to which this structure gives rise. As a function of the rapidities variables the form factors have two kinds of simple poles. The first kind of singularity arises whenever two particles a and b among those appearing in eq.(1.1.6) have a bound state c at $\theta_{ab} = iu_{ab}^c$. Then, in the vicinity of this value of θ_{ab}, the form factor $F_{a_1\dots a_m ab}(\theta_1,\dots,\theta_m,\theta_a,\theta_b)$ can be represented as in fig. 7. Recalling eq.(2.3.17) we obtain

$$i\lim_{\varepsilon\to 0}\varepsilon F_{a_1\dots a_m ab}(\theta_1,\dots,\theta_m,\theta+i\bar{u}_{ab}^c-\frac{\varepsilon}{2},\theta-i\bar{u}_{ab}^c+\frac{\varepsilon}{2}) = g_{ab}^c F_{a_1\dots a_m c}(\theta_1,\dots,\theta_m,\theta)\ , \quad (3.1.17)$$

where we recall that $\bar{u}_{ab}^c \equiv \pi - u_{ab}^c$. This equation establishes a recursive structure between $n+1$- and n-particle form factors.

The poles of second type show up if there are particles and antiparticles among the particles entering the form factor and are of purely kinematical origin. Indeed, suppose that the particle a and the antiparticle $\bar{a}$ have rapidities differing by $i\pi$. Then, through the crossing symmetry relation (3.1.11), the incoming antiparticle $\bar{a}$ can be interpreted as the particle a in the final state with an unchanged value of the rapidity (see fig. 8). To this particular kinematical configuration corresponds a pole whose residue is given by the relation [46]

$$-i\lim_{\tilde{\theta}\to\theta}(\tilde{\theta}-\theta)F_{a_1\dots a_m\bar{a}a}(\theta_1,\dots,\theta_m,\tilde{\theta}+i\pi,\theta) = \left(1-\prod_{i=1}^{m}S_{aa_i}(\theta-\theta_i)\right)F_{a_1\dots a_m}(\theta_1,\dots,\theta_m)\ , \quad (3.1.18)$$

relating form factors with $n+2$ and n particles.

In ref. [46] equations (3.1.11-3.1.18) were used as a set of axioms to show that the operators defined by matrix elements between asymptotic states satisfy proper local commutativity relations.

In order to give some basic informations about the explicit construction of form factors, let's consider for simplicity the case in which the theory contains only one species of particles. Then, as was shown in ref. [48], the general solution of Watson's equations (3.1.13) and (3.1.14) can always be written in the form

$$F_n(\theta_1,\ldots,\theta_n) = K_n(\theta_1,\ldots,\theta_n)\prod_{i<j} F_{min}(\theta_{ij}) \ , \tag{3.1.19}$$

where $F_{min}(\theta)$ has the properties that it satisfies eqs.(3.1.16), is analytic in $0 \leq \mathrm{Im}\theta \leq \pi$, has no zeroes in $0 < \mathrm{Im}\theta < \pi$, and converges to a constant for large values of θ. These requirements uniquely determine $F_{min}(\theta)$ up to a normalization $\mathcal{N}$. In terms of the function $f(x)$ defined through the following integral representation of the two-particle S-matrix

$$S_2(\theta) = \exp\left[\int_0^\infty \frac{dx}{x}\, f(x) \sinh\left(\frac{x\theta}{i\pi}\right)\right] \ , \tag{3.1.20}$$

F_{min} can be written as [48]

$$F_{min}(\theta) = \mathcal{N} \exp\left[\int_0^\infty \frac{dx}{x}\, f(x) \frac{\sin^2 \frac{x\hat{\theta}}{2\pi}}{\sinh x}\right] \ , \tag{3.1.21}$$

where

$$\hat{\theta} \equiv i\pi - \theta \ . \tag{3.1.22}$$

Due to the properties of F_{min} the remaining factor K_n in eq.(3.1.19) should be a solution of Watson's equations with $S_2 = 1$. This implies that K_n is a completely symmetric, $2\pi i$-periodic function of the θ_i's. Moreover, it must contain all the physical poles expected in the form factor. In order to have a power-law bounded ultraviolet behaviour of the two-point function eq.(3.2.1) one has to require that the form factors behave asymptotically at most as $\exp(k\theta_i)$ as $\theta_i \to \infty$, k being a constant independent of i. This means that, once one extract from K_n the denominator which gives rise to the poles, the remaining part has to be a symmetric function of the variables $x_i \equiv e^{\theta_i}$ with a finite number of terms, i.e. a symmetric polynomial in the x_i's. A basis for the space of such polynomials is provided by the *elementary symmetric polynomials* $\sigma_k^{(n)}(x_1,\ldots,x_n)$ generated by [49]

$$\prod_{i=1}^{n}(x+x_i) = \sum_{k=0}^{n} x^{n-k}\sigma_k^{(n)}(x_1,\ldots,x_n) \ , \tag{3.1.23}$$

with the convention that the $\sigma_k^{(n)}$ with $k > n$ and $n < 0$ are zero. For the other cases one finds

$$\sigma_0 \ = \ 1$$

$$\begin{aligned}\sigma_1 &= x_1 + x_2 + \ldots + x_n \\ \sigma_2 &= x_1 x_2 + x_1 x_3 + \ldots + x_{n-1} x_n \\ &\vdots \\ \sigma_n &= x_1 x_2 \ldots x_n \quad .\end{aligned}$$

The $\sigma_k^{(n)}$ are homogeneous polynomials in x_i of total degree k and of degree one in each variable.

Providing this general scheme with the specific informations coming from the pole structure of the theory and using the recursive equations (3.1.17) and (3.1.18) the form factors for several models have been explicitly constructed (see refs. [50, 51, 52].

3.2 Expansion over Intermediate States

As already mentioned, the correlation functions of local operators can be written as an infinite series over multi-particle intermediate states. In particular, the two-point function of an hermitean operator $\mathcal{O}(x)$ in real Euclidean space is given by

$$\begin{aligned}< \mathcal{O}(x)\mathcal{O}(0) > &= \sum_{n=0}^{\infty} \int \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n} < 0|\mathcal{O}(x)|a_1(\theta_1)\ldots a_n(\theta_n) >_{in} \\ &\times \; {}_{in}< a_1(\theta_1)\ldots a_n(\theta_n)|\mathcal{O}(0)|0 > \\ &= \sum_{n=0}^{\infty} \int \frac{d\theta_1 \ldots d\theta_n}{n!(2\pi)^n} |F_{a_1\ldots a_n}(\theta_1,\ldots,\theta_n)|^2 \exp(-r\sum_{i=1}^{n} m_i \cosh \theta_i) \quad (3.2.1)\end{aligned}$$

where $r = \sqrt{x_0^2 + x_1^2}$ denotes the radial distance and the factor $1/n!$ is due to the ordering prescription over the rapidities previously discussed. All the integrals in eq.(3.2.1) are nonsingular and convergent. The series is expected to converge as well. Note that the n-particle term in eq.(3.2.1) behaves as $e^{-n(mr)}$ (m is the typical mass scale of the theory) so that the higher contributions are strongly suppressed in the infrared limit.

Equation (1.5.16) can be integrated to obtain the Zamolodchikov C-function as a function of the separation $r = |x|$ in the two-point correlator $< \Theta(x)\Theta(0) >$. The result is

$$C(R) = c_{IR} + \frac{3}{2}\int_R^{\infty} dr\, r^3 < \Theta(x)\Theta(0) > , \qquad (3.2.2)$$

where $c_{IR} = C(+\infty)$ vanishes for massive theories (the normalization of Θ was changed with respect to sec. 1.5). For $R = 0$ this formula just reproduces the Cardy's sum rule [53] expressing the total variation of the central charge in passing from the ultraviolet to the infrared limit:

$$\Delta c \equiv c_{IR} - c_{UV} = \frac{3}{2}\int_0^{\infty} dr\, r^3 < \Theta(x)\Theta(0) > \quad . \qquad (3.2.3)$$

This expression has been related in refs. [54, 55] to the spectral representation of the two-point function of Θ. Indeed defining the spectral density

$$c_1(\mu) = \frac{6}{\pi^2}\frac{1}{\mu^3}\text{Im}G(p^2 = -\mu^2) , \qquad (3.2.4)$$

where

$$G(p^2) = \int d^2x\, e^{-ipx} < \Theta(x)\Theta(0) > , \tag{3.2.5}$$

it turns out that one can write

$$\Delta c = \int_0^\infty d\mu\, c_1(\mu) \quad . \tag{3.2.6}$$

Inserting a complete set of in-states into eq.(3.2.4), one can express $c_1(\mu)$ in terms of the form factors F_n of the trace of the energy-momentum tensor:

$$\begin{aligned} c_1(\mu) &= \frac{12}{\mu^3}\sum_{n=1}^{\infty}\int \frac{d\theta_1\ldots d\theta_n}{(2\pi)^n n!}|F_n(\theta_1,\ldots,\theta_n)|^2 \\ &\times\delta\left(\sum_{i=1}^{n} m_i \sinh\theta_i\right)\delta\left(\sum_{i=1}^{n} m_i\cosh\theta_i - \mu\right) \end{aligned} \tag{3.2.7}$$

(the vacuum contribution should be subtracted in this approach). Inserting this expression in eq.(3.2.6) and exploiting the δ-functions we obtain the following result for the n-particle contribution to Δc:

$$\Delta c^{(n)} = \frac{12}{m^4}\int \frac{d\theta_1\ldots d\theta_{n-1}}{(2\pi)^n n!}\frac{|F_n(\theta_1,\ldots,\theta_n)|^2}{\cosh\theta_n(\sum_{i=1}^{n}\cosh\theta_i)^3}\,|_{sinh\theta_n=-\sum_{i=1}^{n-1}\sinh\theta_i} \quad . \tag{3.2.8}$$

For $n=1$ and $n=2$ eq.(3.2.8) simply gives

$$\Delta c^{(1)} = \frac{6}{\pi m^4}|F_1|^2 , \tag{3.2.9}$$

$$\Delta c^{(2)} = \frac{3}{8\pi^2 m^4}\int_0^\infty d\theta\,\frac{|F_2(2\theta)|^2}{\cosh^4\theta} \quad . \tag{3.2.10}$$

These formulas were used to test the convergence of the series over intermediate states for models whose form factors are known [55, 49]. It turns out that the first non-vanishing contribution (which is not necessarily $\Delta c^{(1)}$ but depends on internal symmetries of the model) already saturates the sum rule eq.(3.2.3) while the next one gives only small corrections.

Also in light of these results it becomes interesting to have explicit expressions for the first contributions to the expansion over intermediate states of formula (3.2.2). Denoting by $C^{(n)}$ the n-particle contribution to C one easily obtains

$$C^{(1)}(\rho) = c_{IR} + \frac{3}{2\pi}\frac{\rho^3}{m^4}|F_1|^2\left\{K_3(\rho) - \frac{2}{\rho}K_2(\rho)\right\} , \tag{3.2.11}$$

$$C^{(2)}(\rho) = c_{IR} + \frac{3}{4\pi^2}\left(\frac{\rho}{m}\right)^4\int_0^\infty d\theta\,|F_2(2\theta)|^2\left\{\frac{K_3(2\rho\cosh\theta)}{\rho\cosh\theta} - \frac{K_2(2\rho\cosh\theta)}{(\rho\cosh\theta)^2}\right\} , \tag{3.2.12}$$

where we introduced the dimensionless variable $\rho \equiv mR$, m being the mass of the particles entering F_1 and F_2 (eq.(3.2.12) holds in the case the two particles entering F_2 have the same mass); the K_n's are the modified Bessel functions. It can be easily verified that for $\rho = 0$ these formulas correctly reproduce eqs.(3.2.9) and (3.2.10).

3.3 Thermodynamic Bethe Ansatz

In this section we will describe a powerful method, known as *thermodynamic Bethe ansatz* (TBA) [56, 57, 58, 59, 60, 61], which gives the possibility to recover the ultraviolet data of an integrable model using only the on-shell informations contained in the S-matrix. This allows us, in particular, to test if the *minimal* S-matrix obtained using the bootstrap procedure is the correct one to describe the massive theory arising from the relevant perturbation of a CFT.

Let's start by considering a relativistic field theory living on a torus generated by two orthogonal circles B and C of circumference L and R respectively. One can develop an hamiltonian approach to this situation choosing the time direction along the circle B or, alternatively, along the circle C. This leads to the definition of two different Hamiltonian, H_C and H_B respectively. Therefore the partition function of the theory can be written in two different ways:

$$Z(R,L) = \mathrm{tr}_{\mathcal{C}} e^{-LH_C} = \mathrm{tr}_{\mathcal{B}} e^{-RH_B} \ , \tag{3.3.1}$$

where $\mathcal{B}$ ($\mathcal{C}$) is the space of states on B (C).

Consider now the limit $L \to \infty$, $L \gg R$. This corresponds to the thermodynamic limit for the system living on B and we can write

$$lnZ(R,L) \simeq -RLf(R) \ , \tag{3.3.2}$$

where $f(R)$ is the free energy for unit length at temperature $T = 1/R$. On the other hand, in the limit we are considering, the second expression for $Z(R,L)$ in eq.(3.3.1) is dominated by the contribution of the ground state of H_C whose energy $E_0(R)$ depends on the size R of the system:

$$Z(R,L) \simeq e^{-E_0(R)L} \ . \tag{3.3.3}$$

We see from eq.(1.3.9) that in the ultraviolet limit $R \to 0$, where the system is described by a CFT with central charge c, the ground state energy behaves as $E_0(R) \simeq -\pi\tilde{c}/6R$, where

$$\tilde{c} = c - 12d_{min} \ , \tag{3.3.4}$$

d_{min} being the lowest anomalous scale dimension in the CFT. In a unitary theory $d_{min} = 0$ and $\tilde{c} = c$. In light of this result we introduce the scaling function $\tilde{C}(mR)$ (m is the mass scale of the theory) such that $\tilde{C}(0) = \tilde{c}$ and write

$$E_0(R) = -\frac{\pi}{6R}\tilde{C}(mR) \ . \tag{3.3.5}$$

Putting together eqs.(3.3.2), (3.3.3) and (3.3.5) one sees that the determination of $\tilde{C}(mR)$ reduces to the computation of the free energy of a system of relativistic particles living on a line of length $L \to \infty$ at temperature $1/R$. We will discuss this problem for the case in which the system of particles is described by a purely elastic scattering theory (see sec. 2.3) containing n species of particles with masses m_a, $a = 1, \ldots, n$.

