ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Applications of Thermodynamical Formalism to Nearly-circular Julia Sets

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Simonetta Abenda

SUPERVISOR

Prof. Pierre Moussa

October 1991

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE I STUDI AVANZATI

TRIESTE

Strada Costiera 11

Scuola Internazionale Superiore di Studi Avanzati
International School for Advanced Studies

Applications of Thermodynamical Formalism to Nearly-circular Julia Sets

Thesis submitted for the degree of
"Magister Philosophiæ"

CANDIDATE
Simonetta Abenda

SUPERVISOR
Prof. Pierre Moussa

October 1991

Index

Introduction

In chaotic dynamics, experiments as well as numerical simulations produce unstable trajectories from which only statistical information can be extracted. This is the case when the trajectories belong to a strange attractor, which means that they are highly sensitive to initial conditions [**RuEck**].
Although it might be easy to display a graphical representation of such an attractor, one needs a specific method to extract numerical parameters associated to it, in order to try a meaningful comparison with theoretical arguments.

At this aim the multifractal description of measures supported by strange sets [**HJKPS, F, HeP, BoR, BoT, CLP, CDM**] has been introduced in order to analyze actual experimental data in chaotic systems.
We believe that an interpretation using large deviation probability theory [**E**] should clarify many features of this thermodynamical approach for the multifractal description of strange sets and a justification of such an approach on the basis of the thermodynamic formalism has previously been given for Cantor sets invariant under expanding Markov maps by Collet et al [**CLP**].

We [**CDM, AM**] will illustrate the results in a somewhat different case, that is for measures supported on Julia sets associated to complex valued polynomial transformations on the complex plane and, in particular, we will be interested in small perturbations of the monomial case z^q, with q integer not smaller than 2, for which the Julia set is the unit circle.
The Julia sets associated to polynomials T of degree q which may be treated as a small perturbation of the monomial case, are quite untrivial objects: in fact, while

topologically still circles, they have a complicated geometric structure, since they are non-rectifiable Jordan curves [**BR, BL, Sul1, Sul2**].

These sets belong to the class of hyperbolic Julia sets, which have inspired the definition of mixing repellers [**Ru2**]. In the last years many studies have been devoted to understanding the geometric and dynamic properties of strange repellers [**Ru2, Su2, Va, Bor, BoT, BD**].

These are invariant sets under the iteration of a map characterized by the fact that each orbit, which starts in a neighbourhood, moves away from them. Sometimes the repeller arises as a boundary separating the basin of two attractors: a trajectory with initial orbit point close to the repeller will wander for a short time before falling into an attractor and this transient period is characterized by the expanding properties of the repeller itself.

To these mappings the powerful techniques of the thermodynamic formalism [**BowRu, Ru1, Ru3**] can be applied. Such a formalism is a body of ideas and results originated in equilibrium statistical mechanics and which has had a considerable impact on the study of the ergodic theory of hyperbolic differentiable dynamical systems.

Although our presentation will be centered around the Julia set case, the exposition of the multifractal formalism using large deviation arguments is more general; however the existence of the "thermodynamic limit" requires some specific hypotheses, for instance the invariance of the set and of the measure supported by it under an expansive transformation.

The key point in the case of hyperbolic Julia sets, is that the expansive transformation provides a way of comparing different scales in a uniform way, due to the so-called distortion lemma [**Su1, Su2**] which will described in the first chapter.

More precisely, we will use the thermodynamic formalism "á la Collet" [**HJKPS, CLP**] for polynomials close to z^q, [**CDM**] that is we will introduce the uniform $\mathcal{Z}_U(\beta)$ and the dynamical $\mathcal{Z}_D(\beta)$ partition functions associated to coverings of the Julia set. The thermodynamical limit of these functions exists, as the size of the pieces of the coverings goes to zero. The corresponding free energies $\mathcal{F}_U(\beta)$ and $\mathcal{F}_D(\beta)$ may be shown to be analytic functions in β and real analytic in the coefficients occuring in the polynomial T, for T small perturbation of the monomial case.

One of the consequences are the real analyticity properties, as function of the parameters of the polynomial transformation, of the Hausdorff dimension, and of the correlation dimensions of arbitrary order D_β, which form the dimension spectrum [**Ru2, CDM**].

We will show how to extend previous perturbative results [**Ru2, WBKS, CDM**] on the Hausdorff dimension for polynomials of the form $T(z) = z^q + c$, with c small complex constant, to the case of a generic polynomial $T(z) = z^q + c(z)$, with $c(z)$ polynomial of degree less than or equal to $q - 2$, which will be treated as a perturbation of the monomial case z^q [**AM**].

We will describe briefly the procedure which we have followed in order to perform perturbation expansions, that is expansions on powers of the real part and the imaginary part of the coefficients of the polynomial $c(z)$, for the various thermodynamic functions associated with the Julia set J_T and its invariant and balanced measure μ.

We will show that it is possible to express the dynamical free energy in form of a classical statistical model; then the usual perturbative methods apply and permit a perturbative expansion of both $\mathcal{F}_D$ and $\mathcal{F}_U$ and therefore of the spectrum

of dimensions D_β.
The starting point is so the computation of the dynamical free energy, for which we will use the following expression

$$\mathcal{Z}_D^{(n)}(\beta) \quad \approx \quad q^n \int_J \prod_{k=0}^{n-1} \left(\left| T'(T^k(z)) \right| \right)^\beta d\mu(z).$$

Starting from such an expression requires an effective way of computing the perturbation expansion for mean values taken on the perturbed measure μ, besides the usual expansions occuring in statistical mechanics in a case where the interactions are not limited to nearest neighbours.
This has been accomplished using the Boettcher conjugation equation [**Bl**], so to be left with the evaluation of such integrals with respect to the Lebesgue measure on the circle.
The resulting expansion in terms of c agrees with the preceding ones for the case in which the perturbation polynomial $c(z)$ reduces to a constant.

Our aim in affording such a perturbative expansion is to give a device for computation which may be applied to any polynomial, not give numerics, since all these perturbative approaches have an intrinsec limit in what they do not preserve the convexity property of the "free energy" [**CDM**], so that such computations do not reproduce the whole dimension spectrum D_β, but are valid only in a small range of β.

On the other side, our approach, allows a direct evaluation of the thermodynamic quantities, in a somewhat different spirit than the approach based on classical thermodynamic formalism [**BowRu, Ru2**], in the sense that the large deviation argument may clarify the relation between the box counting, which is related to geometric measurements, and true dimensions as the Hausdorff dimension, as it is shown in earlier papers. These considerations apply to many systems and in particular to those to which the distortion lemma applies.

A difficulty remains for the systems for which the thermodynamical function displays discontinuities: an example of this situation occurs when the box counting procedure gives rise to a non convex function $f(t)$. In this case we have to look for a more detailed way of computing partition functions, which will provide, without any doubt another interpretation in the large deviation formalism.

The study of the scaling properties of chaotic systems has also another facet, apart from the "macroscopic part", which we have analyzed whatsoever. In fact, on one side one should seek for global desciptions which enables one to predict the overall properties of the multifractals. This aspect, which has its analog in thermodynamics and has been introduced above, is centered on the study of the generalized dimensions of the set or its spectrum of scaling indeces.

On the other side, one should seek a more detailed type of approach, which enables one to get a more detailed description of the local properties of the strange set. This approach, which may be considered as the "microscopic part" of the program, has its analog in statistical mechanics and is centered in our case on the study of the so-called scaling functions [**FJP, KaP, JKP**].

Such aspect has not yet received a rigorous treatment and, fenomenologically, it has been introduced by Feigenbaum in the case of systems on the borderline of chaos [**F**].

It should be noticed that the limiting scaling function depends on the way it is constructed, and the various approaches not-necessarily in the limit are equivalent, that is give rise to the same thermodynamic functions. In the case of nearly-circular Julia sets it is possible to show that the various approaches converge to equivalent limits [**JKP**], meaning that it is possible to map the theory onto Ising models with finite range interactions. The largest eigenvalue of the associated transfer matrix then furnishes the thermodynamic functions.

We will consider a perturbative approach to the scaling properties of nearly circular Julia sets [**AM**], showing that all the self-similarity structure is already present in the first order expansion in c and show that the first order expansion in c converges to a limit we conjecture is continuos in function of the point on the Julia set. This argument is in favour of the convergence and continuity also of the limiting scaling function itself.

As a last remark, we would like to comment the fact that throughout this thesis we will restrict our discussion to the apparently academic problem of polynomial iterations, but we think of this example as of a non trivial model problem, in a domain which has been growing very fast in recent years.

Moreover we consider a dynamical system in one complex dimension. The invariant set is not an attractor and the transformation is repulsive on the set, a situation analogous to the case of Markov maps or cookie cutters.

In order to get a strange attractor, one needs to consider a non conformal map in at least two real dimensions, that is two complex dimensions for its complexified version. The relevance of our analysis to actual physical situations could be questioned, but the usual argument is that what we model is in fact the Poincaré return map associated to a diffeomorphism and the invariant measure is nothing but what we get by considering only the transverse unstable directions.

Clearly the analysis of the geometrical properties of chaotic dynamics will still receive a large development in the close future.

The plan of the thesis is as follows:

In the first chapter, we will consider the properties of Julia sets, which are needed in the subsequent chapters and which cluster around the topological and geometrical characterization of polynomial nearly circular Julia sets.

On one side the distortion lemma [**Sul1, Sul2**] is the central geometrical property which will be considered, since it allows to characterize the self-similarity of the Julia set and on the other side, since it provides uniform bounds on the expansion properties of the map, it is used throughout in the thermodynamic characterization of the Julia sets under study (existence of the thermodynamic limit, characterization of the dimension spectrum and so on).

On the other side the Boettcher conjugation theorem [**Bl**] tells us that the Julia sets under study are topologically circles; such property will be heavily used in the perturbation expansions in Chapters 3 and 4.

In the second chapter we introduce and discuss the thermodynamic formalism "á la Collet" [**CLP, CDM**] , specializing to the case of hyperbolic Julia sets embedded with the harmonic measure [**Br**].

We will define the dynamical and uniform partition function, give arguments for the existence of the thermodynamic limit and its analytic properties and discuss the relation between the Legendre transform of the uniform free energy and the dimension spectrum [**CDM**].

In the third chapter, after a preliminary discussion of previous results in the perturbative approach [**Ru2, WBKS, CDM**], we explain our perturbation results [**AM**] for the dimension spectrum starting from the dynamical partition function. In the appendix we give some details on the computation.

Finally in the forth chapter, we introduce and discuss the microscopic formalism of the scaling function [**FJP, KaP, JKP**] and, after a brief account on previous results, discuss our perturbative approach [**AM**] and the convergence arguments for the perturbation expansion terms.

The results presented in chapters 3 and 4 and in the appendix are part also of [**AM**].

Acknowledgements

I would like to express my gratitude to Professor Pierre Moussa for his stimulating supervision in the accomplishment of this work. I would like to thank him and Anna for all what they did to make pleasant and enjoyable my stay in Paris, and INFN - sezione di Bologna, for partial financial support.
I would like to thank Professor Giorgio Turchetti for his generous and helpful encouragements.
I would like to thank Professor Gianfausto Dell'Antonio for his careful reading of the manuscript and his comments on the work.
Finally, I wish to thank all the Staff Members and the Students for the active and enjoyable spirit of collaboration, that I daily breath at Sissa and, in particular, Pasquale who represents its perfect realization.

Chapter 1

Some results of the theory of Julia sets

The theory of holomorphic noninvertible dynamical systems of the Riemann sphere is an extremely fascinating and intricate subject of study in itself. The indecomposable, completely invariant sets are often fractals ([**M**]) because they are quasi-self-similar ([**Sul1,Sul2**]), sometimes they are Jordan curves whose Hausdorff dimension is greater than one; yet these sets are determined by a single analitic function $z_{n+1} = R(z_n)$ of a single complex variable.

The study of this subject started with P. Fatou [**Fa1, Fa2**] and G. Julia [**Ju**]. They applied the theory of normal families and, in particular, Montel's theorem, to prove some remarkable results. The fundamental one is that the Riemann sphere splits into two disjoint sets: the closure of the repelling periodic points, which is called the Julia set, and the open set of stable points, which is sometimes referred to as Fatou set.

Recently there has been an explosion of interest in the subject and many mathematicians have made substantial contributions. Among them, we quote, as an example of how a "simple" dynamical system can have a complicated dynamics, Sullivan's complete classification of the dynamics (see [**Sul3,Sul4**]) in the domains of normality, and the work Hubbard and Douady [**DH**] on the dynamics of quadratic polynomials surely must be quoted.

In this chapter we will limit ourselves to present the properties of Julia sets which are relevant for our purposes, since a comprehensive survey of classical and recent results would bring us too far away from the aim of this thesis.
In particular we recall that we are interested in polynomials with complex coefficients $T(z)$ of degree $q \geq 2$ of the form

$$T(z) \quad = \quad z^q + \lambda\, c(z), \tag{1.1}$$

where $c(z)$ is a polynomial of degree at most $q-1$ and λ is a small complex scale parameter.

The definition of Julia set is given in section 1. Section 2 contains some fundamental classical consequences of Montel's theorem. In section 3 we introduce the conjugation function and show that it can be analytically extended outside the Julia set. Section 4 contains some important results about the geometric characterization of nearly circular Julia sets: the distortion lemma gives in fact a meaning to the notion of self-similarity of such sets, while a therem of Ruelle shows that the Hausdorff dimension depends analytically on the map. Finally, in section 5 we give some results about the invariant measure which can be defined on the Julia set.

1.1 Definitions

Let us consider a discrete dynamical system of the Riemann sphere $\mathbb{C}$ generated by a holomorphic transformation

$$R \;:\; \bar{\mathbb{C}} \;\rightarrow\; \bar{\mathbb{C}}. \tag{1.2}$$

The phase space is then the unique, simply connected, closed Riemann surface $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ which is homeomorphic to the 2-dimensional sphere

$$S^2 = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | \ x_1^2 + x_2^2 + x_3^2 = 1\}.$$

If we use the variables z and $w = \frac{1}{z}$ to represent the two standard coordinate charts on $\bar{\mathbb{C}}$ determined by stereographic projection, then any holomorphic map R of $\bar{\mathbb{C}}$ can be written in the form

$$R(z) = \frac{T(z)}{Q(z)}, \tag{1.3}$$

where T and Q are polynomials with complex coefficients and no common factors. Hence, there is a one-to-one correspondence between rational function, (1.3), and holomorphic maps (1.2).

The *degree*, $deg(R)$, of any continuous map $R : S^2 \to S^2$ is a homotopy invariant which measures how many times R applies S^2 into itself and is defined as

$$deg(R) = \max \ \{deg(T), \ deg(Q)\},$$

when $R(z)$ is written in the form of Eq. (1.3).
It is also the number (counted with multiplicity) of the inverse images of any point of $\bar{\mathbb{C}}$. The theory of Fatou and Julia applies to rational maps R whose degree is at least two.

A dynamical system is formed by repeated iteration of the map R from $\bar{\mathbb{C}}$ to itself.

Definition: given $z_0 \in \bar{\mathbb{C}}$, the sequence $\{z_n\}$, inductively defined by

$$z_{n+1} = R(z_n),$$

is called the *forward orbit* of z_0 and denoted by $\mathcal{O}^+(z_0)$.

I denote by R^n the n-fold composition $R^n = R \circ R \circ R \circ \ldots \circ R$ of the function R with itself.

