Soliton Theories and Quantum Integrability

Thesis submitted for the degree

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Contents

1 Introduction

In the last years two dimensional Field Theory has attracted a lot of attention. The approaches where quite different, partly originating from the interest in string theory, and on the other hand from statistical physics. The mergence point turned out to be conformal field theory.

Let's forget for the moment how the interest in this field came up, and what was the motivation. Observing two dimensional systems from its own right, one can try to find out properties and classify this space of theories. In this thesis I will concentrate on one special class of theories, integrable ones. From a practical point of view their importance lies in the fact that they can serve as a starting point for perturbation theory for realistic models (e.g. in surface physics). On the other hand there is the hope that they can be completely classified. For example conformal field theories: they are integrable, and describe the critical point of second order phase-transitions. Classifying those theories would amount in classifying the possible universality classes of statistical models in two dimensions.

But the conformal invariant theories do not exhaust the set of integrable theories. There are also integrable non-conformal theories, well-known for a long time such as for example in statistical mechanics the Ising-model or in soliton-theory the sine-Gordon model. What is their relation to conformal field theory? Since at the critical point they reduce to conformally invariant theories, the inverse process should be possible: starting from a conformal invariant theory one should be able to obtain those models by perturbation in some relevant direction in the space of couplings of the theory.

There are different ways to describe the resulting systems either one looks at them as lattice statistical models or as field theories. If the field theoretical model contains only massive particles, one has the possibility to describe the dynamics by the S-matrix approach. Against the lagrangian formulation of the theory this has the advantage that in the first ones quantization is a hard task, while the latter gives directly a quantum theory; this is particularly true for soliton theories which contain degenerate vacua, where perturbation theory cannot be applied. So not having at hand a direct quantization procedure for those theories one uses S-matrix theory to describe the interaction of solitons quantistically.

There is one structure which appears in all of the above mentioned concepts: quantum groups. They were first noticed in the form of the Yang-Baxter equation ([50] and [72]) which turned up as a consistency-equation for the factorization of the scattering matrix into a two particle one. Really the Yang-Baxter equation was known before in statistical mechanics as the Star-Triangle relation, already mentioned by Onsager [55] in his work about the Ising model in 1944, and later on for finding solutions of other statistical models [3] .

Another line of development was the theory of factorized scattering-matrix in two dimensional quantum field theory [77], where the same mechanism is at work as in the before mentioned models. Both can be related to the quantum inverse method, where basic commutation relations of operators are described by a solution of the Yang-Baxter equation. These works led to the idea of introducing certain deformations of groups and Lie algebras, where from the terminology "Quantum Group" arose [20] . These structures where uncovered in all disciplines of integrable two dimensional field theory, in conformal field theory (see e.g. [32, 51]), in lattice statistical models (e.g. [64]) and the S-matrix approach ([61],[47]), and therefore also in soliton theory. It seems to be the unification scheme of concepts for two-dimensional integrable systems.

This thesis consists of two major parts. The first one is a review of the above mentioned concepts and introduces integrable two dimensional systems and quantum groups. The second part, on the other hand, contains our recent results in this field. It deals with integrable restrictions of certain soliton theories which can be seen as perturbations of conformal field theories (CFT) and in addition we will compare these results with those obtained using S-matrix theory as an axiomatic theory.

Still a lot of work has to be done, many results are still on intuitive grounds. It seems to be like a puzzle, where one already guesses the picture and tries to fit in new stones. The beauty in the subject right now is, that the new 'stones' built in, fit the picture generally believed.

Part I

Concepts of Two Dimensional Physics

2 Introduction to 2D Systems

This chapter tries to give the basic concepts of two-dimensional physics. The subject is very vast and, in addition, it can be seen from many different point of views. Here we will try to give a comprehensive treatment of this topic. It contains mainly an account on systems near a special class of fixed points, called minimal conformal theories. Or to say it in a more descriptive way, it will introduce conformal field theories, perturbations of it, the relation to statistical mechanical models and last but not least S matrix theory. Since the subject is vast and excellent reviews exist (they will be cited along the way), I will restrict myself to give some basic ideas, combined with some results which will be useful later on.

2.1 Phase Transitions in Statistical Mechanics

Let's start with statistical mechanical models. They are usually simplified versions of "matter" (e.g. solids or gases) where values are assigned to be sites of a lattice. The form of the lattice depends on the geometry of the described system, as well as the values assigned to the sites. A well known example is the Ising model, taking two values describing the polarization of a metal, or surface models where the fluctuations of the surface is described by heights, taking integer values.

To introduce dynamics into this system one needs to couple the different sites. In the Ising model one has a nearest neighbour coupling

$$H = \sum_{<i,j>} J s_i s_j + \sum_i H_i s_i \quad ,$$

where J is the spin-spin interaction strength and H_i the magnetic field acting on the spin s_i, $s_i = \pm 1$, and $< i, j >$ indicating that the sum is over nearest neighbours. Though, this is no limitation. We could equally couple any sites, with different distances and also with a different number of sites. Therefore we have infinitely many possible systems, basically characterized by their couplings.

The feature we are interested in is that many of those systems exhibit phase transitions. That is, we observe discontinuities in the physical quantities, varying one or more couplings of our theory. These phenomena are well known, as for example spontaneous magnetization or the liquid gas transition. Let us consider a magnetic system (just to decide the terminology to use). The partition function is $Z = tr\ T$ (tr denoting the trace). $T = e^{\beta H}$ is called the transfer matrix, and β corresponds to the inverse of the temperature. For the Ising system we have

$$Z = \sum_{s_i=\pm 1} e^{\beta \sum_{i=1}^{l} J s_i s_{i+1} + h s_i} \quad ,$$

and the transfer matrix is a 2×2 matrix which can be written as

$$< s'|T|s >= e^{\beta [J s s' + \frac{h}{2}(s+s')]} \quad .$$

In the thermodynamic limit (i.e. $l \to \infty$) the partition function is related to the largest eigenvalue of the transfer matrix, λ_0. This can be calculated for a finite system (in theory) since it corresponds to diagonalizing the transfer matrix. To do this one would like T to be real and symmetric which is achieved restricting the interactions to be isotropic and the lattice to be hypercubic. This on the other hand implies that λ_0 is positive, and the corresponding eigenvector is unique and has strictly positive coefficients. It is a regular function in β.

The free energy W is given in the thermodynamic limit as $W \sim l \ln \lambda_0$. The connected two point spin correlation function behaves in this limit as

$$< s_i s_j >_c \equiv < s_i s_j > - < s_i >< s_j > \sim e^{\frac{-|i-j|}{\xi}} \quad ,$$

wherein $\xi = (\ln \frac{\lambda_0}{\lambda_1})^{-1}$ is the correlation length, and λ_1 is the second largest eigenvalue of T. Therefore for a finite system no phase transition occurs, i.e. the free energy is a regular function of the inverse temperature β, and the correlation length ξ remains finite, except for β infinite.

The situation changes when the transverse size l becomes infinite. Let's consider again the Ising system. Then in the high temperature limit spins up and spins down are equally distributed which in terms of the transfer matrix is seen that there is only one non vanishing eigenvalue. Therefore the correlation length vanishes. On the other hand in the low temperature limit all spins tend to be aligned up or down. At β strictly infinite, both states are eigenvectors of T, corresponding to the twice degenerate largest eigenvalue. We see that our results of finite systems do not apply any more. Notice that this corresponds to a breaking of the symmetry between spin up and spin down in the system (this happens only for $d > 1$). Looking at the correlation length, it diverges as $\xi \sim e^{\beta J l^{d-1}}$. Therefore no analytic continuation between high and low temperature situations is possible and thermodynamical quantities must have at least one singularity in β at some finite value β_c.

Define $\Gamma(M) = MH - W(H)$ where M is the magnetization $M = \frac{\partial W}{\partial H}$. Also let $V = \beta J$ and $H = \beta h$. Then $H = \frac{\partial \Gamma}{\partial M}$. So it follows that if $H = 0$, then the magnetization is given by an extremum of Γ. Furthermore since the partition function in zero field is $\exp(-l\Gamma(M))$ the dominant saddle points are minima of $\Gamma(M)$.

What happens if one varies the temperature and therefore also V ? One can calculate the minima of $\Gamma(M)$ in the mean field limit (which in field theory is the saddle point evaluation). When V increases, in general at some value of V other local minima appear which eventually become the absolute minima of $\Gamma(M)$. At this point, the magnetization makes a discontinuous jump, and the system undergoes a first order phase transition. The correlation length which is related to the second derivative of $\Gamma(M)$, $\xi \sim (\Gamma''(M)^{\frac{1}{2}})$, remains generically finite (see e.g. [82]). If no absolute minimum appears at a finite distance from the origin, then at V_c the origin ceases to be a minimum of the potential, and below this temperature T_c two minima move continuously away from the origin. Since the magnetization remains continuous at V_c, we obtain a second order phase transition at V_c. At the critical point $\Gamma''(M) = 0$ and so the correlation length diverges.

Though the former situation is the more common one, we are interested in the latter one. Because of the divergence of the correlation length, information depending on the local details of the interaction is wiped out at the critical point. Further one encounters universality, that is that differnent lagrangians give the same critical exponents. The criterium when this happens we will analyze in the next section.

2.1.1 Renormalization

Since the correlation length diverges at the critical point, one needs to take in account an 'infinite' amount of degrees of freedom. That is, to get near the critical point one needs to look at larger and larger areas of the system. This is the idea of renormalization.

Assume a statistical system involving a set of couplings $\{K_\alpha\}$. One forms a blockspin, which is an average over spins of a certain area. One supposes that the resulting system, formed by the blockspins, can be described by an interaction of the same type as the old one (which is the property of renormalizability), and tries to determine the new couplings $\{K'_\alpha\}$. This means that there exists a mapping in the space of couplings from $\{K_\alpha\}$ to $\{K'_\alpha\}$. If this mapping is smooth one can study the fixed point structure, i.e. the points where $K^\star_\alpha = R_\alpha(K^\star_\alpha)$, and their neighborhood. For that we linearize the equation around the fixed point

$$\delta K'_\alpha = K'_\alpha - K^\star_\alpha = R_{\alpha\beta} K_\beta \quad , \qquad R_{\alpha\beta} = \frac{\partial R_\alpha(K)}{\partial K_\beta}|_{K=K^\star} \quad ,$$

and determine the eigenvalues λ_i and the eigenvectors $e_{\alpha,i}$ of $R_{\alpha\beta}$. The next step is to change the basis in the space of couplings to that, according to the eigenvectors

$$\delta \tilde{K}_\alpha \equiv \sum_n u_n e^R_{\alpha,n} \quad , \qquad \text{that is} \quad \sum_\alpha < e^L_{\alpha,k}, \delta K_\alpha >= u_k$$

(R (L) denotes the right (left) eigenvector).These now transform under a renormalization group mapping according to the eigenvalues, $u'_k = \lambda_k u_k$. Therefore one can distinguish three classes of couplings:

$$\begin{array}{lll} \lambda_k > 1 & (\lambda_k)^n u_k \to \infty & \text{relevant coupling,} \\ \lambda_k = 1 & (\lambda_k)^n u_k \to ? & \text{marginal coupling,} \\ \lambda_k < 1 & (\lambda_k)^n u_k \to 0 & \text{irrelevant coupling.} \end{array} \qquad (1)$$

Universality can now be reformulated in a more precise way: Systems which differ only by irrelevant parameters will have the same behaviour near the critical point.

2.1.2 Transition to Field Theory

Recall that discretizing a scalar field theory, here e.g. ϕ^4 with

$$S_E = \int d^dx \{\phi(-\Delta\phi) + \mu^2\phi^2 + \lambda\phi^4 + h(x)\phi(x)\} \quad ,$$

we obtain the discretized generating functional Z_{dis}

$$Z_{dis} = \sum_{\phi_i} e^{\sum_{<ij>} \phi_i\phi_j - \sum h_i\phi_i(x)} \left[\prod_i e^{-\bar{\mu}^2\phi_i^2 - \lambda\phi_i^4} d\phi_i\right] \quad ,$$

$\bar{\mu} = \mu^2 + 2$, which is an Ising system with a weighted free spin distribution. Similarly the discretization of different field theoretical models leads to other statistical systems. The inverse limit, starting from a discrete system, is more difficult, because taking the limit of the lattice constant to zero, one obtains singularities which have to be removed by renormalization.

We consider now the critical theory in the field theoretical setting. Let's assume we have an action $S(\phi)$, with ϕ being some kind of local field. Then we know that the Noether current of a translation invariant system is the energy momentum tensor $T_{\mu\nu}$. A scale transformation acts as $x^\mu \to e^\lambda x^\mu = (1 + \lambda + \ldots)x^\mu$. It is a non constant transformation. The action transforms as

$$\delta S = \int d^2x T^{\mu\nu} \partial_\mu \epsilon_\nu \quad .$$

In this case $\epsilon_\nu = \lambda x_\nu$, and so we get

$$\delta S = \int d^2x [\lambda T^\mu_\mu + D^\mu \partial_\mu \lambda] \quad ,$$

where D^μ is defined as the Noether current related to scale transformations, according to the rule $\partial S = \int d^2x D_\mu \partial^\mu \lambda$. So we get the equation $\partial_\mu D^\mu = T^\mu_\mu$, and see that the tracelessness of the energy momentum tensor is a necessary condition for the system to be invariant under scale transformations.

Consider now special conformal transformations (always in two dimensions).

$$x^\mu \to x^\mu + (2x^\mu x^\nu - \delta^{\mu\nu} x^2) b_\nu \quad .$$

Remarkably the corresponding Noether current

$$K^{\nu\lambda} = 2x^\mu x^\nu T_{\mu\lambda} - x^2 T^{\nu\lambda} \quad ,$$

is also conserved if the energy momentum tensor is traceless. But these generators, together with the Poincare generators, form the whole of the conformal group. Therefore we see, that scale invariance in two dimensions implies the invariance under all of the conformal group. Physically this means that the critical points of a second order phase transition, which are a scale invariant system, can be described by a conformal field theory.

Starting from the above observation one can invert the procedure, and try to obtain properties of the statistical system from the information, obtained analyzing the conformally invariant field theory. This will be done in the next paragraph. The hope is that, in this way, one can obtain a classification of all universality classes of second order phase transitions.

2.2 Conformal Field Theory

Conformal transformations in general are defined as coordinate transformations such that the angle between intersecting curves at the intersection point is preserved. In two dimensions this means that $z \to f(z)$ and $\bar{z} \to f(\bar{z})$ with

$$\partial_{\bar{z}} f(z) = 0 \quad , \qquad \partial_z f(\bar{z}) = 0 \quad ,$$

i.e. it is the group of analytic coordinate transformations. Expand

$$f(z) = z + \sum_{n=-\infty}^{\infty} \epsilon_n z^{n+1} \quad , \quad \bar{f}(\bar{z}) = \bar{z} + \sum_{n=-\infty}^{\infty} \bar{\epsilon}_n \bar{z}^{n+1} \quad .$$

This transformation is generated by

$$l_n = z^{n+1} \partial_z \quad \text{and} \quad \bar{l}_n = \bar{z}^{n+1} \partial_{\bar{z}} \quad ,$$

which means that

$$\phi(z+\sum_n \epsilon_n z^{n+1}\,,\,\bar z\sum_n \bar\epsilon_n \bar z^{n+1}) = (1+\sum_n \epsilon_n l_n + \sum_n \bar\epsilon_n \bar l_n)\;\;\phi(z,\bar z)\;\;.$$

The commutation rules of these objects are

$$\begin{gathered}[l_m,l_n]=(n-m)l_{m+n}\;\;,\;\;\left[\bar l_m,\bar l_n\right]=(n-m)\bar l_{m+n}\;\;,\\ \left[\bar l_m,l_n\right]=0\;\;.\end{gathered} \tag{2}$$

Turning to the quantum theory we need operators $L_n,\bar L_n$ which act on the Hilbert space of states. The generator of the transformation is written as

$$U_f = e^{\sum_n \epsilon_n L_n}\quad,$$

or infinitesimally, $U_f = 1+\sum_n \epsilon_n L_n$. The field should transform as

$$U_f\,\Phi(z,\bar z)\;,U_f^{-1} = f\circ\Phi(z,\bar z)\;\;.$$

Let us consider for simplicity a scalar field and only the holomorphic part of the transformation. This means that we impose $(f\circ\Phi)(z)=\Phi(f(z))$. Now

$$\begin{array}{cc} f(z)=z+\epsilon_n z^{n+1} & U_f=1+\sum\epsilon_n L_n\\ \downarrow & \downarrow\\ f\circ\Phi=\Phi(z)+\sum_n\epsilon_n l_n\Phi(z) & U_f\Phi(z)U_f^{-1}=\Phi(z)+\sum_n\epsilon_n[L_n,\Phi(z)]\;\;,\end{array} \tag{3}$$

Comparing the two sides one obtains

$$[L_n,\Phi(z)] = l_n\Phi(z) = z^{n+1}\partial_z\Phi(z)\quad. \tag{4}$$

We compute both $[L_m,[L_n,\Phi]]$ and $[L_n,[L_m,\Phi]]$ and find

$$[[L_m,L_n],\Phi]=[l_n,l_m]\Phi=(m-n)l_m+n\Phi=(m-n)[L_{m+n},\Phi]\quad.$$

The most general equation satisfying the above relation is

$$[L_m,L_n]=(m-n)L_{m+n}+\hat c(m,n)\quad,$$

where $\hat c$ is an operator commuting with all fields. Since we need the consistency of the Jacobi identity we find

$$\hat c(n,m)=\frac{c}{12}(m^3-m)\delta_{m+n,0}\quad.$$

Similarly one can find the algebra for the antiholomorphic part with a different central charge $\bar{c}$. Summarizing, we obtain as the generators of conformal transformations the elements of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0} \quad . \tag{5}$$

One could arrive at this point in a much more general framework using a generic field. One then examines a correlator $\langle X \rangle = \langle \Phi_1 \Phi_2 \ldots \rangle$ and calculates from the generating functional Z the Ward identity, which gives the action of the conformal transformation on the correlator,

$$\langle \delta_\epsilon X \rangle = \frac{1}{2\pi i} \oint_C dz\, \epsilon(z) \langle T(z) X \rangle \quad . \tag{6}$$

If one expands the generators

$$T(z) = \sum_{n=-\infty}^{\infty} z^{-(n+2)} L_n \quad , \tag{7}$$

one finds the Virasoro generators satisfying the algebra (5).

Now the main problem is to use the representation theory of the Virasoro algebra to determine the physical content of the theory. One proceeds like in $SU(2)$ where one has a highest weight state $|j, m = j\rangle$ with the properties $X^+|j,j\rangle = 0$ and $H|j,j\rangle = j|j,j\rangle$. Here the rôle of H is taken by L_0 (see e.g. [30, 35]), and a primary state (or highest weight state) is defined as

$$L_0|\Phi\rangle = h|\Phi\rangle \quad , \quad L_n|\Phi\rangle = 0 \quad , \qquad n > 0, \tag{8}$$

h is the highest weight. Considering also the antiholomorphic part, one get's analogously the highest weight $\bar{h}$. One defines $\Delta = h + \bar{h}$ and the spin $s = h - \bar{h}$. Secondary states (or descendent states) are obtained by applying L_{λ_i} as

$$|\chi\rangle = (L_{-\lambda_1})^{n_1} \ldots L_{-\lambda_r})^{n_r}|0\rangle \quad , \tag{9}$$

with eigenvalue

$$L_0|\chi\rangle = (h + \sum_i \lambda_i n_i)|\chi\rangle \quad . \tag{10}$$

The number $\sum \lambda_i n_i$ is called the level of the state. States of different levels are orthogonal.

The central term causes that in general the commutator in (4) acquires another term. In these representations the general expression [30] is

$$[L_n, \Phi(z)] = z^{n+1}\partial_z \Phi(z) + h(n+1)z^n \Phi(z) \quad . \tag{11}$$

For special values of the central charge and the highest weight h it can happen that at some level a linear combination of descendent fields behaves as a primary field, due to the algebraic relations (5), that is

$$L_0|\Phi_d\rangle = (h+l)|\Phi_d\rangle \quad , \quad L_n|\Phi_d\rangle = 0, \qquad n > 0 \tag{12}$$

But this implies that it is a null-state, i.e. $\langle\Phi_d|\Phi_d\rangle = 0$. Representations of this kind are called degenerate. The null states can be used to restrict the degenerate representations. One can eliminate these states together with all its descendents from a chosen representation, since they decouple from the rest of the states. A sophisticated analysis (see [35] and references therein) shows that such a reduction is possible only for a very restricted set[1] of central charges c and highest weights h. They are called minimal models and are determined by

$$c = 1 - \frac{6(p-q)^2}{pq} \quad , \quad p, q \text{ positive relative prime integers} \tag{13}$$

$$h_{r,s} = \frac{(sp-rq)^2 - (p-q)^2}{4pq} \quad , \quad 1 \leq s \leq q-1 \ , \ 1 \leq r \leq p-1 \quad . \tag{14}$$

These Formulae have far reaching consequences:

- First of all every theory is defined by a central charge. The only ambiguity left, remains from modular invariance, which we will analyze later. So we see that the minimal models can be labeled by two integers. Therefore we can classify them as $\mathcal{M}_{p,q}$. The content of the theory is specified by respective $h_{r,s}$, that is we have a finite number of primary states in the theory given by the rational highest weights $h_{r,s}$.

[1] The set here does not exhaust all minimal theories, which can be constructed using larger symmetry algebras [7]

- Secondly we would like to translate these results into field theory. Now the L_n are operators acting on fields. These can be divided into primary fields and secondary fields as before the states. One example of a secondary field we already encountered. Since we defined the L_n to be the Laurant coefficients of $T(z)$ we have

$$L_n X(z) = \oint_z d\zeta (\zeta - z)^{n+1} T(\zeta) X(z) \quad , \tag{15}$$

and with this, calculating $L_{-2}I$, we find that

$$(L_{-2}I)(z) = T(z) \quad . \tag{16}$$

Now define the operator $\mathcal{L}_n$ to give the action of L_n on the correlator of primary fields as

$$\langle \Phi_1 \ldots \Phi_n L_{-k} \Phi \rangle = \mathcal{L}_{-k} \langle \Phi_1 \ldots \Phi_n \Phi \rangle \quad . \tag{17}$$

Through the Ward identity these can be determined in terms of differential operators. That means that a null relation now becomes a differential equation for the correlator. Using that the L_{-k} are the Laurant coefficients of $T(z)$ (7), and the Ward identity (6) one can also determine differential equations for the correlators of secondary fields [30]. So in theory one can compute all correlators of the theory, though in practise it is not said that it is a straight forward task to solve actually these equations.