Generally speaking, the wave function formalism is inappropriate to describe a system of relativistic particles due to the virtual and real particle creation. But for a system of N particles on a line of length L much larger than the correlation length ξ there exist regions of the configuration space in which the particles are widely separated from each other: $|x_i - x_j| \gg \xi$, $\forall i \neq j$ ($\xi \sim 1/m_1$ if m_1 stays for the mass of the lightest particle in the theory). In these *free regions* the off-shell effects can be neglected and it is sensible to describe the system by a *Bethe wave function* $\psi(x_1, \ldots, x_n)$ proportional to the free wave function $\prod_{i=1}^{N} e^{ip_i x_i}$. Now the important step: the passage from a free region to a different one in which, for example, particles i and j exchanged their positions can be described simply by multiplying the original wave function by the scattering amplitude $S_{ij}(\theta_{ij})$. In particular, if we impose periodic boundary conditions, this criterion leads to the relation

$$e^{ip_i L} \prod_{j:j\neq i} S_{ij}(\theta_{ij}) = 1 \ , \qquad i = 1, \ldots, N \ . \tag{3.3.6}$$

Defining

$$S_{ij}(\theta) = e^{i\delta_{ij}(\theta)} \ , \tag{3.3.7}$$

eq.(3.3.6) can be written as

$$m_i L \sinh \theta_i + \sum_{j:j\neq i} \delta_{ij}(\theta_{ij}) = 2\pi n_i \ , \tag{3.3.8}$$

with N integers numbers n_i. This system of transcendental equations selects admissible sets of rapidities in free regions. Note that in the non-interacting case, where $\delta_{ij} = 0$, eq.(3.3.8) reduces to the usual quantization condition for the momenta of a particle in a box: $p_i = 2\pi n_i / L$.

If identical particles are present, statistical constraints on the wave functions must be taken into account. The unitarity condition eq.(2.3.4) allows two different cases:

$$\text{(a)} \qquad S_{ij}(0) = -1 \ . \tag{3.3.9}$$

Then the wave function is antisymmetric in the coordinates of two identical particles with the same rapidity. This is allowed for fermions but implies that a system of bosons cannot contains identical particles with the same rapidities. Under this respect bosons behaves like fermions since each value of rapidity can be occupied by at most one particle. As a consequence, all the integers n_i in eq.(3.3.8) must be different and we refer to this case as "fermionic". On the other hand, if the identical particles are fermions, the states with coinciding rapidity are allowed and there are no constraints on the numbers n_i; we will refer to this case as "bosonic".

$$\text{(b)} \qquad S_{ij}(0) = 1 \ . \tag{3.3.10}$$

In this case the situation in just inverted, bosons giving rise to the bosonic case and fermions to the fermionic case.

Since each set of N integers n_i for which the system (3.3.8) admits a solution selects a set of rapidities, we can imagine a system of levels in the rapidity space. In the thermodynamic limit $L \to \infty$ the distance between two adjacent levels behaves as $\theta_i - \theta_{i+1} \sim 1/m_1 L$ (think to the free case) and it is useful to introduce continuous level densities $\rho_a(\theta)$, with the index a referring to different species of particles. We also define rapidity densities of particles $\tilde{\rho}_a(\theta)$ as

$$\tilde{\rho}_a(\theta) = \frac{n_a(\theta)}{\Delta\theta} , \tag{3.3.11}$$

where $n_a(\theta)$ is the number of particles of species a contained in an interval $\Delta\theta$ around θ such that $1/m_1 L \ll \Delta\theta \ll 1$. Using these definitions equation (3.3.8) can be rewritten in the integral form

$$m_a L \cosh\theta + \sum_{b=1}^{n} \int d\theta' \, \phi_{ab}(\theta - \theta')\tilde{\rho}_b(\theta') = 2\pi \rho_a(\theta) , \tag{3.3.12}$$

where

$$\phi_{ab}(\theta) \equiv \frac{\partial}{\partial\theta}\delta_{ab}(\theta) = -i\frac{\partial}{\partial\theta} ln S_{ab}(\theta) \ . \tag{3.3.13}$$

The free energy Lf can be computed by the usual thermodynamic relation

$$Lf(\rho, \tilde{\rho}) = H_B(\rho) - \frac{1}{R}\mathcal{S}(\rho, \tilde{\rho}) , \tag{3.3.14}$$

where the total energy $H_B(\rho)$ is given by

$$H_B(\rho) = \sum_{a=1}^{n} \int d\theta \, m_a \tilde{\rho}_a(\theta) \cosh\theta , \tag{3.3.15}$$

and $\mathcal{S}(\rho, \tilde{\rho})$ is the entropy of the system. The number of particles in a rapidity interval $\Delta\theta$ is $\tilde{\rho}_a(\theta)\Delta\theta$ and $\rho_a(\theta)\Delta\theta$ is the number of levels in the same interval for a given species of particles with mass a. Therefore the number of possible distributions of such particles among these levels is

$$\frac{[\rho_a(\theta)\Delta\theta]!}{[\tilde{\rho}_a(\theta)\Delta\theta]! \, [(\rho_a(\theta) - \tilde{\rho}_a(th))\Delta\theta]!} \tag{3.3.16}$$

in the fermionic case, and

$$\frac{[(\rho_a(\theta) + \tilde{\rho}_a(\theta)\Delta\theta]!}{[\rho_a(\theta)\Delta\theta]! \, [\tilde{\rho}(\theta)\Delta\theta]!} \tag{3.3.17}$$

in the bosonic case. Since the entropy is the logarithm of the number of possible distributions for given densities ρ and $\tilde{\rho}$ in the limit $L \to \infty$, we have

$$\mathcal{S}_{Fermi}(\rho, \tilde{\rho}) = \sum_{a=1}^{n} \int d\theta \, [\rho_a ln \rho_a - \tilde{\rho}_a ln \tilde{\rho}_a - (\rho_a - \tilde{\rho}_a) ln(\rho_a - \tilde{\rho}_a)] \tag{3.3.18}$$

and

$$\mathcal{S}_{Bose}(\rho,\tilde{\rho}) = \sum_{a=1}^{n} \int d\theta \, [(\rho_a+\tilde{\rho}_a) ln(\rho_a+\tilde{\rho}_a) - \rho_a ln \rho_a - \tilde{\rho}_a ln \tilde{\rho}_a] \quad . \tag{3.3.19}$$

At this point we have to minimize the free energy in order to determine the densities ρ and $\tilde{\rho}$ at equilibrium. They are related by the dynamical equation (3.3.12) from which we get

$$\frac{\delta \rho_a(\theta')}{\delta \tilde{\rho}_b(\theta)} = \frac{1}{2\pi}\phi_{ab}(\theta'-\theta) \quad . \tag{3.3.20}$$

Define the *pseudoenergies* $\varepsilon_a(\theta)$ by

$$\frac{\tilde{\rho}_a(\theta)}{\rho_a(\theta)} = \frac{e^{-\varepsilon_a(\theta)}}{1 \pm e^{-\varepsilon_a(\theta)}} \tag{3.3.21}$$

and introduce also

$$L_a(\theta) = \pm ln\left(1 \pm e^{-\varepsilon_a(\theta)}\right) \tag{3.3.22}$$

(here and in the following upper and lower signs refer to the particle a being of fermionic or bosonic type, respectively). Using eq.(3.3.20) to compute the variation of H_B and $\mathcal{S}$ with respect to $\tilde{\rho}_a$, the extremum condition for f take the form

$$Rm_a \cosh\theta = \varepsilon_a(\theta) + \sum_{b=1}^{n} \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi} \, \phi_{ab}(\theta-\theta') L_b(\theta') \quad . \tag{3.3.23}$$

These are the TBA equations written in unified form for fermionic and bosonic case. The numerical solution of these integral equations provides the values of the pseudoenergies $\varepsilon_a(\theta)$ which are the necessary ingredients for the determination of the function $\tilde{C}(r)$ $(r \equiv mR)$ through the formula

$$\begin{aligned} \tilde{C}(r) &= -\frac{6R^2}{\pi} f(R) \\ &= \frac{3}{\pi^2} \sum_{a=1}^{n} \int_{-\infty}^{+\infty} d\theta \, L_a(\theta) \frac{m_a}{m_1} r \cosh\theta \quad . \end{aligned} \tag{3.3.24}$$

This expression can be explicitly evaluated in the limit $r \to 0$. In this limit θ has to be taken very large in order to give a non-negligible contribution to the left-hand side of equation (3.3.23). One has

$$r\frac{m_a}{m_1}\cosh\theta \sim \frac{r}{2}\frac{m_a}{m_1}e^{\theta} \sim \frac{m_a}{m_1} exp\left(\theta - ln\frac{2}{r}\right) \quad . \tag{3.3.25}$$

Numerical work shows that the pseudoenergies assume constant values ε_a in the range $-ln(2/r) \ll \theta \ll ln(2/r)$, where the left-hand side of eq.(3.3.23) can be neglected, and grow exponentially at very large values of $|\theta|$. In the interval in which they are constant eq.(3.3.23) reduces to

$$\varepsilon_a = \pm \sum_{b=1}^{n} N_{ab} \, ln(1 \pm e^{-\varepsilon_b}) \,, \tag{3.3.26}$$

with

$$N_{ab} = -\int_{-\infty}^{+\infty} \frac{d\theta}{2\pi}\, \phi_{ab}(\theta) = -\frac{1}{2\pi}(\delta_{ab}(\infty) - \delta_{ab}(-\infty)) \ . \tag{3.3.27}$$

On the other hand this region does not contribute to the integral in eq.(3.3.24) which is sensible only to the large $|\theta|$ limit for $r \to 0$. In this limit the pseudoenergies $\hat{\varepsilon}_a(\theta)$ are determined by eq.(3.3.23) in the form

$$\frac{r}{2}\frac{m_a}{m_1}e^{\theta} = \hat{\varepsilon}_a(\theta) + \sum_{b=1}^{n} \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi}\, \phi_{ab}(\theta - \theta')\hat{L}_b(\theta') \ , \tag{3.3.28}$$

where

$$\hat{L}_a(\theta) = \pm ln(1 \pm e^{-\hat{\varepsilon}_a(\theta)}) \ , \tag{3.3.29}$$

and we have

$$\tilde{C}(0) = \frac{6}{\pi^2} \sum_{a=1}^{n} \lim_{r\to 0} \int_0^{\infty} d\theta\, \hat{L}_a(\theta) \frac{r}{2}\frac{m_a}{m_1}e^{\theta} \tag{3.3.30}$$

(in last line the parity of $\varepsilon_a(\theta)$ was used to change the lower boundary of integration). Taking a derivative of eq.(3.3.28) with respect to θ one obtains another expression for $\frac{r}{2}\frac{m_a}{m_1}e^{\theta}$ which can be substituted in eq.(3.3.30) and, after several integration by parts, allows to arrive at the final result

$$\tilde{c} = \sum_{a=1}^{n} \tilde{c}_a^{\pm}(\varepsilon_a) \ , \tag{3.3.31}$$

where

$$\tilde{c}_a^{\pm}(\varepsilon_a) = \frac{6}{\pi^2} \times \begin{cases} L\left(\frac{1}{1+e^{\varepsilon_a}}\right) \\ L(e^{-\varepsilon_a}) \end{cases} = \frac{6}{\pi^2} \int_0^{\infty} dx\, \frac{x + \varepsilon_a/2}{e^{x+\varepsilon_a} \pm 1} \ . \tag{3.3.32}$$

Here the ε_a are determined by equation (3.3.26) and $L(x)$ is Rogers' dilogarithm function [62]

$$L(x) = -\frac{1}{2}\int_0^x dy \left[\frac{ln\, y}{1-y} + \frac{ln(1-y)}{y}\right] \ . \tag{3.3.33}$$

In conclusion we give an expression for the quantities N_{ab} defined in eq.(3.3.27) and entering eq.(3.3.26). We have seen in sec. 2.3 that the S-matrix element $S_{ab}(\theta)$ for a purely elastic scattering theory can be written as product of the building blocks $f_\alpha(\theta)$ so that $\phi_{ab}(\theta) = \sum_i \phi[f_{\alpha_i}](\theta)$ and $N_{ab} = \sum_i N[f_{\alpha_i}]$, in an obvious notation. Direct computation gives

$$\phi[f_\alpha](\theta) = -i\frac{d}{d\theta} ln f_\alpha(\theta) = -\frac{\sin\alpha\pi}{\cosh\theta - \cos\alpha\pi} \tag{3.3.34}$$

and

$$N[f_\alpha] = (1 - |\alpha|)\mathrm{sgn}\alpha \qquad \text{for } -1 < \alpha < 1 \ , \tag{3.3.35}$$

where sgnα is the sign of α (sgn$0 = 0$). This implies

$$N[F_\alpha] = \mathrm{sgn}\alpha \qquad \text{for } -\frac{1}{2} \leq \alpha \leq \frac{1}{2} \tag{3.3.36}$$

for the composite blocks (2.3.14).

Chapter 4

The Staircase Model

4.1 Roaming Trajectories

We discussed in previous sections how an integrable deformation of a CFT leads in the general case to a massive theory which is characterized by a factorized S-matrix; if the S-matrix is known, off-mass shell informations, including the ultraviolet parameters of the background CFT, can be obtained using the form factors approach or TBA. On the other hand, one can construct infinitely many factorized scattering theories which are perfectly consistent from the S-matrix point of view, but which lack any known field theory interpretation.

In this spirit Al.B. Zamolodchikov proposed in ref. [63] a simple purely elastic scattering theory which under TBA analysis reveals a very remarkable off-shell behavior. The theory contains a single particle, which is chosen to be a boson of mass m, and is defined by the two-particle amplitude

$$S(\theta) = \frac{\sinh\theta - i\cosh\theta_0}{\sinh\theta + i\cosh\theta_0} \ , \tag{4.1.1}$$

where θ_0 is a real parameter. This amplitude satisfies the usual requirements of unitarity and crossing symmetry eqs.(2.3.4) and (2.3.5) which for a single particle theory read simply

$$S(\theta)S(-\theta) = 1 \ , \tag{4.1.2}$$

$$S(\theta) = S(i\pi - \theta) \ . \tag{4.1.3}$$

$S(\theta)$ exhibits two simple zeroes in the physical strip at positions $\theta = \frac{i\pi}{2} \pm \theta_0$, paired via the unitarity relation to two simple poles in the unphysical strip at positions $\theta = -\frac{i\pi}{2} \pm \theta_0$.

