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SEMICONTINUITY AND RELAXATION PROPERTIES OF A CURVATURE DEPENDING FUNCTIONAL IN 2-D

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TRIESTE

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1. Introduction.

The aim of this thesis is to study the functional

$$F(E) := \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z),$$

where $E \subset \mathbf{R}^2$ is a bounded open set of class C^2, $p > 1$ is a real number, $\kappa(z)$ is the curvature of ∂E at the point z, and $\mathcal{H}^1$ denotes the one dimensional Hausdorff measure in $\mathbf{R}^2$.

We are interested in the study of the minimum problem

$$\min_E \{ \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) + \int_E g(z) \, dz \}, \tag{1.1}$$

where $g \in L^\infty(\mathbf{R}^2)$ is non negative for $|z|$ large enough. This can be considered as a simplified version of a variational problem, depending on partitions of $\mathbf{R}^2$ rather than on sets E, proposed by D. Mumford [14] to obtain a good segmentation of images in computer vision. Moreover, it was recently conjectured by E. De Giorgi [4] that problem (1.1) is connected with the asympotic behaviour of some singular perturbations of minimum problems arising in the theory of phase transition.

If we apply the direct method of calculus of variation to problem (1.1), we are led to consider a minimizing sequence $\{E_h\}_h$, which, in our hypothesis on g, satisfies

$$\sup_h \mathcal{H}^1(\partial E_h) < +\infty, \quad E_h \subset B_R(0) \quad \text{for any } h,$$

where $B_R(0) := \{z \in \mathbf{R}^2 : |z| < R\}$. By a well known compactness theorem there exists a subsequence $\{E_{h_k}\}_k$ which converges in $L^1(\mathbf{R}^2)$ to some bounded set E of finite perimeter. We shall prove that the functional F is lower semicontinuous with respect to the convergence in $L^1(\mathbf{R}^2)$. This allows us to show that, if the limit set E is of class C^2, then E is a minimizer of (1.1). Since, in general, it is hard to prove that the limit of a minimizing sequence is of class C^2, we want to extend the functional F to the set $\mathcal{M}$ of all bounded Lebesgue measurable subsets of $\mathbf{R}^2$, in such a way that the extended functional $\overline{F}$ is still lower semicontinuous with respect to the convergence in $L^1(\mathbf{R}^2)$. As usual in the theory of relaxation (see [2]), we define $\overline{F} : \mathcal{M} \to [0, +\infty]$ as the lower semicontinuous envelope of F with respect to the $L^1(\mathbf{R}^2)$-topology, that is

$$\overline{F}(E) := \inf\{\liminf_{h \to +\infty} F(E) : E_h \to E \text{ in } L^1(\mathbf{R}^2)\}.$$

The main purpose of the paper is to study the functional $\overline{F}$ and to determine the family of sets E for which $\overline{F}(E) < +\infty$. The study of the minimum problem (1.1) has led us to regard the functional

$$\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \tag{1.2}$$

as a function of E rather than of ∂E. In fact, the compactness properties of a minimizing sequence of (1.1) ensure a good convergence for the sets E_h, but the corresponding weak convergence of the boundaries ∂E_h seems to be not appropriate for variational purposes. The main difficulties, in this paper, are due essentially to the lack of good continuity properties of the map $E \to \partial E$.

Lower semicontinuity results and existence of minimizers under suitable boundary conditions are much easier for the functional, related to (1.2),

$$\int_\Gamma [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z), \tag{1.3}$$

where Γ varies now over all curves of class C^2 satisfying prescribed boundary conditions. In the case $p = 2$, the problem is classical and the minimizers, discovered by Euler in 1744 [8], are called *elastica* because of their application to the theory of flexible inextensible rods. For a complete treatment of the elastica we refer to [13, 10]. Unfortunately, these results can not be applied directly to the study of (1.1).

In this paper we have considered the problem in the plane. The extension of our results to the n-dimensional case is a difficult open problem and seems to require the methods of geometric measure theory [9, 17]. We want to stress that all our proofs are obtained by using only elementary tools. We cannot exclude that some of these results could also be obtained in a more direct (but less elementary) way by using varifold theory [12, 6, 7].

We describe now in detail the content of the paper.

In Section 2 we give some notation and we introduce the problem.

Section 3 is devoted to the study of the lower semicontinuity of F. Precisely, we prove (Theorem 3.2) that *given a sequence $\{E_h\}_h$ of bounded open sets of class C^2 converging in $L^1(\mathbf{R}^2)$ to a bounded open set E of class C^2, then*

$$\int_{\partial E} (1 + |\kappa(z)|^p) \, d\mathcal{H}^1(z) \leq \liminf_{h \to +\infty} \int_{\partial E_h} (1 + |\kappa_h(z)|^p) \, d\mathcal{H}^1(z),$$

where κ and κ_h are the curvatures of ∂E and of ∂E_h, respectively.

The definition of F makes sense if E is a set whose boundary can be parametrized, locally, by arcs of regular curves of class $H^{2,p}$, and the previuos semicontinuity theorem holds also in this case (Corollary 3.2).

We emphasize that the semicontinuity of F and the definition of $\overline{F}$ are considered with respect to the $L^1(\mathbf{R}^2)$-topology. This means that the sequence $\{E_h\}_h$ approximates E if and only if $|E_h \triangle E| \to 0$ as $h \to +\infty$, where $|\cdot|$ denotes the Lebesgue measure and $\triangle$ is the symmetric difference of sets, and no further condition is required on ∂E_h and ∂E. Simple examples show that there exist sets E with $\overline{F}(E) < +\infty$, whose boundary is not smooth. In particular, let us consider the set E of Figure 1.1. Then ∂E is not smooth because of the cusp points. Figure 1.2a shows that $\overline{F}(E) < +\infty$. In fact the sequence $\{E_h\}_h$

approximates E in $L^1(\mathbf{R}^2)$ and $\sup_h F(E_h) < +\infty$. On the other hand, the approximation of Figure 1.2b is not allowed. Here the cusp points are smoothed by circular arcs, and it is easy to prove that, if $p > 1$, then $\int_{\partial E_h} |\kappa_h(z)|^p \, d\mathcal{H}^1(z) \to +\infty$ as $h \to +\infty$. Note that figure 1.2a shows that, at the limit, the sequence $\{\partial E_h\}_h$ creates a hidden arc (with multiplicity two), given by the segment joining the two cusp points.

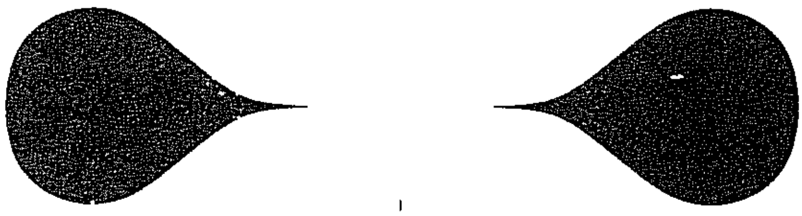

FIGURE 1.1.

FIGURE 1.2.

The main issue is obviously to characterize those sets E such that $\overline{F}(E) < +\infty$. We found some necessary conditions and some different sufficient conditions, but the complete characterization of this class of sets still remains an open problem.

In Section 4 we present some regularity properties of the sets E such that $\overline{F}(E) < +\infty$. To be precise, the following result is proven (Theorem 4.1, Remark 4.4).

Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F}(E) < +\infty$. Then, up to a modification on a set of measure zero, we have that

(i) E is bounded and open;

(ii) $\mathcal{H}^1(\partial E) < +\infty$;

(iii) *there exists a system of curves* $\Lambda = \{\gamma^1, \ldots, \gamma^m\}$ *of class* $H^{2,p}$ *such that* ∂E *is contained in the union* (Λ) *of the traces of the curves* γ^i *and* $E = int(A \cup (\Lambda))$, *where* A *is the set of all points of* $\mathbf{R}^2 \setminus (\Lambda)$ *of index* 1 *with respect to* Λ;

(iv) ∂E *has an unoriented continuous tangent;*

(v) ∂E *can have at most a countable set of cusp points.*

At the end of the section we classify the points of ∂E according to the local properties of the normal line.

In Section 5 we show examples of sets E such that $\overline{F}(E) < +\infty$, despite of their boundary being very irregular. Precisely, in Example 1 a set E is described whose irregular boundary points have positive one dimensional Hausdorff measure. In Example 2 a set having an infinite number of cusp points is shown.

In Section 6 we deal with the following problem: which conditions must satisfy the boundary of a set E in order to have that $\overline{F}(E) < +\infty$? To answer this question, we begin to study which systems of curves can be obtained as limits, in the $H^{2,p}$ norm, of a sequence of boundaries of smooth sets. Hence we introduce the definition of system of curves with multiplicity, without crossings and satisfying the finiteness property (Definition 6.1), and the notion of left and right ordering of a node. Then the following result holds (Theorem 6.1): *let* Γ *be a system of curves of class* $H^{2,p}$ *without crossings and satisfying the finiteness property, and define* E *as the set of all points of* $\mathbf{R}^2 \setminus (\Gamma)$ *of odd index with respect to* Γ. *Then*

$$\partial E \subseteq (\Gamma).$$

Moreover there exists a sequence $\{E_h\}_h$ *of bounded open sets of class* C^∞ *such that*

(i) $E_h \to E$ *in* $L^1(\mathbf{R}^2)$ *as* $h \to +\infty$;

(ii) $\sup_h F(E_h) < +\infty$.

Hence

$$\overline{F}(E) < +\infty.$$

In addition, up to a suitable surgery operation on the parameter space of Γ, *we have that*

$$\partial E_h \to \Gamma \qquad \textit{strongly in } H^{2,p} \quad \textit{as } h \to +\infty.$$

The proof of this theorem relies essentially on two ideas. The first one is that, from the point of view of the energy functional (1.3), we are free to make suitable surgery operations on the parameter space of Γ in such a way that the left and the right orderings on any node coincide. The second idea is to approximate an arc of Γ having integer multiplicity by a sequence of arcs having multiplicity one.

Then, quite surprisingly, using elementary properties of the regular graphs and the previous result, we demonstrate one of the main results of the paper (Theorem 6.4), namely, *let* E *be a subset of* $\mathbf{R}^2$ *with the following properties:*

(i) E is bounded and open;

(ii) ∂E has an unoriented continuous tangent;

(iii) E is bounded by a finite number of closed curves of class $H^{2,p}$ up to the closure;

(iv) $E = \{z \in \mathbf{R}^2 : \exists r > 0 \ |B_r(z) \setminus E| = 0\}$.

Suppose that ∂E is smooth except for a finite number n of cusp points. Then

$$n \text{ is even} \iff \overline{F}(E) < +\infty.$$

In Section 7 we localize the definition of F to all open subsets of $\mathbf{R}^2$, i.e., we consider the functional

$$F(E,\Omega) := \int_{\partial E \cap \Omega} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z), \tag{1.4}$$

where Ω is an open subset of $\mathbf{R}^2$ and E is a bounded open subset of $\mathbf{R}^2$ such that $\partial E \cap \Omega$ is of class C^2. We prove (Theorem 7.1) that $F(\cdot,\Omega)$ is $L^1(\Omega)$-lower semicontinous.

Finally, let $\overline{F}(\cdot,\Omega)$ denote the lower semicontinuos envelope of $F(\cdot,\Omega)$ with respect to the $L^1(\Omega)$-topology. The main result of this section is that, as conjectured by E. De Giorgi [4, Conjecture 5] in a slightly different context, there are sets E such that $\overline{F}(E,\mathbf{R}^2) < +\infty$ but $\overline{F}(E,\cdot)$, considered as a set function, is not a measure. Precisely, we construct an example of a set E, whose boundary is smooth except for two cusp points, such that

$$\overline{F}(E, \mathbf{R}^2 \setminus \overline{Q_1}) + \overline{F}(E, Q_2) < \overline{F}(E,\mathbf{R}^2) < +\infty,$$

where Q_1 and Q_2 are two suitable open squares in $\mathbf{R}^2$ with $Q_1 \subset\subset Q_2$.

This shows that $\overline{F}(E,\Omega)$ can not be represented by an integral of the form (1.2).

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2. Notations and preliminaries.

A plane curve $\gamma : [0,1] \to \mathbf{R}^2$ of class C^1 is said to be regular if $\frac{d\gamma(t)}{dt} \neq 0$ for every $t \in [0,1]$. Each closed regular curve $\gamma : [0,1] \to \mathbf{R}^2$ will be identified, in the usual way, with a map $\gamma : \mathbf{S}^1 \to \mathbf{R}^2$, where $\mathbf{S}^1$ denotes the unit circle with clockwise orientation. By $(\gamma) := \{\gamma(t) : t \in [0,1]\} = \gamma([0,1])$ we denote the trace of γ and by $l(\gamma)$ its length; s will denote the arc length parameter, and $\dot\gamma$, $\ddot\gamma$ the first and the second derivative of γ with respect to s. Let us fix a real number $p > 1$ and let $q > 1$ be its conjugate exponent, i.e., $\frac{1}{p} + \frac{1}{q} = 1$. If the second derivative $\ddot\gamma$ in the sense of distributions belongs to L^p, then the curvature $\kappa(\gamma)$ of γ is given by $|\ddot{\gamma}|$, and $\| \kappa(\gamma) \|_{L^p}^p := \int_0^{l(\gamma)} |\kappa(\gamma)|^p \, ds < +\infty$; in this case we say that γ is a curve of class $H^{2,p}$, and we will write $\gamma \in H^{2,p}$. If $z \in \mathbf{R}^2 \setminus (\gamma)$, $I(\gamma, z)$ will be the index of z with respect to γ [3].

$\mathcal{H}^1$ will denote the 1-dimensional Hausdorff measure in $\mathbf{R}^2$ [9]; for any $z_0 \in \mathbf{R}^2, \varrho > 0$, $B_\varrho(z_0) := \{z \in \mathbf{R}^2 : |z - z_0| < \varrho\}$ is the ball centered at z_0 with radius ϱ. Given a measurable set $E \subseteq \mathbf{R}^2$, χ_E will denote its characteristic function, that is $\chi_E(z) := 1$ if $z \in E$, $\chi_E(z) := 0$ if $z \notin E$; $|E|$ will be the Lebesgue measure of E. We say that E is of class $H^{2,p}$ (resp. C^2) if E is open and is locally the subgraph of a function of class $H^{2,p}$ (resp. C^2) with respect to a suitable orthogonal coordinate system. Note that if the boundary ∂E of E can be parametrized, locally, by arcs of regular curves of class $H^{2,p}$ (resp. C^2), then E is of class $H^{2,p}$ (resp. C^2).

If $\mathcal{M}$ denotes the class of all bounded Lebesgue measurable subets of $\mathbf{R}^2$, we define the map $F : \mathcal{M} \to [0, +\infty]$ by

$$F(E) := \begin{cases} \displaystyle\int_{\partial E} (1 + |\kappa(z)|^p) d\mathcal{H}^1(z) & \text{if } E \text{ is open of class } C^2, \\ +\infty & \text{elsewhere on } \mathcal{M}, \end{cases}$$

where $\kappa(z)$ is the curvature of ∂E at the point z.

Let $g \in L^\infty(\mathbf{R}^2)$ be a function such that $\emptyset \neq \{z \in \mathbf{R}^2 : g(z) < 0\} \subseteq B_R(0)$, for a suitable $R > 0$. Let us consider the minimum problem

$$\min_E \{ \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) + \int_E g(z) \, dz \}, \tag{2.1}$$

where E varies over all bounded open subsets of $\mathbf{R}^2$ with boundary of class C^2. It is clear that (2.1) is equivalent to the problem

$$\min_{E \in \mathcal{M}} G(E), \qquad \text{where} \qquad G(E) := F(E) + \int_E g(z) \, dz, \tag{2.2}$$

in the sense that (2.1) and (2.2) have the same minimum values and the same minimizers. By the assumptions on g, it is immediate to verify that, if problem (2.1) has a solution E, then E could be non empty.

We shall identify $\mathcal{M}$ with a closed subset of $L^1(\mathbf{R}^2)$ by means of the map $E \to \chi_E$. The $L^1(\mathbf{R}^2)$-topology on $\mathcal{M}$ is, therefore, the topology on $\mathcal{M}$ induced by the distance $d(E_1, E_2) = |E_1 \triangle E_2|$, where $E_1, E_2 \in \mathcal{M}$ and $\triangle$ is the symmetric difference of sets.

Let us prove that any minimizing sequence $\{E_h\}_h$ of G is relatively compact in $L^1(\mathbf{R}^2)$. Let $\{E_h\}_h$ be such a minimizing sequence; clearly, we can assume that $\sup_h G(E_h) < +\infty$, hence E_h is of class C^2 for any h. Then, since

$$\int_{\partial E_h} [1 + |\kappa_h(z)|^p] \, d\mathcal{H}^1(z) \leq \sup_h \; G(E_h) + \| g \|_\infty \, |B_R(0)|,$$

it follows that

$$H := \sup_h \mathcal{H}^1(\partial E_h) < +\infty. \tag{2.3}$$

Now we will show that there exist $\varrho > 0$ and $h_0 \in \mathbf{N}$ such that $E_h \subset B_\varrho(0)$ for any $h > h_0$. Suppose by contradiction that for any $\varrho > 0$, for any $h_0 \in \mathbf{N}$ there exists $h > h_0$ such that E_h has a connected component C_h with $C_h \cap \overline{B_\varrho(0)} = \emptyset$. Let us fix such ϱ, h_0 and $h = h(h_0)$. If we consider the set $E_h' := E_h \setminus C_h$, we get $G(E_h') < G(E_h)$. Let us consider the subsequence $\{E_{h(h_0)}\}_h$, and let us denote it by $\{E_h\}_h$. Since this subsequence is a minimizing sequence, it follows that necessarily $G(E_h) - G(E_h') = G(C_h) \to 0$ as $h \to +\infty$. On the other hand, one can show (see (3.2)) that C_h gives a positive and independent of h contribute to the energy F, that is $F(C_h) \not\to 0$. Contradiction.

