SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Thesis submitted for the degree of Magister Philosophiae

On the Classification of Coxeter Simplices

Candidate:
Monique Annie GRADOLATO

Supervisor:
Prof. Bruno ZIMMERMANN

Academic Year 1990/91

TRIESTE

M. A. GRADOLATO

On the Classification of Coxeter Simplices

Thesis submitted for the degree of Magister Philosophiae
Academic Year 1990/91

Math. Subj. Class. (1991): 05C50, 52B

Key Words: polytope, simplex, Gram matrix, Coxeter graph

Introduction

In this work I am dealing with *polytopes* and *simplices*, the higher dimensional generalization of polygon and triangle, and with the classification of simplices whose dihedral angles are integer (non-trivial) submultiples of π, the socalled *Coxeter simplices.*

If we ask for a triangle tesselating the Euclidean plane, then the angles are forced to be integer submultiples of π. So we have to seek for a Coxeter triangle. If the angles are $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$ then the Coxeter triangle is usually referred to as a (p, q, r) triangle. The Euclidean condition on the angle sum of a triangle let (p, q, r) be $(2, 4, 4)$ or $(2, 3, 6)$ or $(3, 3, 3)$. If the Euclidean plane is substituted by the spherical one or by the hyperbolic one, the condition on the angle sum changes, giving other values for (p, q, r). The possible triangles are no more in finite number. In the spherical plane the angle sum is greater than π and the possible spherical (p, q, r) triangles are $(2, 3, 3)$, $(2, 3, 4)$, $(2, 3, 5)$ and $(2, 2, r)$, for any $r \geq 2$. Usually to the list is also added the non Coxeter triangle $(1, r, r)$, for any $r \geq 2$, which is however a degenerate triangle: two sides lie on the same line. In the hyperbolic plane the angle sum is less than π and there are hyperbolic (p, q, r) triangles for any integer positive numbers p, q, r such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1$.

The aim of this work is to present tools for the classification of Coxeter simplices in higher dimensions and to keep, as much as possible, a parallel and simultaneous treatment of the Euclidean, spherical and hyperbolic case.

In the preliminaries I explain what I mean by inner product space structure and which models I choose for the geometric ambient space of the objects of study. The more technical proofs are put in an appendix at the end of the work.

In the first part I give the definition of polytope and root system: roughly speaking a polytope is a non-void convex region, whose boundary decomposes into a finite number of pieces - the faces -, each contained in a hyperplane. Then to each face we can associate the direction orthogonal to the hyperplane containing it. Given the direction we choose a vector of unit length, that placed starting at the face, comes out from the polytope. The vectors so obtained are called *roots.* The finite set they form is called the *root system* of the polytope. It may happen that the root system contains with each root also its opposite: e.g. in an Euclidean hypercube the hyperplanes are two by two parallels. But this will not be the general case.

The root system enable us to associate to the polytope a symmetric real matrix: the *Gram matrix*, whose entries are the products of the roots.

Can the process be reversed?

The answer is given for the class of polytopes free from obtuse dihedral angles, which I call *non-obtuse-angled polytopes*, to whom the Coxeter simplices belong.

The Gram matrix of a non-obtuse-angled polytope has all diagonal entries equal to 1, and the entries off the diagonal non-positive. I denote the class of matrices with such entries by $\mathcal{A}$. The characterization of subclasses of $\mathcal{A}$ is done by the application of the *Perron-Frobenius theorem* on the spectrum of a non-negative matrix.

It is in particular proved that:

A matrix G in the class $\mathcal{A}$ is the Gram matrix of a spherical non-obtuse-angled simplex if and only if it is positive definite; of an Euclidean simplex if and only if it is indecomposable positive semidefinite; of a hyperbolic non-obtuse-angled simplex *if and only if it is indecomposable of index one less of the rank. The simplex is in any of the previuos cases uniquely determined up to a motion of the relative space, and in Euclidean case up to similarity. And the matrix is uniquely determined up to isomorphism of matrices - i.e. a same permutation applied to the rows and to the columns -.*

The effective determination of the polytope is done via the root system. The last can be explicitly obtained, in most of the examples, by the application of an extended version of the *Gram-Schmidt orthogonalization process*, which I have introduced in the appendix.

In the first part it is also proved that the admissible shape of a non-obtuse-angled polytope is heavily imposed by the geometry of the space. The only non-obtuse-angled spherical polytopes are simplices. In Euclidean space the class is enlarged to the products of simplices. In hyperbolic space there are no restrictions. The situation and the reason for such a behaviour can be intuitively understood in two dimensions: the crucial fact is the existence of parallels. Choose a line and the most wide allowed amplitude for the angles: then choose two different orthogonals to the given line. In spherical plane they meet: there are no parallels. In Euclidean plane we can choose one more line to form a non-obtuse-angled polygon: but we cannot obtain more than four sides. In hyperbolic plane there are ultraparallels and there is no constrainment for the lines to close to a polygon in a finite number of steps.

A matrix is a discrete object as a graph: in the second part I classify Coxeter simplices by means of graphs.

To a simplex is associated a graph: to each face a vertex, to each dihedral angle of value $\frac{\pi}{p}$ an edge marked $p-2$. (There are some remarks to be done about parallel faces.) An edge marked 0 means no edge. The graph obtained has no loops. Such a marked graph is called a *Coxeter graph.*

Starting from a Coxeter graph consider its adjacency matrix. Then substitute each entry off the diagonal with its mark and add 2 to it. Then substitute each diagonal entry with 1. Apply to each entry the function $f(*) = -\cos(\frac{\pi}{*})$. I call the resulting matrix the *Gram matrix* of the Coxeter graph and I call the Coxeter graph *spherical* if its Gram matrix is positive definite, *Euclidean* if it is indecomposable positive semidefinite and *hyperbolic* if it is indecomposable non-degenerate but indefinite of index one less the order (which in this case is equal to the rank).

I say that a Coxeter graph satisfies the Coxeter Spherical respectively the Coxeter Spherical Euclidean condition on induced subgraphs if

CS: Each induced subgraph on all vertices but one is spherical.

CSE: Each induced subgraph on all vertices but one is either spherical or Euclidean, but at least one is Euclidean.

Each Coxeter spherical CS-graph corresponds to a spherical simplex. On connected graphs CS and CSE conditions turns out to be sufficient conditions for the existence of the associated simplex: the geometrical type is decided by the sign of the determinant of the Gram matrix:

positive	spherical
null	Euclidean
negative	hyperbolic

The determinant of a Coxeter CSE-graph is negative and the corresponding simplex is hyperbolic. It has vertices at infinity (one for each Euclidean induced subgraph on all vertices but one) and it is unbounded, but of finite volume.

The following scheme collects the considered concepts:

$$\begin{array}{ll} \text{Coxeter polytope} & \\ \updownarrow & \\ \text{root system} & \\ \updownarrow & \\ \text{Gram matrix} & \rightarrow \text{ inner product structure on } \mathbb{R}^N \\ \updownarrow & \\ \text{Coxeter graph} & \end{array}$$

The correspondence between graphs and simplices is bijective up to isomorphisms of graphs and up to motions of the space (in Euclidean case up to similarity).

I introduce a partial order relation in the class of Coxeter CS- and CSE-graphs. I define three type of constructive steps to do consecutives: 1) join two vertices that are not; 2) increase just one mark; 3) take a new vertex and join it by a new edge to a pre-existent vertex. Then we can start from the one vertex graph and perform consecutives of type 3): we do the classification of all unmarked CS-graphs and CSE-graphs that are trees. Then for the trees of given order we perform consecutives of type 1) and 2) and we obtain all the Coxeter graphs of that order. The method introduced allows a simultaneous treatment of the three geometric types of simplices: spherical, Euclidean and hyperbolic, giving a global picture of the three cases. I have added here four tables. Table 1 contains unmarked trees up to order 10. Table 2 contains unmarked graphs that are not trees. Table 3 contains all graphs of order 4 from Tables 1 and 2 and their consecutives of type 2). Table 4: same as Table 3 for the graphs of order 5. The names of the Coxeter graphs are not completely standard in the literature and I have introduced denominations which, to me, recalls as much as possible the shape of the graph. In the Tables the Euclidean CS-graphs are enclosed by a rectangle, the hyperbolic CS-graphs are enclosed by a double rectangle, the hyperbolic CSE-graphs by a triple rectangle. The spherical graphs are not enclosed.

In preparing this work I have primarily consult (in alphabetical order) Bourbaki [4], Coxeter [7], Grove-Benson [13], Thurston [19], Vinberg [20]. The book of Grove and Benson treats the spherical case, and that of Coxeter the spherical and Euclidean cases. Other references (but also those ones) are put in the text.

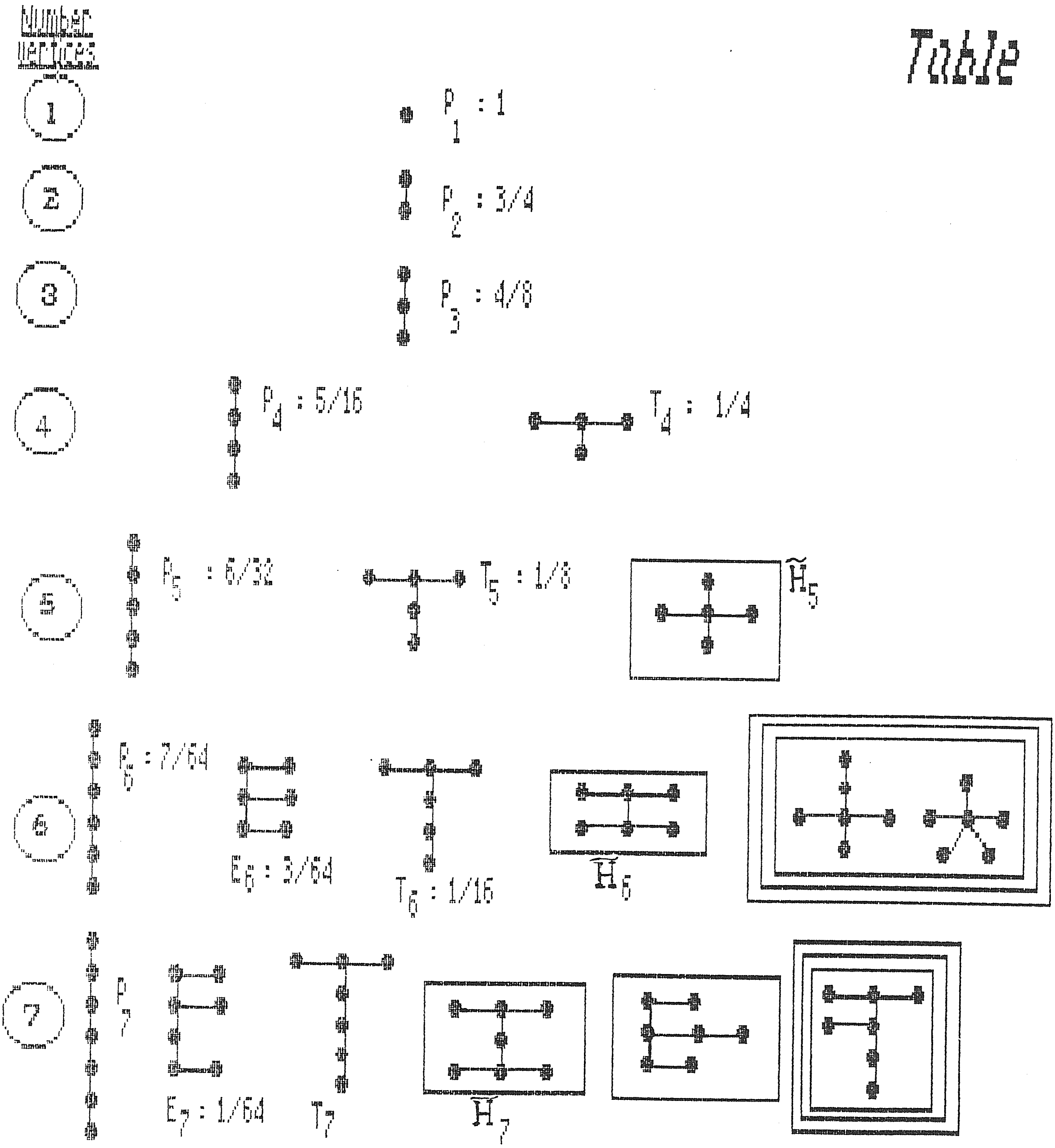
Prerequisites are notions of linear algebra and matrices (e.g. [2], [11]); and some elementary notions of graph theory (e.g. [12]).

It must be noted that there is no universal concordance in the denomination of concepts; so after each definition, if not original, I have put references.

Acknowledgements

I wish to thank prof. Dale Husemoller for discussion and advices about the bibliography.