The TBA analysis of this model goes along the standard lines described in section 3.3 for the "fermionic case" since from eq.(4.1.1) we get $S(0) = -1$ (note that the same off-shell pattern discussed below can be obtained supposing that the particle of the theory is a fermion and changing the sign of the amplitude (4.1.1)). Since only one particle is present, one has to deal with the single TBA equation

$$Rm\cosh\theta = \varepsilon(\theta) + \int_{-\infty}^{+\infty} \frac{d\theta'}{2\pi}\,\phi(\theta - \theta')L(\theta') \ , \tag{4.1.4}$$

where

$$L(\theta) \equiv \ln\left(1 + e^{-\epsilon(\theta)}\right) \tag{4.1.5}$$

and the kernel $\phi(\theta)$ defined in eq.(3.3.13) in this specific case is given by

$$\phi(\theta) = \frac{1}{\cosh(\theta+\theta_0)} + \frac{1}{\cosh(\theta-\theta_0)} \quad . \tag{4.1.6}$$

The ultraviolet effective central charge $\tilde{c}_{UV} = \tilde{C}(mR = 0)$ is easily obtained using the relations (3.3.32), (3.3.27) and (3.3.26); it turns out to be

$$\tilde{c}_{UV} = 1 \quad . \tag{4.1.7}$$

The interesting features of the model under consideration appear when equation (4.1.4) is solved numerically and the effective central charge at intermediate distances is computed through eq.(3.3.24) [63]. The results of this analysis for various values of the parameter θ_0 are shown in figs. 9a-d where the effective central charge $\tilde{C}$ is plotted as a function of the logarithmic scale

$$x = \ln\frac{mR}{2} \quad . \tag{4.1.8}$$

For $\theta_0 = 0$, $\tilde{C}(x)$ shows the usual behaviour smoothly interpolating between the ultraviolet limit $\tilde{C}(x = -\infty) = \tilde{c}_{UV}$ and the value $\tilde{C}(x = +\infty) = 0$ characteristic of massive theories; but for $\theta_0 \neq 0$ the situation becomes highly non-trivial and $\tilde{C}(x)$ develops a "staircase" pattern which becomes more and more visible as θ_0 increases. More precisely, for θ_0 sufficiently large (say $\theta_0 > 20$) $\tilde{C}(x)$ clearly exhibits a series of plateaux at values coinciding with those provided by formula (1.3.6) for the central charges of unitary minimal models $\mathcal{M}_p$; the plateau at $\tilde{C} = 1 - 6/p(p+1)$ lies inside the interval $-(p-2)\theta_0/2 < x < -(p-3)\theta_0/2$ with $p = 3, 4, \ldots$. Since the difference in the heights of the neighbouring steps becomes small as $x \to -\infty$, the numerical resolution becomes insufficient in the deep ultraviolet limit and the picture is slurred. Nevertheless, at $\theta_0 = 50$ one can clearly distinguish 8 steps, the highest being of height 21/22 and corresponding to $\mathcal{M}_{11}$ central charge.

These results unavoidably lead to an interpretation of the model defined by eq.(4.1.1) closely related to the massless RG flows between the theories $\mathcal{M}_p$ and $\mathcal{M}_{p-1}$ induced by the perturbing field $\Phi^{p}_{(1,3)}$. Indeed the characteristic pattern of figs. 9a-d are suggestive for a one-parameter family of *roaming trajectories* interpolating between all the fixed points $\mathcal{M}_p$: according to eq.(4.1.7), each trajectory starts from the limiting fixed point $\mathcal{M}_\infty$ and then, for θ_0 large enough, flows very close to the fixed points $\mathcal{M}_p$ spending approximately the same fraction $\theta_0/2$ of the RG time x near each one. In ref. [63] it was shown that for $\theta_0 \gg 1$ and $x \sim -(p-2)\theta_0/2$ (this is the value corresponding to the switching from c_{p+1} to c_p) the function $\tilde{C}(x)$ reproduces with high accuracy the values obtained by the TBA system proposed in ref. [64] as describing the flow from $\mathcal{M}_{p+1}$ to $\mathcal{M}_p$ (since this is a massless flow, the TBA analysis differs from the standard one presented in sec. 3.3). Thus we can imagine that the limiting $\theta_0 \to \infty$ trajectory starts at $\mathcal{M}_\infty$ and then proceed on the critical surface following the massless trajectories $\mathcal{M}_p \to \mathcal{M}_{p-1}$ until at $\mathcal{M}_3$ it develops a finite correlation length and gives rise to a massive infrared behaviour.

One can use the numerical data for $\tilde{C}(x)$ to compute the beta-functions along the RG trajectories [63]. Indeed, let's call Φ the operator which draws the field theory along the trajectory and g the conjugated coupling constant. Then the expectation value of Φ on the TBA geometry can be written as

$$< \Phi >_R = \frac{\pi}{6R}\frac{\partial \tilde{C}}{\partial g} \quad . \tag{4.1.9}$$

We now eliminate the dependence on R fixing $R = 1$ and normalize the field Φ through

$$< \Phi >_{R=1} = -\frac{\pi}{6} \;, \tag{4.1.10}$$

where the minus sign is due to the fact that $\tilde{C}$ monotonically decreases along the trajectory. Requiring g to be zero at the ultraviolet fixed point, eqs.(4.1.9) and (4.1.10) give

$$g(x) = 1 - \tilde{C}(x) \quad . \tag{4.1.11}$$

On the other hand the beta-function is simply defined as the derivative of the coupling constant with respect to the scale parameter so that

$$\beta(g) = -\frac{\partial \tilde{C}}{\partial x} \quad . \tag{4.1.12}$$

The last two relations give a parametric representation of $\beta(g)$. Figs. 10a-c show the behaviour of the beta-function for different values of the parameter θ_0: $\beta(g)$ develops deep minima in correspondence of the values $g = 6/p(p+1)$, $p = 3, 4, \ldots$ which become progressively indistinguishable from zeroes when θ_0 increases. Note that, while the higher minima turn subsequently to zeroes when θ_0 grows, the beta-function in between to zeroes is stabilized at the corresponding interpolating shape.

An interpretation of the results presented above from the conformal perturbation theory point of view was proposed in ref. [66] where the following hamiltonian density was argued to describe the staircase model:

$$\mathcal{H} = \mathcal{H}_p + \lambda\Phi^p_{(1,3)} - \tilde{\lambda}\Phi^p_{(3,1)} \quad . \tag{4.1.13}$$

Here $\mathcal{H}_p$ stays for the hamiltonian density of the minimal conformal model $\mathcal{M}_p$ and the suffix p for the fields denotes that they belong to $\mathcal{M}_p$. For $\lambda > 0$ and $\tilde{\lambda} = 0$ eq.(4.1.13) simply corresponds to the deformation studied in sec. 1.6 and interpolating between $\mathcal{M}_p$ and $\mathcal{M}_{p-1}$. The aim of the irrelevant perturbation coming from $\Phi^p_{(3,1)}$ is then to deform this interpolating trajectory in such a way to avoid that it stops at $\mathcal{M}_p$ ($\mathcal{M}_{p-1}$) in the ultraviolet (infrared) limit. This would make possible a multiple crossover of the type described above. A detailed perturbative study of the RG equations corresponding to the deformation (4.1.13) was carried out in ref. [66] in the limit of large values of

p. To leading order in $1/p$ it was shown that, for λ and $\tilde{\lambda}$ both positive, there exists a unique one parameter family of solutions exhibiting the characteristic behaviour of roaming trajectories. Indeed, if we denote by x the RG time and by θ_0 the parameter labelling the solutions, it turns out that each trajectory come close to each fixed point $\mathcal{M}_{p'}$ in the time interval $(p-p'-1/2)\theta_0 < \theta < (p-p'+1/2)\theta_0$. In particular, for the RG flow of the function $C(x)$ defined in sec. 1.5 one obtains

$$C(k\theta_0) = c_{p-k} + O(p^{-4}) , \qquad (4.1.14)$$

$$C((k+1/2)\theta_0) = c_{p-k} - \frac{1}{2}(c_{p-k} - c_{p-k-1}) + O(p^{-4}) \qquad (4.1.15)$$

for integer k.

In conclusion we make some remark about the particular deformation of the fixed point $\mathcal{M}_p$ defined by eq.(4.1.13). Both the perturbations $\Phi_{(1,3)}$ and $\Phi_{(3,1)}$ are separately integrable (see sec. 2.1 and ref. [22]). But, while in the general case a linear combination of two integrable perturbation does not generate an integrable field theory off criticality (this is the case, for example, of Ising model under simultaneous thermal ($\Phi_{(1,3)}$) and magnetic ($\Phi_{(1,2)}$) perturbations), the combination of $\Phi_{(1,3)}$ and $\Phi_{(3,1)}$ was argued to be integrable in ref. [65] so that one can expect (4.1.13) to correspond to a factorized scattering theory.

The multiple crossover exhibited by the theory (4.1.13) for λ and $\tilde{\lambda}$ positive gives rise to an interesting critical behaviour as a function of the relevant "temperature-like" parameter λ. Indeed for $\lambda = 0$ the theory is in the universality class of $\mathcal{M}_p$ but the thermodynamic singularities as $\lambda \to 0$ are determined not by $\mathcal{M}_p$ alone, but simultaneously by all the fixed points $\mathcal{M}_p, \mathcal{M}_{p-1}, \ldots, \mathcal{M}_3$. Some exact exponents are obtained in ref. [67].

4.2 Sinh-Gordon Theory

The scattering amplitude (4.1.1) can be considered as an analytical continuation of the S-matrix of the Sinh-Gordon theory, namely the theory of a two-dimensional scalar field $\phi(x)$ with the action

$$A = \int d^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{g^2}\cosh g\phi(x)\right] . \qquad (4.2.1)$$

It can be regarded as a perturbation of the free massless conformal action by means of the relevant operator $\cosh g\phi(x)$ of anomalous dimension $\Delta = -\beta^2/8\pi$ or, alternatively, as a deformation of the conformal Liouville action

$$A = \int d^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \lambda e^{g\phi}\right] \qquad (4.2.2)$$

by the relevant operator $e^{-g\phi}$.

Action (4.2.1) possesses a $\mathbf{Z}_2$-symmetry under the substitution $\phi \to -\phi$ and can be mapped into the action of the Sine-Gordon model by an analytic continuation in g, namely $g \to ig$. In a perturbative approach, ultraviolet divergencies come only from tadpole diagrams and can be removed by a normal ordering prescription. This gives rise to finite wave function and mass renormalization, while the coupling constant g does not renormalize.

The Sinh-Gordon model is the simplest example of a large class of integrable theories, the affine Toda field theories. Integrability allows the determination of the exact S-matrix which is given by [71]

$$S(\theta, B) = \frac{\sinh\theta - i\sin\frac{\pi B}{2}}{\sinh\theta + i\sin\frac{\pi B}{2}} , \tag{4.2.3}$$

where B is the following function of the coupling constant g:

$$B(g) = \frac{2g^2}{8\pi + g^2} . \tag{4.2.4}$$

It is evident from this relation that in the Sinh-Gordon theory $B(g)$ takes values in the range $[0, 2)$. The two-particle amplitude (4.2.3) has no poles in the physical strip (then there are not bound states) and exhibits two zeroes at the crossing symmetric positions $i\pi B/2$ and $i\pi(2-B)/2$.

An interesting feature of (4.2.3) is its invariance under the substitution

$$B \to 2 - B , \tag{4.2.5}$$

corresponding through eq.(4.2.4) to the strong-weak coupling constant duality

$$g \to \frac{8\pi}{g} . \tag{4.2.6}$$

At the *self-dual point* $B(\sqrt{8\pi}) = 1$ the two zeroes of the scattering amplitude collide at $\theta = i\pi/2$. If we now analytically continue the parameter B to the complex values

$$B = 1 \pm \frac{2i}{\pi}\theta_0 \tag{4.2.7}$$

the zeroes split again but along a direction parallel to the real θ-axis and (4.2.3)exactly coincides with the scattering amplitude (4.1.1).

The form factors for the Sinh-Gordon model were computed in ref. [51]. Using eqs.(3.1.20) and (3.1.21) to compute the minimal two-particle form factor F_{min} one obtains

$$F_{min}(\theta, B) = \mathcal{N} \exp\left[8\int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}\left(1-\frac{B}{2}\right)\right)\sinh\frac{x}{2}}{\sinh^2 x} \sin^2\left(\frac{x\hat{\theta}}{2\pi}\right)\right] , \tag{4.2.8}$$

with $\hat{\theta} = i\pi - \theta$. If one fixes the normalization of F_{min} requiring that

$$\lim_{\theta\to\infty} F_{min}(\theta, B) = 1 , \tag{4.2.9}$$

the factor $\mathcal{N}$ in eq.(4.2.8) should be

$$\mathcal{N}(B) = F_{min}(i\pi, B) = \exp\left[-4\int_0^\infty \frac{dx}{x}\frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{2}\left(1-\frac{B}{2}\right)\right)\sinh\frac{x}{2}}{\sinh^2 x}\right] . \tag{4.2.10}$$

The following results were obtained for form factors of the trace Θ of the energy-momentum tensor (this is a $\mathbf{Z}_2$-even operator):

$$F_0(\theta) = \frac{\pi m^2}{2\sin(\pi B/2)} , \tag{4.2.11}$$

$$F_2(\theta) = 2\pi m^2 \frac{F_{min}(\theta)}{F_{min}(i\pi)} , \tag{4.2.12}$$

$$F_{2n}(\theta_1,\ldots,\theta_2 n) = \frac{2\pi m^2}{F_{min}(i\pi)}\left(\frac{4\sin(\pi B/2)}{F_{min}(i\pi)}\right)^{n-1}\sigma_1^{(2n)}\sigma_{2n-1}^{(2n)}P_{2n}(x_1,\ldots,x_{2x})\prod_{i<j}\frac{F_{min}(\theta_{ij})}{x_i+x_j} , \tag{4.2.13}$$

with $n > 1$ in the last line. The P_n's are symmetric polynomials of total degree $n(n-3)/2$ and of degree $n-3$ in each variable whose first representatives are

$$\begin{aligned}
P_3(x_1,\ldots,x_3) &= 1\\
P_4(x_1,\ldots,x_4) &= \sigma_2\\
P_5(x_1,\ldots,x_5) &= \sigma_2\sigma_3 - c_1^2\sigma_5\\
P_6(x_2,\ldots,x_6) &= \sigma_3(\sigma_2\sigma_4-\sigma_6) - c_1^2(\sigma_4\sigma_5+\sigma_1\sigma_2\sigma_6)
\end{aligned} \tag{4.2.14}$$

where $c_1 = 2\cos(\pi B/2)$ (the P_n with n odd enter the expressions for the form factors of the elementary field ϕ). In ref. [51] formula (3.2.10) was used to test the velocity of convergence of the expansion in the number of intermediate particles. Since the Sinh-Gordon model can be regarded as a deformation of the free massless boson theory, the expected value for the ultraviolet central charge is $c_{UV} = 1$; on the other hand $c_{IR} = 0$ as for any massive theory. The numerical results for the two-particle contribution $\Delta c^{(2)}$ are listed in table 2 and clearly show that the sum rule is saturated by this first contribution also for large values of the coupling constant.