Hence, by (2.3), it follows that

$$E_h \subset B_{\varrho+H}(0) \quad \text{for any } h.$$

We deduce that

$$\sup_h [\mathcal{H}^1(\partial E_h) + |E_h|] < +\infty, \qquad E_h \subset B_{\varrho+H}(0) \quad \text{for any } h.$$

Using the Rellich compactness theorem in BV (see [11, Th. 1.19]), it follows that there exist a bounded set E of finite perimeter and a subsequence $\{E_{h_k}\}_k$ such that $E_{h_k} \to E$ in $L^1(\mathbf{R}^2)$ as $k \to +\infty$, and this shows that any minimizing sequence of G is relatively compact in $L^1(\mathbf{R}^2)$.

We denote by $\overline{F}$ the lower semicontinuous envelope of F with respect to the topology of $L^1(\mathbf{R}^2)$. It is known that, for every $E \in \mathcal{M}$, we have

$$\overline{F}(E) := \inf\{\liminf_{h\to+\infty} F(E_h) : E_h \to E \text{ in } L^1(\mathbf{R}^2) \text{ as } h \to +\infty\}. \tag{2.4}$$

For the main properties of the relaxed functional we refer to [2]. In particular, one can prove that

$$\inf_E \{F(E) + \int_E g(z)\ dz\} = \min_{E\in\mathcal{M}} \overline{F}(E) + \int_E g(z)\ dz.$$

In addition every minimizing sequence of $F(E) + \int_E g(z)\ dz$ has a subsequence converging to a minimum point of $\overline{F}(E) + \int_E g(z)\ dz$ and every minimum point of $\overline{F}(E) + \int_E g(z)\ dz$ is the limit of a minimizing sequence of $F(E) + \int_E g(z)\ dz$.

The main purpose of this paper will be to study the properties of the functional $\overline{F}$. From the definition, it follows immediately that $\overline{F}(E) < +\infty$ if and only if there exists a sequence $\{E_h\}_h$ of bounded open sets of class $\mathcal{C}^2$ such that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$ and

$$\sup_h \mathcal{H}^1(\partial E_h) < +\infty, \quad \sup_h \int_{\partial E_h} |\kappa_h(z)|^p\ d\mathcal{H}^1(z) < \infty, \tag{2.5}$$

where, for any h, κ_h denotes the curvature of ∂E_h.

DEFINITION 2.1. *By a system of curves we mean a finite family* $\Lambda = \{\gamma^1, ..., \gamma^m\}$ *of closed simple regular curves of class* C^1 *such that* $|\frac{d\gamma^i}{dt}|$ *is constant on* $[0,1]$ *for any* $i = 1, ..., m$. *Denoting by* S *the disjoint union of* m *unit circles* $\mathbf{S}^1_1, ..., \mathbf{S}^1_m$, *we shall identify* Λ *with the map* $\Lambda : S \to \mathbf{R}^2$ *defined by* $\Lambda_{|\mathbf{S}^1_i} = \gamma^i$, *for* $i = 1, ..., m$. *The trace* (Λ) *of* Λ *is the union of the traces of the curves* γ^i, *i.e.,* $(\Lambda) := \bigcup_{i=1}^m (\gamma^i)$.

DEFINITION 2.2. *If* $\Lambda = \{\gamma^1, ..., \gamma^m\}$ *is a system of curves of class* $H^{2,p}$, *we define*

$$l(\Lambda) := \sum_{i=1}^m l(\gamma^i), \qquad \| \kappa(\Lambda) \|^p_{L^p} := \sum_{i=1}^m \| \kappa(\gamma^i) \|^p_{L^p} = \sum_{i=1}^m \int_0^{l(\gamma^i)} |\ddot{\gamma}^i(s)|^p \, ds.$$

If $z\mathbf{R}^2 \setminus (\Lambda)$ *we define the index of* z *with respect to* Λ *as*

$$I(\Lambda, z) := \sum_{i=1}^m I(\gamma^i, z).$$

As $|\frac{d\gamma^i}{dt}|$ is constant on $[0,1]$, we have $s(t) = t l(\gamma^i)$, hence

$$\int_0^{l(\gamma^i)} |\ddot{\gamma}^i(s)|^p \, ds = l(\gamma^i)^{1-2p} \int_0^1 |\frac{d^2\gamma^i}{dt^2}|^p \, dt. \tag{2.6}$$

3. Semicontinuity of F.

DEFINITION 3.1. *By a disjoint system of curves we mean a system of curves* $\Delta := \{\gamma^1, \ldots, \gamma^m\}$ *such that* $(\gamma^i) \cap (\gamma^j) = \emptyset$ *for any* $i, j = 1, \ldots, m$, $i \neq j$.

Note that if $\{E_h\}_h$ is a sequence of sets satisfying (2.5), then a suitable parametrization $\{\Delta_h\}_h$ of $\{\partial E_h\}_h$ satisfies

$$\sup_h l(\Delta_h) < \infty, \qquad \sup_h \| \kappa(\Delta_h) \|^p_{L^p} < \infty. \tag{3.1}$$

LEMMA 3.1. *Let* $\Lambda = \{\gamma^1, \ldots, \gamma^m\}$ *be a system of curves of class* $H^{2,p}$; *then*

$$m \leq l(\Lambda) \| \kappa(\Lambda) \|^q_{L^p} (2\pi)^{-q}.$$

PROOF: Let γ be a simple closed regular curve of class $H^{2,p}$; let us prove that

$$l(\gamma) \geq (2\pi)^q \| \kappa(\gamma) \|^{-q}_{L^p} . \tag{3.2}$$

Indeed, if $\gamma \in C^2$, then [5, Th. 5.7.3]

$$\int_0^{l(\gamma)} |\ddot{\gamma}(s)| \, ds \geq 2\pi. \tag{3.3}$$

By a standard approximation argument, inequality (3.3) holds for any curve γ of class $H^{2,p}$. Hence, using the Hölder inequality, if $\gamma \in H^{2,p}$ we get (3.2), since

$$2\pi \leq \int_0^{l(\gamma)} |\ddot{\gamma}(s)| \, ds \leq l(\gamma)^{\frac{1}{q}} (\int_0^{l(\gamma)} |\ddot{\gamma}(s)|^p \, ds)^{\frac{1}{p}} = l(\gamma)^{\frac{1}{q}} \parallel \kappa(\gamma) \parallel_{L^p} .$$

Then, recalling Definition 2.2, we obtain that $l(\gamma^i) \geq (2\pi)^q \parallel \kappa(\gamma^i) \parallel_{L^p}^{-q}$ for any $i = 1, \ldots, m$. It follows that

$$l(\Lambda) = \sum_{i=1}^{m} l(\gamma^i) \geq \sum_{i=1}^{m} (2\pi)^q \parallel \kappa(\gamma^i) \parallel_{L^p}^{-q} \geq (2\pi)^q \parallel \kappa(\Lambda) \parallel_{L^p}^{-q} m. \quad \square$$

The following generalization of inequality (3.2), as well as Corollary 3.1, will be very useful in section 7.

LEMMA 3.2. *Let $\gamma : [0,1] \to \mathbf{R}^2$ be a simple regular curve of class $H^{2,p}$ and let us denote by θ_0 and θ_1 the oriented angles between the x-axis and the oriented tangent vectors of γ at $t = 0$ and $t = 1$, respectively. Then*

$$l(\gamma) \geq |\theta_1 - \theta_0|^q \parallel \kappa(\gamma) \parallel_{L^p}^{-q} .$$

PROOF: Let us write with obvious notations $\dot{\gamma}(s) = (\cos\theta(s), \sin\theta(s))$. Then, using the Hölder inequality, it follows that

$$\parallel \kappa(\gamma) \parallel_{L^p}^p = \int_0^{l(\gamma)} |\dot{\theta}(s)|^p \, ds \geq l(\gamma)^{-\frac{p}{q}} (\int_0^{l(\gamma)} |\dot{\theta}(s)| \, ds)^p \geq$$

$$l(\gamma)^{-\frac{p}{q}} (|\int_0^{l(\gamma)} \dot{\theta}(s) \, ds|)^p = |\theta_1 - \theta_0|^p l(\gamma)^{-\frac{p}{q}}. \quad \square$$

COROLLARY 3.1. *Let $\gamma, \theta_0, \theta_1$ be as in Lemma 3.2. Then*

$$l(\gamma) + \parallel \kappa(\gamma) \parallel_{L^p}^p \geq |\theta_1 - \theta_0| [(\frac{p}{q})^{\frac{1}{p}} + (\frac{q}{p})^{\frac{1}{q}}]. \tag{3.4}$$

PROOF: Lemma 3.2 implies that

$$l(\gamma) + \parallel \kappa(\gamma) \parallel_{L^p}^p \geq l(\gamma) + l(\gamma)^{-\frac{p}{q}} |\theta_1 - \theta_0|^p.$$

Hence, since the minimum point of the function $l(\gamma) + l(\gamma)^{-\frac{p}{q}} |\theta_1 - \theta_0|^p$ is reached at $l(\gamma) = |\theta_1 - \theta_0| (\frac{p}{q})^{\frac{1}{p}}$, we obtain

$$l(\gamma) + \parallel \kappa(\gamma) \parallel_{L^p}^p \geq |\theta_1 - \theta_0| (\frac{p}{q})^{\frac{1}{p}} + |\theta_1 - \theta_0|^p |\theta_1 - \theta_0|^{-\frac{p}{q}} (\frac{p}{q})^{-\frac{1}{q}} = |\theta_1 - \theta_0| [(\frac{p}{q})^{\frac{1}{p}} + (\frac{q}{p})^{\frac{1}{q}}].$$

DEFINITION 3.2. *Let $E \subset \mathbf{R}^2$ be a bounded open set of class C^1. We say that a disjoint system of curves Δ is an oriented parametrization of ∂E if $(\Delta) = \partial E$, and, in addition,*

$$E = \{z \in \mathbf{R}^2 : I(\Delta, z) = 1\}, \quad \mathbf{R}^2 \setminus \overline{E} = \{z \in \mathbf{R}^2 : I(\Delta, z) = 0\}.$$

PROPOSITION 3.1. *Each bounded subset E of $\mathbf{R}^2$ of class $H^{2,p}$ (resp. C^2) such that ∂E as a finite number of connected components admits an oriented parametrization of class $H^{2,p}$ (resp. C^2).*

PROOF: Since E is of class $H^{2,p}$ each connected component of ∂E can be parametrized by a regular closed curve of class $H^{2,p}$. The last assertion follows from the Jordan's Theorem. □

DEFINITION 3.3. *Let $m \in \mathbf{N}$, $m \geq 1$; we say that a sequence $\{\Lambda_h\}_h$ of systems of curves of class $H^{2,p}$ converges weakly in $H^{2,p}$ to a system of curves $\Lambda = \{\gamma^1, \dots, \gamma^m\}$ of class $H^{2,p}$ if the number of curves of each system Λ_h equals m for h large enough, i.e., $\Lambda_h = \{\gamma_h^1, \dots, \gamma_h^m\}$, and, in addition, for any $i = 1, \dots, m$, $\gamma_h^i \rightharpoonup \gamma^i$ weakly in $H^{2,p}(0,1)$ as $h \to +\infty$.*

Note that, if $\{\Lambda_h\}_h$ converges to $\Lambda = \{\gamma^1, \dots, \gamma^m\}$ weakly in $H^{2,p}$, then

$$\gamma_h^i \to \gamma^i \text{ in } C^1([0,1]) \text{ as } h \to +\infty, \tag{3.5}$$

for any $i = 1, \dots, m$.

DEFINITION 3.4. *We say that Λ is a limiting system of curves of class $H^{2,p}$ if Λ is the weak $H^{2,p}$ limit of a sequence $\{\Delta_h\}_h$ of oriented parametrization of bounded open sets of class $H^{2,p}$.*

The following remark is an easy consequence of (3.5).

REMARK 3.1. *If Λ is a limiting system of curves of class $H^{2,p}$, then $I(\Lambda, z) \in \{0,1\}$ for any $z \in \mathbf{R}^2 \setminus (\Lambda)$.*

THEOREM 3.1. *Let $\{\Lambda_h\}_h$ be a sequence of systems of curves satisfying (3.1) such that the traces (Λ_h) are bounded in $\mathbf{R}^2$ uniformly with respect to h. Then there exists a subsequence which converges weakly in $H^{2,p}$ to a system of curves Λ.*

PROOF: By (3.1) and by Lemma 3.1, the number m_h of curves of the system Λ_h is bounded uniformly with respect to h. Hence, for a subsequence (still denoted by $\{\Lambda_h\}_h$), there exists $m \in \mathbf{N}$ such that $\Lambda_h = \{\gamma_h^1, \dots, \gamma_h^m\}$ for any h. Fix $i \in \{1, \dots, m\}$; using (3.2) and (3.1) we get that there exist two positive constants α, β such that

$$\alpha \leq l(\gamma_h^i) \leq \beta \quad \text{for any } h.$$

Then, using (2.6) we obtain that there exists a positive constant c such that $\int_0^1 |\frac{d^2\gamma_h^i}{dt^2}|^p\, dt \leq c\, l(\gamma_h^i)^{2p-1} \leq c\,\beta^{2p-1}$, whence, since (Λ_h) are bounded uniformly with respect to h by the hypothesis, the family $\{\gamma_h^i\}_h$ is equibounded in $H^{2,p}$. Then, for a subsequence, there exist m curves $\gamma^1, \ldots, \gamma^m$ of class $H^{2,p}$ such that, for any $i = 1, \ldots, m$, $\gamma_h^i \rightharpoonup \gamma^i$ weakly in $H^{2,p}(0,1)$ as $h \to +\infty$. This shows that $\{\Lambda_h\}_h$ converges to $\Lambda := \{\gamma^1, \ldots, \gamma^m\}$ weakly in $H^{2,p}$. □

In order to prove the semicontinuity Theorem 3.2, we need the following Lemma.

LEMMA 3.3. *Let $E \subseteq \mathbf{R}^2$ be a measurable set, let $\{E_h\}_h$ be a sequence of sets satisfying (2.5), and suppose that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Let us define*

$$E^\star := \{z \in \mathbf{R}^2 : \exists\, r > 0\ |B_r(z) \setminus E| = 0\}.$$

Then $E^\star$ is bounded, open, and $|E \triangle E^\star| = 0$. Moreover there exists a limiting system Λ of curves of class $H^{2,p}$ with the following properties:

(i) $E^\star = int(A \cup (\Lambda))$, where $A := \{z \in \mathbf{R}^2 \setminus (\Lambda) : I(\Lambda, z) = 1\}$;

(ii) $\partial E^\star \subseteq \partial A \subseteq (\Lambda)$;

(iii) $\{z \in \mathbf{R}^2 \setminus (\Lambda) : I(\Lambda, z) = 0\} \subseteq \mathbf{R}^2 \setminus \overline{E^\star}$;

(iv) $\partial E^\star \subseteq \overline{\mathbf{R}^2 \setminus \overline{E^\star}}$.

PROOF: It follows immediately from the definition that $E^\star$ is open. For any h, by (3.1) and by Lemma 3.1 ∂E_h admits an oriented parametrization Δ_h of class C^2 and the number m_h of curves of the system Δ_h is bounded uniformly with respect to h. Hence, for a subsequence (still denoted by $\{\Delta_h\}_h$), there exists $M \in \mathbf{N}$ such that $\Delta_h = \{\gamma_h^1, \ldots, \gamma_h^M\}$ for any h. In order to apply compactness arguments, we shall transform the sequence $\{\Delta_h\}_h$, which is not necessarily bounded (see Figure 3.1), into a sequence $\{\widetilde{\Delta}_h\}_h$ such that $\{(\widetilde{\Delta}_h)\}_h$ is bounded.

FIGURE 3.1.

Let us consider first the sequence $\{\gamma_h^1\}_h$ of curves. If the sequence of real numbers $\{a_h^1 := \sup_{t\in[0,1]} |\gamma_h^1(t)|\}_h$ converges to $+\infty$, then we eliminate γ_h^1, i.e., we replace Δ_h by the system $\Delta_h^1 := \{\gamma_h^2, \dots, \gamma_h^M\}$. Note that, as $l(\gamma_h^1)$ is uniformly bounded with respect to h, the behaviour of $\{a_h^1\}_h$ gives that, for any $R > 0$, there exists $h_R \in \mathbf{R}$ such that $I(\gamma_h^1, z) = 0$ (hence $I(\Delta_h^1, z) = I(\Delta_h, z)$) for any $h \geq h_R$ and for any $z \in B_R(0)$. If $\{a_h^1\}_h$ does not tend to $+\infty$, there exists a subsequence, still denoted by $\{\gamma_h^1\}_h$, such that the traces (γ_h^1) are bounded uniformly with respect to h. In this case we define $\Delta_h^1 := \Delta_h$. Starting from $\{\Delta_h^1\}_h$, we repeat the same procedure, obtaining a new sequence of systems of curves $\{\Delta_h^2\}_h$. After M steps, we end up with a sequence of systems $\{\Delta_h^M\}_h$, which we shall denote by $\{\widetilde{\Delta}_h\}_h$. By construction, for every h, $\widetilde{\Delta}_h$ is a disjoint system of curves of class C^2, and for every $R > 0$ there exists $h_R \in \mathbf{R}$ such that

$$I(\widetilde{\Delta}_h, z) = I(\Delta_h, z) \tag{3.6}$$

for any $h \geq h_R$ and for any $z \in B_R(0)$. It is clear also by construction that the traces $(\widetilde{\Delta}_h)$ are bounded uniformly with respect to h, i.e., there exists $R_0 > 0$ such that

$$(\widetilde{\Delta}_h) \subset B_{R_0}(0) \qquad \text{for any } h. \tag{3.7}$$

From (3.6) and (3.7) it follows easily that, for any $z \in \mathbf{R}^2 \setminus (\widetilde{\Delta}_h)$ we have $I(\widetilde{\Delta}_h, z) \in \{0,1\}$ for h large enough. Let us define $A_h := \{z \in \mathbf{R}^2 \setminus (\widetilde{\Delta}_h) : I(\widetilde{\Delta}_h, z) = 1\}$. Then, for any h, A_h is a bounded open set of class C^2, and, since $\widetilde{\Delta}_h$ is a disjoint system, we have $\partial A_h = (\widetilde{\Delta}_h)$. Using Theorem 3.1, there exists a subsequence $\{\widetilde{\Delta}_h\}_h$ which converges weakly in $H^{2,p}$ to a limiting system Λ of curves of class $H^{2,p}$. By Remark 3.1, $I(\Lambda, z) \in \{0,1\}$ for any $z \in \mathbf{R}^2 \setminus (\Lambda)$. Let us define the open set $A := \{z \in \mathbf{R}^2 \setminus (\Lambda) : I(\Lambda, z) = 1\}$. Since $\chi_{A_h}(z) = I(\widetilde{\Delta}_h, z)$ for any $z \in \mathbf{R}^2 \setminus (\widetilde{\Delta}_h)$, and $\chi_A(z) = I(\Lambda, z)$ for any $z \in \mathbf{R}^2 \setminus (\Lambda)$, by the continuity property of the index and by the dominated convergence theorem we have that $A_h \to A$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Let us prove that

$$|E \triangle A| = 0. \tag{3.8}$$

By (3.6) and (3.7), for any $R \geq R_0$, we have that $A_h = E_h \cap B_R(0)$ for h large enough; passing to the limit as $h \to +\infty$, we obtain that $|A \triangle (E \cap B_R(0))| = 0$. Since R is arbitrary, we get (3.8).