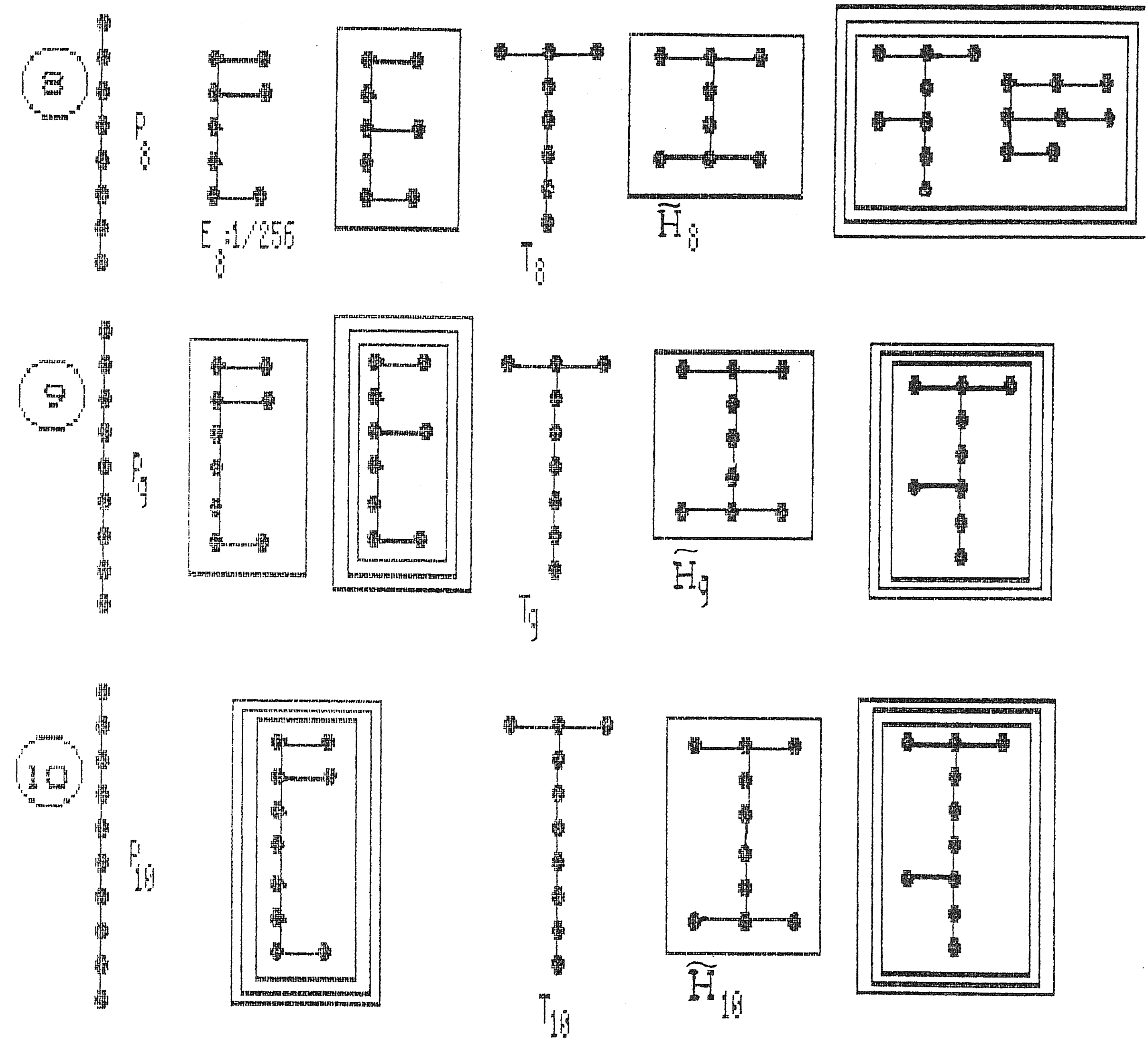
Table 1



COXETER UNMARKED CS and CSE TREES

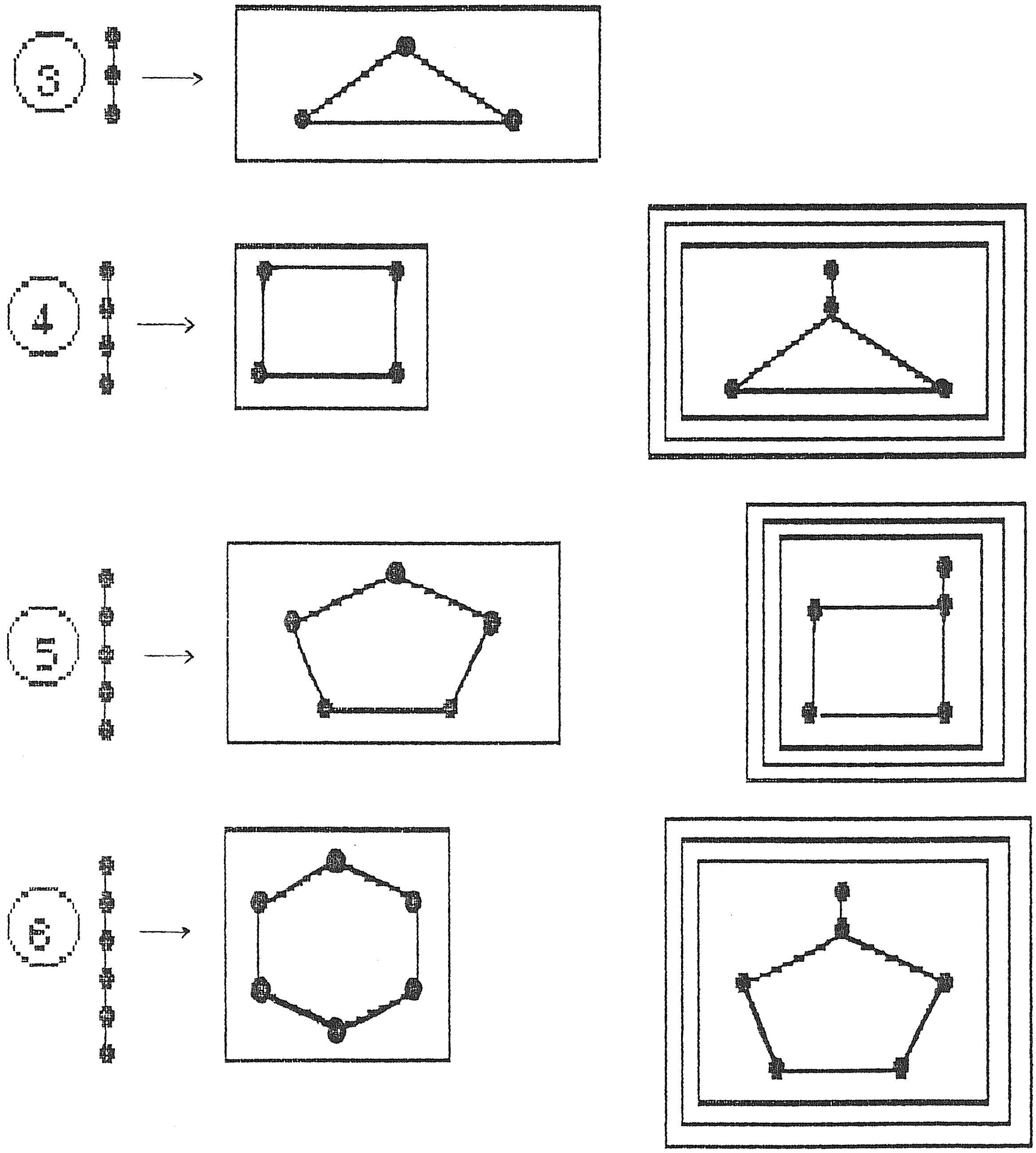
Number vertices

continuation Table 1



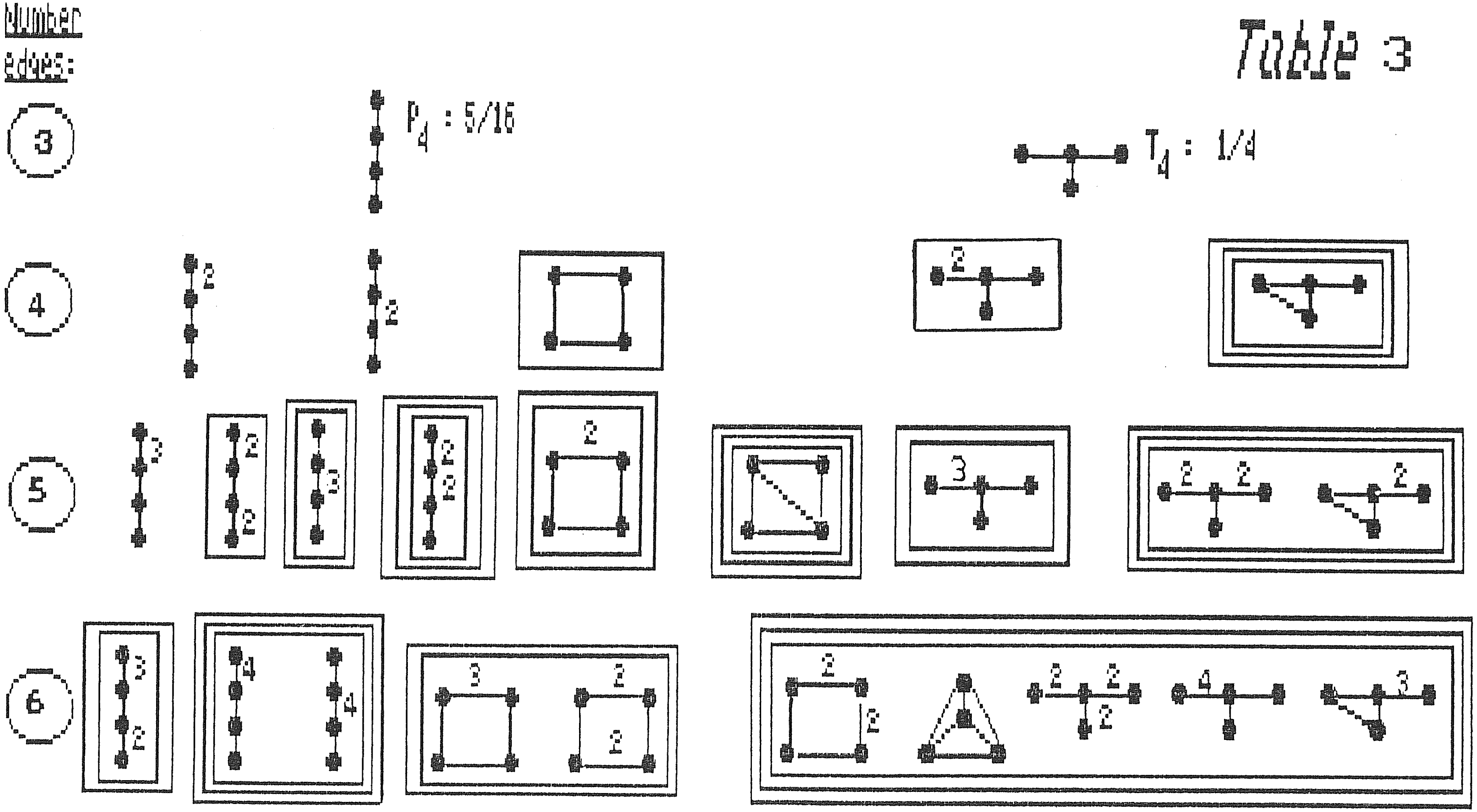
Number
vertices

Table 2



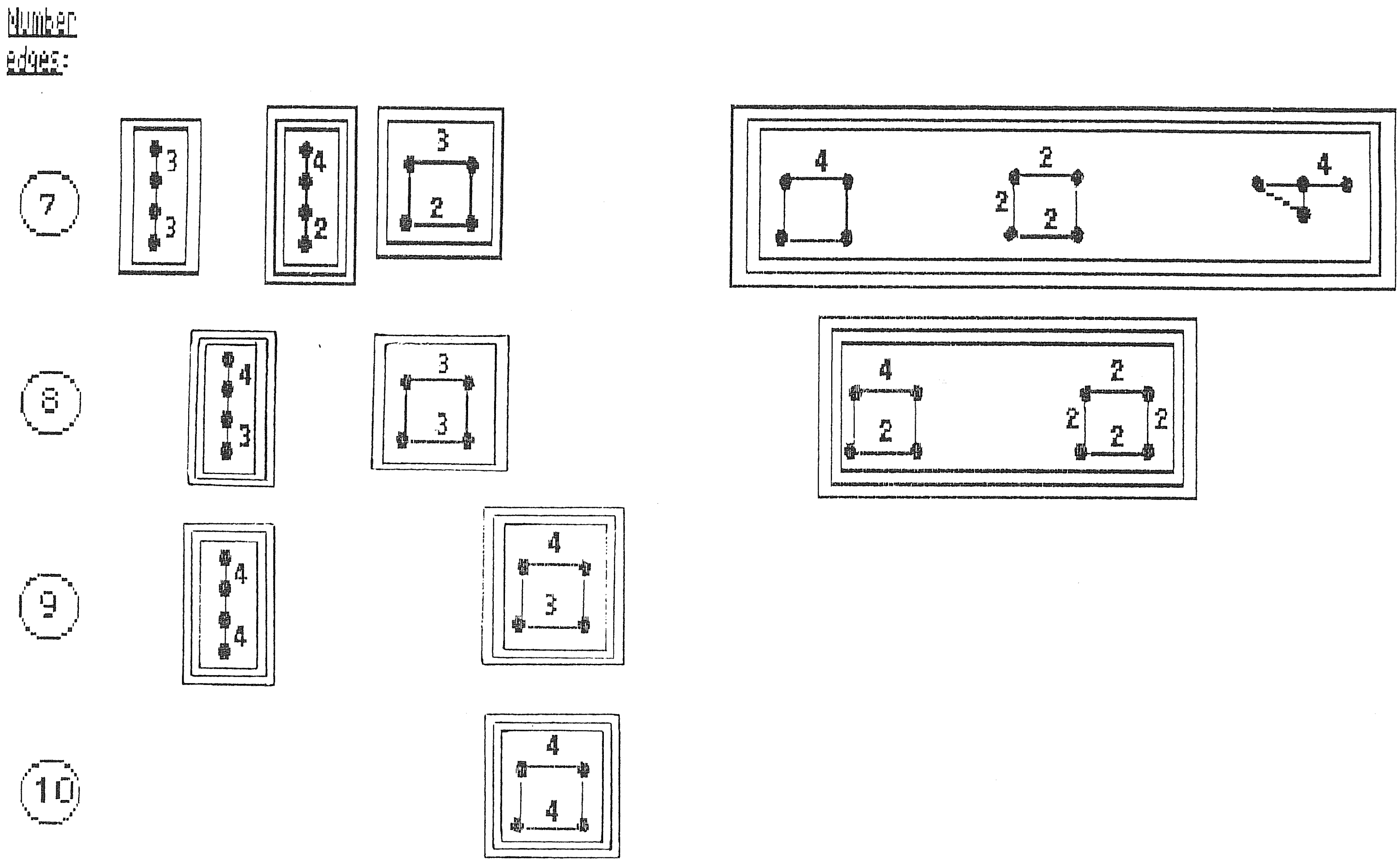
COXETER UNMARKED CS and CSE-GRAPHS not CYCLE FREE

Table 3



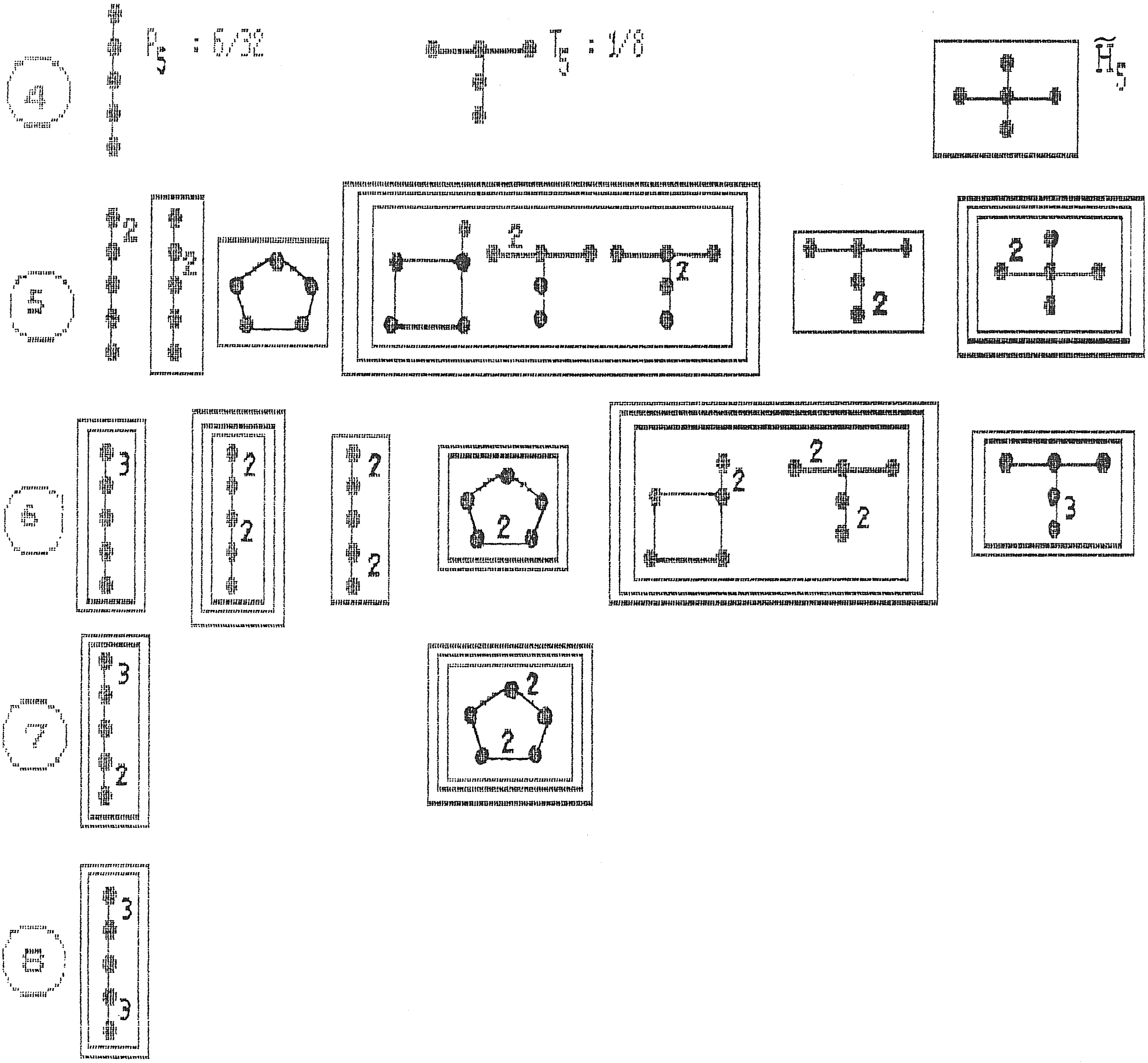
COXETER CS and CSE-GRAPHS ON 4 VERTICES

continuation **Table 3**



Number edges:

Table 4



COXETER CS and CSE-GRAPHS
ON 5 VERTICES

Contents

I Coxeter unmarked CS and CSE-trees - Table 1 iv

II Coxeter unmarked CS and CSE-graphs not cycle free - Table 2 vi

III Coxeter CS and CSE-graphs on 4 vertices - Table 3 vii

IV Coxeter CS and CSE-graphs on 5 vertices - Table 4 ix

0 Preliminaries 1

I Polytopes and simplices 5

1 Hyperplanes, roots and angles 5

2 Non-obtuse-angled polytopes 8

II Coxeter simplices and Coxeter graphs 26

1 Coxeter spherical graphs 31

2 Coxeter Euclidean graphs 34

3 Coxeter hyperbolic graphs 38

4 Construction and classification 38

III Appendix 42

A Gram matrices of system of vectors 42

B Existence and uniqueness theorems 44

C Gram-Schmidt extended process 46

D Example 49

References 51

Index 53

0 Preliminaries

Let us recall some general algebraic and geometrical facts and give at the same time the notations that we will use throughout the work.

Let $\mathbb{R}^N$ be the real N-dimensional vector space. The null vector $(0, ..., 0)$ will be denoted by 0. Consider a real bilinear symmetric form B on $\mathbb{R}^N$

$$B \, : \, \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$$

The symbol B will denote also the matrix associated to the form when the basis is the standard one:

$$e_1 = (1, 0, ..., 0), \; e_2 = (0, 1, 0, ..., 0), \; ... \; , \; e_N = (0, ..., 0, 1).$$

If we start with a real symmetric matrix $B = (b_{ij})$, we define the form on the elements of the basis

$$B(e_i, e_j) = b_{ij}$$

and then extend by linearity.

Instead of $B(x, y)$ the symbol $x \cdot_B y$ will be used. If x and y are considered as column matrices, then $x \cdot_B y$ is just the matricial row-column product $x^t \; B \; y$.

We say that B defines an *inner product* on $\mathbb{R}^N$ and $(\mathbb{R}^N, \cdot_B)$ is an *inner product space* [14].

Let I_m be the identity matrix of order m and 0_m the null matrix of order m. Let us denote the matrix $\begin{Vmatrix} I_m & 0 & 0 \\ 0 & -I_l & 0 \\ 0 & 0 & 0_t \end{Vmatrix}$ by $I(m, l, t)$; in particular if $t = 0$, we will denote $I(m, l, 0)$ just by $I(m, l)$.

Example 0.1

Let consider $B = I_N$, *the inner product is just the usual* scalar product *on* $\mathbb{R}^N$*:*

$$x \; \cdot_{I_N} \; y \; = \; \sum_{i=1}^{N} \; x_i \; y_i.$$

This gives to $\mathbb{R}^N$ *the usual Euclidean (vector space) structure,* E^N, *so it will be written* $\cdot_E$ *instead of* $\cdot_{I_N}$.

Consider $x \in E^N$ then $x \cdot_E x > 0$, unless $x = 0$. A bilinear form B with this property is called *positive*; so the inner product $\cdot_B$. If instead, $x \cdot_B x < 0$ unless $x = 0$, the inner product is called *negative*.

In a positive inner product space we have the socalled *Schwarz inequality* [11, ch.9.5]

$$(x \cdot_B y)^2 \leq (x \cdot_B x)(y \cdot_B y).$$

Example 0.2

Let be $B = I(N-1, 1)$*; then*

$$x \; \cdot_{I(N-1,1)} \; y \; = \; \sum_{i=1}^{N-1} \; x_i \; y_i \; - \; x_N \; y_N$$

and $(\mathbb{R}^N, \cdot_{I(N-1,1)})$ *is the* pseudo-Euclidean vector space of signature $(N-1,1)$ [11] *usually denoted by* $E^{N-1,1}$.

Henceforward instead of the symbol $\cdot_{I(N-1,1)}$ *we will use the symbol* $\cdot_\Lambda$ *since the hyperbolic space is represented in* $E^{N-1,1}$, *as we will soon see.*

The inner product $\cdot_\Lambda$ *is found in relativistic studies, so terms there costumary will be also used:*

$x \ : \ x \cdot_\Lambda x > 0$ *is called spacelike*

$x \ : \ x \cdot_\Lambda x = 0$ *lightlike*

$x \ : \ x \cdot_\Lambda x < 0$ *timelike*

The inner product $\cdot_\Lambda$ is not positive. In fact

$$e_N \cdot_\Lambda e_N = -1 \text{ and even } (e_{N-1} + e_N) \cdot_\Lambda (e_{N-1} + e_N) = 0.$$

Inner products which admits both vectors with positive and negative "square" are called *indefinite.* Non-null vectors with null selfproduct, as $e_{N-1} + e_N$, are called *isotropic.*

The *B-lenght* of $x \in \mathbb{R}^N$ is $|x|_B = (|x \cdot_B x|)^{\frac{1}{2}}$. The vector x is a B-unit vector if $|x|_B = 1$ (this definition extends the one given by Ryan for $(\mathbb{R}^3, \cdot_\Lambda)$ in [17, ch. 7]).

Given two vectors in $\mathbb{R}^N$ they are said to be *orthogonal* relatively to $\cdot_B$ or B-orthogonal if $x \cdot_B y = 0$. The *B-orthogonal complement* for a subspace W of $\mathbb{R}^N$ is

$$W^{\perp_B} = \{x \in \mathbb{R}^N |\ x \cdot_B y = 0, \ \ \forall y \in W\}.$$

If $\cdot_B$ is positive then for each subspace W we have $W \cap W^{\perp_B} = \{0\}$. If the inner product is indefinite this is not always the case. A subspace $W \subset \mathbb{R}^N$ is said *isotropic* if there exists a non-null vector of W B-orthogonal to W, i.e. $W \cap W^{\perp_B} \neq \{0\}$. W is said *totally isotropic* if $\cdot_B|_W \equiv 0$ [3, §4]. A non-null vector is isotropic if and only if it generates an isotropic subspace or equivalently a totally isotropic subspace, as the two definitions coincide for 1-dimensional subspaces.

In examples 0.1 and 0.2, $\mathbb{R}^N$ is non-isotropic. This happens if and only if the determinant of the matrix B is non-degenerate. Then the inner product is said *non-degenerate.*

Consider an $(N-1)$-dimensional linear subspace H of $\mathbb{R}^N$ and its linear algebraic defining equation

$$a_1 x_1 \ + \ \dots \ + \ a_N x_N \ = \ 0, \qquad (a_1, ..., a_N) \neq (0, ..., 0).$$

In E^N the orthogonal direction to H is spanned by the vector $\mathbf{n_E} = (a_1, ..., a_N)$, while in $E^{N-1,1}$ the orthogonal direction is spanned by $\mathbf{n_\Lambda} = (a_1, ..., a_{N-1}, -a_N)$. (N.B. $\mathbf{n_E}$ and $\mathbf{n_\Lambda}$ are not necessarily unit vectors.)

A linear homomorphism ϕ of $\mathbb{R}^N$, which also preserves the inner product $\cdot_B$, is called a *metric homomorphism* [3, §4, n.3]:

$$\phi(x) \cdot_B \phi(y) = x \cdot_B y, \ \ \forall x, y \in \mathbb{R}^N.$$

As

$$x \cdot_B y = \frac{1}{2}((x+y) \cdot_B (x+y) - x \cdot_B x - y \cdot_B y)$$

the condition can be equivalently stated

$$\phi(x) \cdot_B \phi(x) = x \cdot_B x, \quad \forall x \in \mathbb{R}^N.$$

This happens if and only if the matrix Φ associated to ϕ, relatively to the fixed basis of $(\mathbb{R}^N, \cdot_B)$, is such that

$$\Phi^t \, B \, \Phi \; = \; B.$$

If $\cdot_B$ is non-degenerate each metric homomorphism is an isomorphism and will be called a *motion.*

Given any symmetric real matrix B of order N there exists a basis of $\mathbb{R}^N$ such that $\cdot_B$ expressed via this basis has matrix $I(index(B), rank(B) - index(B), N - rank(B))$. This basis gives a decomposition of $\mathbb{R}^N$ as an orthogonal direct sum $S_1 \oplus S_2 \oplus S_3$ where $(S_1, \cdot_B|_{S_1})$ is a positive subspace, $(S_2, \cdot_B|_{S_2})$ is a negative subspace and $(S_3, \cdot_B|_{S_3})$ is a totally isotropic subspace. The *Sylvester law of inertia* [11, ch.IX,§2], [14, ch.1-4] says that the dimensions of S_1, S_2 and S_3 are uniquely determined. It implies that, up to congruence, there are $N+1$ different non-degenerate inner product space structures on $\mathbb{R}^N$.

The objects I am interested in, the polytopes, either live in the n-dimensional *Euclidean (affine) space* E^n, or in the n-dimensional *sphere* S^n or in the n-dimensional *hyperbolic space* Λ^n. The spaces S^n and Λ^n will be briefly introduced hereinafter. To avoid any confusion each geometrical name will be prefixed with a letter to denote in which geometry it is situated, e.g. an S-line in S^2 is a great E-circle of S^2. The letter S will stand for spherical, E for Euclidean and the Cyrillic Λ of Lobačevskij for hyperbolic.

Let consider Euclidean geometry. The Euclidean space as a set of points is an affine space, i.e. no point singles out with special properties [8, ch. 1,7]. We can nevertheless treat points as vectors [15, ch.1]: the coordinates of a point x are the components of the vector from the origin to x. An Euclidean polytope is a set of points of the affine Euclidean space, its roots are elements of the Euclidean vector space.

Let consider spherical geometry. The set of points is S^n, the n-dimensional sphere contained in E^{n+1}:

$$S^n = \{x \in E^{n+1} | \; x \cdot_E x = 1\}.$$

The metric structure is that inherited from E^{n+1} and $\cdot_S$ is just $\cdot_E$. A hyperplane H in S^n is the intersection of S^n with a linear n-dimensional subspace $\mathcal{H}$ of E^{n+1}, i.e. an E-hyperplane through the origin. The E-orthogonal direction to $\mathcal{H}$ will be called the S-orthogonal direction of H. The S-dihedral angles formed by two hyperplanes H_1 and H_2 are the E-dihedral angles formed by $\mathcal{H}_1$ and $\mathcal{H}_2$.

Let now consider hyperbolic geometry: we will introduce the hyperboloid model of Λ^n. Consider the set

$$Q = \{x \in E^{n,1} | \; x \cdot_\Lambda x = -1\}.$$

It is an E-hyperboloid: its equation motivates the name *sphere of imaginary radius.* Q has two connected components

$$Q^+ = \{x \in Q|\ x_{n+1} > 0\} \qquad \text{and} \qquad Q^- = \{x \in Q|\ x_{n+1} < 0\}.$$

The upper sheet Q^+ (with the restriction of the inner product $\cdot_\Lambda$) is chosen as a model for the hyperbolic n-dimensional space Λ^n.

Let $x, y \in Q$. Then $x \cdot_\Lambda y < 0$ if and only if x and y belong to the same component of Q [11, 9.22], [16, ch.5,lemma 29].

A subspace W of $E^{n,1}$ is called *spacelike* [16, ch.5] [11] or *elliptic*[19] if the inner product restricted to W is positive; W is called *timelike* or *hyperbolic* if the inner product restricted to it is non-degenerate and of index one less the dimension; W is called *lightlike* or *parabolic* if the restricted inner product is degenerate. The Λ-orthogonal hyperplane of a timelike (spacelike) vector is spacelike (timelike) [16, ch.5, lemma 26]. Each isotropic direction of $E^{n,1}$ determines a point at infinity of Λ^n or as it is also called an ideal point of Λ^n. In the contrast the points of Q^+ will be called ordinary points.

A hyperplane H in Λ^n is the intersection of Q^+ with a linear n-dimensional hyperbolic subspace $\mathcal{H}$ of $E^{n,1}$. The Λ-orthogonal direction, in $E^{n,1}$, to $\mathcal{H}$ has the generating vector spacelike. I will called it the Λ-orthogonal direction of H. The angles formed by two hyperplanes H_1 and H_2, intersecting in Λ^n, are the Λ-angles formed by $\mathcal{H}_1$ and $\mathcal{H}_2$.

A motion of Λ^n is a motion of $E^{n,1}$ which preserves Q^+.

The hyperboloid model is isometric to the Poincaré disc model and to the upper half-space model [21, sec. 2.4].

The symbol X^n will stand for anyone of the three geometric spaces, when there will no distinction to be done. And the prefix X- or the X used as an index, will denote a concept or a symbol referred to the geometry in X^n.

Few lines else about notations on matrices and submatrices.

Let M be a matrix. Its entries will be either denoted by m_{ij} or $(M)_{ij}$. M^i_j will denote the submatrix obtained deleting the i-th row and the j-th column; $M^{i_1 i_2 \dots i_s}_{j_1 j_2 \dots j_s}$ the submatrix obtained deleting the rows $i_1, i_2, \dots, i_s$ and the columns $j_1, j_2, \dots, j_s$. $M^{[i_1 i_2 \dots i_s]}_{[j_1 j_2 \dots j_s]}$ will denote the submatrix obtained taking only the entries with row index $i_1, i_2, \dots, i_s$ and column index $j_1, j_2, \dots, j_s$. E.g. $M^{[i]}_{[j]} = m_{ij}$. $|M|$ will denote the determinant of M.

Part I
Polytopes and simplices

The two dimensional notion of *polygon*, bounded intersection of halfplanes, can be extended in any dimension to that of *polytope*, intersection of half-hyperspaces. In three dimensions a polytope is also called *polyhedron*.

The definition can be given in spherical, Euclidean and hyperbolic geometry.

The formal definition is

Definition I.1 *A* polytope *is a convex non-void region of the space X^n enclosed by a finite number of hyperplanes.*

The following special requirements are put

1) *Finite volume.*
2) *Each hyperplane determines an half-hyperspace which contains the polytope: none half-hyperspace contains the intersecion of the others.*

Every polytope has finite volume in spherical geometry. But in Euclidean case the finite volume request implies that the polytope is bounded. In the hyperbolic case there are allowed vertices at infinity. The given definition does not exclude, in spherical case, that a polytope contains antipodal points.