4.3 Analytic Continuation of Form Factors

The fact that the Zamolodchikov's scattering amplitude (4.1.1) can be obtained as an analytical continuation of the Sinh-Gordon S-matrix naturally suggests to consider an analogous analytical continuation of the form factors of the Sinh-Gordon model in order to see if the results of the TBA analysis can be reproduced through this alternative approach.

Let's perform an analytical continuation in the parameter B according to eq. (4.2.7) and define

$$\alpha = \frac{2\theta_0}{\pi} . \tag{4.3.1}$$

Then the minimal two-particle form factor (4.2.8) can be rewritten in the form

$$F_{min}(\theta,\alpha) = \mathcal{N}(\alpha) h_1(\theta) h_2(\theta,\alpha) \tag{4.3.2}$$

where

$$\mathcal{N}(\alpha) = \exp\left\{-2\int_0^\infty \frac{dx}{x}\left(\cosh\frac{x}{2} - \cos\frac{\alpha}{2}x\right)\frac{\sinh\frac{x}{2}}{\sinh^2 x}\right\}\ , \tag{4.3.3}$$

$$h_1(\theta) = \exp\left\{2\int_0^\infty \frac{dx}{x}\frac{\sin^2\frac{x\hat{\theta}}{2\pi}}{\sinh x}\right\} = -i\sinh\frac{\theta}{2}\ , \tag{4.3.4}$$

$$h_2(\theta,\alpha) = \exp\left\{-4\int_0^\infty \frac{dx}{x}\frac{\cos\frac{\alpha}{2}x}{\sinh^2 x}\sinh\frac{x}{2}\sin^2\frac{x\hat{\theta}}{2\pi}\right\}\ . \tag{4.3.5}$$

Formula (3.2.10) takes the form

$$\Delta c^{(2)}(\alpha) = \frac{3}{2}\int_0^\infty d\theta\,\frac{|h_1(2\theta)h_2(2\theta,\alpha)|^2}{\cosh^4\theta} \tag{4.3.6}$$

and can be used to determine the two-particle contribution to the ultraviolet central charge as a function of the parameter α. The numerical results of this computation are listed in table 3 and show that $\Delta c^{(2)}(\alpha)$ monotonically decreases from the value very close to 1 corresponding to the Sinh-Gordon self-dual point ($\alpha = 0$) to the asymptotic value 1/2. This asymptotic limit can be reproduced analytically along the following lines. An analysis of the integrals (4.3.3) and (4.3.5) gives

$$\mathcal{N}(\alpha) \sim \exp\left(-\frac{\pi}{4}|\alpha|\right) \qquad \text{for } |\alpha| \gg 1\ , \tag{4.3.7}$$

$$h_2(\theta,\alpha) \simeq 1 \qquad \text{for } |\alpha| \gg 1 \text{ and } \theta < |\alpha|\ . \tag{4.3.8}$$

Due to the normalization condition eq.(4.2.9), relation (4.3.7) determines the asymptotic behaviour in θ of the product $h_1 h_2$ and permits to conclude that the integral in eq.(4.3.6) receives negligible contribution for $\theta > |\alpha|$. Therefore, we can use the relation (4.3.8) to write

$$\Delta c^{(2)}(|\alpha| \to \infty) = \frac{3}{2}\int_0^\infty d\theta\,\frac{\sinh^2\theta}{\cosh^4\theta} = \frac{1}{2}\ . \tag{4.3.9}$$

The explicit evaluation of higher particles contributions to Δc becomes a non-trivial numerical problem since it requires multiple integrations. Nevertheless we can try to estimate the asymptotic behaviour in α of $\Delta c^{(n)}$ in order to establish if a non-vanishing contribution can be expected which permits to reproduce the value $\Delta c = 1$. The n-particle form factor entering the formula (3.2.8) for $\Delta c^{(n)}$ is given by eq.(4.2.13) and after the analytic continuation can be written as (n even, $n > 2$)

$$F_n(\theta_1,\ldots,\theta_n) = 2\pi m^2 g_n(\alpha)\sigma_1^{(n)}\sigma_{n-1}^{(n)} P_n(x_1,\ldots,x_n)\prod_{i<j}\frac{h_1(\theta_{ij})h_2(\theta_{ij},\alpha)}{x_i + x_j}\ , \tag{4.3.10}$$

where we introduced

$$g_n(\alpha) = (4\cosh\pi\alpha/2)^{n/2-1}\mathcal{N}^{n(n/2-1)}(\alpha) \ . \tag{4.3.11}$$

Basing on a reasoning similar to that presented above we expect the α-dependence coming from h_2 to be strongly suppressed after the integration over rapidities so that the asymptotic behaviour in α of $\Delta c^{(n)}$ should be determined by the exponential factors contained in g_n and P_n. In the large α limit we can use relation (4.3.7) to obtain

$$g_n(\alpha) \sim \exp\left\{-\frac{\pi}{2}\left(\frac{n}{2}-1\right)^2|\alpha|\right\} \ . \tag{4.3.12}$$

On the other hand the examination of the symmetric polynomials P_n whose first representatives are listed in (4.2.14) shows that they are not sufficient to cancel the exponential decreasing behaviour coming from eq. (4.3.12). Thus we are led to conclude

$$\Delta c^{(n)}(|\alpha|\to\infty) = 0 \ , \qquad n > 2 \ . \tag{4.3.13}$$

The results of this section, which at first sight appear inconsistent with the requirement $\Delta c = 1$, have a natural interpretation once the nontrivial interplay between the two scales of the problem, θ and α, is correctly taken into account. We already noted in sec. 3.2 that the n-particle contribution to the expansion over intermediate states of a two-point correlation function behaves as $e^{-n(mr)}$, r denoting the radial distance. As a consequence, once a scale r is fixed, there will exist a number n_r such that the states with a number of particles $n \geq n_r$ give a negligible contribution to the series. Clearly $n_r \to \infty$ as $r \to 0$. Reversing the argument we conclude that the partial sum

$$\Delta c_T^{(n)} \equiv \sum_{m=1}^{n} \Delta c^{(m)} \tag{4.3.14}$$

reproduces the variation of the C-function from the infrared limit $r = \infty$ to a certain scale r_n such that the contribution to $C(r)$ coming from m-particle states with $m > n$ can be neglected for $r > r_n$ ($r_\infty = 0$).

In usual situations, when $C(r)$ is a smooth function which stays constant in the deep ultraviolet, the first contributions are sufficient to give a very good approximation of Δc. The case of the staircase model is very different. Indeed, let's consider a scale r' such that $C(r', \alpha = 0) > 1/2$. According with the results of the TBA analysis, after the first jump from 0 to 1/2 the function $C(r, \alpha)$ stays constant at 1/2 until a value r^* proportional to $e^{-\pi|\alpha|/4}$ is reached and the second jump takes place. Therefore, for every r' there exists a value α' such that $C(r', |\alpha| > |\alpha'|) = 1/2$. This in turn amounts to say that

$$\lim_{|\alpha|\to\infty} \Delta c_T^{(n)}(\alpha) = \frac{1}{2} \qquad \text{for any finite } n \ . \tag{4.3.15}$$

Such conclusion is exactly reproduced by eqs.(4.3.9) and (4.3.13).

Further clarification of the picture presented above is obtained noting that in the limit θ fixed, $|\alpha| \to \infty$ the scattering amplitude (4.1.1) tends to -1. This value coincides with the S-matrix [79] of the scaling Ising model (SIM), namely the deformation of the minimal model $\mathcal{M}_3$ with central charge 1/2 through the energy operator $\varepsilon = \Phi_{(1,2)}$. In the same limit the only non-vanishing form factor is the two-particle one, in agreement with the result which can be obtained exploiting the well known equivalence of SIM with a theory of free neutral fermions. One obtains

$$F_2^{SIM}(\theta) = \lim_{|\alpha|\to\infty} F_2(\theta,\alpha)|_{\theta \text{ fixed}} = -2\pi i m^2 \sinh\frac{\theta}{2} \quad . \tag{4.3.16}$$

Using eq.(3.2.1) we get the exact two-point correlation function for the trace of the energy-momentum tensor in SIM

$$< \Theta(r)\Theta(0) >= m^4\{K_1^2(mr) - K_0^2(mr)\} \quad . \tag{4.3.17}$$

This result exactly coincides with that obtained by the free fermion formalism (see, for example, [78]).

4.4 Generalization of the Model

The Sinh-Gordon model is the simplest example of the so-called affine Toda field theories. Generally speaking the affine toda theory associated to a semisimple Lie algebra $\mathcal{G}$ of rank r is a theory of r bosonic field ϕ^j with a Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi^j)^2 - \frac{m_0^2}{g^2}\sum_{i=0}^{r} q_i \exp\left(g\sum_{j=1}^{r}\alpha_i^j\phi^j\right) \quad , \tag{4.4.1}$$

where m_0 is a mass scale, g the coupling constant, α_i $(i = 1,\ldots,r)$ the simple roots of $\mathcal{G}$ and α_0 the maximal negative root; the set of integers $\{q_i\}$ is specific for each algebra. The Sinh-Gordon Lagrangian is recovered from (4.4.1) for $\mathcal{G} = A_1$. The affine Toda theories are integrable at the quantum level [68, 69, 70] and give rise to purely elastic scattering theories [71, 29, 32, 72]. The S-matrix contains the coupling constant dependence through the function $B(g)$ defined in eq.(4.2.4) and is invariant under the substitution $B \to 2 - B$.

The TBA systems for the affine Toda theories based on simply laced algebras were studied numerically and analytically in refs. [73, 74, 75, 76] in order to determine the effect of the analytical continuation eq.(4.2.7) in this general case. It was shown that the effective central charge $\tilde{C}(R)$ exhibits a staircase pattern completely analogous to that observed in the A_1 case. More precisely, the function $\tilde{C}(R)$ for a theory based on the Lie algebra $\mathcal{G}$ of a group G interpolates between the values of the central charges of the coset models $G_k \times G_1/G_{k+1}$. These values are easily obtained from the general formula (1.4.14); for the case $\mathcal{G} = A_r$ one obtains

$$c_p^r = r\left(1 - \frac{h(h+1)}{p(p+1)}\right) \quad , \qquad p = h+1, h+2, \ldots \tag{4.4.2}$$

where $h = r + 1$ is the Coxeter number and $p = h + k$ labels the values of each infinite series. Thus the staircase pattern discussed in previous sections, whose steps are generated by formula (4.4.2) with $r = 1$, appears as the simplest example of a phenomenon which extends to a large class of theories. In the following we will apply the form factor approach to the next simplest case, $\mathcal{G} = A_2$.

The A_r affine Toda theory is invariant under the action of the cyclic group Z_h. It is interesting to note that the requirement that a conformal theory is also Z_h-symmetric leads [77] to an infinite series of solutions corresponding to models whose central charges are exactly those given by formula (4.4.2). The conformal theory with central charge c^r_{h+s} is interpreted as describing the $(s+1)$-critical point of the Z_h-symmetric generalization of Ising model.

The minimal solution of the bootstrap procedure for the S-matrices of Z_h-invariant models was found in ref. [44] and we already considered in sec. 2.4 the case $h = 3$ corresponding to the scaling 3-state Potts model. These minimal S-matrices are not suitable for the A_r affine Toda theories since they do not contain the coupling constant dependence. Such dependence is introduced [71] multiplying the minimal solutions for a so called Z-factor which does not change the bound state structure and the bootstrap equations but introduces some zeroes in the physical strip, ensuring that the S-matrix tends to unity as the coupling constant g tends to zero. For $h = 3$ the complete solution reads

$$S_{aa}(\theta) = f_{2/3}(\theta) f_{-B/3}(\theta) f_{(B-2)/3}(\theta) \ , \tag{4.4.3}$$
$$S_{a\bar{a}}(\theta) = -f_{1/3}(\theta) f_{B/3-1}(\theta) f_{-(B+1)/3}(\theta) \ , \tag{4.4.4}$$

where f_α are the building blocks defined in sec. 2.3. In the limit θ fixed, $|\alpha| \to \infty$ the Z-factors disappear and the minimal scattering amplitudes are recovered. Therefore, according to the reasoning of previous section, we expect that in such limit the partial sums $\Delta c_T^{(n)}$ reproduce the value of the central charge of the first model visited by roaming trajectories, namely the 3-state Potts model. Contrary to the scaling Ising model, the scaling 3-state Potts model does not correspond to a free theory so that we expect non-vanishing contributions from all the form factors allowed by symmetry arguments.