Definition: Let $\mathcal{U}$ be an open subset of $\bar{\mathbb{C}}$, with the spherical metric, $\mathcal{F} = \{f_i | i \in I\}$ a family of meromorphic functions defined on $\mathcal{U}$ with values in $\bar{\mathbb{C}}$, where I is any index set. $\mathcal{F}$ is a *normal family* if every sequence f_n contains a subsequence f_{n_i} which converges uniformly on compact subsets of $\mathcal{U}$.

The condition of normality can be shown to be equivalent to that of equicontinuity on every compact subset of $\mathcal{U}$ (Arzela's theorem, see [**Ahl**] for details).

Definition: a point $z \in \bar{\mathbb{C}}$ is an element of the *Fatou set* $F(R)$ of R if there exists a neighborhood $\mathcal{U}$ of z in $\bar{\mathbb{C}}$ such that the family of iterates $\{R^n | \mathcal{U}\}$ is a normal family. The *Julia set* $J(R)$ is the complement of the Fatou set.

The Fatou set is open by definition and, since R is continuous and an open mapping, F is completely invariant, that is $z \in F$ implies $R(z) \in F$ and $R^{-1}(z) \in F$. Consequently, $J(R)$ is also completely invariant and compact (see [**Br, Bl**] for instance).

It is easy to show by contradiction that the Julia set is never empty.

Definition: Let z_0 be a periodic point of period n. Then the number $\gamma_{z_0} = (R^n)'(z_0)$ is the *eigenvalue* of the periodic orbit.

Since by the chain rule γ_{z_0} is the product of the derivatives of the map R along the orbit, then it is an invariant of the orbit $\mathcal{O}^+(z_0)$.

Definition: A periodic orbit $\mathcal{O}^+(z_0)$ is:

(1) *attractive* if $0 < |\gamma_{z_0}| < 1$;

(2) *superattracting* if $\gamma_{z_0} = 0$;

(3) *repelling* if $|\gamma_{z_0}| > 1$;

(4) *neutral* if $|\gamma_{z_0}| = 1$.

From the Mean Value Theorem and Arzela's theorem it follows that attractive and superattractive periodic orbits $\mathcal{O}^+(z_0)$ are contained in F, repelling periodic orbits in J ([Bl]).

1.2 Basic properties of Julia sets

In this section we quote some of the classical basic results of the theory of Julia sets (see theorem 1.2 and corollary 1.3), which will be used in the following. In particular the fact that the Julia set is the closure of repulsive periodic orbits and a perfect set is a direct consequences of the following theorem of Montel, which gives a sufficient condition for normality of families of meromorphic functions.

Theorem 1.1

Let $\mathcal{F}$ be a family of meromorphic functions defined on a domain $\mathcal{U}$. Suppose there exist points $a, b, c \in \bar{\mathbb{C}}$ such that $[\cup_{f\in\mathcal{F}} f(\mathcal{U})] \cap \{a, b, c,\} = \emptyset$. Then $\mathcal{F}$ is a normal family on $\mathcal{U}$.

Definition: Let $\mathcal{U}$ be a any neighborhood of z $J(R)$ then the *set of exceptional points for z* is defined as $E_z = \cup E_{\mathcal{U}}$, where $E_{\mathcal{U}} = \bar{\mathbb{C}} - \cup_{n>0} R^n(\mathcal{U})$.

Notice that as $\mathcal{U}$ becomes smaller, then $E_{\mathcal{U}}$ "grows" and it can also be considered as the inductive limit of $E_{\mathcal{U}}$, as $\mathcal{U}$ goes to z. From Montel theorem, it immediately follows that $E_{\mathcal{U}}$ at most contains two points. It then follows that $E_{\mathcal{U}}$ is indipendent of $\mathcal{U}$ for sufficiently small $\mathcal{U}$ and E_z contains at most two points (see for example

[Bl]).

In the case of polynomials $T(z)$, the point at ∞ is very special: it is a superattractive fixed point whose only inverse image is itself; hence $J(T)$ is contained in $\mathbb{C}$ and ∞ is an exceptional point.

Theorem 1.2

The Julia set J has the following properties:

(i) *J is a perfect set;*

(ii) *J equals the closure of the repelling periodic points;*

(iii) *If $z \in \bar{\mathbb{C}} - E$, then $J \subset$ {accumulation points of $\cup_{n\geq 0} R^{-n}(z)$}. Consequently, if $z \in J$, then $J =$* closure $\{\cup_{n\geq 0} R^{-n}(z)\}$;

(iv) *Let p be an attractive fixed point of R and call the stable set of p the set $\mathcal{W}^s(p) = \{z|\ R^n(z) \to p \text{ as } n \to \infty\}$, then the frontier of $\mathcal{W}^s(p)$ is J.*

Idea of the proof:(see [Bl] or [Br] for details). *(i)*: since we have already seen that $J(R)$ is a non empty, closed, compact set, it is sufficient to show that it is dense in itself. This follows from Montel's theorem and the fact that if $a \in J$, then there exists $b \in J$ such that $a \in \mathcal{O}^+(b)$, but $b \notin \mathcal{O}^+(a)$.

For *(iii)* one uses the definition of exceptional set and property *(i)* of this theorem. The proof of *(ii)* is made up of two fundamental steps: one first shows that J is contained in the closure of the set of periodic points using Montel's theorem; then one bounds the number of attracting periodic orbits plus half of the number of neutral periodic orbits by a finite number $2q-2$, where q is the degree of R. Then (ii) easily follows.

(iv) is a consequence of the observation that J does not contain proper, closed, completely invariant subsets of J.

One important consequence of *(ii)* is that neighbourhoods of points in J are eventually surjective, that is J is "locally eventually onto" (leo),

Corollary 1.3

Let A be a closed subset of $\bar{\mathbb{C}}$ such that $A \cap E = \emptyset$. Given a neighbourhood $\mathcal{U}$ of a point $p \in J$, there exists an integer N such that $A \in R^N(\mathcal{U})$.
Therefore, if D is a domain such that $D \cap J \neq \emptyset$, then there exists N such that $R^N(D \cap J) = J$.

The proof follows from the previous theorem and the fact that A is compact (see [Bl] for details). It may be considered as a first justification of the term "fractal" applied to Julia sets, we will comment the notion of self-similarity for Julia sets later in the present chapter.

1.3 The conjugation function

Both superattracting and attracting periodic orbits are locally conjugated to simple maps of disks through analytic homeomorphisms. In the attractive case $R|\mathcal{U}$ is conjugated to the rotation $\gamma\ z$, (see [SM]) so that the neighbourhood $\mathcal{U}$ is forward invariant, i.e. $R^n(\mathcal{U}) \subset \mathcal{U}$ and the orbit of every point in $\mathcal{U}$ is asymptotic to the periodic orbit $\mathcal{O}^+(z_0)$.

The following theorem, known as Boettcher conjugacy theorem (see [Bl] for the proof), shows that the dynamics in the superattracting case is much more interesting. It applies to any analytic function, not just to rational maps. We denote by $(R^n)^{(k)}(z_0)$ the k-th derivative of R^n.

Theorem 1.4

Let $\mathcal{O}^+$ be a superattracting periodic orbit. Suppose $k \geq 2$, $(R^n)^{(k)}(z_0) \neq 0$, *and*

$$(R^n)'(z_0) = (R^n)^{(2)}(z_0) = \ldots = (R^n)^{(k-1)}(z_0) = 0.$$

Then there exists a neighbourhood $\mathcal{U}$ *of* z_0 *and an analytic homeomorphisn* Φ : $\mathcal{U} \to D_r$ *(for some* r*) such that* $\Phi(z_0) = 0$ *and* $\Phi'(z_0) = 1$, *and the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{\quad R^n \quad} & \mathcal{U} \\ \Phi \big\downarrow & & \big\downarrow \Phi \\ D_r & \xrightarrow[\quad z \to z^k \quad]{} & D_r \end{array}$$

As in the periodic case $\mathcal{U}$ is a forward invariant neighbourhood and every orbit in it is asymptiotic to $\mathcal{O}^+(z_0)$, but the map $z \to z^k$ is not locally invertible: this gives a great deal of information which is particularly helpful in the analisys of the dynamics of polynomials.

Let $T(z)$ be a polynomial of degree q. As we have already mentioned, the point at infinity plays a special role since it is an exceptional superattractive fixed point such that $T^{-1}(\infty) = \{\infty\}$. Consequently, the Fatou set is never empty, the Julia set is contained in a bounded set of the complex plane and the stable set $\mathcal{W}^s(\infty)$ coincides with the immediate basin of attraction $\mathcal{A}(\infty)$, that is the maximal domain containing ∞ on which the family $\{T^n\}$ is normal. Moreover, Boettcher theorem establishes the existence of a neighbourhood $\mathcal{U}$ of infinity and of a real number $r \geq 1$ such that $T|\mathcal{U}$ is analytically conjugated to the map $z \to z^q$ restricted to the set $\bar{\mathbb{C}} - D_r$.

Since this conjugancy equation will be extensively used in the following, it is important to know what is its maximal domain of definition. The following theorem ([Br] and [Bl]) gives the answer:

Theorem 1.5

The following statements are equivalent:

(i) The map $T|\mathcal{A}(\infty)$ is analytically conjugated to the map $z \to z^q$ restricted to the exterior of the unit circle;

(ii) The set $\mathcal{A}(\infty)$ is simply connected;

(iii) The Julia set J is connected;

(iv) The sets $\mathcal{A}(\infty)$ and $C' \equiv \{$ finite critical points $\}$ are disjoint. In other words, for every finite critical point z, the sequence of successive images $T^n(z)$ remains uniformly bounded.

Proof: $(i) \Rightarrow (ii)$. This is due to the fact that the conjugagy is a homeomorphism between a disk and $\mathcal{A}(\infty)$.

$(ii) \Rightarrow (iii)$. A plane set is simply connected if and only if its boundary is connected, and in the present case $J(T) = \partial\mathcal{A}(\infty) = \partial\mathcal{W}^s(\infty)$.

$(iii) \Rightarrow (iv)$. It can be shown that $\mathcal{A}(\infty) \cap C' = \emptyset$ implies that $J(T)$ is disconnected. At this purpose define $D = \{z| \ |z| > r\}$ such that $T(D) \subset D \subset \mathcal{A}(\infty)$ and $D_n = T^{-n}(D)$.

Let $h : \mathcal{U} \to D$ be the conjugacy between $T(z)$ and $z \to z^q$ in a neighbourhood of infinity and call $A_0 = h^{-1}(D_{r^d} - D_r)$.

Construct an orthogonal coordinate system on $\mathcal{U}$ where circles concentric to $\partial\mathcal{U}$ are wrapped by $T(z)$ onto other such circles in a d-to-1 fashion.

Try to extend the conjugacy by taking inverse images by $T(z)$ and define $A_{-1} = T^{-1}(z)$. If $A_0 \cap T(C') = \emptyset$, then $T : A_{-1} \to A_0$ is a d-fold covering and A_{-1} an annulus. So one can pull back the coordinate system and define the conjugacy $h : A_{-1} \to D_s$ with $s = r^{\frac{1}{d}}$ and can go on iteratively to define $A_{-k} = T^{-1}(A_{-k+1})$ as long as $A_{-k+1} \cap T(C') = \emptyset$.

But, since $\mathcal{A}(\infty) \cap C' \neq \emptyset$, there exists an integer N such that $T(c) \in A_{-N}$ for some $c \in C'$. In this case $T : T^{-1}(A_{-N}) \to A_{-N}$ is not a covering space, but a branched cover.

The inverse images of the Jordan curve l trough $T(c)$ is a pinched curve which bounds at least two finite open sets and these sets disconnect $J(T)$.

$(iv) \Rightarrow (i)$. Use the same notations as before to denote the annuli A_{-k} and the fact that $T : A_{-k} \to A_{-k+1}$ is always a d-fold covering space. Then $\mathcal{A}(\infty) = \mathcal{U} \cup (\cup_{n=0}^{\infty} A_{-n})$.

1.4 Fractal properties of nearly circular Julia sets

In this section we will give a sufficient condition of fractality for the Julia set of a rational map. The word "fractal" has been introduced by Mandelbrot [**M**] to indicate sets which "look alike everywhere" and whose Hausdorff dimension is different from their Euclidean one. D. Sullivan [**Sul1, Sul2**] has given a simple description of self-similarity when the map $R|J$ is expanding.

Definition: $J(R)$ is *quasi-self-similar* if there exist constants K and r_0 such that, for all $x \in J(R)$ and for all $r < r_0$, $\frac{1}{r}[J \cap D_r(x)]$ is mapped onto J by a K-quasi-isometry, that is a bijection that distorts distances between $1/K$ and K.

That means that each small piece of J can be expanded to a standard size and then mapped into J by a K-quasi isometry.

Proposition 1.6

If $|R'(z)| > 1$ *for all* $z \in J(R)$, *then* $J(R)$ *is quasi-self-similar.*

This proposition is called distortion lemma and is due to D. Sullivan. We will prove an equivalent result in the case of small perturbations of the polynomial mapping $z \longrightarrow z^q$.

At this aim let $T(z) = z^q + \lambda\, c(z)$ be a polynomial of degree $q \geq 2$, with $deg(c) \leq q-1$. Choose $\rho > 1$ in such a way that $|T'(z)| > \eta > 1$ in the annular region $A_\rho = \{z \in \mathbb{C} | \frac{1}{\rho} < |z| < \rho\}$, for some η, that is T is expansive on A_ρ.

Finally take δ to be approximately equal half the distance between two consecutive roots of unity of order q. Then under the previous assumptions, the following proposition holds:

Proposition 1.7

Consider two orbits of length n, x_i *and* y_i, *for* $i = 1, ..., n$, *defined for all* $i \geq 1$ *by* $x_{i+1} = T(x_i)$ *and* $y_{i+1} = T(y_i)$, *such that* $|x_i - y_i| < C < \delta$ *and* $x_i, y_i \in A_\rho$, $\forall i$. *Then*

(i) $|x_i - y_i| < C\, \eta^{i-n}, \quad i = 1, ..., n;$

(ii) *there exists* $\gamma > 1$ *such that for all* $j = 1, ..n$:

$$\frac{1}{\gamma} \quad < \quad \left| \frac{(T^j)'(x_1)}{(T^j)'(y_1)} \right| \quad < \quad \gamma; \tag{1.4}$$

(iii) *for any sets* I *and* J *such that* $I_n, J_n \subset A_\rho$ *where* $I_n = T^n(I)$ *and* $J_n = T^n(J)$ *with* $\operatorname{diam}(I), \operatorname{diam}(J) < \delta$ *and such that* I *and* J *are obtained from* $T^n(I)$ *and* $T^n(J)$, *respectively, by application of the same inverse branch of* T^n, *we have:*

$$\frac{1}{\gamma} \quad < \quad \left(\frac{\text{diam } (I)}{\text{diam } (J)} \right) \Big/ \left(\frac{\text{diam } (T^j(I))}{\text{diam } (T^j(J))} \right) \quad < \quad \gamma, \qquad \forall j = 1, ..., n. \tag{1.5}$$

Proof: *(i)* follows immediately by the hypotheses; to prove *(ii)*, observe that

$$\left| \log \left| \frac{(T^j)'(x_1)}{(T^j)'(y_1)} \right| \right| \quad \leq \quad \sum_{k=1}^{j} |\log |T'(x_k)| - \log |T'(y_k)||$$

$$\leq \quad K \sum_{k=1}^{j} |x_k - y_k| \quad \leq \quad K \sum_{k=1}^{j} \eta^{k-j} \quad \leq \quad \frac{\eta}{\eta - 1},$$

where K is a constant (remind that $\log(T')$ is holomorphic in the region A_ρ.
(iii) is also immediate: recall that $\mathrm{diam}(I) = \sup_{x,y\in I} |x-y|$; then choose $x, y \in I$ and $u, v, \in J$, such that they realize the sup up to epsilon. Then from (ii) the assertion follows.