The second requirement will be referred to as *the economical choice*, since we consider the minimum number of hyperplanes needed to determine the polytope.

Let us consider the generalized notion of triangle:

Definition I.2 *A* simplex *is a polytope of X^n enclosed by $n+1$ hyperplanes such that each hyperplane does not contain the intersection of the remaining n.*

1 Hyperplanes, roots and angles

Let denote by H_i, $i = 1, ..., N$, the hyperplanes bounding the polytope $\mathcal{P} \subset X^n$.

Then $H_i \cap \mathcal{P}$ is an $(n-1)$-*face* of the polytope $\mathcal{P}$. Suppose $s \geq 1$, then $\bigcap_{i=1}^{s} H_i \cap \mathcal{P}$, if non-void, is an $(n-s)$-face of $\mathcal{P}$. The term *face* will denote an $(n-1)$-face, the term *edge* a 1-face and the term *vertex* a 0-face.

Consider for H_i its X-orthogonal direction and let it be generated by v_i. Then $v_i \cdot_X v_i > 0$ and

$$H_i = \{x \in X^n |\ x \cdot_X v_i = d_i\}, \qquad \text{for some real number } d_i,$$

in the case $X^n = S^n$ or Λ^n, $d_i = 0$, $i = 1, ..., N$. In the sequel we wil suppose $v_i \cdot_X v_i = 1$.

H_i determines two closed half-hyperspaces

$$H_i^- = \{x \in X^n |\ x \cdot_X v_i \leq d_i\} \quad \text{and} \quad H_i^+ = \{x \in X^n | x \cdot_X v_i \geq d_i\}.$$

The polytope $\mathcal{P}$ is contained either in H_i^- or in H_i^+. Suppose v_i has been chosen such that $\mathcal{P} \subset H_i^-$.

The finite set of vectors $\{v_i\}_{i=1,...,N}$ is called the *system of roots* or *root system* of $\mathcal{P}$.

Note I.3 *It is to be mentioned that in other contexts where root systems are introduced, e.g. in Lie groups context [4], the roots have to verify integral conditions so they are no more chosen of unit length; the root system is supposed to generate the whole space and to be symmetric, so each vector belongs to the root system with its opposite.*

The intersection, in E^3, of three halfspaces, whose roots form a basis, is called *trihedron.* It is an unbounded polyhedron. More generally, a *polyhedral angle* is an unbounded polyhedron obtained as the intersection of more than three halfspaces, such that the bounding planes have a (but only one) common point and the root system contains a basis. I give the following

Definition I.4 *Consider n hyperplanes in E^n, such that their roots form a basis. Choose for each hyperplane one of the two hyper-halfspaces it determines. The intersection of the n hyper-halfspaces is an* n-hedron. *If we have more hyperplanes than the dimension, then the intersection might be void. If non-void we will call it a* polytopal angle.

Let $\mathcal{P} = \bigcap_{i=1}^{n} H_i^-$ be an n-hedron in E^n. We want $\mathcal{P}$ to be non-void. Then the lines containing the edges generate the whole space E^n. Choose a (non null) vector f_i on each edge $\bigcap_{k\neq i}^{1,n} H_k \cap \mathcal{P}$, $i = 1, ..., n$. Then $\{f_i\}_{i=1,...,n}$ is a basis of E^n. Let v_i be the root of $\mathcal{P}$ corresponding to the hyperplane H_i:

$$v_i \cdot_E f_k = 0, \quad k \neq i, \ 1 \leq k \leq n, \quad \text{and} \quad v_i \cdot_E f_i \neq 0.$$

Then the roots are linearly independent: given a null combination, multiplication with f_i shows that the i-th coefficient is null, $i = 1, ..., n$. So they form also a basis of E^n. In the case $\mathcal{P}$ is a polytopal angle the system of roots have more than n vectors, but contains in any case a basis of E^n.

If $X^n = S^n$ or Λ^n a hyperplane H is obtained intersecting the model of the space with a subspaces $\mathcal{H}$ of E^{n+1} respectively $E^{n,1}$. In the Euclidean case consider $\mathcal{H}$ to be the translated of H through the origin of E^n.

Proposition I.5 *Let $\mathcal{P} = \bigcap_{i=1}^{N} H_i^-$. Then the root system of the polytope $\mathcal{P} \subset S^n$ (Λ^n respectively E^n) contains a basis of E^{n+1} ($E^{n,1}$ respectively E^n) if and only if $\bigcap_{i=1}^{N} \mathcal{H}_i = \{0\}$.*

Proof The orthogonal of the null subspace is the whole space and

$$(\{0\})^{\perp_X} = (\bigcap_{i=1}^{N} \mathcal{H}_i)^{\perp_X} = \left\langle \bigcup_{i=1}^{N} (\mathcal{H}_i)^{\perp_X} \right\rangle = \left\langle \bigcup_{i=1}^{N} \langle v_i \rangle \right\rangle.$$

Then $\{v_i\}_{i=1,...,N}$ contains a basis of the respective inner product space if and only if $\bigcap_{i=1}^{N} \mathcal{H}_i = \{0\}$.

■

There are polytopes which do not satisfy the condition, e.g. the $(1, r, r)$ triangles cited in the introduction. But the condition is satisfied by polytopes in Euclidean and hyperbolic space (remember that for a polytope we mean a finite volume one).

A polytope $\mathcal{P}$ is called *non-degenerate* [20] if $\bigcap_{i=1}^{N} \mathcal{H}_i = \{0\}$, i.e. equivalently if the root system contains a basis. This is enough since we have the finite volume condition on polytopes. In general it is necessary to ask that the bounding hyperplanes H_i have no common ordinary point nor (in the hyperbolic case) a common point at infinity.

Each n-simplex in X^n is non-degenerate.

A spherical degenerate polytope contains at least a couple of antipodal points. In hyperbolic case it has to be considered the character of $\bigcap_{i=1}^{N} \mathcal{H}_i$: it can be spacelike, lightlike or timelike. If it is spacelike the polytope has a face at infinity so infinite volume. If it is lightlike or timelike all the hyperplanes has a common point at infinity or at least an ordinary common point, it follows that it has infinite volume (the situation can be pictured in the upper halfspace model).

The matrix

$$G_v = (v_i \cdot_X v_j)_{i,j=1,\ldots,N}$$

is real symmetric and it is called the *Gram matrix* of $\mathcal{P}$. Its determinant is called the *Gramian* of $\mathcal{P}$.

Corollary I.6 *If the polytope $\mathcal{P}$ is non-degenerate then its Gram matrix G admits a non-degenerate submatrix of order $n+1$, if $X^n = S^n$ or Λ^n, and of order n if $X^n = E^n$.*

Proof By the previous proposition the root system contains a basis of the respective space. Consider the submatrix of the G which contains only the inner products of the elements of the basis. This submatrix is just the Gram matrix of the basis. And the Gramian of a set of linearly independent vectors is non null [prop. III.1].

■

Given a polytope $\mathcal{P}$ in X^n, let H_i and H_j, $i \neq j$, be two bounding hyperplanes. Then H_i and H_j are *adjacent* in $\mathcal{P}$ if their intersection contains an $(n-2)$-face L_{ij} of $\mathcal{P}$. In such a case the dihedral angle they form and which contains $\mathcal{P}$, is the *dihedral angle* α_{ij} of $\mathcal{P}$ at L_{ij}.

Let $V = \bigcap_{s=1}^{n} H_{i_s}$ be a vertex of $\mathcal{P}$. Then V is the vertex of an X-polytopal angle containing $\mathcal{P}$: $\bigcap_{i=1}^{n_V} H_i^-$, where $n_V \geq n$. Consider an $(n-1)$-X-sphere Σ centered at V of small enough radius, such that no other vertex of $\mathcal{P}$ belongs to Σ. Then $\bigcap_{s=1}^{n_V} H_{i_s}^- \cap \Sigma$ is a polytope on Σ, which has the same dihedral angles of the unbounded polytopal angle $\bigcap_{s=1}^{n_V} H_{i_s}^-$: recall that an X-sphere is X-orthogonal to the X-lines through its center. This polytope is called the *vertex polytope* at V. If V is an ordinary point then the vertex polytope is spherical. If V is a point at infinity (hyperbolic case) the vertex polytope is Euclidean, since an horosphere is isometric to the Euclidean space of same dimension.

Suppose that the vertex polytope at each vertex V of $\mathcal{P}$ is a simplex. Then $\mathcal{P}$ is called *polytope of simplicial type* [1]. The simpler example of polytope of simplicial type is a simplex. There are three Platonic solids of simplicial type: tetrahedron, cube, dodecahedron.

For dimensional reasons the minimum number of hyperplanes that intersect at a vertex is n, giving rise to at least n edges concurrent at the vertex. This minimum is attained at each vertex of a polytope of simplicial type:

Proposition I.7 *In a polytope of simplicial type $\mathcal{P} \subset X^n$, at each vertex the number of concurrent edges is equal to the dimension n.*

Proof Each vertex polytope is an $(n-1)$-simplex and it has n vertices. The edges of $\mathcal{P}$ concurrent to a vertex are in bijective correspondence with the vertices of the vertex polytope, so are in number of n.

■

This does not happen in the octahedron and in the icosahedron (4 respectively 5 edges concurrent at each vertex).

2 Non-obtuse-angled polytopes

A polytope is called *non-obtuse-angled* if all its dihedral angles are acute or right, *acute-angled* [5] if all its dihedral angles are acute.

The non-obtuse-angled polytopes are constrained by the geometry in which they live to assume particular forms [5, thm.1,thm.2]:

Theorem I.8 *A spherical non-obtuse-angled polytope is a simplex.*

Proof The proof goes by induction, with base $n = 2$: every non-obtuse-angled spherical polygon is a triangle.

Consider a non-obtuse-angled spherical polygon $\mathcal{P}$ with $N \geq 3$ vertices and let V be one of this. Join V, with S-segments, to all vertices but its adjacents. In such a way we obtain $N-2$ S-triangles. The sum of the angles of $\mathcal{P}$ is just the sum of the angles of all the $N-2$ (sub)triangles. On S^2 the angle sum of a triangle is greater than π. So the angle sum of $\mathcal{P}$ is greater than $(N-2)\pi$. The polygon $\mathcal{P}$ has N angles; if all were less than or equal to $\frac{N-2}{N}\pi$ a contradiction will follow. So at least one, say α is greater than $\frac{N-2}{N}\pi$. Then by the non-obtuse-angled hypothesis made on $\mathcal{P}$ we have that

$$\frac{N-2}{N}\pi < \alpha \leq \frac{\pi}{2}$$

which implies

$$N < 4.$$

As $N \geq 3$, it follows $N = 3$.

Now assume the thesis true in dimension $n-1$.

Let $\mathcal{P} \subset S^n$ be the non-obtuse-angled polytope. Then by inductive hypothesis, $\mathcal{P}$ is of simplicial type. We want to prove that each 2-face is a triangle.

Suppose $n = 3$. There are three S-planes meeting at each vertex and forming a trihedral angle. The vertex polytope $\mathcal{P}'$ at V is an S-triangle. Denote by α, β, γ its angles and by a, b, c its sides. The angles α, β, γ coincide with dihedral angles of $\mathcal{P}$; whereas a, b, c are (dihedral) angles of 2-faces of $\mathcal{P}$. By the Law of Cosines in spherical geometry, e.g. [17, Ch.4,thm.38],

$$\cos\gamma = \sin\alpha\sin\beta\cos c - \cos\alpha\cos\beta.$$

Recall that $0 \leq \alpha, \beta \leq \frac{\pi}{2}$; then $0 \leq \sin\alpha, \sin\beta, \cos\alpha, \cos\beta \leq 1$. So

$$\cos\gamma \leq \cos c, \quad \text{and a priori we know that} \quad 0 \leq \gamma, c < \pi.$$

The cosinus function is decreasing in $[0,\pi]$ and $\gamma \leq \frac{\pi}{2}$ then

$$\frac{\pi}{2} \geq \gamma \geq c.$$

Similarly, from

$$\cos\alpha = \sin\beta\sin\gamma\cos a - \cos\beta\cos\gamma$$

and

$$\cos\beta = \sin\alpha\sin\gamma\cos b - \cos\alpha\cos\gamma$$

it follows

$$a, b \leq \frac{\pi}{2}.$$

Suppose now be $n \geq 4$. We want to reduce to the previous case. Let us denote by $\mathcal{H}_i$ the hyperplane in E^{n+1} such that $H_i = S^n \cap \mathcal{H}_i$. Consider an $(n-3)$-face

$$L_{i,j,k} = H_i \cap H_j \cap H_k \cap \mathcal{P}$$

and the $(n-2)$-dimensional Euclidean subspace

$$\mathcal{L}_{i,j,k} = \mathcal{H}_i \cap \mathcal{H}_j \cap \mathcal{H}_k.$$

Let W be an internal point of $L_{i,j,k}$. Consider the roots v_i, v_j, v_k and the space they generate in E^{n+1} together with the vector from the origin to W:

$$\mathcal{W}_{i,j,k} = \langle v_i, v_j, v_k, \vec{OW}\rangle \text{ and } \quad W_{i,j,k} = \mathcal{W}_{i,j,k} \cap S^n.$$

Then $\mathcal{L}_{i,j,k}$ is an $(n-2)$-dimensional subspace of E^{n+1} orthogonal to the 3-dimensional subspace $\langle v_i, v_j, v_k\rangle$ and $\mathcal{L}_{i,j,k} \cap \langle v_i, v_j, v_k\rangle = \{0\}$. It follows that $\mathcal{L}_{i,j,k} \cap \mathcal{W}_{i,j,k} = \langle \vec{OW}\rangle$ so $L_{i,j,k} \cap W_{i,j,k} = \{W\}$ and $L_{i,j,k}$ is an $(n-3)$-dimensional S-hyperplane S-orthogonal to the 3-dimensional S-hyperplane $W_{i,j,k}$.

The Grassmann relation applied to $\mathcal{H}_i$ and $\mathcal{W}_{i,j,k}$ gives

$$dim\,(\mathcal{H}_i \cap \mathcal{W}_{i,j,k}) = dim\,\mathcal{H}_i + dim\,\mathcal{W}_{i,j,k} - dim\,(\mathcal{H}_i \cup \mathcal{W}_{i,j,k}) = n + 4 - (n+1) = 3$$

so $dim\,(H_i \cap W_{i,j,k}) = 2$. Also $dim\,(H_j \cap W_{i,j,k}) = 2$ and $dim\,(H_k \cap W_{i,j,k}) = 2$. Then in the intersection with $W_{i,j,k}$ the three hyperplanes H_i, H_j, H_k bound a trihedral angle. The dihedral angles are preserved, since we have intersect by an S-orthogonal 3-dimensional S-hyperplane. The previous conclusions obtained for $n = 3$ can be applied.

Then the 2-faces of $\mathcal{P}$ are non-obtuse-angled and so S-triangles.

Let V_0 be a vertex of $\mathcal{P}$. There are n edges concurrent to V_0 since $\mathcal{P}$ is of simplicial type [prop. I.7]; denote $V_1, ..., V_n$ all the vertices of $\mathcal{P}$ joined to V_0 by edges. The 2-faces through V_0V_1 are:

$$V_0V_1V_2, ..., V_0V_1V_n.$$

Among the edges concurrent at V_1, there are

$$V_1V_0,\ V_1V_2, ...,\ V_1V_n;$$

there cannot be others as the number of n is already reached. We can find in the same way that the edges concurrent at any other vertex V_i are only

$$V_iV_0,\ V_iV_1, ...,\ V_iV_{i-1},\ V_iV_{i+1}, ...,\ V_1V_n.$$

So $\mathcal{P}$ has $n+1$ vertices and is so an n-simplex.

■

Note that an n-X-sphere

$$\Sigma_{x_0,r} = \{x \in X^{n+1} |\ d_X(x, x_0) = r\}, \quad r > 0$$

is a model for spherical geometry, so theorems proved for polytopes in S^n are valid in any n-X-sphere.

Corollary I.9 *Each non-obtuse-angled polytope with ordinary vertices is of simplicial type.*

Proof Each vertex polytope is spherical and non-obtuse-angled, so by the theorem above it is a simplex.

■

Let now consider the Euclidean case.

Lemma I.10 *An Euclidean acute-angled polytope is a simplex.*

Proof Let proceed by reductio ad absurdum: consider an Euclidean polytope $\mathcal{P} \subset E^n$ that is not a simplex. The idea consists in projecting the polytope onto an n-sphere of sufficiently large radius (consider in E^{n+1} an osculating n-sphere to E^n in a vertex of the polytope; project as in the stereographic projection but from the center of the sphere. In this way hyperplanes bounding $\mathcal{P}$ are mapped to S-hyperplanes). The image is a spherical polytope whose dihedral angles are as close as it is need to those of $\mathcal{P}$. Choose the radius so large that the image results a non-obtuse-angled spherical polytope and the previous theorem gives the absurd.

■

A polytope which is the product of mutually orthogonal simplices, with one common vertex, is called *simplicial prism*. (By definition also a simplex is considered to be a simplicial prism.) The "factors" of this product are called *constituents*. The symbol $\mathcal{P} = [\mathcal{P}_1, \mathcal{P}_2, ..., \mathcal{P}_s]$ will denote that $\mathcal{P}$ is the product of $\mathcal{P}_1$,...,$\mathcal{P}_s$. The k-simplex will be denoted by $\triangle^k$.

Theorem I.11 *An Euclidean non-obtuse-angled polytope is a simplicial prism.*

Proof Let $\mathcal{P} \subset E^n$ and N be the number of bounding hyperplanes.

In dimension 2, just follow the proof of spherical case: now the angle sum is equal to $(N-2)\pi$. Then $3 \leq N \leq 4$. If $N = 3$ then $\mathcal{P}$ is a simplex $\triangle^2$; if $N = 4$ then $\mathcal{P}$ has all angles right and is a rectangle $[\triangle^1, \triangle^1]$.

In dimension greater or equal to 3 the property of being non-obtuse-angled transfers as in the spherical case to the faces of $\mathcal{P}$: in fact, let $n = 3$, an angle of a 2-face is the side of some vertex simplex; let $n > 3$, we consider once again the vertex simplices. They are spherical, so we can proceed as in the proof of the previous theorem and reduct to the 3-dimensional case. So the 2-faces of $\mathcal{P}$ are Euclidean non-obtuse-angled polygons, so either triangles or rectangles.

Suppose $n = 3$.

The polytope $\mathcal{P}$ is non-obtuse-angled, so of simplicial type [cor. I.9] and at each vertex there are 3 concurrent edges [prop. I.7]. Let f_i denote the number of faces with i sides. Then the number of edges is $\frac{1}{2}(3f_3+4f_4)$ as each edge is the side of two faces. And the number of vertices is $\frac{1}{3}(3f_3+4f_4)$ as each vertex belongs to three faces. A substitution in the Euler formula for polyhedra:

$$\text{number of vertices} \; - \; \text{number of edges} \; + \; \text{number of faces} \; = 2$$

leads

$$\frac{1}{2}f_3 + \frac{1}{3}f_4 = 2, \quad \text{with } \frac{1}{2}f_3, \frac{1}{3}f_4 \text{ non-negative integers.}$$

An arithmetical analysis shows that there are only 3 possibilities $(f_3, f_4) = (4,0)$ or $(2,3)$ or $(0,6)$ which correspond respectively to the tetrahedron $\triangle^3$, the triangular prism $[\triangle^2, \triangle^1]$ and the rectangular solid $[\triangle^1, \triangle^1, \triangle^1]$.

Suppose $n \geq 4$.

Each bounding $(n-1)$-face is non-obtuse-angled, just by the same considerations which prove us that the 2-faces are non-obtuse-angle. So by inductive hypothesis an $(n-1)$-bounding face is a simplicial prism: it decomposes in the product of one or more simplices. Let fix attention on a vertex V_0; its vertex polytope is an $(n-1)$-simplex $\mathcal{P}'$. The 1-skeleton of $\mathcal{P}'$ -the vertex 1-skeleton- is a complete graph on n vertices K_n, a socalled n-clique. Let $\mathcal{F}$ be an $(n-1)$-face containing V_0. Then $\mathcal{F}$ intersects $\mathcal{P}'$ in an $(n-2)$-simplex $\mathcal{F}'$: let colour in red the edges of the 1-skeleton of $\mathcal{F}'$ that are contained in the constituent factors of $\mathcal{F}$. Let colour in blue the remaining edges: they correspond to right angles of bounding 2-faces. So they remain blue taking the spanned subgraph on any other set of $n-1$ vertices of $\mathcal{P}'$ and repeating the procedure of colouring according to another $(n-1)$-face of $\mathcal{P}$ at V_0. This means that the colouring is consistent. Moreover it is only necessary to fix colours on three (but not less than three) different $(n-1)$-cliques of the vertex 1-skeleton, to colour all the edges. Take now an edge V_0V_1 of $\mathcal{P}$. Each $(n-1)$-face through V_0V_1 allows colours to travel from one $(n-1)$-clique of the 1-skeleton of $\mathcal{P}'_0$ to an $(n-1)$-clique of the 1-skeleton of $\mathcal{P}'_1$. The $(n-1)$-faces are in number of $n-1$. Since $n-1 \geq 3$, the coloured vertex 1-skeleton is the same at each vertex and characterizes $\mathcal{P}$. So the assumption $n \geq 4$, i.e. $n-1 \geq 3$, turns out to be crucial to prove that the same colouring occurs at each vertex 1-skeleton.

Note that from the previous analysis, the coloured 1-skeletons in dimension 3 are: *red-red-red* (tetrahedron), *red-blue-blue* (triangular prism) and *blue-blue-blue* (rectangular solid).

To conclude the inductive step, it remains to prove that the red connected components of the coloured vertex 1-skeleton are cliques. Consider the 1-skeleton of $\mathcal{P}'$. Remove a vertex of it, say v. Then the red connected components of the complete subgraph on all vertices of $\mathcal{P}'$ but v, are cliques as by inductive hypothesis each $(n-1)$-face is a simplicial prism. If all edges concurrent in v are blue, there is nothing to prove. If v is connected by one red edge to a red component, then it must be red-connected to each vertex of this component, since otherwise there would be a 3-cycle coloured *red-red-blue* which is impossible as observed above. There can be a red edge from v to a 1-vertex (red) component. So in any case the red components of the vertex 1-skeleton are cliques and they single out the constituents of $\mathcal{P}$.

N.B. Each K_{s+1} determines an s-simplex $\triangle^s$ constituent of $\mathcal{P}$.