- A third very important consequence is related to statistical physics. The dimensions of the primary fields are intrinsically related to the critical exponents of statistical models [10, 83] . For example in conformal field theory (CFT) the two point function of primary fields can be determined to be

$$\langle 0 | \Phi_h(z) \Phi_{h'}(0) | 0 \rangle = \frac{1}{z^{2h+2h'}} \delta_{h,h'} \langle 0 | \Phi_h(1) | \Phi \rangle \quad . \tag{18}$$

On the other hand take a statistical system, for example once again the Ising model. The correlator of the order parameter σ has at criticality the behaviour

$$\langle \sigma_r \sigma_0 \rangle \sim \frac{1}{|r|^{\frac{1}{4}}}$$

(for the Ising model the exponent η has the value $\eta = \frac{1}{4}$). Therefore we can assign [10] the values $\Delta_\sigma = \bar{\Delta}_\sigma = \frac{1}{16}$ to this field. But this in the above classification is the field $\Phi_{1,2}$ (i.e. the field with weight $\Delta_{1,2}$) of the CFT $\mathcal{M}_{3,4}$. Also the other operators of the Ising model can be brought in correspondence to those of the theory $\mathcal{M}_{3,4}$. This renders the identification of the critical Ising model with the model $\mathcal{M}_{3,4}$ complete.

Similar other systems have been identified , e.g. the models $\mathcal{M}_{4,5}$, $\mathcal{M}_{5,6}$, $\mathcal{M}_{6,7}$ with $c = \frac{7}{11}, \frac{4}{5}, \frac{6}{7}$ were identified with the tricritical Ising model, 3-state Potts model and tricritical 3-state Potts model respectively. Therefore, classifying all CFT (which are not exhausted by the minimal models), would amount in classifying all possible critical behaviours. From the above it follows, that the critical behaviour should be determined by rational exponents.

Above we only mentioned some examples of statistical models. Really all minimal models have been found to be critical points of statistical models [2, 34]. But our examples have another peculiarity: they were all of the type $\mathcal{M}_{p,p+1}$. The reason for that is the following:
In order to select our models we did not include unitarity as a constraint on our theory. One can show ([30, 10]) that unitarity restricts c and h to be positive, and so only models of the type $\mathcal{M}_{p,p+1}$ survive. The problematic feature of a non unitary theory is that it contains correlators which diverge for large distances (compare (18)), since the theory contains at least one weight h smaller than zero. This, in the field theoretical setting, is an obvious inconsistency, but has applications in statistical systems as for example in polimer physics [21].

The simplest non unitary system is the Yang-Lee model. It corresponds to the singularity in the Ising model above its critical temperature in a non zero purely imaginary magnetic field ih. If one describes this as a field theory, the action becomes that of a single scalar field, with

$$S = \int d^2r(\frac{1}{2}(\nabla\Phi)^2 + i(h-h_c)\Phi + \frac{1}{3}ig\Phi^3) \quad . \tag{19}$$

The imaginary coupling renders the theory non unitary. At the critical point only one relevant operator survives, which is the field itself, and can be shown [11] to have $\Delta_\Phi = -\frac{1}{5}$. The central charge of the system is $c = -\frac{22}{5}$. These are, on the other hand, the conformal data of the minimal model $\mathcal{M}_{2,5}$.

2.2.1 Integrals of Motion

Since our theories posses an infinite number of degrees of freedom, we expect also an infinite number of conserved quantities. A first one we already found, since $\partial_{\bar{z}}T(z) = 0$, because of conformal invariance. On the other hand we saw in (16) that $T(z) = L_{-2}I$. Similar one finds that

$$(L_{-n}I)(z) = \frac{1}{(n-2)!}\partial_z^{n-2}T(z) \quad . \tag{20}$$

Applying successively operators L_{-n} one obtains

$$T_4(z) \equiv (L_{-2}L_{-2}I)(z) = \oint d\zeta(\zeta - z)^{-1}T(\zeta)T(z) \quad , \tag{21}$$

which is interpreted as the regularized square of T, $T_4(z) =: T^2 : (z)$. Let $\mathbb{L}$ be the module spanned by the descendents of I. The highest weight is zero. We can decompose this module into

$$\mathbb{L} = \bigoplus_{s=0}^{\infty} \mathbb{L}_s \quad ,$$

since application of L_{-n} increases the spin. But these objects do not contain a $\bar{z}$ dependence, which makes them all integrals of motion $\partial_z\mathbb{L} = 0$. Every one of them, labeled as T_s^α, α numbering the elements at given spin s, gives rise to infinitely many operators

$$\oint d\zeta T_s^\alpha(\zeta)(\zeta - z)^{n+s-1} \quad , \qquad n = 0, \pm 1, \pm 2$$

which are integrals of motion.

But, these operators are not algebraically independent, since they are related to the Virasoro generators. A basis must be formed modulo the Virasoro algebra. But there is a further linear dependence, because a very simple reason. We saw (11) that L_{-1} corresponds to a derivative ∂_z. Therefore states formed by applying L_{-1} are linearly depending. Since $\mathbb{L}$ contains all possible derivatives, one has that $L_{-1}\mathbb{L} \subset \mathbb{L}$. Then a linear independent basis can be formed, constructing the factor space $\hat{\mathbb{L}} \equiv \mathbb{L}/L_{-1}\mathbb{L}$.

For example let us try to construct the first elements of $\hat{\mathbb{L}}$. For spin zero we have the identity operator I. At spin one we have $L_{-1}I$, which is linearly dependent. At spin two we could apply twice L_{-1}, which again we discard, and we remain with $L_{-2}I = T$. At level 3 we have $L_{-3}I$ which gets rewritten in term of the Virasoro algebra and becomes a derivative, ... So for the lowest spins we obtain the following base elements

spins	0	1	2	3	4	5	6
basis vectors	I	$\emptyset$	$T_2 = L_{-2}I$	$\emptyset$	$T_4 = L_{-2}^2 I$	$\emptyset$	$T_6^{(1)} = L_{-2}^3 I$ $T_6^{(2)} = L_{-3}^2 I$

(22)

For conformal systems defined on a torus there exists a general formula to calculate the dimensionality of a submodule (for any primary field), at level k. Since we are interested in this situation (which means periodic boundary conditions in both directions of the lattice of the statistical models) we can use this result. On the torus the partition function takes a simple form [10]

$$Z = q^{-\frac{c}{24}} \bar{q}^{-\frac{c}{24}} tr\left(q^{L_0} \bar{q}^{L_0}\right) \quad . \tag{23}$$

Let's decompose this as

$$Z(q, \bar{q}) = \sum_{h,\bar{h}} N_{h,\bar{h}} \chi_h(q) \chi_{\bar{h}}(\bar{q}) \quad , \tag{24}$$

where we define the character χ as

$$\chi_h(q) \equiv q^{-\frac{c}{24}} tr_{\,h} q^{L_0} = q^{-\frac{c}{24}+h} \sum_{n=0}^{\infty} d_h(n) q^n \quad , \tag{25}$$

$d_h(n)$ is the degeneracy of states in a representation at level n. Note that the quantity we are interested in is given by $d_0(n) - d_o(n-1)$. The numbers $N_{h,\bar{h}}$ is fixed by modular invariance. This means that one requires Z to be invariant under modular transformations, which characterize special base changes on the torus. For a given central charge there can exist more than one theory satisfying these requirements. They are classified in the so ADE series, found in [9].

If there were no null states we would have $d_h(n) = P(n)$ (the partition of the integer n), because all states of the form (9) would be independent. But for minimal models the character expansion is not so simple. It is given [62] by

$$\chi_{r,s}(c) = q^{-\frac{c}{24}} \frac{1}{P(q)} \sum_{k=-\infty}^{\infty} \left[q^{[2kpp'+rp-sp']^2-(p-p')^2]/4pp'} - (s \leftrightarrow -s) \right] \quad , \tag{26}$$

where p, p' are relative coprime, and $1 \leq r \leq p'-1$, $1 \leq s \leq p-1$ and $rp - sp' > 0$. $P(q)$ is given as $P(q) = \prod_{n=1}^{\infty}(1-q^n)$. So, to get $d_h(n)$ one needs to calculate all relevant terms in this expansion, and count the number of times the term q^n appears. It is useful to let a computer do this work.

2.3 Perturbations

In the last section we found that conformal minimal field theories are integrable. This corresponds to the critical point in a statistical model. But there exist also statistical models which are integrable also away from criticality. For example the Ising model: it is integrable away from criticality for a zero magnetic field as well as in a magnetic field but at the critical temperature T_c. This means that we have special directions in our space of couplings, for which the system remains integrable.

Let's turn back to CFT. We know that we can get away from the critical point only by adding a relevant field [2]. So we will make the ansatz

$$H = H_{CFT} + \lambda \int \Phi(x) d^2x \quad , \tag{27}$$

with some relevant field $\Phi(x)$.

What about integrability? The quantities T_s^α which we have in CFT (see page 14) cannot serve any more as integrals of motion, because since we have left the critical point, the energy momentum tensor is not any more traceless. At lowest order we have

$$\partial_{\bar{z}} T = [H_p, T(z)] \quad , \quad H_p = \lambda \int \Phi(x) d^2x \quad . \tag{28}$$

[2]In minimal CFT there appear no marginal fields.

But these are quantities of the CFT, and therefore the commutator can be determined ([76, 22]), giving

$$\partial_{\bar{z}} T = \partial_z (\lambda(1-h)\Phi) \quad . \tag{29}$$

This can be symbolically written as $\partial_{\bar{z}} L_{-2} I = L_{-1}\Phi$. So let's try to construct a deformed current for the next current of CFT, namely $T_4 = L_{-2}^2 I$. That is, we want to obtain

$$\partial_{\bar{z}} L_{-2}^2 I = \text{element of } \Phi_s \quad , \quad \text{of derivative form} \ .$$

This could be $L_{-1}L_{-2}$ or L_{-1}^3. If L_{-3} appears T_4 can not be deformed to give a conserved quantity. But assume we have a null state at level three. Then L_{-3} can be expressed in terms of the other two elements. But this means that we have a conservation law at level three.

This analysis can be generalized, and is called the 'counting argument'. For, let $\widehat{\mathbb{L}}$ be again, as in CFT, the space of independent descendents of I. Now $\partial_{\bar{z}}\mathbb{L}_{s+1}$ must take values in the space $\mathbb{P}_s$ (as before in the example). $\mathbb{P}$ and $\widehat{\mathbb{P}}$ are constructed analogously to $\mathbb{L}$ and $\widehat{\mathbb{L}}$ but constructed over Φ instead of I. If the dimension of $\widehat{\mathbb{L}}_{s+1}$ is larger than that of $\widehat{\mathbb{P}}$, one can find a quantity which maps onto $L_{-1}\Phi_{s-1}$ and therefore defines a conserved current. The respective dimensions can be computed using the character expansion (26). For example for the perturbations of the type $\Phi_{1,3}$ one gets

s	1	2	3	4	5	6	7
$dim\ (\widehat{\mathbb{L}}_{s+1})$	1	0	1	0	2	0	3
$dim\ (\widehat{\mathbb{P}}_s)$	0	1	0	2	1	3	2

(30)

We obtain conserved charges for spin $s = 1, 3, 5, 7$.

Simply as it is, the counting argument has also its limits. It works only for low spins. Of course if $dim\,(\widehat{\mathbb{L}}_{s+1}) < dim\,(\widehat{\mathbb{P}}_s)$, this does not mean that there cannot exist a conserved charge at this level, but it has to be found by other means. Nevertheless it is very powerful. Doing a similar analysis for other operators $\Phi_{r,s}$ one finds that only 3 of them are selected to have conserved quantities: $\Phi_{1,3}\Phi_{1,2}$ and $\Phi_{2,1}$. For the series $\Phi_{1,2}$, for example, one finds the conserved spins $s = 1, 5, 7, 11$.

On the other hand it is not a conclusive analysis. First of all we need an infinite set of conserved quantities to get an integrable theory. The existence of an infinite number of conserved quantities, for example all s uneven for $\Phi_{1,3}$, has to be conjectured at this point. Secondly the integrals of motion should be in involution. There is a plausibility argument that this is indeed the case. Since $[T_{odd}, T_{odd}] = T_{even}$, but we don't have any T_{even} in the set of our conserved quantities, their commutator should be zero.

We have constructed integrable field theories perturbing CFT, constructing three series of noncritical models, using only the information of the critical point. Following this trajectory we should arrive at another fixed point in the infrared limit.

There exists a general result for unitary theories. It is called the c-Theorem [78], and shows that there exists a function $C(\lambda)$ (λ being the couplings), which is decreasing along the renormalization group trajectory and is stationary at the fixed points corresponding to $\mathcal{M}_{p,p+1}$. Moreover this function becomes the central charge at the fixed point $C(\lambda^*) = c$. Trying to construct such trajectories is a more difficult and results are known for the flow between the Ising model and the tricritical Ising model [81]. Also one should note, that there is no corresponding theorem for non-unitary theories, and it is not clear, how renormalization group trajectories evolve.

We know that perturbed conformal field theory corresponds to some statistical model, but how can we describe it in a field theoretical language? For this scope we introduce Liouville theory, which can be interpreted as a realization of CFT, and is a special example of a wider class of theories, Toda field theories.

2.4 Toda Field Theory

Let's analyze the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{\beta^2}\sum_{j=1}^{r} e^{\beta\alpha_i.\phi} \quad . \tag{31}$$

Herein the notation is as follows: The action is built intrinsically over a Lie algebra. Let r be the rank of a finite Lie algebra $\mathcal{G}$ [18] and let ϕ be an r - component scalar field. The exponential interaction is determined by the α_i which are chosen to be the positive simple roots of $\mathcal{G}$. The equations of motion in light cone coordinates are

$$\partial_+\partial_-\phi_j = -\frac{1}{\beta}\sum_{i=1}^{r}\alpha_i e^{\beta\alpha_i.\phi} \quad ,$$

or redefining $\varphi = \alpha_i\phi$

$$\partial_+\partial_-\varphi_j = -\frac{1}{\beta}\sum_{i=1}^{r}a_{ij}e^{\beta\varphi_j} \quad , \tag{32}$$

a_{ij} is the Cartan matrix of $\mathcal{G}$, $a_{ij} = \frac{\langle\alpha_i.\alpha_j\rangle}{\langle\alpha_j.\alpha_j\rangle}$. The energy momentum tensor, in light cone coordinates takes the form

$$T_{\pm\pm} = -\frac{1}{2}\partial_\pm\phi\partial_\pm\phi + \frac{1}{\beta}\rho\partial_\pm^2\phi \quad , \qquad T_{\pm,\mp} = 0 \quad .$$

Herein $\rho = \sum$ fundamental weights . So it is traceless and we have a classically conformal invariant theory. This property survives after quantization. Computing the conformal algebra generated by the quantized energy momentum tensor [33], one finds that the central charge is related to the coupling as

$$c = r + 12\rho^2(\frac{1}{\beta}+\beta)^2 = r\left(1 + h(h+1)(\frac{1}{\beta}+\beta)^2\right) \quad , \tag{33}$$

h being the dual coxeter number of $\mathcal{G}$ [18]. We will mainly examine A_n for which $h = n+1$. Our aim is to obtain realizations of minimal models. We see in (33) that this is impossible for a real coupling constant, so we perform an analytic continuation to imaginary coupling constant $\gamma = i\beta$ [33]. The simplest algebra is A_1, for which $h = 2$. Let $\gamma^2 = \frac{p}{q}$ in (33), which becomes

$$c = 1 - \frac{6(p-q)^2}{pq} \quad ,$$

the value we found in section 2.2 . Using different algebras, not A_1, there are only single cases how to obtain a minimal model as for example for E_8 with the coupling $\gamma^2 = \frac{31}{32}$ which gives $c = \frac{1}{2}$, or A_2 with $\gamma^2 = \frac{4}{5}$ giving $c = \frac{4}{5}$. Leaving aside these exceptional cases we will concentrate on the algebra A_1. In that case the equation of motion become

$$\partial^2\phi = -\frac{2}{\beta}e^{\beta\phi} \quad ,$$

which is the Liouville equation.

What happens if we try to use an affine Kac-Moody algebra instead of a finite algebra ? We can write down the action in the same way, but now including also the 0^{th} root. That is, we add a term

$$\delta V(\phi) = \frac{\lambda}{\beta^2} e^{\beta\alpha_0.\phi} \quad , \tag{34}$$

to our lagrangian (31). The minimum of the potential becomes now

$$\sum_i \alpha_i e^{\beta\alpha_i.\phi} = \lambda \alpha_0 e^{\beta\alpha_0\phi} \quad ,$$

which is finite, where before, for the conformal invariant theory it was zero. Call the solution of this equation $\phi^{(0)}$, and look for perturbations around the vacuum, i.e. analyze

$$V = \frac{1}{\beta^2}\sum_{i=1}^{r} e^{\beta\alpha_i(\phi^{(0)}+\phi)} + \frac{\lambda}{\beta^2} e^{\beta\alpha_0(\phi^{(0)}+\phi)} \quad ,$$

and use the fact that α_0 can be expressed in terms of the other roots $\alpha_0 = -\sum_{n=1}^{r} n_i\alpha_i$. With this one can rewrite V as

$$V = \frac{k^2}{\beta^2}\sum_{i=0}^{r} n_i e^{\beta\alpha_i.\phi} \quad ,$$

where the constant k is given by $k^2 = \lambda e^{\beta\alpha_0\phi^{(0)}}$. Expanding around the minimum $\phi = 0$ we get

$$V(\phi) = \frac{k^2}{\beta^2}\sum_{i=0}^{r} n_i + \frac{1}{2}(M^2)^{ab}\phi^a\phi^b + c^{abc}\phi^a\phi^b\phi^c + \dots \quad , \tag{35}$$

summing over indices a,b,c. We have therefore acquired a mass matrix

$$(M^2)^{ab} = \sum_{i=0}^{r} n_i \alpha_i^a \alpha_i^b \quad ,$$

and three point couplings

$$c^{abc} = k^2\beta \sum_i n_i \alpha_i^a \alpha_i^b \alpha_i^c \quad .$$

The relevance of this construction is that it preserves integrability. That is, as well the conformally invariant, as the perturbed Toda field theories, are classically integrable [54].

Let's consider again A_1. Now we have two possibilities to get affine Toda field theories. Or we take $A_1^{(1)}$, corresponding to the Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, or $A_2^{(2)}$, with

$(a) = \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix}$. Let us write down the corresponding equations of motion. For $A_1^{(1)}$ we find $\partial^2\varphi = \frac{2}{\beta}\sinh\beta\varphi$ or for complex coupling, which we are interested in

$$\partial^2\varphi = \frac{2}{\gamma}\sin\gamma\varphi \quad .$$

This is the sine-Gordon equation, well known as a soliton equation, since it exhibits degenerate vacua. Analyzing this perturbation in terms of the conformal operators for $\gamma^2 = \frac{p}{q}$ one finds that this is equivalent to perturbing the Liouville lagrangian with the conformal field $\phi_{1,3}$. For $A_2^{(2)}$ we obtain

$$\partial^2\varphi = \frac{1}{\beta}(e^{-2\beta\varphi} - 2e^{\beta\varphi}) \quad , \tag{36}$$

which is called the Bullough-Dodd equation, and it admits also solitons for imaginary coupling.

We just briefly mention here the case of real coupling Toda field theory. They have been analyzed in detail [13, 8], and the masses and couplings for the respective algebras have been calculated. But since they do not provide perturbations of minimal models, we will not consider them further.

Let us look at $A_1^{(1)}$. The classical analysis would say that one gets one scalar massive field, which is twice degenerate. This is true for real coupling constant, but for imaginary coupling constants this analysis proves to be superficial. In this case these particles are a soliton and anti-soliton respectively and can build bound states. So we need to give a closer look to the system in order to decide its particle content.

2.4.1 The Sine-Gordon Model

To get some idea of the basic features of soliton theories, let us review some facts about the sine-Gordon model. The minima are infinitely degenerate, with the values $\phi_n = \frac{n\pi}{\gamma}$. Rewriting the lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - U(\phi) \quad , \qquad \left(= \frac{1}{2}(\partial_0\phi)^2 + \frac{1}{2}(\partial_1\phi)^2 - U\right)$$

the equations of motion are

$$\phi_{00} - \phi_{11} + U'(\phi) = 0 \quad .$$

Solving this equation one finds the time-independent solution

$$\phi = \frac{4}{\beta} \tan^{-1}(e^{\pm\sqrt{2}x}) \quad .$$

The solution with the + (-) is called the (anti-) soliton.

These are not the only classical solutions. There exist periodic solutions of the form

$$\phi(x,t) = \frac{4}{\beta} \text{arctan} \left(\frac{\eta \sin \omega t}{\cosh \eta\omega t} \right) \quad ,$$

wherein $\eta = \frac{\sqrt{2-\omega^2}}{\omega}$ and ω is a parameter characterizing the solution. Analyzing this solution one finds that this state can be interpreted as a soliton and an anti-soliton oscillating such that no translational motion occurs. These states are called breathers.

Both, solitons and breathers, can be found also in quantum theory. In a semi-classical approach one finds the solitons, and the breathers are given by the DHN-formula, which is a generalization of the Sommerfeld quantization (see [14] and references therein), which yields in our case

$$E_n = 2M \sin \frac{\gamma'^2 n}{16} \qquad \text{where } n = 0, 1, 2, \ldots < \frac{8\pi}{\gamma'^2} \tag{37}$$

$$\gamma'^2 = \frac{\gamma^2}{1 - \frac{\gamma^2}{8\pi}} \quad , \tag{38}$$

The sine-Gordon equation is the simplest soliton equation we encounter in Toda field theory. Already here, one meets difficulties to quantize the theory, and the best one can do remains on a semiclassical level.

To improve this situation we will introduce the S matrix approach. Though this is an indirect way to deal with the soliton theory, it will lead us to a quantum mechanical analysis of the problem.

2.5 The S Matrix Approach

We are posed with the following problem. We are dealing with a theory, which is integrable, classically and quantistically. Now the scope is to write down the solution of the theory, i.e. the correlation functions. In "ordinary" quantum field theory we are used to analyze the symmetry of the lagrangian, and further to proceed by perturbation theory. But this can never unravel the structure we have to deal with. Since solitons connect different vacua of the theory, they are intrinsically non perturbative objects.

There are principally two possibilities: one is to go to the lattice, and study in this regularized scheme the corresponding statistical system (which for the sine-Gordon system should be the six vertex model). Alternatively one studies the scattering matrix, which shows the reaction of particles, when they interact. In this approach we postulate the integrability of the theory and derive a set of conditions to fulfil. Using symmetry arguments one can then conjecture the S matrix of the theory under investigation.

Only then we compare the results with the semiclassical analysis obtained in the last section. There is no direct way to verify our results. This on the other hand does not mean that we cannot compare our theories. In the S matrix approach one can, in theory (modulo serious technical problems), compute the correlation functions and therefore critical exponents [3]. But for sure, calculating the S matrix is the first step in direction to resolve the integrable quantum theory.

Since our work deals basically in this field, this section will introduce some more technical details as the previous ones. The S matrix is introduced into quantum field theory most easily in the Lehmann-Simanzik-Zimmermann formulation. Suppose we have a massive quantum field theory, which we assume (for simplicity) has a physical spectrum with only one excitation. There are two sets of creation and annihilation operators

$$a_{in}(\beta), a^{\star}_{in}(\beta);\ a_{out}(\beta), a^{\star}_{out}(\beta) \quad ;$$

which describe the physical particle asymptotically for $t \to -\infty, t \to \infty$, where they are supposed to be free. We parameterize the momentum p_μ in terms of the rapidity β, . defined as

$$p_0(\beta) = m\cosh\beta \qquad p_1(\beta) = m\sinh\beta \quad .$$

[3]This program unfortunately has been carried out only for the field theoretic correspondence of the Ising model [75]

These operators satisfy canonical commutation relations. One also supposes that there is a common vacuum for these operators

$$a_{in}(\beta)|0\rangle = a_{out}(\beta)|0\rangle = 0 \quad .$$

These states should be connected by a unitary transformation, which is the scattering matrix, i.e.

$$a^{\star}_{in}(\beta_1)\dots a^{\star}_{in}(\beta_1)|0\rangle = \sum_m S(\beta'_1\dots\beta'_m|\beta_1\dots\beta_n)a^{\star}_{out}(\beta'_1)a^{\star}_{out}(\beta'_m)|0\rangle d\beta'_1 d\dots\beta'_m$$

Further one needs locality in the theory. These properties restrict the form of the functions $S(\beta'_1\dots\beta'_m|\beta_1\dots\beta_n)$.