In the previous section, it has been shown that the Julia set associated to a polynomial T of degree q which can be treated as a small perturbation of the map $z \rightarrow z^q$ topologically is a circle. However, geometrically it differs from a circle, since it can be shown to be a non rectifiable Jordan curve.
This last statement is a consequence of the following theorems (see [**Br**] for instance):

Theorem 1.8

Let J(T) be the Julia set associated to a polynomial; then, if T has a (super)attractive fixed point α such that the set of finite critical points of $T^{-1}(z)$ is contained in the stable set $\mathcal{W}^s(\alpha)$, then $J(T)$ is a Jordan curve.

Theorem 1.9

Let α be a (super)attractive fixed point of the rational map $R(z)$. Suppose that $\mathcal{W}^s(\alpha)$ is simply connected and that $\partial\mathcal{W}^s = J_\alpha$ has empty intersection with the closure of the set of critical points of R. Then, if J_α is not a circle or a straight line, J_α does not have a tangent at any point.

Corollary 1.10

Let $T(z) = z^q + \lambda\ c(z)$, be a polynomial of degree q, where deg(c) $\leq q-1$, then there exists $\epsilon > 0$ such that for $0 < |\lambda| < \epsilon$ the Julia set $J(T)$ is a non-rectifiable Jordan curve.
The Hausdorff dimension δ_H of $J(T)$ is $1 < \delta_H < 2$.

D. Ruelle [**Ru2**] has then proved the following conjecture by D. Sullivan on the dependence of the Hausdorff dimension $\delta_H(R)$ from R, as R varies in an analytic family of maps.

Theorem 1.11

The Hausdorff dimension of $J(R)$ is a real analytic function of the coefficients of $z \to R(z)$ in any open connected set where each such map is expanding.

1.5 The invariant measure on the Julia set

Let E be a bounded closed set in the z-plane and let μ be a positive mass distribution on E of finite total mass. The logarithmic potential is then defined by

$$u(z) \quad = \quad \int_E \log \frac{1}{|z-\zeta|}\, d\mu(\zeta)$$

and the energy integral by

$$I(\mu) \quad = \quad \int_E \int_E \log \frac{1}{|z-\zeta|}\, d\mu(\zeta)\, d\mu(z).$$

Set $V = \inf_{\mu,\mu(E)=1} I(\mu)$, then the capacity $\gamma(E)$ of E is defined by

$$\gamma(E) \quad = \quad \exp(-V).$$

Throughout this section we will restrict ourselves to polynomials of the form $T(z) = z^q + a_{q-1}z^{q-1} + \cdots + a_0$, with $q \geq 2$.

It is then possible to show (see [Br] for details) that the capacity of the Julia set $J(T)$ is equal to $\gamma(J) = 1$ and that the support of μ^*, the equilibrium distribution of J (i.e. $\mu^*(J) = 1$ and $I(\mu^*) = V$,) is $J(T)$ itself.

We can then introduce a sequence $\{\mu_n\}$ of mass distributions defined as follows [**Br**]: μ_0 places the mass 1 at a fixed point z_0 in the plane, except the exceptional points.

μ_1 places the mass q^{-1} at the q predecessors of order 1 of z_0 and, in general, μ_n places the mass q^{-n} at the q^n predecessors of order n of z_0. Then it is possible to show that this sequence converges weakly to the equilibrium distribution $\mu^*(J)$, (see [**Br**] for instance).

Moreover measure μ^* is invariant under T, ([**Br**]) that is

$$\mu^*\big(T^{-1}(B)\big) \quad = \quad \mu^*(B), \tag{1.6}$$

and balanced ([**BD, BGH**]), which means

$$\mu^*(T_i^{-1}(B)) \quad = \quad (1/q)\,\mu^*(B), \qquad \text{for } i = 1, ..., q, \tag{1.7}$$

where B is any Borel set in the complex plane.

A consequence of balancedness is that, if B is sufficiently small and if the various preimages of $B' \equiv T(B)$ are well separated, then

$$\mu^*\big(T(B)\big) \quad = \quad q\,\mu^*(B) \tag{1.8}$$

or for two different sufficiently small Borel sets I and J with $\mu^*(J) \neq 0$, we have

$$\left(\frac{\mu^*(T(I))}{\mu^*(T(J))}\right) \quad = \quad \left(\frac{\mu^*(I)}{\mu^*(J)}\right), \tag{1.9}$$

where diam(I), diam$(J) < \delta$, with δ as in Proposition 1.7, so that the preimages of I, J will be disjoint at least for ϵ small enough.

Chapter 2

Application of thermodynamic formalism to expanding maps

In the last few years, several authors ([**Ru1, Ru2, BD, BoR, CLP, Va, Ru3**]) have considered the ergodic properties of a transformation on a repeller in connection with its geometric structure. If the transformation is sufficiently regular in a neighbourhood of the repeller and uniformly hyperbolic on it (mixing repeller), one can apply the powerful techniques of the axiom A systems [**BowRu**] to understand the dynamics on the repeller and its fractal properties.

In this chapter, we will especially consider conformal transformations on the repeller, i.e. in every point the tangent map is a scalar times an isometry. Sullivan (see [**S1, S2**] and Chapter one for what concerns the case of hyperbolic Julia sets) has already pointed out the importance of these systems which include all the one-dimensional differentiable maps, the rational endomorphisms of the Riemann sphere and every group of Möbius transformations of the n-sphere.

The plan of the chapter is as follows: in section 1, we introduce the uniform partition function $\mathcal{Z}_U$. In section 2, using large deviation arguments we recover the multifractal spectrum and relate it to the measure geometric dimensions. In section 3, we give the relation between the dimension spectrum and the set of generalized correlation dimensions. In section 4, we introduce the dynamical partition function

$\mathcal{Z}_D$ and show how the "free energy" $\mathcal{F}_D$ is related to $\mathcal{F}_U$. Finally in section 5, we discuss the analyticity properties of the thermodynamic functions, from which the analyticity properties of the geometric dimensions immediately follows.

2.1 The uniform partition function $\mathcal{Z}_U$

Let us consider a normalized probability measure μ with support contained in a bounded set S in the complex plane, that is $\mu(S) = 1$. The model we have in mind is the harmonic measure μ supported on a nearly-circular Julia set associated to a map T of the form

$$T(z) \;=\; z^q + \lambda\, c(z) \tag{2.1}$$

which has been introduced in the previous chapter.

Let us cover S with squares of size 2^{-n} forming a partition for any n over a bounded set in the complex plane containing S. We will call $\mathcal{P}_n$ such a partition. For n large, the number of square boxes needed to cover S is bounded by $A\ 2^{2n}$, where A is the area of some bounded set including S. Now select boxes b such that for $t > 0$:

$$t \;<\; \frac{(-1)}{n}\log_2(\mu(b)) \;<\; t+\epsilon, \tag{2.2}$$

and call $N_n(t)$ their number. We then make the following fundamental ansatz, known under the name of "box counting" assumption, on the behaviour of $N_n(t)$ for n large:

$$\lim_{n\to\infty} \frac{1}{n}\log_2\left(N_n(t)\right) \;=\; f(t). \tag{2.3}$$

In the case of hyperbolic Julia sets, for which the thermodynamic limit exists and is regular, the box counting "ansatz" receives a rigorous treatment in the frame of large deviations, as we will see in the next section.

In order to introduce the thermodynamic formalism, it is now convenient to introduce the uniform partition function $\mathcal{Z}_U$ defined by:

$$\mathcal{Z}_U^{(n)}(\beta) \quad = \quad \sum_b (\mu(b))^\beta. \tag{2.4}$$

The previous sum extends over all boxes b needed to cover S, such that $\mu(b) \neq 0$, so to allow negative values for β.

The box counting assumption allows to evaluate the behaviour for large n of $\mathcal{Z}_U^{(n)}(\beta)$. Indeed we have:

$$\mathcal{Z}_U^{(n)}(\beta) \quad \simeq \quad \int_0^\infty N_n(t)\, 2^{-n\beta t} dt. \tag{2.5}$$

Since $N_n(t) \sim 2^{nf(t)}$, we can estimate the partition function as:

$$\mathcal{Z}_U^{(n)}(\beta) \quad \simeq \quad \int_0^\infty 2^{n(f(t)-\beta t)} dt \qquad (2.6).$$

Therefore $\mathcal{Z}_U^{(n)}(\beta) \sim 2^{-n\mathcal{F}_U^{(n)}(\beta)}$, that is we have the "thermodynamic limit":

$$\lim_{n\to\infty} \big(-1/n\big) \log_2 \big(\mathcal{Z}_U^{(n)}(\beta)\big) \quad = \quad \mathcal{F}_U(\beta), \tag{2.7}$$

where $\mathcal{F}_U(\beta)$, the "free energy" associated to the measure μ, is defined by the following condition:

$$\mathcal{F}_U(\beta) \quad = \quad \inf_{t\geq 0} \; (\beta\, t - f(t)). \tag{2.8}$$

The relation between $f(t)$ and $\mathcal{F}_U(\beta)$ is the usual Legendre transform, similar to the relation between entropy and free energy in classical statistical mechanics.

Now we will show the existence of the thermodynamic limit for the uniform partition function. We first observe that if (2.7) is valid with $b \in \mathcal{P}_n$ in (2.4), the same limit in (2.7) is obtained if instead we assume b in (2.4) to belong to a covering $\mathcal{R}_n$- made with sets not necessarily square shaped and not necessarily forming a partition - such that the following properties hold:

1) there exists an integer h such that for any n and any b in $\mathcal{P}_n$, there exists sets $b_>$ and $b_<$ with $b_> \in \mathcal{R}_{n-h}$ and $b_< \in \mathcal{R}_{n+h}$ such that $b_< \subset b \subset b_>$;

2) the same property holds if we exchange the roles of the partitions $\mathcal{R}_n$ and $\mathcal{P}_n$.

Theorem 2.1

For the uniform partition function $\mathcal{Z}_U^{(n)}(\beta)$ which has been defined in (2.4), the expression: $(-1/n)\log_2(\mathcal{Z}_U^{(n)}(\beta)))$ has a finite limit when n goes to infinity.

Idea of the proof: The proof is simple and makes use of the classical argument given below, which derives the result from a subadditivity assumption on the logarithm of the partition function:

Lemma 2.2

Suppose that the sequence of positive numbers Z_n fulfils the inequality:

$$Z_{n+m} \quad \leq \quad c\, Z_n\, Z_m, \tag{2.9}$$

where the constant C is independent of n, then $(1/n)\log_2(Z_n)$ has a limit when n goes to infinity. Without additional assumptions this limit can be $-\infty$.
If we also assume a reversed inequality such as: $Z_{n+m} \geq C' Z_n\, Z_m$, where the constant $C' > 0$ is also independent of n, then the above mentioned limit is finite.

To prove that the uniform partition function satisfies the assumptions of the lemma, one has to use the distortion lemma and the "leo property", that is Proposition 1.7 and Corollary 1.5 of Chapter 1, as a way to compare small scales to normal order one scales, and Eq. (1.9) of Chapter 1 to control the corresponding scaling properties of the measure μ. In fact what is really needed is a kind of homogeneity property for the set and the measure supported by it, which results from the expanding character of the map. For the details, see for instance [**CDM**].

In an analogous way one can prove that, if we restrict the covering to a small open box $\mathcal{O}$ instead of S, we only need to iterate the covering sufficiently many times to get a covering of S and check that the distortion lemma allows to compare the final covering of S with the initial covering of $\mathcal{O}$. The comparison yields the same thermodynamic limit $\mathcal{Z}_U(\beta)$.

2.2 The dimension spectrum $f(t)$

The existence of the thermodynamic limit (2.7) for the uniform partition function, together with its regularity, which will be proven for the case under study in the following sections, ensures some kind of box counting statement. More precisely, we define $N_n^<(t)$, (resp. $N_n^>(t)$), as the number of boxes b with size 2^{-n}, such that $\mu(b) > 2^{-nt}$ (resp. $\mu(b) < 2^{-nt}$). Then we have the following proposition, which is the adaptation of classical arguments of large deviation theory [**E**] to our case:

Proposition 2.3

Given the measure μ, we assume that the uniform partition function defined by equation (2.4) fulfils the thermodynamic limit (2.7) and that the resulting $\mathcal{F}_U(\beta)$ is differentiable. We then define the function $f(t)$ as:

$$f(t) = \inf_\beta \left(t\,\beta - \mathcal{F}_U(\beta)\right). \tag{2.10}$$

The function $f(t)$ is thus a convex function, so there is a value t_m such that for $t < t_m$, the function $f(t)$ is non-decreasing and for $t > t_m$ non increasing. Then *we have:*

$$f(t) = \begin{cases} \lim_{n\to\infty} \frac{1}{n} \log_2 \left(N_n^{<}(t)\right), & \textit{for} \quad t < t_m; \\ \lim_{n\to\infty} \frac{1}{n} \log_2 \left(N_n^{>}(t)\right), & \textit{for} \quad t > t_m. \end{cases} \tag{2.11}$$

Idea of the proof: An easy proof of the previous statement follows directly by a proposition by D. Plachky et al. [**Pl, PlS**]; let us consider the following sequence of random variables:

$$W_n(b) = \begin{cases} 0 & \text{if } \mu(b) = 0; \\ \log(\mu(b)) & \text{otherwise.} \end{cases}$$

Then, using as associated probability measure the counting measure it follows immediately that

$$\mathbb{E}(\beta W_n) = \frac{\mathcal{Z}_U^{(n)}(\beta)}{A_n},$$

where A_n denotes the cardinality of the sets b which cover S and which is bounded by

$$c'\, 2^n \leq A_n \leq c\, 2^n.$$

Then from the usual arguments of large deviation theory the proposition follows.

We will end this section stating a result which gives a meaning to the name dimension spectrum given to the function $f(t)$, relating it to the geometrical dimensions associated to the measure μ through the local density exponent $\theta(x)$, which is defined as:

$$\theta(x) = \limsup_{r\to 0} \left[\frac{\log_2 \left(\mu(B_r(x))\right)}{\log_2(r)}\right], \tag{2.12}$$

where $B_r(x)$ is a ball of radius r centered at point x in the complex plane. Then the following statement holds [**CLP**]:

Proposition 2.4

Assume that the thermodynamic limit exists not only for a covering of the full set S, but also that the same limit is obtained when in (2.4) we restrict the covering to the intersection of S with any small open set. Then, defining f by Eq. (2.10), and assuming that the distortion lemma is verified (see Proposition (1.7) in the first Chapter), we have, for $f(t) \geq 0$:

$$f(t) \quad = \quad \dim(B(t)), \tag{2.13}$$

where $B(t)$ is the set of points x such that the local exponent $\theta(x) = t$. Moreover, the same result (2.13) holds if we replace in (2.12) the superior limit with an inferior one.