■

Corollary I.12 *The number N of bounding hyperplanes for an Euclidean non-obtuse-angled polytope $\mathcal{P} \subset E^n$ is between*

$$n+1 \leq N \leq 2n.$$

Proof Let $\mathcal{P} = [\triangle^p, \triangle^q]$, $\triangle^i \subset E^i$. Where $E^p \times E^q = E^n$ and $n = p + q$. An $(n-1)$-face of $\mathcal{P}$ is obtained either from the product of a $(p-1)$-face of $\triangle^p$ and $\triangle^q$ or from the product of $\triangle^p$ and a $(q-1)$-face of $\triangle^q$. So they are in number of $N = (p+1) + (q+1) = p + q + 2$. In general let $\mathcal{P} = [\triangle^{n_1}, ..., \triangle^{n_s}]$, $n_i \geq 1$, then $n_1 + ... + n_s = n$ and $N = n_1 + ... + n_s + s = n + s$. The number N increases with s, $1 \leq s \leq n$. So $N \geq n+1$ (n-simplex) and $N \leq 2n$ (rectangular hypersolid $[\triangle^1, ..., \triangle^1]$).

■

The Gram matrix of an non-obtuse-angled polytope cannot have arbitrary entries. Since the roots are chosen unitary the entries of the principal diagonal are all 1.

Proposition I.13 *The Gram matrix of a non-obtuse-angled polytope has all entries off the diagonal non-positive.*

Proof Let $\mathcal{P} = \bigcap_{i=1}^N H_i^-$ be the non-obtuse-angled polytope, $\{v_i\}$ its root system and α_{ij} the dihedral angles.

Let X^n be E^n: then if H_i and H_j are adjacent they form a non-obtuse angle α_{ij}, $0 < \alpha \leq \frac{\pi}{2}$. The angle between v_i and v_j is the supplementary of α_{ij} as v_i and v_j are orthogonal to the faces and external to the dihedral angle of the polytope:

$$v_i \cdot_E v_j = -\cos(\alpha_{ij}) \leq 0.$$

It may also happen that two hyperplanes are parallel; by the economical choice of hyperplanes and since the polytope is not a void set we have that $H_i^- \cap H_j^- \neq \emptyset$ and $H_i^+ \cap H_j^+ = \emptyset$. So v_i and v_j are parallel but opposite:

$$v_i \cdot_E v_j \;=\; -1.$$

Let X^n be S^n: then the angles of the spherical simplex are the angles of the polytopal angle centered at the origin of E^{n+1} and the root systems coincide. Then we conclude by the Euclidean case.

Let X^n be Λ^n: remember that Λ^n is represented in $E^{n,1}$ by the hyperboloid upper sheet Q^+. The hyperplanes H_i and H_j are obtained intersecting Q^+ with the n-dimensional linear subspaces $\mathcal{H}_i$ and $\mathcal{H}_j$. Then $\mathcal{H}_i \cap \mathcal{H}_j$ is an $(n-1)$-dimensional subspace of $E^{n,1}$ and $\langle v_i, v_j \rangle$ is its Λ-orthogonal complement. There can occur three possibilities according to the fact that $\mathcal{H}_i \cap \mathcal{H}_j$ can be hyperbolic, parabolic or elliptic.

$\mathcal{H}_i \cap \mathcal{H}_j$ **hyperbolic** H_i and H_j intersect in an $(n-2)$-dimensional hyperplane of Λ^n (are adjacent faces) and v_i, v_j generate an elliptic (spacelike) plane of $E^{n,1}$. The restriction of the inner product to a spacelike subspace is positive definite. Then the Schwarz inequality is available, so $|v_i \cdot_\Lambda v_j| \leq 1$. The Λ-angle formed by v_i and v_j is supplementary to the Λ-dihedral angle α_{ij}

$$v_i \cdot_\Lambda v_j = -\cos(\alpha_{ij}) \leq 0.$$

$\mathcal{H}_i \cap \mathcal{H}_j$ **parabolic** H_i and H_j are parallel; let w be a non null vector generating the isotropic direction of $\mathcal{H}_i \cap \mathcal{H}_j$; then $\langle w \rangle \subset \langle v_i, v_j \rangle$; v_i, v_j are Λ-orthogonal to w and stand in different halfplanes determined by $\langle w \rangle$: then $v_i + v_j$ is isotropic which implies that

$$v_i \cdot_\Lambda v_j \;=\; -1.$$

$\mathcal{H}_i \cap \mathcal{H}_j$ **elliptic** the hyperplanes H_i and H_j are not adjacent, they do not intersect and are ultraparallel. The 2-dimensional subspace $\langle v_i, v_j \rangle$ is hyperbolic (timelike) and intersect Q^+ in the common perpendicular to H_i and H_j. Then by a formula relating the inner product to geometrical quantities [6, ch.10], [19, ch.2] we have that

$$v_i \cdot_\Lambda v_j = -\cosh(d_\Lambda(H_i, H_j)) < 0.$$

■

We are primarily interested in n-simplices in X^n, $n \geq 2$. (A 0-simplex is just a point; a 1-simplex is a segment.) The root system of an n-simplex has $n+1$ vectors, and its Gram matrix is an $(n+1) \times (n+1)$ square matrix.

Before stating general results we will consider in dimension 2 three propositions, which are provable by elementary techniques.

Proposition I.14 *Let $\mathcal{P}$ be a non-obtuse-angled spherical 2-simplex. Then its Gram matrix G is positive definite.*

Proof Let α, β and γ denote the angles of $\mathcal{P}$. The positivity of the Gram matrix will follow from the inequality about angle sum $\alpha + \beta + \gamma > \pi$.

$$G = \left\| \begin{array}{rrr} 1 & -\cos\alpha & -\cos\beta \\ -\cos\alpha & 1 & -\cos\gamma \\ -\cos\beta & -\cos\gamma & 1 \end{array} \right\| .$$

The matrix G is positive definite if and only if its leading principal minors are positive [2, ch.17]. There are 3 principal minors to check:

$$\begin{aligned} |G^{[1]}_{[1]}| &= 1; \\ |G^{[12]}_{[12]}| &= \left| \begin{array}{rr} 1 & -\cos\alpha \\ -\cos\alpha & 1 \end{array} \right| = 1 - (-\cos\alpha)^2 = \sin^2\alpha; \\ |G^{[123]}_{[123]}| &= |G| = \\ &= 1 - 2\cos\alpha\cos\beta\cos\gamma - \cos^2\alpha - \cos^2\beta - \cos^2\gamma \end{aligned}$$

The first two minors are obviously positive. Let us prove that this is so also for the third.

The 2-simplex $\mathcal{P}$ is spherical non-obtuse-angled; so its angles α, β and γ fulfil the following inequalities:

$$\begin{aligned} \alpha, \beta, \gamma &\leq \frac{\pi}{2} \\ \alpha + \beta + \gamma &> \pi \end{aligned}$$

$$\text{then} \qquad \frac{\pi}{2} \geq \alpha > \pi - (\beta + \gamma) \geq 0.$$

Applying the cosine function, decreasing on $[0, \frac{\pi}{2}]$, to the above inequalities we have

$$0 \leq \cos\alpha < \cos(\pi - (\beta+\gamma)) = -\cos(\beta+\gamma) < 1$$

$$\text{and} \qquad 0 \leq \cos^2\alpha < (-\cos(\beta+\gamma))^2 < 1$$

so $\quad -\cos^2\alpha > -\cos^2(\beta+\gamma) \quad$ and $\quad -\cos\alpha > \cos(\beta+\gamma).$
It follows that

$$\begin{aligned}
|G| &> 1 + 2\cos(\beta+\gamma)\cos\beta\cos\gamma - \cos^2(\beta+\gamma) - \cos^2\beta - \cos^2\gamma \\
&= 1 + 2(\cos\beta\cos\gamma - \sin\beta\sin\gamma)\cos\beta\cos\gamma - (\cos\beta\cos\gamma - \sin\beta\sin\gamma)^2 \\
&\quad - \cos^2\beta - \cos^2\gamma \\
&= 1 + 2\cos^2\beta\cos^2\gamma - 2\sin\beta\sin\gamma\cos\beta\cos\gamma - \cos^2\beta\cos^2\gamma \\
&\quad + 2\sin\beta\sin\gamma\cos\beta\cos\gamma - \sin^2\beta\sin^2\gamma - \cos^2\beta - \cos^2\gamma \\
&= 1 + \cos^2\beta\cos^2\gamma - \sin^2\beta\sin^2\gamma - \cos^2\beta - \cos^2\gamma \\
&= 1 + \cos^2\beta\cos^2\gamma - (1-\cos^2\beta)(1-\cos^2\gamma) - \cos^2\beta - \cos^2\gamma \\
&= 1 + \cos^2\beta\cos^2\gamma - 1 + \cos^2\beta + \cos^2\gamma - \cos^2\beta\cos^2\gamma - \cos^2\beta - \cos^2\gamma \\
&= 0.
\end{aligned}$$

■

Proposition I.15 *Let $\mathcal{P}$ be a non-obtuse-angled Euclidean 2-simplex. Then its Gram matrix G is degenerate of rank and index 2.*

Proof Do the same computations as above, but substitute inequalities by equalities -in Euclidean case $\alpha + \beta + \gamma = \pi$ -.
■

Proposition I.16 *Let $\mathcal{P}$ be a non-obtuse-angled hyperbolic 2-simplex. Then its Gram matrix is non-degenerate of index 2.*

Proof The angle sum for a hyperbolic triangle is less than π:

$$\alpha + \beta + \gamma < \pi$$

Suppose there exist at least two angles, say β and γ, with sum $\geq \frac{\pi}{2}$. Then

$$0 < \alpha < \pi - (\beta+\gamma) \leq \frac{\pi}{2}$$

and the previous computations apply with reversed inequality signs.
If this does not happen, the above procedure fails as

$$\cos(\beta+\gamma) < 0.$$

But observe that in this case $\beta+\gamma<\frac{\pi}{2}$, $\quad \alpha+\gamma<\frac{\pi}{2}$ then

$$\beta<\frac{\pi}{2}-\gamma,\ \alpha<\frac{\pi}{2}-\gamma$$

$$\cos\beta>\cos(\frac{\pi}{2}-\gamma)\ =\ \sin\gamma\ >0,\qquad \cos\alpha>\cos(\frac{\pi}{2}-\gamma)\ =\ \sin\gamma\ >\ 0$$

$$\cos\beta\cos\alpha>\sin^2\gamma$$

$$-\cos^2\alpha<-\sin^2\gamma,\qquad -\cos^2\beta<-\sin^2\gamma,\qquad -\cos\alpha\cos\beta<-\sin^2\gamma$$

and

$$\begin{aligned} |G| &= 1+2(-\cos\alpha\cos\beta)\cos\gamma+(-\cos^2\alpha)+(-\cos^2\beta)-\cos^2\gamma \\ &< 1-2\sin^2\gamma\cos\gamma-\sin^2\gamma-\sin^2\gamma-\cos^2\gamma \\ &= -2\sin^2\gamma\cos\gamma-\sin^2\gamma \\ &= -\sin^2\gamma(2\cos\gamma+1)<0 \end{aligned}$$

So G is non-degenerate and has index 2.

∎

Let now return to the general case and first of all to the spherical one.

Proposition I.17 *The Gram matrix of a spherical non-obtuse-angled n-simplex is positive definite.*

Proof The root system of a spherical simplex $\mathcal{P}\subset S^n$ is a collection of $n+1$ vectors of E^{n+1}. As we have already seen in proposition I.5, the root system contains a basis. In this case the root system itself is a basis of E^{n+1}. So the Gram matrix of the simplex $\mathcal{P}$ is the Gram matrix of a basis. Then it is congruent to the Gram matrix of any other basis in the given inner product $\cdot_E$ [prop. III.3]. In particular it is congruent to I_{n+1} the Gram matrix of the standard basis of E^{n+1} and therefore positive definite.

∎

Proposition I.18 *Let G be the Gram matrix of a non-obtuse-angled n-simplex $\mathcal{P}$. If $\mathcal{P}$ is Euclidean then $|G|=0$. If $\mathcal{P}$ is spherical then $rank(G)=index(G)=n+1$. If $\mathcal{P}$ is hyperbolic then $rank(G)=n+1$ but $index(G)=n$.*

Proof

If $\mathcal{P}\subset E^n$ the $n+1$ roots are $n+1$ vectors of an n-dimensional Euclidean space, so linearly dependent. The Gramian of a set of linearly dependent vectors is null: $|G|=0$ [prop.s III.1 and III.2]. The simplex is bounded. Let V_i be the vertex opposite to the i-th face H_i

$$V_i=\bigcap_{k,k\neq i}^{1,n+1} H_k,\quad i=1,...,n+1.$$

The vertex polytope at each V_i is a spherical $(n-1)$-simplex, and its Gram matrix is a submatrix of the Gram matrix of the simplex $\mathcal{P}$. So by proposition I.17 the rank and the index of G are at least n.

If $\mathcal{P}\subset S^n\subset E^{n+1}$ we conclude by proposition I.17.

If $\mathcal{P}\subset\Lambda^n\subset E^{n,1}$ the $n+1$ roots form a basis of $E^{n,1}$ then G is congruent to the Gram matrix of the standard basis of $E^{n,1}$ [prop. III.3], so $rank(G)=n+1$ and $index(G)=n$.

∎

Lemma I.19 *Let G be the Gram matrix of a non-obtuse-angled $(n-1)$-simplex. Then the cofactor of each entry is non-negative.*

Proof [5, lemma 9.1]

We will proceed by induction. Start with the simply case of a 1-simplex: $n = 2$. Then $|G_1^2| = |G_2^1| = a_{12} \leq 0$, so $(-1)^{1+2}|G_1^2| \geq 0$.

Suppose now the assertion to be true for each s-simplex $s \leq n-2$: we want to prove it for an $(n-1)$-simplex $\mathcal{P}$. The Gram matrix of $\mathcal{P}$ has order n. Each principal submatrix of order $n-1$ is positive semi-definite since it is the Gram matrix of a vertex polytope. So $|G_i^i| \geq 0, \quad i = 1, ..., n$. Apply the inductive hypothesis to the submatrix G_n^n of order $n-1$:

$$\text{then} \qquad (-1)^{i+j} G_{j,n}^{i,n} \geq 0, \quad 1 \leq i, j \leq n-1.$$

Let consider the submatrix G_n^i of order $n-1$. Compute the determinant by the cofactor formula on the $(n-1)$-th row of G_n^i (the n-th row of G as $1 \leq i \leq n-1$):

$$|G_n^i| = \sum_{j=1}^{n-1} g_{n,j} (-1)^{j+n-1} |G_{n,j}^{i,n}|.$$

So

$$\begin{aligned}
(-1)^{i+n}|G_n^i| &= \sum_{j=1}^{n-1} (-g_{n,j})(-1)^{j+n+i+n} |G_{n,j}^{i,n}| \\
&= \sum_{j=1}^{n-1} (-g_{n,j})((-1)^{j+i} |G_{j,n}^{i,n}|) \\
&\geq 0
\end{aligned}$$

as all the terms in the sum are non-negative.

The general case can be proved from this just applying the transposition (jn) to the columns and rows of G.

■

Lemma I.20 *Let G be the Gram matrix of a non-obtuse-angled $(n-1)$-simplex. Suppose an entry (and its symmetric) is decreased. Then the determinant of the resulting matrix is also decreased.*

Proof Let $g_{ij}, \quad i < j$ is decreased by $k \in \mathbb{R}^+$. Denote by G' the new matrix:

$$G' = \left\| \begin{array}{ccccccc}
g_{11} & \cdots & g_{1i} & \cdots & g_{1j} & \cdots & g_{1n} \\
\vdots & & \vdots & & \vdots & & \vdots \\
g_{1i} & \cdots & g_{ii} & \cdots & g_{ij} - k & \cdots & g_{in} \\
\vdots & & \vdots & & \vdots & & \vdots \\
g_{1j} & \cdots & g_{ij} - k & \cdots & g_{jj} & \cdots & g_{jn} \\
\vdots & & \vdots & & \vdots & & \vdots \\
g_{1n} & \cdots & g_{in} & \cdots & g_{jn} & \cdots & g_{nn}
\end{array} \right\|$$

Consider two matrices that coincide except for the i-th row. The sum of their determinants is equal to the determinant of a third matrix, which coincides with the first two except for the i-th row: this is, entry by entry, the sum of the i-th rows of the initial two matrices. Applying this result in the opposite sense, we have:

$$|G'| = \begin{vmatrix} g_{11} & \dots & g_{1i} & \dots & g_{1j} & \dots & g_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ g_{1i} & \dots & g_{ii} & \dots & g_{ij} & \dots & g_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ g_{1j} & \dots & g_{ij} & \dots & g_{jj} & \dots & g_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ g_{1n} & \dots & g_{in} & \dots & g_{jn} & \dots & g_{nn} \end{vmatrix} + \begin{vmatrix} g_{11} & \dots & g_{1i} & \dots & g_{1j} & \dots & g_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ g_{1i} & \dots & g_{ii} & \dots & g_{ij} & \dots & g_{in} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & -k & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ g_{1n} & \dots & g_{in} & \dots & g_{jn} & \dots & g_{nn} \end{vmatrix}$$

$$+ \begin{vmatrix} g_{11} & \dots & g_{1i} & \dots & g_{1j} & \dots & g_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & -k & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ g_{1j} & \dots & g_{ij} & \dots & g_{jj} & \dots & g_{jn} \\ \vdots & & \vdots & & \vdots & & \vdots \\ g_{1n} & \dots & g_{in} & \dots & g_{jn} & \dots & g_{nn} \end{vmatrix} + \begin{vmatrix} g_{11} & \dots & g_{1i} & \dots & g_{1j} & \dots & g_{1n} \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & 0 & \dots & -k & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \dots & -k & \dots & 0 & \dots & 0 \\ \vdots & & \vdots & & \vdots & & \vdots \\ g_{1n} & \dots & g_{in} & \dots & g_{jn} & \dots & g_{nn} \end{vmatrix} =$$

$$= \quad |G| + (-k)(-1)^{j+i}|G^i_j| + (-k)(-1)^{i+j}|G^j_i| + (-1)^{2i+2j}(-1)(-k)^2|G^{ij}_{ij}|.$$

Then by the hypothesis on G and by previous lemma

$$|G'| - |G| = -k(-1)^{j+i}|G^i_j| - k(-1)^{i+j}|G^j_i| - k^2|G^{ij}_{ij}| \leq 0,$$

which proves the assertion for $i \neq j$. If $i = j$ then $|G'| = |G| - k|G^i_i|$ and $|G'| - |G| \leq 0$ as before.

■

Coxeter calls *a-forms* [7, §10.2] the quadratic forms whose associated matrices have non-positive entries off the (principal) diagonal. Henceforth $\mathcal{A}(n)$ will denote the set of square symmetric matrices of order n with all the (principal) diagonal entries equal to 1, and non-positive entries off the diagonal; $\mathcal{A}_S(n)$ the subset of positive definite matrices of $\mathcal{A}(n)$; $\mathcal{A}_E(n)$ the subset of degenerate matrices of $\mathcal{A}(n)$ with $rank = index = n-1$; $\mathcal{A}_\Lambda(n)$ the subset of non-degenerate matrices of index $n-1$.

Lemma I.21 *Let $G \in \mathcal{A}_S(n)$ then each entry off the diagonal has absolute value strictly less than* 1.

Proof Since G is positive definite each principal minor $|G^{[ij]}_{[ij]}|$ is greater than zero.

$$0 < |G^{[ij]}_{[ij]}| = \begin{vmatrix} 1 & g_{ij} \\ g_{ij} & 1 \end{vmatrix}.$$

So $1 - g^2_{ij} > 0$ and $|g_{ij}| < 1$.

■

Lemma I.22 *Let $G \in \mathcal{A}(n)$ be positive semi-definite. Then each entry has absolute value less or equal to* 1.

Proof Just repeat the previous proof: now the principal minors $|G^{[ij]}_{[ij]}|$ are non-negative.
■

Lemma I.23 *Let $G \in \mathcal{A}_\Lambda(n)$ and suppose $G^k_k \in \mathcal{A}_S(n-1)$, $k = 1, ..., n$. If $n > 2$ then each entry off the diagonal has absolute value less than* 1.

Proof For each i, j, $i \neq j$ $G^{[ij]}_{[ij]}$ is the submatrix of G^k_k, for some k, so lemma I.21 concludes.
■

The previous lemma is never true if $n = 2$: if $G \in \mathcal{A}_\Lambda(2)$ then $|G| = 1 - g^2_{12} < 0$, but $G^i_i \in \mathcal{A}_S(1)$.

The positive semi-definite matrices of $\mathcal{A}(n)$ share a very special property which will be proved by the use of the theorem of *Perron-Frobenius.*

Recall that a matrix A is said *decomposable* [20] if it can be written, after maybe a possible permutation of rows and the same permutation of columns, as the direct sum of matrices of strictly less order (i.e. an isomorphic image of it, is written as a direct sum). It is said *indecomposable* if this does not happen. This definition has an interesting interpretation in the language of graphs. Let A' be the $(0, 1)$-matrix detecting the non null entries of A: $a'_{ij} = 1$ if and only if $a_{ij} \neq 0$. Interpret A' as the adjacency matrix of a graph Γ on n vertices. Then A is indecomposable if and only if Γ is connected.

Note I.24 *The definition of indecomposable matrix can be given for non-symmetric matrices and then results slightly different* [10, ch.1].

The Perron-Frobenius theorem is proved for irreducible *matrices* [9], *which in general are different from indecomposable ones, but which coincide in the symmetric case.*

Let us state the theorem directly in the case we are interested in, that of indecomposable symmetric matrices:

Theorem I.25 *(*Perron-Frobenius*) An indecomposable symmetric matrix with non-negative entries admits a positive eigenvalue, which has geometric multiplicity one and it is not exceeded by any absolute value of other eigenvalues. Any eigenvector of it has components all positive or all negative; moreover algebraic multiplicity is also one.*

Proof [9],[18]
■

In the following proposition we will show that the hypothesis of indecomposibility made on a positive semi-definite (but not definite, since otherwise the result would be trivial) matrix of $\mathcal{A}(n)$ constrains it to belong to the class $\mathcal{A}_E(n)$. Before, we will prove a characterization lemma for the eigenvectors of the zero eigenvalue of a positive semi-definite matrix.

Lemma I.26 *Let G be a symmetric positive semi-definite matrix of order n and q_G the quadratic form on $\mathbb{R}^n$ associated to it: $q_G(x) = x \cdot_G x = x^t\, G\, x$. Then the quadratic form q_G vanishes on a vector x if and only if x is an eigenvector of the zero eigenvalue of G.*

Proof Since G is positive semi-definite (but not definite) its determinant vanishes and 0 is an eigenvalue of G.

Let x be an eigenvector of the zero eigenvalue: $G\,x = 0$. Then $q_G(x) = x^t\, G\, x = 0$.

To prove the other implication we will need the positive semi-definite hypothesis. Let $a \in \mathbb{R}$ and $y \in \mathbb{R}^n$. Suppose $q_G(x) = 0$, then

$$0 \leq q_G(y + ax) = q_G(y) + 2a(y \cdot_G x), \quad \forall a \in \mathbb{R}.$$

Since the above inequality is true for all $a \in \mathbb{R}$, then

$$0 = y \cdot_G x = y^t\, G\, x = \sum_{i=1}^{n} y_i (G\, x)_i.$$

We can choose arbitrarily the vector $y \in \mathbb{R}^n$ so $G\,x$ must be the null vector and x is an eigenvector of the zero eigenvalue of G.