We will not discuss further the general formalism, but turn to the case of integrability. The main characteristic feature is the existence of infinitely many local conservation laws. The first of them are $I_1 = p_0 + p_1$ and $I_{-1} = p_0 - p_1$ being the integral of $T(z)$ and $\bar{T}(\bar{z})$, with eigenvalues Me^{β} and $Me^{-\beta}$ respectively, where M is the mass. Higher conservation laws I_s have eigenvalues of the form $M_s e^{\beta s}$, $M_s e^{-\beta s}$, M_s being a constant depending on the spin s. Due to locality of $I_{\pm s}$, their eigenvalues on multi-particle states are the sums of one particle eigenvalues.

The S matrix must commute with I_s, I_{-s} and therefore eigenvalues of "in" and "out" states coincide. So if we have the rapidities $\{\beta'_1\dots\beta'_n\}$ after scattering, we should satisfy the infinite set of equations

$$\sum_{j=1}^{n} e^{\pm s\beta_j} M_s = \sum_{j=1}^{m} e^{\pm s\beta'_j} M_s \quad ,$$

for any conserved spin s. The only solution, consistent with analyticity, is $n = m$ and even more, $\{\beta'_1\dots\beta'_n\} = \{\beta_1\dots\beta_n\}$. This means that the scattering is purely elastic. This implies factorized scattering, which means that the n-particle scattering matrix reduces to a product of $n(n-1)/2$ two particle scattering one. The argument for this is conceptually the following:

Since we have massive particles the interaction should be short-range. Now split the interaction region into domains such that

$$|x_1 - x_2| \lesssim R \ , \quad |x_1 - x_i| \gg R \ , \quad |x_i - x_2| \gg R \ , \ |x_i - x_j| \gg R \ , \quad i,j = 3,4\dots$$

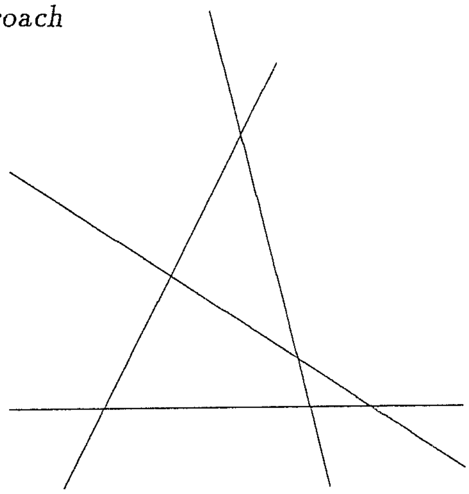


Figure 1: Factorization of multi-particle scattering

In this region we can describe the process as a two particle scattering while the others behave approximately as free ones, compare figure 1. Because the conservation laws must hold after every one of these processes, factorized scattering follows (for a more precise argument see [77]).

We have reduced the problem to calculating the two particle S matrix. This can be represented as follows

$$S^{\epsilon'_1\epsilon'_2}_{\epsilon_1\epsilon_2} = \langle 0|a^{\epsilon'_1}_{out}(\beta'_1)a^{\epsilon'_2}_{out}(\beta'_2)a^{\star}_{in,\epsilon_1}(\beta_1)a^{\star}_{in,\epsilon_2}(\beta_2)|0\rangle = \delta(\beta_1-\beta'_1)\delta(\beta_2-\beta'_2)S^{\epsilon'_1\epsilon'_2}_{\epsilon_1\epsilon_2}$$

with $\beta_2 > \beta_1$, $\beta'_2 > \beta'_1$ and the ϵ_i labeling possible internal degrees of freedom. $S^{\epsilon'_1\epsilon'_2}_{\epsilon_1\epsilon_2}(\beta_1 - \beta_2)$ depends only on the difference of rapidities, because of Lorentz invariance.

Denote $\beta_{ij} = (\beta_i - \beta_j)$. Now consider a three particle scattering. This can be decomposed in two different ways into two particle scattering, both of which should be equivalent. The result is

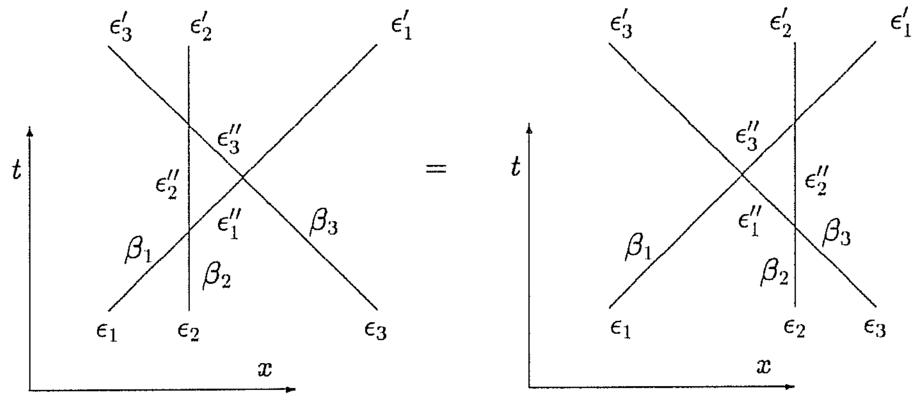
$$\sum_{\epsilon'''_1\epsilon'''_2\epsilon'''_3} S^{\epsilon'_1\epsilon'_2}_{\epsilon_1\epsilon_2}(\beta_1-\beta_2)S^{\epsilon''_1\epsilon'_3}_{\epsilon_1\epsilon'''_3}(\beta_1-\beta_3)S^{\epsilon''_2\epsilon''_3}_{\epsilon_2\epsilon_3}(\beta_2-\beta_3) \tag{39}$$

$$= \sum_{\epsilon'''_1\epsilon'''_2\epsilon'''_3} S^{\epsilon'_2\epsilon'_3}_{\epsilon''_2\epsilon''_3}(\beta_2-\beta_3)S^{\epsilon'_1\epsilon''_3}_{\epsilon''_1\epsilon_3}(\beta_1-\beta_3)S^{\epsilon''_1\epsilon''_2}_{\epsilon_1\epsilon_2}(\beta_1-\beta_2) \tag{40}$$

(41)

Figure 2: Cuts and poles of the S matrix in the s plane

or in graphical terms, drawing the trajectories in space time



(42)

which is the Yang Baxter equation.

2.5.1 Analytic Structure

Let's analyze the analytic structure of the S matrix amplitudes in terms of the invariant energy squared

$$s = 4p^\mu p_\mu = 4m^2 \sinh(\frac{\beta_{12}}{2}) \ . \tag{43}$$

The matrix $S_{\epsilon_1\epsilon_2}^{\epsilon_1'\epsilon_2'}$ is analytic in the complex s-plane, with two cuts along the real axes $s \leq 0$ and $s \geq 4m^2$. This si pictured in fig. 2, wherein dots indicate poles, which correspond to bound states. The cut for $s \leq 0$ corresponds to the threshold in the variable $t = (p_1 - p_4)^2$, which we keep fixed. For example consider a diagonal S-matrix, which means that our particles do not posses internal degrees of freedom. Then our Riemann surface is composed of two sheets and the two cuts can be replaced by a single one going from $s = 0$ to $s = 4m^2$. The transformation to the rapidity

$$\beta = \ln\left(\frac{s - 2m^2 + \sqrt{s(s-4m^2)}}{2m^2}\right) \ ,$$

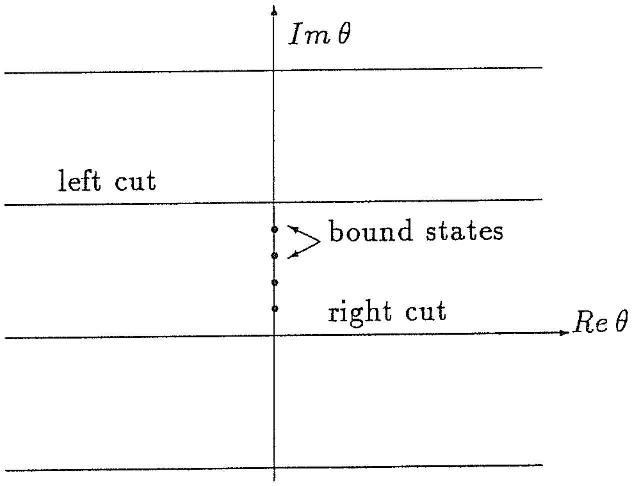


Figure 3: Analytic structure of the S matrix in the θ-plane

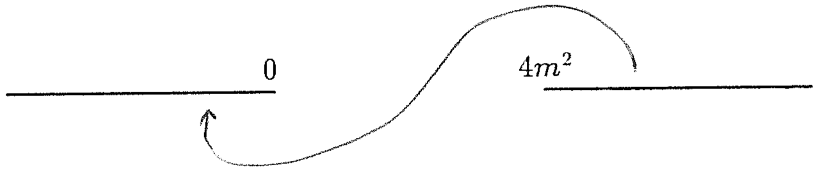


Figure 4: Analytic continuation corresponding to crossing

which is the inverse of (43), transforms the physical sheet of the s-plane into the strip $0 \le \operatorname{Im}\theta \le \pi$. The edges of the right and left cuts of the physical sheet get mapped into the axes $\operatorname{Im}\theta = 0$ and $\operatorname{Im}\theta = \pi$ respectively. The axes $\operatorname{Im}\theta = l\pi$, $l = -1, \pm 2\ldots$ correspond to the edges of cuts of the other complex s plane sheets (see fig. 3). The unitarity requirement can be written as

$$S^{\epsilon_1'\epsilon_2'}_{\epsilon_1\epsilon_2}(\beta) S^{\epsilon_1\epsilon_2}_{\epsilon_1''\epsilon_2''}(-\beta) = \delta^{\epsilon_1'}_{\epsilon_1''}\delta^{\epsilon_2'}_{\epsilon_2''} \quad .$$

The last important ingredient is crossing symmetry: if we regard both s and t as complex variables, we can reach the region $t \ge 4m^2$, $s \le 0$ by analytic continuation, which describes the 'crossed' scattering process(see fig. 2.5.1). The corresponding transformation in s is $s \longrightarrow 4m^2 - s$ or in terms of the rapidity, $\beta \longrightarrow i\pi - \beta$. This yields the crossing symmetry relation

$$S^{\epsilon_1'\epsilon_2'}_{\epsilon_1\epsilon_2}(\beta) = c_{\epsilon_1\epsilon_1''} S^{\epsilon_1''\epsilon_2'}_{\epsilon_1'''\epsilon_2}(i\pi - \beta) c^{\epsilon_1'\epsilon_1'''} \quad , \tag{44}$$

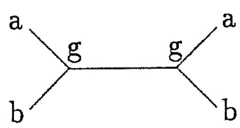
where c is the matrix of charge conjugation.

2.5.2 Bound States and the Bootstrap Principle

Let's analyze, for simplicity, only diagonal S matrices of self-conjugate particles. That is, we deal with scalar particles. Therefore our S matrix can be written as S_{ab} a,b indicating the kind of particles which scatter. If we require spatial reflection symmetry we obtain that $S_{ab} = S_{ba}(-\theta)$, i.e. the order of the indices is irrelevant. Now we enumerate our poles as $\theta = iu^c_{ab}$ to indicate that it is the pole of $S_{ab}(\theta)$ forming the bound state c in the s channel. But this means that in this channel $s = m_c^2$ or

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u^c_{ab} \quad ,$$

wherefrom we can determine the mass of the new particle. Because of crossing symmetry there must be also the pole $\theta = i\bar{u}^c_{ab}$ where $\bar{u}^c_{ab} = \pi - u^c_{ab}$. How can one decide which is the physical one ? For a unitary theory one decides on the grounds of the residue at the pole. This can be seen roughly because of the following argument: Consider the Feynman diagram



$$\sim \frac{g^2}{s - m_c^2} \quad , \tag{45}$$

which should correspond to

$$S_{ab} = \frac{R^c}{\theta - iu^c_{ab}} \quad .$$

Therefore R^c, the residue, should be positive. This is not any more true if we consider imaginary couplings, so we loose this requirement and have an arbitrariness in our theory.

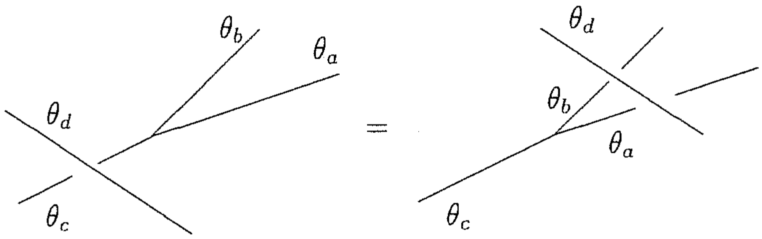
Assume we know the scattering amplitudes S_{ad} and S_{bd}. Now we would like to compute the scattering amplitude for S_{cd}, c being a particle corresponding to a pole in the amplitude S_{ab}. For that we analyze

$$S_{abd}(\theta_1\theta_2, \theta_3) = S_{ab}(\theta_{12})S_{ad}(\theta_{13})S_{bd}(\theta_{23}) \quad ,$$

and at the location of the pole

$$S_{abd}(\theta_1\theta_2, \theta_3) = S_{ab}(\theta_{12})S_{cd}(\theta_c - \theta_3) \quad ,$$

which diagrammatically corresponds to



Analyzing the rapidities appearing in this equation, one obtains

$$S_{cd}(\theta) = S_{ad}(\theta + i\bar{u}_{ac}^{b}) S_{bd}(\theta - i\bar{u}_{bc}^{a}) \quad , \tag{46}$$

which is called the bootstrap equation. In principle this allows us to start from a given amplitude S_{ab} and calculate all amplitudes of the particles appearing in our theory. For scalar particles unitarity and crossing read as

$$S_{ab}(\theta) = S_{ab}(i\pi - \theta) \quad , \qquad S_{ab}(\theta) S_{ab}(-\theta) = 1 \quad . \tag{47}$$

There is a general solution to these equations, which is

$$S(\theta) = \prod_x \frac{\tanh(\frac{\theta}{2} + i\pi x)}{\tanh(\frac{\theta}{2} - i\pi x)} \equiv \prod_x f_x(\theta) \quad . \tag{48}$$

This means that without any input from the model we want to examine we have restricted the possible form of the S matrix to that given in (48) .

In order to examine models with internal degrees of freedom, i.e. non diagonal S matrices we need some more technology. We will see that factorizable S matrices can be assigned a quantum symmetry. Then crossing- and bootstrap equations have a natural explanation in terms of quantum groups. Therefore we need some basic information in this field, which we will introduce in the next chapter.

3 Introduction to Quantum Groups

The purpose of this chapter is to collect the basic items in the theory of quantum groups. Since I would like to keep it self-contained, but on the other hand oriented at the applications I have in mind, I'll not put too much emphasis on mathematical peculiarities. I'll introduce the notation of a Hopf algebra and then go over to analyze the structure of quantum universal enveloping algebras. Most examples will be given for the algebra $U_q(sl(2))$.

3.1 Hopf Algebras and the Appearance of the Yang-Baxter Equation

Here I will just flash how the R-matrix is defined in the mathematical framework of Hopf-algebras [71] [48]. Recall the concept of an algebra: The basic ingredients are a multiplication map m,

$$\mathrm{A} \otimes \mathrm{A} \xrightarrow{\mathrm{m}} \mathrm{A} \quad : \quad (\mathrm{a}, \mathrm{b}) \longrightarrow \mathrm{a.b} \quad , \tag{1}$$

which is associative,

$$\begin{tikzcd} & \mathrm{A} \otimes \mathrm{A} \arrow[dr, "m"] & \\ \mathrm{A} \otimes \mathrm{A} \otimes \mathrm{A} \arrow[ur, "m \otimes id"] \arrow[dr, "id \otimes m"'] & & \mathrm{A} \\ & \mathrm{A} \otimes \mathrm{A} \arrow[ur, "m"'] & \end{tikzcd} \quad , \tag{2}$$

id being the identity map. In terms of elements the relation is $(a.b)c = a(b.c)$. Further it has a unit map $a = a.1 = 1.a$, which in terms of diagrams can be viewed as

$$\begin{tikzcd} \mathrm{A} = \mathrm{A} \otimes \mathrm{C} \arrow[rr, "id"] \arrow[dr, "id \otimes i"'] & & \mathrm{A} \\ & \mathrm{A} \otimes \mathrm{A} \arrow[ur, "m"'] & \end{tikzcd} \quad , \tag{3}$$

wherein i denotes the inclusion map $i : C \rightarrow \mathrm{A}$.

Let's now consider the dual space $\mathrm{A}^\star$ with the opposite structures, i.e. since $m : \mathrm{A} \otimes \mathrm{A} \longrightarrow \mathrm{A}$ we introduce $\Delta : \mathrm{A}^\star \longrightarrow \mathrm{A}^\star \otimes \mathrm{A}^\star$ which we define as $\Delta(l)(a \otimes b) =$

$l(a.b)$ $l \in \mathbf{A}^*$, $a, b \in \mathbf{A}$. Where, since it is an object of the dual space, l is meant to be a map $l : \mathbf{A} \to \mathbf{C}$, s.t. $l(\alpha a + \beta b) = \alpha l(a) + \beta l(b)$. Δ is called the co-product. Co-associativity is defined by the inverse of diagram (1.2)

$$\begin{tikzcd} & \mathbf{A}^* \otimes \mathbf{A}^* \arrow[dr, "\Delta \otimes id"] & \\ \mathbf{A}^* \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & & \mathbf{A}^* \otimes \mathbf{A}^* \otimes \mathbf{A}^* \\ & \mathbf{A}^* \otimes \mathbf{A}^* \arrow[ur, "id \otimes \Delta"'] & \end{tikzcd} . \tag{4}$$

Also the unit has an analog, called the co-unit ϵ, satisfying

$$\begin{tikzcd} \mathbf{A}^* \arrow[rr, "id"] \arrow[dr, "\Delta"'] & & \mathbf{A}^* \otimes C = \mathbf{A}^* \\ & \mathbf{A}^* \otimes \mathbf{A}^* \arrow[ur, "{id \otimes \epsilon \ (\text{ or } \epsilon \otimes id\ )}"'] & \end{tikzcd} , \tag{5}$$

or in formulae $(id \otimes \epsilon) \circ \Delta = (\epsilon \otimes id) \circ \Delta = id$. The vectorspace equipped with co-product and co-unit is called a co-algebra.

What is the motivation for introducing these structures? We will have to deal with objects which are neither algebras nor groups. Although many things can be generalized from the theory of algebras, the direct product of two irreducible representations for example, will no longer be a representation itself, but has to be substituted by the co-product.

We would like to merge the algebra and its dual, the co-algebra, into one object, which we will call bi-algebra. But in $\mathbf{A}$ there are only products and in $\mathbf{A}^*$ there are only co-products defined, so we need an additional axiom to make the two structures compatible. This can be done defining Δ and ϵ to be algebra homomorphisms, i.e.

$$\Delta(a.b) = \Delta(a).\Delta(b), \tag{6}$$

$$\epsilon(a.b) = \epsilon(a)\epsilon(b) . \tag{7}$$

To enlighten those concepts we will give two examples.

(I) Group: or more precisely $C(G)$, which is the vector-space of all smooth (continuous) complex-valued functions on G, a Lie-group. The product is defined by

$(f,f')(g) = f(g)f'(g)$. The co-product is $\Delta(f)(g_1,g_2) = f(g_1g_2)$, $g_1,g_2 \in G$ and the co-unit is given by $\epsilon(f) = f(e)$, wherein e is the unit in G. From the multiplication law we see that this constitutes a commutative algebra (i.e. $m = m\circ\sigma$, σ being the permutation-map $\sigma : \mathbf{A}\otimes\mathbf{A}' \rightarrow \mathbf{A}'\otimes\mathbf{A}$). Now let's see whether the algebra is co-commutative, i.e. $\sigma\circ\Delta = \Delta$. Decompose $\Delta(f)(g_1,g_2) = \sum f'\otimes f''(g_1,g_2) = f(g_1g_2)$. But $\sigma\circ\Delta = f''\otimes f'(g_1,g_2) = f(g_2g_1)$. Therefore the algebra is co-commutative if and only if G is abelian.

(II) Algebra: Let $\mathcal{G}$ be a Lie algebra and $U_{\mathcal{G}}$ its universal enveloping algebra (i.e. the tensor-algebra on C generated by 1 and the elements of $\mathcal{G}$ modulo the defining relations of the algebra), one can define the co-product by

$$\Delta(x) = x\otimes 1 + 1\otimes x \quad , \qquad \epsilon(x) = 0 \quad . \tag{8}$$

From the basic commutation relations $[x_i,x_j] = c^k_{ij}x_k$ the algebra is commutative if and only if $\mathcal{G}$ is abelian. It is co-commutative from the definition (8).

In our bi-algebra structure we are missing an ingredient which we have in groups: the inverse element. Introducing a similar concept, called antipode, one obtains a Hopf algebra. The antipode S is a bijective map $S : \mathbf{A} \rightarrow \mathbf{A}$, such that

$$S(a.b) \;=\; S(b).S(a) \qquad \text{and} \tag{9}$$

$$m\circ(S\otimes id)\circ\Delta \;=\; m\circ(id\otimes S)\circ\Delta = i\circ\epsilon \quad . \tag{10}$$

To understand its importance we give its form for the two examples above, which become Hopf-algebras with

$$\begin{aligned} S(f)(g) = f(g^{-1}) \quad &\text{for } C(G) \ , \\ S(x) = -x \qquad &\text{for } U_q(\mathcal{G}) \ , \end{aligned} \tag{11}$$

where the analog to the inverse element is visible.

We finish this series of definitions of mathematical objects by introducing the concept of a quasitriangular Hopf algebra. This is a Hopf algebra, containing an element R which is invertible and satisfies

$$(\Delta\otimes id)R \;=\; R_{13}R_{23} \quad , \tag{12}$$

$$(id \otimes \Delta)R = R_{13}R_{12} \ , \tag{13}$$
$$(\sigma \circ \Delta)h = R(\Delta h)R^{-1}, \ \ \forall h \in \mathrm{A} \quad . \tag{14}$$

R is an object acting in the tensor-space, i.e. $R = \sum R^{(1)} \otimes R^{(2)}$, or in general

$$R_{ij} = \sum 1 \otimes \ldots \otimes R^{(1)} \otimes \ldots \otimes R^{(2)} \otimes \ldots \otimes 1 \ .$$

Here $R^{(1)}$ is on the i^{th} position in the tensor-product and $R^{(2)}$ at the j^{th}. The first two equations give as a consequence the Yang-Baxter equation

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12} \ . \tag{15}$$

The last equation (14) can be viewed as a substitution of co-commutativity.Though co-commutativity is not requested (this would give an ordinary universal enveloping algebra), the non co-commutativity is controlled by the R- matrix.