Let us prove that $|E \triangle E^\star| = 0$. By (3.8), it is enough to prove that

$$|A \triangle E^\star| = 0. \tag{3.9}$$

Since A is open and $|E \triangle A| = 0$, for any $z \in A$ there exists $r > 0$ such that $B_r(z) \subset A$ and $|B_r(z) \setminus E| \leq |B_r(z) \triangle E| \leq |A \triangle E| = 0$; hence $z \in E^\star$. Then we get

$$A \subseteq E^\star. \tag{3.10}$$

To prove (3.9), being $|(\Lambda)| = 0$, it is sufficient to show that

$$E^\star \subseteq A \cup (\Lambda). \tag{3.11}$$

If $z \notin A \cup (\Lambda)$, then $I(\Lambda, z) = 0$. Since the index is locally constant, there exists $r > 0$ such that $I(\Lambda, w) = 0$ for any $w \in B_r(z)$. This implies that $0 = |B_r(z) \cap A| = |B_r(z) \cap E|$, hence $z \notin E^\star$. This proves (3.11), so we can conclude that $|E \triangle E^\star| = 0$.

Let us prove (i). The inclusion $E^\star \subseteq \text{int}(A \cup (\Lambda))$ follows from (3.11) and from the fact that $E^\star$ is open. To prove the opposite inclusion, let $z \in \text{int}(A \cup (\Lambda))$. Then there exists $r > 0$ such that $B_r(z) \subset A \cup (\Lambda)$. Hence $|B_r(z) \setminus A| = |B_r(z) \setminus E| = 0$, and therefore $z \in E^\star$. This concludes the proof of (i). In particular, it follows that $E^\star$ is bounded.

Now we will prove (ii). Let us show that

$$\partial A \subseteq (\Lambda). \tag{3.12}$$

If $z \notin (\Lambda)$, then either $I(\Lambda, z) = 1$ or $I(\Lambda, z) = 0$. In the former case $z \in A$, hence $z \notin \partial A$. In the latter case, there exists a neighbourhood U of z such that $I(\Lambda, w) = 0$ for every $w \in U$. This implies $U \cap A = \emptyset$, hence $z \notin \partial A$. This concludes the proof of (3.12). Note that the inclusion (3.12) might be strict (see Figure 1.2 a).

To prove (ii) it remains to show that $\partial E^\star \subseteq \partial A$. Let $z \in \partial E^\star$ and let U be a neighbourhood of z_0. Clearly there exists a point $z \in U \setminus E^\star$; from (3.10) it follows that

$$z \in U \setminus A. \tag{3.13}$$

Moreover, since $U \cap E^\star$ is open, using the definition of $E^\star$ and (3.8), we can find a point $w \in U \cap E^\star$ with the property that there exists $r > 0$ such that $B_r(w) \subset U$ and $|B_r(w) \setminus A| = |B_r(w) \setminus E| = 0$. Hence we can choose a point w' such that

$$w' \in A \cap U. \tag{3.14}$$

Since U is an arbitrary neighbourhood of z_0, (3.13) and (3.14) imply that $z_0 \in \partial A$, hence $\partial E^\star \subseteq \partial A$.

Let us prove (iii). Let $z_0 \in \{z \in \mathbf{R}^2 \setminus (\Lambda) : I(\Lambda, z) = 0\}$; then there esists $r > 0$ such that $B_r(z_0) \subset \{z \in \mathbf{R}^2 \setminus (\Lambda) : I(\Lambda, z) = 0\}$. Moreover, from (3.9), it follows that $|B_r(z_0) \cap E^\star| = 0$; since $E^\star$ is open we get $\emptyset = B_r(z_0) \cap E^\star = B_r(z_0) \cap \overline{E^\star}$. This gives (iii).

It remains to prove (iv). From (iii), it is enough to show that

$$\partial E^\star \subseteq \overline{\{z \in \mathbf{R}^2 \setminus (\Lambda) : I(\Lambda, z) = 0\}}.$$

Let $z_0 \in \partial E^\star$. If by contradiction there exists a ball $B_r(z_0)$ such that $B_r(z_0) \cap \{z \in \mathbf{R}^2 \setminus (\Lambda) : I(\Lambda, z) = 0\} = \emptyset$, then $|B_r(z_0) \cap (\mathbf{R}^2 \setminus E^\star)| = 0$; hence $|B_r(z_0) \setminus E^\star| = 0$, that gives $z_0 \in E^\star$. $\square$

If $\overline{F}(E) < +\infty$, we shall consider $E^\star$ as the best representative of E in its equivalence class, therefore, to simplify the notations, in the sequel we always shall assume $E = E^\star$. Note that, with this convention, ∂E has no isolated points, and cannot have any one dimensional subset lying inside or outside of E. Obviously, if E is a bounded open set of class C^2, then $E^\star = E$.

THEOREM 3.2. *The functional $F(\cdot)$ is $L^1(\mathbf{R}^2)$-lower semicontinuous on the class of all bounded open subsets of $\mathbf{R}^2$ of class $\mathcal{C}^2$, i.e., given a sequence $\{E_h\}_h$ of bounded open sets of class $\mathcal{C}^2$ converging in $L^1(\mathbf{R}^2)$ to a bounded open set E of class $\mathcal{C}^2$, then*

$$\int_{\partial E}(1+|\kappa(z)|^p)\, d\mathcal{H}^1(z) \leq \liminf_{h\to+\infty}\int_{\partial E_h}(1+|\kappa_h(z)|^p)\, d\mathcal{H}^1(z), \tag{3.15}$$

where κ and κ_h are the curvatures of ∂E and of ∂E_h, respectively.

PROOF: Let $\{E_h\}_h$ be a sequence of bounded open sets of class $\mathcal{C}^2$, $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h\to+\infty$. We can suppose that the right hand side in (3.15) is finite, otherwise the result is trivial. Let $\{E_{h_k}\}_k$ be a subsequence of $\{E_h\}_h$ with the property that

$$\lim_{k\to+\infty} F(E_{h_k}) = \liminf_{h\to+\infty} F(E_h) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. For any k, ∂E_k has a finite number of connected components. Let Δ_k be an oriented parametrization of ∂E_k. Since $\{E_k\}_k$ satisfies (2.5), the sequence $\{\Delta_k\}_k$ satisfies (3.1). As in the proof on Theorem 3.1, we can suppose that, for any k, $\Delta_k = \{\gamma_k^1,\dots,\gamma_k^M\}$ with M independent of k. Let $\{\tilde{\Delta}_k\}_k = \{\gamma_k^{i_1},\dots,\gamma_k^{i_m}\}$ be the equibounded sequence and let Λ be the limiting system of curves of class $H^{2,p}$ constructed in the proof of Lemma 3.3. Then, by construction, $m\leq M$, $\Lambda = \{\gamma^{i_1},\dots,\gamma^{i_m}\}$, and for any $j=1,\dots,m$ we have that $\gamma_k^{i_j} \rightharpoonup \gamma^{i_j}$ weakly in $H^{2,p}$ as $k\to+\infty$. Using (ii) of Lemma 3.3 and the weak lower semicontinuity of the L^p norm, we have

$$\liminf_{h\to+\infty} F(E_h) = \lim_{k\to+\infty} F(E_k) = \lim_{k\to+\infty}\sum_{i=1}^{M} l(\gamma_k^i) + \lim_{k\to+\infty}\sum_{i=1}^{M} l(\gamma_k^i)^{1-2p}\int_0^1 |\frac{d^2\gamma_k^i}{dt^2}|^p\, dt$$

$$\geq \lim_{k\to+\infty}\sum_{j=1}^{m} l(\gamma_k^{i_j}) + \lim_{k\to+\infty}\sum_{j=1}^{m} l(\gamma_k^{i_j})^{1-2p}\int_0^1 |\frac{d^2\gamma_k^{i_j}}{dt^2}|^p\, dt$$

$$= \sum_{j=1}^{m} l(\gamma^{i_j}) + \sum_{j=1}^{m} l(\gamma^{i_j})^{1-2p}\lim_{k\to+\infty}\int_0^1 |\frac{d^2\gamma_k^{i_j}}{dt^2}|^p dt \geq \mathcal{H}^1(\partial E) + \sum_{j=1}^{m} l(\gamma^{i_j})^{1-2p}\int_0^1 |\frac{d^2\gamma^{i_j}}{dt^2}|^p dt$$

$$= \mathcal{H}^1(\partial E) + \sum_{j=1}^{m}\int_0^{l(\gamma^{i_j})} |\ddot{\gamma}^{i_j}(s)|^p\, ds = \mathcal{H}^1(\partial E) + \| \kappa(\Lambda) \|_{L^p}^p \geq F(E),$$

that gives the assertion. □

Note that the inequality (3.15) might be strict. In fact, let $r\in\mathbf{R}^+$, $E := B_r(0)$, $E_k := B_{r-\frac{1}{k}}(0) \cup \{z\in\mathbf{R}^2 : r+1-\frac{1}{k} \leq |z| \leq r+1+\frac{1}{k}\}$ for any k. Then ∂E is strictly contained in $(\Lambda) = \partial B_r(0)\cup\partial B_{r+1}(0)$, and

$$\lim_{k\to+\infty} F(E_k) = F(E) + 2F(B_{r+1}(0)).$$

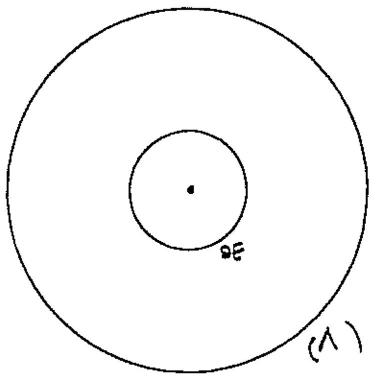

FIGURE 3.2.

The following result generalizes Theorem 3.2.

COROLLARY 3.2. *Let $E \subset \mathbf{R}^2$ be a bounded open set of class $H^{2,p}$. Then*

$$\overline{F}(E) = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z). \tag{3.16}$$

In particular, $\overline{F}(E) < +\infty$.

PROOF: Theorem 3.2 holds with the same proof if E is of class $H^{2,p}$, hence, passing to the infimum with respect to the approximating sequence $\{E_h\}_h$ in (3.15), we infer that

$$\int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z) \leq \overline{F}(E). \tag{3.17}$$

Let us prove the opposite inequality. Proposition 3.1 implies that there exists an oriented parametrization $\Delta = \{\gamma^1, \dots, \gamma^m\}$ of ∂E of class $H^{2,p}$. Hence in particular $\partial E = \cup_{i=1}^m (\gamma^i)$, where $\gamma^i : [0,1] \to \mathbf{R}^2$ are closed simple regular disjoint curves of class $H^{2,p}$. For any $i = 1, \dots, m$, let us consider a sequence $\{\gamma_h^i\}_h$ of curves of class $C^\infty([0,1])$ such that $\gamma_h^i \to \gamma^i$ strongly in $H^{2,p}$ as $h \to +\infty$. It follows that, for h large enough, the approximating system $\Delta_h := \{\gamma_h^1, \dots, \gamma_h^m\}$ is a disjoint system of curves. Moreover, by construction, (Δ_h) are equibounded uniformly with respect to h. For any h, let us define $E_h := \{z \in \mathbf{R}^2 \setminus (\Delta_h) : I(\Delta_h, z) = 1\}$. Then $\partial E_h = \cup_{i=1}^m (\gamma_h^i)$, $E_h \to \{z \in \mathbf{R}^2 \setminus (\Delta) : I(\Delta, z) = 1\} = E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$, and $\int_{\partial E_h} [1 + |\kappa_h(z)|^p] \, d\mathcal{H}^1(z) = \sum_{i=1}^m \int_{\gamma_h^i} [1 + |\kappa(\gamma_h^i)|^p] \, dt$. It follows that

$$\overline{F}(E) \leq \lim_{h \to +\infty} \sum_{i=1}^m \int_{\gamma_h^i} [1 + |\kappa(\gamma_h^i)|^p] = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z),$$

by construction. This concludes the proof of (3.16). $\square$

In particular, from the preceding result, it follows that if $E \subset \mathbf{R}^2$ is a bounded open set of class $H^{2,p}$, then there exists a sequence $\{E_h\}_h$ of bounded open subsets of $\mathbf{R}^2$ of class C^∞ such that $E_h \to E$ in $L^1(\mathbf{R}^2)$, $\partial E_h \to \partial E$ in C^1, $\kappa_h \to \kappa$ strongly in L^p, as $h \to +\infty$.

4. Some properties of the sets E such that $\overline{F}(E) < +\infty$.

We recall the definition of tangent cone for an arbitrary subset of $\mathbf{R}^2$ (see for instance [9, 3.1.21]).

DEFINITION 4.1. *Whenever $A \subseteq \mathbf{R}^2$, $z_0 \in \overline{A}$, we define the tangent cone of A at z_0, denoted by $T_A(z_0)$, as the set of all $v \in \mathbf{R}^2$ such that for every $\epsilon > 0$ there exist $z \in A, 0 < r \in \mathbf{R}$ with $|z - z_0| < \epsilon$, $|r(z - z_0) - v| < \epsilon$. Such vectors v will be called tangent vectors of A at z_0.*

By the definition, it follows that $T_A(z_0)$ is a closed subset of $\mathbf{R}^2$.

If $T_A(z_0)$ is a straight line we can write $T_A(z_0) = \{\alpha\tau(z_0) : \alpha \in \mathbf{R}\}$, where $\tau(z_0)$ is a unit vector determined up to the sign that we will call a tangent unit vector of A at z_0. In this case we define the normal line $N_A(z_0)$ of A at z_0 as the straight line through the origin orthogonal to $T_A(z_0)$. We will make the following convention: once $\tau(z_0)$ is chosen, $\nu(z_0)$ is the orthogonal unit vector to $\tau(z_0)$ taken in such a way that $\{\tau(z_0), \nu(z_0)\}$ is oriented as the canonical basis of $\mathbf{R}^2$.

Note that if Λ is a system of curves, $z_0 \in (\Lambda)$ and $\Lambda^{-1}(z_0) = \{t_1, \ldots, t_n\}$, then

$$T_{(\Lambda)}(z_0) = \bigcup_{i=1}^{n} \{\alpha \frac{d\Lambda}{dt}(t_i) : \alpha \in \mathbf{R}\}. \tag{4.1}$$

DEFINITION 4.2. *We say that a system of curves Λ has no crossings if, whenever $\gamma^i(t_1) = \gamma^j(t_2)$ for some $i \neq j$ and $t_1, t_2 \in [0,1]$, then $\frac{d\gamma^i(t_1)}{dt}$ and $\frac{d\gamma^j(t_2)}{dt}$ are parallel.*

Note that if Λ is a limiting system of curves of class $H^{2,p}$ then Λ has no crossings. This follows easily from the fact that Λ is limit in C^1 of systems of *simple* curves (see (3.5)).

Let Λ be a system of curves without crossings, and let $z_0 \in (\Lambda)$; by (4.1) the tangent cone $T(z_0) := T_{(\Lambda)}(z_0)$ of (Λ) at z_0 is a straight line.

DEFINITION 4.3. *Let Λ be a system of curves without crossings, let $z_0 \in (\Lambda)$ and let $\tau(z_0)$ be a tangent unit vector of (Λ) at z_0. If there exists $t \in S = \mathbf{S}^1_1 \cup ... \cup \mathbf{S}^1_m$ such that $\Lambda(t) = z_0$, and $\frac{d\Lambda}{dt}(t) \cdot \tau(z_0) > 0$ (resp. $\frac{d\Lambda}{dt}(t) \cdot \tau(z_0) < 0$), then we say that Λ is positively (resp. negatively) oriented with respect to $\tau(z_0)$ in t, or equivalently, that Λ points to the right (resp. to the left) with respect to $\nu(z_0)$ in t.*

We want to list some regularity properties of those sets E such that $\overline{F}(E) < +\infty$. We shall identify the real projective space $\mathbf{P}^1$ with the set of all one dimensional linear subspaces of $\mathbf{R}^2$.