Idea of the proof: The inequality $f(t) \geq \dim B(t))$ results immediately from the existence of the covering of the set S by boxes. For the proof in the other direction one has to find a lower bound to the Hausdorff dimension $\dim(B(t))$. Such a lower bound is given by an argument due to Frostman [**Fal, CLP**]:

Lemma 2.5

Suppose there exist a measure ν and a set $\mathcal{D}$, such that $\nu(\mathcal{D}) = 1$ and, for some k and some positive constant c, $\nu(B_\epsilon) \leq c\,\epsilon^k$, for any ball B_ϵ of small radius ϵ. Then we get the inequality:

$$\dim(\mathcal{D}) \quad \geq \quad k \tag{2.14}$$

The strategy is then to construct explicitly a measure with support on $B(t)$ satisfying the conditions of Lemma 2.5, with $k = f(t)$ [**CLP, CDM**].

2.3 The spectrum of generalized correlation dimensions D_q

Let β be integer and positive. Then the generalized correlation dimensions are defined as

$$D_q = \frac{1}{q-1} \lim_{n\to\infty} -(1/n) \log\left(\mathcal{Z}_U^{(n)}(q)\right). \tag{2.15}$$

This formula extends to the case $q = 0$ and, in the case of nearly circular Julia sets Eq. (2.1), it can be proven that D_0 coincides with the Hausdorff dimension δ_H. In fact a relation between $\mathcal{F}_U(0)$ and the Hausdorff dimension $\delta_H(S)$ is expected, since, for $\beta = 0$, in (2.4), we just count how many boxes are needed to cover the support S of μ. Indeed we have $-\mathcal{F}_U(0) \geq \delta_H(S)$; for the equality one has to use Frostman's lemma (Lemma 2.5) and the distortion lemma (Proposition 1.7, of Chapter 1).

For $q = 1$ one recovers the information dimension σ, which is usually defined as

$$\sigma = \lim_{n\to\infty} -\frac{1}{n} \sum_b \mu(b) \log\left(\mu(b)\right). \tag{2.16}$$

To prove the equality between D_1 and σ one uses analogous arguments as for the Hausdorff dimension. Heuristically one can see that indeed this is the expected result, by expressing (2.15), as

$$D_q = \lim_{n\to\infty} -\frac{1}{n} \log_2 \left(\sum \mu(b) \exp\left\{(q-1)\log\left(\mu(b)\right)\right\}\right),$$

and expanding the exponential in the limit $q \to 1$.

The name of generalized correlation dimensions is due to the fact that for $q = 2$, one recover the correlation dimension, which is defined as the following limit, when it exists,

$$\nu = \lim_{l\to 0} \frac{\log C(l,\mu)}{\log(l)},$$

where

$$\mathcal{C}(l,\mu) = \int_{S\times S} \theta(l - \|x-y\|)\,d\mu(x)\,d\mu(y).$$

It is interesting to evaluate the total measure of the reunion of boxes b satisfying (2.2) for a given t. Using (2.3), the result is that the total measure vanishes forall but one value t_1 of t, such that $t - f(t)$ is minimum, which corresponds to $\beta = 1$ in equation (2.8). This particular value is, in general, different from the value t_m for which $f(t)$ is maximum and equal to the Hausdorff dimension of the support S. If we discard some particular, but very interesting, cases corresponding to strictly self-similar fractals (as the original Cantor set), we see that, in fact, almost all the measure is contained in a set of Hausdorff dimension t_1 strictly smaller than the dimension of the support. As discussed above t_1 is nothing else than the information dimension $\sigma = D_1$. The number $t_1 = D_1$ is the so called Hausdorff dimension of the measure, that is the smallest possible dimension of sets with full measure. So, by removing from the support S, itself of dimension t_m, sets of measure zero, one can get a resulting set of dimension t_1.

2.4 The dynamical partition function $\mathcal{Z}_D$

In this section we will consider the dynamical partition function associated to a nearly circular Julia sets, which is defined by the following property:

$$\mathcal{Z}_D^{(n)}(\alpha) = \sum_{b\in\mathcal{D}_n} \left(|b|\right)^{-\alpha}, \tag{2.17}$$

where the sum has to be taken over the sets b in the partition $\mathcal{D}_n$ defined recursively as follows: the pieces of the partition $\mathcal{D}_n$ are the preimages under T of the pieces of the partition $\mathcal{D}_{(n-1)}$, where, as usual, $|b|$ denotes the diameter of the set b. It is more convenient to introduce a more general mixed kind of partition functions:

$$\mathcal{Z}_C^{(n)}(\alpha,\beta) \quad = \quad \sum_{b\in\mathcal{C}_n} (|b|)^{\alpha}\,(\mu(b))^{\beta}, \tag{2.18}$$

where the sum has to be taken over the pieces b of a partition $\mathcal{C}_n$, with sizes $|b|$ decreasing approximately exponentially-like in n. Obvious arguments show that, when we take for $\mathcal{C}_n$ the uniform partition $\mathcal{P}_n$ defined previously, we have, up to a multiplicative constant indipendent of n:

$$\mathcal{Z}_P^{(n)}(\alpha,\beta) \quad = \quad 2^{-n\alpha}\,\mathcal{Z}_U^{(n)}(\beta), \quad \text{if} \quad \mathcal{C}_n = \mathcal{P}_n. \tag{2.19}$$

On the other hand, when we take for $\mathcal{C}_n$ the dynamical partition $\mathcal{D}_n$ defined above, we have in view of the balancedness property of the harmonic measure μ, Eq. (1.7) of Chapter 1:

$$\mathcal{Z}_D^{(n)}(\alpha,\beta) \quad = \quad q^{-n\beta}\,\mathcal{Z}_D^{(n)}(-\alpha), \quad \text{if} \quad \mathcal{C}_n = \mathcal{D}_n \tag{2.20}$$

Now consider the partition function $\mathcal{Z}_P$ associated to the uniform partition function given in Eq. (2.19), then the following result holds:

Proposition 2.6

The dynamical partition function $\mathcal{Z}_D^{(n)}(\alpha)$ has a thermodynamic limit:

$$\lim_{n\to\infty} (1/n)\,\log_q\left(\mathcal{Z}_D^{(n)}(\alpha)\right) \quad = \quad \mathcal{F}_D(\alpha), \tag{2.21}$$

which satisfies:

$$\mathcal{F}_D(\mathcal{F}_U(\beta)) \quad = \quad \beta \tag{2.22}$$

Idea of the proof: The argument for the existence of the thermodynamic limit of the dynamical partition function is very close to the one given for Theorem 2.1.

Starting from the mixed partition function associated to $\mathcal{P}_n$, choosing n and m large, but with $n \gg m$ and using the distortion lemma, one gets:

$$\mathcal{Z}_P^{(n)}(\alpha,\beta) \quad \sim \quad 2^{-n\alpha}\, 2^{-n\mathcal{F}_U(\beta)}\, \mathcal{Z}_D^{(m)}\big(-\mathcal{F}_U(\beta),\beta\big). \tag{2.23}$$

The previous relation is in fact a way to show the existence of the thermodynamic limit, that is the limit of $(1/n)\log_q(\mathcal{Z}_D^{(n)}(\alpha))$, as n goes to infinity, which is a well known result in the thermodynamic formalism. As in (2.21), let us denote $\mathcal{F}_D(\alpha)$ this limit. Then, using Eqs. (2.19) and (2.20), one gets:

$$2^{-n\alpha}\, 2^{-n\mathcal{F}_U(\beta)} \quad \sim \quad 2^{-n\alpha}\, 2^{-n\mathcal{F}_U(\beta)}\, q^{-m\beta}\, q^{-m\mathcal{F}_D(\mathcal{F}_U(\beta))}, \tag{2.24}$$

and finally

$$\mathcal{F}_D\big(\mathcal{F}_U(\beta)\big) \quad = \quad \beta, \tag{2.25}$$

from which the proposition follows. The relation between $\mathcal{F}_U$ and $\mathcal{F}_D$ is surprisingly simple and seemed unnoticed before the work of Collet et al. [**CLP**].

2.5 Analytical properties in the thermodynamic formalism

In the previous section, a simple connection has been established between the uniform and the dynamical partition functions. Here we will use this relation in order to deduce analyticity properties for $\mathcal{F}_U$ from analyticity properties of $\mathcal{F}_D$.

Theorem 2.7

The limit $\mathcal{F}_D(\beta)$ of $(1/n)\log_q(\mathcal{Z}_D^{(n)}(\beta))$ when n goes to infinity as well as the limit $\mathcal{F}_U(\beta)$ of $(-1/n)\log_2(\mathcal{Z}_U^{(n)}(\beta))$ exist, are analytic in β and real analytic in the coefficients occuring in the polynomial T, for λ sufficiently small in Eq. (2.1).

Idea of the proof: One can use the following alternate expression for the dynamical partition function:

$$\mathcal{Z}_D^{(n)}(\beta) \quad \sim \quad \sum_{\{i\}} \prod_{k=1}^{n} \left(|T'(x_{i_1,\dots,i_k})|\right)^{\beta},$$

(see [**CDM**] or Chapter 3) and observe that this expression is the same as for a partition function for a one dimensional Ising like system, with q possible states at each site, and an interaction at all sites, which decreases sufficiently fast with the distance for the application of the usual results. The theorem for $\mathcal{F}_D$ then follows as a consequence of the existence of the limit and of a uniform bound on $\log \mathcal{Z}_U^{(n)}(\beta)$ which can be provided along the lines of [**Do**], by applying the well-known compactness criterium for anlytic functions.

The analyticity properties for $\mathcal{F}_U$ result from the implicit function theorem which can be applied to Eq. (2.25) relating $\mathcal{F}_U$ and $\mathcal{F}_D$. In fact no problem arises from the functional inversion in (2.25) for polynomials T close to z^q, since in the unperturbed case we have the following expressions for the "free energies":

$$\begin{aligned} \mathcal{F}_U(\beta) &= (\beta - 1), \\ \mathcal{F}_D(\beta) &= (\beta + 1). \end{aligned}$$

Various regularity results in β are avaliable in earlier literature [**Bow, Ru1**], including real analyticity properties for the Hausdorff dimension [**Ru2**], which will be commented in the next chapter. In view of the expression for the correlation dimensions:

$$D_r \quad = \quad \frac{1}{r-1}\, \mathcal{F}_U(\beta),$$

one can deduce from the previous theorem, the analyticity properties of the Hausdorff dimension and of the various higher order correlation dimensions. So we have now completed our program which consisted in considering the analyticity properties of the multifractal properties of Julia sets close to the unit circle. Of course, if we consider polynomials far from z^q, one expects interesting singularities in the thermodynamic functions which remain to be analized. Such a situation seems to occur for the polynomial $z^2 + 1/4$, as shown by numerical calculations **[KaP]**.

Chapter 3

Perturbative computation of thermodynamical quantities

In the previous chapter we have introduced the thermodynamic formalism for nearly circular Julia sets and we have shown that the thermodynamic limit exists and is real analytic in the perturbation parameter. Moreover, we have noticed that, once one is able to compute the dynamical free energy $\mathcal{F}_D(\beta)$, then it is possible to compute the dimension spectrum D_q, defined in equation (2.15) of Chapter 2.

Once $\mathcal{F}_D(\beta)$ is expressed in form of a classical statistical mechanics model, the usual perturbative methods can be used and permit a perturbative expansion of $\mathcal{F}_D(\beta)$ and $\mathcal{F}_U(\beta)$, and therefore of D_β. We will concentrate on the generic polynomial case

$$T(z) \quad = \quad z^q + \lambda\, c(z), \tag{3.1}$$

where λ is a sufficiently small parameter and $c(z)$ a polynomial of degree less than q. In this way we are able to compute $\mathcal{F}_D(\beta)$, up to the third order in β.

Our aim is to give a device for computation which may be applied to any polynomial and generalizes the previous perturbative computations which were restricted to the case $T(z) = z^q + c$. We do not claim that the perturbative approach is numerically competitive for the estimation of the Hausdorff dimension and more generally for the set of generalized dimensions D_β. It should also be noticed that the perturbative approach has an intrinsec limitation in what it does not preserve the convexity property of the "free energy" [CDM], so that our computations cannot reproduce the whole dimension spectrum, but are valid only in a small range of β. In next Chapter we will see an alternative approach which makes use of the scaling function.

The plan of the Chapter is as follows: in section 1, we review previous analogous perturbative computations, which treated the simplified perturbative case $T(z) = z^q + c$ with c constant.

In section 2, we justify our approach and then, in section 3, we show that it is possible to treat perturbatively the generic case Eq. (3.1), obtaining results which are in agreement with the previous ones. In the appendix, finally, we give the details of the computations. The results presented in section 2 and 3 and in the appendix have also appeared in S. Abenda et al. [AM].

3.1 Review of previous results in perturbative computation of thermodynamic quantities for nearly circular Julia sets

We are going to review briefly the methods of perturbative computation of the dimension spectrum due to D. Ruelle [Ru2], M. Widom et al. [WBKS] and to P. Collet et al. [CDM]. They are similar and permit to compute the Hausdorff dimension up to order 2 [Ru2] and 3 [WBKS] and the set of correlation dimensions for small β up to order 4 [CDM], for the family of perturbed maps of the form:

$$T(z) \quad = \quad z^q + c, \tag{3.2}$$

with c small complex constant parameter. In particular the starting point in [**Ru2, WBKS**] is a theorem of Ruelle [**Ru2**], which provides a useful formula for computing the Hausdorff dimension:

Theorem 3.1

Let J_T be the repeller for a nearly circular Julia set of the form Eq. (3.2). Let

$$\Psi(x) \quad = \quad -\log\left|T'(x)\right|$$

and let δ_H be the Hausdorff dimension of the set J_T. Then the series

$$\zeta(u) \quad = \quad \exp\left\{\sum_{n=1}^{\infty}\frac{u^n}{n}\sum_{x\in \mathrm{FixT}^n}\exp\left\{\sum_{k=0}^{n-1}\delta_H\,\Psi\left(T^k x\right)\right\}\right\} \tag{3.3}$$

has a non-vanishing convergence radius and extends to a meromorphic function of u, which it is denoted again by $\zeta(u)$.
This function has a simple pole at $u = 1$ and no other zeroes and poles inside the unit disk.

Then, taking the logarithm of Eq. (3.3), we see that a necessary condition for δ to be the Hausdorff dimension is that

$$\lim_{n\to\infty}\sum_{x\in \mathrm{FixT}^n}\exp\left\{\sum_{k=0}^{n-1}\delta\,\Psi(T^k x)\right\} \quad = \quad 1. \tag{3.4}$$

With the aim of approximating the Hausdorff dimension, one first compute perturbatively

$$\mathcal{A}_D^{(n)}(\beta) \quad \equiv \quad \sum_{z\in \mathrm{Fix}T^n}\left|(T^n)'(z)\right|^{-\beta} \tag{3.5}$$

using the Boettcher conjugation equation, which was introduced and discussed in Chapter one:

$$\Phi(z^q) \quad = \quad T(\Phi(z)), \tag{3.6}$$

which can be continuously extended to the Julia set. And then, one uses such an expansion for computing δ_H. (For details on the perturbative computation using the conjugation equation, see Appendix 1 and [**Ru2, WBKS, AM**]). The resulting expression for the Hausdorff dimension is then:

$$\delta_H \quad = \quad 1 + \frac{|c|^2}{4\log(q)} + \delta_{q,2}\,\frac{3\,(c+\bar{c})\,|c|^2}{16\log q}. \tag{3.7}$$

The second order term was established by Ruelle, while the third order term is due to Widom et al. Before passing to the work of [**CDM**], observe that $\mathcal{A}_D^{(n)}(\beta)$ is an equivalent expression of the dynamical partition function.