■

Proposition I.27 *An indecomposable positive semi-definite (non positive definite) matrix in $\mathcal{A}(n)$ has rank $n-1$ and all submatrices of order $n-1$ are positive definite.*

Proof Let $G \in \mathcal{A}(n)$ be positive semi-definite. Then G is degenerate and has 0 among its eigenvalues.

All the eigenvalues are real, as it is symmetric, and are non-negative, as it is positive semi-definite. Since all the entries of G off the diagonal are non-positive then the matrix $I_n - G$ has non-negative entries. Moreover it is indecomposable, since the principal diagonal entries, also in a possible permutation of rows and columns, remain diagonal entries. I.e. the zeroes on the principal diagonal do not bring decomposability to $I_n - G$, if this is not brought by G.

The theorem of Perron-Frobenius applies to it.

Now, λ is an eigenvalue of G if and only if $1 - \lambda$ is an eigenvalue of $I_n - G$:

$$\begin{aligned} |G - \lambda I_n| &= (-1)^n \,|-G + \lambda I_n| \\ &= (-1)^n \,|-G + I_n - I_n + \lambda I_n| \\ &= (-1)^n \,|-G + I_n - (1-\lambda) I_n|. \end{aligned}$$

So the greatest eigenvalue of $I_n - G$ is the least of G. Then $\lambda = 0$ is a simple root of the characteristic polynomial of G and all the others eigenvalues are positive.

We have now to prove that any principal submatrix of order $n-1$ is positive definite.

Let us consider the quadratic form q_G on $\mathbb{R}^n$ associated to G: $q_G(x) = x^t G x = x \cdot_G x$. Then q_G is positive semi-definite:

$$q_G(x) \geq 0 \quad \text{for any} \quad x \in \mathbb{R}^n.$$

Fix $j \in \{1, ..., n\}$. Consider the quadratic form associated to the principal submatrix G_j^j:

$$q_{G_j^j}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) = q_G(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n)$$

$q_{G_j^j}$ is positive semi-definite since so is q_G. Let $x \in \mathbb{R}^n$ and $q_G(x) = 0$. Then either $x = 0$ or x is an eigenvector of the zero eigenvalue of G. The Perron-Frobenius theorem implies that the coordinates of x are all positive or all negative. So if $q_{G_j^j}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n) = 0$ then $q_G(x_1, ..., x_{j-1}, 0, x_{j+1}, ..., x_n) = 0$ and so $x_1 = ... = x_{j-1} = x_{j+1} = ... = x_n = 0$. This means that $q_{G_j^j}$ is positive definite and so is G_j^j. Since j was any number between 1 and n, we have proved also the second assertion.
■

In the following lemma we will see what happens if we allow the principal submatrices of order less by one, to be Euclidean.

Lemma I.28 *Let $G \in \mathcal{A}(n)$ be indecomposable and suppose $n > 2$. Suppose G_i^i either belongs to $\mathcal{A}_S(n-1)$ or is indecomposable and belongs to $\mathcal{A}_E(n-1)$, $i = 1, ..., n$; suppose moreover that there exists at least one k, $0 \leq k \leq n$ such that $|G_k^k| = 0$. Then $G \in \mathcal{A}_\Lambda(n)$.*

Proof Since G_i^i is either Euclidean or spherical then $-1 \leq g_{ij} \leq 0$.

We have to determine the rank and the index of G. Let us first verify that the determinant is negative. To this aim we will consider a particular congruent matrix of G.

Suppose $|G_n^n| = 0$ and $|G_{n-1,n}^{n-1,n}| \neq 0$; the order of G_n^n is $n-1$. Let $e_1, ..., e_n$ be the standard basis in $\mathbb{R}^n$. Let denote the submatrix G_n^n by A. The matrix A is the Gram matrix of $e_1, ..., e_{n-1}$ in $(\mathbb{R}^{n-1}, \cdot_A)$.

There exist $n-1$ vectors $\varepsilon_1, ..., \varepsilon_{n-1}$ [app. C] such that their Gram matrix in $(\mathbb{R}^{n-1}, \cdot_A)$ is $\left\| \begin{matrix} I_{n-2} & 0 \\ 0 & 0 \end{matrix} \right\|$. The new basis is obtained by the use of the extended version of the Gram-Schmidt orthogonalization process, so each vector ε_i is a linear combination of $e_1, ..., e_i$, $i = 1, ..., n-1$. The vector ε_{n-1} is isotropic for $\cdot_A$: $\varepsilon_{n-1} \cdot_A \varepsilon_{n-1} = 0$. Since the associated quadratic form q_A on $\mathbb{R}^{n-1}$ is positive semi-definite, the vector ε_{n-1} is an eigenvector of the null eigenvalue of A. Since A is indecomposable, by the Perron-Frobenius theorem we are sure that its components $(a_1, ..., a_{n-1})$ are all positive or all negative. Since $\{e_i\}$ is the standard basis and ε_{n-1} is a linear combination of $e_1, ..., e_{n-1}$, we have $\varepsilon_{n-1} = \sum_{i=1}^{n-1} a_i e_i$. We want to show that $\varepsilon_{n-1} \cdot_G e_n \neq 0$. In fact

$$\varepsilon_{n-1} \cdot_G e_n = \sum_{i=1}^{n-1} a_i (e_i \cdot_G e_n) = \sum_{i=1}^{n-1} a_i g_{in}.$$

Since G is indecomposable at least one among $g_{1n}, ..., g_{n-1,n}$ is different from 0. So all the terms in the sum are all negative or all positive (according to the sign of the a_i) and at least one is different from 0: then $\varepsilon_{n-1} \cdot_G e_n \neq 0$.

Let C be the non-singular matrix of the change of basis from $e_1, ..., e_n$ to $\varepsilon_1, ..., \varepsilon_{n-1}, e_n$. Then G is congruent, via C, to the Gram matrix of the new basis $\varepsilon_1, ..., \varepsilon_{n-1}, e_n$

[prop. III.3]

$$C^t G C = \left\| \begin{array}{cccccc} 1 & 0 & \ldots & & 0 & \varepsilon_1 \cdot_G e_n \\ 0 & 1 & 0 \ldots & & 0 & \varepsilon_2 \cdot_G e_n \\ \vdots & \vdots & & & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 & \varepsilon_{n-2} \cdot_G e_n \\ 0 & 0 & \ldots & & 0 & \varepsilon_{n-1} \cdot_G e_n \\ \varepsilon_1 \cdot_G e_n & \varepsilon_2 \cdot_G e_n & & \ldots & \varepsilon_{n-1} \cdot_G e_n & 1 \end{array} \right\| .$$

So $|A| = -(\varepsilon_{n-1} \cdot_G e_n)^2$ and it is negative as $\varepsilon_{n-1} \cdot_G e_n \neq 0$. The sign of the determinant of two congruent matrices is the same then also $|G| < 0$.

Since $|G| < 0$ then the index (the number of positive terms in the canonical form) is less then the order, and the difference $rank(G) - index(G)$ must be odd. Then we have $index(G) = rank(G) - 1$.

■

We will now consider the problem of the existence of the non-obtuse-angled polytope given the matrix G. Obviously G must belong to the class $\mathcal{A}$.

Let G be a symmetric matrix of order N and of rank $n \leq N$. The *Sylvester's law of inertia* tell us that the index of G is well defined, i.e. there exists a well defined integer s, $0 \leq s \leq n$, such that G is congruent to $I(s, n-s, N-n)$. Then there exists a collection of N vectors in $\mathbb{R}^n$ with Gram matrix G, and defined up to a motion of $(\mathbb{R}^n, \cdot_{I(s,n-s)})$ [theorem III.5]. This collection of vectors will be called a *root system* of the matrix. Given the roots of the matrix we will interpret them as the root system of a polytope. I have developed an extended version of the *Gram-Schmidt orthogonalization process* to explicitly construct, even if G is singular, a collection of vectors whose Gram matrix is G [see app. C]: the condition on G is to possess a submatrix of maximal rank with all leading principal minors different from zero.

Theorem I.29 *Let $G \in \mathcal{A}_S(n+1)$. Then G is the Gram matrix of a spherical non-obtuse-angled n-simplex.*

Proof The rank and the index of G are both equal to $n+1$. Then G is congruent to I_{n+1}. There exist $n+1$ roots v_i, $i = 1, ..., n+1$ for G [thm. III.5]. Since their Gram matrix is non-singular, the $n+1$ roots form a basis of E^{n+1} [prop. III.1].

Consider the $n+1$ hyperplanes E-orthogonal to the v_i's:

$$\mathcal{H}_i = \{x \in E^{n+1} | \; x \cdot_E v_i = 0\}, \qquad i = 1, ..., n+1.$$

We will prove that

$$\bigcap_{i=1}^{n+1} \mathcal{H}_i^- \neq \emptyset$$

is a simplicial unbounded open $(n+1)$-hedron.

Since $\{v_i\}_{i=1,...,n+1}$ is a basis of E^{n+1}, each vector x can be written uniquely as $x = \sum x_i v_i$. Substitute the expression of x in the system of inequalities

$$\{x \cdot_E v_i \leq 0 \,, \qquad i = 1, ..., n+1$$

we obtain a system of linear inequalities in the variables x_i

$$\left\{\sum_{i=1}^{n+1} x_j g_{ij} \leq 0\ , \quad j = 1, ..., n+1\right.$$

which has infinite solutions. In fact there is a bijective correspondence with the $(n+1)$-hedron of E^{n+1} defined by

$$\{y_j \leq 0\ , \quad j = 1, ..., n+1 :$$

for each $(n+1)$-uple $(y_1, ..., y_{n+1})$, $y_i \leq 0$, the system

$$\left\{\sum g_{ij} x_j = y_i\ , \quad i = 1, .., n+1\right.$$

has a unique solution by the non-degeneracy of G. The intersection with S^n gives the spherical n-simplex, whose root system is $\{v_i\}_{i=1,...,n+1}$.

We have $-1 < g_{ij} < 0$ [lemma I.21] and the dihedral angle of the $(n+1)$-hedron enclosed by $\mathcal{H}_i$ and $\mathcal{H}_j$ is $\alpha_{ij} = \arccos(-g_{ij})$. And the angle between $H_i = \mathcal{H}_i \cap S^n$ and $H_j = \mathcal{H}_j \cap S^n$ is by definition the angle between $\mathcal{H}_i$ and $\mathcal{H}_j$. The simplex is determined up to a motion of E^{n+1} (thought as a vector space).

■

Theorem I.30 *Let $G \in \mathcal{A}(n+1)$ be positive semi-definite (but non definite) and indecomposable. Then there exists a non-obtuse-angled Euclidean n-simplex determined up to similarity, whose Gram matrix is G.*

Proof By the hypothesis we have $rank(G) = index(G) = n$ [prop. I.27]. There exists a root system $\{v_i\}_{i=1,...,n+1} \subset E^n$ [thm. III.5] and we can suppose that the first n roots $v_,\ i = 1, ..., n$, form a basis of E^n.

Consider the hyperplanes

$$H_i = \{x \in E^n |\ x \cdot_E v_i = 0\},\ \ i = 1, ..., n$$

and the half-hyperspaces

$$H_i^- = \{x \in E^n |\ x \cdot_E v_i \leq 0\}.$$

Then $\bigcap_{i=1}^n H_i^-$ is an n-hedron with vertex at the origin [thm. I.29]. A root determines the E-orthogonal hyperplane up to translation; the last hyperplane $H_{n+1} = \{x \in E^n |\ x \cdot_E v_{n+1} \leq d\}$ has to be chosen in the pencil of parallels hyperplanes determined by v_{n+1}, i.e. we have to choose d.

The eigenvector $c = (c_1, ..., c_{n+1})$ of the 0 eigenvalue, gives the coefficients of the null linear combination of the $n+1$ dependent roots: in fact $G\,c = 0$ represents globally the null vector so we have

$$0 = \sum_{j=1}^{n+1} g_{ij} c_j = \sum_{j=1}^{n+1} v_i \cdot_E v_j c_j = v_i \cdot_E (\sum_{j=1}^{n+1} c_j v_j),\ \ i = 1, ..., n,$$

just considering the first n components. Then as $\{v_i\}_{i=1,...,n}$ is a basis,

$$c_1 v_1 + ... + c_{n+1} v_{n+1} = 0.$$

By the theorem of Perron-Frobenius c has components all positive or all negative. So

$$v_{n+1} = \nu_1 v_1 + ... + \nu_n v_n, \quad \text{and} \quad \nu_i < 0, \ i = 1, ..., n.$$

Suppose x be an interior point of $\bigcap_{i=1}^{n} H_i^-$ then $x \cdot_E v_i < 0$, $i = 1, ..., n$ and

$$x \cdot_E v_{n+1} = \nu_1(x \cdot_E v_1) + ... + \nu_n(x \cdot_E v_n) > 0.$$

The origin is the vertex of the n-hedron $\bigcap_{i=1}^{n} H_i^-$, and we want it to belong to the hyper-halfspace H_{n+1}^-: so we have to choose $d \geq 0$. Since we want also interior points of the n-hedron to belong to the hyper-halfspace, we have to choose $d > 0$. So d can be any positive real number. The hyperplane H_{n+1} is not parallel to any of the hyperplanes H_i, $i = 1, ..., n$, since the eigenvector c has all components different from zero. Then $\bigcap_{i=1}^{n+1} H_i^-$ is an Euclidean n-simplex. The dihedral angles are determined as in the previous theorem.

Different values of d, give rise to similar n-simplices.

■

Let drop the assumption of indecomposibility on G.

If

$$G = G_1 \oplus G_2 \oplus ... \oplus G_k$$

$G_i \in \mathcal{A}_E(N_i)$, $N_i \geq 2$, G_i indecomposable: to G_i corresponds an Euclidean non-obtuse-angled $(N_i - 1)$-simplex; then the direct product of the k simplices gives a non-degenerate Euclidean non-obtuse-angled polytope in E^n, $n = \sum_{i=1}^{k} N_i - k$. The rank of G is n and the index is $n - k$.

But if $G \in \mathcal{A}_E(n+1)$ the indecomposibility is constrained by finite volume request on the polytope. Let consider an example.

Example I.31 *Let $G \in \mathcal{A}(3)$ be the following matrix:*

$$G = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{Vmatrix}$$

this does not correspond to a finite volume Euclidean 2-simplex.

With the procedure described in the appendix, we find that

$$C^tGC = \begin{Vmatrix} I_2 & 0 \\ 0 & 0 \end{Vmatrix} \qquad \text{where} \qquad C = \begin{Vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{Vmatrix}.$$

A root system for G is $v_1 = (1,0)$, $v_2 = (0,1)$ and $v_3 = (0,-1)$. Seek for the polytope as in theorem I.30. The result is an infinite half-strip.

This happens because the index is one less the order and among the G_i, $i = 1, ..., k$, only one can have index less by one of the order (Euclidean simplex), all the others must be positive definite (unbounded s-hedrons, for some s). Then the direct product polytope results of infinite volume.

In the hyperbolic case there are no more restrictions on the order of the matrix [20].

Theorem I.32 *Let $G \in \mathcal{A}(N)$ be indecomposable of rank $n+1$ and index n, $(n+1 \leq N)$. Then there exists a convex non-obtuse-angled non-degenerate hyperbolic polytope $\mathcal{P} \subset \Lambda^n$ whose Gram matrix is G. The polytope $\mathcal{P}$ is uniquely determined up to a motion.*

Proof

The matrix G has $rank(G) = n+1$ and $index(G) = n$. There exists a root system $\{v_i\}_{i=1,...,N} \subset E^{n,1}$ [thm. III.5] such that the first $n+1$ roots v_i form a basis of $E^{n,1}$.

Each root is spacelike, as $v_i \cdot_\Lambda v_i = 1$ by construction. So the linear subspaces of codimension 1

$$\mathcal{H}_i = \{x \in E^{n,1} | \ x \cdot_\Lambda v_i = 0\}, \quad i = 1, ..., N$$

are hyperbolic (timelike) and they intersect the sphere of imaginary radius Q in the hyperplanes $H_i, \quad i = 1, ..., N$.

Let consider the half-hyperspaces

$$\mathcal{H}_i^- = \{x \in E^{n,1} | \ x \cdot_\Lambda v_i \leq 0\}, \quad i = 1, ..., N.$$

We have to prove that their intersection $\mathcal{K} = \bigcap_{i=1}^N \mathcal{H}_i^-$ is a non-void convex polytopal angle with vertex at the origin of $E^{n,1}$ and that we will obtain a polytope in the hyperbolic space, i.e. that $\mathcal{K}$ intersect in an open subset one of the two sheets of Q and that the hyperplanes H_i contain the faces of this polytope.

Apply Perron-Frobenius theorem to G: let $\lambda (< 0)$ be the least eigenvalue and $c = (c_1, ..., c_N), \quad c_i > 0, \quad i = 1, ..., N$ a corresponding eigenvector. Consider the vector $x = \sum_j c_j v_j$. Then

$$x \cdot_\Lambda v_i = \sum_j c_j v_j \cdot_\Lambda v_i = \sum_j c_j g_{ji} = \sum_j g_{ij} c_j = (Gc)_i = (\lambda c)_i = \lambda c_i < 0$$

$$\text{so} \quad x \in \dot{\mathcal{K}}; \qquad \text{moreover} \qquad x \cdot_\Lambda x = \sum_j c_i (x \cdot_\Lambda v_i) < 0.$$

Then $\xi = \frac{x}{|x|_\Lambda} \in Q = Q^+ \cup Q^-$ and $\xi \in \mathcal{K}$. Let us suppose $\xi \in Q^+$. (This is not restrictive, since if $\xi \in Q^-$ then $\tilde{\xi} = -\xi \in Q^+$, and this can be achieved just replacing all the roots v_i with their opposites $-v_i$ or c with $-c$).

Take $w \in Q^-$. Then w and ξ belong to different components of the hyperboloid Q and so $\xi \cdot_\Lambda w > 0$. Let $y \in \mathcal{K}$: by definition of $\mathcal{K}$ we have $v_j \cdot_\Lambda y \leq 0$. Then

$$x \cdot_\Lambda y = \sum_j c_j v_j \cdot_\Lambda y \leq 0, \text{ as } c_j > 0. \qquad \text{So } \mathcal{K} \cap Q \subset Q^+.$$

Consider the Λ-orthogonal projection x' of x on $\mathcal{H}_j$:

$$x' = x - \frac{x \cdot_\Lambda v_j}{v_j \cdot_\Lambda v_j} v_j = x - (x \cdot_\Lambda v_j) v_j.$$

It follows that $x' \neq 0$ since x is timelike and v_j is spacelike. Suppose $i \neq j$, then

$$x' \cdot_\Lambda v_i = x \cdot_\Lambda v_i - (x \cdot_\Lambda v_j)(v_j \cdot_\Lambda v_i) = x \cdot_\Lambda v_i - g_{ij}(x \cdot_\Lambda v_j) < 0 \quad \text{so} \quad x' \in \mathcal{K}.$$

We have that $\xi' = \frac{x'}{|x'|_\Lambda}$ is the Λ-orthogonal projection of ξ on $H_j = \mathcal{H}_j \cap Q^+$ (ξ' belongs to the plane generated by ξ and v_j, that intersect Q^+ in the Λ-line Λ-orthogonal to H_j: ξ' is the foot of the Λ-perpendicular to H_j through ξ). Let $\mathcal{P} = \mathcal{K} \cap Q^+$. The Λ-orthogonal projection of $\mathcal{P}$ on H_j is in fact contained in $H_j \cap \mathcal{P}$ and contains a non-empty open subset of H_j, this implies that H_j bounds $\mathcal{P}$.

Then $\mathcal{P}$ is the wanted polytope. Its Gram matrix is G. The polytopal angle $\mathcal{K}$ is unique up to a motion of $E^{n,1}$ [thm. III.6]. A motion of Λ^n is a motion of $E^{n,1}$ which preserves Q^+. Then also $\mathcal{P}$ is unique up to a motion, as $\mathcal{K}$ intersects only one component of Q.

■

Let consider the case of a simplex: $N = n+1$ and $G \in \mathcal{A}_\Lambda(n+1)$. Then the simplex is bounded if $G_k^k \in \mathcal{A}_S(n)$, $k = 1, ..., n+1$; it has finite volume if $G_k^k \in \mathcal{A}_S(n) \cup \mathcal{A}_E(n)$, $k = 1, ..., n+1$ [20, thm.4.1].

Denote by $\{v_i\}_{i=1,...,n+1}$ the roots. Let suppose $G_k^k \in \mathcal{A}_S(n)$. Then

$$\mathcal{W}_k = \bigcap_{i,\ i\neq k}^{1,n+1} \mathcal{H}_i$$

is 1-dimensional in $E^{n,1}$. The subspace orthogonal to $\mathcal{W}_k$ is generated by all the roots but v_k, and it is spacelike since G_k^k is supposed positive definite. Then $\mathcal{W}_k$ is timelike and it intersects the sphere of imaginary radius in an (ordinary) point of the hyperbolic space. Let suppose $G_k^k \in \mathcal{A}_E(n)$. Then the subspace generated by all roots but v_k

$$\langle v_1, ..., \widehat{v_k}, ..., v_{n+1} \rangle$$

is parabolic (lightlike) and contains one isotropic direction

$$\mathcal{W}_k = \bigcap_{i,\ i\neq k}^{1,n+1} \mathcal{H}_i.$$

Then $\mathcal{W}_k$ is isotropic so determines an ideal point. So the simplex has a vertex at infinity.

Part II
Coxeter simplices and Coxeter graphs

A non-obtuse-angled simplex $\mathcal{P} = \bigcap_{i=1}^{n+1} H_i^- \subset X^n$ (of finite volume) is given the appellative of *Coxeter*'s if any two hyperplanes H_i, H_j, $i \neq j$, bounds $\mathcal{P}$ in a dihedral angle of value $\frac{\pi}{n_{ij}}$, $n_{ij} \in \mathbb{Z}$, $n_{ij} \geq 2$. I.e. all dihedral angles are integer (non trivial) submultiples of π. If H_i and H_j are disjoint then we put $n_{ij} = \infty$. In a simplex this happens in Euclidean case for $n = 1$ and in hyperbolic case for $n = 2$ (parallel edges with a common vertex at infinity).

To each Coxeter simplex $\mathcal{P}$ is associated a discrete object Γ: a simple graph marked on edges or alternatively a multiple graph with allowed parallel edges [4, ch. IV,§1,9], [5, p.619], [19, ch.13].

We will denote the set of vertices with $\mathcal{V}(\Gamma)$ and the set of edges with $\mathcal{E}(\Gamma)$.