The minimal two-particle form factors obtained from equations (3.1.20) and (3.1.21) are given by

$$F^{aa}_{min}(\theta, B) = \mathcal{N}_{aa}(B) \exp\left\{ 2 \int_0^\infty \frac{dx}{x} q_B(x) \frac{\sinh 2x/3}{\sinh^2 x} \sin^2 \frac{x\hat{\theta}}{2\pi} \right\} \ , \tag{4.4.5}$$
$$F^{a\bar{a}}_{min}(\theta, B) = \mathcal{N}_{a\bar{a}}(B) \exp\left\{ 2 \int_0^\infty \frac{dx}{x} q_B(x) \frac{\sinh x/3}{\sinh^2 x} \sin^2 \frac{x\hat{\theta}}{2\pi} \right\} \ , \tag{4.4.6}$$

where we introduced the function

$$q_B(x) = 4 \sinh \frac{xB}{6} \sinh\left(\frac{x}{3}\left(1 - \frac{B}{2}\right)\right) \ . \tag{4.4.7}$$

The minimal two-particle form factors satisfy the following functional relation which will be used later on

$$\frac{F^{aa}_{min}(\theta + i\pi/3) F^{aa}_{min}(\theta - i\pi/3)}{F^{a\bar{a}}_{min}(\theta)} = \frac{\cosh\theta - 1/2}{\cosh\theta - \cos\pi(1-B)/3} \ . \tag{4.4.8}$$

Imposing the asymptotic condition eq.(4.2.9) the normalization factors are fixed to be

$$\mathcal{N}_{aa}(B) = F^{aa}_{min}(i\pi, B) = \exp\left\{-\int_0^\infty \frac{dx}{x} q_B(x) \frac{\sinh 2x/3}{\sinh^2 x}\right\} , \qquad (4.4.9)$$

$$\mathcal{N}_{a\bar{a}}(B) = F^{a\bar{a}}_{min}(i\pi, B) = \exp\left\{-\int_0^\infty \frac{dx}{x} q_B(x) \frac{\sinh x/3}{\sinh^2 x}\right\} , . \qquad (4.4.10)$$

Since the trace Θ of the energy-momentum tensor is a $\mathbf{Z}_3$-invariant operator (it is a piece of the Lagrangian), its expectation values between the vacuum and an n-particle state $|n>$ (the form factors) will be non vanishing only if $|n>$ has a total $\mathbf{Z}_3$-charge equal to zero. For $n = 2$ the only neutral states are $|a(\theta_1)\bar{a}(\theta_2)>$ and $|\bar{a}(\theta_1)a(\theta_2)>$ (the two states should be distinguished due to the ordering prescription over the rapidities). Since a and $\bar{a}$ do not form any bound state in the direct scattering channel, we conclude that the two particle form factors must coincide with the minimal one, up to a numerical constant. Such constant is easily fixed taking the limit $g = 0$ in which the theory becomes free. Therefore we can write

$$F_{a\bar{a}}(\theta, B) = F_{\bar{a}a}(\theta, B) = 2\pi m^2 \frac{F^{a\bar{a}}_{min}(\theta, B)}{\mathcal{N}_{a\bar{a}}(B)} . \qquad (4.4.11)$$

We give in table 4 some numerical results for $\Delta c^{(2)}$ for real values of B and for values corresponding to the analytic continuation $B = 1 + i\alpha$. The situation appears similar to that observed in the A_1 case. Indeed the two-particle contribution reproduces with good approximation the expected value $c = 2$ of the ultraviolet central charge for real values of the coupling constant; in the analytic continued region $\Delta c^{(2)}(\alpha)$ exhibits a monotonically decreasing behaviour toward the stable asymptotic value 0.79217. Note that the 3-state Potts model has central charge $c = 0.8$

Let's pass to the computation of $\Delta c^{(3)}$. The non-vanishing three-particle form factors are F_{aaa} and $F_{\bar{a}\bar{a}\bar{a}}$ and, obviously, they give the same contribution to $\Delta c^{(3)}$. To be specific we will refer to the first one. The amplitude S_{aa} has a pole in the direct channel at $\theta = 2\pi i/3$ corresponding to the fusion $aa \to \bar{a}$. According to equation (2.3.17), in the vicinity of this pole the S-matrix is expressed as

$$S_{aa}(\theta) \simeq \frac{i(g^{\bar{a}}_{aa})^2}{\theta - 2\pi i/3} . \qquad (4.4.12)$$

Direct computation gives

$$(g^{\bar{a}}_{aa})^2 = \sqrt{3}\frac{\sin\frac{\pi B}{6}\sin\frac{\pi}{6}(2-B)}{\sin\frac{\pi}{6}(4-B)\sin\frac{\pi}{6}(2+B)} . \qquad (4.4.13)$$

For the case under consideration the general equations (3.1.17) and (3.1.19) read

$$F_{aaa}(\theta_1, \theta_2, \theta_3) = K(\theta_1, \theta_2, \theta_3) \prod_{i<j} F^{aa}_{min}(\theta_{ij}) , \qquad (4.4.14)$$

$$2i \lim_{\varepsilon\to 0} \varepsilon F_{aaa}\left(\theta + i\frac{\pi}{3} - \varepsilon, \theta - i\frac{\pi}{3} + \varepsilon, \theta'\right) = g^{\bar{a}}_{aa} F_{a\bar{a}}(\theta, \theta') . \qquad (4.4.15)$$

The factor K in the first equation contains the poles at positions $\theta_{ij} = 2\pi i/3$; they can be extracted writing

$$K(\theta_1, \theta_2, \theta_3) = H \left[\prod_{i<j} (\omega x_i + x_j)(\omega^{-1} x_i + x_j) \right]^{-1} Q(x_1, x_2, x_3) \ , \tag{4.4.16}$$

where $x_i \equiv e^{\theta_i}$, $\omega = e^{i\pi/3}$, H is a normalization factor and Q is a symmetric polynomial in the x_i's. Using the functional relation eq.(4.4.8) simple algebra gives

$$\begin{aligned} 2i \lim_{\varepsilon \to 0} \varepsilon F_{aaa}(\theta + i\pi/3 - \varepsilon, \theta - i\pi/3 + \varepsilon, \theta_3) &= -\frac{H}{\sqrt{3}} Q(x\omega, x\omega^{-1}, x_3) \\ &\times \frac{F^{aa}_{min}(2\pi i/3) F^{a\bar{a}}_{min}(\theta - \theta_3)}{x^2 (x + x_3)^2 (x^2 + x_3^2 - \gamma x x_3)} \end{aligned} \tag{4.4.17}$$

where

$$\gamma \equiv 2 \cos \frac{\pi}{3}(1 - B) \ . \tag{4.4.18}$$

Comparing with eq.(4.4.15) and recalling eq.(4.4.11) we get

$$H = -\frac{2\sqrt{3} g^{\bar{a}}_{aa} \pi m^2}{F^{aa}_{min}(2\pi i/3) \mathcal{N}_{a\bar{a}}} \ , \tag{4.4.19}$$

$$Q(x\omega, x\omega^{-1}, x_3) = x^2 (x + x_3)^2 (x^2 + x_3^2 - \gamma x x_3) \ . \tag{4.4.20}$$

The last equation shows that Q is a polynomial of total degree 6 and maximal degree 4 in each variable. Moreover conservation of energy-momentum tensor implies that Q should be proportional to $\sigma_1 \sigma_2$, where the σ_n's are the elementary symmetric polynomials introduced in sec. 3.1. These requirements and the constraint provided by eq.(4.4.20) permit the complete determination of Q; it is given by

$$Q(x_1, x_2, x_3) = \sigma_1 \sigma_2 (\sigma_1 \sigma_2 - (2 + \gamma) \sigma_3) \ . \tag{4.4.21}$$

Having determined the three-particle form factors we can apply the asymptotic criterion discussed in the previous section in order to establish if a non-vanishing value can be expected for $\Delta c^{(3)}(\alpha)$ when $\alpha \to \infty$. We write the form factor in the form

$$F_{aaa}(\theta_i) = \frac{g^{\bar{a}}_{aa}(\alpha) \gamma(\alpha) \mathcal{N}^3_{aa}(\alpha)}{F^{aa}_{min}(2\pi i/3, \alpha) \mathcal{N}_{a\bar{a}}(\alpha)} h(\theta_i, \alpha) \ , \tag{4.4.22}$$

where, according to the considerations of previous section, the function $h(\theta_i, \alpha)$ is expected to give rise to a weak (possibly zero) α-dependence (for $\alpha \to \infty$) after integration over rapidities. In the large α limit we have

$$\mathcal{N}_{aa}(\alpha) \sim F^{aa}_{min}(2\pi i/3) \sim \exp\left(-\frac{2\pi}{9}|\alpha|\right) \ , \tag{4.4.23}$$

$$\mathcal{N}_{a\bar{a}}(\alpha) \sim \exp\left(-\frac{\pi}{9}|\alpha|\right) \ , \tag{4.4.24}$$

$$g_{aa}^{\bar{a}}(\alpha) \sim 1 \ , \tag{4.4.25}$$

$$\gamma(\alpha) \sim \exp\left(\frac{\pi}{3}|\alpha|\right) \ , \tag{4.4.26}$$

so that we immediately obtain

$$F_{aaa}(\theta_i, \alpha) \sim h(\theta_i, \alpha) \ . \tag{4.4.27}$$

The numerical analysis confirms that $\Delta c^{(3)}(\alpha)$ approaches a constant value when $|\alpha|$ becomes large (see table 5). Moreover, the asymptotic value of the partial sum $\Delta c_T^{(3)} = \Delta c^{(2)} + \Delta c^{(3)}$ coincides with remarkable precision with the 3-state Potts model central charge.

Chapter 5

Conclusion

In this thesis we reviewed some important developments in the domain of QFT in two dimensions which represent significant progresses toward the ambitious aim of a classification of all the possible local QFT's. From the RG point of view this problem amounts to studying, at last qualitatively, the infinite-dimensional space of local interactions and the topology of the RG flow on it. In this picture every local QFT corresponds to a RG trajectory which typically starts from a fixed point of the RG. In two dimensions the field-theory solutions corresponding to fixed points exhibits a much more extended symmetry than scale invariance, i.e. the infinite-dimensional conformal symmetry. The use of this symmetry provided us with an enormous number of explicit constructions describing possible ultraviolet behaviours in two-dimensional QFT and it seems reasonable to hope in a complete classification of all the fixed points.

A non scale-invariant theory associated to a RG trajectory flowing from a fixed point can be obtained as a deformation of the conformal action by a combination of scalar operators present in the CFT which describes the fixed point. Two different infrared behaviours are possible for the resulting theory. The trajectory may flows to another fixed point, and in this case the corresponding theory is massless, or it may develop a finite correlation length giving rise to a massive theory which can be characterized in terms of its scattering data.

In the known CFT's there are particular relevant fields (the integrable operators) which generate off-critical theories containing an infinite number of integrals of motion. For a massive theory in two dimensions the presence of non-trivial integrals of motion leads to a drastic simplification in the scattering theory. The S-matrix is shown to be factorizable in terms of the two-particle scattering amplitudes which in turn can be determined in many cases by a bootstrap approach based on unitarity and analyticity.

It is commonly believed that the whole structure of a massive QFT in hidden in its scattering theory. An important step toward the complete reconstruction of off-shell properties is represented by the computation of the matrix elements of local operators between asymptotic states (form factors). Such computation has been explicitly performed for several integrable theories and immediately permits to write down the correlation functions as an infinite sum over multiparticle intermediate states. Another powerful technique which allows us to extract informations on the ultraviolet conformal theory corresponding to an integrable relativistic model starting from its on-mass-shell data is the thermodynamic Bethe ansatz (TBA). This approach is based on the observation that the finite-temperature free energy predicted by the scattering theory can be interpreted as the ground state energy of the (euclidean) theory on a finite periodic geometry. Since the small volume behaviour of the ground state energy is described by the corresponding ultraviolet CFT, the last can be identified by studying the high-temperature limit of the TBA equations. TBA played an important role in verifying the correctness of many S-matrices which in the general case are constructed by the bootstrap procedure under certain minimality assumptions.

In the last part of this thesis we considered a particular factorized scattering theory which under TBA analysis reveals an extremely peculiar and rich off-shell pattern characterized by a multiple crossover among the infinite fixed points corresponding to the minimal unitary conformal models. While a field theory interpretation of this *staircase* model is still lacking, the most suggestive feature of the scattering amplitude under consideration remains the fact that it can be seen as an analytic continuation to complex values of the coupling constant of the S-matrix of the Sinh-Gordon theory. Since the form factors for this model were recently constructed, it seems quite natural to study the effects of the analytic continuation on the form factors themselves. We showed that this procedure leads to results which are completely consistent with the TBA analysis so that the form factor approach appears as a very promising tool for future investigation of the staircase model.

Bibliography

[1] D.J. Amit, *Field Theory, the Renormalization Group and Critical Phenomena* (Mc-Graw Hill, Inc. 1978).

[2] K.G. Wilson and J. Kogut, *Phys. Rep.* **12** (1974), 200.

[3] P. Pfeuty and G. Toulouse, *Introduction to the Renormalization Group and to Critical Phenomena* (Wiley, 1977).

[4] A.M. Polyakov, *JETP Lett.* **12** (1970), 381; *Sov. Phys. JETP* **39** (1974), 10.

[5] K.G. Wilson, Phys. Rev. **179** (1969), 1499.

[6] L.P. Kadanoff, " Scaling, universality and operator algebras", in *Phase Transition and Critical Phenomena*, Vol 5A, edited by C.Domb and M.S. Green, Academic Press, London, 1976.

[7] A.M. Polyakov, *Sov. Phys. JEPT* **30** (1970) 151.

[8] A.A. Belavin, A.M. Polyakov and A.B. Zamolodchikov, *Nucl. Phys.* **B241** (1984), 333.

[9] V. Kac, *Lecture Notes in Phys.*, vol. 94 (Springer, 1979), 441.

[10] B.L. Feigin and D.B. Fuchs, *Funct. Analysis and Appl.* **16** (1982), 114.

[11] D. Friedan, Z. Qiu and S. Shenker, *Phys. Rev. Lett.* **52** (1984), 1575.

[12] J.L. Cardy, *Nucl. Phys.* **B270** [FS 16] (1986), 186; *Nucl. Phys.* **B275** [FS 17] (1986), 200.

[13] A. Cappelli, C. Itzykson and J.B. Zuber, *Nucl. Phys.* **B280** [FS 18] (1987), 445.

[14] A.B. Zamolodchikov, *Sov. J. Nucl. Phys.* **44** (1986), 529.

[15] D. Huse, *Phys. Rev.* **B30** (1984), 3908.

[16] G.E. Andrew, R.J. Baxter and P.J. Forrester, *Journ. Stat. Phys.* **35** (1984), 193.