P. Collet et al. [**CDM**] start with the perturbative computation of $\mathcal{F}_D$ using an equivalent expression of the dynamical partition function $\mathcal{Z}_D^{(n)}(\beta)$, Eq. (2.17) of Chapter 2, in terms of the preimages of a starting point ξ, which may be any point in or close to the Julia set J_T:

$$\mathcal{Z}_D^{(n)}(\beta) \quad \sim \quad \sum_{\{i\}} \prod_{k=0}^{n-1} \left(|T'(T^k(x_{\{i\}}))|\right)^{\beta}. \tag{3.8}$$

To pass from equation (2.17)

$$\mathcal{Z}_D^{(n)}(\beta) \quad = \quad \sum_{b\in\mathcal{D}_n} \left(|b|\right)^{-\beta} \quad = \quad \sum_{i=1}^{q^n} \left(|T_i^{-n}(\mathcal{A})|\right)^{-\beta},$$

where $\mathcal{A}$ is any bounded open set including the Julia set J and which does not contain any of the exceptional points, to Eq. (3.8) requires the use of the distortion lemma (Proposition 1.7 of Chapter 1) The indipendence of Eq. (3.8) on the starting point is also an easy consequence of the same lemma. Starting from (3.8) leads to the usual expansions occuring in statistical mechanics, in a case where the interactions are not limited to nearest neighbours. Some of the tricks [**CDM**] use are very similar to those used in the statistics of the fixed points by [**Ru2**], and permit to compute the pertubative expansion of the dynamical free energy up to order 4 in powers of c, which we report here for comparison with our results:

$$\begin{aligned}\mathcal{F}_D(\beta) \quad &= \quad \beta + \frac{1}{\log(q)}\left\{\frac{1}{4}\beta^2|c|^2 + \delta_{q,2}\frac{1}{16}\beta^2(2-\beta)(c+\bar{c})|c|^2 + \right.\\ &\left[\frac{\delta_{q,2}}{32}(2-\beta) + \frac{\delta_{q,3}}{96}(4-\beta)\right]\beta^2(2-\beta)(c^2+\bar{c}^2)|c|^2\\ &\left. + \left[\frac{3+q^2}{16\,(q^2-1)}\beta^2 - \frac{3+q}{16\,(q-1)}\beta^3 - \frac{1}{64}\beta^4\right]|c|^4\right\}\\ &+ O\big(|c|^5\big).\end{aligned} \tag{3.9}$$

Then, with the help of (2.22)

$$\mathcal{F}_D\big(\mathcal{F}_U(\beta)\big) \quad = \quad \beta, \tag{3.10}$$

they compute the expansion of the uniform free energy $\mathcal{F}_U$ in powers of c and then use (2.15)

$$D_\beta \quad = \quad \frac{1}{\beta-1}\,\mathcal{F}_U(\beta) \tag{3.11}$$

to expand the correlation dimensions D_β on powers of c for β sufficiently small. In this way, they obtain the following perturbative expansion for the Hausdorff dimension:

$$\begin{aligned}\delta_H \quad = \quad 1 + \frac{1}{\log(q)}&\left\{\frac{|c|^2}{4} + \frac{3\delta_{q,2}}{16}(c+\bar{c})|c|^2 + \frac{9\delta_{q,2}+5\delta_{q,3}}{32}(c^2+\bar{c}^2)|c|^2 + \right.\\ &\left.\left(\frac{25+16q+7q^2}{64(q^2-1)} + \frac{1}{8\log(q)}\right)|c|^4\right\} + O(|c|^5).\end{aligned} \tag{3.12}$$

Finally, they use the Legendre transform (2.10)

$$f(t) \quad = \quad t\,\beta - \mathcal{F}_U(\beta); \text{ where } \beta \text{ is given by : } \frac{\partial \mathcal{F}_U(\beta)}{\partial \beta} = t;$$

to expand the dimension spectrum $f(t)$ in powers of c. There is a specific difficulty in this last step, due to the fact that the unperturbed function $f(t)$ corresponding to the unit circle with the Lebesgue measure, is singular and takes only one value different from $-\infty$, that is $f(1) = 1$. But there is no difficulty to compute the perturbed inverse function $t(f)$, the unperturbed one being taken as the constant function with value equal to one.

3.2 Perturbation of the dynamical free energy $\mathcal{F}_D(\beta)$

We will now present our results in the perturbative computation of the thermodynamic quantities [AM]. We will use an alternate expression for the dynamical partition function $\mathcal{Z}_D^{(n)}(\beta)$, defined in Eq. (2.17)

$$\mathcal{Z}_D^{(n)}(\beta) \quad = \quad \sum_{b\in\mathcal{D}_n} \left(|b|\right)^{-\beta} \quad = \quad \sum_{i=1}^{q^n} \left(\left|T_i^{-n}(\mathcal{A})\right|\right)^{-\beta}, \qquad (3.13)$$

where the second sum runs over the q^n inverse branches of the function T^n and $\mathcal{A}$ is a somewhat arbitrary bounded open set including J_T.

With the help of the distortion lemma one can pass from (3.13) to

$$\mathcal{Z}_D^{(n)}(\beta) \quad \sim \quad \sum_{x,T^n(x)=\xi} \left|(T^n)'(x)\right|^{\beta}, \qquad (3.14)$$

where the sum runs over the preimages of ξ, where ξ be any point on the Julia set or sufficiently nearby to it, so that the mapping is still expanding. Defining the preimages of ξ by the following labelling procedure:

$$T\left(x_{i_1,\dots,i_n}\right) \quad = \quad x_{i_1,\dots,i_{n-1}}, \qquad i_k = 1,\dots,q, \ \text{ for } \ k = 1,\dots,n, \qquad (3.15)$$

where, of course, $T(x_{i_1}) = \xi$, we can rewrite Eq. (3.14) as:

$$\begin{aligned} \mathcal{Z}_D^{(n)}(\beta) &\sim \sum_{\{i\}} \prod_{k=0}^{(n-1)} \left(\left|T'(T^k(x_{\{i\}}))\right|\right)^{\beta} = \\ &= \sum_{\{i\}} \prod_{k=1}^{n} \left(\left|T'(x_{i_1,\dots,i_k})\right|\right)^{\beta}, \end{aligned} \tag{3.16}$$

where the sum has to be taken over all possible values of multi-indices $\{i\}$ which run for $i_1, \dots, i_n$, and T^n denotes the n-th iterate of T, such that $T^n(z) = T(T^{(n-1)}(z))$ and $T^0(z) = z$. Moreover it is convenient to write Eq. (3.16) for any arbitrary starting point and then take the mean value using the invariant and balanced measure μ introduced in section 2, Eqs. (1.6) and (1.7), that is $\mu(T^{-1}(B)) = \mu(B)$ and $\mu(T_i^{-1}(B)) = (1/q)\mu(B)$, for any inverse branch T_i^{-1} and any Borel set B. Then we get:

$$\begin{aligned} \mathcal{Z}_D^{(n)}(\beta) \sim \mathcal{Z}_D^{(n)}(\beta)' &= \int_J d\mu(\xi) \sum_{x:T^n(x)=\xi} \left|(T^n)'(x)\right|^{\beta} \\ &= q^n \int_J d\mu(x) \prod_{k=0}^{(n-1)} \left(\left|T'(T^k(x))\right|\right)^{\beta}. \end{aligned} \tag{3.17}$$

Equation (3.17) is the expression of the dynamical partition function which we use as starting point in our expansions, where

$$T(z) = z^q + \lambda\, c(z), \qquad \text{with} \qquad c(z) = \sum_{j=0}^{q-2} a_j z^j. \tag{3.18}$$

Then, applying the Boettcher conjugation equation, $\Phi(z^q) = T(\Phi(z))$, we get:

$$\mathcal{Z}_D^{(n)}(\beta)' = q^{n(1+\beta)} \int_0^{2\pi} \frac{d\theta}{2\pi} \prod_{r=0}^{n-1} \left| \Phi\left(e^{i\theta q^r}\right)^{q-1} + \frac{\lambda}{q} \sum_{j=1}^{q-2} a_j j\, \Phi\left(e^{i\theta q^r}\right)^{j-1} \right|^{\beta}. \tag{3.19}$$

The fact that we use the expansion for the measure after having used Boettcher function, does not a justification of the consistency of this method, since the Boettcher conjugation function is just continuous on the Julia set; indeed there are arguments in favour of its validity (A. Douady, private communication). Let us denote by $\vec{a} = \{a_0, \dots, a_{q-2}\}$, the vector of coefficients of the polynomial $c(z)$. The resulting formula for the third order expansion in the coefficients of the polynomial $c(z)$, of the dynamical free energy $\mathcal{F}_D(\beta) = \lim_{n\to\infty}(-1/n)\log_q \mathcal{Z}_D^{(n)}(\beta)'$ is:

$$\begin{aligned}\mathcal{F}_D(\beta) &= 1 + \beta + \frac{1}{\log(q)}\left\{\frac{\beta^2}{4}\sum_{j=0}^{q-2}|a_j|^2\left(1-\frac{j}{q}\right)^2 + \right. \\ &\left. \frac{\beta^2}{4}\mathcal{I}_1(\vec{a}) + \frac{\beta^3}{16}\mathcal{I}_2(\vec{a})\right\} + O(\max_{0\le j\le q-2}|a_j|^3),\end{aligned} \tag{3.20}$$

where

$$\begin{aligned}\mathcal{I}_1(\vec{a}) &= \frac{1}{q}a_0^2\bar{a}_{q-2} \\ &+ a_0\sum_{i=1}^{q-2} a_i\bar{a}_{i-1}\left[\frac{1}{q}\left(1-\frac{i-1}{q}\right) + \frac{1}{q^2}\left(2-\frac{1}{q}\right)\left(\frac{i-1}{q}-i\right)\right] \\ &+ \sum_{i+j\ge q} a_i a_j \bar{a}_{i+i-q}\left[\frac{1}{2} - \frac{i}{q}\left(\frac{2q-1}{q}\right) + \frac{i\,j}{2q^2}\right] \\ &+ \sum_{i+j>q} a_i a_j \bar{a}_{i+j-q}\left[\frac{-1}{2q} + \left(2-\frac{1}{q}\right)\frac{i+j-q}{q} - \frac{i\,j\,(i+j-q)}{q^3}\right];\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_2(\vec{a}) &= -\frac{2}{q}a_0^2\bar{a}_{q-2} + a_0\sum_{i=1}^{q-2} a_i\bar{a}_{i-1}\left[2\left(-1+\frac{i}{q}\right) + 2\left(\frac{i-1}{q}\right)\left(1-\frac{i}{q}\right)\right] \\ &+ \sum_{i+j\ge q} a_i a_j \bar{a}_{i+j-q}\left[\left(\frac{2i}{q}-\frac{ij}{q^2}-1\right) - \left(\frac{i+j-q}{q^2}\right)\left(2i-1+\frac{ij}{q}\right)\right].\end{aligned}$$

It can be easily checked that, if we restrict ourselves to the case

$$a_0 = c,\ \ a_1 = \dots = a_q = 0,$$

our perturbative expansion of the dynamical partition function reduces to the one computed by [CDM], which was denoted above as Eq. (3.9), up to the third order in $a_0 \equiv c$.

3.3 Perturbative computation of $\mathcal{F}_U(\beta)$ and D_β

It is now easy to derive the expansion for the uniform free energy and for the correlation dimensions using Eqs. (3.10) and (3.11) which have been commented in Chapter 2.

In fact, replacing $c(z)$ by $\lambda\, c(z)$, with λ real, we get:

$$\mathcal{F}_D(\beta) = 1 + \beta + \lambda^2\phi_2(\beta) + \lambda^3\phi_3(\beta) + O(\lambda^4), \tag{3.21}$$

so that, inserting it into the expansion of the uniform free energy:

$$\mathcal{F}_U(\beta) = -1 + \beta - \lambda^2\varphi_2\beta - \lambda^3\varphi_3(\beta) + O(\lambda^4), \tag{3.22}$$

one gets

$$\varphi_2(\beta+1) = \phi(\beta); \qquad \varphi_3(\beta+1) = \phi(\beta). \tag{3.23}$$

So that, we can now compute the correlation dimensions using Eq. (3.11)

$$D_\beta = \frac{1}{(\beta-1)}\, \mathcal{F}_U(\beta).$$

Observe that $D_1 = 1$ for any q, since, at any order, there is at least a coefficient $(\beta-1)^2$ in the expansion of $\mathcal{F}_U(\beta)$, as it has been shown in the appendix. Now the Hausdorff dimension of the Julia set is D_0, so that:

$$\delta_H \equiv D_0 = 1 + \frac{1}{\log(q)} \left\{ \frac{1}{4} \sum_{j=0}^{q-2} |a_j|^2 \left(1 - \frac{j}{q}\right)^2 + \frac{1}{4}\,\mathcal{I}_1(\vec{a}) - \frac{1}{16}\,\mathcal{I}_2(\vec{a}) \right\},$$

which is in agreement with the formulas previously obtained by [**Ru2, WBKP, CDM**] for the case $a_0 = c$, $a_1 = \cdots = a_{q-2} = 0$.

It should be noticed that the second order term is a positive definite quadratic form, which is in agreement with the theoretical prediction of D. Sullivan (Corollary 1.10 of Chapter 1); the fact that the quadratic form is diagonal is in some sense unexpected and has no interpretation yet.

In the next chapter we will see an alternative approach to the computation of the thermodynamic spectrum, based on the microscopic scaling structure associated to the Julia set.

Appendix to Chapter 3

Exp - Log perturbative expansion of the partition function

We will give in this appendix some detail on the perturbative computation of the dynamical partition function $\mathcal{Z}_D^{(n)}(\beta)$ up to third order. We shall calculate the perturbation expansion around the $c(z) = 0$ for the dynamical partition function associated to the Julia set defined by the polynomial

$$T(z) \quad = \quad z^q \, + \, \lambda \, c(z), \tag{3.24}$$

where $c(z)$ is a polynomial of degree $\leq q-2$ and

$$c(z) \quad = \quad \sum_{j=0}^{q-2} a_j z^j. \tag{3.25}$$

For arbitrary integer $q \geq 2$, we will give the expansion up to third order on powers of c.