3.2 Quantum Universal Enveloping Algebras

Up to now we did not explain yet the term Quantum Group. It arises from deforming groups or algebras to become quasitriangular Hopf algebras, which are neither commutative nor co-commutative. This terminology appeared since this can be viewed as quantizing a classical algebra [26, 27], by introducing into the commutation relations a quantization parameter h (like the Planck's constant).

Let's take any simple Lie algebra and write its commutation relations in the Chevalley-basis [18]

$$\begin{gathered}[H_i, H_j] = 0 \ , \qquad [X_i^+, X_j^-] = \delta_{ij} H_j \ , \\ \left[H_i, X_j^+\right] = a_{ij} X_j^+ \ , \ \ [H_i, X_j^-] = -a_{ij} X_j^- \ , \end{gathered} \tag{16}$$

$$\sum_{n=0}^{1-a_{ij}} (-1)^n \begin{pmatrix} 1-a_{ij} \\ n \end{pmatrix} \left(X_i^{\pm}\right)^{1-a_{ij}-n} X_j^{\pm} \left(X_i^{\pm}\right)^n = 0 \ , \quad i \neq j \ ,$$

$$\text{wherein} \quad i, j = 1, 2, \ldots, l$$
$$\begin{pmatrix} n \\ m \end{pmatrix} = \frac{n!}{m!(n-m)!} \quad .$$

The elements $X_j^\pm, H_j$ span an $sl(2)$ subalgebra. Remember that for a simple Lie algebra $\mathcal{G}$ with rank l there are l simple roots r_j and l fundamental weights λ_j related by the Cartan matrix

$$a_{ij} = \frac{2(r_i, r_j)}{(r_j, r_j)} \quad , \qquad r_j = \sum_{i=1}^{l} \lambda_i a_{ij} \quad ,$$

and (r_i, r_j) denotes the inner product of the root vectors.
An irreducible representation $D(N)$ is denoted by its highest weight N and the states in the representation space V_N by the weight m. N and m are expressed in terms of the fundamental weights,

$$N = \sum_{j=1}^{l} (N)_j \lambda_j \quad , \qquad m = \sum_{j=1}^{l} (m)_j \lambda_j \quad .$$

Example: Consider the Algebra $sl(2)$ defined by the commutation relations

$$[X^+, X^-] = H \quad , \qquad [H, X^\pm] = \pm 2X^\pm \quad .$$

The representations are labelled by $N = j = 0, \frac{1}{2}, 1\ldots$, and m goes from -$j, -j+1, \ldots, j$, i.e. a representation with highest weight j is $2j+1$ dimensional.

Into the algebra one can introduce a complex parameter q to obtain the deformed algebra $U_q(\mathcal{G})$. The new commutation relations are

$$\begin{gathered}
[H_i, H_j] = 0 \quad , \qquad k_i k_j = k_j k_i \quad , \\
\left[H_i, X_j^\pm\right] = \pm a_{ij} X_j^+ \quad , \qquad k_i X_j^\pm = q^{\pm \frac{a_{ij}}{2}} X_j^\pm k_i \quad , \\
[X_i^+, X_j^-] = \delta_{ij} [H]_q \quad , \\
\sum_{n=0}^{1-a_{ij}} (-1)^n \left[\begin{array}{c} 1 - a_{ij} \\ n \end{array} \right]_q \left(X_i^\pm\right)^{1-a_{ij}-n} X_j^\pm \left(X_i^\pm\right)^n = 0 \quad i \neq j \quad ,
\end{gathered} \tag{17}$$

but now with the symbols

$$[n]_q \equiv \frac{q^n - q^{-n}}{q - q^{-1}} \qquad (\xrightarrow{q \to 1} n),$$

$$\left[\begin{array}{c} n \\ m \end{array} \right]_q \equiv \frac{[n]_q [n-1]_q \ldots [n-m+1]_q}{[m]_q [m-1]_q \ldots [1]_q} \quad ,$$

and the abbreviations

$$k_j = q_j^{\frac{H_j}{2}} \quad , \quad q_j = q^{(r_j, r_j)} \quad .$$

This algebra can be given a Hopf algebra structure by means of the co-product

$$\Delta(H_i) = H_i \otimes 1 + 1 \otimes H_i \quad , \quad \Delta(X_i^{\pm}) = X_i^{\pm} \otimes q^{\frac{H_i}{2}} + q^{-\frac{H_i}{2}} \otimes X_i^{\pm} \; , \tag{18}$$

and the antipode

$$\gamma(H_i) = -H_i \quad , \quad \gamma(X_i^{\pm}) = -q^{\rho} X_i^{\pm} q^{-\rho} \quad ,$$

$$\rho = \sum_{r_i \in \Delta_+} H_i \quad ,$$

wherein Δ_+ is the set of positive roots of $\mathcal{G}$. This can be seen as a deformation of our algebra, since in the limit $q \to 1$, the commutation relations (17) go over into those of the algebra (16).

3.3 Representations

For q not a root of unity (that case we will treat later), representations of $U_q(\mathcal{G})$ can be obtained as deformation of representations of the classical algebras $U(\mathcal{G})$. The reason for that is that both algebras have a common Cartan subalgebra, formed by the H_i. The consequences are:

- The finite dimensional reducible representations of $U_q(\mathcal{G})$ are completely reducible.
- The irreducible representations are parameterized by the highest weights λ of the algebra $\mathcal{G}$. The modules can be decomposed into the sum of weight spaces $V^{\lambda} = \oplus_{\mu} V^{\lambda}(\mu)$, and the dimensions of $V^{\lambda}(\mu)$ are the same as for the irreducible representations of $\mathcal{G}$. Therefore any irreducible representation can be considered as some irreducible component of corresponding tensorial powers of basic representations.

As an illustrative example how this works, let's consider $U_q(sl(2))$. The R-matrix is known (see e.g. [20, 40, 63]). It is

$$R = q^{\frac{H \otimes H}{2}} \sum \frac{(1 - q^{-2})^n}{[n]_q!} (q^{\frac{H}{2}} X^+ \otimes q^{-\frac{H}{2}} X^-)^n q^{\frac{n(n-1)}{2}} \quad , \tag{19}$$

To find irreducible representations we can deform the Borel-Weyl construction [1]. The fundamental representation has two states: spin up and spin down. These two states are represented by two variables u and v. Irreducible representations of spin j are given by homogeneous polynomials of degree $2j$. A basis is given by

$$e^j_m = \frac{u^{j+m}v^{j-m}}{\sqrt{(j+m)!(j-m)!}} \quad , \qquad m = -j, -j+1, \ldots, j \quad , \tag{20}$$

and the operators $X^\pm, H$ are represented by differential operators

$$X^+ = u\frac{\partial}{\partial v} \quad , \qquad X^- = v\frac{\partial}{\partial u} \quad , \qquad H = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \quad . \tag{21}$$

This construction is embedded in a geometrical framework [1]. The q-deformation is constructed using the same polynomials in u and v as a basis, but deforming the generators as q-derivatives, i.e.

$$D_u f(u) = \frac{f(qu) - f(q^{-1}u)}{(q-q^{-1})u} \quad . \tag{22}$$

In terms of this new derivatives the generators for $sl(2)_q$ are

$$X^+ = uD_v \quad , \qquad X^- = vD_u \quad , \qquad H = u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} \quad ; \tag{23}$$

explicitly we then find

$$X^+e^j_m = \sqrt{[j+m+1]_q[j-m]_q}\ \ e^j_{m+1} \quad , \qquad X^-e^j_m = \sqrt{[j-m+1]_q[j+m]_q}\ \ e^j_{m-1} \tag{24}$$

$$He^j_m = 2m\, e^j_m \quad .$$

Inserting these expressions into (19) one can also find the matrix elements of the R-matrix[40]. We give here only as an example the R-matrix for the two dimensional representation.

$$R = q^{-\frac{1}{2}} \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q-q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \tag{25}$$

Another property of this matrix is crossing symmetry: Let w be the generalization of the Weyl element. It can be represented as

$$we^j_m = (-1)^{j-m}q^{-j(j+1)+m}e^j_{-m} \quad . \tag{26}$$

Introduce the linear antiautomorphism $\tau(X^{\pm}) = X^{\mp}$, and $\tau(H) = H$. Then

$$waw^{-1} = \tau S(a) \qquad \forall a \in sl(2)_q \quad ,$$

Therefore also R has a definite transformation behaviour, which is

$$(\tau \otimes id)R^{-1} = (w \otimes 1)R(w^{-1} \otimes 1) \quad , \tag{27}$$

or for the representations we considered

$$\left(\left(R^{j_1 j_2}\right)^{-1}\right)^{t_1} = w_1 R^{j_1 j_2} w_1^{-1} \quad , \qquad w_1 \equiv w^{j_1} \otimes 1 \quad ,$$

wherein t_1 means transposition in the first space.

3.4 Clebsch-Gordan Coefficients and 6j-Symbols

Since the $U_q(\mathcal{G})$ is a tensor algebra we want to study the properties of tensor-products of $\mathcal{G}$. For that we introduce the representation of the R-matrix in the form

$$R^{j_1 j_2} = P^{j_1 j_2}(D_{j_1} \otimes D_{j_2})(R) \quad . \tag{28}$$

$P^{j_1 j_2}$ is the permutation matrix $P^{j_1 j_2} : V_{j_1} V_{j_2} \rightarrow V_{j_2} V_{j_1}$, and D_{j_i} denotes the representation. From now on we restrict ourselves to $sl(2)_q$. Many formulas are valid also for other algebras, and others are easily generalizable [59] [60]. But since our applications will be purely for the case of $sl(2)_q$, we keep all of this part restricted to it.

For the above representation the Yang-Baxter equation takes the form of the braid-equation [29]

$$(R^{j_1 j_2} \otimes 1)(1 \otimes R^{j_1 j_3})(R^{j_2 j_3} \otimes 1) = (1 \otimes R^{j_2 j_3})(R^{j_1 j_3} \otimes 1)(1 \otimes R^{j_1 j_2}) \quad , \tag{29}$$

where the action of the operators takes from $V^{j_1} \otimes V^{j_2} \otimes V^{j_3} \rightarrow V^{j_3} \otimes V^{j_2} \otimes V^{j_1}$. Let's take the tensor space $V^{j_1} \otimes V^{j_2}$, and assume that V^j is a subspace of it. Then the Clebsch-Gordan coefficient

$$K_j^{j_1 j_2}(q, a) : \ V^{j_1} \otimes V^{j_2} \rightarrow V^j,$$

is a projection-matrix onto the subspace V^j. Herein a labels the multiplicity with which V^j appears in $V^{j_1} \otimes V^{j_2}$. Our normalization is such that

$$K_j^{j_1 j_2}(q,a) \left(K_{j'}^{j_1 j_2}(q,a)\right)^t = \delta_{jj'} \quad . \tag{30}$$

Since in the applications we have in mind we will not encounter multiplicities larger than 1, we will skip the general theory [59], and restrict the discussion to $a = 1$.

$R^{j_1 j_2}$ can be viewed as 'diagonal' matrix with the eigenvectors $K_j^{j_1 j_2}$. That is

$$\begin{aligned} \left(R^{j_1 j_2}(q) \;\; K_j^{j_1 j_2}(q)\right)^t &= (-1)^{j_1+j_2-j} \;\; q^{c_j - c_{j_1} - c_{j_2}} \left(K_j^{j_1 j_2}(q)\right)^t , \\ K_j^{j_1 j_2}(q) \;\; R^{j_1 j_2}(q) &= (-1)^{j_1+j_2-j} \;\; q^{c_j - c_{j_1} - c_{j_2}} K_j^{j_1 j_2}(q) \quad . \end{aligned} \tag{31}$$

The c_j are the eigenvalues of the Casimir operator in the respective representation, and have the value $j(j+1)$. On the contrary one can decompose $R^{j_1 j_2}$ in terms of the Clebsch-Gordan coefficients

$$\begin{aligned} R^{j_1 j_2}(q) &= \sum_{j \in D_{j_1} \otimes D_{j_2}} (-1)^{j_1+j_2-j} \;\; q^{c_j - c_{j_1} - c_{j_2}} \;\; \mathcal{P}_j^{j_1 j_2}(q) \;, \\ \left(R^{j_1 j_2}(q)\right)^{-1} &= \sum_{j \in D_{j_1} \otimes D_{j_2}} (-1)^{j_1+j_2-j} \;\; q^{-c_j - c_{j_1} - c_{j_2}} \;\; \mathcal{P}_j^{j_2 j_1}(q) \quad , \end{aligned} \tag{32}$$

defining

$$\mathcal{P}_j^{j_2 j_1}(q) = \left(K_j^{j_1 j_2}(q)\right)^t K_j^{j_1 j_2}(q) \;\; .$$

The $\mathcal{P}$ are projection operators on the respective representations.

Example: Let's analyze the three dimensional representation. Using (24) we get

$$X^+ = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \sqrt{[2]_q} \;, \qquad X^- = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \sqrt{[2]_q} \;, \qquad H = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix} \;.$$

Putting this into the expression (19) we obtain

$$R = q^{\frac{H \otimes H}{2}} (1 + (q^4 - 1) E \otimes F + (q^2 - 1)^2 (q^2 + 1) E^2 \otimes F^2) \quad , \tag{33}$$

where we have introduced

$$q^{\frac{H}{2}} X^+ = q \sqrt{[2]_q} E \;\;, \qquad q^{-\frac{H}{2}} X^- = q^2 \sqrt{[2]_q} F \quad .$$

Since for $sl(2)$ we have "$3\otimes 3 = 5\oplus 3\oplus 1$" we can find the projectors simply from inverting the relations

$$\begin{aligned} R_{12} &= \rho_0\mathcal{P}_0 + \rho_1\mathcal{P}_1 + \rho_2\mathcal{P}_2 \ , \\ (R_{21})^{-1} &= \rho_0'\mathcal{P}_0 + \rho_1'\mathcal{P}_1 + \rho_2'\mathcal{P}_2 \ , \\ P &= \mathcal{P}_0 + \mathcal{P}_1 + \mathcal{P}_2 \quad . \end{aligned}$$

Recall that P is the permutation operator and the last relation equals (30). The ρ_i are obtained from (32). Therefore the projection operators onto the representations are given as

$$\begin{aligned} \mathcal{P}_0 &\sim R_{12} - (R_{21})^{-1} - (q^4 - q^{-4})P \ , \\ \mathcal{P}_1 &\sim -q^2 R_{12} - q^{-2}(R_{21})^{-1} + (q^6 - q^{-6})P \ , \\ \mathcal{P}_2 &\sim q^6 R_{12} - q^{-6}(R_{21})^{-1} + (q^2 - q^{-2})P \ , \end{aligned} \tag{34}$$

where we omitted the normalization factors. Note, that we are using in this example an R matrix which is not multiplied with the permutation matrix (compare (28))

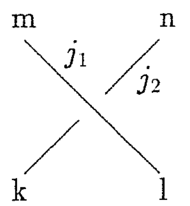
Another useful identity results from restricting the defining identity (12) to an irreducible representation. The result we will called pentagon identity

$$\begin{aligned} R^{jj'}\left(K_{j'}^{j_1 j_2}\right)_{23} &= \left(K_{j'}^{j_1 j_2}\right)_{12}(R^{jj_2})_{23}(R^{jj_1})_{12} \ , \\ R^{j'j}\left(K_{j'}^{j_1 j_2}\right)_{12} &= \left(K_{j'}^{j_1 j_2}\right)_{23}(R^{j_1 j})_{12}(R^{j_2 j})_{23} \ , \end{aligned} \tag{35}$$

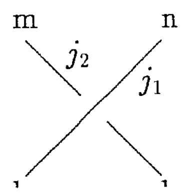
with sums over repeated indices j_k.

3.4.1 Graphical Representation

Let's represent the R-matrix as

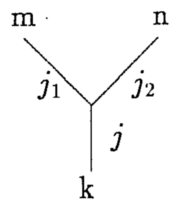
$$\left(R^{j_1 j_2}\right)^{kl}_{mn} \longrightarrow$$ 

, (36)

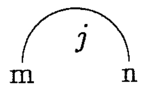
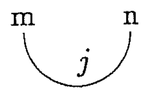
$$\left(\left(R^{j_1 j_2}\right)^{-1}\right)^{kl}_{mn} \longrightarrow$$ 

. (37)

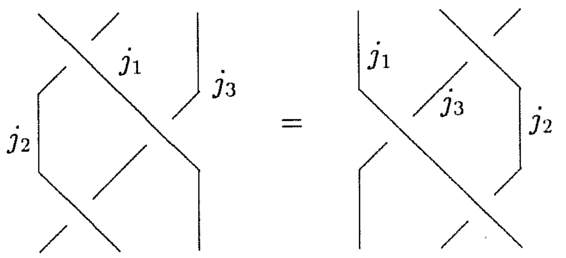
The Clebsch-Gordan-Coefficients are drawn as vertices

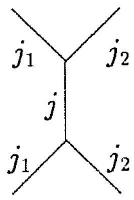
$$\left(K_j^{j_1 j_2}\right)_k^{mn} \equiv \begin{bmatrix} j_1 & j_2 & j \\ m & n & k \end{bmatrix} \longrightarrow$$  , (38)

and the transposed coefficient turning the diagram (38) upside-down. The crossing-matrix we represent as

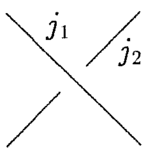
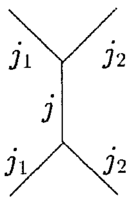
$(w_j)_{mn} \longrightarrow$  , $\left(w_j^{-1}\right)_{mn} \longrightarrow$  . (39)

Now we can translate the previous equations into graphical language. For example the Yang-Baxter Equation

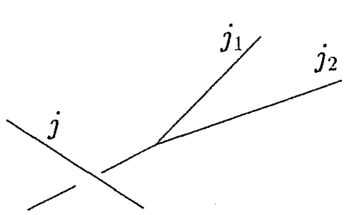
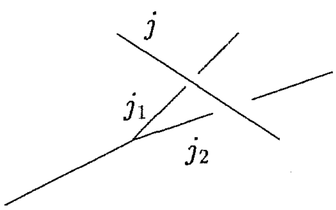
 . (40)

The projector $\mathcal{P}_j^{j_1 j_2}$ becomes  ,

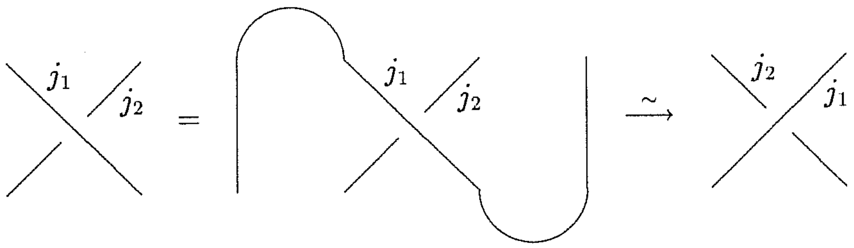
and therefore the decomposition of the R-matrix (32) is drawn as

 $= \sum_j$  $(-1)^{j_1+j_2-j}\, q^{c_j - c_{j_1} - c_{j_2}}$, (41)

and similar for R^{-1}. The pentagon identity takes the form

 =  . (42)

The crossing symmetry amounts in rotating the diagram about 90 degrees, as shown below.



$$\quad . \qquad (43)$$

3.4.2 6j-Symbols

We want to entangle still further our representations. How can we get an irreducible representation out of a three-fold product of spaces $V^{j_1} \otimes V^{j_2} \otimes V^{j_3}$? We obtain different bases in this space if we start by decomposing the first two or the last two spaces. Let $V^{j_2} \otimes V^{j_3} = \oplus_{j_{23}} V^{j_{23}}$, and the other one $V^{j_1} \otimes V^{j_2} = \oplus_{j_{12}} V^{j_{12}}$, then the two bases become

$$e_m^{j_{12}j}(j_1 j_2 | j_3) = \sum_{m_1 m_2 m_3} \begin{bmatrix} j_{12} & j_3 & j \\ m_{12} & m_3 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j_3 & j_{12} \\ m_1 & m_3 & m_{12} \end{bmatrix}_q e_{m_1}^{j_1} \otimes e_{m_2}^{j_2} \otimes e_{m_3}^{j_3} \quad , \qquad (44)$$

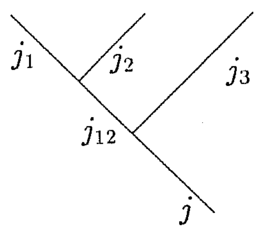
and

$$e_m^{j_{23}j}(j_1 | j_2 j_3) = \sum_{m_1 m_2 m_3} \begin{bmatrix} j_2 & j_{23} & j \\ m_1 & m_{23} & m \end{bmatrix}_q \begin{bmatrix} j_2 & j_3 & j_{23} \\ m_2 & m_3 & m_{23} \end{bmatrix}_q e_{m_1}^{j_1} \otimes e_{m_2}^{j_2} \otimes e_{m_3}^{j_3} \quad , \qquad (45)$$

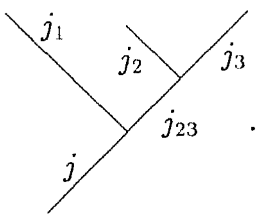
where we have used the matrix-elements of the Clebsch-Gordan coefficients (38).
The elements of these two bases are connected by the $6j$-symbols, i.e.

$$e_m^{j_{12}j}(j_1 j_2 | j_3) = \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q e_m^{j_{23}j}(j_1 | j_2 j_3) \quad , \qquad (46)$$

or graphically



$$= \sum_{j_{23}} \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j & j_{23} \end{Bmatrix}_q$$



$$\quad . \qquad (47)$$

In our representation (24) they are given as

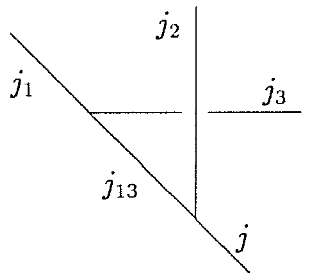
$$
\begin{aligned}
\left\{ \begin{array}{ccc} a & b & e \\ d & c & f \end{array} \right\}_q &= \sqrt{[2e+1][2f+1]}\ (-1)^{c+d+2e-a-b} \ \times \\
&\Delta(abe)\Delta(acf)\Delta(cef)\Delta(dbf)\ \textstyle\sum_z (-1)^z [z-1]! \ \times \\
&([z-a-b-e]![z-a-c-f]![z-b-d-f]![z-d-c-e]! \ \times \\
&[a+b+c+d-z]![a+d+e+f-z]![b+c+e+f-z]!)^{-1}
\end{aligned} \tag{48}
$$

wherein we use the conventions that $[0]! = 1$ and $[x] = 0$ if and only if $x < 0$ in the sum.

$$
\Delta(abc) = \left(\frac{[-a+b+c]![a-b+c]![a+b-c]!}{[a+b+c+1]!} \right)^{\frac{1}{2}} \quad .
$$

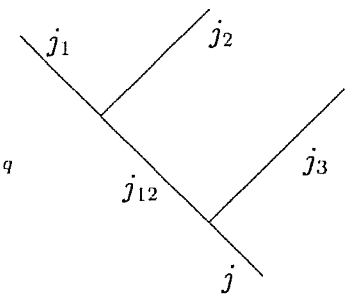
3.5 The Shadow-World Representation

Another identity involving the $6j$-symbols will be the starting-point for introducing a "shadow-world", which will be a different representation of the same algebra. We will also exam the analog in statistical mechanics. The identity [40], which can be proven using the graphical formulation of the last section is written as

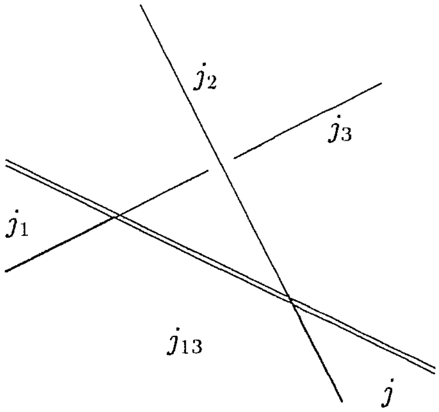


$$
= \sum_{j_{12}} (-1)^{j_{13}+j_{12}-j-j_1}\ q^{c_j + c_{j_1} - c_{j_{13}} - c_{j_{12}}} \ \times
$$

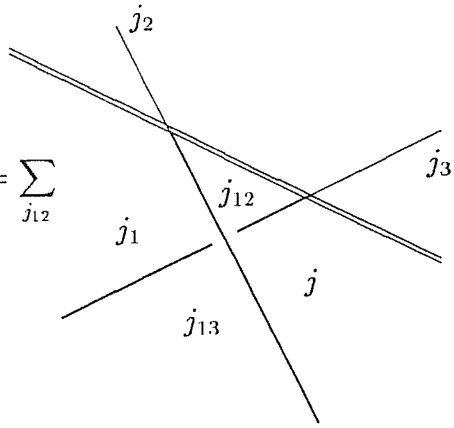
$$
\left\{ \begin{array}{ccc} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{array} \right\}_q \tag{49}
$$



Now we redraw this identity in the following form:



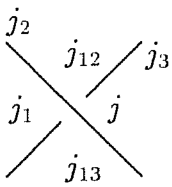
$$= \sum_{j_{12}}$$



(50)

The double line, called the "horizon", parts the 'world' from the 'shadow-world'. The graphic equations seem much more comprehensible, since they amount to simply moving the horizon.