[17] P. Goddard and D. Olive, *Int. J. Mod. Phys.* **A1** (1986), 304.

[18] D. Friedan, Z. Qiu and S. Shenker, *Phys. Lett.* **151B** (1985), 37.

[19] A.B. Zamolodchikov, *Sov. J. Nucl. Phys.* **46** (1987), 1090.

[20] J.L. Cardy, A.W.W. Ludwig, *Nucl. Phys.* B285[FS19] (1987), 687.

[21] Vl.S. Dotsenko and V.A. Fateev, *Nucl. Phys.* **B240** [FS 12] (1984), 312; *Nucl. Phys.* **B251** [FS 13] (1985), 691; *Phys. Lett.* **B 154** (1985), 291.

[22] A.B. Zamolodchikov, *Advanced Studies in Pure Mathematics* **19** (1989), 1; *Int. J. Mod. Phys.* A4 (1989), 4235.

[23] S. Coleman, J. Mandula, *Phys. Rev.* **159** (1969), 1251.

[24] A.B. Zamolodchikov and Al.B. Zamolodchikov, *Ann.Phys.* **120** (1979), 253.

[25] A.M. Polyakov, *Phys. Lett.* **72B** (1977), 224; S. Parke, *Nucl. Phys.* **B174** (1980), 166.

[26] R. Shankar, E. Witten, *Phys. Rev.* **D17** (1978), 2134.

[27] A.B. Zamolodchikov, *Int. J. Mod. Phys.* **A3** (1988), 743.

[28] J.L. Cardy and G. Mussardo, *Phys. Lett.* **B225** (1989), 275.

[29] P. Christe and G. Mussardo, *Nucl. Phys.* **B330** (1990), 465.

[30] V.A. Fateev and A.B. Zamolodchikov, *Int. J. Mod. Phys.* **A5** (1990), 1025.

[31] G. Sotkov and C.J. Zhu, *Phys. Lett.* B229 (1989),391.

[32] H.W. Braden, E. Corrigan, P.E. Dorey and R. Sasaki, *Nucl. Phys.* **B338** (1990), 689.

[33] K. Hepp, *Helv. Phys. Acta* **32** (1964), 639.

[34] P. Mitra, *Phys. Lett.* **72B** (1977), 62.

[35] G. Mussardo, *Phys. Rep.* **218** (1992), 215.

[36] M. Karowski, H.J. Thun, *Nucl. Phys.* **B130** (1977), 295; M. Karowski, *Nucl. Phys.* **B153** (1979), 244; B. Schroer, T.T. Truong and P. Weisz, *Phys. Lett.* **B63** (1976), 422.

[37] Vl.S. Dotsenko, *Nucl. Phys.* **B235** [FS11] (1984), 54.

[38] F.Y. Wu, *Rev. of Mod. Phys.* **54-1** (1982), 235.

[39] R.J. Baxter, *J. Phys.* **C6** (1973), L445.

[40] R.J. Baxter, *J. Phys.* **A13** (1980), L61.

[41] A.B. Zamolodchikov and V.A. Fateev, *Sov. Phys. JETP* **62** (1984), 215.

[42] A.B. Zamolodchikov and V.A. Fateev, *Sov. Phys. JETP* **63** (1986), 913.

[43] V.A. Fateev and A.B. Zamolodchikov, *Nucl. Phys.* **B280** [FS18] (1987), 644.

[44] R. Köberle and J.A. Swieca, *Phys. Lett.* **86B** (1979), 209.

[45] B.M. McCoy and T.T. Wu, *The two dimensional Ising model* (Harvard Univ. Press, 1982).

[46] F.A. Smirnov, *J. Phys.* **18** (1984) L 873; *J. Phys.* **19** (1986), L 575; A.N. Kirillov and F.A. Smirnov, *Phys. Lett.* **B198** (1987), 506; *Int. J. Mod. Phys.* **A3** (1988), 731.

[47] K.M. Watson, *Phys. Rev.* **95** (1954), 228.

[48] B. Berg, M. Karowski and P. Weisz, *Phys. Rev.* **D19** (1979), 2477; M. Karowski and P. Weisz, *Nucl. Phys.* **B139** (1978), 445; M. Karowski, *Phys. Rep.* **49** (1979), 229.

[49] I.G. MacDonald, Symmetric Functions and Hall Polynomials (Clarendon Press, Oxford, 1979).

[50] Al. B. Zamolodchikov, *Nucl. Phys.* **B348** (1991), 619.

[51] A. Fring, G. Mussardo, P. Simonetti, "Form factors for integrable lagrangian field theories, the Sinh-Gordon model", preprint SISSA-EP-92-146, Imperial-TP/91-92/31.

[52] A. Fring, G. Mussardo, P. Simonetti, "Form factors of the elementary field in the Bullough-Dodd model", preprint SISSA-EP-92-208, USP-IFQSC/TH/92-51.

[53] J.L. Cardy, *Phys. Rev. Lett.* **60** (1988), 2709.

[54] A. Cappelli, D. Friedan and J.I. Latorre, *Nucl. Phys.* **B352** (1991), 616.

[55] D. Freedman, J.I. Latorre and X. Vilasis, *Mod. Phys. Lett.* **A6** (1991), 531.

[56] Al.B. Zamolodchikov, *Nucl. Phys.* **B342** (1990), 695.

[57] C.N. Yang and C.P. Yang, *J. Math. Phys.* 10, N7 (1969), 1115.

[58] T.R. Klassen and E. Melzer, *Nucl. Phys.* **B338** (1990), 485.

[59] P. Christe and M.J. Martins, *Mod. Phys. Lett.* **A5** (1990), 2189.

[60] Al. B. Zamolodchikov, *Phys. Lett.* **B253** (1991), 391;

[61] M.J. Martins, *Phys. Rev. Lett.* **65** (1990), 2091; M.J. Martins, *Phys. Lett.* **B240**, 404; H. Itoyama and P. Moxhay, *Phys. Rev. Lett.* **65** (1990), 2102;

[62] L. Lewin, *Dilogarithms and associate functions* (MacDonald, London, 1958); *Polylogarithms and associated functions* (Elsevier-North Holland), New York and Oxford, 1981.

[63] Al.B. Zamolodchikov, *Resonance Factorized Scattering and Roaming Trajectories*, ENS-LPS-335, 1991.

[64] Al.B. Zamolodchikov, *Nucl. Phys.* **B358** (1991), 524.

[65] Al.B. Zamolodchikov, *Nucl. Phys.* **B358** (1991), 497; Al. B. Zamolodchikov, *Nucl. Phys.* **B366** (1991), 122.

[66] M. Lassig, "Multiple crossover phenomena and scale hopping in two dimensions", Julich preprint, November 1991.

[67] M. Lassig, "Exact critical exponents of the staircase model", Julich preprint, January 1992.

[68] P. Mansfield, *Nucl. Phys.* **B208** (1982), 277; *Nucl. Phys.* **B222** (1983), 419.

[69] D. Olive and N. Turok, *Nucl. Phys.* **B215** [FS7] (1983), 470.

[70] J.L. Gervais and A. Neveu, *Nucl. Phys.* **B224** (1983), 329; A. Bilal and J.L. Gervais, *Phys. Lett.* **B206** (1988), 412; A. Bilal and J.L Gervais *Nucl. Phys.* **B314** (1989), 646.

[71] A.E. Arinshtein, V.A. Fateyev and A.B. Zamolodchikov, *Phys.Lett.* **B87** (1979), 389.

[72] C. Destri and H.J. de Vega, *Nucl. Phys.* **B358** (1991), 251.

[73] M.J. Martins, *Phys. Rev. Lett.* **69** (1992), 2461.

[74] M.J. Martins, "Exact resonance A-D-E S-matrices and their renormalization group trajectories", preprint SISSA-EP-85. SISSA preprint (1992)

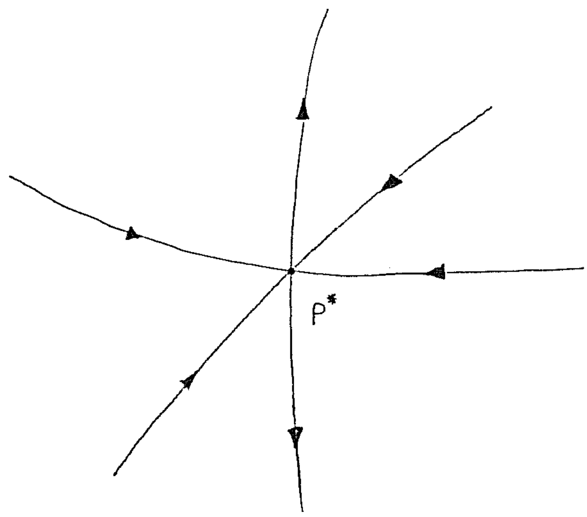
[75] P.E. Dorey and F. Ravanini, "Staircase models from affine Toda field theory", preprint SPhT/92-065, DFUB-92-09, Int. J. Mod. Phys. A in press.

[76] P.E. Dorey and F. Ravanini, "Generalising the staircase models", preprint CERN-TH.6739/92, DFUB-92-21.

[77] V.A. Fateev and S. Lykyanov, *Int. Jour. Mod. Phys.* **A3** (1988), 507; Lectures given in II Spring school *Contemporary Problems in Theoretical Physics* Kiev 1988, Preprint ITP-88-74-76P, Kiev (in russian).

[78] J.L. Cardy, *Conformal invariance and statistical mechanics*, in *Champs, Cordes, et Phénomènes Critiques*, proceedings of the 1988 Les Houches Summer School, ed. E. Brezin and J. Zinn-Justin, North Holland, Amsterdam, 1989.

[79] M. Sato, T. Miwa and M. Jimbo, *Proc. Jap. Acad.* **53**A (1977) 147,153, 183.



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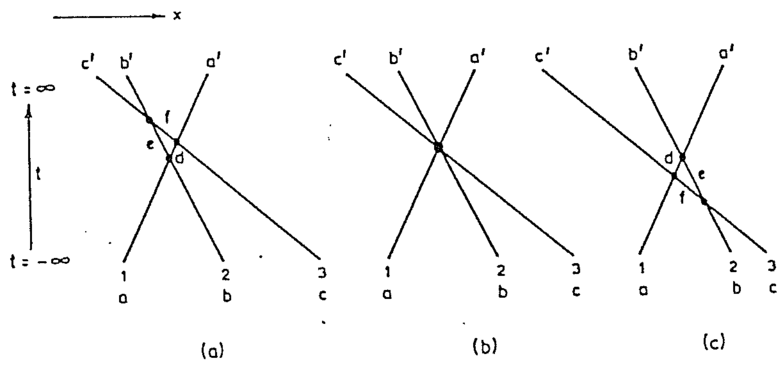
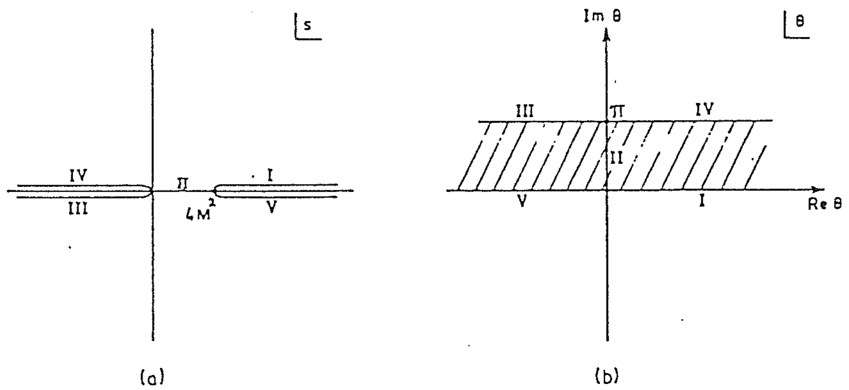


fig. 2



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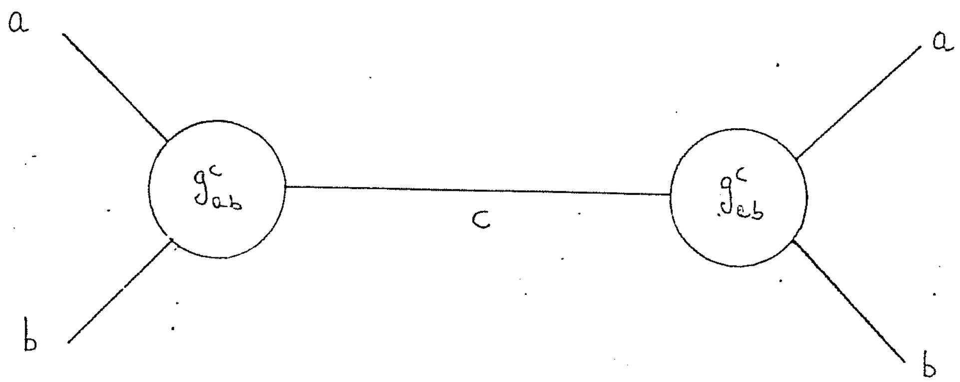

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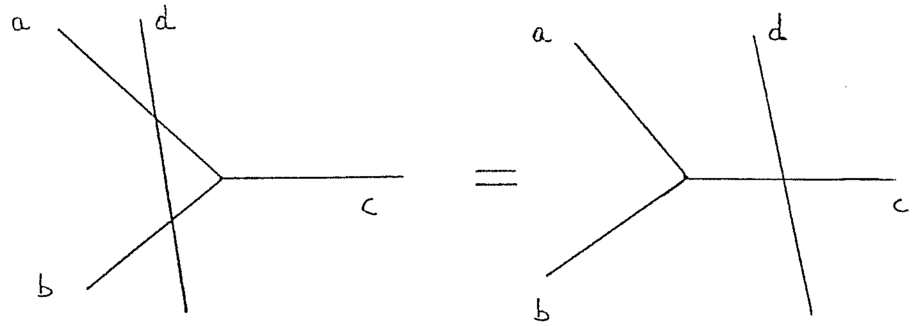

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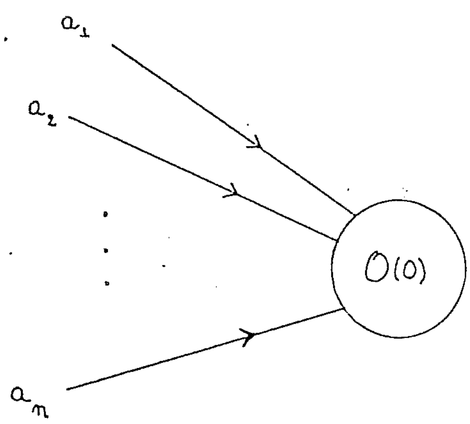