DEFINITION 4.4. *Let A be a subset of $\mathbf{R}^2$. We say that A has an unoriented continuous tangent if at each point $z \in A$ the tangent cone $T_A(z)$ of A at z_0 is a straight line and the map $T_A : z \to T_A(z)$ from A into $\mathbf{P}^1$ is continuous.*

PROPOSITION 4.1. *Let Λ be a system of curves of class C^1 without crossings; then the set (Λ) has an unoriented continuous tangent.*

PROOF: We must prove that the map $T_{(\Lambda)}$ is continuous. Let $z_0 \in (\Lambda)$, $\Lambda^{-1}(z_0) = \{t_1, \dots, t_n\}$. As $\frac{d\Lambda}{dt}(t_1) = \cdots = \frac{d\Lambda}{dt}(t_n)$ are parallel, it is easy to prove that the only unoriented tangent vector of (Λ) at z_0 is $T_{(\Lambda)}(z) = \pi(\frac{d\Lambda}{dt}(t_1)) = \cdots = \pi(\frac{d\Lambda}{dt}(t_n))$, where π denotes the canonical projection of $\mathbf{R}^2$ into $\mathbf{P}^1$. Let $U \subseteq \mathbf{P}^1$ be an open neighbourhood of $T_{(\Lambda)}(z_0)$. The map $t \to T_{(\Lambda)}(\Lambda(t))$ is continuous because $T_{(\Lambda)}(\Lambda(t)) = \pi(\frac{d\Lambda}{dt}(t))$. Hence for any $\varepsilon > 0$, there exist $\delta_1, \dots, \delta_n$ positive numbers such that, if $|t - t_i| < \delta_i$, then $T_{(\Lambda)}(\Lambda(t)) \in U$, so that if $V := \cup_{i=1}^n \Lambda(t_i - \delta_i, t_i + \delta_i)$, then $T_{(\Lambda)}(V) \subset U$. Let us take δ_i so small in such a way that the intervals $]t_i - \delta_i, t_i + \delta_i[$ are pairwise disjoint; since Λ is a system of curves parametrized with constant velocity, the implicit function theorem implies that V is a neighbourhood of z_0 in (Λ) for the induced topology from $\mathbf{R}^2$. This concludes the proof. □

THEOREM 4.1. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F}(E) < +\infty$. Then, up to a modification on a set of measure zero, we have that*

(i) E is bounded and open;

(ii) $\mathcal{H}^1(\partial E) < +\infty$;

(iii) there exists a limiting system of curves Λ of class $H^{2,p}$ such that $(\Lambda) \supseteq \partial E$ and $E = int(A \cup (\Lambda))$, where $A = \{z \in \mathbf{R}^2 \setminus (\Lambda) : I(\Lambda, z) = 1\}$;

(iv) ∂E has an unoriented continuous tangent;

(v) $\overline{F}(E) \geq \inf\{l(\Lambda) + \| \kappa(\Lambda) \|_{L^p}^p : \Lambda \in \mathcal{A}(E)\}$, where $\mathcal{A}(E)$ is the collection of those limiting systems of curves Λ of class $H^{2,p}$ such that $(\Lambda) \supseteq \partial E$.

PROOF: Assertions (i), (ii) and (iii) follow from (2.5) and Lemma 3.3. Assertion (iv) follows from Proposition 4.1, and (ii) of Lemma 3.3. Let us prove (v). Since $\overline{F}(E) < +\infty$, there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^2 satisfying (2.5) and such that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Using the same notations as in the proof of Theorem 3.2, for a subsequence $\{E_k\}_k$ we have, from the weak lower semicontinuity of the L^p norm,

$$\liminf_{h \to +\infty} F(E_h) = \lim_{k \to +\infty} F(E_k) = l(\Lambda) + \lim_{k \to +\infty} \int_{\partial E_k} |\kappa_k(z)|^p \, d\mathcal{H}^1(z)$$

$$\geq l(\Lambda) + \| \kappa(\Lambda) \|_{L^p}^p \geq \inf\{l(\Lambda) + \| \kappa(\Lambda) \|_{L^p}^p : \Lambda \in \mathcal{A}(E)\}.$$

Taking the infimum with respect to the approximating sequence $\{E_h\}_h$, we get (v). □

DEFINITION 4.5. *Let $E \subseteq \mathbf{R}^2$ be a measurable set, let $z \in \partial E$, and suppose that the tangent cone of ∂E at z is a straight line. Let $\nu(z)$ be a normal unit vector of ∂E at z, and, for any $\varrho > 0$, let*

$$N_\varrho^+(z) := \{z + s\nu(z) : 0 < s < \varrho\}, \quad N_\varrho^-(z) := \{z - s\nu(z) : 0 < s < \varrho\}.$$

Then we define

$$\partial_{00}E := \{z \in \partial E : \exists r > 0 \ N_r^-(z) \cup N_r^+(z) \subseteq \mathbf{R}^2 \setminus \overline{E}\},$$

$$\partial_{11}E := \{z \in \partial E : \exists r > 0 \ N_r^-(z) \cup N_r^+(z) \subseteq E\},$$

$$\partial_{01}E := \{z \in \partial E : \exists r > 0 \ N_r^-(z) \subseteq \mathbf{R}^2 \setminus \overline{E},\ N_r^+(z) \subseteq E \text{ or conversely }\}.$$

REMARK 4.1. *Suppose that $E \subseteq \mathbf{R}^2$ is such that at any point of ∂E the tangent cone is a straight line; then, for any $z \in \partial E$ there exists $r > 0$ such that $N(z) \cap B_r(z) \cap \partial E = \{z\}$.*

PROOF: Follows immediately from the definition and the uniqueness of the tangent line at z. □

REMARK 4.2. *Let $E \subseteq \mathbf{R}^2$ be a measurable set having an unoriented continous tangent. Then*

$$\partial E = \partial_{00}E \cup \partial_{11}E \cup \partial_{01}E. \tag{4.2}$$

In particular, if $\overline{F}(E) < +\infty$, then (4.2) is fulfilled.

PROOF: The inclusion $\partial_{00}E \cup \partial_{11}E \cup \partial_{01}E \subseteq \partial E$ is obviuos. The opposite inclusion follows from Remark 4.1. □

REMARK 4.3. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F}(E) < +\infty$, and let A be as in (iii) of Theorem 4.1. Then the definitions of $\partial_{00}E$, $\partial_{11}E$ and $\partial_{01}E$ remains unchanged if we replace E by A in Definition 4.5.*

PROOF: Let $z \in \partial E$ and suppose for instance that $z \in \partial_{11}E$. We will show that $N_r^+(z) \subset A$ and $N_r^-(z) \subset A$. Let (Λ) be as in (iii) of Theorem 4.1. By Remark 4.1 there exists $r > 0$ such that $B_r(z) \cap N(z) \cap (\Lambda) = \{z\}$. Hence, since $E = \text{int}(A \cup (\Lambda))$, we have the assertion. The other cases are analogous. □

A particular case of the next Lemma ($r = 0, m = 1$) can be found in [1, Lemma 9.2.5]. The general case can be proved using the same methods.

LEMMA 4.1. *Let Λ be a system of curves without crossings, $z_0 \in (\Lambda)$, and $z_1, z_2 \neq z_0$ be points on $N(z_0)$, one on each side of z_0, and close enough to z_0. Suppose that Λ is positively oriented r-times, and negatively m-times with respect to $\tau(z_0)$ in z_0. Then $|I(\Lambda, z_1) - I(\Lambda, z_2)| = |r - m|$.*

Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F}(E) < +\infty$, and let $\Lambda = \{\gamma^1, \ldots, \gamma^m\}$ be the system of curves of class $H^{2,p}$ of Theorem 4.1. Suppose that $z_0 \in \partial_{01}E$; from Lemma 4.1, we deduce that, if $\{t_1, \ldots, t_n\} = \Lambda^{-1}(z_0)$, then $n = 2k + 1$, $k \geq 0$. More precisely, we get that Λ points k-times to the right with respect to $\nu(z_0)$, and $(k+1)$-times to the left (or viceversa). If $z_0 \in \partial_{00}E$ or $z_0 \in \partial_{11}E$, we get $n = 2k$, $k \geq 1$, and Λ points k-times to the right with respect to $\nu(z_0)$, and k-times to the left.

Let $p \in (\Lambda) \setminus \partial E$; then from (iii) of Theorem 4.1 it follows that there exists $r > 0$ such that either $I(\Lambda, z) = 0$ for any $z \in N_{(\Lambda)}(p) \cap B_r(p) \setminus (\Lambda)$, or $I(\Lambda, z) = 1$ for any $z \in N_{(\Lambda)}(p) \cap B_r(p) \setminus (\Lambda)$. This shows that (Λ) has even multiplicity at any point $z \in (\Lambda) \setminus \partial E$.

Suppose now that $z_0 \in \partial E$ has a neighbourhood where ∂E is of class C^1. Reparametrize Λ by the arc lenght. If there exist $t_i, t_j \in \Lambda^{-1}(z_0)$ such that $\frac{d\gamma^r}{dt}(t_i) = \frac{d\gamma^s}{dt}(t_j)$ for some $r \neq s$, $s, r \in \{1, \dots, m\}$, then from the uniqueness of the parametrization by arc lenght, we infer that there exists $\eta > 0$ such that, for any $t \in]-\eta, \eta[$ we have $\gamma^r(t_i + t) = \gamma^s(t_j + t)$. This means that when two different branches of Λ meet together with the same orientation at a point z_0 where ∂E is locally of class C^1, then they must glue together at least for a short time.

In the following figures we show some examples of sets E with $\overline{F}(E) < +\infty$, their approximation in the $L^1(\mathbf{R}^2)$ norm, and the limiting system of curves Λ obtained.

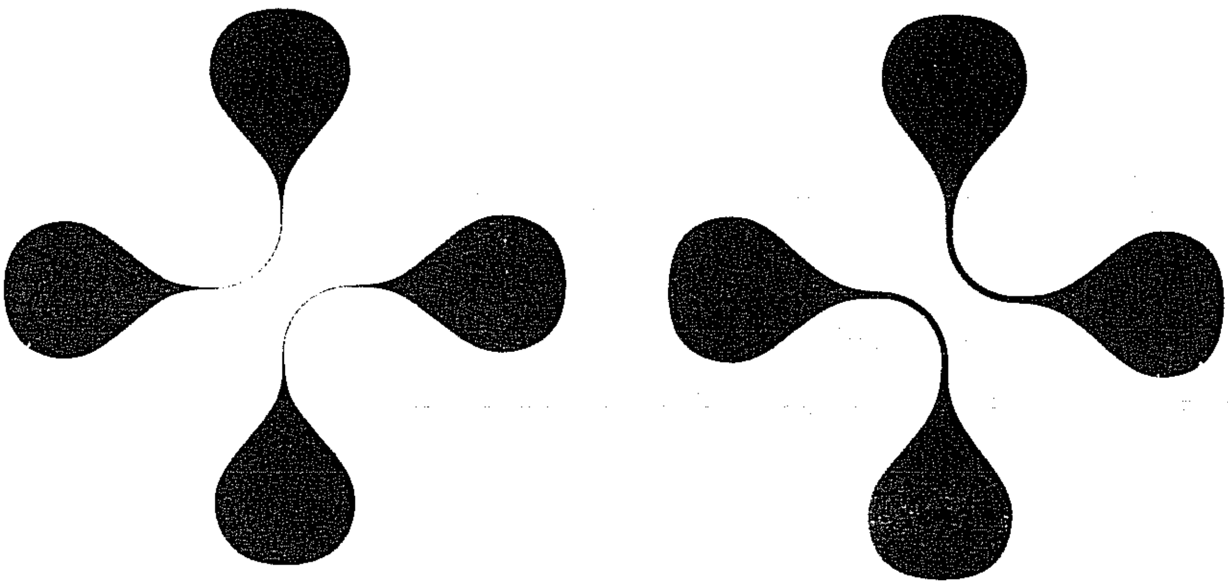

FIGURE 4.1.

DEFINITION 4.6. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F}(E) < +\infty$, $z_0 \in \partial E$, and let $\tau(z_0)$ be a tangent unit vector of ∂E at z_0. We say that z_0 is a cusp point of ∂E if there exists $r > 0$ such that*

$$\textit{either} \quad B_r^+(z_0) \cap \partial E = \emptyset, \quad \textit{or} \quad B_r^-(z_0) \cap \partial E = \emptyset,$$

where

$$B_r^+(z_0) := \{y \in B_r(z_0) : (y - z_0) \cdot \tau(z_0) > 0\}, \ B_r^-(z_0) := \{y \in B_r(z_0) : (y - z_0) \cdot \tau(z_0) < 0\}.$$

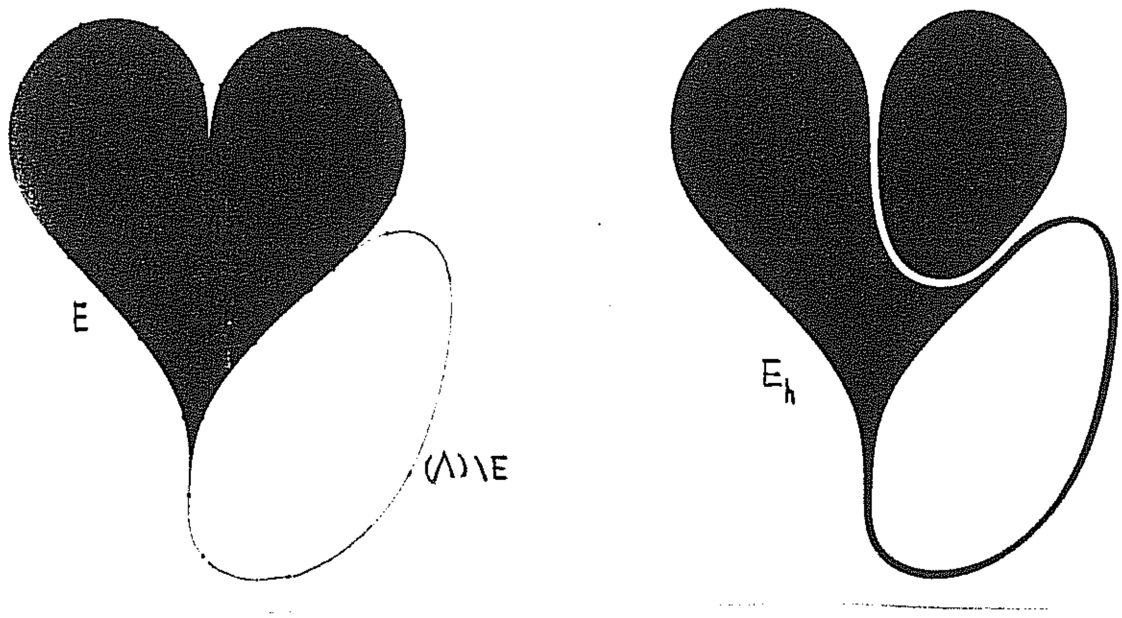

FIGURE 4.2.

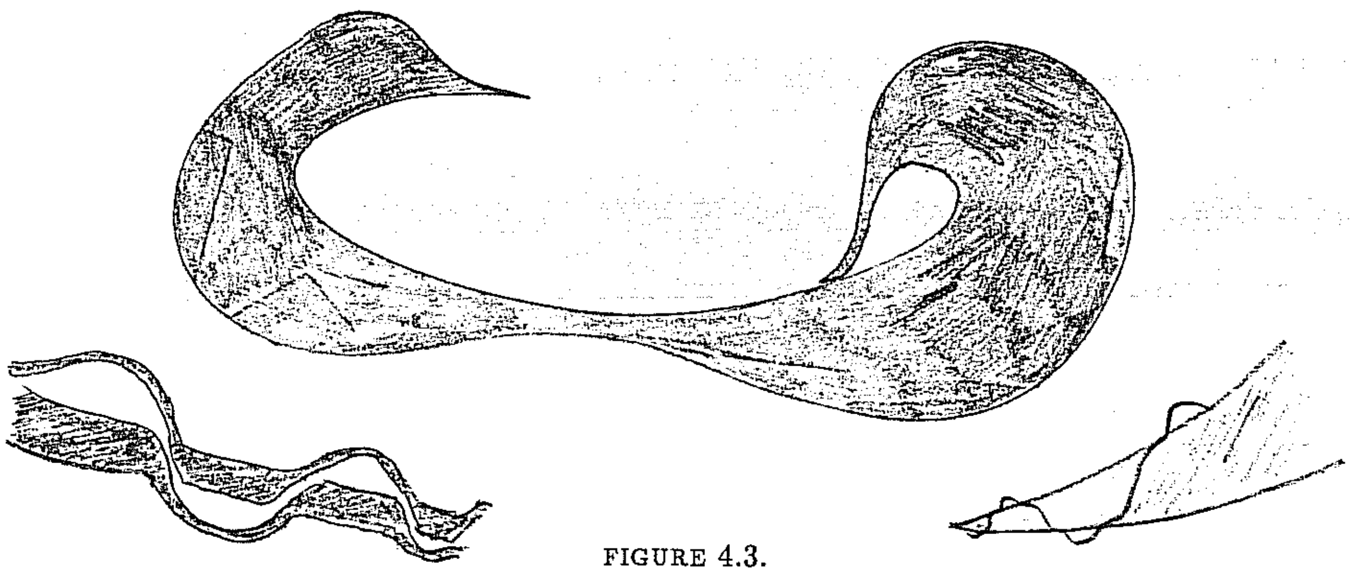

FIGURE 4.3.

REMARK 4.4. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F}(E) < +\infty$. Then ∂E have at most a countable number of cusp points.*

PROOF: Let $\Lambda = \{\gamma^1, \dots, \gamma^m\}$ be the limiting system of curves of class $H^{2,p}$ as in (iii) of Theorem 4.1, and let C be the set of the cusp points of ∂E belonging to (γ^1). For any $r \in C$ let $t_r \in [0,1]$ be such that $\gamma^1(t_r) = r$. Assume for simplicity that $t_r \in]0,1[$. By the definition of cusp point and since γ^1 is parametrized with constant velocity, there exists $\epsilon_r > 0$ such that either $\gamma^1(I_r^-) \cap \partial E = \emptyset$, or $\gamma^1(I_r^+) \cap \partial E = \emptyset$, where $I_r^- :=]t_r - \epsilon_r, t_r[$,

$I_r^+ :=]t_r, t_r+\epsilon_r]$. Let $C^- := \{r \in C : \gamma^1(I_r^-) \cap \partial E = \emptyset\}$, $C^+ := \{r \in C : \gamma^1(I_r^+) \cap \partial E = \emptyset\}$. Note that in particular $\gamma^1(I_r^-) \cap C^- = \emptyset$ for any $r \in C^-$, and $\gamma^1(I_r^+) \cap C^+ = \emptyset$ for any $r \in C^+$. Hence, for any $r, s \in C^+$, $r \neq s$, $I_s^+ \cap I_r^+ = \emptyset$, and the same holds for C^-. It follows the assertion. □

We will see at the end of the next section that there exist sets E such that $\overline{F}(E) < +\infty$ having a countable number of cusp points.