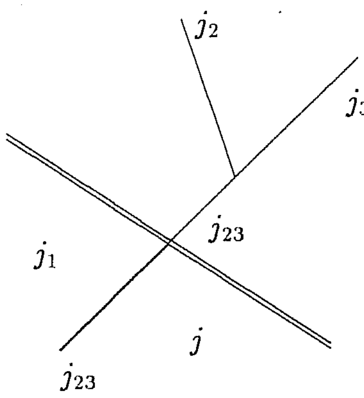
The rules assigned to the elements of the graphs though, are different in the shadow-world. First of all, we place j's onto the strings and in them. In the world instead we place (as before) j's at the strings and the matrix-indices m on the end of the strings (which we have omitted in most equations for simplicity). j's assigned to lines remain the same crossing the horizon. So we can extract the the rules for the shadow-world graphs, comparing with the corresponding real-world graph. For example comparing (49) and (50) we extract



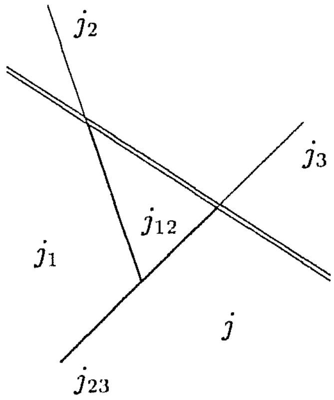
$$\longrightarrow \quad (-1)^{j_{13}+j_{12}-j-j_1}\, q^{c_j+c_{j_1}-c_{j_{13}}-c_{j_{12}}} \quad \times$$

$$\left\{ \begin{array}{ccc} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{array} \right\}_q \tag{51}$$

Or let's redraw graph (47) as

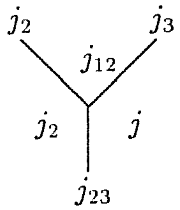


$$= \sum_{j_{12}}$$



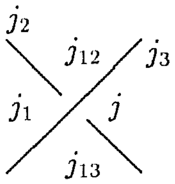
(52)

wherefrom we extract



$$\longrightarrow \left\{ \begin{matrix} j_3 & j_1 & j_{23} \\ j_1 & j & j_{12} \end{matrix} \right\}_q \quad . \tag{53}$$

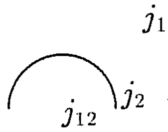
Similarly one gets



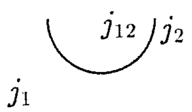
$$\longrightarrow \quad (-1)^{-j_{13}-j_{12}+j+j_1} \; q^{-c_j - c_{j_1} + c_{j_{13}} + c_{j_{12}}} \quad \times$$

$$\left\{ \begin{matrix} j_3 & j_1 & j_{13} \\ j_2 & j & j_{12} \end{matrix} \right\}_q \tag{54}$$

or the analog of the crossing matrices



$$\longrightarrow (-1)^{j_1+j_2-j_{12}} \sqrt{\frac{[2j_{12}+1]}{[2j_1+1]}}\, \delta(j_1 j_2 j_{12}) \quad , \tag{55}$$



$$\longrightarrow (-1)^{j_1+j_2-j_{12}} \sqrt{\frac{[2j_{12}+1]}{[2j_1+1]}}\, \delta(j_1 j_2 j_{12}) \quad . \tag{56}$$

In this way many identities can be transported into the shadow-world , like the Yang-Baxter equation (15) and the pentagon identity (35). So the shadow-world appears to be a different representation of the same algebra. This is true, in fact. It corresponds to the transition from vertex- to IRF-models in statistical mechanics [3] . Note that the R-matrix in this representation has become proportional to a $6j$-symbol. Let's elaborate a little bit on this point.

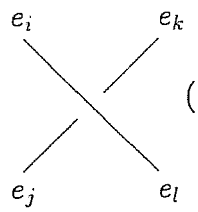
Take a square lattice, rotated about 45° with $N \times M$ sites. Assign to each link a 'state' taking $n+1$ possible values. Present these by a vector e_i, $1 \leq i \leq n+1$ among the weights of the fundamental representation of sl(n+1). The partition function is

$$Z = tr\,(X_1 X_2)^{\frac{M}{2}} \quad , \tag{57}$$

where

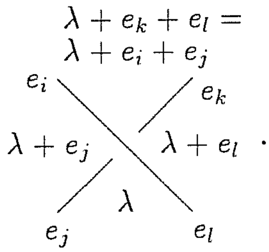
$$\begin{aligned} X_1 &= T_{12}T_{34}\ldots T_{N-1,N} \ , \\ X_2 &= T_{23}T_{45}\ldots T_{N-2,N-1} \ , \end{aligned} \tag{58}$$

are the column to column transfer-matrix. The R are the Boltzmann weights of the vertex-configuration specified by the states

$$T(e_i \otimes e_j) = \sum_{e_k, e_l}$$ 

$$(e_k \otimes e_l) \quad . \tag{59}$$

These can be transformed to Face Models assigning values at the dual lattice as



Consider for examples $sl(2)$. The weights are e_1 and $e_2 = e_1$. Since there are only 2 values, they are usually represented by arrows, and the heights on the dual lattice take their values in Z. This is the usual correspondence between the six vertex model and BCSOS (Body-centered Solid on solid) model [3] [4].

The T-matrix in the vertex basis is written in a direct product basis

$$T : V_{j_1} \otimes V_{j_2} \to V_{j_2} \otimes V_{j_1} \quad ,$$

i.e. the matrix elements are

$$\langle j_2 m_2 | \langle j_1 m_1 | T | j_1 m_1' \rangle | j_2 m_2' \rangle \quad .$$

The transition to the face-basis amounts in taking direct products and decomposing them into irreducible representations. The transformation formula amounts to be exactly relation (49) [56] [57]. This is the foundation of the shadow-world formalism, hereby proving that our thumb-rule of assigning rules to shadow-world graphs and re-drawing graphs in the shadow-world are really allowed. The generalized vertex-model characterized by the weights of $sl(n)$ introduced before, is connected with the A_n quantum group.

The use of this formalism is twofold: first of all with the help of shadow-world diagrams one can find new identities for 6j-symbols,which otherwise can be proved by rather involved algebraic calculations. Secondly in representing our algebra we can change the basis, to the one which is more convenient for describing the physical problem at hand.

3.6 Representations when q is a root of unity

Up to now we always assumed that q should not be a root of unity. We'll try now to clarify the peculiarities happening in this case.
We will use for this discussion a spin-basis, i.e. we represent

$$X^{\pm} = \sum_i X_i^{\pm} = \sum_i q^{\frac{\sigma_z}{2}} \otimes \ldots \otimes q^{\frac{\sigma_z}{2}} \otimes \frac{\sigma^{\pm}}{2} \otimes \ldots \otimes q^{-\frac{\sigma_z}{2}} , \tag{60}$$

where the term $\frac{\sigma^{\pm}}{2}$ is sited at the i^{th} place, and let these act on the usual spin states

$$|\uparrow\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad .$$

For example take the representation $j = 1$ for which

$$X^{+} = q^{\frac{\sigma_z}{2}} \otimes \frac{\sigma^{+}}{2} \quad , \qquad X^{-} = \frac{\sigma^{-}}{2} \otimes q^{-\frac{\sigma_z}{2}} \quad .$$

The representation takes the form

$$\begin{array}{ccc} |a\rangle = |\uparrow\uparrow\rangle & & \\ \downarrow\uparrow & & \\ X^-|a\rangle = q^{\frac{1}{2}}|\uparrow\downarrow\rangle + q^{-\frac{1}{2}}|\downarrow\uparrow\rangle & & |b\rangle = q^{-\frac{1}{2}}|\uparrow\downarrow\rangle - q^{\frac{1}{2}}|\downarrow\uparrow\rangle \\ \downarrow\uparrow & & \\ (X^-)^2|a\rangle = |\downarrow\downarrow\rangle & & \end{array} \tag{61}$$

We get, as in the case of $sl(2)$, a triplet and a singlet. Now let $q^p = \pm 1$. Note that $[p]_q = 0$ and therefore $[x+np]_q = [x]_q$. Further $(X^{\pm})^p = 0$ [58]. So, looking at a representation with dimension larger than p, we have certainly some null-vectors in our representation space.

Consider the simplest case: $q = e^{\frac{i\pi}{3}}$, and take the product space $(C_2)^3$. In analogy with the Lie algebra case this should decompose into "$2 \otimes 2 \otimes 2 = 4 \oplus 2 \oplus 2$". But take the state $|\uparrow\uparrow\uparrow\rangle$, and apply $(X^-)^3$. We cannot reach $|\downarrow\downarrow\downarrow\rangle$ since it gives 0.
So the structure becomes more difficult in this case. Reducible representations are not any more completely reducible. But one can avoid these complications. Because up to the point $j < \frac{1}{2}(p-1)$ everything is fine. One can restrict the representations to those ones. For this purpose one needs to restrict also the tensor product to the "good" representations. This is done by keeping only the highest weight vectors which are annihilated by X^- and at the same time are not in the image of $(X^+)^{p-1}$, that is: only representations with spin $j \leq \frac{p-2}{2}$ appear. The remaining representations have the following features [1] :

- they are highest weight representations
- $|j\rangle \neq (X^+)^{p-1}|\,anything\,\rangle$ for any highest weight vector $|j\rangle$.
- The highest weight vectors are in the space $|j\rangle \in Ker\, X^+ / \, Im\,(X^+)^{p-1}$

Analyzing the decomposition of representations one finds that

$$V^{j_1} \otimes V^{j_2} = \sum_{j=|j_1-j_2|}^{\min(j_1+j_2,p-2-j_1-j_2)} V^j \tag{62}$$

3.7 Introducing a Spectral Parameter

Recall the Kac-Moody algebras and the corresponding current-algebra [31].

$$[J^a(x), J^b(y)] = i\hbar f^{abc} J^c(x)\delta(x-y) \quad .$$

The commutation relations of our previous algebra have gained another continuous parameter. This is a motivation to study affine quantum groups, to arrive at solutions of the Yang-Baxter equation with a spectral parameter. These are the R-matrices which appear in statistical mechanics context [64].

As for $U_q(\mathcal{G})$, $\mathcal{G}$ a simple algebra, we construct it from the Chevalley basis of our algebra. But now we have an additional root in our Cartan matrix, usually denoted by ψ equals minus the highest root of $\mathcal{G}$. The commutation relations (17) are now valid for all generators of the affine group. Let $\hat{U} = \hat{U}(A_N^{(1)})$ be the algebra defined by taking (a_{ij}) the Cartan matrix of $A_N^{(1)}$ with generators $q^{\pm H_i}, X_i^{\pm}$, with $0 \leq i \leq N$. One can define a mapping $\hat{U}' \xrightarrow{\xi} \hat{U} \otimes C(\lambda, \lambda^{-1})$, which is an algebra homomorphism. This maps

$$\begin{array}{llll} q^{\pm\frac{h_i}{2}} \xrightarrow{\xi} q^{\pm\frac{h_i}{2}} \ , & i = 0, 1, \ldots N, & & \\ (X_0^+)' \xrightarrow{\xi} \lambda X_0^+ \ , & (X_i^+)' \xrightarrow{\xi} X_i^+ \ , & i = 1, \ldots N, & \\ (X_0^-)' \xrightarrow{\xi} \lambda X_0^- \ , & (X_i^-)' \xrightarrow{\xi} X_i^- \ , & i = 1, \ldots N. & \end{array} \tag{63}$$

Though, this is not a Hopf algebra homomorphism, i.e. does not commute with the co-product

$$(\xi \otimes \xi)\Delta' = \Delta\xi \quad ,$$

where we abbreviated $\sigma \circ \Delta = \Delta'$. Now let's make an ansatz [37] for the new co-product in our affine algebra by taking

$$(X_0^+)' \otimes 1 = \lambda X_0^+ \otimes 1 \quad 1 \otimes (X_0^+)' = 1 \otimes \mu X_0^+ \tag{64}$$

and taking the co-product given before (18). To make this an Hopf-algebra we need to satisfy (14)

$$\Delta' R = R\Delta \quad . \tag{65}$$

Written explicitly, these relations become

$$R(\lambda,\mu)\Delta(X_i^{\pm}) = \Delta'(X_i^{\pm})R(\lambda,\mu), \tag{66}$$

$$R(\lambda,\mu)(\lambda X_0^{\pm}\otimes q^{-\frac{H_0}{2}} + q^{\frac{H_0}{2}}\otimes \mu X_0^{\pm}) = (q^{-\frac{H_0}{2}}\otimes \lambda X_0^{\pm} + \mu X_0^{\pm}\otimes q^{\frac{H_0}{2}})R(\lambda,\mu) \ . \tag{67}$$

The solution space of these equations is one-dimensional and the solutions satisfy the Yang-Baxter equation (as they should do). The result depends only on the ratio of λ and μ, i.e. $x \equiv \frac{\lambda}{\mu}$.

For example for the fundamental representation of $A_N^{(1)}$, the R-matrix becomes

$$R(x,q) = q^{\frac{1}{2}}xR_{12} - q^{-\frac{1}{2}}x^{-1}R_{21}^{-1} \ . \tag{68}$$

Or for example the R matrix for $A_2^{(2)}$ in the fundamental representation can be written as

$$R(x) = (x^{-1}-1)q^3R_{12}(q) + (1-x)q^{-3}R_{21}^{-1}(q) + q^{-5}(q^4-1)(q^6+1)I \quad , \tag{69}$$

with R_{12} being the R matrix of the spin 1 representation of $U_q(sl(2)$. Many properties carry over to affine quantum groups. For example the decomposition into projectors is now

$$R(x,q) = \sum_{\nu}\rho_\nu(x,q)\mathcal{P}_\nu \quad . \tag{70}$$

This is the same decomposition as before, but now with factors depending on the spectral parameter. For example for (68) this becomes

$$R(x,q) = (xq + x^{-1}q^{-1})\mathcal{P}_1 + (qx^{-1} - xq^{-1})\mathcal{P}_0 \quad . \tag{71}$$

The R-matrices with spectral parameter have been found and classified in [37]. We will not need more than the two examples of above. For details on affine quantum groups see [38] and references therein.

3.7.1 The Fusion Procedure

Up to now, we have constructed the R-matrices for a representation directly from the requirements of the quasitriangularity of the Hopf algebra (12 - 14). There is though, an alternative approach, called the fusion procedure [38] . If one knows $R_{V''V}$ and $R_{V''V'}$, one can write down a new R matrix, which is an intertwiner between the spaces V'' and $V \otimes V'$ as

$$R_{V''V\otimes V'}(x) = (I \otimes R_{V''V'}(xy_1))(R_{V''V}(xy_2) \otimes I) \quad , \tag{72}$$

and

$$R_{V\otimes V'V''}(x) = (R_{VV''}(xy_1) \otimes I)(I \otimes R_{V'V''}(xy_2)) \quad . \tag{73}$$

With an appropriate choice of y_1 and y_2 these new solutions can be restricted to a subspace of the product space $V \otimes V'$. The condition for this is, that $R_{V'V}(y_2/y_1)$ is a projection operator. Let $R_{V'V}(y_2/y_1) \sim \mathcal{P}_W$, that is $W \in V \otimes V'$. Then the restriction of R to W, i.e. $R_{V''W}(x) = R_{V''V\otimes V'}\,|_{V''\otimes W}$ is the R matrix intertwining the space $V'' \otimes W$.

Part II

S Matrices for Integrable Restrictions of Soliton Equations

4 The S Matrix of the Sine-Gordon Model

The sine-Gordon S matrix has been derived in [77]. We will discuss the quantum group symmetry and how the shadow-world representation can be used to determine the particle content at rational values of the coupling constant (see section 2.4.1). On the other hand we will use the constraints provided by factorized scattering, to select consistent S matrix systems. The exciting result of confronting the two approaches is, that for the analyzed class of systems, only the ones, deriving from the sine Gordon S matrix survive.

4.1 The Hidden Quantum Group Symmetry of the Sine-Gordon S Matrix

The S matrix of the sine-Gordon model was derived using the assumption of factorized scattering and the $O(2)$ symmetry of the model [4].
Denote the soliton as A and the anti-soliton as $\bar{A}$. Then the S matrix amplitudes can be written as

$$\begin{aligned} A(\beta_1)A(\beta_2) &= S(\beta_{12})A(\beta_2)A(\beta_1) \ , \\ \bar{A}(\beta_1)\bar{A}(\beta_2) &= S(\beta_{12})\bar{A}(\beta_2)\bar{A}(\beta_1) \ , \\ A(\beta_1)\bar{A}(\beta_2) &= S_T(\beta_{12})\bar{A}(\beta_2)A(\beta_1) + S_R(\beta_{12})A(\beta_2)\bar{A}(\beta_1) \ . \end{aligned} \tag{1}$$

The amplitudes are given in [77]

$$S(\beta) = S_0 \sinh\left(\frac{\pi}{\xi}(\beta - \pi i)\right) \tag{2}$$

[4]This symmetry is not obvious, but corresponds to the rotational symmetry of the disorder parameter [77]

$$\begin{aligned} S_T(\beta) &= -S_0 \sinh\left(\frac{\pi}{\xi}\beta\right) && (3) \\ S_R(\beta) &= -S_0 \sinh\left(\frac{\pi^2 i}{\xi}\right) && (4) \end{aligned}$$

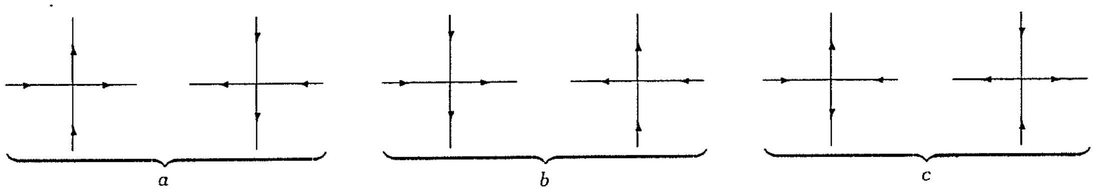
The function S_0 is an infinite product of Γ functions (see page 54). Here we use the notation $\xi \equiv \frac{1}{8}\gamma'^2$, where γ'^2 denotes the renormalized coupling constant (38). Taking $S \equiv \begin{pmatrix} A \\ \bar{A} \end{pmatrix}$ one can write this in matrix form as

$$(S \otimes 1)(\beta_1)\,(1 \otimes S)(\beta_2) = \begin{pmatrix} S & & & \\ & S_T & S_R & \\ & S_R & S_T & \\ & & & S \end{pmatrix} (1 \otimes S)(\beta_2)\,(S \otimes 1)(\beta_1) \qquad (5)$$

Crossing symmetry takes the form

$$S = S_T(i\pi - \beta) \ , \quad S_R = S_R(i\pi - \beta) \ . \qquad (6)$$

One notices that the scattering matrix is identical with the T matrix of the six vertex model (see page 45). We have assigned weights to the vertex configurations, as



(7)

and the T matrix which we defined in section 3.5 , is written as

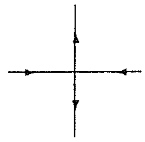
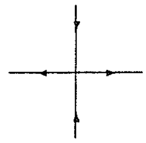
$$T = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix} \ . \qquad (8)$$

In solving the model one is led to the parameterization [3]

$$a = \rho \sin(\lambda - u) \ , \quad b = \rho \sin u \ , \quad c = \rho \sin \lambda \quad ,$$

which are essentially the functions we found above.

We impose periodic boundary conditions, as usual. Then we must have the same amount of weights

$c_1 =$  and $c_2 =$  in each row.

Since the partition function (57) depends only on the products of the weights, one may give c_1 and c_2 different weights, i.e.

$$c_1 = ce^{-xu} \quad \text{and} \quad c_2 = ce^{xu} \quad ,$$

and write the corresponding T matrix as

$$T = \begin{pmatrix} a & & & \\ & b & ce^{xu} & \\ & ce^{-xu} & b & \\ & & & a \end{pmatrix} \quad .$$

This matrix corresponds to the R-matrix (68) [64]. We would like to apply an analogous transformation to the S matrix, in order to unravel the quantum group structure[5].