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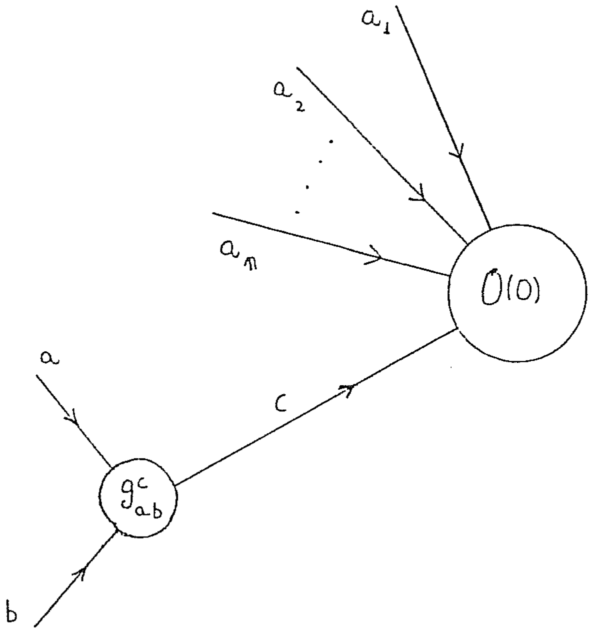

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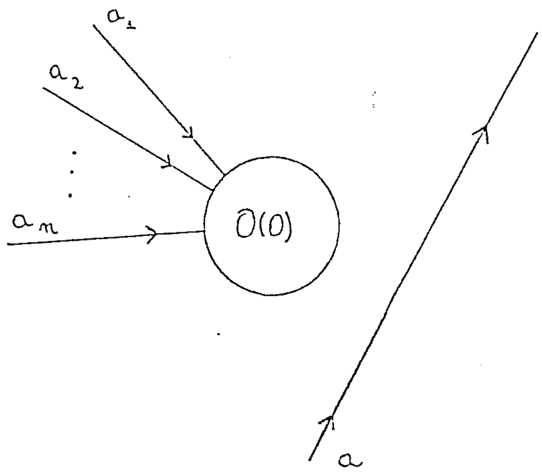
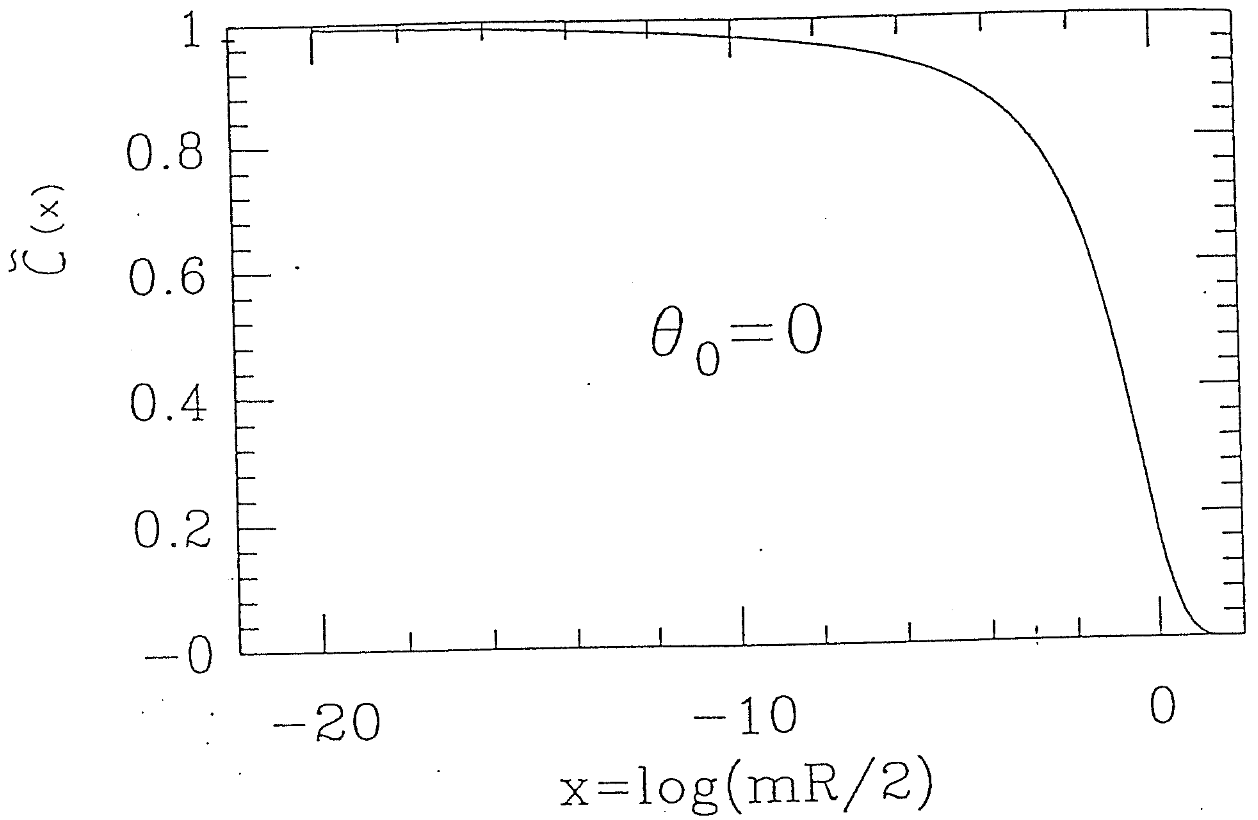


fig. 8



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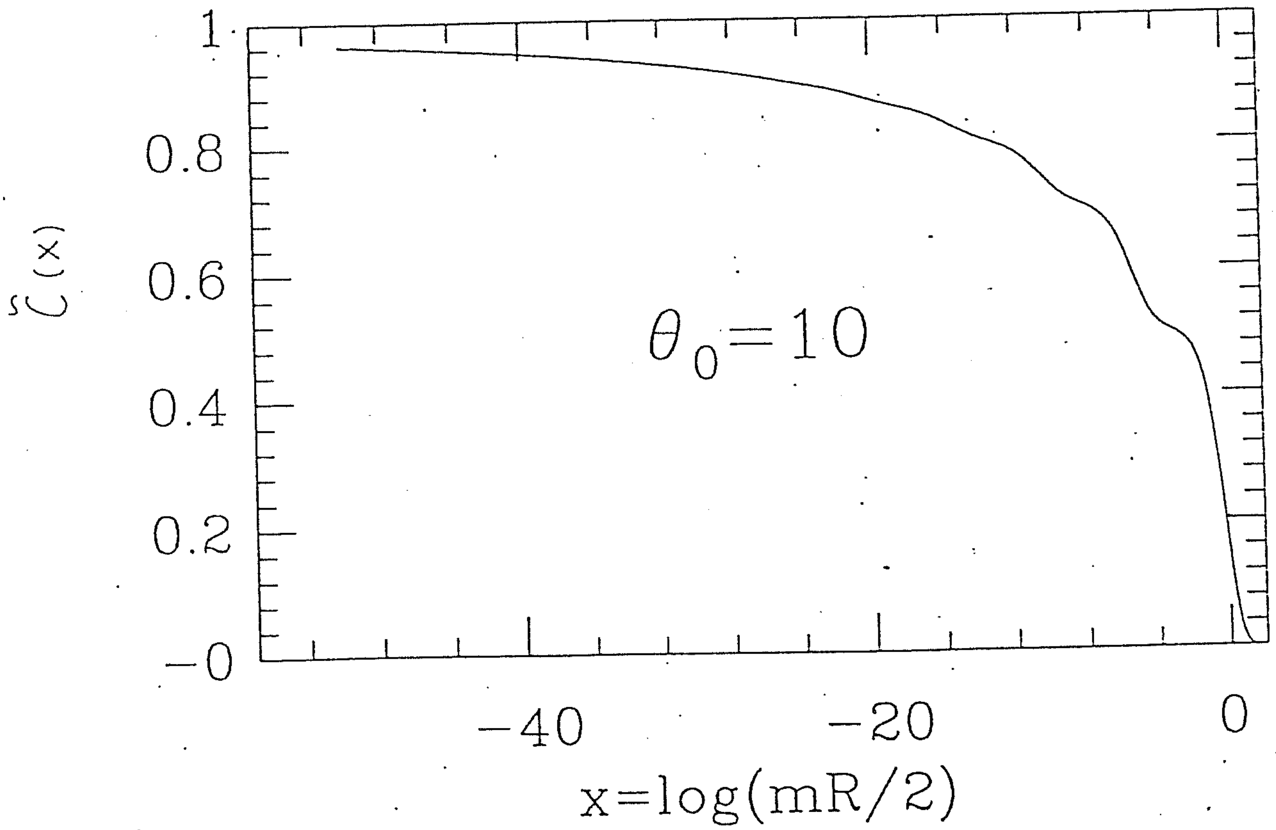
0.6
0.4
0.2
−0
$\tilde{C}(x)$
$\theta_0=0$
−20
−10
0
x=log(mR/2)

fig. 9a



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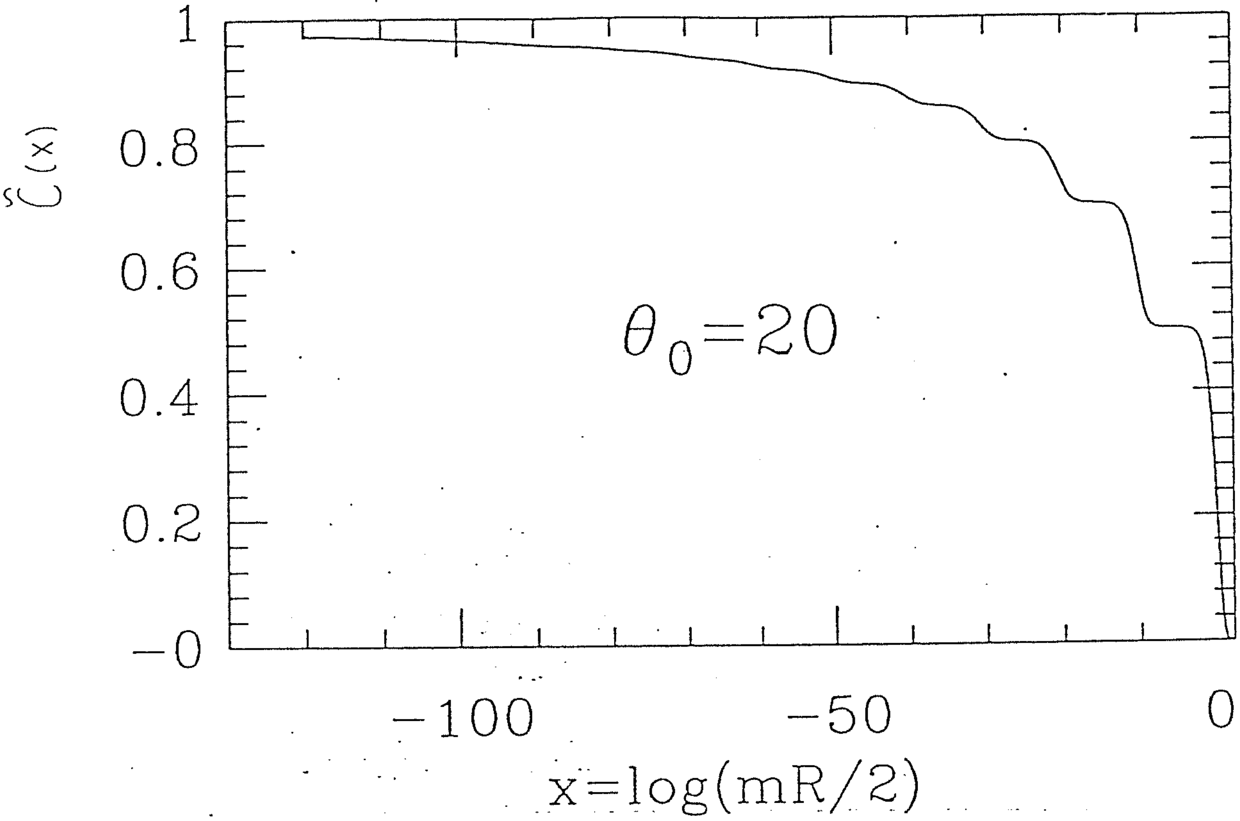
0.4
0.2
−0
$\tilde{C}(x)$
$\theta_0=10$
−40
−20
0
x=log(mR/2)