Definition 4.7. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F}(E) < +\infty$, $z_0 \in \partial E$, and let $T(z_0)$ be the tangent line of ∂E at z_0. We say that z_0 is a **branch point** of ∂E if there exists $r > 0$ and there exists a tangent unit vector $\tau(z_0)$ of ∂E at z_0 such that $B_r^+(z_0) \cap \partial E$ is a cartesian graph with respect to $T(z_0)$ and $B_r^-(z_0) \cap \partial E$ is not a cartesian graph with respect to $T(z_0)$.*

An example of branch point is shown in Figure 4.3.

Corollary 4.1. *Let $E \subseteq \mathbf{R}^2$ be a measurable set such that $\overline{F}(E) < +\infty$. Then ∂E can have at most a countable number of branch points.*

Proof: We will follow the notations of Remark 4.4. Let C be the set of all branch points of ∂E belonging to (γ^1). For any $r \in C$ there exist $\varepsilon_r > 0$ such that $\gamma^1(I_r^+)$ or $\gamma^1(I_r^-)$ does not meet any other branch point. The assertion then follows reasoning as in Remark 4.4. □

5. Some critical examples.

In this section we show some pathological examples of measurable sets E whose boundary ∂E is very irregular and, despite of this, $\overline{F}(E) < +\infty$.

Example 1.

There exists a measurable set E such that $\overline{F}(E) < +\infty$ and $\mathcal{H}^1(\partial_{00}E \cup \partial_{11}E) > 0$.

Proof: Let A be an open subset of $[0,1]$ with the following properties:

(i) A is dense in $[0,1]$;

(ii) $A = \bigcup_{k=1}^{+\infty} I_k$, where $I_k =]a_k, b_k[$, I_k pairwise disjoint;

(iii) $\mathcal{H}^1([0,1] \setminus A) > 0$.

For any k, let $x_k := \frac{b_k - a_k}{2}$, and let $\phi : \mathbf{R} \to [0,1]$ be a function of class C^∞, such that $\phi(x) = \phi(-x)$, $\phi(x) = 0$ if $|x| \geq 1$, $\phi(0) = 1$. For any $x \in \mathbf{R}$, let $\phi_k(x) := \phi(\frac{2}{b_k - a_k}(x - x_k))$. Then there exists a sequence $\{\eta_k\}_k$ of real numbers such that the function

$$\Phi(x) := \sum_{k=1}^{\infty} \eta_k \phi_k(x)$$

is of class $C^\infty(\mathbf{R})$. In fact, for any l, denote by $\|\cdot\|_{C^l}$ the C^l norm, and suppose that the sequence of sequences $\{\eta_k^{(1)}\}_k, \ldots, \{\eta_k^{(l)}\}_k, \ldots$ has been constructed in such a way that

$\sum_{k=1}^{\infty} \eta_k^{(l)} \parallel \phi_k(z) \parallel_{C^l} < +\infty$. Then, if we define

$$\eta_k := \min(\eta_k^{(1)}, \eta_k^{(2)}, ..., \eta_k^{(k)}),$$

we get the convergence in any norm $\| \cdot \|_{C^l}$ of the series defining Φ. Finally, we define (see Figure 5.1

$$E := \{(x, y) : x \in A, -\Phi(x) < y < \Phi(x)\}.$$

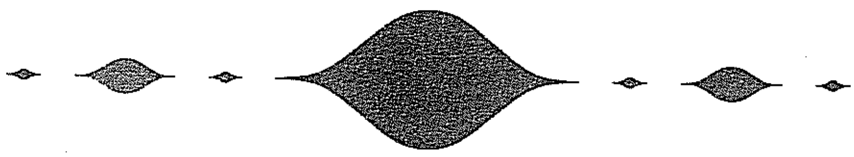

FIGURE 5.1.

By construction, $\partial_{00}E = ([0,1] \setminus A) \times \{0\}$, $\partial_{11}E = \emptyset$; hence, by (iii) we get $\mathcal{H}^1(\partial_{00}E \cup \partial_{11}E) = \mathcal{H}^1([0,1] \setminus A) > 0$. Moreover, as $\partial_{01}E = \partial E \cap (A \times \mathbf{R})$, by (i) we obtain $\overline{\partial_{01}E} = \partial E$.

To check that $\overline{F}(E) < +\infty$, we must approximate in the $L^1(\mathbf{R}^2)$ norm the set E with a sequence $\{E_h\}_h$ of bounded open sets of class C^2 satisfying (2.5). It is easy to see that it is possible to find $\{E_h\}_h$ such that $E_h \to E$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$, and

$$\partial E_h := \{(x, \Phi(x) + \frac{1}{h}) : x \in]0,1[\} \bigcup \{(x, -\Phi(x) - \frac{1}{h}) : x \in]0,1[\} \cup (C_h^1) \cup (C_h^2),$$

where (C_h^1) and (C_h^2) are the traces of two simple regular curves C_h^1, C_h^2 of class C^∞ joining, respectively, the points $(0, -\frac{1}{h}), (1, -\frac{1}{h})$ and the points $(0, \frac{1}{h}), (1, \frac{1}{h})$, and with the following properties:

(i) $(C_h^1) \cap (C_h^2) = \emptyset$;

(ii) $\sup_h [l(C_h^1) + \int |\kappa(C_h^1)|^p \, ds < +\infty$, $\sup_h [l(C_h^2) + \int |\kappa(C_h^2)|^p \, ds < +\infty$;

(iii) $\sup_{x \in C_h^1, y \in C_h^2} |\mathrm{dist}(x, y)| \to 0$ as $h \to +\infty$.

Then $\{E_h\}_h$ satisfies the required properties (see Figure 5.2). □

Example 2.

There exists a measurable set E such that $\overline{F}(E) < +\infty$ and ∂E has a countable number of cusp points.

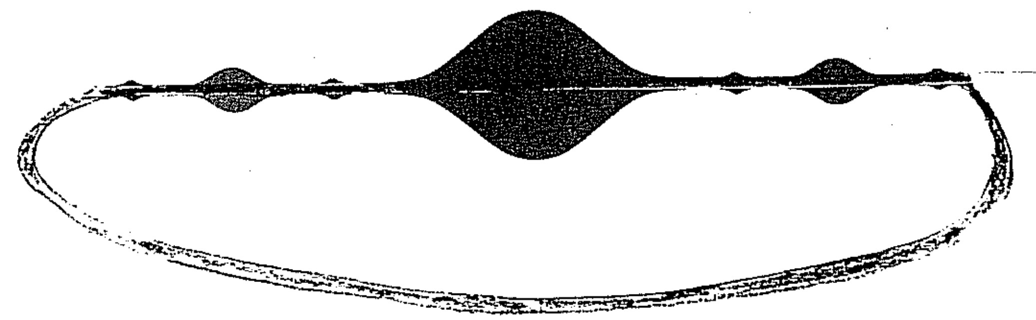

FIGURE 5.2.

PROOF: Consider the family of intervals $I_2 :=]\frac{1}{4}, \frac{1}{2}[, I_3 :=]\frac{7}{12}, \frac{2}{3}[, I_k :=]1 - \frac{2k-1}{2k(k-1)}, 1 - \frac{1}{k}[:=]a_k, b_k[$, $k \geq 2$, $k \in \mathbf{N}$. Using the same notations of Example 1, we can construct the functions ϕ_k, Φ and we can define E as

$$E := \{(x, y) : x \in \cup_{k=1}^{\infty} I_k, -\Phi(x) < y < \Phi(x)\}.$$

Then E verifies the required properties. □

6. Systems of curves that can be approximated by boundaries of sets. Let $E \subset \mathbf{R}^2$ be a bounded open set. The main problem of this section is to understand which kind of conditions must satisfy ∂E in order to have that $\overline{F}(E) < +\infty$. To do that, we need the following definition.

DEFINITION 6.1. *Let S be the disjoint union of m unit circles $\mathbf{S}_1^1, \ldots, \mathbf{S}_m^1$, and let $\Gamma : S \to \mathbf{R}^2$ be a system of curves. We say that Γ satisfies the finiteness property if there exists a finite number of points $t_1, \ldots, t_M$ in S such that the unique finite partition $\mathcal{P}$ of $S \setminus \{t_1, \ldots, t_M\}$ composed of connected open sets satisfies the following properties:*

(i) any element of $\mathcal{P}$ can not be a unit circle;

(ii) for any $I, H \in \mathcal{P}$ either $\Gamma(I) \cap \Gamma(H) = \emptyset$ or $\Gamma(I) \equiv \Gamma(H)$;

(ii) $\Gamma_{|I} : I \to \Gamma(I)$ is an omeomorfism for any $I \in \mathcal{P}$.

Condition (i) in Definition 6.1 is uninfluent in order to prove the results of this section. Clearly, if Γ is a system of curves satisfying (ii) and (iii), condition (i) is fulfilled by suitably inserting a finite number of new points t_j in S.

DEFINITION 6.2. *Let Γ be a system of curves satisfying the finiteness property. For every $I \in \mathcal{P}$ the set $\Gamma(I)$ will be called a branch of Γ, and for every $i = 1, \ldots, M$ the point $\Gamma(t_i)$ will be called a node of Γ. The set of all nodes will be denoted by $\mathcal{N}(\Gamma)$.*

DEFINITION 6.3. *Let Γ be a system of curves satisfying the finiteness property, and let J be an open and connected subset of $S \setminus \{t_1, \ldots, t_M\}$. The set $\Gamma(J)$ will be called and arc of Γ.*

Let J be an arc of Γ, let $z \in \Gamma(J)$. Then the cardinality of the set $\Gamma^{-1}(z)$ depends only on J and does not depend on z. This cardinality will be called the multiplicity of the arc $\Gamma(J)$. In particular, since $\Gamma(J)$ is contained in a branch $\Gamma(I)$, the multiplicity of $\Gamma(J)$ is the multiplicity of $\Gamma(I)$.

Unless specified, throughout this section the symbol Γ will denote a system of curves without crossings (see Definition 4.2) and satisfying the finiteness property.

The problem will be to approximate Γ in the $H^{2,p}$ norm by a sequence whose elements are *boundaries* of smooth bounded open sets ([16], [18, 8.9.4]).

We begin with the following Lemma.

LEMMA 6.1. *Let $\Gamma : S \to \mathbf{R}^2$ be a system of curves of class $H^{2,p}$ without crossings and satisfying the finiteness property, $\{t_1, \ldots, t_M\} = \Gamma^{-1}(\mathcal{N}(\Gamma))$. There exists a sequence $\{\Gamma_\varepsilon\}_{\varepsilon>0}$ of system of curves of class $H^{2,p}$ without crossings, satisfying the finiteness property, and there exists $\varepsilon_0 > 0$ such that*

(i) $\Gamma_\varepsilon : S \to \mathbf{R}^2$ *for any* $0 < \varepsilon < \varepsilon_0$;

(ii) $\Gamma_\varepsilon^{-1}(\mathcal{N}(\Gamma_\varepsilon)) = \{t_1^\varepsilon, \ldots, t_M^\varepsilon\}$, *and* $t_i^\varepsilon \to t_i$ *as* $\varepsilon \to 0$, *for any* $i = 1, \ldots, M$;

(iii) $\Gamma_\varepsilon \to \Gamma$ *strongly in* $H^{2,p}(S)$ *as* $\varepsilon \to 0$;

(iv) $\Gamma_\varepsilon(t_i^\varepsilon) = \Gamma(t_i)$ *for any* $i = 1, \ldots, M$, *for any* $0 < \varepsilon < \varepsilon_0$;

(v) $\Gamma_{\varepsilon|S\setminus\{t_1^\varepsilon,\ldots,t_M^\varepsilon\}} : S \setminus \{t_1^\varepsilon, \ldots, t_M^\varepsilon\} \to \Gamma_\varepsilon(S \setminus \{t_1^\varepsilon, \ldots, t_M^\varepsilon\})$ *is an omeomorfism, for any* $0 < \varepsilon < \varepsilon_0$.

The meaning of Lemma 6.1 is the following. The system of curves Γ can be approximated in the $H^{2,p}$ norm by a sequence of systems of curves $\{\Gamma_\varepsilon\}_\varepsilon$ of class $H^{2,p}$ satisfying the following properties:

(1) the systems Γ_ε are defined on the same parameter space S;

(2) in the approximation the nodes of Γ are kept fixed, i.e., $\mathcal{N}(\Gamma) = \mathcal{N}(\Gamma_\varepsilon)$ (condition (iv));

(3) all the branches of Γ_ε have multiplicity 1 (condition (v)). Hence if A is a branch of Γ with multiplicity n, it is approximated by n different branches of Γ_ε with multiplicity 1.

Condition (ii) says essentially that $\Gamma_\varepsilon^{-1}(\mathcal{N}(\Gamma_\varepsilon))$ is a perturbation of $\Gamma^{-1}(\mathcal{N}(\Gamma))$, and it is due to the fact that, if we want that the approximating curves have constant velocity, we need to make a reparametrization of Γ_ε.

Moreover, as we will see in the proof, it follows that the approximating branches of different branches of Γ are non overlapping.

PROOF OF LEMMA 6.1: We will construct the approximation for a couple (q_1, q_2) of consecutive nodes of Γ, and for one of the branches joining q_1 and q_2. Then it will be sufficient to repeat this procedure for all the couples of consecutive nodes and for all the branches of Γ.

Let $q_1, q_2 \in \mathcal{N}(\Gamma)$ be two consecutive nodes, let A be one branch joining q_1 and q_2, and suppose that the multiplicity of A is n. Then $\Gamma^{-1}(A) = \cup_{k=1}^{n} I_k$, where $I_k \in \mathcal{P}$ for any $k = 1, \ldots, n$. Let $\nu : A \to \mathbf{R}^2$ be a continuous unit normal vector field on A, and let us extend ν to a continuous vector field defined on $\mathbf{R}^2$ such that $\frac{1}{2} \le |\nu(z)| \le 1$ for any $z \in \mathbf{R}^2$. To overcome problems on the regularity of the approximating curves, we regularize ν as follows. We take a function $\widetilde{\nu} \in \mathcal{C}^\infty(\mathbf{R}^2, \mathbf{R}^2)$ such that $|\widetilde{\nu}| \le 1$ and $\widetilde{\nu}(z) \cdot \nu(z) \ge \frac{1}{2}$ for any $z \in \mathbf{R}^2$. Such a function exists: in fact, defining $\nu_\eta := \nu \star \varrho_\eta$ where $\{\varrho_\eta\}_\eta$ is a sequence of mollifiers, we get $\nu_\eta \in \mathcal{C}^\infty(\mathbf{R}^2, \mathbf{R}^2)$ and $\nu_\eta \cdot \nu \to 1$ uniformly on the compact sets of $\mathbf{R}^2$ as $\eta \to 0$. It is enough then to define $\widetilde{\nu} := \nu_\eta$ for η small enough.

Let $d : \overline{A} \to [0, +\infty[$ be the function defined as

$$d(z) := \min\{\text{dist}(z, B) : B \text{ is a branch of } \Gamma \text{ joining } q_1 \text{ and } q_2, B \neq A\}.$$

From the definition it follows that d is continuous, $d(z) \ge 0$ for any $z \in A$, and $d(z) = 0$ if and only if $z = q_1$ or $z = q_2$. To be sure that the approximating sequence has the same regularity of the original system of curves Γ, and to guarantee that approximations of different branches do not overlap, we introduce a function $h : \overline{A} \to [0, +\infty[$ of class $\mathcal{C}^\infty$ having the following properties: $h(q_1) = h(q_2) = 0$, $0 < h(z) < d(z)$ for any $z \in A$, and all the derivatives of h at the points q_1, q_2 vanish. We define the n approximating branches $A_\varepsilon^1, \ldots, A_\varepsilon^n$ of A as follows: for any $i = 1, \ldots, n$

$$A_\varepsilon^i := \Gamma_\varepsilon(I_i), \qquad \text{where } \Gamma_\varepsilon(t) := \Gamma(t) + \varepsilon(\frac{2i}{n} - 1) h(\Gamma(t)) \widetilde{\nu}(\Gamma(t)), \quad t \in I_i,$$

and ε is sufficiently small. Then we repeat this procedure for all couples of consecutive nodes of Γ and for all branches. Finally, if $\Gamma_\varepsilon = \{\gamma_\varepsilon^1, \ldots, \gamma_\varepsilon^k\}$, we reparametrize all the closed curves γ_ε^i in such a way that their velocity is constant. This concludes the proof. □

Let $q \in \mathcal{N}(\Gamma)$, let $\tau(q)$ be a tangent unit vector of (Γ) at q and $r > 0$. The normal line $N(q)$ of (Γ) at q divides the ball $B_r(q)$ in two half balls,

$$B_r^-(q) := \{z \in B_r(q) : (z - q) \cdot \tau(q) < 0\}, \qquad B_r^+(q) := \{z \in B_r(q) : (z - q) \cdot \tau(q) > 0\}.$$

The set $B_r^-(q)$ (resp. $B_r^+(q)$) will be called the left (resp. the right) half ball of centre q and radius r. If $\Gamma^{-1}(q) = \{s_1, \ldots, s_l\}$, using the implicit function theorem, one can prove that, when r is sufficiently small, the set $B_r^-(q) \cap \Gamma(S)$ is composed by a finite number $n \le l$ of arcs, that we will call the local left arcs of Γ at q. Moreover, $\Gamma^{-1}(B_r^-(q) \cap \Gamma(S))$ is composed by a finite number $J_1, \ldots, J_l$ of intervals (in S), and each interval J_i has one of its end points belonging to $\{s_1, \ldots, s_l\}$. These intervals will be called local left intervals at q. Analogously, by considering the set $B_r^+(q) \cap \Gamma(S)$, we define the local right arcs of Γ at q and the local right intervals at q.

Let $q \in \mathcal{N}(\Gamma)$, let $\nu(q)$ be a unit normal vector of (Γ) at q, and let $A_1, \ldots, A_n$ be the local left arcs of Γ at q. We want to give an ordering to the set $\{A_1, \ldots, A_n\}$.