To each hyperplane corresponds a vertex: $\gamma_i \leftrightarrow H_i$, $i = 1, ..., n+1$, then

$$\mathcal{V}(\Gamma) = \{\gamma_1, ..., \gamma_{n+1}\}.$$

Two vertices γ_i and γ_j are joined by an edge $\eta_k = [\gamma_i\gamma_j] \in \mathcal{E}(\Gamma)$ if the corresponding hyperplanes H_i and H_j enclose an angle of $\frac{\pi}{n_{ij}}$, and $n_{ij} \geq 3$. To η_k is associated the mark $n_{ij} - 2$; mark ∞ if the hyperplanes are parallel, but we do not consider directions on edges. Usually the mark is written above the edge, mark 1 is omitted and an edge with mark 1, in the text will be referred to as an unmarked edge.

The marked graph Γ is called *Coxeter diagram* [19] or *Coxeter graph* [4] of $\mathcal{P}$. The cardinality of $\mathcal{V}(\Gamma)$ will be called the *order* of Γ. The unmarked underlying graph will be called the *support* of Γ.

There is an alternative description, in which the edge η_k of mark $n_{ij}-2$ is substitued by $n_{ij} - 2$ parallel edges connecting γ_i and γ_j. So an edge marked k is the same as k parallel edges.

The given definition follows the one given by Thurston in [19]. It allows to collect more cases in one single notation. Note that Coxeter [7], Bourbaki [4] and others define the mark to be n_{ij} and omit marks 3.

Proposition II.1 *Given a spherical Coxeter simplex, its Coxeter graph is cycle free.*

Proof [4, ch.V,§4,8]

Let us proceed by reductio ad absurdum: suppose there exists a cycle

$$\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_k}\gamma_{i_1}$$

of length k greater or equal to 3. The inner product space structure given by the Gram matrix G of the simplex on $\mathbb{R}^{n+1}$ is positive [prop. I.17]. This means that each non-null vector has positive length: let $x = e_{i_1} + e_{i_2} + ... + e_{i_k}$ then $x \cdot_G x > 0$. But

$$x \cdot_G x \;=\; (e_{i_1} + e_{i_2} + ... + e_{i_k}) \cdot_G (e_{i_1} + e_{i_2} + ... + e_{i_k})$$

$$
\begin{aligned}
&= \sum_{j=1}^{k} e_{i_j} \cdot_G e_{i_j} + 2 \sum_{j<s}^{1,k} e_{i_j} \cdot_G e_{i_s} \\
&= \sum_{j=1}^{k} 1 + 2 \sum_{j<s}^{1,k} g_{i_j i_s} \\
&= k + 2 [\ g_{i_1 i_2} + g_{i_2 i_3} + \dots + g_{i_{k-1} i_k} + g_{i_1 i_k} + \text{ negative terms }]
\end{aligned}
$$

We have pointed out in the last row the entries of G corresponding to the edges of the supposed existent cycle: $[\gamma_{i_1}\gamma_{i_2}]$, $[\gamma_{i_2}\gamma_{i_3}]$, ... $[\gamma_{i_{k-1}}\gamma_{i_k}]$ and $[\gamma_{i_1}\gamma_{i_k}]$. Their marks are greater or equal to 1, so $n_{i_j i_s} \geq 3$. The function $f(*) = -\cos\frac{\pi}{*}$ is decreasing for $* \geq 3$ and $-\frac{1}{2}$ is an upper bound in the given domain. Then $g_{i_j i_s} \leq -\frac{1}{2}$

$$
\text{so} \qquad x \cdot_G x \leq k - 2k\frac{1}{2} = k - k = 0.
$$

This is an absurd as $x \neq 0$.

■

In the alternative description, with multiple edges, the proposition proves that the only possible cycles are the 2-cycles.

We want start from a marked graph Γ. Let the order of Γ be $n+1$ and let m_{ij} be the mark of the edge $[\gamma_i\gamma_j]$. Construct a matrix C of order $n+1$ with on the principal diagonal all entries equal to 1; if $i \neq j$ define $c_{ij} = m_{ij} + 2$ (i.e. the marks added up with 2). Remember that if γ_i and γ_j are not joined, then we can suppose $m_{ij} = 0$. The matrix C is called the Coxeter matrix [4] of the graph Γ. Apply to the Coxeter matrix (to all its entries) the function $f(x) = -\cos\frac{\pi}{x}$ and $f(\infty) = -1$. We obtain a new matrix which I call the Gram matrix of the graph.

For short, in the sequel a Coxeter graph will be said *spherical*, respectively *Euclidean* or *hyperbolic* if its Gram matrix is positive definite, respectively indecomposable degenerate positive semi-definite or indecomposable non-degenerate but indefinite of index one less the order.

Note II.2 *Let me point out that in topological graph theory* spherical graph [12, 1.4,1.5] *is sometimes employed, in place of* planar graph [12, 1.6], *for a graph which admits an imbedding in the 2-sphere.*

Remember that a graph map [12, 1.1.6] is given by a function on vertices and a function on edges such that incidence is preserved. For marked graphs on edges, just require that the mark of an edge is less or equal to the mark of its image.

Proposition II.3 *If two Coxeter graphs are isomorphic then the Gram matrices are isomorphic.*

Proof An isomorphism of graphs is an isomorphism on vertices and an isomorphism on edges which preserves incidence relations and marks. The isomorphism on vertices gives for one matrix a permutation on rows and the same permutation on columns. This shows that an isomorphic image of one matrix, is equal entry by entry to the other.

■

Proposition II.4 *Given a Coxeter simplex $\mathcal{P}$, the Coxeter graph Γ is connected if and only if the Gram matrix G is indecomposable.*

Proof

The proposition will be proved if we will show that: the matrix is decomposable if and only if the graph is not connected.

Γ is not connected

if and only if

$\mathcal{V}(\Gamma)$ splits in two disjoint subsets $\mathcal{V}_1$ and $\mathcal{V}_2$, such that for every possible choice of $\gamma_i \in \mathcal{V}_1$ and $\gamma_j \in \mathcal{V}_2$, γ_i and γ_j are not joined

if and only if

there exist two subsets $\mathcal{I}_1$, $\mathcal{I}_2$ of $\mathcal{I} = \{1, ..., n+1\}$ such that $\mathcal{I}_1 \cap \mathcal{I}_2 = \emptyset$, $\mathcal{I}_1 \cup \mathcal{I}_2 = \mathcal{I}$ and $g_{ij} = 0$ whenever $i \in \mathcal{I}_1$, $j \in \mathcal{I}_2$

if and only if

there exists a permutation σ on $\mathcal{I}$ such that $\sigma(\mathcal{I}_1) = \{1, ..., |\mathcal{I}_1|\}$, $\sigma(\mathcal{I}_2) = \{|\mathcal{I}_1| + 1, ..., n+1\}$, $|\mathcal{I}_i|$ cardinality of $\mathcal{I}_i$, and applied to the rows and columns of G shows it, as a direct sum of two submatrices.

The subsets $\mathcal{I}_1$ and $\mathcal{I}_2$ are determined by $\mathcal{V}_1$ and $\mathcal{V}_2$ and viceversa.

■

The Coxeter graph is a visual tool: the existence and the type of the corresponding simplex is checked via the Gram matrix developed theory.

Lemma II.5 *A Coxeter graph of order $n+1$ corresponds to a Coxeter spherical n-simplex if an only if its Gram matrix is positive definite.*

Proof Start from a Coxeter spherical n-simplex $\mathcal{P}$. Construct, as illustrated, the Coxeter graph Γ associated, up to isomorphism of graphs, to $\mathcal{P}$. The Gram matrix of Γ is the Gram matrix of $\mathcal{P}$. So it is positive definite.

Start from the Coxeter graph Γ. Compute its Gram matrix G. If it is positive definite it determines, up to a motion, a spherical n-simplex $\mathcal{P}$ [thm. I.29]. The Coxeter graph associated to $\mathcal{P}$ is isomorphic to Γ.

■

The notion of submatrix of a matrix has its correspondent in the notion of induced subgraph of a graph [12, 1.3.1].

The subgraph Γ' is an *induced subgraph* of Γ if $\mathcal{V}(\Gamma') \subset \mathcal{V}(\Gamma)$ and for each couple of vertices $\gamma_i, \gamma_j \in \Gamma'$, $[\gamma_i\gamma_j] \in \mathcal{E}(\Gamma')$ if and only if $[\gamma_i\gamma_j] \in \mathcal{E}(\Gamma)$ and the mark is the same.

A general Coxeter graph does not correspond to a Coxeter simplex: we introduce the following conditions.

A Coxeter graph is said to satisfy the Coxeter Spherical respectively the Coxeter Spherical Euclidean condition on induced subgraphs if

CS: Each induced subgraph on all vertices but one is spherical.

CSE: Each induced subgraph on all vertices but one is either spherical or Euclidean, but at least one is Euclidean.

Proposition II.6 *A connected Coxeter graph of order $n+1$ corresponds to a Coxeter bounded n-simplex if and only if it is a CS-graph.*

Proof In a Coxeter bounded n-simplex all bounding hyperplanes but one intersect in a(n ordinary) vertex and each vertex polytope is a spherical simplex. So, in the corresponding Coxeter graph, any induced subgraph on all vertices but one is spherical.

Viceversa, the Gram matrix G of a connected Coxeter CS-graph of order $n+1$ is an indecomposable matrix in $\mathcal{A}(n+1)$ such that each submatrix of order n is in $\mathcal{A}_S(n)$. So there exists a Coxeter bounded n-simplex $\mathcal{P}$ which admits G as its Gram matrix and: $\mathcal{P}$ is spherical if $|G|>0$; $\mathcal{P}$ is Euclidean if $|G|=0$; $\mathcal{P}$ is hyperbolic if $|G|<0$ [thms. I.29, I.30, I.32, lemma I.28]. So the geometrical type is decided by the sign of the Gramian.

■

Proposition II.7 *A connected Coxeter graph corresponds to an unbounded hyperbolic simplex of finite volume if and only if it is a CSE-graph.*

Proof The difference with the above proposition stays in ideal vertices, i.e. vertices of the simplex which are points at infinity of Λ^n. A sphere around such a vertex is a horosphere which is isometric to Euclidean space E^{n-1} [6, p.197,p.251], [19, ch.3]; then the vertex polytope is an Euclidean simplex.

■

The classification of Coxeter n-simplices can be done by the classification of all possible Coxeter CS and CSE graphs. It will be done by induction.

Let introduce a partial ordering in the set of marked graphs.

Definition II.8 *Given two graphs, Γ and Γ', we say that Γ* precedes *Γ'*

$$\Gamma \preceq \Gamma' \quad \Leftrightarrow \quad \exists \varphi : \Gamma \to \Gamma', \quad \text{such that } \Gamma \approx \varphi(\Gamma)$$

i.e. if and only if Γ may be embedded in Γ' as a subgraph.

Two graphs, $\Gamma \preceq \Gamma'$, are said to be consecutives *if one of the following operations tranforms Γ in Γ'*

1) *adjunct of a new edge (marked 1) between two non adjacent vertices*
$\mathcal{V}(\Gamma) = \mathcal{V}(\Gamma')$ *and there exists γ_i and γ_j, $i \neq j$, such that $\mathcal{E}(\Gamma) \not\ni [\gamma_i\gamma_j]$ and* $\mathcal{E}(\Gamma') = \mathcal{E}(\Gamma) \cup \{[\gamma_i\gamma_j]\}$;

2) *increasing of just one mark*
$\mathcal{V}(\Gamma) = \mathcal{V}(\Gamma')$, $\mathcal{E}(\Gamma) = \mathcal{E}(\Gamma')$ *and there exists γ_i and γ_j, $i \neq j$, such that* $m'_{ij} = m_{ij}+1$, *all the other marks being unchanged;*

3) *adjunt of a new vertex to be joined (by an unmarked edge) to a pre-existent vertex*
$\mathcal{V}(\Gamma') = \mathcal{V}(\Gamma) \cup \{\gamma'\}$ *and* $\mathcal{E}(\Gamma') = \mathcal{E}(\Gamma) \cup \{[\gamma_i\gamma']\}$ *for some $\gamma_i \in \mathcal{V}(\Gamma)$.*

If we think to an unexistent edge between two non-adjacent vertices as an edge of mark 0, then case 1) is just a particular case of 2).

To apply induction on Coxeter graphs it will useful to have formulas to evaluate the Gramian of a graph by the values of its precedents.

Lemma II.9 *Let Γ' be obtained from Γ by the adjunct of a vertex γ' joined to the vertex $\gamma_i \in \mathcal{V}(\Gamma)$ by an edge $[\gamma_i\gamma']$ of mark k. Then $|G'| = |G| - |G_i^i| \cos^2 \frac{\pi}{k}$.*

Proof Let n be the order of Γ. The submatrix G_i^i is the Gram matrix of the graph obtained deleting the vertex γ_i and all edges incident to it.

Since the Gramian is unchanged by an isomorphism of graphs [prop. II.3], it can always be supposed that $i = n$ and $\gamma' = \gamma_{n+1}$. Then

$$|G'| = \begin{vmatrix} 1 & \dots & g_{1n} & 0 \\ \vdots & & \vdots & 0 \\ g_{1n} & \dots & 1 & -\cos\frac{\pi}{k} \\ 0 & \dots & -\cos\frac{\pi}{k} & 1 \end{vmatrix}.$$

To obtain the desired result just evaluate the determinant by the expansion formula with respect to the last column and then with respect to the last row.

■

The removal of a vertex in a graph means also the removal of all edges incident to it. So if Γ is a graph and $\gamma_i \in \mathcal{V}(\Gamma)$, with the symbol $\Gamma \setminus \gamma_i$ we denote the graph, whose vertex set is

$$\mathcal{V}(\Gamma \setminus \gamma_i) := \mathcal{V}(\Gamma) \setminus \{\gamma_i\}$$

and whose edge set is $\mathcal{E}(\Gamma\setminus\gamma_i) := \mathcal{E}(\Gamma)\setminus\{\eta \in \mathcal{E}(\Gamma) \mid \exists\gamma \in \mathcal{V}(\Gamma), \gamma \neq \gamma_i \text{ and } \eta = [\gamma_i\gamma]\}$.

Corollary II.10 *Let the hypothesis be as in the above lemma. Suppose that $\Gamma \setminus \gamma_i$ splits in two or more connected components:*

$$\Gamma \setminus \gamma_i = \Gamma_1 \cup \Gamma_2 \cup \dots \cup \Gamma_s.$$

Then

$$|G'| = |G| - \cos^2 \frac{\pi}{k} |G_1|\,|G_2| \dots |G_s|.$$

Proof The submatrix G_i^i, or a suitable isomorphic image of it, decomposes as the direct sum of the Gram matrices corresponding to the connected components of $\Gamma \setminus \gamma_i$. And the determinant of a direct sum matrix is the product of the determinants of the indecomposable components

$$|G_1 \oplus \dots \oplus G_s| = |G_1| \cdot \dots\dots \cdot |G_s|,$$

just by the Laplace theorem on determinant expansion.

■

These results will help us to conclude that when an edge is added or a mark is increased, the value of the determinant decreases:

Theorem II.11 *Let Γ and Γ' be two Coxeter graphs, G and G' the respective Gram matrices. Suppose $\Gamma \prec \Gamma'$ and $G \in \mathcal{A}_S \cup \mathcal{A}_E \cup \mathcal{A}_\Lambda$ then $|G'| \leq |G|$.*

Proof The lemma will be proved, if it is achieved for consecutive graphs.

If we have added an unmarked edge between two disconnected vertices (consecutives of type 1)) or if we have increased the mark of just one edge (consecutives of type 2)) then the conlusion follows by lemma I.20; if we have added one vertex and joined it, by a new edge, to a pre-existent vertex (consecutive of type 3)) the proof follows by lemma II.9.

■

Henceforth the Gram matrix of the graph Γ will be denoted by the same letter Γ and the Gramian by $|\Gamma|$.

In the following three section we will discuss the existence of infinite families of spherical and Euclidean Coxeter graphs and the low dimensional cases ($n \leq 2$). The case of dimension $n \geq 3$ is treated in section 4.

1 Coxeter spherical graphs

There exist Coxeter spherical and Euclidean graphs of arbitrarily high order. In the spherical case there are three families of Coxeter spherical graphs.

Let us denote by P_n the standard path on n vertices [12, 1.2.1]

o———o———o———o— . . . —o.

Theorem II.12 *The n-path P_n is the Coxeter graph of a spherical $(n-1)$-simplex for each n.*

Proof Start from the one vertex graph P_1: each P_n can be obtained as a successive of P_1 and we have the chain

$$P_1 \prec P_2 \prec \dots \prec P_n \prec \dots.$$

moreover P_{n+1} is a consecutive, of type 3), of P_n.

The first graph, P_1 is the one vertex graph; its Gram matrix is the 1×1 matrix with entry 1, so $|P_1| = 1$.

The consecutive P_2 is made up of two vertices joined by one edge; its Coxeter matrix is $\begin{Vmatrix} 1 & 3 \\ 3 & 1 \end{Vmatrix}$ and its Gram matrix is $\begin{Vmatrix} 1 & -\frac{1}{2} \\ -\frac{1}{2} & 1 \end{Vmatrix}$. So $|P_2| = 1 - (-\frac{1}{2})^2 = \frac{3}{4}$.

Then for the consecutive of P_2 we apply the formula of lemma II.9: and going on we obtain, in general, $|P_{n+1}| = |P_n| - \frac{1}{4}|P_{n-1}|$.

$$\text{CLAIM:} \qquad |P_n| = \frac{n+1}{2^n}.$$

This is true for $n = 1, 2$. Suppose, by induction, it is true for $k \leq n$, $n \geq 3$. Then

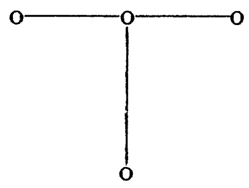
$$|P_{n+1}| = \frac{n+1}{2^n} - \frac{1}{4}\frac{n}{2^{n-1}} = \frac{2n+2}{2^{n+1}} - \frac{n}{2^{n+1}} = \frac{n+2}{2^{n+1}}.$$

The claim is proved: so for each n we have $|P_n| > 0$ and the chain on graphs gives a reversed chain on Gramians:

$$\begin{array}{ccccccccc} |P_1| & > & |P_2| & > & |P_3| & > \dots > & |P_n| & > \dots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 1 & > & \frac{3}{4} & > & \frac{4}{8} & > \dots > & \frac{n+1}{2^n} & > \dots \end{array}$$

If we remove any of the vertices of P_n, we obtain a graph or a union of graphs of the same type; so each induced subgraph on all vertices but one is spherical. The graph P_n is then the graph of a spherical $(n-1)$-simplex.
■

Let T_4 denote the Coxeter graph on 4 vertices, which can be drawn in form of a "T" [10, 1.4,I.1]:



and suppose to extend the stem of the "T" by the adjunct of one vertex and one edge at time (consecutives of type 3)). Let T_n, $n \geq 4$, denote the graph of this type on n vertices. Then

$$T_4 \prec T_5 \prec .... \prec T_n \prec ....$$

Theorem II.13 *The Coxeter graph T_n corresponds to a spherical $(n-1)$-simplex for each $n \geq 4$.*

<u>Proof</u> The graph T_4 is obtained as a consecutive P_3: adjunct a vertex and join it, by a new edge, to the central vertex of P_3. The removal of the central vertex from P_3 leaves the totally disconnected graph on two vertices, i.e. two copies of $I_1 = P_1$, so:

$$|T_4| = |P_3| - \frac{1}{4}|I_1|\,|I_1| = \frac{4}{2^3} - \frac{1}{4} = \frac{1}{4}.$$

T_5 is obtained from T_4 by the adjunct of a vertex joined to any of the vertices of valence 1:

$$|T_5| = |T_4| - \frac{1}{4}|P_3| = \frac{1}{4} - \frac{1}{4}\frac{4}{2^3} = \frac{1}{8}.$$

Let $n > 5$: $\qquad |T_{n+1}| = |T_n| - \frac{1}{4}|T_{n-1}|.\qquad$ Then

$$\text{CLAIM:} \qquad |T_n| = \frac{1}{2^{n-2}}.$$

It follows by induction: it is true for $n = 4, 5$. Suppose it is true for any $k \leq n$, $n \geq 6$, then:

$$|T_{n+1}| = \frac{1}{2^{n-2}} - \frac{1}{4}\frac{1}{2^{n-3}} = \frac{2}{2^{n-1}} - \frac{1}{2^{n-1}} = \frac{1}{2^{n-1}}.$$

The induced subgraphs on all vertices but one of T_n can be

- P_{n-1}
- $I_1 \cup I_1 \cup P_{n-3}$
- $P_3 \cup P_{n-4}$
- $T_s \cup P_{n-s-1}, \; 4 \leq s \leq n-2$
- T_{n-1}

so in each case spherical. Then T_n is spherical.

■

Let B_n, $n \geq 2$, be the graph which support is P_n, but such that one of the two terminal edge has mark 2

$$\text{o}\overset{2}{\text{———}}\text{o}\text{———}\text{o}\text{—} \ldots \text{—o}.$$

Theorem II.14 *The Coxeter graph B_n corresponds to a spherical simplex for each $n \geq 2$.*

Proof Also the B_n form an ordered chain of consecutive elements, each obtained from the precedent by an adjunct of type 3):

$$B_2 \prec B_3 \prec \ldots . \prec B_n \prec \ldots .$$

The Coxeter matrix of B_2 is $\begin{Vmatrix} 1 & 4 \\ 4 & 1 \end{Vmatrix}$ and its Gram matrix is $\begin{Vmatrix} 1 & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & 1 \end{Vmatrix}$. Then $|B_2| = 1 - (-\frac{\sqrt{2}}{2})^2 = \frac{1}{2}$.

Then
$|B_3| = |B_2| - \frac{1}{4}|I_1| = \frac{1}{4}$
and
$|B_4| = |B_3| - \frac{1}{4}|B_2| = \frac{1}{8}$.
In general
$|B_n| = |B_{n-1}| - \frac{1}{4}|B_{n-2}|$
and as before, by induction, we can prove a formula valid for each $n \geq 2$

$$\text{CLAIM:} \qquad |B_n| = \frac{1}{2^{n-1}}.$$

It is true for $n = 2, 3$; so suppose $n \geq 4$ and the claim true for any $k \leq n$, then

$$|B_{n+1}| = \frac{1}{2^{n-2}} - \frac{1}{4}\frac{1}{2^{n-3}} = \frac{2}{2^{n-1}} - \frac{1}{2^{n-1}} = \frac{1}{2^{n-1}}.$$

The induced subgraphs of B_n on all vertices but one can be one of the followings

- P_{n-1}
- $I_1 \cup P_{n-2}$
- $B_s \cup P_{n-s-1}$, $\quad n-2 \geq s \geq 2$
- B_{n-1}

so in each case spherical.

Then each B_n corresponds to spherical $(n-1)$-simplex.

■

We want now to determine all possible Coxeter spherical CS graphs for order less or equal to 3: this means that we will determine the Coxeter spherical simplices in dimension $n \leq 2$. Note in fact that the dimension is one less the order: if n is the dimension of the space, the number of vertices of the n-simplex is $n+1$. So the number of vertices, which is for a simplex also the number of faces, is equal to the order of the corresponding marked graph. A Coxeter graph is spherical if and only if so are all its connected components. Then we will determine only connected graphs.