We will use the Boettcher conjugation function Φ, and the following conjugation equation

$$T(\Phi(\zeta)) \quad = \quad \Phi(\zeta^q), \tag{3.26}$$

which can be be continuously extended to the Julia set J, as we have shown in Theorem 1.5 of Chapter 1. The Boettcher function admits the following expansion, analytic outside the Julia set,

$$\Phi(z) \equiv z\left\{1+\tilde{\Phi}(z)\right\} = z\left\{1+\sum_{n\geq 1}\lambda^n\varphi_n(z)\right\}, \tag{3.27}$$

which, inserted in the conjugation equation (3.26), leads to the following recursive equation for the computation of the coefficients $\varphi_n(z)$,

$$\varphi_n(z^q) - q\,\varphi_n(z) = R_n(z), \qquad n\geq 1, \tag{3.28}$$

where $R(z)$ is a polynomial in $\varphi_1(z),...,\varphi_{n-1}(z)$.
For instance, the first terms read

$$\begin{aligned}\varphi_1(z^q) - q\,\varphi_1(z) &= \sum_{j=0}^{q-2} a_j\, z^{-(q-j)};\\ \varphi_2(z^q) - q\,\varphi_2(z) &= \frac{q(q-1)}{2}\varphi_1^2(z) + \sum_{j=0}^{q-2} a_j\, j\, z^{-(q-j)}\varphi_1(z).\end{aligned} \tag{3.29}$$

The solution is then given by

$$\varphi_n(z) = -\sum_{p\geq 0}\frac{1}{q^{p+1}}\,R_n(z^{q^p}), \tag{3.30}$$

as it can be easily proven.
As we will see below, in our expansion we will deal with sums of $\varphi_n(z)$, with z of the form

$$z = \exp\{2i\pi\theta q^l\}, \qquad l=0,...N, \tag{3.31}$$

so that it is convenient to express such sums with the help of Eq. (3.28), as

$$\begin{aligned}\sum_{l=0}^{N-1}\varphi_n\left(e^{i\theta q^l}\right) &= \\ &= \frac{q^{N-1}+\ldots+1}{q^{N-1}}\varphi_n\left(e^{i\theta q^{N-1}}\right) - \sum_{l=0}^{N-2}\frac{q^l+\ldots+1}{q^{l+1}}R_n\left(e^{i\theta q^l}\right).\end{aligned} \tag{3.32}$$

We use the following expansion for the dynamical partition function $\mathcal{Z}_D^{(n)}(\beta)$, which has been commented in Chapter 3,

$$\mathcal{Z}_D^{(N)}(\beta) = q^N \int_J d\mu(z) \prod_{l=0}^{N-1} \left| T'\left(T^l(z)\right)\right|^{\beta}; \tag{3.33}$$

Substituing the expression of the derivative in the formula above, we get

$$\begin{aligned}\mathcal{Z}_D^{(N)}(\beta) &= q^{N(1+\beta)} \int_J d\mu(z) \prod_{l=0}^{N-1} \left| \left(T^l(z)\right)^{q-1} + \frac{\lambda}{q}\sum_{j=1}^{q-2} a_j\, j \left(T^l(z)\right)^{j-1} \right|^{\beta} \\ &\equiv q^{N(1+\beta)} \mathcal{I}^{(N)}(\beta).\end{aligned} \tag{3.34}$$

Using the Boettcher conjugation equation (3.26) and denoting by $\zeta = \exp\{i\theta\}$, we get

$$\begin{aligned}\mathcal{I}^{(N)}(\beta) &= \int_0^{2\pi} \frac{d\theta}{2\pi} \prod_{l=0}^{N-1} \left\{ 1 + \left[\left(1+\tilde{\Phi}(\zeta^{q^l})\right)^{q-1} - 1\right]\right. \\ &\left. + \frac{\lambda}{q}\sum_{j=1}^{q-2} j a_j \zeta^{-q^l(q-j)} \left(1+\tilde{\Phi}(\zeta^{q^l})\right)^{j}\right\}^{\frac{\beta}{2}} \left\{c.c.\right\}^{\frac{\beta}{2}} \\ &\equiv \int_0^{2\pi} \frac{d\theta}{2\pi} \prod_{l=0}^{N-1} \left\{ 1 + \tilde{\Psi}(\zeta^{q^l})\right\}^{\frac{\beta}{2}} \left\{c.c.\right\}^{\frac{\beta}{2}} \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left\{ \sum_{l=0}^{N-1} \frac{\beta}{2} \left[\log(1+\tilde{\Psi}(\zeta^{q^l}) + \log(1+c.c.)\right]\right\} \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left\{ \sum_{l=0}^{N-1} \frac{\beta}{2} \left(\tilde{U}_l^- + \tilde{U}_l^+\right)\right\} \\ &= \int_0^{2\pi} \frac{d\theta}{2\pi} \exp\left\{ \frac{\beta}{2} U^- + U^+\right\},\end{aligned} \tag{3.35}$$

where, we have denoted by

$$\tilde{U}^-(z) = \log(1+\tilde{\Psi}(z)), \tag{3.36}$$

the expansion of the logarithm.

From (3.30) and the formula above, it follows immediately that U^- contains only negative frequencies and that $U^+ = \bar{U}^-$.

Expanding now the exponential, we get

$$\mathcal{I}^{(N)}(\beta) = \int_0^{2\pi} \frac{d\theta}{2\pi} \left\{ 1 + \frac{\beta^2}{4} U^- U^+ + \frac{\beta^3}{16} \left(U^- (U^+)^2 + c.c \right) + O(\lambda^4) \right\}. \tag{3.37}$$

Notice that there is no term of first order in β; this is due to the fact that U^- (resp. U^+) contains only negative (positive) frequencies, so that the corresponding zero Fourier expansion term is null.

For the same reason, it follows also that there is no first order term in λ, that is no first order term in the coefficients of the polynomial $a_0, ..., a_{q-2}$, since the expansion of U^- and U^+ start with λ.

The strategy is now to express conveniently the sum of $\tilde{U}_l^-$ and the one of complex conjungate terms, and then to multiply the resulting terms. At this aim, we use Eq. (3.32).

Second order coefficient in c(z)

For what concerns the coefficient of second order in λ, we observe that the only contribution can come from the product $U^- U^+$. Manipulating U^- and keeping only the first (lowest) expansion term in λ, we get

$$\begin{aligned} U^- = \lambda \Bigg\{ & \frac{q^{N+1} - 1}{q^N} \varphi_1 \left(e^{i\theta q^{N+1}} \right) - \sum_{l=0}^{N-1} \frac{q^{l+1} - 1}{ql + 1} \sum_{j=0}^{q-2} a_j \, e^{-i\theta q^l (q-j)} \\ & + \sum_{l=0}^{N-1} \frac{1}{q} \sum_{j=1}^{q-2} a_j \, j \, e^{-i\theta q^l (q-j)} \Bigg\}. \end{aligned} \tag{3.38}$$

We then multiply the first order expansion of U^- and of U^+, and integrate in θ. At

this aim, we have to compute which are the possible combinations of frequencies which give rise to the zero Fourier expansion term in θ.
They are those for which

$$q^n(q-j) = q^m(q-i), \qquad 0 \leq n,m \leq N, \qquad 0 \leq i,j \leq q-2,$$

that is the only contribution comes from the terms for which $n = m$ and $i = j$. We then get that the second order coefficient of the expansion of $\mathcal{Z}_D^{(N)}(\beta)$ in $c(z)$ is

$$\frac{\beta^2}{4} U^- U^+ = \frac{\beta^2}{4} N \left\{ \sum_{j=0}^{q-2} |a_j|^2 \left(1 - \frac{j}{q}\right)^2 + O\left(N^{-1}\right) \right\}, \tag{3.39}$$

from which we immediately get the second order expansion in $c(z)$ in Eq. (3.20).

Third order coefficient in c(z)

The only contributions to the third order expansion coefficient of $\mathcal{Z}_D^{(N)}(\beta)$ in $c(z)$ come from the terms U^-U^+ and $U^-(U^+)^2$; that is they contain only second and third order powers in β.
The startegy is as before; we are not going to give details on the computation, which, while lenghty, is straightforward. We just observe that the frequencies which contribute to the expansion must satisfy

$$q^n(q-i) + q^m(q-j) = q^p(q-k), \qquad 0 \leq n,m,p \leq N, \qquad 0 \leq i,j,k \leq q-2.$$

The possible solutions are

$$p = n = m+1, \quad j = 0, \quad k = i - 1;$$
$$p = n = m, \quad i + j \geq k \quad k = i + j - q;$$
$$p = n+1 = m+1, \quad i = j = 0 \quad k = q - 2.$$

It is then easy to understand that the resulting terms in the expansions, which were denoted by $\mathcal{I}_1(\vec{a})$ and $\mathcal{I}_2(\vec{a})$ in Eq. (3.20) in Chapter 3, have the form

$$
\begin{aligned}
\mathcal{I}_1(\vec{a}) \;&=\; \frac{1}{q} a_0^2 \bar{a}_{q-2} \\
&+\; a_0 \sum_{i=1}^{q-2} a_i \bar{a}_{i-1} \left[\frac{1}{q}\left(1 - \frac{i-1}{q}\right) + \frac{1}{q^2}\left(2 - \frac{1}{q}\right)\left(\frac{i-1}{q} - i\right)\right] \\
&+\; \sum_{i+j\geq q} a_i a_j \bar{a}_{i+i-q} \left[\frac{1}{2} - \frac{i}{q}\left(\frac{2q-1}{q}\right) + \frac{i\,j}{2q^2}\right] \\
&+\; \sum_{i+j>q} a_i a_j \bar{a}_{i+j-q} \left[\frac{-1}{2q} + \left(2 - \frac{1}{q}\right)\frac{i+j-q}{q} - \frac{i\,j\,(i+j-q)}{q^3}\right];
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{I}_2(\vec{a}) \;&=\; -\frac{2}{q} a_0^2 \bar{a}_{q-2} + a_0 \sum_{i=1}^{q-2} a_i \bar{a}_{i-1} \left[2\left(-1 + \frac{i}{q}\right) + 2\left(\frac{i-1}{q}\right)\left(1 - \frac{i}{q}\right)\right] \\
&+\; \sum_{i+j\geq q} a_i a_j \bar{a}_{i+j-q} \left[\left(\frac{2i}{q} - \frac{ij}{q^2} - 1\right) - \left(\frac{i+j-q}{q^2}\right)\left(2i - 1 + \frac{ij}{q}\right)\right].
\end{aligned}
$$

Expanding the logarithm of $\mathcal{Z}_D^{(N)}(\beta)$ in powers of λ and taking the limit for $N \to \infty$, in

$$
\mathcal{F}_D(\beta) \;=\; -\lim_{N\to\infty} \frac{1}{N} \log \mathcal{Z}_D^{(N)}(\beta)
$$

we then get

$$
\begin{aligned}
\mathcal{F}_D(\beta) \;&=\; 1 + \beta + \frac{1}{\log(q)} \left\{ \frac{\beta^2}{4} \sum_{j=0}^{q-2} |a_j|^2 \left(1 - \frac{j}{q}\right)^2 + \right. \\
&\left. \frac{\beta^2}{4} \mathcal{I}_1(\vec{a}) + \frac{\beta^3}{16} \mathcal{I}_2(\vec{a}) \right\},
\end{aligned}
$$

where it can be easily checked that, if we restrict ourselves to the case

$$
a_0 = c, \;\; a_1 = \ldots = a_q = 0,
$$

our perturbative expansion of the dynamical partition function reduces to the one computed by [**CDM**], Eq. (3.21), up to the third order.

Chapter 4

Microscopic scaling structure and thermodynamics

We will present a quantitative perturbative theory of the scaling properties of nearly circular Julia sets. Such a theory has a "macroscopic" part which consists of the generalized dimensions of the set or its spectrum of scaling indeces, which have been already discussed in Chapter 2 and 3, and a "microscopic" part consisting of scaling functions, which will be considered in this chapter and were first introduced by M. Feigenbaum see, for instance, [**FJP**, **KaP** , **JKP**].

These two facets are formally and computationally equivalent to thermodynamics and statistical mechanics in the theory of many body systems. M. Jensen et al [**JKP**] have constructed such scaling functions for Julia sets, showing that there are at least two different approaches to this construction, one starting from the backwards iterates of the repulsive fixed point and the other from the set of repulsive periodic orbits, which they have termed Feigenbaum and Ruelle-Bowen-Sinai approach, respectively.

In the case of nearly-circular Julia sets both approaches seem to converge, meaning that it should be possible to map the theory onto Ising models with finite range interactions. The largest eigenvalue of the associated transfer matrix would then furnish the thermodynamic functions.

We will in particular concentrate on the approximation of the so-called Feigenbaum scaling function and give arguments for its convergence using the perturbative approach, computing the transfer matrix eigenvalue for big n and discuss the self-similarity of the scaling function (see also [**AM**]).
Some numerics on the scaling function, using a distinct perturbative approach based on the so-called derivative method of Withers [**W**], is also presented in [**OS**]. In section 1, we introduce the scaling function, while in section 2, we discuss the result previously obtained by Jensen et al. [**JKP**] for the case of nearly circular Julia sets. In section 3, we use our perturbative approach to approximate the scaling function and discuss its self-similarity and convergence properties.

4.1 The scaling function

Let J_T be a hyperbolic Julia set associated to some polynomial mapping T, carrying the harmonic measure, as already dicussed in the previous chapters. Let us consider the partitions of the set J_T into "balls" of radius l_i such that the measure, p_i, associated to each of the balls is the same. Call N_n the number of boxes in the n-th step of refinement of the set, so that $p_i = N_n^{-1}$ and consider the partition function $\mathcal{Z}_C^{(n)}(\alpha, \beta)$ associated to such a partition, which was introduced in Chapter 2. Call $\alpha = -\tau$ and $\beta = q$, so that

$$\mathcal{Z}_C^{(n)}(-\tau, q) \quad = \quad \sum_i |l_i|^{-\tau} \, p_i^q, \tag{4.1}$$

and define $\tau(q)$ as the value of τ for which $\mathcal{Z}_C^{(n)}(-\tau, q) = 1$. Then from relation (4.1), we get

$$N_n^{q(\tau)} \quad = \quad \sum_i (l_i)^{-\tau}. \tag{4.2}$$

In general N_n grows exponentially in n, as $N_n \sim a^n$, for some positive costant a. For instance in the case of nearly circular Julia sets of polynomials of degree s, we have $a = s$.

Then, we can write the index i as a q-adic sequence of numbers $(\epsilon_1, ..., \epsilon_n)$. Eq. (4.2) then reads

$$a^{nq(\tau)} = \sum_{\epsilon_1,...,\epsilon_n} \left[l(\epsilon_n, ..., \epsilon_1)\right]^{-\tau}. \qquad (4.3)$$

Performing two steps of refinement, we get

$$\sum_{\epsilon_{n+1},...,\epsilon_1} \left|l(\epsilon_{n+1}, ..., \epsilon_1)\right|^{-\tau} = a^{q(\tau)} \sum_{\epsilon_n,...,\epsilon_1} \left|l(\epsilon_n, ..., \epsilon_1)\right|^{-\tau}. \qquad (4.4)$$

The microscopic information is then carried by the so-called scaling function $\sigma(\epsilon_{n+1}, ..., \epsilon_1)$, which is the daughter-to-mother ratio

$$\sigma(\epsilon_{n+1}, ..., \epsilon_1) = \frac{l(\epsilon_{n+1}, ..., \epsilon_1)}{l(\epsilon_n, ..., \epsilon_1)}. \qquad (4.5)$$

The scaling function σ depends in principle by the whole hystory $(\epsilon_1, ..., \epsilon_{n+1})$; however, making appropriate choices of the quantities $l(\epsilon_n, ..., \epsilon_1)$, one can produce a scaling function which depends most strongly upon the high-order digits of the symbol sequence, and in this case we write

$$\sigma(\epsilon_n, ..., \epsilon_1) = \bar{\sigma}(\epsilon_n, \epsilon_{n-1}, ...),$$

or we may consider a formalism in which the l's are chosen to make the function σ depend most strongly upon the first elements of the sequence: in such case we will write

$$\sigma(\epsilon_n, ..., \epsilon_1) = \tilde{\sigma}(\epsilon_1, \epsilon_2, ...).$$

Inserting (4.5) in (4.4), we get

$$\sum_{\epsilon_{n+1},...,\epsilon_1} \sigma(\epsilon_{n+1}, ..., \epsilon_1) \left|l(\epsilon_n, ..., \epsilon_1)\right|^{-\tau} = a^{q_n(\tau)} \sum_{\epsilon_n,...,\epsilon_1} l(\epsilon_n, ..., \epsilon_1)\big|. \qquad (4.6)$$

However, in the cases that we will consider, the scaling function will depend, for large n, only on one end of the symbol sequence and, as n goes to infinity $q_n(\tau) \to q(\tau)$. To see how this works in a heuristic way, note that Eq. (4.6) may be brought to an eigenvalue equation by adding summations and Kronecker δ functions:

$$\sum_{\epsilon_{n+1},\ldots,\epsilon_1;\epsilon'_n,\ldots,\epsilon'_2} \delta_{\epsilon_n,\epsilon'_n} \cdots \delta_{\epsilon_2,\epsilon'_2}\, \sigma(\epsilon_{n+1},\ldots,\epsilon_1)^{-\tau} \left|l(\epsilon'_n,\ldots,\epsilon'_2,\epsilon_1)\right|^{-\tau} = a^{q_n(\tau)} \sum_{\epsilon_n,\ldots,\epsilon_1} \left|l(\epsilon_n,\ldots,\epsilon_1)\right|^{-\tau}. \tag{4.7}$$

Let us now define the transfer matrix $\mathcal{T}$ by

$$< \epsilon_{n+1},\ldots,\epsilon_2 \left| \mathcal{T} \right| \epsilon'_n,\ldots,\epsilon'_2,\epsilon_1 > = \sigma(\epsilon_{n+1},\ldots,\epsilon_1)^{-\tau}\delta_{\epsilon_n,\epsilon'_n}\,\delta_{\epsilon_2,\epsilon'_2}. \tag{4.8}$$

For large n, Eq. (4.7) can be considered an eigenvalue equation. If σ depends weakly upon either end of its symbol sequence, one can truncate the transfer matrix by simply neglecting the matrix indeces that do not appear in σ. In the large-n limit, $a^{q_n(\tau)}$ becomes the largest eigenvalue of this truncated matrix:

$$a^{q(\tau)} = \lambda(\tau). \tag{4.9}$$

This kind of formalism involving a transfer matrix and eigenvalues is the usual way of expressing problems in one-dimensional statistical mechanics. In this way we have mapped the process of refinement of the set onto an Ising model problem where the lenght or memory in σ is the range of interaction and the number of values, that ϵ_n takes on, is the number of spin states. The thermodynamic information $q(\tau)$ is then calculable from the largest eigenvalue of the transfer matrix.