For any $i = 1, \ldots, n$ let $z_i := (\partial B_r^-(q) \cap \partial B_r(q)) \cap A_i$. Suppose for simplicity that $q = 0$ and that $\nu(q)$ coincides with the second vector of the canonical basis of $\mathbf{R}^2$. Then we say that $z_i \prec z_j$ if and only if $z_{i_y} \prec z_{j_y}$. This correspond to the ordering in which the local left arcs meet $\partial B_r^-(q) \cap \partial B_r(q)$, once that the set $\partial B_r^-(q) \cap \partial B_r(q)$ has been oriented as $\nu(q)$. Then we will say that

$$A_i \prec A_j \quad \text{if and only if} \quad z_i \prec z_j.$$

Now, let us suppose that, for any $i = 1, \ldots, n$ the multiplicity of A_i is 1. In this case, the cardinality of $\Gamma^{-1}(q) = \{s_1, \ldots, s_l\}$ equals the cardinality of the local left arcs, that is $l = n$. With this assumptions, the previuos ordering on the local left arcs induces an ordering on $\{s_1, \ldots, s_l\}$ simply defined by

$$s_i \prec s_j \quad \text{if and only if} \quad A_i \prec A_j.$$

DEFINITION 6.4. *This ordering on the set $\Gamma^{-1}(q) = \{s_1, \ldots, s_l\}$ will be called a left ordering on the node q.*

Analogously, using the ordering in which the local right arcs of Γ at q meet $\partial B_r^+(q) \cap \partial B_r(q)$, once that the set $\partial B_r^+(q) \cap \partial B_r(q)$ has been oriented as $\nu(q)$, and supposing that any local right arc at q has multiplicity 1, we get an other ordering on the set $\{s_1, \ldots, s_l\}$, that we will call a right ordering on q. Note that all these orderings depend only on the choice of $\nu(q)$.

DEFINITION 6.5. *Let Γ be a system of curves without crossings and satisfying the finiteness property. Let $\mathcal{G}_\Gamma$ be the undirected graph whose vertices are the nodes of Γ, and whose edges are the branches of Γ. Moreover, let us associate to any edge of $\mathcal{G}_\Gamma$ the positive integer given by the multiplicity of the corresponding branch of Γ. The graph $\mathcal{G}_\Gamma$ will be called the graph with multiplicity associated to Γ. The set of all vertices will be denoted by $\mathcal{V}(\mathcal{G}_\Gamma)$.*

Obviously the ordering of the local left (resp. right) arcs of Γ at a node q is an ordering of the set $\mathcal{G}_\Gamma \cap B_r^-(q)$ (resp. $\mathcal{G}_\Gamma \cap B_r^+(q)$) of the local left (resp. right) edges. If any branch of Γ has multiplicity 1 it follows that any vertex of $\mathcal{G}_\Gamma$ inherits the left and the right ordering of the corresponding node.

If q is a vertex of $\mathcal{G}_\Gamma$, the number of edges counted with their multiplicity having a given endpoint q we shall designate by

$$\varrho(q).$$

This number is called the local degree at q. In this definition, any loop at q will be counted as a double edge.

Analogously, we can define the number of local left (resp. right) edges $\varrho^-(q)$ (resp. $\varrho^+(q)$) counted with their multiplicity, that we will call the local left (resp. right) degree at q. Of course, here no loops are present.

Note that since $\mathcal{G}_\Gamma$ is the graph associated to a system of curves, the following property holds:

$$(6.1) \qquad \varrho^-(q) = \varrho^+(q) \qquad \text{for any } q \in \mathcal{V}(\mathcal{G}_\Gamma).$$

The graph $\mathcal{G}_\Gamma$ will be called *regular*.

DEFINITION 6.6. *Suppose that any branch of* Γ *has multiplicity* 1. *We say that* Γ *verifies the compatibility condition if, for any* $q \in \mathcal{N}(\Gamma)$, *the left ordering on* q *coincides with the right ordering on* q.

Note that the compatibility condition does not depend on the choice of the unit normal vector $\nu(q)$.

We now prove the following crucial result.

REMARK 6.1. *Let* $\Gamma : S \to \mathbf{R}^2$ *be a system of curves without crossings and satisfying the finiteness property, and suppose that any branch of* Γ *has multiplicity* 1. *Then there exists a system of curves* $\widetilde{\Gamma} : \widetilde{S} \to \mathbf{R}^2$ *satisfying the following properties:*

(i) $\widetilde{\Gamma}$ *has the same nodes and the same branches of* Γ, *i.e.,* $\mathcal{G}_{\widetilde{\Gamma}} = \mathcal{G}_\Gamma$;

(ii) $\widetilde{\Gamma}$ *verifies the compatibility condition.*

PROOF: For any $q \in \mathcal{N}(\Gamma)$ let us fix a unit normal vector $\nu(q)$ of (Γ) at q. This implies that the left and the right orderings on any node of Γ are assigned, as well as the left and the right orderings of any vertex of $\mathcal{G}_\Gamma$. Without loss of generality, we can suppose that $\mathcal{G}_\Gamma$ is connected, otherwise we repeat the reasoning for any connected component. To prove the assertion, one has to show that there exists a closed path on $\mathcal{G}_\Gamma$ whose trace is $\mathcal{G}_\Gamma$ and having the following property: *if* q *is a vertex of* $\mathcal{G}_\Gamma$ *and if* A *is a local left arc on* q *whose ordering number is* α, *then* A *passes through* q *and continues on* $\mathcal{G}_\Gamma$ *along the local right arc having the same ordering number* α. This closed path can be obviously constructed by "annihilating" the couples of the corresponding left and right ordering numbers on any vertex.

This strategy corresponds to a surgery operation on the parameter space S of Γ, that we shall briefly describe. Let us write the partition $\mathcal{P}$ of $S \setminus \{t_1, \ldots, t_M\}$ as $\cup_{i=1}^d I_i$, where $I_i = (a_i, b_i)$ (recall that any unit circle is oriented clockwise). Let us consider I_1; suppose that b_1 comes from the left (with respect to $\nu(q)$) on a node q_1. Then, since the pre-images of q_1 have a (left) ordering, the parameter b_1 has a (left) ordering number $n_l(b_1)$. Next, let us consider the right branch, parametrized by (a_j, b_j), such that a_j (or b_j) has $n_l(b_1)$ as right ordering number on q. Suppose that, for instance, a_j verifies this property. In this way we have created a couple (b_1, a_j). We repeat this reasoning for every I_i, obtaining a set of couples containing all the pre-images of the nodes of Γ.

Consider (b_1, a_j): if $b_1 = a_j$ the algorithm stops. If $a_1 \neq b_j$ we substitute to (a_1, b_1) and (a_j, b_j) a unique parametrization, by gluing b_1 with a_j. There are two possibilities: either (a_1, b_1) is different from (a_j, b_j), or we have glued together two end points of the same interval. In the first case we have decreased the number of the intervals composing the disjoint union $\mathcal{P}$. Otherwise we create a unit circle, and this circle remains unchanged. In any case, reasoning by induction, we end up the algorithm by getting, as new parameter space $\widetilde{S}$, a finite number of unit circles. □

We remark that the parameter spaces S and $\widetilde{S}$ could be different. We stress also that

$$l(\Gamma) = l(\widetilde{\Gamma}), \quad \text{and} \quad \| \kappa(\Gamma) \|_{L^p} = \| \kappa(\widetilde{\Gamma}) \|_{L^p} .$$

DEFINITION 6.7. *The system of curves Γ and $\widetilde{\Gamma}$ of Remark 6.1 will be called two equivalent systems of curves.*

Let $\{\Gamma_\varepsilon\}_\varepsilon$ be the sequence of Lemma 6.1, $\Gamma_\varepsilon : S \to \mathbf{R}^2$. For any $\varepsilon > 0$, let $\widetilde{\Gamma_\varepsilon} : \widetilde{S_\varepsilon} \to \mathbf{R}^2$ be a system of curves equivalent to Γ_ε. Then, by construction, if ε is sufficiently small, $\widetilde{S_\varepsilon}$ does not depend on ε, and it will be denoted by $\widetilde{S}$.

LEMMA 6.2. *Let $\{\Gamma_\varepsilon\}_\varepsilon$ be the sequence of Lemma 6.1, and for any $\varepsilon > 0$ let $\widetilde{\Gamma_\varepsilon} : \widetilde{S} \to \mathbf{R}^2$ be a system of curves equivalent to Γ_ε. Then, for any ε sufficiently small, there exists a sequence $\{\Gamma_\varepsilon^\eta\}_{\eta>0}$ of systems of curves of class C^∞ and there exists $\eta_0 > 0$ such that*

(i) $\Gamma_\varepsilon^\eta : \widetilde{S} \to \mathbf{R}^2$ is a disjoint system of curves (i.e., $\mathcal{N}(\Gamma_\varepsilon^\eta) = \emptyset$), for any $0 < \eta < \eta_0$;

(ii) $\Gamma_\varepsilon^\eta \to \widetilde{\Gamma_\varepsilon}$ strongly in $H^{2,p}(\widetilde{S})$ as $\eta \to 0$.

PROOF: We shall suppose, for simplicity, that Γ_ε is not parametrized with constant velocity, in such a way that $\mathcal{N}(\Gamma_\varepsilon) = \{t_1, \ldots, t_M\}$ is independent of ε. A suitable reparametrization will be made at the end of the proof. Moreover, the construction is shown for a fixed ε sufficiently small, for a fixed node of $\widetilde{\Gamma_\varepsilon}$, and it must be repeated for any element of $\mathcal{N}(\widetilde{\Gamma_\varepsilon})$.

Let $q \in \mathcal{N}(\widetilde{\Gamma_\varepsilon})$, $\widetilde{\Gamma_\varepsilon}^{-1}(q) = \{s_1, \ldots, s_l\}$, and let $R(q)$ be a small rectangle centered at q, $R(q) := \{z \in \mathbf{R}^2 : z = q + \alpha\tau(q) + \beta\nu(q) : |\alpha| \leq \alpha_0, |\beta| \leq \beta_0\}$, in such a way that $R(q) \cap (\widetilde{\Gamma_\varepsilon})$ is composed only by arcs of $\widetilde{\Gamma_\varepsilon}$. Let $A_1, \ldots, A_n$ be the local left arcs of $\widetilde{\Gamma_\varepsilon}$ belonging to $R(q) \cap (\widetilde{\Gamma_\varepsilon})$, and let $J_1, \ldots, J_n$ be the corresponding local left intervals. Let $\pi R(q) \to T(q) \cap R(q)$ be the canonical projection of $R(q)$ on the tangent line $T(q)$ of Γ at q. Then we define the local left approximation of $\widetilde{\Gamma_\varepsilon}$ by

$$\Gamma_\varepsilon^\eta(t) := \widetilde{\Gamma_\varepsilon}(t) + \eta\nu(q)(\frac{2k}{n} - 1)\sigma(\pi(\widetilde{\Gamma_\varepsilon}(t))), \quad t \in A_k, \ k = 1, \ldots, n, \ 0 < \eta < \eta_0,$$

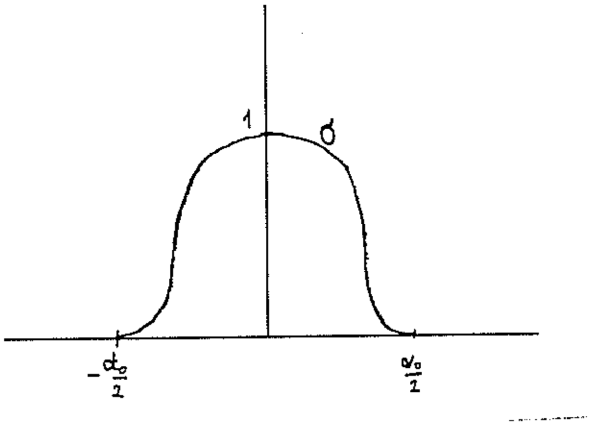

FIGURE 6.1.

where η_0 is a positive number sufficiently small, and σ is a suitable y-symmetric function of the form pictured in figure 6.1.

Making the same construction for the local right arcs of $\widetilde{\Gamma_\varepsilon}$ in $R(q)$, and using the fact that $\widetilde{\Gamma_\varepsilon}$ verifies the compatibility condition, we glue together in an ordered way the local left arcs with the local right arcs, without creating local self-intersections. We repeat this argument for any node, and we get a sequence of disjoint family of curves of class $H^{2,p}$ converging to $\widetilde{\Gamma_\varepsilon}$ strongly in $H^{2,p}$. Then, we regularize by convolution, and we obtain an approximating family of curves of class C^∞. Finally, with a reparametrization, we get a sequence of disjoint systems of curves verifying the required properties. □

The notion of left and right ordering of a node can be given also when Γ is a system of curves whose branches have multiplicity larger than 1, as well as the definitions of compatibility condition and of equivalence. Since this approach seems to be quite complicated, we prefer to give the following definition of equivalence between systems of curves whose branches have multiplicity larger that 1.

DEFINITION 6.8. *The (strong) $H^{2,p}(\widetilde{S})$ limit of the sequence $\{\widetilde{\Gamma_\varepsilon}\}_\varepsilon$ is a system of curves $\widetilde{\Gamma}$ defined on $\widetilde{S}$, and it will be called a system of curves equivalent to Γ.*

Note that

$$\mathcal{G}_{\widetilde{\Gamma}} = \mathcal{G}_\Gamma, \quad l(\widetilde{\Gamma}) = l(\Gamma), \quad \| \kappa(\widetilde{\Gamma}) \|_{L^p} = \| \kappa(\Gamma) \|_{L^p} .$$

THEOREM 6.1. *Let $\Gamma : S \to \mathbf{R}^2$ be a system of curves of class $H^{2,p}$ without crossings and satisfying the finiteness property, and define*

$$E := \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \ (mod.\ 2)\}.$$

Then

$$\partial E \subseteq (\Gamma).$$

Moreover there exists a sequence $\{E_h\}_h$ of bounded open sets of class C^∞ such that *(i)* $E_h \to E$ *in* $L^1(\mathbf{R}^2)$ *as* $h \to +\infty$*;*

(ii) $\sup_h F(E_h) < +\infty$.

Hence

$$\overline{F}(E) < +\infty.$$

In addition, there exists a system of curves $\widetilde{\Gamma} : \widetilde{S} \to \mathbf{R}^2$ *equivalent to* Γ *such that, if* Γ_h *is an oriented parametrization of* ∂E_h *for any* h*, then*

$$\Gamma_h \to \widetilde{\Gamma} \qquad \textit{strongly in } H^{2,p}(\widetilde{S}) \quad \textit{as } h \to +\infty.$$

Finally,

$$\overline{F}(E) \leq \inf\{l(\Gamma) + \| \kappa(\Gamma) \|_{L^p}^p : \Gamma \in \mathcal{B}(E)\}, \tag{6.2}$$

where $\mathcal{B}(E)$ *is the collection of those systems of curves* Γ *of class* $H^{2,p}$ *without crossings, satisfying the finiteness property and such that* $(\Gamma) \supseteq \partial E$.

Proof: By the definition, E is open. Let us prove that $\partial E \subseteq (\Gamma)$. Let $z \notin (\Gamma)$; if $I(\Gamma, z) \equiv 1 \pmod{2}$, then $z \in E$, hence $z \notin \partial E$. If $I(\Gamma, z) \equiv 0 \pmod{2}$, there exists $r > 0$ such that $I(\Gamma, z') \equiv 0 \pmod{2}$ for any $z' \in B_r(z)$. Hence $B_r(z) \cap E = \emptyset$, so that $z \notin \partial E$.

Using a diagonal argument, let us call $\{\Gamma_h\}_h$ the sequence of disjoint systems of curves of Lemma 6.2, $\Gamma_h : \widetilde{S} \to \mathbf{R}^2$. Then, using Definition 6.8 we get that there exists a system of curves $\widetilde{\Gamma} : \widetilde{S} \to \mathbf{R}^2$ equivalent to Γ such that $\Gamma_h \to \widetilde{\Gamma}$ strongly in $H^{2,p}(\widetilde{S})$. For any h let us define

$$E_h := \{z \in \mathbf{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) \equiv 1 \pmod{2}\}.$$

Note that this definition is independent of the orientation of any $\mathbf{S}^1_i$, that $\widetilde{S}$ can be reoriented in such a way that

$$E_h = \{z \in \mathbf{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) = 1\},$$

and that, since Γ_h is a disjoint system, $\partial E_h = \Gamma_h$. From the fact that

$$\chi_{\{z \in \mathbf{R}^2 \setminus (\Gamma_h) : I(\Gamma_h, z) \equiv 1 \pmod{2}\}} \to \chi_{\{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod{2}\}}$$

for every $z \notin (\Gamma)$ as $h \to +\infty$, using the dominated convergence theorem, we get $E_h \to \{z \in \mathbf{R}^2 \setminus (\Gamma) : I(\Gamma, z) \equiv 1 \pmod{2}\}$ in $L^1(\mathbf{R}^2)$ as $h \to +\infty$. Since by construction $\sup_h F(E_h) < +\infty$, it follows $\overline{F}(E) < +\infty$. Moreover

$$l(\Gamma) + \| \kappa(\Gamma) \|_{L^p}^p = \lim_{h \to +\infty} \int_{\partial E_h} [1 + |\kappa_h(z)|^p] \, d\mathcal{H}^1(z) \geq \overline{F}(E).$$

Passing to the infimum with respect to $\Gamma \in \mathcal{B}(E)$, we get (6.2). □

Definition 6.9. *Let E be a subset of $\mathbf{R}^2$ with the following properties:*

(i) E is bounded and open;

(ii) ∂E has an unoriented continuous tangent;

(iii) E is bounded by a finite number of closed curves of class $H^{2,p}$ up to the closure;

(iv) $E = \{z \in \mathbf{R}^2 : \exists r > 0 \ |B_r(z) \setminus E| = 0\}$.

We say that a point $q \in \partial E$ is an exceptional point if there exists $r > 0$ and there exists a tangent unit vector $\tau(q)$ of ∂E at q such that $B_r^+(q) \cap \partial E$ is either a cartesian graph with respect to $T(q)$ or is empty and $B_r^-(q) \cap \partial E$ is not a cartesian graph.