fig. 9b



1
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0.6
0.4
0.2
−0
$\tilde{C}(x)$
$\theta_0=20$
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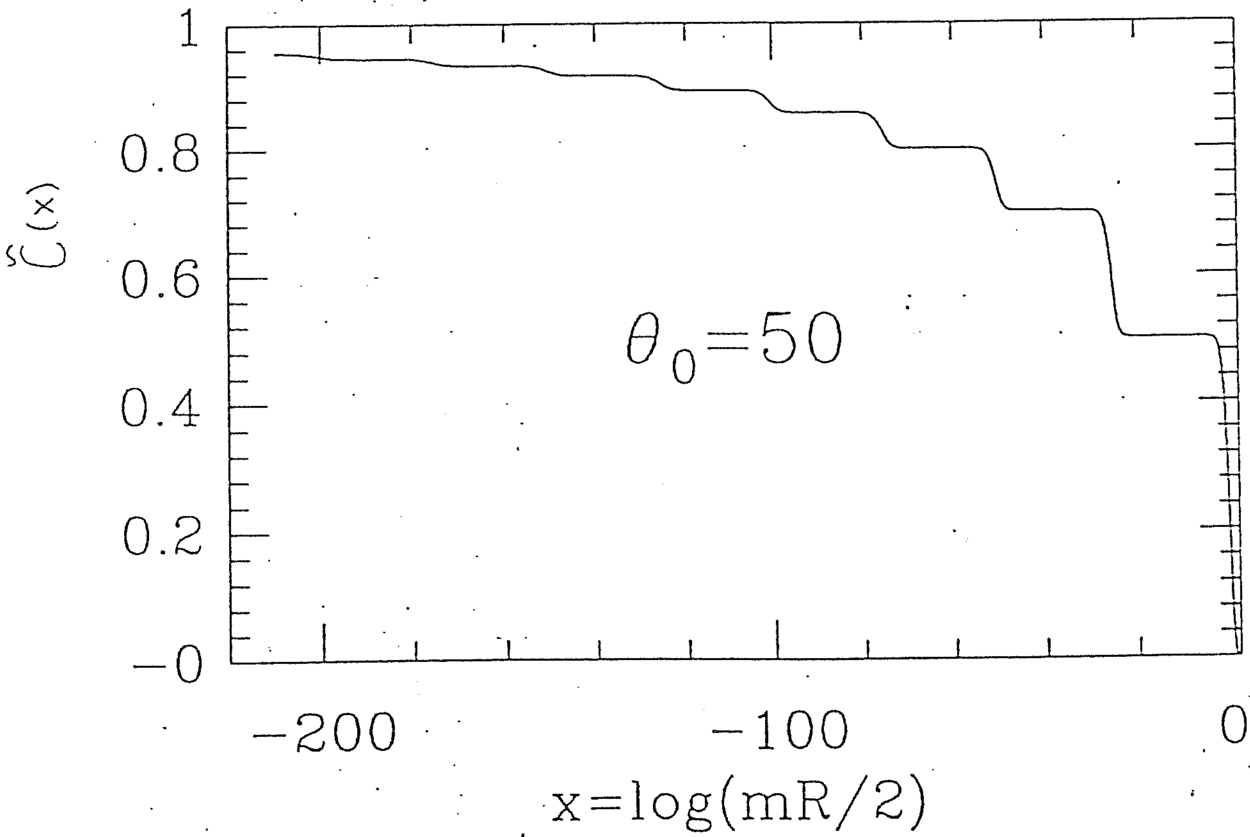
−50
0
x=log(mR/2)

fig. 9c



1
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0.2
−0
$\tilde{C}(x)$
$\theta_0=50$
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0
x=log(mR/2)

fig. 9d

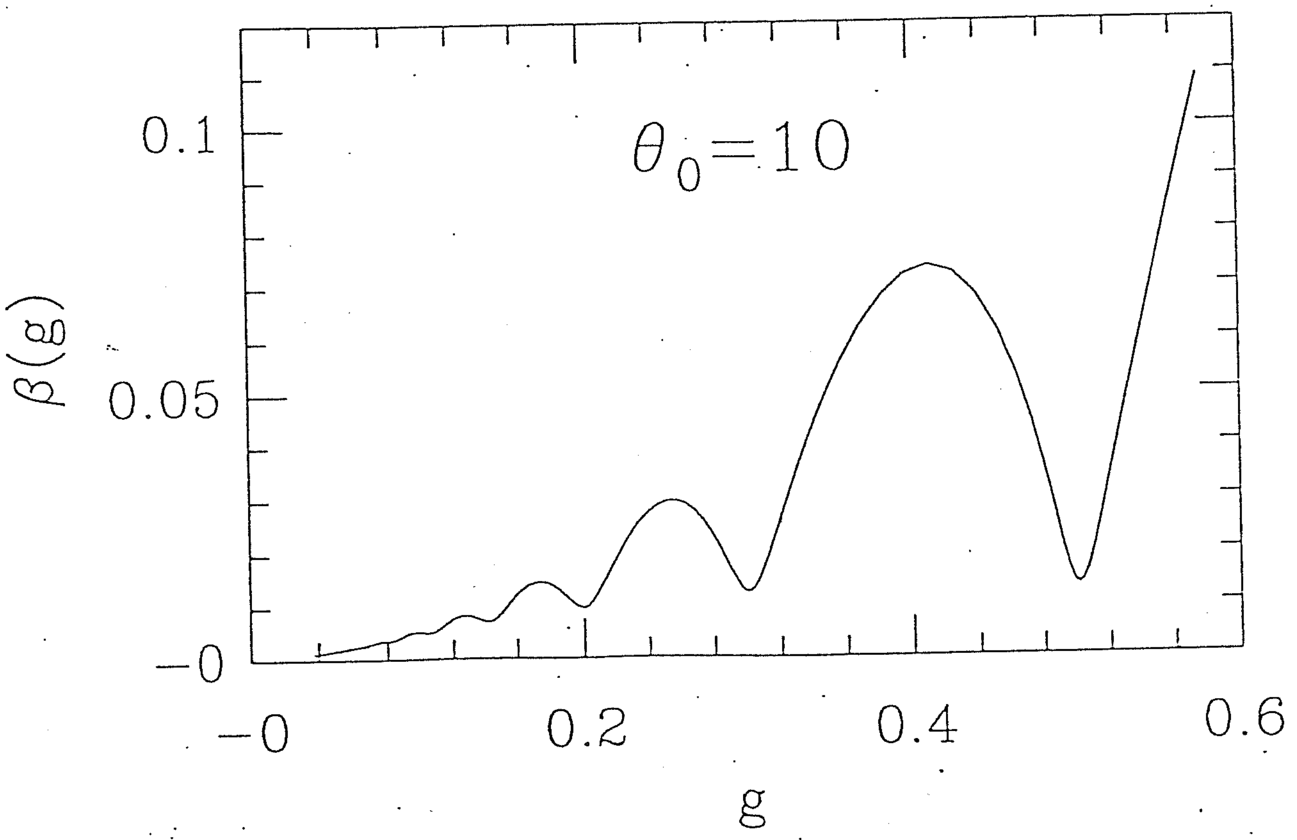

fig. 10 a

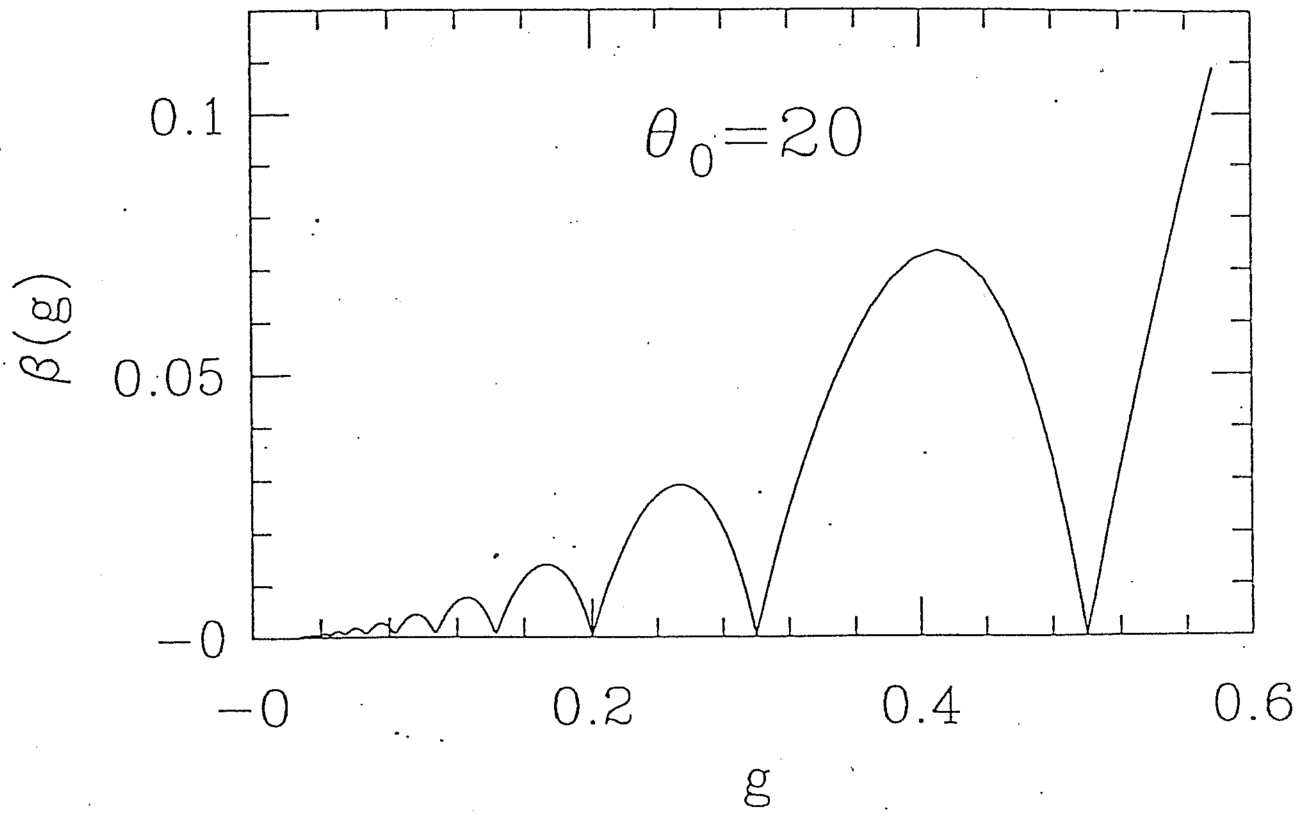

fig. 10 b

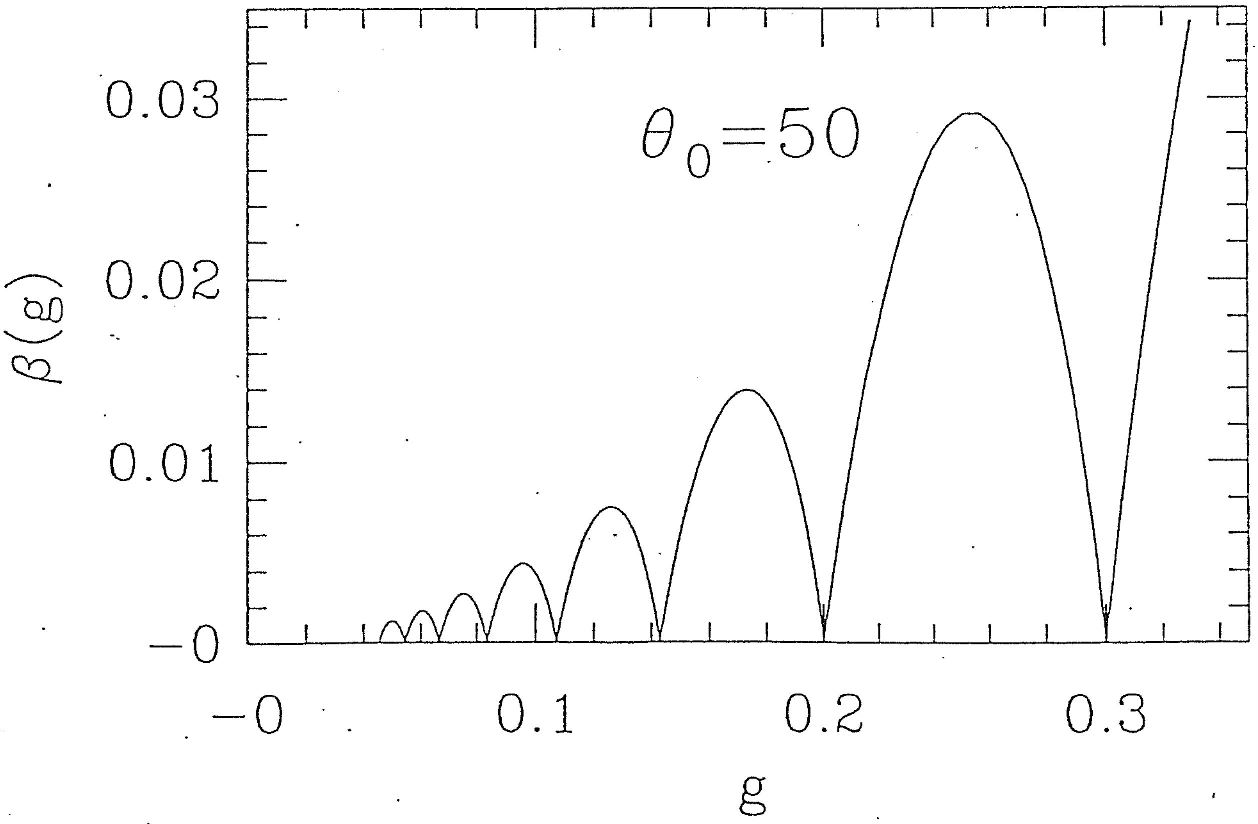

fig. 10c

3	13/8	2/3	1/8	0
7/5	21/40	1/15	1/40	2/5
2/5	1/40	1/15	21/40	7/5
0	1/8	2/3	13/8	3

table 1

B	$\frac{g^2}{4\pi}$	$\Delta c^{(2)}$
$\frac{1}{500}$	$\frac{2}{999}$	0.9999995
$\frac{1}{100}$	$\frac{2}{199}$	0.9999878
$\frac{1}{10}$	$\frac{2}{19}$	0.9989538
$\frac{3}{10}$	$\frac{6}{17}$	0.9931954
$\frac{2}{5}$	$\frac{1}{2}$	0.9897087
$\frac{1}{2}$	$\frac{2}{3}$	0.9863354
$\frac{2}{3}$	1	0.9815944
$\frac{7}{10}$	$\frac{14}{13}$	0.9808312
$\frac{4}{5}$	$\frac{4}{3}$	0.9789824
1	2	0.9774634

table 2

α	$\Delta c^{(3)}$
1	0.931891
2	0.787725
3	0.639326
4	0.554173
5	0.518329
6	0.505659
7	0.501640
8	0.500453
9	0.500121
10	0.500031
15	0.500000

table 3

B	$\Delta c^{(2)}$
0.1	1.939886
0.2	1.891538
0.3	1.852521
0.4	1.821125
0.5	1.796131
0.6	1.776660
0.7	1.762084
0.8	1.751959
0.9	1.745996
1	1.744026
1+i	1.583121
1+2i	1.316131
1+3i	1.106650
1+4i	0.965532
1+5i	0.880283
1+6i	0.834015
1+7i	0.811020
1+8i	0.800326
1+9i	0.795592
1+10i	0.793571
1+12i	0.792391
1+14i	0.792201
1+16i	0.792172
1+18i	0.792168
1+20i	0.792168

table 4

B	$\Delta c^{(3)}$
0.1	0.0218
1	0.1693
1+5i	0.1200
1+10i	0.0072
1+15i	0.0072
1+20i	0.0071

table 5