4.2 Review of previous results on scaling function

We will now describe Jensen et al. [**JKP**] construction of a scaling function depending mostly on the tail of the symbol sequence; that is what they call the Feigenbaum approach. They consider the quadratic mapping

$$T(z) \quad = \quad z^2 + c \tag{4.10}$$

where c is a small real constant. In the range $\frac{-3}{4} < c < \frac{1}{4}$, the map has an unstable fixed point which equals $\xi = \frac{1}{2} + (\frac{1}{4} - c)^{1/2}$. Throughout this chapter the following convention will be used for the sign of the square root: the imaginary part of the the square-root is always non-negative and, if the imaginary part vanishes, the real part must be positive. This convention will be used below to define the symbol sequence.

We have seen in Chapter 1 (see Theorem 1.2, part *(iii)*), that the Julia set may be obtained from the preimages of the unstable fixed point; at every level of construction there are then 2^n points, whose set is denoted by $\mathcal{P}_n$. We can construct recursivley such points, as

$$x(\epsilon_1, \epsilon_2, ..., \epsilon_n) \quad \equiv \quad T^{-n}_{(\epsilon_1, \epsilon_2, ..., \epsilon_n)}(\xi) \quad = \quad (1 - 2\epsilon_1)\left[x(\epsilon_2, ..., \epsilon_n) - c\right]^{1/2}, \tag{4.11}$$

where $\epsilon_i = 0$ when the positive branch is used and $\epsilon_i = 1$ for a negative branch. It is evident that the position of a point mostly depends on ϵ_1 and only weakly on ϵ_n. We also notice that, by interpreting $\epsilon_1, ..., \epsilon_n$ as the bynary expansion of a number

$$t \quad = \quad \sum_{k=1}^{n} \epsilon_k \, 2^{-k}, \tag{4.12}$$

the points $x(t) \equiv x(\epsilon_1, ..., \epsilon_n)$ are well ordered on the Jordan curve around the origin. For brevity of notation, we shall denote by Σ_r^m the sequence

$$\Sigma_r^m \quad = \quad (\epsilon_r, ..., \epsilon_m). \tag{4.13}$$

Let us now define the distances $l^{(n)}(\Sigma_1^n)$ by

$$l^{(n)}(\Sigma_1^n) \quad = \quad \left|x\big(t+1/2^n\big) \; - \; x(t)\right|, \tag{4.14}$$

where t is the binary fraction (4.12). The distances defined in (4.14) are the nearest-neighbour distances in $\mathcal{P}_n$.

Next we define the "local mothers" of these distances, denoted by $l^{(n-1)}(\Sigma_1^n)$, to be simply the nearest-neighbour distances of the previous generation

$$l^{(n-1)}(\Sigma_1^n) \quad = \quad l^{(n-1)}(\Sigma_1^{n-1}). \tag{4.15}$$

Finally, we define the daughter-to-mother ratio, or the scaling function, by

$$\sigma(\Sigma_1^n) \quad = \quad \frac{l^{(n)}(\Sigma_1^n)}{l^{(n-1)}(\Sigma_1^n)}. \tag{4.16}$$

If one now plots σ in function of t, one gets an highly oscillating non-converging structure, since this scaling function depends mostly on the tail of the symbolic sequence. To enphatize the dependence on the tail, one must reorder the numbers (4.16) by reading the binary number backwards, according to

$$t' \quad = \quad \sum_{k=1}^{n} \epsilon_k \, 2^k/2^{n+1}. \tag{4.17}$$

Numerically, Jensen et al. show that the scaling function converges exponentially fast, computing the average difference between $\sigma(\Sigma_1^n)$ and $\sigma(\Sigma_1^{n-1})$ as well as the largest deviation, as a function of n of $\mathcal{P}_n$. Estimating the leghts $l^{(n)}$ by using the derivative of the map

$$l^{(n)}(\Sigma_1^n) \quad = \quad \left|T'(x(\Sigma_1^n))\right|^{-1} l^{(n-1)}(\Sigma_2^n), \tag{4.18}$$

one gets the following estimate the scaling function:

$$\sigma(\Sigma_1^n) \quad \simeq \quad \sigma(\Sigma_2^n), \tag{4.19}$$

since the derivative factors are aproximately equal; this is in agreement with the fact that $\sigma(\Sigma_1^n)$ depends only very weakly upon the leading digits of Σ_1^{n-1}.
On the other hand, using (4.18), one also finds

$$s(\Sigma_1^n) = \left[l^{(n-1)}(\Sigma_1^{n-1})\right]^{-1} \left|T'(x(\Sigma_1^n))\right|^{-1} l^{(n-1)}(\Sigma_2^n). \tag{4.20}$$

Then inserting, in the definition of transfer matrix, we get

$$< \Sigma \mid \mathcal{T} \mid \Sigma' > = \left|\sigma(\Sigma)\right|^{-\tau} \delta(\Sigma', \tilde{S}(\Sigma)), \tag{4.21}$$

where $\tilde{S}(\epsilon_n, ..., \epsilon - 1) = (\epsilon_{n-1}, ..., \epsilon_1)$. One can then extract the thermodynamic information from there.

4.3 Perturbative approach to the scaling function

In this section we will present our perturbation results in c complex [**AM**] on the Feigenbaum scaling function. We will consider in particular the case of nearly circular quadratic Julia sets, but the extension to the generic case of a polynomial of degree q is straightforward.
We will use perturbative methods in the complex constant c, which preserve the symmetry properties of the scaling function which are a direct consequence of the symmetry properties of the underlying Julia set.
As above, we will denote by

$$t = \sum_{k=0}^{n} \frac{\epsilon_k}{2^k} \tag{4.22}$$

the dyadic expansion of the binary code associate to the $\epsilon_1, ..., \epsilon_n$-th preimage of the repulsive fixed point $\xi = \frac{1}{2} + (\frac{1}{4} - c)^{1/2}$, as in [**JKP**], that is

$$x(\epsilon_1, ...\epsilon_n) = T^{-n}_{\epsilon_1, ..., \epsilon_n}(\xi). \tag{4.23}$$

Let us take $\zeta = \alpha + i\beta$ on the Julia set J then, $\zeta' = -\zeta \in J$, by the invariance of the Julia set and the fact that $T(\zeta) = T(\zeta')$. The corresponding bynary coding for ζ and ζ' are $(\epsilon_1, \epsilon_2, ..., \epsilon_n)$ and $(1 - \epsilon_1, \epsilon_2, ..., \epsilon_n)$; so that it can be easily checked that

$$\sigma(\epsilon_1, \epsilon_2, ..., \epsilon_n) = \sigma(1 - \epsilon_1, \epsilon_2, ..., \epsilon_n), \tag{4.24}$$

which is our first symmetry property for the scaling function. Notice that when we plot σ against $x = (\epsilon_n, ..., \epsilon_1)$, we have that points x with the same "head" $(\epsilon_n, ..., \epsilon_2)$ have the same value for the scaling function.

In the case of real c, another symmetry property is true: in fact observe that if $\zeta = \alpha + i\beta$ is in J, then also its complex conjugate $\zeta' = \bar{\zeta}$ is in J, by a similar argument as above. The corresponding bynary coding for ζ and ζ' then read $(\epsilon_1, ..., \epsilon_n)$ and $(1 - \epsilon_1, ..., 1 - \epsilon_n)$, so that the corresponding scaling functions are equal:

$$\sigma(\epsilon_1, \epsilon_2, ..., \epsilon_n) = \sigma(1 - \epsilon_1, 1 - \epsilon_2, ..., 1 - \epsilon_n). \tag{4.25}$$

It follows that it is sufficient to compute the scaling function for $0 \leq t \leq 1/2$, since $\sigma(t) = \sigma(1 - t)$.

We will now pass to the first order perturbative computation of the scaling function σ_n in the parameter c, using two perturbative methods which give rise to the same limiting first order perturbative scaling function. As we have mentioned in the previous section, the arguments in favour of the convergence of the scaling function come just from numerical considerations and the regularity properties of the limiting scaling function have not been characterized yet, even if it is reasonable to believe that the outcoming scaling function will be a fractal curve associated to a Hölder continuous function; so in particular there is no proof that in doing perturbative computations, we are allowed to exchange the limits in c going to zero and n to infinity. The motivation in doing perturbative computations on the scaling function is that, even stopping at first order, we get an untrivial fractal structure which seems to contain all the complexity structure of the unperturbed scaling function at least for very small $c \neq 0$.

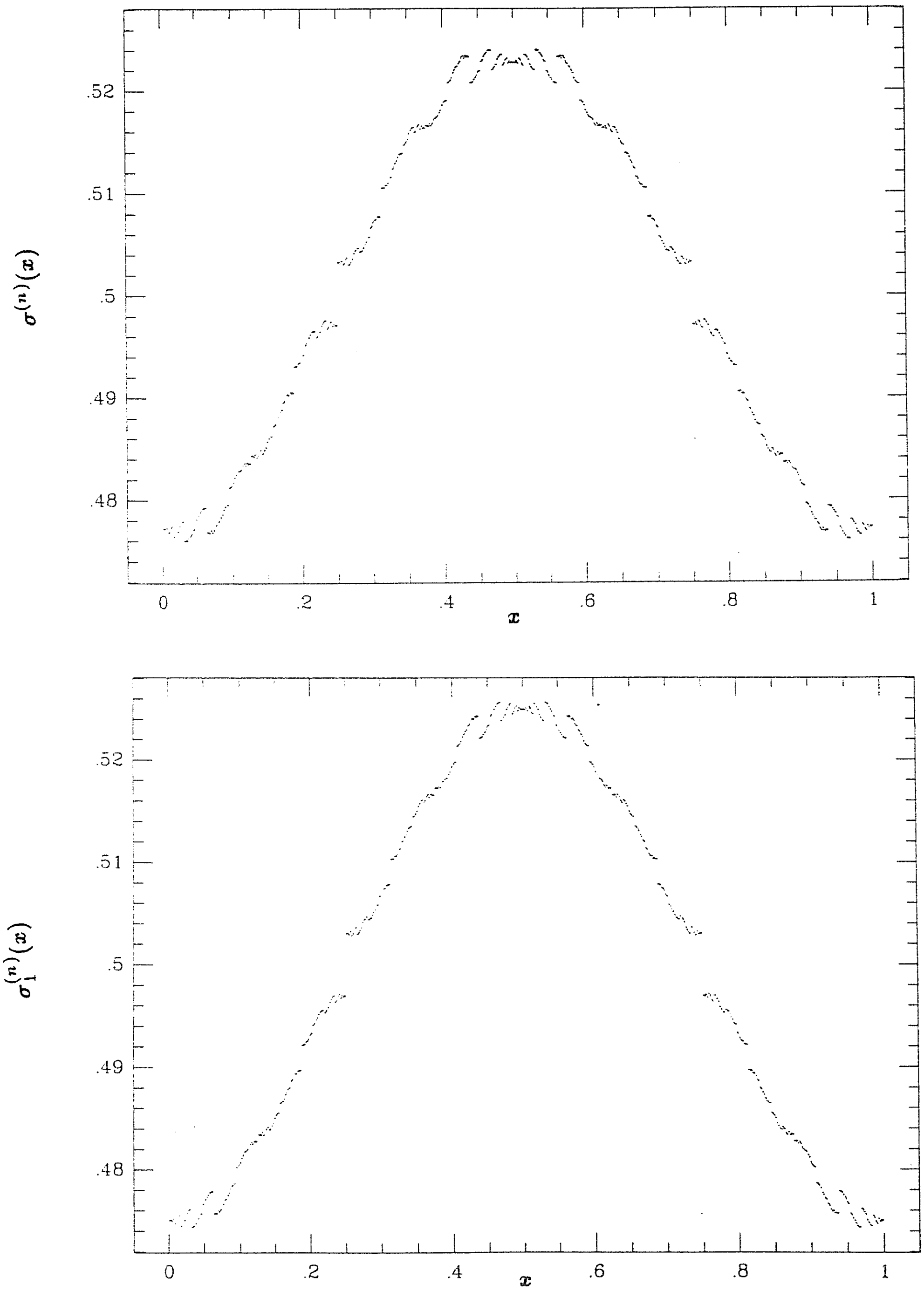

Figura 1: The Feigenbaum scaling function $\sigma^{(n)}(x)$ for the map $T(z) = z^2 + c$ for $c = -.05$ and $n = 12$, on the left, and its first order approximation $\sigma_1^{(n)}(x)$ in c for $n = 12$ on the right.