We represent our n-particle soliton state as $|\beta_1, \ldots, \beta_n\rangle_{SG}$, corresponding to the operators S which get exchanged by the S matrix. Scattering applied to these states acts as

$$\begin{aligned} &|\beta_n \ldots \beta_i, \beta_{i+1}, \ldots \beta_1\rangle_{SG} S_{i,i+1}(\beta_i - \beta_{i+1}) = \\ &|\beta_n \ldots \beta_{i+1}, \beta_i, \ldots \beta_1\rangle_{SG} P_{i,i+1} \quad , \end{aligned} \tag{9}$$

and similar for states $_{SG}\langle\beta_1, \ldots, \beta_n|$ which are dual i.e.

$$_{SG}\langle\beta_1, \ldots, \beta_n|\beta_1', \ldots, \beta_m'\rangle_{SG} = \delta_{mn} \prod \delta(\beta_i - \beta_i')\, I \quad .$$

[5]Also the form 8 can be interpreted as an R matrix in the so called principle gradation. It is not clear yet, whether it fulfills the conditions of the quantum group in higher representations [25]

Then the base transformation is written as [69]

$$|\beta_1,\ldots,\beta_n\rangle = \prod_{i=1}^{n} e^{-\frac{\pi}{2\xi}\sigma_i^3\beta_i}|\beta_1,\ldots,\beta_n\rangle_{SG} \quad ,$$

$$_{SG}\langle\beta_1,\ldots,\beta_n| = \langle\beta_1,\ldots,\beta_n|\prod_{i=1}^{n} e^{\frac{\pi}{2\xi}\sigma_i^3\beta_i} \quad . \tag{10}$$

The new bases $\langle\beta_1,\ldots,\beta_n|$ and $|\beta_1,\ldots,\beta_n\rangle$ are again dual, but do not have any more an apparent physical interpretation. The states cannot any more be interpreted as doublets of solitons and anti-solitons. The S-matrix on the other hand has now the form

$$S = S_0\begin{pmatrix} \sinh(\frac{\pi}{\xi}(\beta - i\pi) & & & \\ & -\sinh\left(\frac{\pi}{\xi}\beta\right) & -\sinh\left(\frac{\pi^2 i}{\xi}\right)e^{\frac{\pi\beta}{\xi}} & \\ & -\sinh\left(\frac{\pi^2 i}{\xi}\right)e^{-\frac{\pi\beta}{\xi}} & -\sinh\left(\frac{\pi}{\xi}\beta\right) & \\ & & & \sinh\left(\frac{\pi}{\xi}(\beta - i\pi)\right) \end{pmatrix} \tag{11}$$

Defining $\hat{S} = PS$ this can be related to the R matrix as $\hat{S} = S_0R(x,q)$ with the parameters $x = e^{\frac{\pi\beta}{\xi}}$ and $q = -e^{-\frac{i\pi^2}{\xi}}$ and $R(x,q)$ given by (68).

Now we could adopt the opposite way of reasoning: we start out with an R matrix and try to construct the corresponding S matrix. Since the Yang-Baxter equation is satisfied one needs to obtain unitarity and crossing, i.e. $S(x,q)S(x^{-1}q) = 1$ and $S(x,q) = S(-\frac{1}{xq},q)$. From (32) and (68) we derive that

$$R(x,q)R(x^{-1},q) = (x^{-1}q - xq^{-1})(qx - x^{-1}q^{-1}) \quad . \tag{12}$$

Therefore we require for the multiplier S_0 that

$$S_0(x)S_0(x^{-1}) = \frac{1}{(x^{-1}q - xq^{-1})(qx - x^{-1}q^{-1})} \quad .$$

Further since the R matrix already satisfies crossing symmetry (27) we require

$$S_0(x) = S_0(-\frac{1}{xq}) \quad .$$

The solution to these equations is

$$S_0(\beta) = \frac{1}{\sinh\frac{\pi}{\xi}(\beta - \frac{i\pi}{\xi})}\times$$

$$\times \prod_{k=0}^{\infty} \frac{\Gamma(\frac{2k\pi}{\xi}+1+\frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi}+\frac{\pi}{\xi}-\frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi}+\frac{\pi}{\xi}+1-\frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi}+\frac{2\pi}{\xi}+\frac{i\beta}{\xi})}{\Gamma(\frac{2k\pi}{\xi}+1-\frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi}+\frac{\pi}{\xi}+\frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi}+\frac{\pi}{\xi}+1+\frac{i\beta}{\xi})\Gamma(\frac{2k\pi}{\xi}+\frac{2\pi}{\xi}-\frac{i\beta}{\xi})} \tag{13}$$

which can be represented as an integral [67]

$$S_0(\beta) = \frac{1}{\sinh\frac{\pi}{\xi}(\beta - \frac{i\pi}{\xi})} \exp\left(-i\int_0^\infty \frac{\sin k\pi \sinh(\frac{\pi-\xi}{2})k}{k\cosh\frac{\pi k}{2}\sinh\frac{\xi k}{2}}dk\right) \tag{14}$$

We give an outline of the derivation of the solution:

Introduce

$$u(x) = (q^{-1}x^{-1} - qx)S_0(x) \quad ,$$

then $u(x)u(x^{-1}) = 1$ and

$$u(x^{-1})u(-x^{-1}q^{-1}) = \frac{x^{-1}-x}{q^{-1}x^{-1}-qx} = \frac{\Gamma(\lambda+\mu)\Gamma(1-\lambda-\mu)}{\Gamma(\mu)\Gamma(1+\mu)} \quad , \tag{15}$$

where we parameterized $x = e^{-i\pi\mu}$ and $q = -e^{-i\pi\lambda}$. Now take every term and expand it that that one divide it into two pieces: one belonging to μ, the other one to $\mu+\lambda$, in order to to satisfy (15). So for example

$$\Gamma(\lambda+\mu) = \frac{\Gamma(\lambda+\mu)\Gamma(2\lambda+\mu)\dots}{\Gamma(2\lambda+\mu)\Gamma(3\lambda+\mu)\dots} =$$

$$= \prod_{k=0}^{\infty} \frac{\Gamma(2k\lambda+\lambda+\mu)\Gamma(2k\lambda+2\lambda+\mu)}{\Gamma(2k\lambda+2\lambda+\mu)\Gamma(2(k+1)\lambda+\lambda+\mu)}$$

and similar for the other terms. Then we find

$$u(\mu) = \prod_{k=0}^{\infty} \frac{\Gamma(2k\lambda+\lambda-\mu)\Gamma(2(k+1)\lambda+\mu)\Gamma(2k\lambda+1+\mu)\Gamma(2k\lambda+\lambda+1-\mu)}{\Gamma(2k\lambda+\lambda+\mu)\Gamma(2(k+1)\lambda-\mu)\Gamma(2k\lambda+1-\mu)\Gamma(2k\lambda+\lambda+1+\mu)}$$

Putting in the physical values and transforming back to S_0 we find the result (13).

This result is not unique. There is a whole class of solutions, which differ one from each other by the so called CDD-factors, which are of the form (48). Here in this case we knew the answer ahead, but if we didn't, the situation would get more complicated.

A hint to decide about physical solutions is, whether the resulting poles correspond to particles which transform under a definite representation of the quantum group. That is, that the R matrix at the corresponding value of x degenerates into a projector. Let's see whether this is here the case.

Analyzing the poles of S_0 we find that they are located at

$$\beta = i\pi - in\xi \qquad n \geq 0 \ ,$$

$$\beta = in\xi \qquad n \geq 0 \ ,$$

in the physical strip. Choosing $\beta = i\pi - in\xi$ as s channel pole this corresponds to $x = (-1)^n q^{-1}$, and from (71) we find that the R matrix degenerates into a 1 dimensional projector at these points. Physically we interpret this result, that scalar particles are created, which we expected from our analysis of section (2.4.1).

4.1.1 Bootstrap, and S-matrices involving Breathers

Now we would like to compute the complete S matrix, i.e. also the S- matrices for the breathers. In [77] they have been calculated using information for the semi-classical analysis. We would like to calculate them from the fusion procedure introduced in section (3.7.1). Let's calculate the S matrix of the soliton and the first breather. The pole corresponds to $\beta_b = i\,\pi - i\xi$. Therefore in terms of x and q we need to shift by the values $y_1 \longrightarrow ixq^{-\frac{1}{2}}$ and $y_2 \longrightarrow -ixq^{\frac{1}{2}}$. But now we deal with degenerate particles. Therefore the bootstrap applied to the matrix part, which is represented by R, we take in account using the fusion procedure. The way how the variables have to be shifted, we determine from 46 . So, first, we can apply the bootstrap to S_0, which becomes

$$S_0(\beta + \frac{1}{2}\beta_b)\; S_0(\beta - \frac{1}{2}\beta_b) \;=\; \frac{1}{\sinh(\frac{\pi}{\xi}\frac{\beta}{} - \frac{3i\pi^2}{2\xi} + \frac{i\pi}{2})} \frac{1}{\sinh(\frac{\pi}{\xi}\frac{\beta}{} + \frac{i\pi^2}{2\xi} + \frac{i\pi}{2})} \tag{16}$$

$$\times\; \frac{\sinh(\frac{\beta}{2} - \frac{i\xi}{4} + \frac{i\pi}{4})\sinh(\frac{\beta}{2} + \frac{i\xi}{4} + \frac{i\pi}{4})}{\sinh(\frac{\beta}{2} + \frac{i\xi}{4} - \frac{i\pi}{4})\sinh(\frac{\beta}{2} - \frac{i\xi}{4} - \frac{i\pi}{4})}\;. \tag{17}$$

Then we calculate the matrix term, calculating

$$(\mathcal{P}_0 \otimes I)\left(I \otimes R(\beta + \frac{i\pi}{2} - \frac{i\xi}{2})\right) \times \left(R(\beta - \frac{i\pi}{2} + \frac{i\xi}{2}) \otimes I\right)(\mathcal{P}_0 \otimes I)\;\;.$$

In the second term we can use the decomposition (71) directly while in the first part we need to compute the product explicitly. We find that the first two factors in (17) cancel out and the S matrix for scattering of soliton and breather reads

$$S_{S,b_1} = \frac{\sinh(\frac{\beta}{2} - \frac{i\xi}{4} + \frac{i\pi}{4})\sinh(\frac{\beta}{2} + \frac{i\xi}{4} + \frac{i\pi}{4})}{\sinh(\frac{\beta}{2} + \frac{i\xi}{4} - \frac{i\pi}{4})\sinh(\frac{\beta}{2} - \frac{i\xi}{4} - \frac{i\pi}{4})}\;. \tag{18}$$

Now we can apply directly (46) to obtain the scattering of the breathers which is

$$S_{b_1 b_1} = \frac{\sinh(\frac{\beta}{2} - \frac{i\xi}{2} + \frac{i\pi}{2})\sinh(\frac{\beta}{2} + \frac{i\xi}{2})}{\sinh(\frac{\beta}{2} + \frac{i\xi}{2} - \frac{i\pi}{2})\sinh(\frac{\beta}{2} - \frac{i\xi}{2})}\;.$$

4.2 Reduction at rational values of the coupling constants

We know that for rational values of the coupling constant, which enters into q, we can restrict our representations of the quantum group. For this we turn to the shadow world representation [47, 6]. That is, we simply substitute in (68), i.e.

$$R(x,q) = xq^{\frac{1}{2}}R_{12} - q^{-\frac{1}{2}}x^{-1}R_{21}^{-1} \quad ,$$

instead of the R matrix the shadow-world objects

$$\left(R_{\frac{1}{2}\frac{1}{2}}\right)^{ab}_{cd} \longrightarrow (-1)^{a+c-b-d}\; q^{c_d+c_b-c_a-c_c} \left\{ \begin{matrix} \frac{1}{2} & d & c \\ \frac{1}{2} & b & a \end{matrix} \right\}_q \quad , \tag{19}$$

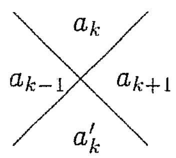
and similarly for $R^{-1}_{\frac{1}{2}\frac{1}{2}}$.

There are several points to clarify: first of all the bases. We do not have solitons as a basis any more, but the new objects can be interpreted as kinks interpolating between different vacua. That is, our bases become

$$|\beta_1, j_1|a_1|\beta_2, j_2|a_2|\dots|a_{n-1}|\beta_n k_n\rangle$$

β_i are again the rapidities, j_i are the $U_q(sl(2))$ spins which also automatically distinguish breathers from kinks and a_i are the values assigned to the dual lattice. Then we can interpret the S-matrix as scattering of kinks. The Kink-Kink amplitudes can be pictured as

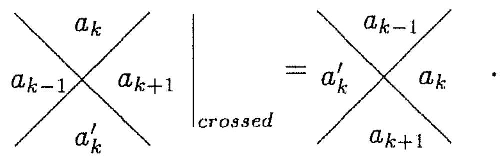
$$S\left(\beta \left| \begin{matrix} a_{k-1} & a_k \\ a_{k+1} & a'_k \end{matrix} \right|\right) =$$



Recall that we applied a gauge transformation to obtain the formulation in terms of the R-matrix. This mechanism introduce a non-trivial crossing symmetry (compare (6) and (44)), so to get a trivial crossing symmetry we have to do another gauge transformation, but now in the shadow world basis. This can be accomplished by ([5]) including a factor

$$\left(\frac{[2a_k+1][2a'_k+1]}{[2a_{k-1}+1][2a_{k+1}+1]}\right)^{-\frac{\beta}{2\pi i}} \quad .$$

into the amplitudes. Only after this transformation we obtain the crossing relations



Unitarity is automatically satisfied since the R-matrix also in the shadow world representation satisfies (12). To complete, one has to clarify how one can restrict the representation of the quantum group symmetry and how to relate, in the case of rational value of the coupling constant, this to physics. Let $\frac{\xi}{\pi} = \frac{p}{q}$. The mathematical recipe is very simple . In section 3.6 we found that the values of spin cannot exceed $\frac{1}{2}(p-2)$ if $q^{\frac{p}{2}} = 1$, therefore the values $a, b, \ldots$ are constrained by $j_{max} = \frac{p}{2} - 1$. On the other hand for the decomposition of representation (62) we have the constraint that

$$|a_k - \frac{1}{2}| \leq a_{k+1} \leq \min(a_k + \frac{1}{2}, p - \frac{5}{2} - a_k) \quad .$$

In a physical way, this can be interpreted as the kinks are restricted in their form, that is, that they can only connect certain vacua. So in some way the other vacua are effectively decoupled from the theory. This idea was formulated in [24], showing that in the sine-Gordon theory there exists a BRST symmetry which decouples the higher soliton sectors. Recall that the values of the coupling constant coincide with those for which the Liouville theory is equivalent to a minimal model (see section 2.4), and on the other hand, sine-Gordon is interpreted as a $\phi_{1,3}$ perturbation of the minimal models. So we conclude that the obtained scattering matrices describe integrable perturbations of minimal models [61]

.

4.3 The Series $\mathcal{M}_{2,2n+3}$

Let us analyze what result one would obtain looking at $\Phi_{1,3}$ perturbations of $\mathcal{M}_{2,2n+3}$. The scattering matrices become particularly simple, since $\xi = \frac{2\pi}{2n+1}$. Because of the restriction $a_k \leq \frac{p-2}{2}$, we see, since $p = 2$ that the solitons completely decouple from the theory and only breathers survive. From [28] the S matrix of the breathers takes the form

$$S_{ab} = f_{\frac{|a-b|}{2n+1}} f_{\frac{a+b}{2n+1}} \prod_{k=1}^{min(a,b)-1} \left(f_{\frac{|a-b|+2k}{2n+1}}\right)^2 \quad , \tag{20}$$

$$a,b = 1,\ldots,n \quad \text{and} \quad x = \frac{2\pi}{2n+1} \quad .$$

We would like to take now a completely different point of view. We have found with the above method scattering matrices corresponding to integrable field theories. They satisfy all the requirements deriving from factorized scattering. But now we want to see how strong these requirements are: that is, do these requirements allow a larger class of S-matrices than the ones, we found ? This analysis has been started in [44], and we will here resume the results found there.

Let's summarize the constraints found in section 2.5. They were: unitarity and crossing

$$\begin{aligned} S_{ab}(\beta)S_{ab}(-\beta) &= 1 \quad , \\ S_{ab}(\beta) &= S_{ab}(i\pi - \beta) \quad , \end{aligned} \tag{21}$$

the bootstrap equations

$$S_{cd}(\beta) = S_{bd}(\beta - i\overline{u}_{bc}^{a}) S_{ad}(\beta + i\overline{u}_{ac}^{b}) \tag{22}$$

where the resonance angles u_{ab}^c were related to the masses of the particles A_a, A_b and A_c by

$$\cos u_{ab}^c = \frac{m_c^2 - m_a^2 - m_b^2}{2m_a m_b} \quad , \tag{23}$$

and the consistency equations

$$\gamma_s^a \, e^{is\overline{u}_{ac}^b} + \gamma_s^b \, e^{is\overline{u}_{bc}^a} = \gamma_s^c \quad , \tag{24}$$

for which we required a non-trivial solutions $\gamma_s^c \neq 0$ in order to obtain a conserved current at spin s. Suppose that there are nontrivial solutions of (24). Normalizing the nonzero eigenvalues of the lightest particle A_1 to 1, it is easy to show by induction that all other eigenvalues are real and eqs. (24) can be written in terms of two equations

$$\gamma_s^a = \frac{\sin(s\overline{u}_{bc}^a)}{\sin(s\overline{u}_{ac}^b)} \gamma_s^b \quad , \tag{25}$$

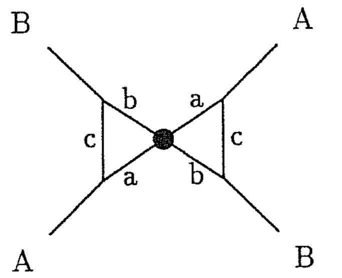


Figure 5: Multiscattering process responsible for higher order pole singularities in the S-matrix

and

$$(\gamma_s^c)^2 = (\gamma_s^a)^2 + (\gamma_s^b)^2 + 2\gamma_s^a\gamma_s^b \cos(s u_{ab}^c) \quad . \tag{26}$$

Eq. (25) is particularly useful because in order to have a non zero value for γ_s^a and γ_s^b the above ratio of sines should be independent of any bound state A_c appearing in the channel $\mid A_a A_b >$. Therefore, knowing the resonance angle of one bound state in this channel we can use this equation (a) to correctly identify the location of the other ones or (b) to prove that it is not possible to have higher order conserved charges compatible with the bootstrap.

In order to have a consistent set of elastic S-matrices, one has also to analyze the additional constraints related to the the higher order singularities introduced by the bootstrap equations. The basic idea is due to Coleman and Thun [15] and has been generalized in [13, 8]. In two dimensions, a box diagram corresponding to multiparticle scattering is singular if it can be drawn as a geometrical figure with all internal and external lines on-shell (fig. 5). This is equivalent to evaluating the discontinuity of this graph by the Cutcowski rules: the point interactions correspond to S-matrix elements and the lines to the on-mass shell propagators. The higher order poles are located at

$$\beta_{AB} = 2\pi - u_{Ac}^a - u_{Bc}^b \quad . \tag{27}$$

If S_{ab} in the middle is regular at this value of the rapidity, we obtain a double pole, otherwise if S_{ab} is itself singular at β_{AB}, we get a higher order singularity. Of course this explanation only works if we can actually draw such graph, i.e. if

$$u_{ac}^{A} + u_{bc}^{B} < \pi \quad . \tag{28}$$

If this condition does not hold, it is not possible to explain the appearance of higher order poles in terms of the principles of analytic S-matrix theory.

In particular, as it was noticed in [8], the scattering amplitude S_{11} of the lightest particle cannot have higher order poles because the resonance angle of two heavy particles with the lightest one is greater than $2\pi/3$ and therefore it is impossible to draw a figure like fig.2 with the particle A_1 on all four external legs and the internal ones on-shell.

Let us consider a bootstrap system with

$$S_{11} = f_x(\beta) \quad . \tag{29}$$

Our approach consists in applying eqs.(22) as far as there are singularities in the functions S_{ab} identifiable as bound states[6]. We prove that there is only one possible way to implement the bootstrap which satisfies the consistency equations. (25) with the spectrum given by

$$m_k = 2m \sin \frac{kx}{2} \quad . \tag{30}$$

Moreover, if we also require a consistent explanation of the higher order poles, we have to put the mass of the lightest particle A_{2n+1} produced by the bootstrap equal to zero, and decouple this particle from the massive sector of the theory. This is equivalent to have the following quantization condition for x

$$m_{2n+1} = 0, \qquad \rightarrow x = \frac{2\pi}{2n+1} \quad . \tag{31}$$

In order to get familiar how this works, let us first study the cases when x is close to $2\pi/3$.

[6]We will not make the distinction between real and virtual states which appears for instance in the discussion of sine-Gordon. We base our analysis only on the bootstrap eqs. (22) and on the requirements we discuss in the previous paragraphs.

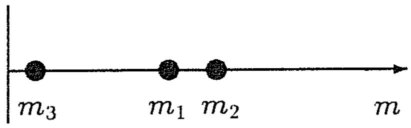


Figure 6: Mass spectrum generated by $S_{11} = f_x$ with x close to $2\pi/3$

a) $x > 2\pi/3$. In this case the singularity at $\beta = i\pi x$ corresponds to a bound state A_2 with mass m_2 less than m_1

$$\frac{m_2}{m_1} = 2\cos\frac{x}{2} \quad . \tag{32}$$

and therefore contradicts the assumption that A_1 was the lightest particle. This fact alone is not necessarily a drawback since we were aware that the bootstrap allows computing scattering amplitudes choosing any arbitrary particle as starting point. Therefore it could only mean that our initial identification of the lightest particle was wrong. But the real difficulty comes when we compute

$$S_{22} = f_{2x}(f_x)^2 \quad . \tag{33}$$

because we see that in this amplitude (which is now that of the lightest particle) appears a double pole which cannot occur. Hence there is no consistent set of S-matrices starting from $S_{11} = f_x$ when $x > 2\pi/3$.

b) x slightly less than $2\pi/3$, $x = (2/3 - \epsilon)\pi$, $(\epsilon \to 0)$. In this case the bootstrap produces three bound states with masses (fig.3)

$$m_k = 2m\sin\frac{kx}{2} \quad , \qquad k = 1,2,3 \quad , \tag{34}$$

and S-matrices

$$\begin{array}{lll} S_{11} = f_x \ , & S_{12} = f_{\frac{3x}{2}} f_{\frac{x}{2}} \ , & S_{13} = f_x f_{2x} \ , \\ S_{22} = f_{2x}(f_x)^2 \ , & S_{23} = f_{\frac{5x}{2}} f_{\frac{x}{2}} (f_{\frac{3x}{2}})^2 \ , & S_{33} = f_{3x}(f_x f_{2x})^2 \ . \end{array} \tag{35}$$

As in the case a), we obtain a particle A_3 with mass m_3 less than m_1 we started with. S_{33} contains as well unwanted double poles. The only way to make this system consistent is to push $m_3 \to 0$ and correspondingly decouple A_3 from the rest of the theory. In this limit all S-matrices involving A_3 go to the identity and the other particle state A_2 becomes identical to A_1. The initial three-particle system collapses to that one with only one particle state and S-matrix

$$S_{11} = f_{\frac{2}{3}}(\beta) \quad . \tag{36}$$

This corresponds to the S-matrix of Yang-Lee [12].