$n=0$ The order is 1; loops are not admitted, so there is only one graph: P_1.

$n=1$ The order is 2. We have a spherical segment, i.e. a plane non-obtuse angle: there are infinitely many:

$$\circ \overset{k}{\text{——}} \circ \;, \quad k \in \mathbb{Z},\; k \geq 0.$$

$n=2$ The graph cannot be a cycle [prop. II.1], and it is connected by assumption. Suppose one edge unmarked and the other marked k. By the inequality $\frac{\pi}{2}+\frac{\pi}{3}+\frac{\pi}{k+2} > \pi$ it follows $k \leq 3$. So:

$$\circ\text{——}\circ\text{——}\circ \quad P_3$$

$$\circ\text{——}\circ\overset{2}{\text{——}}\circ \quad B_3$$

$$\circ\text{——}\circ\overset{3}{\text{——}}\circ$$

Suppose that one edge is marked 2 and the other k; then we have to analyse the inequality $\frac{\pi}{2}+\frac{\pi}{4}+\frac{\pi}{k+2} > \pi$. Which implies $k \leq 1$. This happens also if the marks are 3 and k. So we do not get new graphs. There are three connected Coxeter spherical graphs on 3 vertices.

2 Coxeter Euclidean graphs

Let now consider Coxeter CS graphs, whose Gramian is null: the Coxeter Euclidean graphs.

The matrix of an Euclidean simplex is indecomposable, since the system of roots does not decompose in two subset of mutually orthogonal roots, then the corresponding Coxeter graph is connected. This means that we have also to start by a connected Coxeter graph.

Moreover in the defined order on Coxeter graphs, an Euclidean one can only be the consecutive of a spherical one; never of an Euclidean graph, since the value of the Gramian decreases. So in an ordered chain of graphs, if there is an Euclidean graph this is unique.

The formulas for the Gramians of the spherical families P_n, T_n and B_n suggest that there would be also infinite families of Coxeter Euclidean graphs.

Let $\widetilde{P_n}$ be obtained from P_n, $n \geq 3$, connecting by an edge the two ends of the path: it is the n-cycle C_n.

Theorem II.15 *The Coxeter graph $\widetilde{P_n}$ is, for each $n \geq 3$, a Coxeter Euclidean graph.*

Proof

The Gram matrix of $\widetilde{P_n}$ is almost the Gram matrix of P_n: but both the entries g_{1n} and g_{n1} are decreased from 0 to $-\frac{1}{2}$.

Apply the computations performed in lemma I.20 (N.B. $k = \frac{1}{2}$), we have

$$|\widetilde{P_n}| = |P_n| - \frac{1}{2}(-1)^{n+1}|(P_n)^1_n| - \frac{1}{2}(-1)^{n+1}|(P_n)^n_1| - \frac{1}{4}\,|(P_n)^{1n}_{1n}|.$$

The submatrix $(P_n)^1_n$ is an upper triangular matrix of order $n-1$ with all the diagonal entries equal to $-\frac{1}{2}$. And the submatrix $(P_n)^n_1$ is the transposed of $(P_n)^1_n$. Finally $(P_n)^{1n}_{1n} = P_{n-2}$. Then

$$\begin{aligned} |\widetilde{P_n}| &= \frac{n+1}{2^n} - \frac{1}{2}(-1)^{n+1}(-\frac{1}{2})^{n-1} - \frac{1}{2}(-1)^{n+1}(-\frac{1}{2})^{n-1} - \frac{1}{4}\frac{n-1}{2^{n-2}} \\ &= \frac{n+1}{2^n} - \frac{1}{2}\frac{1}{2^{n-1}} - \frac{1}{2}\frac{1}{2^{n-1}} - \frac{1}{4}\frac{n-1}{2^{n-2}} \\ &= 0. \end{aligned}$$

Since the induced subgraph on $n-1$ vertices for each possible choice is P_{n-1}, then the Coxeter graphs $\widetilde{P_n}$, $n \geq 3$, all correspond to Euclidean simplices.

∎

The graphs B_n, $|B_n| = \frac{1}{2^{n-1}}$, have the property that

$$|B_{n+1}| - \frac{1}{2}|B_n| = 0.$$

Let construct a graph, which would have, by lemma II.9, the above expression for the Gramian.

We have to do a successive of B_{n+1}: add a new vertex and join it by an edge of mark 2 or equivalently by two parallel edges (N.B. $\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$) to a pre-existent vertex whose removal leaves the graph B_n. If $n = 1$ join the new vertex to any of the two vertices of B_2.

This graphs will be denoted by $\widetilde{B_{n+2}}$.

Theorem II.16 *The graphs $\widetilde{B_n}$, $n \geq 3$, are Coxeter Euclidean CS graphs.*

Proof The Gramian of $\widetilde{B_n}$, $n \geq 4$, is null by construction. We might have some doubt only on $\widetilde{B_3}$, but it will soon disappear:

$$|\widetilde{B_3}| = |B_2| - \frac{1}{2}\,|I_1| = \frac{1}{2} - \frac{1}{2}1 = 0.$$

The induced subgraphs on all vertices but one have as connected components graphs of type B_t, P_m and/or I_k , for some integers t, m, k, so are spherical.

This implies that the graphs $\widetilde{B_n}$ are Euclidean.

∎

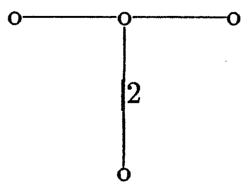
The same happens to T_n:

$$|T_{n+1}| - \frac{1}{2}|T_n| = 0.$$

We have to construct a successive of T_{n+1}: add a new vertex and join it by an edge of mark 2 or equivalently by two parallel edges, to a pre-existent vertex whose removal leaves the graph T_n. If $T_{n+1} = T_4$ join the new vertex to any of the three peripheral vertices.

These graphs will be denoted by $\widetilde{T_{n+2}}$.

We will denote by $\widetilde{T_4}$ the graph obtained by the adjunct of new vertex done as above but joined to the central vertex of P_3:



Theorem II.17 *The graphs $\widetilde{T_n}$, $n \geq 4$, are Coxeter Euclidean CS graphs.*

Proof Let start with $\widetilde{T_4}$ and $\widetilde{T_5}$:

$$|\widetilde{T_4}| = |P_3| - \frac{1}{2}|I_1|^2 = \frac{4}{8} - \frac{1}{2} = 0;$$

$$|\widetilde{T_5}| = |T_4| - \frac{1}{2}|P_3| = \frac{1}{4} - \frac{1}{2}\frac{1}{2} = 0.$$

The general case $n \geq 6$ follows by construction.

The induced subgraphs on all vertices but one have the connected components among T_k, B_m, for some k, m, P_3 and/or I_1. So the CS condition is satisfied: $\widetilde{T_n}$ is a new family of Coxeter Euclidean CS graphs.

■

The graphs $\widetilde{T_n}$ can also be obtained as consecutives of B_n: join a new vertex by an unmarked edge to the last but one vertex counting from the unmarked end. This gives the desired result since

$$|B_{n+1}| - \frac{1}{4}|B_{n-1}| = 0.$$

Suppose join a new vertex by an unmarked edge to the last but one vertex of the stem (the unramified end) of T_n, $n \geq 5$: we call $\widetilde{H_{n+1}}$ the graph so obtained [10, I.1]. We consider $\widetilde{H_5}$ to be obtained from T_4 joining the new vertex to the central vertex of T_4.

Theorem II.18 *The graphs $\widetilde{H_n}$, $n \geq 5$, are Coxeter Euclidean CS graphs.*

Proof Let compute the Gramian of $\widetilde{H_5}$ and $\widetilde{H_6}$:

$$|\widetilde{H_5}| = |T_4| - \frac{1}{4}\,|I_1|^3 = \frac{1}{4} - \frac{1}{4} = 0.$$

$$|\widetilde{H_6}| = |T_5| - \frac{1}{4}|P_3|\,|I_1| = \frac{1}{8} - \frac{1}{4}\frac{1}{2} = 0.$$

Now if $n \geq 7$ we have the general formula

$$|\widetilde{H_n}| = |T_{n-1}| - \frac{1}{4}|T_{n-3}|\,|I_1| = \frac{1}{2^{n-3}} - \frac{1}{4}\frac{1}{2^{n-5}} = 0.$$

The CS condition is satisfied, since the induced subgraphs on all vertices but one are among T_k, for some integer k, P_3 and/or I_1. So $\widetilde{H_n}$ are Coxeter Euclidean CS graphs, as claimed.

■

We have obtained four infinite families of Coxeter Euclidean graph.

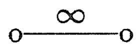
Let now look at what happens in low dimensions:

$n = 0$ There are no Coxeter Euclidean graphs. (The one vertex graph is spherical since its Gramian is positive.)

$n = 1$ The only Euclidean simplex is a segment, the roots are opposite linearly dependent vectors; the Gram matrix is

$$\left\| \begin{matrix} 1 & -1 \\ -1 & 1 \end{matrix} \right\|$$

and the Coxeter graph is

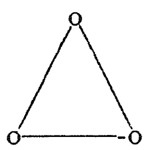


.

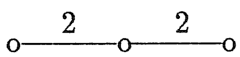
$n = 2$ There are three Coxeter graphs on 3 vertices: the graph must be connected and the marks (k, m, t) must verify the equation

$$\frac{\pi}{k+2} + \frac{\pi}{m+2} + \frac{\pi}{t+2} = \pi.$$

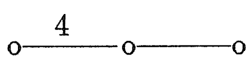
The only non-negative integer solutions are $(1,1,1)$, $(0,2,2)$ and $(0,1,4)$: the 3-cycle $\widetilde{P_3}$



and



$\widetilde{B_3}$



3 Coxeter hyperbolic graphs

The graph of a hyperbolic simplex is connected, since it follows in the marked graph partial order Euclidean ones (maybe not Coxeter's, if we suppose admissible to drop the condition on integral marks).

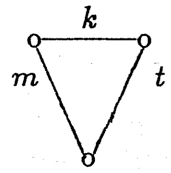
There are no infinite families of Coxeter hyperbolic graphs.

Let, as before, n denote the dimension of the space X^n where we seek for n-simplices: recall that the order of the graph is then $n+1$.

$n = 0$ There are no Coxeter hyperbolic graphs.

$n = 1$ There are infinitely many hyperbolic bounded segments corresponding to Λ-angles with vertex at the origin of $E^{1,1}$. But the two vertices of the Coxeter graph would have to be joined by a dotted edge since this situation is a particular case of ultraparallels hyperplanes, although such a concept makes no much sense in dimension one as hyperplanes are points. Moreover this situation does not appear for the simplices in dimension $n \geq 2$.

$n = 2$ There are infinitely many hyperbolic Coxeter graphs on 3 vertices:



those whose marks (k, m, t) satisfy

$$\frac{\pi}{k+2} + \frac{\pi}{m+2} + \frac{\pi}{t+2} \quad < \quad \pi.$$

They correspond to bounded hyperbolic 2-simplices. The unbounded hyperbolic 2-simplices of finite volume are obtained if we allow k and/or m and/or t to be ∞.

4 Construction and classification

The developed theory allow to classify Coxeter simplices by the classification of Coxeter graphs. We will do it for connected graphs. Let briefly recall the proved results.

If a connected Coxeter graph of order $n+1$ has all induced subgraphs of order n spherical it is the graph of a bounded n-simplex; the geometrical type is determined by the sign of the Gramian:

positive	$\Longrightarrow$	spherical
null	$\Longrightarrow$	Euclidean
negative	$\Longrightarrow$	hyperbolic

if induced subgraphs are allowed to be Euclidean, then the simplex is hyperbolic unbounded but of finite volume.

For each order the condition requires the knowledge of the classification for less orders: an inductive process will be performed. Keeping mind of the various Gramians, the consecutives are easily computed by the formula of lemma II.9.

There are added 4 Tables to the present work, from page iv to page ix.

The following method treats simultaneously the three geometrical cases; to distinguish the geometrical type, in the Tables, we have enclosed Euclidean graphs with a rectangle, hyperbolic bounded graphs with two rectangles, hyperbolic unbounded graphs with three rectangles. The graphs without picture-frame are the spherical ones.

To perform the inductive classification we might have started for each dimension n from the totally disconnected graph on $n+1$ vertices, I_{n+1} (as we use to denote it the symbol of its Gram matrix). The Gramian of I_{n+1} is 1; I_{n+1} is a Coxeter spherical CS graph. If we construct all its possible CS- and CSE-consecutives without increasing the order, we would obtain also disconnected spherical graphs; since we are interested in connected graphs we will proceed differently.

We will first at all start from P_1, the one vertex graph. Then make consecutives by adjuncts of type 3): all graphs will be unmarked. The process looks like an algorithm:

put P_1 in the Table;

<u>repeat</u> $\Gamma :=$ a spherical or Euclidean graph without consecutives from the Table;

  <u>while</u> there are vertices $\gamma \in \mathcal{V}(\Gamma)$ which have not been considered <u>do</u>

   add a new vertex to Γ;

   join it to γ;

   <u>if</u> the new graph Γ' is a CS or a CSE graph (the induced subgraphs are unmarked and must be precedent elements in the Table);

    <u>then</u> add Γ' to the Table;

     compute the Gramian and enclose Γ' in the relative number of rectangles

<u>until</u> we have reached the desired order.

Doing such an algorithmic construction a lot of situations repeat by symmetry.

The first steps are very easy since P_1 has only one consecutive: P_2.

Also P_2 has only one consecutive: P_3.

Then there are different possibilities: but the CS and CSE graphs are quickly recognized.

The CS and CSE graphs so obtained up to order 10 are contained in Table 1.

Note II.19 *Since an induced subgraph must be spherical or Euclidean, the condition applies also to its subgraphs. All induced subgraphs on all vertices but one* *of the induced subgraph of the inital graph must be spherical, i.e. all induced subgraph on all but two vertices of the initial graph must be spherical. Going on in this process it follows that each induced subgraph on $n-s$ vertices, for $2 \leq s < n$, must be spherical.*

About the values of the Gramians, we see that all are less than $|P_1| = 1$. Note that a graph can be the consecutive of more than one and also more than two graphs: its Gramian will be less than the minimum of the Gramians of its precedents.

Recall that a vertex is said to be of *ramification* if at least three vertices are incident to it, i.e. it is of valence at least 3. If the graph is a tree the connected components originated by the removal of a vertex are called *arms*. The *length* of an arm is the maximum of the distances (measured in number of edges) between the vertex and a peripheral vertex of the arm.

With the Table 1 completed up to order 10 it can be observed that if a graph is spherical then

- a vertex has valence at most three
- there cannot be two vertices of ramification
- if there is a vertex of ramification and two arms have length greater or equal to 2 then the third must be of lenght 1
- if there is a ramification vertex, one arm has length one and another length two, then the third must have length less than five.

Let illustrate the other Tables.

Table 2 contains graphs obtained by an adjuct of type 1) from the graphs of Table 1.

Table 3 contains the graphs consecutive to the ones of Tables 1 and 2 with 4 vertices, Table 4 those with 5 vertices. In these Tables it is increased the number of edges.

In spite of a first impression, got from the low dimension examples, the number of hyperbolic simplices does not grow widely with dimension, on the contrary there are no bounded hyperbolic Coxeter simplices in dimension greater than 4. And there are no Coxeter hyperbolic simplices of finite volume in dimension greater than 9.

This is because the CS and CSE conditions soon become very restrictive.

Let prove for example the following:

Theorem II.20 *The Coxeter graph*

$$o\text{———}o\overset{2}{\text{———}}o\text{———}o\text{—}\ \ldots\ \text{—}o$$

on n, $n \geq 3$, vertices is spherical for $n \leq 4$, is Euclidean for $n = 5$ and is hyperbolic unbounded for $n = 6$.

Proof If $n = 3$ the graph is just B_3, which we know it is spherical.

Let $n \geq 4$ and denote by Γ_n our graph. Then

$$|\Gamma_n| = |B_{n-1}| - \frac{1}{4}|P_{n-2}| = \frac{1}{2^{n-2}} - \frac{1}{4}\frac{n-1}{2^{n-2}};$$

so

$$|\Gamma_n| = \frac{4}{2^n} - \frac{n-1}{2^n} = \frac{5-n}{2^n}$$

and $|\Gamma_n| > 0$ if and only if $5 > n$, $|\Gamma_n| = 0$ if and only if $n = 5$. If $n = 6$ then $|\Gamma_n| < 0$ and an induced subgraph on all vertices but one is Euclidean.

For $n \geq 7$ the CSE condition is no more satisfied.

■

Part III

Appendix

A Gram matrices of system of vectors

Let $(\mathbb{R}^n, \cdot_B)$ be the inner product space structure fixed on $\mathbb{R}^n$. To a finite set of vectors $\{v_i\}_{i=1,\dots,p}$ we associate the real symmetric matrix of their inner products

$$G(v_1, \dots, v_p) \quad = \quad (v_i \cdot_B v_j)_{i,j=1,\dots,p};$$

It is called the *Gram matrix* of the set of vectors $\{v_i\}_{i=1,\dots,p}$. The determinant of $G(v_1, \dots, v_p)$ is called the *Gramian* of $\{v_i\}_{i=1,\dots,p}$.

The Gram matrix of the standard basis $\{e_1, \dots, e_n\}$ in $(\mathbb{R}^n, \cdot_B)$ is just B.

Recall that a subspace of an inner product space is non-isotropic if the only common vector with its orthogonal subspace is the null vector.

Proposition III.1 *The Gramian of $\{v_i\}_{i=1,\dots,p}$ is different from zero if and only if the vectors v_i are linearly independent and the subspace they generate is non-isotropic.*

Proof Let $G = G(v_1, \dots, v_p)$ and let W be the subspace generated by the v_i's. Suppose $|G(v_1, \dots, v_p)| \neq 0$. If we suppose that there exists a null linear combination of the v_i's

$$\xi = \sum_{i=1}^{p} \xi_i v_i = 0, \qquad \text{with} \qquad (\xi_1, \dots, \xi_p) \neq (0, \dots, 0)$$

then the identities $\quad 0 = 0 \cdot_B v_j = \xi \cdot_B v_j, \quad j = 1, \dots, p \quad$ show that $(\xi_1, \dots, \xi_p)$ is a non-trivial solution of the homogeneous linear system, in the indeterminates λ_i,

$$(*) \quad \left\{ \sum_{i=1}^{p} g_{ji} \lambda_i = 0 \ , \ j = 1, \dots, p. \right.$$

But this contradicts the supposed non-degeneracy of G. So we have proved that the p vectors are linearly independent and so provide a basis of W. In particular $p \leq n$.

Let now prove that W is non-isotropic, again by reductio ad absurdum.

Suppose there exists $\xi \in W \cap W^{\perp_B}$ and $\xi \neq 0$. Then

$$\xi = \sum_{i=1}^{p} \xi_i \, v_i, \qquad \text{and} \qquad \xi \cdot_B v_j = 0, \quad j = 1, \dots, p.$$

Substitute the expression of ξ in the last p identities. This shows that $(\xi_1, .., \xi_p)$ is a non-trivial solution for the homogeneous system $(*)$. Which is once again an absurd.

Conversely, let $v_1, ..., v_p$ be linearly independent - a basis for W - and W non-isotropic. If we suppose $|G| = 0$ then the system $(*)$ has a non-trivial solution $(\xi_1, ..., \xi_p)$. The vector $\xi = \xi_1 v_1 + ... + \xi_p v_p \in W \setminus \{0\}$ is such that

$$\xi \cdot_B v_i = 0, \quad i = 1, ..., p.$$

So $\xi \in W^{\perp_B}$. This yields $\xi = 0$, since the supposed non-isotropicity tells that $W \cap W^{\perp_B} = \{0\}$. And an absurd has been obtained.

■

Proposition III.2 *If the inner product $\cdot_B$ is positive then the Gramian of any set of vectors is non-negative.*

Proof

Let $\{v_i\}_{i=1,...,p}$ be the set of vectors in $(\mathbb{R}^n, \cdot_B)$. Since $\cdot_B$ is positive all subspaces are non-isotropic, so the Gramian is null if and only if the vectors are linearly dependent.

Suppose the vectors are linearly independent. Denote by G the Gram matrix. Then whenever $x = (x_1, ..., x_p)^t \in \mathbb{R}^p \setminus \{0\}$ the vector $\sum_{i=1}^p x_i v_i \in \mathbb{R}^n$ is non null. So

$$\left(\sum_{i=1}^{p} x_i v_i\right) \cdot_B \left(\sum_{j=1}^{p} x_j v_j\right) > 0$$

and the quadratic form q_G defined on $\mathbb{R}^p$ by $q_G(x) = x^t\, G\, x$ is positive definite. Hence G has positive determinant.

■

Proposition III.3 *Let $\{v_i\}_{i=i,...,n}$ be a set of n vectors in the inner product space structure $(\mathbb{R}^n, \cdot_B)$. Suppose that their Gram matrix $G = G(v_1, ..., v_n)$ is non-degenerate; then G is congruent to the matrix B. In particular if $\{v_1, ..., v_n\}$ becomes the chosen basis of $\mathbb{R}^n$ the matrix B is replaced by G.*

Proof The set of vectors $\{v_1, ..., v_n\}$ is a basis since G is non-degenerate.

Let $v = (v_1, ..., v_n)$ and $e = (e_1, ..., e_n)$ collect the basis as row vectors. Let C be the matrix of the change of basis. Then $v = e\,C$. Let x_v be the column matrix of the components of the vector x in the base v, and similarly x_e in e. Then $x_e = C\, x_v$ and $x \cdot_B y = x_e^t\, B\, y_e = (C\, x_v)^t\, B\, (C\, y_v) = x_v^t\, C^t\, B\, C\, y_v = x_v^t\, A\, y_v = x \cdot_A y$.

Let use the matricial row-column product also with the inner product, whenever the entries are vectors: then

$$G = v^t \cdot_B v = (e\,C)^t \cdot_B (e\,C) = C^t\, e^t \cdot_B e\, C = C^t\, B\, C = A.$$

■

Corollary III.4 *Let $\{v_i\}_{i=i,...,N}$ be a set of $N \geq n$ vectors in the inner product space $(\mathbb{R}^n, \cdot_B)$. Suppose that their Gram matrix $G = G(v_1, ..., v_N)$ has the same rank as the matrix B defining the inner product in the standard basis of $\mathbb{R}^n$. Then $index(G) = index(B)$.*

Proof Apply the previous proposition to a submatrix of maximal rank. It is the Gramian of a basis, and is congruent to B. So they have the same index.

■

B Existence and uniqueness theorems

To determine a polytope starting from a matrix we need an existence and a uniqueneness theorem about collection of vectors having that matrix as their Gram matrix.