The straightforward way of doing the perturbative computation in c is to start from the definition of scaling function $\sigma(t)$, apply the Boettcher conjugation equation and compute the perturbative expansion of the coefficients of the resulting formula. Since t has a finite dyadic expansion, it easily follows that the infinite series which are associated to the coefficients of the expansion of Φ reduce to polynomials in $\exp(2i\pi t)$. Then, denoting $c = |c| \exp 2i\pi\gamma$, we get for the n-th nearest-neighbour distances

$$\begin{aligned} l^{(n)}(\epsilon_1, ..., \epsilon_n) &= \left| T^{-n}_{\epsilon_1,...,1+\epsilon_n}(\xi) - T^{-n}_{\epsilon_1,...,\epsilon_n}(\xi) \right| \\ &= \left| \Phi\left(\exp\left\{2i\pi\frac{\epsilon_1}{2} + \cdots + \frac{1+\epsilon_n}{2^n}\right\}\right) - \right. \\ &- \left. \Phi\left(\exp\left\{2i\pi\frac{\epsilon_1}{2} + \cdots + \frac{\epsilon_n}{2^n}\right\}\right) \right| \\ &= \left[2\left(1 - \cos\frac{2\pi}{2^n}\right)\right]^{\frac{1}{2}} \left\{1 - \frac{\mathrm{Re}(c)}{2^{n-1}} \right. \\ &- \left. \sum_{m=1}^{n-1} \frac{|c|}{2^m} \cos(2\pi t 2^m + \pi 2^{m-n} - 2\pi\gamma) \frac{\sin(\pi 2^{-n} - \pi 2^{m-n})}{\sin(\pi 2^{-n})} \right\}. \end{aligned} \tag{4.26}$$

Then substituting in the definition of the scaling function, we get

$$\begin{aligned} \sigma(e_1, ..., \epsilon_n) &= \left[\left(1 - \cos\frac{2\pi}{2^n}\right)\left(1 - \cos\frac{2\pi}{2^{n-1}}\right)^{-1}\right]^{\frac{1}{2}} \left\{ 1 + \mathrm{Re}(c)2^{-n} \right. \\ &- |c| \sum_{m=1}^{n-1} 2^{-m} \cos(\pi t 2^{m+1} + \pi 2^{m-n} - 2\pi\gamma) \frac{\sin(\pi 2^{-n} - \pi 2^{m-n})}{\sin(\pi 2^{-n})} \\ &+ |c| \sum_{m=1}^{n-2} 2^{-m} \cos(\pi t' 2^{m+1} + \pi 2^{m-n+1} - 2\pi\gamma) \\ &\left. \frac{\sin(\pi 2^{-n+1} - \pi 2^{m-n+1})}{\sin(\pi 2^{-n+1})} \right\}. \end{aligned} \tag{4.27}$$

Another way of doing the perturbative computation is by estimating the lenghts

$l^{(n)}(\Sigma_1^n)$ using the derivative of the map T, that is

$$l^{(n)}(\epsilon_1,...,\epsilon_n) \quad \approx \quad |T'(\eta)|^{-1}\, l^{(n-1)}(\epsilon_2,...,\epsilon_n), \tag{4.28}$$

where η is any point in the "domain" set $(T^{-n}_{\epsilon_1,...,\epsilon_n}(\xi), (T^{-n}_{\epsilon_1,...,1+\epsilon_n}(\xi)) \subset J$. It should be noticed that first we have to take the derivative of the map T and, then, to apply the conjugation function Φ, and that we cannot invert the two operations, since, the derivative of the conjugation function is singular on the Julia set.

In order to preserve the symmetry properties discussed at the beginning of this section we will choose $\eta = T^{-n-1}_{\epsilon_1,...,\epsilon_n,1}(\xi)$, so that, proceeding as above, we obtain the following first order expansion in c for σ:

$$\begin{aligned}
\sigma'(\epsilon_1,...,\epsilon_n) &= \left|T'(T^{-n-1}_{\epsilon_1,...,\epsilon_n,1}(\xi))\right|^{-1} \frac{l^{(n-1)}(\epsilon_2,...,\epsilon_n)}{l^{(n-1)}(\epsilon_1,...,\epsilon_{n-1})} \\
&= \frac{1}{2}\left\{1 + \frac{\mathrm{Re}(c)}{2^n} + |c|\sum_{m=1}^{n} 2^{-m}\cos(2\pi t 2^m + \pi 2^{m-n} - 2\pi\gamma)\right. \\
&- |c|\sum_{m=1}^{n-2} 2^{-m}\Big[\cos(2\pi t'' 2^m + \pi 2^{m-n+1} - 2\pi\gamma) \\
&- \left.\cos(2\pi t' 2^m + \pi 2^{m-n+1} - 2\pi\gamma)\Big]\frac{\sin(\pi 2^{-n+1} - \pi 2^{m-n+1})}{\sin(\pi 2^{-n+1})}\right\}.
\end{aligned} \tag{4.29}$$

Where in the formula above we have denoted by $t' = (\epsilon_1,...,\epsilon_{n-1})$ and $t'' = (\epsilon_2,...,\epsilon_n)$.

The resulting expansion for σ is different for any finite n from the previous one, but it can be easily seen that they converge to the same limit.

In fact, if the previous two expansions for the scaling function are equivalent, then they have to be equivalent also to the following one

$$\sigma_a(\epsilon_1,...,\epsilon_n) \quad = \quad \frac{l_a^{(n)}(\epsilon_1,...,\epsilon_n)}{l_a^{(n-1)}(\epsilon_1,...,\epsilon_{n-1})}. \tag{4.30}$$

where

$$l_a^{(n)} = |T'(T^{-n}(\epsilon_1, ..., \epsilon_n, 1))|^{-1} \, l^{(n-1)}(\epsilon_2, ..., \epsilon_n). \tag{4.31}$$

It is then quite easy to show, from the direct comparison of the first order expansions in c, that

$$\begin{aligned} \frac{\sigma_a^{(n)}(\Sigma_1^n)}{\sigma^{(n-1)}(\Sigma_2^{n-1})} &= 1 + |c| O\left(\frac{n}{2^n}\right) \\ \frac{l^{(n)}(\Sigma_1^n)}{l^{(n-1)}(\Sigma_2^{n-1})} &= \frac{1}{2} + |c| O\left(\frac{n}{2^n}\right). \end{aligned} \tag{4.32}$$

From (4.32) then it easily follows, on one side, that the first order expansions in c of σ, σ_a and σ' are equivalent in the limit of n large. On the other side, denoting by $\eta_n = \epsilon_1, ..., \eta_1 = \epsilon_n$ the dyadic expansion of $x = \sum_{j=1}^n \eta_j / 2^j$, which was discussed above, it follows that the first order expansions in c of σ admit a limit for $n \to \infty$, uniformly in $x = (\eta_1, ..., \eta_n, ...)$.

In particular if

$$\begin{aligned} x^1 &= (\eta_1, ..., \eta_n, \eta_{n+1}^1, ..., \eta_{n+m}^1) \\ x^2 &= (\eta_1, ..., \eta_n, \eta_{n+1}^2, ..., \eta_{n+m}^2) \end{aligned}$$

are two dyadic expansions, with the first n digits in common, it follows from (4.32) that the two first order expansions' difference satisfies

$$\left| \sigma(x^1) - \sigma(x^2) \right| = |c| \, O\left(\frac{n}{2^n}\right); \tag{4.33}$$

that is, the difference of the first order expansions of the scaling functions of points with arbitrary near dyadic expansions can be made arbitrarily small. That means that the first order scaling function admits a limit as n goes to infinity uniformly in the "reversed" dyadic expansion x for $|c|$ sufficiently small.

From this argument, one cannot deduce immediately that the the limit of the first order expansion in c of the scaling function is a continuos function in x.

In fact, continuity would be true if we were able to show that, for the case of finite dyadic numbers, for which there are two limiting dyadic expansions $(\eta_1,...,\eta_l,1,0,0,...)$ and $(\eta_1,...,\eta_l,0,1,1,...)$, the corresponding scaling functions have the same limit. But this is true if, for each fixed dyadic number $x_l = (\eta_1,...,\eta_l)$, after truncating the two dyadic possible expansions to finite lenght $n+l$, we were able to show that the difference of the corresponding first order expansions of the scaling function in c decreases as n increases, that is we have to estimate

$$\begin{aligned} \left|\sigma(\Sigma_1^{n+l+1}) - \sigma({\Sigma'}_1^{n+l+1})\right| &\equiv \\ \equiv \quad & \left|\sigma(0,0,...,0,1,\epsilon_l,...,\epsilon_1) - \sigma(1,1,...,1,0,\epsilon_l,...,\epsilon_1)\right| \quad = \\ = \quad & \sigma(\Sigma_1^{n+l+1}) \left| 1 - \frac{l^{n+l+1}({\Sigma'}_1^{n+l+1})}{l^{n+l+1}(\Sigma_1^{n+l+1})} \frac{l^{n+l}(\Sigma_1^{n+l})}{l^{n+l}({\Sigma'}_1^{n+l})} \right| . \end{aligned} \tag{4.34}$$

Since in correspondance of finite dyadic numbers both the exact and the first order perturbation expansion in c of the scaling function show "gaps" (see figure 1), which numerically seem to decrease quite slowly as n goes to infinity, we expect that, correspondingly, the left and right limit of the scaling function in that point is the same and that the convergence rate is quite slow in comparison with the rate of convergence $O(n/2^n)$ for the points with infinite dyadic expansion. That would give a positive answer to the question of continuity of the limiting perturbative scaling function for all "reversed" dyadic numbers x.

Indeed, from (4.32), we get the rather crude estimate

$$\left|\sigma(x^1) \; - \; \sigma(x^2)\right| \quad = \quad |c| \; O(l/2^l),$$

for truncations of any order $n+l$, which gives a possible estimate of the constant γ in the distortion lemma. We believe that is indeed possible to show that a more careful estimate of the first order perturbative truncation of the scaling function gives $O(l/(2^l n))$.

References

[**Ahl**] L. Ahlfors: *"Complex analysis"* – Mc Graw, Hill (1979)

[**AM**] S. Abenda and P. Moussa: *"Multifractal dimensions of and thermodynamical description of nearly circular Julia sets"* – preprint, submitted to J. Stat. Phys. (1991)

[**BD**] M.F. Barnsley and S. Demko: *"Iterated Function Systems and the Global Reconstruction of Fractals"* – Proc. R. Soc. London **A399**, 243 (1985)

[**BGH**] M. Barnsley, J. Geronimo and A. Harrington: *"Orthogonal polynomials associated with invariant measures of Julia sets"* – Bull. Amer. Math. Soc. 7, 381 (1982)

[**Bl**] P. Blanchard: *"Complex analytic dynamics on the Riemann sphere"* – Bull. Amer. Math. Soc. **11**, 85 (1984)

[**BoR**] T. Bohr and D. Rand: *" The entropy function for characteristic exponents"* – Physica **25D**, 387 (1987)

[**BoT**] T. Bohr and T. Tel: *"The thermodynamics of fractals"* – appeared in "Directions in Chaos", Hao Bai Lin ed., World Scientific (1988)

[**Bow**] R. Bowen: *"Equilibrium states and ergodic theory of Anosov diffeomorphisms"* – Lect. Notes in Math. v. 470 (1975)

[**BowRu**] R. Bowen and D. Ruelle: *"The ergodic theory of Axiom A flows"* – Invent. Math. 29, 181 (1975)

[**Br**] H. Brolin: *"Invariant sets under iteration of rational functions"* – Ark. Math. 6, 103 (1965)

[CDM] P. Collet, R. Dobbertin and P. Moussa: “ *Multifractal analysis of nearly circular Julia sets and thermodynamic formalism*” – submitted to “Annales de l'Inst. H. Poincarè, Service de Physique Théorique de Saclay preprint, SPhT/90/150 (1990)

[CLP] P. Collet, J. Lebowitz and A. Porzio: *“The dimension spectrum of some dynamical systems”* – J. Stat. Phys. 47, 609 (1987)

[DH] A. Douady, J. Hubbard: *“Iteration des polynomes quadratiques complexes ”* – C. R. Acad. Sci. Paris 294, 123 (1982)

[Do] R. L. Dobrushin: *“Induction on volume and no cluster expansion”* – appeared in Proceedings to the VIII International Congress in Mathematical Physics, M. Mebkhout and R. Seneor eds, World Scientific (1987)

[E] R. S. Ellis: *“Entropy, large deviations and statistical mechanics”* – Springer, New York (1985)

[Fa1] P. Fatou: *“Sur les equations fonctionelles”* – Bull. Soc. Math. France 47, 161 (1919)

[Fa2] P. Fatou: *“Sur les equations fonctionelles”* – Bull. Soc. Math. France 48, 33 & 208 (1920)

[Fal] K. J. Falconer: *“The geometry of fractal sets”* – Cambridge Univ. Press, Cambridge (1985)

[F] M. Feigenbaum: *“Some characterization of strange sets”* – J. Stat. Phys. 46, 919 (1987)

[FJP] M. Feigenbaum, M. Jensen and I Procaccia: *“Time ordering and the thermodynamics of strange sets: theory and experimental results”* – Phys. Rev. A 57, 1503 (1986)

[HJKPS] T Halsey, M. Jensen, L. Kadanoff, I. Procaccia and B. Shraiman: “ *Fractal* *measures and their singularities: the characterization of strange sets"* – Phys. Rev **A33**, 1141 (1986)

[HeP] H. Hentschel and I. Procaccia: *"The infinite number of generalized dimensions of fractal and strange sets"* – Physica **8D**, 435 (1983)

[JKP] M. Jensen, L. Kadanoff and I. Procaccia: *"Scaling structure of strange sets"* – Phys. Rev. A **36**, 1409 (1987)

[Ju] G. Julia: *"Memoire sur l'iteration des fonctions rationelles"* – J. Math. 8, 47 (1918)

[KaP] D. Katzen and I. Procaccia: *"Phase transitions in the thermodynamic formalism of multifractals"* – Phys. Rev. Lett. **58**, 1169 (1987)

[M] B.B. Mandelbrot: *"The fractal geometry of nature"* – W.H. Freeman , S. Francisco (1982)

[OS] A. Osbaldestin and M Sarkis: *"Taylos series approximation to Julia set scaling functions"* – Loughborough University preprint, submitted to Physica D (1991)

[Pl] D. Plachky: *"On a theorem of G. L. Sievers"* – Ann. Math. Stat. **42**, 1442 (1971)

[PlS] D. Plachky and J. Steinebach: *"A theorem about probabilities of large deviations with an application to queuing theory"* – Period. Math. Hungar. 6, 343 (1975)

[Ru1] D. Ruelle: *"Thermodynamic formalism"* – Addison–Wesley, Reading (1978)

[Ru2] D. Ruelle: *"Repellers for real analytic maps"* – Erg. Th. & Dynam. Sys. 2, 99 (1982)

[Ru3] D. Ruelle: *"The thermodynamic formalism for expanding maps"* – Commun. Math. Phys. **125**, 239 (1988)

[**RuEck**] D. Ruelle and J.P. Eckmann: *"Ergodic theory of chaos and strange attractors"* – Rev. Mod. Phys. **57**, 617 (1985)

[**SM**] C. Siegel and J. Moser: *"Lectures on celestial mechanics"* – Springer-Verlag (1971)

[**Sul1**] D. Sullivan: *"Conformal and hyperbolic geometry"* – IHES Seminar Notes , (March 1981)

[**Sul2**] D. Sullivan: *"Conformal dynamical systems"* – appeared in "Geometric dynamics", Lect. Notes in Math. v. 1007 edited by J. Palis , Springer-Verlag (1983)

[**Sul3**] D. Sullivan: *"Quasiconformal homeomorphisms and dynamics I"* – Annals of Math. **122**, 401 (1985)

[**Sul4**] D. Sullivan: *"Quasiconformal homeomorphisms and dynamics III: topological conjugacy classes of analytic endomorphisms"* – Acta Mathem. **155**, 243 (1985)

[**Va**] S. Vaienti: *"Some properties of mixing repellers"* – J. Phys. A: Math. Gen **21**, 2023 (1987)

[**WBKS**] M. Widom, D. Bensimon, L. Kadanoff and S. Shenker: *"Strange objects in the complex plane"* – J. Stat. Phys. **32**, 443 (1983)

[**W**] W. Withers: *"Calculation of Taylor series for Julia sets in powers of a parameter"* – in "Chaotic dynamics of fractals", eds. M.F. Barnsley and S. Demko, Academic Press (1986)