Let us consider now the general case and let us prove that there exists only one path in the bootstrap tree which respect the consistency equations. The proof is by induction. Starting with $S_{11} = f_x$, we obtain a new bound state whose mass can be written as

$$\frac{m_2}{m_1} = 2\cos\frac{x}{2} = \frac{2m\sin x}{2m\sin\frac{x}{2}} \quad , \tag{37}$$

where m is an arbitrary mass scale. We can compute S_{12} by applying eq. (22)

$$S_{12} = S_{11}(\beta - i\frac{x}{2})S_{11}(\beta + i\frac{x}{2}) = f_{\frac{3x}{2}}f_{\frac{x}{2}} \quad . \tag{38}$$

We get a function with four singularities: those at $\beta = i\pi x/2$ and $\beta = (1 - x/2)\pi$, from the $f_{\frac{x}{2}}$ term, and those at $\beta = i3/2\pi x$ and $\beta = i(1 - 3/2x)\pi$ from the $f_{\frac{3x}{2}}$ term. Among these, that one at $\beta = i(1 - x/2)\pi$ corresponds to the bound state A_1. Therefore we have correctly identified this angle as the resonance angle due to a bound state. We can now apply eq. (25) in order to decide which of the two poles in $f_{\frac{3x}{2}}$ corresponds to a new bound state A_3. The answer turns out to be that one at $\beta = i3/2\pi x$. This means that we cannot use the other singularity at $\beta = (1 - 3/2x)\pi$ to implement further the bootstrap if we require a non zero solution of the consistency equations but we are obliged to follow the path defined by the resonance angle $u_{12}^3 = \frac{3\pi}{2}x$. The mass of the new particle is

$$\frac{m_3}{m_1} = \frac{2m\sin\frac{3\pi x}{2}}{2m\sin\frac{\pi x}{2}} \quad . \tag{39}$$

We can compute

$$S_{13} = S_{12}(\beta - i\overline{u}_{23}^1)S_{11}(\beta + i\overline{u}_{13}^2) = f_x f_{2x} \quad . \tag{40}$$

Repeating here the same reasoning of before, we can identify the pole at $\beta = i(1-x)\pi$ as u_{13}^2 and in this way fix the ratio of the conserved quantities γ_1/γ_3 in (25). The singularity due to a new bound state A_4 is that at $\beta = iu_{13}^4 = i2\pi x$. The mass of this new bound state is

$$\frac{m_4}{m_1} = \frac{2m\sin(2\pi x)}{2m\sin\frac{\pi x}{2}} \quad . \tag{41}$$

The process can be continued up to the particle A_{2n+1} where n is defined by

$$\frac{2\pi}{2n+3} < x \leq \frac{2\pi}{2n+1} \ , \tag{42}$$

and has to be completed by the computation of the remaining S-matrices. The mass spectrum is given by

$$m_k = 2m\sin\frac{kx}{2}, \qquad k = 1, 2, \ldots 2n+1 \ . \tag{43}$$

The particle A_{2n+1} is the lightest one and its S_{ll}-matrix has a plethora of double poles. We can get a consistent set of S-matrices only if we put $m_{2n+1} = 0$ and decouple this particle from the theory. In this limit the remaining $2n$ particles become identical in couple and we end up with a n-particle system with generic S-matrix [28, 67]

$$S_{ab} = f_{\frac{|a-b|}{2n+1}} f_{\frac{a+b}{2n+1}} \prod_{k=1}^{min(a,b)-1} (f_{\frac{|a-b|+2k}{2n+1}})^2 \quad . \tag{44}$$

$(a, b = 1, 2, \ldots n)$. All double poles have now explanation in terms of multiscattering processes and the conserved spins are all odd numbers but multiple of $2n+1$

$$s = 1, 3, \ldots, 2n-1, 2n+3, \ldots, 4n+1 \qquad (\mathrm{mod}\, 4n+2) \ . \tag{45}$$

The price to be paid is that these S-matrices are not one-particle unitary and as we know they correspond to the $\phi_{1,3}$ deformation of the non-unitary minimal models $\mathcal{M}_{2,2n+3}$ [12, 28, 67].

4.4 Going Back to the Critical Point

We would like to have a criteria to check whether our S matrices are the right ones. To do this, there are mainly two approaches. The truncation space approach and the thermodynamic bethe ansatz.

The former one relies on the fact that one can form a basis in the Hilbert space using the primary fields of the conformal field theory and their descendents. Then the energy spectrum can be computed, since the matrix elements of the potential (being the conformal field) are the three point functions of the conformal theory. Then the method consists in truncating the base of the Hilbert space and diagonalizing the resulting hamiltonian, giving upper bounds to the energy levels [74, 45, 46].

The thermodynamic Bethe ansatz goes the other way around, using the S- matrix as input. In this sense it is the inversion of the ordinary Bethe ansatz. We noted in section 2.1 that the partition function is related to the largest eigenvalue of the transfer matrix T. The Bethe ansatz consists in explicitly solving the corresponding eigenvalue problem $Tg = \Lambda g$ writing down an ansatz for the wave function [3]

$$g(x_1, \dots x_n) = \sum_P A_{p_1,\dots p_n} e^{ik_1x_1} \dots e^{ik_nx_n} \quad ,$$

P denoting the set of permutations of the numbers p_i. From this one obtains an equation for the wavenumber of g. In the thermodynamic limit this equation can be written as an integral equation, and can be solved.

In the S matrix approach we do not have as input the hamiltonian of the theory but only the S matrix. So the problem is to write down a Bethe ansatz for the system using the information available from the S matrix. This problem has been understood only in the case of diagonal S matrices. The main condition is that the scattering must be factorized, so that we can assume that the wave function of our particles are well described by a free wave function in the intermediate region of two scatterings. We make the ansatz for the wave function as

$$\psi(x_1 \dots \psi_n) = e^{i\sum p_j x_j} \sum_P A(P)\Theta(x_P) \quad ;$$

$A(P)$ are coefficients of the momenta whose ordering is specified by

$$\Theta(x_P) = \begin{cases} 1 & \text{if } x_{p_1} < \dots x_{p_n} \\ 0 & \text{otherwise} \end{cases} \quad .$$

The $A(P)$ are determined by the S matrix of the theory. Let the permutation P differ from P' by the exchange of the indices i and j. Then

$$A(P') = S_{ij}(\beta_i - \beta_j)A(P) \ . \tag{46}$$

We impose antiperiodic boundary conditions for our wave functions, which provides that two particles cannot have equal momenta [80]. This leads to a condition on the coefficients,

$$A(i, p_2, \ldots, p_n) = -e^{ip_iL}A(p_2, \ldots, p_n, i) \ , \tag{47}$$

L being the length of the strip on which we consider the theory. If one now compares (46) and (47), one can find the condition

$$e^{iLm_i \sinh\beta_i}\prod_{j\neq i} S_{ij}(\beta_i - \beta_j) = -1 \quad \text{for} \quad i = 1, 2, \ldots N \ . \tag{48}$$

We introduce the phaseshifts $\delta_{ij}(\beta_i - \beta_j) \equiv -i\ln S_{ij}(\beta_i - \beta_j)$. In terms of these the equation become

$$Lm_i \sinh\beta_i + \sum_{j\neq i}\delta_{ij}(\beta_i - \beta_j) = 2\pi n_i \quad \text{for} \ \ i = 1, 2, \ldots, N \ , \tag{49}$$

n_i being some integers. These coupled transcendental equations for the rapidities are called the Bethe ansatz equations. Now one proceeds as in the case of the ordinary Bethe ansatz. One tries to solve those equations in the thermodynamic limit introducing a density of rapidities and transferring the equations into integral equations. Then one can study the thermodynamics of the theory and examine the ultraviolet limit, calculating the effective central charge and the scaling dimensions of the operators.

We will not pursue the general theory further (see [80, 41]), but use only one particular application of the method. It is to calculate the central effective charge, in order to tighten the link of the S matrices as describing perturbations of conformal field theories.

The link is the ground state energy E_0. On a cylinder (i.e. thermodynamic limit) it has the form

$$E_0(R) = Rf(R) = -\frac{\pi\tilde{c}(r)}{6R}$$

f is the free energy and R the circumference of the cylinder. The quantity $r = \frac{R}{R_c}$, where $R_c = \frac{1}{m_1}$ is the largest correlation length corresponding to the smallest mass m_1 in the theory. The fact we will use is, that $\tilde{c}(0)$ is the central effective charge given as

$$\tilde{c}(0) = 1 - \frac{6}{pq} \quad . \tag{50}$$

We would like to calculate this quantity from the TBA. We give here just the result:

As in the Bethe ansatz, one introduces the rapidity density $\rho_r^{(a)}(\beta)$ for any particle species a. To obtain $\tilde{c}(0)$ two limit processes are involved. First of all a minimization process, in order to obtain E_0 and secondly the limit process $r \to 0$. In this limit the effective central charge can be decomposed into a sum $\tilde{c} = \sum_{a=1}^{n} \tilde{c}_a$, corresponding to the different particle species. The $\tilde{c}_a$ can be obtained by integration

$$\tilde{c}_a = \frac{6}{\pi^2} \int_0^\infty dx \frac{x + \frac{\epsilon_a}{2}}{e^{x+\epsilon_a} + 1} \quad .$$

The values ϵ_a are obtained solving the nonlinear system of equations

$$\epsilon_a = - \sum_{b=1}^{N} N_{ab} \ln(1 + e^{-\epsilon_b}) \quad . \tag{51}$$

Let $\varphi_{ab}(\beta) = \frac{d}{d\theta}\delta_{ab}(\beta)$. Then the matrix N_{ab} in (51) is given as

$$N_{ab} = \int_{-\infty}^{\infty} \frac{d\beta}{2\pi} \varphi_{ab}(\beta) = \frac{1}{2\pi}(\delta_{ab}(\infty) - \delta_{ab}(-\infty)) \quad . \tag{52}$$

So the recipe is first to solve the system (51) for the ϵ_a, then to calculate the partial central charges $\tilde{c}_a$, and sum over them.

The matrix N_{ab} is very simple to determine. Since we deal with diagonal S matrices we know that the general form is $S_{ab}(\beta) = \prod_i f_{x_i}(\beta)$. Analyzing (52) for this situation one sees that $\varphi_{ab}(\beta) = \sum_i \varphi[f_{x_i}](\beta)$ so that one can sum up individual contributions coming from the single factors $f_{x_i}(\beta)$. Then $N_{ab} = \sum N[f_{x_i}]$ which is $N_{ab} = \sum \mathrm{sgn}(x_i)$ for $-1 < x \leq 1$.

4.4.1 $\tilde{c}$ for $M_{2,2n+3}$ Theories

The solution of (51) can be obtained in closed form. One finds [41]

$$e^{\epsilon_a}(\mathcal{M}_{2,2n+3}) = \frac{\sin(\frac{a\pi}{2n+3})\sin(\frac{(a+2)\pi}{2n+3})}{\sin^2(\frac{\pi}{2n+3})} \quad . \tag{53}$$

To do the summation over the integral one uses the identity

$$\int_0^\infty dx \frac{x+\frac{\epsilon}{2}}{e^{x+\epsilon}+1} = L(\frac{1}{1+e^\epsilon}) \ , \tag{54}$$

where $L(x)$ is the so called Roger's dilogarithmic function, given by

$$L(x) = -\frac{1}{2}\int_0^x dy[\frac{\ln y}{1-y} + \frac{\ln(1-y)}{y}] \ . \tag{55}$$

There exist sum-rules for these functions which allow one to calculate $\tilde{c}$ exactly. One finds that $\tilde{c} = \frac{2n}{n+3}$ which is the right result.

Conceptually the TBA provides a non-perturbative approach to discuss the ultraviolet limit of S matrix theory. So to say, we have closed the cycle. We started from the critical point and perturbed the corresponding conformal field theory. For the resulting model we calculated (or better conjectured) the S matrix. A strong confirmation that the approach is right, we found in calculating the effective central charge, which can be done exactly.

For the sine-Gordon model this seems to be a double verification of things we already knew. In the next chapter we will construct S matrices for which less information is available. There these techniques will be necessary to support our conjectures.

5 S Matrices for restrictions of the Izergin Korepin Model

We have seen in chapter 2.3 that perturbations by the operators $\Phi_{1,2}$, $\Phi_{1,3}$ and $\Phi_{2,1}$ lead to massive integrable models. We have treated the case of $\Phi_{1,3}$ perturbations in the last section. The $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbations are related to the Izergin-Korepin model, which describes the field theoretic action whose equation of motion is the Bullough-Dodd equation 36. We will undertake an analysis similar to that in the last chapter, in order to construct the S matrix using the quantum group symmetry.

5.1 The Izergin Korepin Model

We mentioned in section 2.4 that the complex Liouville theory can be understood as a realization of minimal models at the quantum level. To understand the rôle of the $\Phi_{1,2}$ and $\Phi_{2,1}$ perturbations, we need to examine this approach in more detail.

The identification of the Liouville model with minimal models goes across the Coulomb gas approach (for example see [19]). This approach describes a massless scalar field embedded in a space with a charge, placed at infinity, whose value is $-2\ \alpha_0$, where α_0 is a rational number. The primary fields are not the fields itself but vertex-operators

$$V_{p,q} = e^{i\alpha_{p,q}\Phi(x)} \quad ,$$

with

$$\alpha_{p,q} = \frac{[(1-p)\alpha_+ + (1-q)\alpha_-]}{2} \tag{56}$$

and p, q positive integers and $\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2+1}$. The central charge is given by $c = 1 - 24\alpha_0^2$. The weights of the vertex operators are those given in (14). Taking the classical limit which corresponds to take $c \longrightarrow \infty$ we would like to identify the corresponding classical objects. To do this we use the relation

$$\alpha_\pm = \frac{\sqrt{1-c} \pm \sqrt{25-c}}{\sqrt{24}} \quad .$$

We see that any operator $V_{p,q}$ with $p > 1$ will explode in this limit. So, only operators of the kind $V_{1,n}$ can be seen in a classical analog of the theory. These operators have classical dimensions $-\frac{n-1}{2}$, and can be identified as the operator

$$e^{-i\frac{n-1}{2}\beta\phi} \tag{57}$$

in the Liouville theory. Now we perturb the Liouville Theory by these operators. Taking $n = 3$ we find the sine-Gordon model, and with $n = 2$ the Izergin-Korepin model [36], corresponding to the lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 + \frac{1}{\beta^2}e^{i\beta\phi} - \frac{\lambda}{\beta^2}e^{-\frac{i\beta}{2}\phi} \tag{58}$$

These two perturbations can be found as classical lagrangians. But what about $\phi_{2,1}$ perturbations which should be integrable, as well? We need to understand this perturbation directly in a quantum formulation. Note that interchanging $\phi_{1,2}$ and $\phi_{2,1}$ corresponds to interchanging α_+ nd α_- in (56), or changing the rôle of r and s. These changes should be made only at a quantum level of the theory. We will examine the consequences of this hypothesis [66] in chapter 5.6.

Now that we have written down the lagrangian of our theory, we find ourselves immediately in troubles. Obviously the lagrangian is not real and the corresponding hamiltonian not self-adjoint. That is, our model as it stands is not properly defined. The corresponding S matrix is non-unitary. The way out of these troubles is the quantum symmetry of the theory. Even though the whole theory is inconsistent, one can try to use the approach, developed in the last section in order to analyze the restricted models at rational values of the coupling constant. The resulting scattering matrices should then be examined for their consistency and be interpreted as perturbations of minimal models.

5.2 Constructing the S Matrix

Our approach will go along the same line as in the last section. We write down the S matrix as

$$S = S_0 R_{12}(x, q)$$

but now R_{12} being the R matrix of $A_2^{(2)}$. Note: In this chapter we will not work with R matrices defined as in (28), but the ones which are not multiplied by the permutation matrix, we need to modify identities of chapter 3 in order to utilize them here. This, just to clarify that differences arise from this fact.

The parameters x and q have to be related to the parameters in our lagrangian. Here we rely on the fact that we study the affine extension of the quantum group $U_q(sl(2))$. Therefore the parameter q should be the same for both algebras. Using this fact one can fix $q^2 = e^{\frac{2\pi^2 i}{\chi}}$, $\chi = \frac{r}{s}\pi$. The parameter x we can fix by crossing symmetry. Using (26) and (69) one can find that

$$R_{12}(-q^6 x^{-1}, q) = c_1 R_{21}^{t_1} c_1 \tag{59}$$

But the crossing relation for the S matrix (44) should read

$$S_{12}(i\pi - \beta) = c_1 S_{21}^{t_1} c_1 \tag{60}$$

We make the ansatz $x = e^{\frac{2\pi}{\xi}\beta}$, and find from comparing the two equations that

$$\frac{2\pi^2}{\xi} = \frac{6\pi^2}{\chi} + (2n+1) \qquad n \in \mathbb{Z} \tag{61}$$

One chooses $n = -2$. Even though at this point this choice seems arbitrary, it turns out [66] that only in this way one can interpret the resulting S matrix as an analytic continuation of the real coupling Toda field theory.

So our S matrix should be defined as

$$S_{12}(\beta) = S_0(\beta) R_{12}(e^{\frac{2\pi}{\xi}\beta}, e^{\frac{2\pi^2 i}{\chi}}) \tag{62}$$

The function S_0 can be derived by the same techniques used in section 4.1. It is constructed in order to guarantee that $S(\beta)S(-\beta) = 1$. The R matrix satisfies now the identity

$$R(x,q)R(x^{-1},q) = (x^{\frac{1}{2}}q^{-3} + q^3 x^{-\frac{1}{2}})(x^{-\frac{1}{2}}q^2 - q^{-2}x^{\frac{1}{2}}) \tag{63}$$

S_0 becomes a more complicated function now. It takes the form

$$S_0(\beta) = \frac{1}{\pi^2}\Gamma\left(\frac{\pi + i\beta}{\xi}\right)\Gamma\left(\frac{\xi - \pi - i\beta}{\xi}\right)\Gamma\left(\frac{\frac{2}{3}\pi + i\beta}{\xi}\right)\Gamma\left(\frac{\xi - \frac{2}{3}\pi - i\beta}{\xi}\right) \;\; \Xi(\beta) \;\; , \tag{64}$$

where $\xi(\beta)$ is the following infinite product

$$\begin{aligned}\Xi(\beta) &= \prod_{k=0}^{\infty}\left\{\frac{\Gamma(\frac{\pi}{\xi}+\frac{2k\pi-i\beta}{\xi})\Gamma(\frac{2\pi}{\xi}+\frac{2k\pi+i\beta}{\xi})}{\Gamma(\frac{\pi}{\xi}+\frac{2k\pi+i\beta}{\xi})\Gamma(\frac{2\pi}{\xi}+\frac{2k\pi-i\beta}{\xi})}\frac{\Gamma(1+\frac{2k\pi+i\beta}{\xi})\Gamma(\frac{\xi+\pi}{\xi}+\frac{2k\pi-i\beta}{\xi})}{\Gamma(1+\frac{2k\pi-i\beta}{\xi})\Gamma(\frac{\xi+\pi}{\xi}+\frac{2k\pi+i\beta}{\xi})}\right.\\ &\times\left.\frac{\Gamma(\frac{\pi}{3\xi}+\frac{2k\pi+i\beta}{\xi})\Gamma(\frac{4\pi}{3\xi}+\frac{2k\pi-i\beta}{\xi})}{\Gamma(\frac{\pi}{3\xi}+\frac{2k\pi-i\beta}{\xi})\Gamma(\frac{4\pi}{3\xi}+\frac{2k\pi+i\beta}{\xi})}\frac{\Gamma(\frac{2\pi+3\xi}{3\xi}+\frac{2k\pi-i\beta}{\xi})\Gamma(\frac{5\pi+3\xi}{3\xi}+\frac{2k\pi+i\beta}{\xi})}{\Gamma(\frac{2\pi+3\xi}{3\xi}+\frac{2k\pi+i\beta}{\xi})\Gamma(\frac{5\pi+3\xi}{3\xi}+\frac{2k\pi-i\beta}{\xi})}\right\} \qquad (65)\end{aligned}$$

which again has an integral representation

$$\begin{aligned}S_0(\beta) &= \left(\sinh\frac{\pi}{\xi}(\beta-i\pi)\sinh\frac{\pi}{\xi}\left(\beta-\frac{2\pi i}{3}\right)\right)^{-1}\\ &\times\exp\left(-2i\int_0^\infty\frac{dx}{x}\frac{\sin\beta x\sinh\frac{\pi x}{3}\cosh\left(\frac{\pi}{6}-\frac{\xi}{2}\right)x}{\cosh\frac{\pi x}{2}\sinh\frac{\xi x}{2}}\right) \quad . \qquad (66)\end{aligned}$$

It seems that the objections about the physicality made above are not true, since we could satisfy the requirement $S(\beta)S(-\beta)=1$. But even if this relation is satisfied we do not necessarily have unitarity, simply because for $|q|=1$ and $x\in\mathbb{R}$, which is a physically interesting situation, one has that $R^*_{12}(x)\neq R_{21}(x^{-1})$, so in that case the S matrix is not unitary.

Nevertheless we continue to examine the S matrix. A first check whether the so defined S matrix is senseful, is to examine the pole structure. That is, whether the S matrix degenerates into projectors at the corresponding values. The poles are given at

$$\beta=\begin{cases} i\pi-i\xi m, & i\xi m, \qquad m>0\\ \frac{2\pi i}{3}-i\xi m, & \frac{\pi i}{3}+i\xi m, \qquad m\geq 0\end{cases} \qquad (67)$$

Note that the first line corresponds exactly to that of the sine-Gordon model pole structure. Also here the poles $i\pi-i\xi m$ correspond to breathers. Since $x=\exp(\frac{2\pi}{\xi}\beta)$ and $q^2=\exp(\frac{2\pi^2 i}{\chi})$ these poles correspond to $x=-q^6$. Using (34) we find that the R matrix degenerates into a 1-dimensional projector.The S matrix of the fundamental breather can be calculated using the fusion procedure, in the same way as for the sine-Gordon model. The result is [66]

$$S_{b_1,b_1}(\beta)=f_{\frac{2}{3}}(\beta)f_{\frac{\xi}{\pi}}(\beta)f_{\frac{\xi}{\pi}-\frac{1}{3}}(\beta) \quad , \qquad (68)$$

where $f_x(\beta)=\tanh\frac{1}{2}(\beta+ix\pi)\coth\frac{1}{2}(\beta-ix\pi)$.

Now consider the second set of poles in (67). For the poles $\frac{2\pi i}{3}-i\xi m$, X takes the value q^{4} and using (34) and (69) we find that the R matrix degenerates into a 3-dimensional projector at these points. Hence, these poles can be interpreted as those corresponding to the creation of higher kinks.