This definition generalizes Definitions 4.6 of cusp point and 4.7 of branch point. Suppose that ∂E has a finite number of exceptional points. Let $\mathcal{G}_{\partial E}$ be the undirected graph whose vertices are the exceptional points, whose edges are the curves surrounding ∂E, and having all edges with multiplicity 1. The multiplicity $m(q)$ of an exceptional point q is defined as follows. Suppose that $B_r^-(q) \cap \partial E$ is not a cartesian graph. Then we define

$$m(q) := \begin{cases} \frac{\varrho^-(q)}{2} & \text{if } \varrho^-(q) \text{ is even,} \\ \frac{\varrho^-(q)-1}{2} & \text{if } \varrho^-(q) \text{ is odd.} \end{cases}$$

Theorem 6.2. *Let E be a subset of $\mathbf{R}^2$ verifying the assumptions of Definition 6.9, and suppose that ∂E has a finite number $\{q_1, \dots, q_n\}$ of exceptional points. Then*

$$\sum_{i=1}^{n} m(q_i) \quad \text{even} \Rightarrow \overline{F}(E) < +\infty.$$

Proof: We shall describe an algorithm that permits to join in a suitable way the couples of vertices, possibly by inserting a finite number of new vertices. Suppose for simplicity that $B_r^-(q_i) \cap \partial E$ is not a cartesian graph for any $i = 1, \dots, n$.

S1. Let us consider q_1.

S11. if $\varrho^-(q_1) = 2$ we join q_1 and q_2 with a smooth curve γ such that

(i) (γ) does not intersect the set $\{q_3, \dots, q_n\}$;

(ii) if (γ) meets ∂E at a point z, we impose that the tangent line of (γ) at z coincides with the tangent line of ∂E at z. This point z will be considered as a new vertex. We associate to (γ) the multiplicity 2.

S12. If $\varrho^-(q_1)$ is even > 2, we attack to q_1 a smooth closed curve (a loop) such that

(i) (γ) does not intersect the set $\{q_2, \dots, q_n\}$;

(ii) as in (ii) of S11.

We associate to (γ) the multiplicity $\frac{\varrho^-(q)}{2}$.

S13. If $\varrho^-(q)$ is odd (obviously $\varrho^-(q) \geq 3$) we reason as in S12 with the difference that the multiplicity associated to (γ) is $\frac{\varrho^-(q)-1}{2}$.

Then reason as in S11.

Note that after these steps we get that $\varrho^-(q) = \varrho^+(q)$ and that, if $z \in \partial E \cap (\gamma)$, then $\varrho^-(z) = \varrho^+(z)$.

S2. Define $\mathcal{G}_{\partial E} := \mathcal{G}_{\partial E} \cup (\gamma)$. This means that $\mathcal{V}(\mathcal{G}_{\partial E}) = \mathcal{V}(\mathcal{G}_{\partial E}) \cup \{z \in \mathbf{R}^2 : z \in (\gamma) \cap \partial E\}$ and that all the edges of $\mathcal{G}_{\partial E}$ have multiplicity 1 except for (γ) that has the multiplicity previously assigned.

S3. $q_1 := q_2$, $q_2 := q_3$, $\ldots$, $q_{n-1} := q_n$, $q_n = \emptyset$. GO TO S1.

In view of the hypothesis, the algorithm stops after a finite number of steps. Then from Theorem 6.1 it follows that $\overline{F}(E) < +\infty$. □

COROLLARY 6.1. *Let E be a subset of $\mathbf{R}^2$ verifying the assumptions of Definition 6.9, and suppose that ∂E is smooth except for a finite number n of cusp points. Then*

$$n \ \text{ even} \Rightarrow \overline{F}(E) < +\infty.$$

Conversely, we can prove the following result.

THEOREM 6.3. *Let E be a subset of $\mathbf{R}^2$ verifying the assumptions of Definition 6.9, and suppose that ∂E is smooth except for a finite number n of cusp points. Then*

$$\overline{F}(E) < +\infty \Rightarrow n \ \text{ even}.$$

PROOF: From Theorem 4.1 (iii) there exists a limiting system of curves Λ of class $H^{2,p}$ such that $(\Lambda) \supseteq \partial E$. We divide the proof in two cases.

FIRST CASE:

suppose that Λ has a finite number of points where, locally, is not a cartesian graph, i.e., let us suppose that Λ satisfies the finiteness property, and let us denote Λ with the letter Γ. Let us associate to the regular graph $\mathcal{G}_\Gamma$ a new graph $\widetilde{\mathcal{G}_\Gamma}$ as follows. Let A denote an edge of $\mathcal{G}_\Gamma$, $\omega(A)$ its multiplicity and $\widetilde{\omega}(A)$ its multiplicity in $\widetilde{\mathcal{G}_\Gamma}$. Then

$$\text{If } \omega(A) = m,\ m \text{ odd, then } \widetilde{\omega}(A) := \frac{m-1}{2} \ (\text{hence, if } \omega(A) = 1, A \text{ is deleted});$$
$$\text{If } \omega(A) = m,\ m \text{ even, then } \widetilde{\omega}(A) := \frac{m}{2}.$$

Note that $\widetilde{\mathcal{G}_\Gamma}$ is not necessarily regular, and that $\mathcal{V}(\widetilde{\mathcal{G}_\Gamma}) = \mathcal{V}(\mathcal{G}_\Gamma)$.

For any $q \in \mathcal{V}(\mathcal{G}_\Gamma)$ let us denote by $\varrho(q)$, $\varrho^-(q)$, $\varrho^+(q)$ (resp. $\widetilde{\varrho}(q)$, $\widetilde{\varrho}^-(q)$, $\widetilde{\varrho}^+(q)$) the local degree and the local left and right degrees of q in $\mathcal{G}_\Gamma$ (resp. in $\widetilde{\mathcal{G}_\Gamma}$). Suppose that

$$q \notin \partial E.$$

Then, since all edges of $\mathcal{G}_\Gamma$ meeting at q have even multiplicity, it follows that $\varrho(q) = 4k$, $k \geq 1$. Hence $\widetilde{\varrho}(q)$ is even.

Suppose that

$$q \in \partial E \quad \text{and } q \text{ is regular in a neighbourhood of } q.$$

Then, two of the edges of $\mathcal{G}_\Gamma$ meeting at q have odd multiplicity and lie one on each side of q with respect to $\nu(q)$, and all the other edges have even multiplicity. Hence $\widetilde{\varrho}(q)$ is even.

The last possibility is when

$$q \in \partial E \quad \text{and } q \text{ is a cusp point of } \partial E.$$

Then, by the definition of cusp point, there exists two edges with odd multiplicity lying either on the left or on the right of q. Let us suppose that there are two left edges A, B with odd multiplicity $2k+1$, $2k'+1$ respectively. Let $\varrho^+_{(q)} = 2n$. Then (see Figure 6.2) we have that, since $\varrho^-(q) = \varrho^+(q)$,

$$2n = 2m + 2k + 1 + 2k' + 1,$$

that gives $n = m+k+k'+1$. Moreover $\widetilde{\varrho}(q) = n+m+k+k'$, hence $\widetilde{\varrho}(q) = 2(m+k+k')+1$ is odd. Since [15, Th. 1.2.1] in a finite graph there is an even number of vertices for which the local degree is odd, we have the assertion.

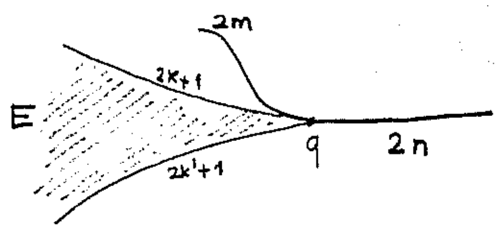

FIGURE 6.2.

SECOND CASE:

Suppose that Λ does not satisfy the finiteness property. Let K be the closure of the set of the accumulation points of C, where

$$C := \{q \in (\Lambda) : \Lambda \text{ is not a cartesian graph in a neighbourhood of } q\}.$$

Then K is a compact set. Let $\Lambda = \{\gamma^1, \ldots, \gamma^{m'''}\}$, let $q \in K$. We will distinguish three possibilities.

First possibility: $q \notin \partial E$.

We can assume $q = 0$, $T(q) = \{(x,y) \in \mathbf{R}^2 : y = 0\}$. Let $R(q) = [-r,r] \times [-s,s]$ be a small closed rectangle centered at q such that $R(q) \cap \partial E = \emptyset$, and such that any γ^i is a cartesian graph in $R(q)$ with respect to $T(q)$. Then $(\Lambda) \cap \{(x,y) \in \mathbf{R}^2 : x = -s, y \in [-r,r]\}$ is a finite number of points, as well as the set $(\Lambda) \cap \{(x,y) \in \mathbf{R}^2 : x = s, y \in [-r,r]\}$. Let $\{z_1, \ldots, z_n\} = (\Lambda) \cap \{(x,y) \in \mathbf{R}^2 : x = -s, y \in [-r,r]\}$, $\{w_1, \ldots, w_l\} = (\Lambda) \cap \{(x,y) \in \mathbf{R}^2 : x = s, y \in [-r,r]\}$ ordered by their y coordinate.

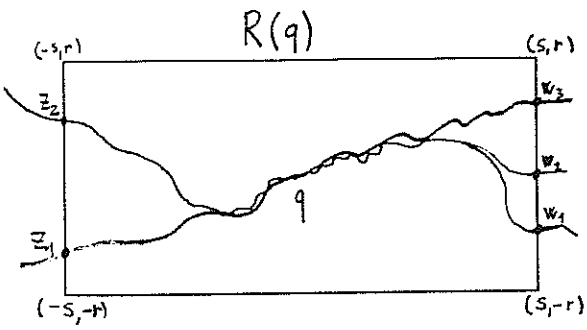

FIGURE 6.3.

Let $2m_{z_i} :=$ cardinality of $\Lambda^{-1}(z_i)$, $i = 1, \ldots, n$, $2m_{w_j} :=$ cardinality of $\Lambda^{-1}(w_j)$, $j = j, \ldots, l$. Let us ridefine Λ in $R(q)$ as follows. We join w_1 and q with a smooth arc α_1 having the following properties:

(i) $(\alpha_1) \subset R(q)$;

(ii) α_1 is a cartesian graph in $R(q)$;

(iii) the first and the second derivative of α_1 at w_1 coincide with the first and the second derivative of γ^1 at w_1, respectively;

(iv) the first and the second derivative of α_1 at q are 0;

(v) (α_1) has multiplicity $2m_{w_1}$.

Then consider w_2 and repeat the same procedure with a curve α_2 such that $(\alpha_1) \cap (\alpha_2) = \{q\}$. Make the same operation for w_j and z_i, $j = 1, \ldots, l$, $i = 1, \ldots, n$.

Then, after this local ridefinition, we have the following facts:

(1) Λ is a system of curves of class $H^{2,p}$ without crossings;

(2) $l(\Lambda) + \| \kappa(\Lambda) \|_{L^p} < +\infty$;

(3) we have eliminated all the points of $\overline{C} \cap \mathrm{int}(R(q))$;

(4) we have created a node q;

(5) q is not a cusp point of ∂E;

(6) ∂E is unchanged;

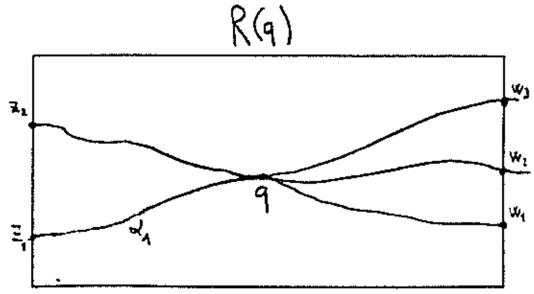

FIGURE 6.4.

(7) the multiplicities are all even and well defined in $R(q)$.

Second possibility: $q \in \partial E$ and ∂E is smooth in a neighbourhood of q.

Take a rectangle $R(q)$ as before (obviously $\partial E \cap R(q) \neq \emptyset$), and such that $R(q) \cap \partial E$ is smooth. Then we adopt the same rules keeping fixed $\partial E \cap R(q)$ but with the following modifications:

(a) all the modified curves could pass, for simplicity, through a point $q' \in R(q) \setminus \partial E$;

(b) the intersection points between the modified curves and ∂E are smooth.

Then also in this case properties 1)-7) of the first possibility hold.

Third possibility: $q \in \partial E$ and q is a cusp point.

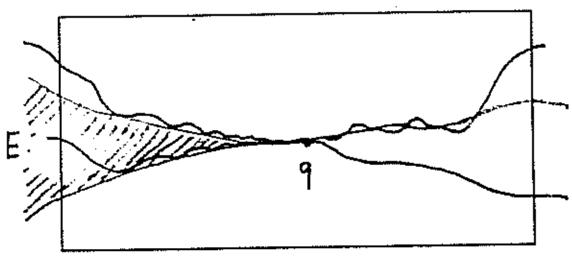

FIGURE 6.5.

This case is a slight modification of the previuos one.

Then, by compactness, we can cover K with a finite number of rectangles verifying the required properties, and we can modify Λ, as previously explained, in each rectangle. What we get is the following: ∂E is unchanged, Λ is a system of curves of class $H^{2,p}$ without crossings and verifying the finiteness property, $(\Lambda) \supseteq \partial E$. Hence, using the preceding result, the number of cusp points of ∂E is even. □

Putting together Corollary 6.1 and Theorem 6.3, we get one of the main results of the paper, namely,

THEOREM 6.4. *Let E be a subset of $\mathbf{R}^2$ verifying the assumptions of Definition 6.9, and suppose that ∂E is smooth except for a finite number n of cusp points. Then*

$$n \text{ is even} \Leftrightarrow \overline{F}(E) < +\infty.$$

In view of Theorem 4.1 (v) and in view of (6.2) we conjecture that if $E \subseteq \mathbf{R}^2$ is a set verifying Definition 6.9 having an even number of cusp points, then

$$\overline{F}(E) = \inf\{l(\Lambda) + \| \kappa(\Gamma) \|_{L^p}^p : \Lambda \in \mathcal{A}(E)\} = \inf\{l(\Gamma) + \| \kappa(\Gamma) \|_{L^p}^p : \Gamma \in \mathcal{B}(E)\},$$

where $\mathcal{A}(E)$ and $\mathcal{B}(E)$ are defined in (v) of Theorem 4.1 and in the statement of Theorem 6.1, respectively. Note that the last infimum does not seem to be reached, as the following example suggests (Figure 6.6).

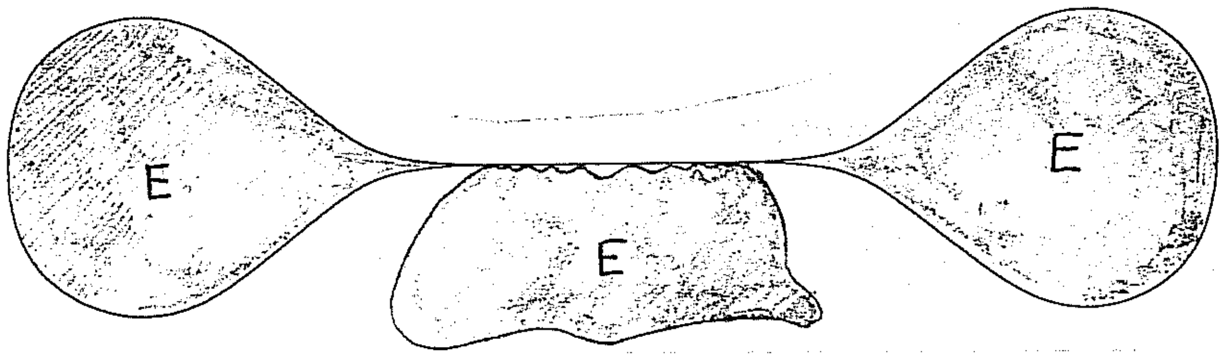

FIGURE 6.6.

7. Localization.

Some results of sections 3, 4 can be localized to an open subset of $\mathbf{R}^2$. Let Ω be an open subset of $\mathbf{R}^2$ and let $\mathcal{M}$ be the set of all bounded measurable subsets of $\mathbf{R}^2$. The $L^1(\Omega)$-topology on $\mathcal{M}$ is the topology induced by the pseudo-distance $d(E_1, E_2) := |(E_1 \Delta E_2) \cap \Omega|$. We say that a subset E of $\mathbf{R}^2$ is of class $\mathcal{C}^2(\Omega)$, and we will write $E \in \mathcal{C}^2(\Omega)$, if E is open and $\partial E \cap \Omega$ is of class $\mathcal{C}^2$, i.e., $E \cap \Omega$ is locally the subgraph of a function of class $\mathcal{C}^2$ with respect to a suitable orthogonal coordinate system.

We define the map $F(\cdot, \Omega) : \mathcal{M} \to [0, +\infty]$ by

$$F(E, \Omega) := \begin{cases} \displaystyle\int_{\partial E \cap \Omega} (1 + |\kappa(z)|^p)\, d\mathcal{H}^1(z) & \text{if } E \in \mathcal{C}^2(\Omega), \\ +\infty & \text{elsewhere on } \mathcal{M}. \end{cases}$$

Note that

$$F(E,\Omega) = \sup_{n\in \mathbf{N}} F(E,\Omega_n) \tag{7.1}$$

for every sequence $\{\Omega_n\}_n$ of open sets invading Ω. Moreover $F(E,\cdot)$ is monotone as a set function, i.e., if Ω_1,Ω_2 are open,

$$\Omega_1 \subseteq \Omega_2 \quad \Rightarrow F(E,\Omega_1) \leq F(E,\Omega_2). \tag{7.2}$$

By $\overline{F}(\cdot,\Omega)$ we denote the lower semicontinuous envelope of $F(\cdot,\Omega)$ with respect to the topology of $L^1(\Omega)$. It is known that, for every $E \in \mathcal{M}$, we have

$$\overline{F}(E,\Omega) := \inf\{\liminf_{h\to+\infty} F(E_h,\Omega) : E_h \to E \text{ in } L^1(\Omega) \text{ as } h \to +\infty\}.$$

Note that $\overline{F}(E,\Omega) < +\infty$ if and only if there exists a sequence $\{E_h\}_h$ of bounded open subsets of $\mathbf{R}^2$ of class $\mathcal{C}^2(\Omega)$ such that $E_h \to E$ in $L^1(\Omega)$ as $h \to +\infty$, and

$$\sup_h \mathcal{H}^1(\partial E_h \cap \Omega) < +\infty, \quad \sup_h \int_{\partial E_h\cap\Omega} |\kappa_h(z)|^p \, d\mathcal{H}^1(z) < +\infty. \tag{7.3}$$

THEOREM 7.1. *Let E be an open subset of class $\mathcal{C}^2(\Omega)$. Then*

$$F(E,\Omega) \leq \liminf_{h\to+\infty} F(E_h,\Omega) \tag{7.4}$$

for any sequence $\{E_h\}_h$ of open subsets of $\mathbf{R}^2$ of class $\mathcal{C}^2(\Omega)$ such that $E_h \to E$ in $L^1(\Omega)$ as $h \to +\infty$.