Theorem III.5 *Let G be a symmetric real matrix of order N, rank n and index s. Then there exist N vectors in $(\mathbb{R}^n, \cdot_{I(s,n-s)})$ whose Gram matrix is G.*

Proof

Start by the standard vectors $e_1, ..., e_N$ in $\mathbb{R}^N$ and consider the inner product space structure given by G. Then there exists a non-singular matrix $\mathcal{C}$ such that $\mathcal{C}^t G \mathcal{C} = I(s, n-s, N-n)$ [2]. Let $\varepsilon_1, ..., \varepsilon_N$ be the new basis obtained by the transformation given by $\mathcal{C}$:

$$\varepsilon = e\mathcal{C}.$$

The isotropic vectors of the new basis are the last $N-n$.

Then $e = \varepsilon\, \mathcal{C}^{-1}$; let c_{ij}^{-1} denote the entries of $\mathcal{C}^{-1}$.

The totally isotropic part can be forgotten taking the following quotient map

$$\Upsilon : \mathbb{R}^N \to \mathbb{R}^N / \langle \varepsilon_{n+1}, ..., \varepsilon_N \rangle$$

$$\text{where } \Upsilon(e_j) = \Upsilon(\sum_{i=1}^{N} \varepsilon_i c_{ij}^{-1}) = (c_{1j}^{-1}, ..., c_{nj}^{-1}), \quad j = 1, ..., N.$$

$\Upsilon(e_1), ..., \Upsilon(e_N) \in \mathbb{R}^n$ and it results

$$\Upsilon(e_i) \cdot_{I(s,n-s)} \Upsilon(e_j) = e_i \cdot_G e_j = g_{ij}.$$

Note that $\langle \varepsilon_{n+1}, ..., \varepsilon_N \rangle = \langle \varepsilon_1, ..., \varepsilon_n \rangle^{\perp_B} = (\mathbb{R}^N)^{\perp_B}$.

■

The question of uniqueness is answered as follows:

Theorem III.6 *Let $(\mathbb{R}^n, \cdot_B)$ be a non-degenerate inner product space. Two system of vectors, with the same cardinality N, both containing a basis of $\mathbb{R}^n$, $n \leq N$, have the same Gram matrix if and only if they are mapped bijectively one into the other by a motion.*

Let before state and prove a lemma, which is a restricted version:

Lemma III.7 *Let $(\mathbb{R}^n, \cdot_B)$ be a non-degenerate inner product space. Two basis have the same Gram matrices if and only if there is a motion which transforms one into the other.*

Proof Let $\{v_i\}_{i=1,\dots,n}$ and $\{u_i\}_{i=1,\dots,n}$ be the two bases.

Suppose first that there exists a motion

$$\phi : (\mathbb{R}^n, \cdot_B) \to (\mathbb{R}^n, \cdot_B)$$

$$\phi(x) \cdot_B \phi(y) = x \cdot_B y, \quad \forall x, y \in \mathbb{R}^n$$

and

$$\phi(v_i) = u_i, \quad i = 1, \dots, n.$$

Then $(G_u)_{ij} = u_i \cdot_B u_j = \phi(v_i) \cdot_B \phi(v_j) = v_i \cdot_B v_j = (G_v)_{ij}$. So the Gram matrices coincide.

Suppose now conversely that $G_v = G_u$. There is, uniquely determined, a linear map ϕ from $\mathbb{R}^n$ to $\mathbb{R}^n$ which maps v_i to u_i. It can be determined by linear algebra techniques on the vectors of the two bases and then extended by linearity.

The assumption on the Gram matrices implies that

$$v_i \cdot_B v_j = u_i \cdot_B u_j = \phi(v_i) \cdot_B \phi(v_j).$$

Let express a vector $x \in \mathbb{R}^n$ in the basis $\{v_i\}$: let $x = \sum_{i=1}^n x_i v_i$. The linearity of ϕ and the invariance of the inner product $\cdot_B$ on the basis imply that

$$\phi(x) \cdot_B \phi(x) = x \cdot_B x, \quad \forall x \in \mathbb{R}^n.$$

Then ϕ is a motion of $(\mathbb{R}^n, \cdot_B)$.

■

Let now prove theorem III.6

Proof Let $\{v_i\}_{i=1,\dots,N}$ and $\{u_i\}_{i=1,\dots,N}$ be the two system of vectors and suppose the bases are formed by the first n vectors in both the systems.

If the motion ϕ exists the Gram matrices coincide, just by a matter of computation as before.

For the converse, by the lemma there exists a motion ϕ such that

$$\phi(v_i) = u_i, \quad i = 1, \dots, n.$$

$$\text{Let} \qquad v_k = \sum_{i=1}^n \nu_i^k v_i \quad \text{and} \quad u_k = \sum_{i=1}^n \mu_i^k u_i, \quad k = n+1, \dots, N.$$

The coincidence of the Gram matrices implies that

$$\nu_i^k = \mu_i^k, \quad i = 1, \dots, n, \quad k = n+1, \dots, N.$$

Then for $k = n+1, ..., N$:

$$\phi(v_k) = \phi(\sum_{i=1}^{n} \nu_i^k v_i) = \sum_{i=1}^{n} \nu_i^k \phi(v_i) = \sum_{i=1}^{n} \mu_i^k u_i = u_k.$$

So ϕ sends one system of vectors to the other.

■

All the system of vectors that will be considered will contain a basis of the space. But if this would not be the case, just consider the generated subspaces.

Somebody would maybe be tempted to ask: "If two systems of vectors in $\mathbb{R}^n$ have the same Gram matrix in one inner product space structure, would they have the same Gram matrix also in another?" The answer is evident taking degenerate inner products of different ranks. But it is also negative in the non-degenerate case, as shown by the following counterexample:

Example III.8 $n = 2$

$$v_1 = (1,0), v_2 = (0,1); \quad u_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}), u_2 = (\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2})$$

Then in $E^2 = (\mathbb{R}^2, \cdot_E)$ *we have* $G_v = G_u = I_2$. *But in* $E^{1,1} = (\mathbb{R}^2, \cdot_\Lambda)$, *we have* $G_v = \begin{Vmatrix} 1 & 0 \\ 0 & -1 \end{Vmatrix}$ *and* $G_u = \begin{Vmatrix} 0 & 1 \\ 1 & 0 \end{Vmatrix}$.

C Gram-Schmidt extended process

The *Gram-Schmidt orthogonalization process* can be extended to the degenerate case (with a non restrictive assumption verified in most of the examples) following the steps presented in [3, §4] for the non-degenerate case. It will provide us a practical computational tool.

Let B be a square $N \times N$ symmetric real matrix of rank n. Consider the inner product space structure defined on $\mathbb{R}^N$ by B:

$$e_i \cdot_B e_j = b_{ij}, \quad 1 \leq i, j \leq N.$$

We want to determine an orthogonal basis, such that the non-isotropic vectors of it are of unit B-length. Suppose reordering of the standard basis has already taken place such that the first principal submatrix of order n (cancel the last $N - n$ rows and columns) is non-degenerate and all its leading principal minors are different from zero. Let denote in the proof the reordered basis still with $\{e_1, ..., e_N\}$.

Let, for short, B^j denote

if $j \leq n$, the first leading principal submatrix of order j, $\quad B_{[1...j]}^{[1...j]}$

otherwise the $(n+1)\times(n+1)$ submatrix

$$\left\| \begin{matrix} b_{11} & b_{12} & \dots & b_{1n} & b_{1j} \\ b_{12} & b_{22} & \dots & b_{2n} & b_{2j} \\ \vdots & \vdots & & \vdots & \vdots \\ b_{1n} & b_{2n} & \dots & b_{nn} & b_{nj} \\ b_{1j} & b_{2j} & \dots & b_{nj} & b_{jj} \end{matrix} \right\|.$$

Denote by $(B^j)_{sj}$ the cofactor of b_{sj} in B^j.

Define

$$o_j = e_j + \sum_{s=1}^{j-1}(B^j)^{-1}_{jj}(B^j)_{sj}e_s \qquad \text{if } j \leq n$$

$$o_j = e_j + \sum_{s=1}^{n}(B^j)^{-1}_{jj}(B^j)_{sj}e_s \qquad \text{if } j > n.$$

We will prove that $\{o_j\}$ is an orthogonal basis for $(\mathbb{R}^N, \cdot_B)$. The extension to what said by Bourbaki in [3] is referred to the isotropic vectors, so let suppose $j > n$ and $1 \leq k \leq n$, then

$$\begin{aligned} o_j \cdot_B e_k &= \textstyle\sum_{s=1}^n (B^j)^{-1}_{jj}(B^j)_{sj}e_s \cdot_B e_k + (B^j)^{-1}_{jj}(B^j)_{jj}e_j \cdot_B e_k \\ &= (B^j)^{-1}_{jj}\left[\textstyle\sum_{s=1}^n (B^j)_{sj}b_{sk} + (B^j)_{jj}b_{jk}\right]. \end{aligned}$$

Since $\{b_{1k}, b_{2k}, ..., b_{nk}, b_{jk}\}$ is the k-th column of B^j and $\{(B^j)_{1j}, (B^j)_{2j}, ..., (B^j)_{nj}, (B^j)_{jj}\}$ are the cofactors of the $(n+1)$-th column of B^j and $k \leq n$, it follows that o_j is B-orthogonal to e_k.

As the subspace spanned by $\{e_1, ..., e_n\}$ coincides with that spanned by $\{o_1, ..., o_n\}$, the new vector o_j is B-orthogonal to o_k for $\;1 \leq k \leq n$.

Now suppose still $j > n$ and also $k > n$:

$$\begin{aligned} o_j \cdot_B o_k &= \left[\textstyle\sum_{s=1}^n (B^j)^{-1}_{jj}(B^j)_{sj}e_s + e_j\right] \cdot_B \left[\textstyle\sum_{s=1}^n (B^j)^{-1}_{jj}(B^j)_{sj}e_s + e_j\right] \\ &= b_{jk} + (B^j)^{-1}_{jj}\left[\textstyle\sum_{s=1}^n (B^j)_{sj}b_{sk}\right] + (B^k)^{-1}_{kk}\left[\textstyle\sum_{l=1}^n (B^k)_{lk}b_{lj}\right] \\ &\quad + (B^j)^{-1}_{jj}(B^k)^{-1}_{kk}\left[\textstyle\sum_{s,l}^{1,n} (B^j)_{sj}(B^k)_{lk}b_{ls}\right] \end{aligned}$$

Use properties of determinants, in particular the expansion formula of a determinant by the cofactors (first theorem of Laplace) [11, ch.4.16], and keep in mind that B is supposed of rank n: then

$$\sum_{l=1}^n (B^k)_{lk}b_{lj} = -(B^k)_{kk}b_{kk}$$

$$\sum_{s=1}^n (B^j)_{sj}b_{sk} = -(B^j)_{jj}b_{jj}.$$

Note that $(B^j)_{jj} = (B^k)_{kk}$.

Consider the submatrix of B obtained taking the rows $1, 2, ..., n, j$ and the columns $1, 2, ..., n, k$:

$$\beta = \begin{Vmatrix} b_{11} & b_{12} & \dots & b_{1n} & b_{1k} \\ b_{12} & b_{22} & \dots & b_{2n} & b_{2k} \\ \vdots & \vdots & & \vdots & \vdots \\ b_{1n} & b_{2n} & \dots & b_{nn} & b_{nk} \\ b_{1j} & b_{2j} & \dots & b_{nj} & b_{jk} \end{Vmatrix}.$$

Then

$$\sum_{s=1}^{n} \beta_{sk} b_{sk} + \beta_{n+1,n+1} b_{jk} = |\beta| = 0$$

but the cofactors of b_{sk} in β are just the cofactors of b_{sj} in B^j, so

$$\sum_{s=1}^{n} (B^j)_{sj} b_{sk} + (B^j)_{jj} b_{jk} = 0.$$

Substituting in $o_j \cdot_B o_k$ all cancel and it results equal to zero.

Suppose that $\{o_j\}$ has been reordered such that the positive vectors all come first. Remember that we have define $|o_j|_B = \sqrt{|o_j \cdot_B o_j|}$; then put

$$\begin{array}{rcll} \varepsilon_j & = & \frac{o_j}{|o_j|_B} & \text{for } j = 1, ..., n \\ \varepsilon_j & = & o_j & \text{for } j = n+1, ..., N \end{array}.$$

$\{\varepsilon_j\}$ is an "orthonormal" basis for $(\mathbb{R}^N, \cdot_B)$.

The formulas for the o_j's can be obtained in an inductive approach:

put

$$o_1 = e_1$$

then (for the first step it is just a matter of rewriting) the set $\{o_1, e_2, ..., e_N\}$ is a basis and its Gram matrix differs from B in rows $1, 2$ and in columns $1, 2$. In particular the leading principal submatrix of order 2 is

$$\begin{Vmatrix} o_1 \cdot_B o_1 & o_1 \cdot_B e_2 \\ o_1 \cdot_B e_2 & e_2 \cdot_B e_2 \end{Vmatrix}.$$

So

$$o_2 = -\frac{e_2 \cdot_B o_1}{o_1 \cdot_B o_1} o_1 + e_2.$$

Then $\{o_1, o_2, e_3, ..., e_n\}$ is a new basis and its Gram matrix differs from B in rows $1, 2, 3$ and columns $1, 2, 3$. The leading principal submatrix of order 3 is (we have supposed $n \geq 3$)

$$\begin{Vmatrix} o_1 \cdot_B o_1 & 0 & o_1 \cdot_B e_3 \\ 0 & o_2 \cdot_B o_2 & o_2 \cdot_B e_3 \\ e_3 \cdot_B o_1 & e_3 \cdot_B o_2 & e_3 \cdot_B e_3 \end{Vmatrix}.$$

(Note that the product is $\cdot_B$ since the vectors o_i are expressed by the e_j.)
Then computing the minors we obtain

$$o_3 = -\frac{e_3 \cdot_B o_1}{o_1 \cdot_B o_1} o_1 - \frac{e_3 \cdot_B o_2}{o_2 \cdot_B o_2} o_2 + e_3.$$

Going on we will obtain

$$\text{for } j \leq n \qquad o_j = e_j - \sum_{s=1}^{j-1} \frac{e_j \cdot_B o_s}{o_s \cdot_B o_s} o_s$$

$$\text{for } j > n \qquad o_j = e_j - \sum_{s=1}^{n} \frac{e_j \cdot_B o_s}{o_s \cdot_B o_s} o_s.$$

D Example

Consider the following example of a hyperbolic quadrilater in Λ^2:
the hyperplanes $H_1, ..., H_4$ are the intersections of the following linear subspaces of $E^{2,1}$ with Q^+

$$\begin{array}{lcrcrcl} \mathcal{H}_1 & : & 2y & - & z & = & 0 \\ \mathcal{H}_2 & : & 2x & - & z & = & 0 \\ \mathcal{H}_3 & : & 2y & + & z & = & 0 \\ \mathcal{H}_4 & : & 2x & + & z & = & 0 \end{array}$$

the roots are

$$\begin{aligned} v_1 &= \frac{1}{\sqrt{3}}(0,2,1) \\ v_2 &= \frac{1}{\sqrt{3}}(2,0,1) \\ v_3 &= \frac{1}{\sqrt{3}}(0,-2,1) \\ v_4 &= \frac{1}{\sqrt{3}}(-2,0,1) \end{aligned}$$

The polygon is $\mathcal{P} = \bigcap_{i=1}^4 H_i^-$.
The Gram matrix is

$$G = \begin{Vmatrix} 1 & -\frac{1}{3} & -\frac{5}{3} & -\frac{1}{3} \\ -\frac{1}{3} & 1 & -\frac{1}{3} & -\frac{5}{3} \\ -\frac{5}{3} & -\frac{1}{3} & 1 & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{5}{3} & -\frac{1}{3} & 1 \end{Vmatrix}.$$

The determinant of G is zero; G defines a degenerate inner product on $\mathbb{R}^4$.

We can suppose have started with G, and forget about the polygon we were started with.

The extended Gram-Schmidt method of orthogonalization applies to the previous matrix G (without need of reordering) and gives

$$\begin{aligned} o_1 &= e_1 \\ o_2 &= \frac{1}{3}\, e_1 + e_2 \\ o_3 &= 2\, e_1 + e_2 + e_3 \\ o_4 &= -e_1 + e_2 - e_3 + e_4. \end{aligned}$$

In this case there is no need nor to reorder the basis obtained. And the coordinates of the vectors ε_i are

$$\begin{aligned} \varepsilon_1 &= (\; 1, \; 0, \; 0, \; 0 \;) \\ \varepsilon_2 &= (\; \tfrac{\sqrt{2}}{4}, \; \tfrac{3\sqrt{2}}{4}, \; 0, \; 0) \\ \varepsilon_3 &= (\; \tfrac{\sqrt{6}}{2}, \; \tfrac{\sqrt{6}}{4}, \; \tfrac{\sqrt{6}}{4}, \; 0 \;) \\ \varepsilon_4 &= (\; -1, \; 1, \; -1, \; 1 \;) \end{aligned}$$

These are the coordinates of the ε_j's in the base $\{e_i\}$ so the product is still $\cdot_B$. If we express the vectors of $\mathbb{R}^4$ in the basis $\{\varepsilon_j\}$ then the product to be used is $\cdot_{I(2,1,0)}$.

In practice, for our purposes, it is not useful to compute explicitly the coordinates of ε_j, i.e. the matrix $\mathcal{C}$. We need $\mathcal{C}^{-1}$ and this can be obtained easily from $\mathcal{O}$, which is triangular with all diagonal entries equal to 1:

$$\begin{cases} o_i &= |o_i|_G\, \varepsilon_i = (e\,\mathcal{O})_i, \qquad i = 1, ..., 3 \\ o_4 &= \varepsilon_4 = (e\,\mathcal{O})_4 \end{cases}$$

Then

$$\begin{aligned} e_1 &= \varepsilon_1 \\ e_2 &= -\frac{1}{3}\, \varepsilon_1 + \frac{2\sqrt{2}}{3}\, \varepsilon_2 \\ e_3 &= -\frac{5}{3}\, \varepsilon_1 - \frac{2\sqrt{2}}{3}\, \varepsilon_2 + \frac{2\sqrt{6}}{3}\, \varepsilon_3 \\ e_4 &= -\frac{1}{3}\, \varepsilon_1 - \frac{4\sqrt{2}}{3}\, \varepsilon_2 + \frac{2\sqrt{6}}{3}\, \varepsilon_3 + \varepsilon_4 \end{aligned}$$

So as ε_4 is isotropic we can forget about it and just consider the vectors in $(\mathbb{R}^3, \cdot_{I(2,1)})$, which is done by the previously defined map Υ. The root system $\{\Upsilon(e_i)\}_{i=1,\dots,4}$ has the same Gram matrix of $\{v_i\}_{i=1,\dots,4}$. The motion that transforms one system in the other has matrix

$$\Phi = \left\| \begin{matrix} 0 & \frac{2\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{2} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} \\ -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{6}}{6} \end{matrix} \right\| .$$

References

[1] E.M. Andreev. On convex polyhedra in l obačevskij spaces. *Math. USSR Sbornik*, Vol. 10(No.3):413–440, 1970.

[2] F. Ayres Jr. *Matrices.* McGraw-Hill Book Company, New York, 1962.

[3] N. Bourbaki. *Éléments de mathématique,XXIV,Formes sesquilinéaires et formes quadratiques.* Hermann, Paris, 1959.

[4] N. Bourbaki. *Éléments de mathématique,XXXIV,Groupes et algébres de Lie.* Hermann, Paris, 1968.

[5] H.S.M. Coxeter. Discrete groups generated by reflections. *Ann. Math.*, Vol 35, No. 3:588–620, 1934.

[6] H.S.M. Coxeter. *Non-Euclidean Geometry.* Univ. of Toronto Press, Toronto, 5th ed. edition, 1965.

[7] H.S.M. Coxeter. *Regular polytopes.* Dover Publications, Inc., New York, 3rd ed. edition, 1973.

[8] M. Crampin and F.A.E. Pirani. *Applicable differential geometry.* Cambridge University Press, Cambridge, london mathematical society, lecture notes series 59 edition, 1986.

[9] F.R. Gantmacher. *The theory of matrices, volume two.* Chelsea Publishing Company, New York,N.Y., 1959.

[10] F.M. Goodman, P. de la Harpe, and V.F.R. Jones. *Coxeter graphs and towers of algebras.* Springer-Verlag, New York, 1989.

[11] W.H. Greub. *Linear algebra.* Springer-Verlag, Berlin-Heidelberg, 3rd ed. edition, 1967.

[12] J.L. Gross and T.W. Tucker. *Topological Graph Theory.* Wiley-Interscience, New York, 1987.

[13] L.C. Grove and C.T. Benson. *Finite reflection groups.* Springer-Verlag, New York, 2nd ed. edition, 1985.

[14] I. Kaplansky. *Linear algebra and geometry: a second course.* Chelsea Publishing Company, New York,N.Y., 2nd ed. edition, 1974.

[15] B. O'Neill. *Elementary differential geometry.* Academic Press, London, 1966.

[16] B. O'Neill. *Semi-Riemannian geometry.* Academic Press, London, 1983.

[17] P.J. Ryan. *Euclidean and non-Euclidean geometry: an analytic approach.* Cambridge University Press, Cambridge, 1986.

[18] E. Seneta. *Non-negative matrices.* George Allen & Unwin Ltd, London, 1973.

[19] W.P. Thurston. *The geometry and topology of 3-manifolds.* Princenton University, Princenton, lecture notes edition, 1978-79.

[20] E.B. Vinberg. Hyperbolic reflection groups. *Russian Math. Surveys*, 40:1:31–75, 1985.

[21] J.A. Wolf. *Spaces of constant curvature.* McGraw-Hill Book Company, New York, 1967.

Index

adjacent faces 7
arm 40

consecutive graphs 29
Coxeter diagram 26
Coxeter graph 26
Coxeter simplex 26

decomposable matrix 18
dihedral angle 7

elliptic subspace 4
Euclidean (affine) space E^n 3

face 5

Gram matrix 7 42
Gramian 42
Gram-Schmidt orthogonalization process 21 46

hyperbolic space Λ^n 3
hyperbolic subspace 4

indecomposable matrix 18
inner product 1
inner product, indefinite - 2
inner product, non-degenerate - 2
inner product, positive - 1
inner product space 1
irreducible matrix 18
isotropic subspace 2
isotropic vectors 2

lenght 2
length of an arm 40
lightlike subspace 4

matrix, decomposable - 18
matrix, irreducible - 18
metric homomorphism 2
motion 3

n-hedron 6

orthogonal complement 2
orthogonal vectors 2

parabolic subspace 4
polygon 5
polytopal angle 6
polytope 5
polytope, acute-angled - 8
polytope of simplicial type 7
polytope, non-degenerate - 6
polytope, non-obtuse-angled - 8
product, inner - 1
product, scalar - 1
pseudo-Euclidean vector space 2

ramification vertex 40
root system 5 21

scalar product 1
simplex, Coxeter - 26
simplicial prism 10
space, inner product - 1
spacelike subspace 4
sphere S^n 3
Sylvester's law of inertia 3 21
system of roots 5

timelike subspace 4
trihedron 6

vertex polytope 7