PROOF: Let $\{E_h\}_h$ be a sequence of open subsets of $\mathbf{R}^2$ of class $\mathcal{C}^2(\Omega)$, $E_h \to E$ in $L^1(\Omega)$ as $h \to +\infty$. We can suppose that the right hand side of (7.4) is finite, otherwise the result is trivial. Hence $\{E_h\}_h$ verifies (7.3). Given a sequence $\{\Omega_n\}_n$ of relatively compact open sets invading Ω, it will be sufficient to prove that

$$F(E,\Omega_n) \leq \liminf_{h\to+\infty} F(E_h,\Omega_n), \tag{7.5}$$

for every n. In fact, (7.4) follows from (7.5), since, by (7.1) and (7.2),

$$F(E,\Omega) = \sup_n F(E,\Omega_n) \leq \sup_n \liminf_{h\to+\infty} F(E_h,\Omega_n) \leq \liminf_{h\to+\infty} F(E_h,\Omega).$$

Fix $n \in \mathbf{N}$. Let $\{E_{h_k}\}_k$ be a subsequence of $\{E_h\}_h$ with the property that

$$\lim_{k\to+\infty} F(E_{h_k},\Omega_n) = \liminf_{h\to+\infty} F(E_h,\Omega_n) < +\infty.$$

For simplicity, this subsequence (and any further subsequence) will be denoted by $\{E_k\}_k$. From (7.3) it follows that, for any k,

$$\partial E_k \cap \Omega_n \subseteq \bigcup_{i=1}^{m_k}(\gamma_k^i) \cup \bigcup_{j=1}^{r_k}(\beta_k^j),$$

where

(i) $\{\gamma_k^1, \dots, \gamma_k^{m_k}\}$ is a disjoint system of (closed) curves of class C^2 such that $(\gamma_k^i)\cap\overline{\Omega_n} \neq \emptyset$;

(ii) $\{\beta_k^1, \dots, \beta_k^{r_k}\}$ is a finite family of simple regular curves of class C^2 such that $|\frac{d\beta_k^j}{dt}|$ is constant on $[0,1]$, $\beta_k^j(0), \beta_k^j(1) \in \partial\Omega$ and $(\beta_k^j) \cap \partial\Omega_n \neq \emptyset$ for any $j = 1, \dots, r_k$;

(iii) all the curves are pairwise disjoint;

(iv)

$$\sum_{i=1}^{m_k} l(\gamma_k^i) + \sum_{j=1}^{r_k} l(\beta_k^j) + \sum_{i=1}^{m_k} \int |\kappa(\gamma_k^i)|^p \, ds + \sum_{j=1}^{r_k} \int |\kappa(\beta_k^j)|^p \, ds < +\infty.$$

Note that Lemma 3.1 implies that $\{m_k\}_k$ is uniformly bounded with respect to k. Moreover $\{r_k\}_k$ is also uniformly bounded with respect to k. This follows from (iv) and from the fact that $l(\beta_k^j) \geq 2\mathrm{dist}(\partial\Omega_n, \partial\Omega)$ for any $j = 1, \dots, r_k$. Passing to a suitable subsequence, we can assume that m_k and r_k are independent of k. These numbers will be denoted by m and r, respectively. Using (iv) and repeating the arguments of Theorem 4.1, by compactness we get a family $\Lambda := \{\gamma^1, \dots, \gamma^m, \beta^1, \dots, \beta^r\}$ of regular curves of class $H^{2,p}$, such that

$\gamma_k^i \rightharpoonup \gamma^i$ weakly in $H^{2,p}$ as $k \to +\infty$ for any $i = 1, \dots, m$;

$\beta_k^j \rightharpoonup \beta^j$ weakly in $H^{2,p}$ as $k \to +\infty$ for any $j = 1, \dots, r$.

Then, from (3.5) and the fact that $E_k \to E$ in $L^1(\Omega)$ as $k \to +\infty$, using the hypothesis that $E \in C^2(\Omega)$, it follows that $(\Lambda)\cap\Omega_n \supseteq \partial E\cap\Omega_n$. In fact, suppose by contradiction that there exists $p \in (\partial E\setminus(\Lambda))\cap\Omega_n$. Let r so small such that $B_r(p) \subseteq \Omega_n$ and $B_r(p)\cap\partial E\cap(\Lambda) = \emptyset$. Then from (3.5) there exists $k_0 \in \mathbf{N}$ such that $\partial E_k \cap B_{\frac{r}{2}}(p) = \emptyset$ for any $k \geq k_0$, and this contradicts the fact that $E_k \to E$ in $L^1(\Omega)$ as $k \to +\infty$. Then, repeating the last part of the proof of Theorem 3.2 relatively to Ω_n and passing to the limit as $n \to +\infty$, we get the assertion. □

COROLLARY 7.1. *Let $E \subset\subset \Omega$ be an open subset of $\mathbf{R}^2$ of class $H^{2,p}$ such that E is relatively compact in Ω. Then*

$$\overline{F}(E,\Omega) = \int_{\partial E} [1 + |\kappa(z)|^p] \, d\mathcal{H}^1(z).$$

In particular, $\overline{F}(E,\Omega) < +\infty$.

PROOF: Theorem 7.1 holds with the same proof if E is of class $H^{2,p}$, hence, passing to the infimum with respect to the approximating sequence in (7.4) we infer that $\int_{\partial E}[1 +$

$|\kappa(z)|^p]\ d\mathcal{H}^1(z) \leq \overline{F}(E,\Omega)$. The opposite inequality can be proved as in Corollary 3.2, using the hypothesis that E is relatively compact in Ω. □

The last part of this section is devoted to prove that $\overline{F}(E,\cdot)$ is not subadditive (see [4, Conjecture 5]). This is the main result of this section, and shows that $\overline{F}(E,\cdot)$ cannot be a measure and in particular that $\overline{F}(E,\Omega)$ can not be represented as in integral of the form (1.2) for a suitable choice of the function $\kappa(z)$.

THEOREM 7.2. *There exists a bounded open set $E \subset \mathbf{R}^2$ such that*

$$\overline{F}(E,\mathbf{R}^2 \setminus \overline{Q_1}) + \overline{F}(E,Q_2) < \overline{F}(E,\mathbf{R}^2) < +\infty,$$

where Q_1 and Q_2 are two suitable open squares in $\mathbf{R}^2$, and $Q_1 \subset\subset Q_2$.

PROOF: Let E be the set of Figure 7.1.

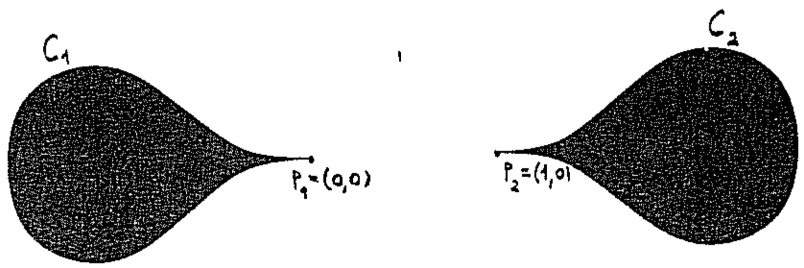

FIGURE 7.1.

Let p_1, p_2 be the two cusp points of ∂E, and let C_1, C_2 be the two connected components of ∂E. We fix $p_1 = (0,0)$ and $p_2 = (1,0)$ and we assume that the unoriented tangent line in p_1 and p_2 is the x-axis. Here it is important that C_1 lies on the left with respect to the y-axis, and that C_2 lies on the right with respect to the normal line $N(p_2)$ of ∂E at p_2. Clearly $\overline{F}(E) < +\infty$ (see for instance Section 1). From Theorem 4.1 (v) it follows that

$$(7.6) \qquad \overline{F}(E) = \overline{F}(E,\mathbf{R}^2) \geq \inf\{l(\Lambda) + \parallel \kappa(\Lambda) \parallel_{L^p}^p : \Lambda \in \mathcal{A}(E)\}.$$

Let $\Lambda \in \mathcal{A}(E)$, $\Lambda = \{\gamma^1, \ldots, \gamma^m\}$; since $(\Lambda) \supseteq \partial E$, there exist $i \in \{1, \ldots, m\}$ and $t_0 \in [0,1]$ such that $\gamma^i(t_0) = p_1$. Up to a reparametrization of the curve γ^i, we can suppose that $t_0 = 0$ and that the normal tangent unit vector to γ^i at $t = 0$ is $(1,0)$. We will suppose also that γ^i is parametrized by arc lenght s.

We can consider the decomposition $[0, l(\gamma^i)] = A \cup B$, where $A := \{s \in [0, l(\gamma^i)] : \gamma^i(s) \in \partial E\}$, and $B := \{s \in [0, l(\gamma^i)] : \gamma^i(s) \notin \partial E\}$. Clearly $B \neq \emptyset$. There are two possibilities:

(i) $(\gamma^i) \cap \partial C_2 \neq \emptyset$;

(ii) $(\gamma^i) \cap \partial C_2 = \emptyset$.

In case (i) one obviously gets

$$\mathcal{H}^1(B) + \int_B |\ddot{\gamma}^i(s)|^p \, ds \geq |p_2 - p_1| = 1. \tag{7.7}$$

Let us consider case (ii). Let us denote by γ_1^i the x-component of the curve γ^i. By construction, we have that $\dot{\gamma}_1^i(0) = 1$. Since γ^i is closed, we can consider the smallest parameter $\overline{s} \in]0, l(\gamma^i)[$ such that $\gamma_1^i(\overline{s}) > 0$ and $\dot{\gamma}_1^i(\overline{s}) = 0$. Then it is easy to prove that $\gamma_1^i(s) > 0$ for any $s \in [0, \overline{s}]$. Hence, since C_1 lies on the left with respect to the y-axis, we get $\partial C_1 \cap \gamma^i([0, \overline{s}]) = \emptyset$. Moreover, by (ii), we have also that $\partial C_2 \cap \gamma^i([0, \overline{s}]) = \emptyset$. Hence, $[0, \overline{s}] \subseteq A$, and using (3.4) we get

$$\mathcal{H}^1(B) + \int_B |\ddot{\gamma}^i(s)|^p \, ds \geq \overline{s} + \int_0^{\overline{s}} |\ddot{\gamma}^i(s)|^p \, ds \geq \frac{\pi}{2}[(\frac{p}{q})^{\frac{1}{p}} + (\frac{q}{p})^{\frac{1}{q}}], \tag{7.8}$$

where it is crucial that the constant in the right hand side does not depend on Λ. Then, from (7.6), using (7.7) and (7.8), we infer that

$$\overline{F}(E, \mathbf{R}^2) \geq \mathcal{H}^1(\partial E) + \int_{\partial E \setminus \{p_1, p_2\}} |\kappa(z)|^p \, d\mathcal{H}^1(z) + c, \tag{7.9}$$

where $c := \min(1, \frac{\pi}{2}[(\frac{p}{q})^{\frac{1}{p}} + (\frac{q}{p})^{\frac{1}{q}})$.

Let us consider two squares Q_1 and Q_2 centered at the point $(\frac{1}{2}, 0)$. Suppose that $Q_1 \subset\subset Q_2$ and that the side b of Q_2 is such that $c > 2(1-b) > 0$. Then $E \cap Q_1 = \emptyset$ implies that

$$\overline{F}(E, Q_1) = 0, \tag{7.10}$$

and from (7.9) we have

$$\overline{F}(E, \mathbf{R}^2) > \mathcal{H}^1(\partial E) + \int_{\partial E \setminus \{p_1, p_2\}} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2(1-b). \tag{7.11}$$

Moreover one easily construct a sequence $\{E_h\}_h$ of bounded open subsets of $\mathbf{R}^2$ of class $C^2(\mathbf{R}^2 \setminus \overline{Q_2})$ such that $E_h \to E$ in $L^1(\mathbf{R}^2 \setminus \overline{Q_2})$ as $h \to +\infty$, and such that (see Figure 7.2)

$$\mathcal{H}^1(\partial E) + \int_{\partial E \setminus \{p_1, p_2\}} |\kappa(z)|^p \, d\mathcal{H}^1(z) + 2(d-b) = \lim_{h \to +\infty} F(E_h, \mathbf{R}^2 \setminus \overline{Q_2}).$$

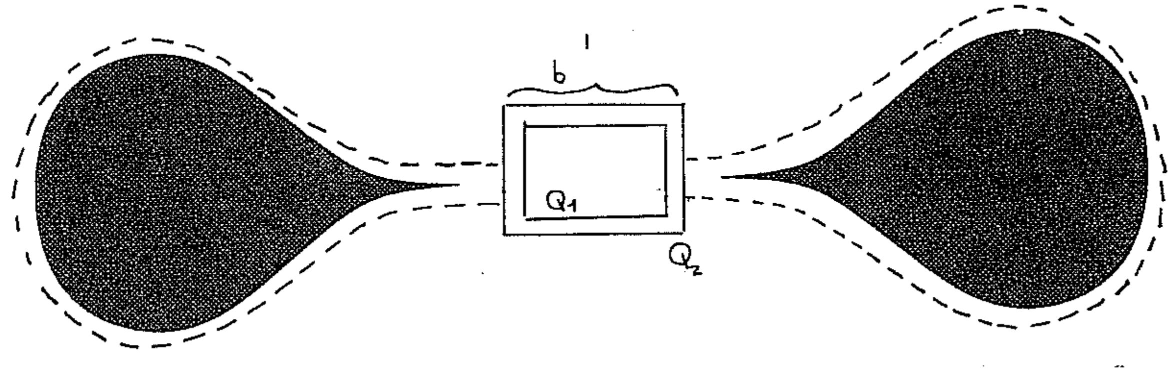

FIGURE 7.2.

This concludes the proof, since from (7.11) and (7.10) we get

$$\overline{F}(E,\mathbf{R}^2) > \lim_{h\to+\infty} F(E_h,\mathbf{R}^2\setminus\overline{Q_2}) \geq \overline{F}(E,\mathbf{R}^2\setminus\overline{Q_2}) = \overline{F}(E,\mathbf{R}^2\setminus\overline{Q_2}) + \overline{F}(E,Q_1). \quad \square$$

8. Appendix: connections with the elastica.

For the definition and the main properties of the elastica we refer to [13, 10]. Let us fix two points $z_0,\ z_1 \in \mathbf{R}^2$, and two angles $\theta_0, \theta_1 \in [0, 2\pi]$. Consider the problem

$$\min\{l(\gamma) + \parallel \kappa(\gamma) \parallel_{L^p}^p : \gamma \in \mathcal{D}\}, \tag{8.1}$$

where $\mathcal{D}$ is the set of all curves $\gamma : [0,1] \to \mathbf{R}^2$ of class $H^{2,p}$ parametrized with constant velocity, and such that

$$\gamma(0) = z_0,\ \gamma(1) = z_1,\ \frac{d\gamma}{dt}(0) = \theta_0,\ \frac{d\gamma}{dt}(1) = \theta_1. \tag{8.2}$$

PROPOSITION 8.1. *If $z_0 \neq z_1$ or $\theta_0 \neq \theta_1$, the problem (8.1) admits a solution. Moreover, if $p = 2$, this solution is an elastica.*

PROOF: Let $\{\gamma_h\}_h$ be a minimizing sequence, and, for any h, let $\lambda_h := |\frac{d\gamma_h}{dt}|$. Since $\{\lambda_h\}_h$ is uniformly bounded with respect to h, passing to a suitable subsequence $\{\gamma_k\}_k$, we have that $\lambda_{h_k} \to \lambda_0 \geq 0$ as $k \to +\infty$. Let us use the index k for this subsequence, and for any further subsequence. If $z_0 \neq z_1$, we get that $\lambda_0 > 0$, since, for any k,

$$|z_1 - z_0| = |\gamma_k(1) - \gamma_k(0)| \leq \int_0^1 |\frac{d\gamma_k}{dt}| dt = \lambda_k.$$

If $\theta_0 \neq \theta_1$ we get the same conclusion, by using (3.4). Hence $\lambda_0 > 0$. By the uniform boundedness of $\{\lambda_k\}_k$ and $\{\lambda_k + \lambda_k^{1-2p} \int_0^1 |\ddot{\gamma}_k|^p\, dt\}_k$ with respect to k, we get that

$$\sup_k \parallel \kappa(\gamma_k) \parallel_{L^p} < +\infty.$$

Hence, using (8.2) and the fact that $\sup_k l(\gamma_k) < +\infty$, we get that

$$\sup_k \| \gamma_k \|_{Htp} < +\infty.$$

Passing to a suitable subsequence, we obtain that there exists a curve $\gamma_0 \in H^{2,p}$ verifying (8.2) such that $\gamma_k \rightharpoonup \gamma_0$ weakly in $H^{2,p}$ as $k \to +\infty$. Using the weak lower semicontinuity of the L^p norm and the fact that $\lambda_0 > 0$, we get easily that γ_0 is a solution of problem (8.1).

If $p = 2$ and γ_0 is a solution of problem (8.1), writing $\dot{\gamma}(s) = (\cos\theta(s), \sin\theta(s))$, we have that γ_0 solves

$$\min_{\lambda,\theta}\{\lambda + \lambda^{-3}\int_0^1 (\frac{d\theta}{dt})^2\, dt\},$$

under the prescribed conditions (8.2). If $|\frac{d\gamma_0}{dt}| = \lambda_0$, then γ_0 is in particular the solution of the minimum problem

$$\min_{\theta}\{\lambda_0 + \lambda_0^{-3}\int_0^1 (\frac{d\theta}{dt})^2\, dt\},$$

under (8.2), i.e., γ_0 is an elastica. □

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