So the hope is that the RSOS restriction of the R-matrix yields S-matrices which have a sensible physical interpretation. For that we change to the IRF basis and take $q^r = 1$. The RSOS states appearing in the reduced model

$$| \beta_1, j_1, k_1, | a_1 | \quad \beta_2, j_2, k_2, \dots | a_{n-1} | \quad \beta_n, j_n, k_n > \tag{69}$$

are now characterized by their rapidity β_i, by their type k (which distinguishes the kinks from the breathers), by their $U_q(sl(2))$ spin j and by the variables a_i characterizing the dual lattice, constrained by the limitations

$$a_i \leq \frac{r-2}{2} \quad , \qquad | a_k - 1 | \leq a_{k+1} \leq \min(a_k + 1, r - 3 - a_k) \ . \tag{70}$$

The S-matrix of these RSOS states is given by replacing in (69) the R matrix by the $6j$-symbols as in (19). One finds

$$\begin{aligned} & S\left(\beta_k - \beta_{k+1} \left| \begin{array}{cc} a_{k-1} & a_k \\ a_{k+1} & a'_k \end{array} \right.\right) = \\ & \frac{1}{4i} \ S_0(\beta_k - \beta_{k+1}) \left[\left\{ \begin{array}{ccc} 1 & a_{k-1} & a_k \\ 1 & a_{k+1} & a'_k \end{array} \right\}_q \right. \\ & \times \left(\left(\exp\left(\frac{2\pi}{\xi}(\beta_{k+1} - \beta_k)\right) - 1 \right) \ q^{c_{a_{k+1}} + c_{a_{k-1}} - c_{a_k} - c_{a'_k} + 3} (-1)^{\nu} \right. \\ & - \left. \left(\exp\left(-\frac{2\pi}{\xi}(\beta_{k+1} - \beta_k)\right) - 1 \right) \ q^{-(c_{a_{k+1}} + c_{a_{k-1}} - c_{a_k} - c_{a'_k} + 3)} (-1)^{-\nu} \right) \\ & \qquad \left. + \, q^{-5}(q^6 + 1)(q^4 - 1)\delta_{a_k, a'_k} \right] . \end{aligned} \tag{71}$$

Herein, c_a are as usual Casimir of the representation a, $c_a = a(a+1)$, $\nu = a_k + a'_k - a_{k+1} - a_{k-1}$ and the expression of the $6j$-symbols is given in (48).

Now we again analyze our S matrices for unitarity. In the IRF basis the relation is written as

$$\sum a'_k S\left(\beta \left| \begin{array}{cc} a_{k-1} & a_k \\ a_{k+1} & a'_k \end{array} \right.\right) S\left(-\beta \left| \begin{array}{cc} a_{k-1} & a'_k \\ a_{k+1} & a''_k \end{array} \right.\right) = \delta_{a''_k a_k}$$

Hence we need to check that

$$S\left(\beta \left| \begin{matrix} a_{k-1} & a_k \\ a_{k+1} & a'_k \end{matrix} \right.\right) = \overline{S\left(\beta \left| \begin{matrix} a_{k-1} & a'_k \\ a_{k+1} & a_k \end{matrix} \right.\right)}$$

This requirement is fulfilled if and only if the $6j$-symbols are real. This happens for the following values of γ: (a) $\gamma = \frac{r\pi}{r+1}$, which correspond to the minimal unitary models $\mathcal{M}_{r,r+1}$; (b) $\gamma = \frac{2\pi}{2n+1}$ and $\gamma = \frac{3\pi}{3n\pm 1}$, which are related to the nonunitary minimal models $\mathcal{M}_{2,2n+1}$ and $\mathcal{M}_{3,3n\pm1}$. For these values of γ, the maximal allowed spin is 0 and $\frac{1}{2}$. Hence, the kinks disappear from the reduced space and only breathers remain as asymptotic states in the spectrum; (c) $\gamma = \frac{4\pi}{4n\pm1}$, which correspond to the nonunitary minimal model $\mathcal{M}_{4,4n\pm1}$. For this series the maximal allowed spin is equal to 1 and, according to the RSOS restriction, the kinks behave as scalar particles.

5.3 S-Matrices of the $\phi_{1,2}$ Perturbed Minimal Models $\mathcal{M}_{2,2n+1}$

The Kac table of the minimal models $\mathcal{M}_{2,2n+1}$ extends along one row. The counting argument and an explicit computation for the $\phi_{1,2}$ perturbed models show the existence of a conserved current with spin $s = 5$ but not that one with spin $s = 3$ [76]. The fusion rules of these CFT do not have any internal symmetry. These two facts together allow the possibility to have the "Φ^3"- property in the S-matrices of the $\phi_{1,2}$ perturbed models.

From the analysis made by Smirnov, we know that in these models the kinks play the role of quarks, in the sense that they form bound states which can occur as asymptotic states but themselves they cannot [69]. How many breathers are in the spectrum? We claim that for the models $\mathcal{M}_{2,2n+1}$, their number is $(n-1)$. The reason is the following. For these models, $\xi = \frac{\pi}{3n}$ and a very special situation happens at these values (see appendix). The $(n-1)$ poles between $i\pi$ and $\frac{2\pi i}{3}$ (and the crossing ones), which are those of the breathers b_i $(i = 1, 2, \dots n-1)$, are now third order poles whereas all other poles relative to the kinks become fourth order poles (see figure 5.3). According to the interpretation of the odd and even order poles put forward in [13, 8], this precludes the possibility of creating higher kinks. Therefore, all breathers (which, generally, are bound states of kinks) are just the $(n-1)$'s relative to the third order poles in the amplitude

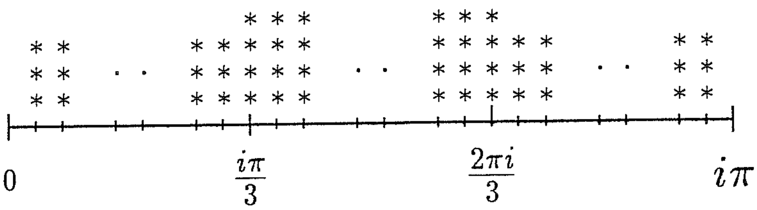


Figure 7: Pole-structure of the kink-kink S matrix

of the fundamental kink[7]. This conclusion is further supported by the analysis of the S-matrices of the breathers b_i with the fundamental kink: in these amplitudes there appears only the pole of the fundamental kink and no other singularities.

We now construct the S-matrix in the sector of the $(n-1)$ breathers b_i. Using (68) one finds for the fundamental particle b_1

$$S_{b_1 b_1}(\beta) = f_{\frac{1}{3n}}(\beta) f_{\frac{2}{3}}(\beta) f_{-\frac{n-1}{3n}}(\beta) \quad . \tag{72}$$

We identify as the physical poles $u_{11}^1 = \frac{2\pi}{3}$ and $u_{11}^2 = \frac{\pi}{3n}$. The first one is interpreted as a bound state corresponding to the fusion $b_1 b_1 \to b_1 \to b_1 b_1$. This means that this S-matrix has the "Φ^3"- property and therefore we cannot have a spin $s = 3$ current in the set of conserved quantities [76]. The second pole we assign to the breather b_2. Its mass is given by

$$\frac{m_2}{m_1} = \frac{\sin \frac{2\pi}{6n}}{\sin \frac{\pi}{6n}} \quad . \tag{73}$$

Using the bootstrap equations [76, 77], we can compute the amplitude $S_{b_1 b_2}$

$$S_{b_1 b_2}(\beta) = f_{\frac{3}{6n}}(\beta) f_{\frac{1}{6n}}(\beta) f_{\frac{2n+1}{6n}}(\beta) f_{\frac{-2n+3}{6n}}(\beta) \quad . \tag{74}$$

Herein the pole $u_{12}^1 = \frac{5\pi}{6n}$ corresponds to the particle 1. Only if n is larger than 3, we get a new particle b_3 at the pole $u_{12}^3 = \frac{\pi}{2n}$. Otherwise, for $n = 3$, this factor cancels with

[7] Although S_{12} is not really sensible of a physical interpretation, as we discuss in the next section, its analytic properties enter the structure of singularities in the RSOS physical sector.

the zero $f_{\frac{-2n+3}{6n}}(\beta)$. Treating similarly the scattering of b_1 and b_3, a new bound state b_4 appears and so on. By induction we obtain the whole sequence S_{b_1,b_k} ($k = 1, 2, \ldots < n-1$)

$$S_{b_1 b_k} = f_{\frac{k+1}{6n}} f_{\frac{k-1}{6n}} f_{\frac{2n+k-1}{6n}} f_{\frac{-2n+k+1}{6n}} \quad . \tag{75}$$

The remaining scattering amplitudes are obtained again by induction, applying the bootstrap equations. Finally the general S-matrix S_{b_p,b_k} (for $k \geq p = 1, 2, \ldots n-1$) is given by

$$\begin{aligned} S_{b_p b_k} &= f_{\frac{k+p}{6n}} (f_{\frac{k+p-2}{6n}} \cdots f_{\frac{k-p+2}{6n}})^2 f_{\frac{k-p}{6n}} \\ &\quad \times f_{\frac{2n+k+p-2}{6n}} \cdots f_{\frac{2n+k-p+2}{6n}} f_{\frac{2n+k-p}{6n}} \\ &\quad \times f_{\frac{-2n+k-p+2}{6n}} \cdots f_{\frac{-2n+k+p-2}{6n}} f_{\frac{-2n+k+p}{6n}} \quad . \end{aligned} \tag{76}$$

The exact mass spectrum is

$$m_k = \sin \frac{k\pi}{6n} \quad , \qquad k = 1, 2, \ldots n-1 \quad . \tag{77}$$

Postponing the discussion on the analytic structure to the next section, here we make some comments about the above S matrices. First of all, notice that the first line in (76) corresponds exactly to the structure of the S matrices found for the $\phi_{1,3}$ deformation of these models [12, 53, 28]. Further, these are the poles identified as physical ones, and therefore also the mass-spectrum has the identical structure as in the $\phi_{1,3}$ case. Secondly, the number of poles in the physical sheet given by the functions in the second line of (76), coincides with the number of zeros given by the functions of the third lines. Therefore, for what concerns the computation of the effective central charge in the ultraviolet regime of these scattering theories, the matrix N_{ab} which enters the thermodynamical Bethe ansatz (TBA) coincides with that of the deformation $\phi_{1,3}$ (see section 4.4). Then it is not surprising to find also in this case the correct values $\tilde{c} = \frac{2(n-1)}{2n+1}$. In [43] the TBA and the Truncation space approach have been applied. The truncation spectrum is in good agreement with the S matrices. Also the scaling region of the system was examined in the TBA, and the exponents of the theory calculated. All results obtained (which involve complex numerical techniques) confirm that the above S matrices describe perturbations of the models $\mathcal{M}_{2,2n+1}$ in the direction $\Phi_{1,2}$.

5.4 Pole Structure of the S-Matrix

Looking at equation (76), it seems that the bootstrap program has not been carried out to the end. In fact, there are still poles in the S-Matrix which have not been identified as particles. This, because above we only analyzed those poles which give rise to the breathers, corresponding to the first line of the general amplitude (76). Besides the argument we already gave for the truncation of the spectrum to the $(n-1)$ breathers only, the unphysical origin of these remaining poles also shows up in the conservation laws. If interpreted as singularities due to new particles, these spurious poles would not be consistent with conserved quantities of higher spin [76], and therefore the entire theory would be spoiled.

The domain of analyticity of an elastic S matrix consists of a two-sheet Riemann surface with square-root singularities at the threshold points of the s and u channel, respectively at $(m_1+m_2)^2$ and at $(m_1-m_2)^2$. The mapping

$$\beta = \ln\left(\frac{s-m_1^2-m_2^2+\sqrt{((s-(m_1+m_2)^2)(s-(m_1-m_2)^2)}}{2m_1m_2}\right) \tag{78}$$

transforms the physical sheet of the s plane into the strip $0 \leq \text{Im } \beta \leq \pi$. The second sheet is mapped into the strip $-\pi \leq \text{Im } \beta \leq 0$, and both repeat with period $2\pi i$. In order to understand the origin of the spurious poles in the S matrices (76), it is better to interpret the singularities in a function f_{-x} not as zeros on the physical strip but as poles on the second sheet of the Riemann surface. Concerning this point, let us observe the following facts. The expression of the mass of a bound state A_c in a scattering state $\mid A_a A_b\rangle$ is an even function of the resonance angle u_{ab}^c

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u_{ab}^c \quad . \tag{79}$$

Hence, reversing the sign of u_{ab}^c, the value of m_c does not change. Moreover, suppose we have given a closed bootstrap system with a generic S matrix of the form[8]

$$S_{ab}(\beta) = \prod_{x_i} f_{x_i}(\beta) \quad , \tag{80}$$

[8]For simplicity, we consider here the case of purely elastic diagonal S matrices. The argument given in the text can be easily generalized to the other cases.

$u_{11}^2 = \frac{2\pi}{5}$	$u_{12}^1 = \frac{4\pi}{5}$ $u_{12}^2 = \frac{3\pi}{5}$
	$u_{22}^3 = \frac{4\pi}{5}$

Table 1: Resonance angles of the $A_4^{(2)}$ model.

where all $x_i > 0$. Let us change all factors f_{x_i} into f_{-x_i}. If we now apply the bootstrap equation to the zeros instead of the poles we again end up with a closed system with the same spectrum as the original one.

Hence, if one has an S-matrix with poles only in one sheet, the interpretation is the usual one. All odd-order poles must correspond to bound states. These, according to the bootstrap-principle, have to be followed and must give rise to conserved quantities of higher spin. The interesting situation though occurs, when poles appear in both sheets of the Riemann surface. For special values of their positions, it may happen that, through the bootstrap, they overlap each other and produce spurious poles. In these cases it is also possible that expected particles disappear from the spectrum and reappear as zeros. In order to understand this mechanism better, let us consider a particularly simple and illustrative example, i.e. the second model of the $A_{2n}^{(2)}$ Affine Toda Field Theories [13, 8]. The whole set of S matrices of this system is given by

$$\begin{aligned} S_{11} &= f_{-b} f_{\frac{2}{5}} f_{b-\frac{2}{5}} \\ S_{12} &= f_{\frac{3}{5}} f_{b-\frac{3}{5}} f_{\frac{4}{5}} f_{b-\frac{4}{5}} \\ S_{22} &= f_{-b} f_{\frac{4}{5}} f_{b-\frac{4}{5}} f_{\frac{2}{5}} f_{b-\frac{2}{5}} f_{\frac{2}{5}} f_{-b-\frac{2}{5}} \ . \end{aligned} \tag{81}$$

The poles corresponding to the bound states are given by the b-independent terms (which are the minimal S matrices). The values are in the Table 5.4. The mass spectrum is

$$m_1 = M \quad , \qquad m_2 = 2M \quad \cos\frac{\pi}{5} \quad . \tag{82}$$

The remaining functions in (81) introduce zeros on the physical sheet. The conjectured expression of $b(g)$, as function of the coupling constant g of the Lagrangian, reads [13, 8]

$$b(g) = \frac{1}{5}\,\frac{g^2}{1+g^2}\;. \tag{83}$$

For a finite value of g, the terms containing $b(g)$ do not modify the spectrum. But, increasing g, the zeros move around and, for $g \to \infty$, they overlap with the poles, producing the following set of S matrices

$$S_{11} = f_{\frac{2}{5}}(f_{-\frac{1}{5}})^2 \;, \qquad S_{12} = f_{-\frac{3}{5}} f_{\frac{1}{5}} \;, \qquad S_{22} = f_{-\frac{4}{5}}\;. \tag{84}$$

If we retained the usual interpretation of the bound states as poles in the physical strip of the amplitudes, we would conclude, that in the above system the bound state A_2 has disappeared from the amplitude S_{12} as well as A_1 from S_{22}. Actually, as result of the collision of the zeros with the poles, we see that these particles have been moved onto the second sheet.

The same pattern is easily established for all other models of the Affine Field Theories $A_{2n}^{(2)}$. On the other hand, using the analysis made in [13, 8], it is possible to see that the S matrices of the Affine Toda Field Theories of the simply laced algebras (ADE) do not show this overlapping behaviour. A natural interpretation of the peculiar features of the series $A_{2n}^{(2)}$ comes from its group origin. This series is obtained as a folding of the simply-laced models $A_{2n}^{(1)}$ under the Z_2 automorphism of their Dynkin diagram. This folding projects the $2n$ fields of the original theory onto a n-dimensional subspace. n particles of the original $A_{2n}^{(1)}$ theories rearrange themselves as the new particles of the reduced models $A_{2n}^{(2)}$ (and then they appear as poles in the physical strip[9]), but we may think of the other n's of the initial model as particles living on the second sheet of the Riemann surface of the latter one. They only show up in the S matrices in the strong coupling limit $g \to \infty$.

A similar mechanism is responsible for the spurious poles in the S-matrix (76) of the $\phi_{1,2}$ deformation of the $M_{2,2n+1}$ models. The only difference is that the locations of the zeros are now not adjustable parameters but they are fixed from the beginning. The first

[9]Here we also remind that the $A_{2n}^{(2)}$ theories are the only non-simply laced theories which seem to be consistent without inclusion of other fields in the Lagrangian. For instance, the one loop corrections do not spoil the classical mass ratios [13, 8].

one occurs in the amplitude $S_{b_1b_1}$ through the term $f_{-\frac{n-1}{3n}}$. If we calculate the mass of the particle corresponding to it, we find $m_x = \frac{1}{2} - \sin(\frac{n-2}{6n}\pi)$. But we also get the same mass at the spurious pole in $S_{b_1b_2}$, namely at $u_{12}^x = \frac{4n-1}{6n}\pi$. Therefore this singularity can be interpreted as a zero which through the action of the bootstrap appears as a pole in the physical sheet.

The whole analysis of the analytic structure of the S matrices is based on some basic steps which we clarify through the first non trivial model of our system[10], that one corresponding to $M_{2,7}$. This model has two physical particles with S-matrices given by

$$S_{b_1b_1} = f_{\frac{1}{9}} f_{\frac{2}{3}} f_{-\frac{2}{9}} \quad , \quad S_{b_1b_2} = f_{\frac{1}{18}} f_{\frac{7}{18}} \quad , \quad S_{b_2b_2} = f_{\frac{2}{3}} f_{\frac{1}{9}} f_{\frac{5}{9}} \quad . \tag{85}$$

Let's now follow the above interpretation and calculate the S-matrix relative to the pole at $\beta = -i\frac{2\pi}{9}$ on the second sheet of the Riemann surface of $S_{b_1b_1}$. Denoting this spurious particle by a_1, we have

$$S_{b_1a_1} = (f_{\frac{2}{9}})^2 f_{\frac{5}{9}} f_{-\frac{1}{9}} f_{-\frac{1}{3}} \quad , \quad S_{b_2a_1} = (f_{\frac{1}{6}})^2 f_{\frac{1}{2}} f_{\frac{5}{18}} f_{-\frac{1}{18}} \quad , \quad S_{a_1a_1} = (f_{\frac{2}{3}})^3 (f_{\frac{1}{9}})^2 (f_{-\frac{2}{9}})^2 \quad . \tag{86}$$

In the amplitude $S_{b_1a_1}$ we can easily identify the particles b_1 (relative to $u_{b_1a_1}^{b_1} = \frac{10\pi}{9}$) and b_2 (with $u_{b_1a_1}^{b_2} = \frac{5\pi}{9}$). The pole at $\beta = -\frac{\pi}{3}$ on the second sheet gives rise to a new spurious particle a_2 and turns up as a pole in the physical sheet of the amplitude $S_{b_2b_2}$, i.e. that one at $u_{b_2b_2}^{a_2} = \frac{5\pi}{9}$.

Still we have not finished the analysis of this model. There remains one spurious pole in the amplitude $S_{b_2b_2}$ to be explained. This is the singularity at $\beta = i\frac{\pi}{9}$. But, looking at the general amplitude, we see that it arose from a cancellation of a zero with a double pole coming from the multi-scattering graph of the figure 5.4. This is exactly the same mechanism we encountered in the example of ATFT $A_{2n}^{(2)}$, i.e. fine tuning the value of the coupling constant to a special value, one can boost a pole from one sheet of the Riemann surface into the other.

The analysis in the case of $\Phi_{1,2}$ perturbations is much more involved than in the $\Phi_{1,3}$

[10]Notice that for $n = 2$ we have the Yang-Lee model, in which holds the identification $\phi_{1,2} \equiv \phi_{1,3}$. Therefore, the S-matrix for this system reduces to that one discussed in [12].

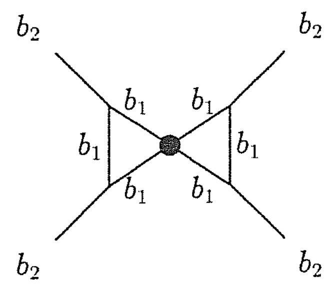


Figure 8: Multiscattering process responsible for higher order pole singularities in the S-matrix

case but the basic mechanisms explained in the example above, successfully applied to all the models defined by (76).

5.5 $\phi_{1,4}$ Deformation of $\mathcal{M}_{2,9}$

In a recent paper, Martins [49] pointed out the connection between the models $\mathcal{M}_{2,9}+\phi_{1,4}$ and $\mathcal{M}_{8,9}+\phi_{1,2}$. Basically, the argument relies on the identification of the fields $\phi_{1,k}$ with the vertex operators given in (57). This equation implies that the field $\phi_{1,2}(\gamma)$ is related to the field $\phi_{1,5}(\tilde{\gamma})$, provided that $\gamma = 4\tilde{\gamma}$ [11]. In the model $\mathcal{M}_{2,9}$ we have $\phi_{1,4} \equiv \phi_{1,5}$. Hence, it should be possible to recover the $\phi_{1,4}$ deformation of this model using the analysis of the $\phi_{1,2}$ deformation of the unitary model $\mathcal{M}_{8,9}$. The above observation also makes less mysterious the origin of the integrability of the $\phi_{1,4}$ deformation, which is usually prevented by counting argument and null-vector considerations.

Smirnov [69] found, that for the model $\mathcal{M}_{8,9}$, the spectrum consists of four particles: two kinks with the masses

$$M \quad , \qquad 2M\cos\frac{\pi}{15} \quad , \tag{87}$$

and two breathers with the masses

$$2M\sin\frac{4\pi}{15} \quad , \qquad 4M\sin\frac{4\pi}{15}\cos\frac{3\pi}{5} \quad . \tag{88}$$

[11]It is also necessary to make a corresponding rescaling in the exponential term of the Liouville action.

The S matrices of the fundamental kinks and of the fundamental breather are given by eqs. (71) and (68) respectively. If we would like to construct the S-matrix proposed by Martins from the massive theory $M_{8,9} + \phi_{1,2}$, we need to restrict the space of states such that it contains only particles with scalar behaviour and not kink-like. Hence we have to find a combination of the kink amplitudes which give rise to the S-matrix of the fundamental particle of the ones given below (91). The combination of the kink S matrices we are looking for is

$$S(\beta) = S\left(\beta \left| \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right.\right) + S\left(\beta \left| \begin{array}{cc} 1 & 1 \\ 2 & 2 \end{array} \right.\right) = S_0(\beta) \sinh\frac{\pi}{\xi}(\beta + i\pi) \sinh\frac{\pi}{\xi}\left(\beta + \frac{2\pi i}{3}\right) \quad . \tag{89}$$

Simplifying the expression (66) for S_0, i.e. with $\xi = \frac{8\pi}{15}$, one obtains

$$S_0(\beta) = \frac{f_{\frac{2}{3}}(\beta) f_{\frac{2}{15}}(\beta) f_{\frac{7}{3}}(\beta) f_{-\frac{1}{15}}(\beta) f_{-\frac{2}{5}}(\beta)}{\sinh\frac{\pi}{\xi}(\beta + i\pi) \sinh\frac{\pi}{\xi}(\beta + \frac{2\pi i}{3})} \quad . \tag{90}$$

As explained in the next, it gives rise to the S-matrix of the fundamental particle of the perturbed system $M_{2,9} + \phi_{1,4}$. Using the above expression (90), the expression (89) reduces to the S matrix proposed by Martins for the fundamental particle of $\mathcal{M}_{2,9} + \phi_{1,4}$ [49]. It satisfies the usual requirement of unitarity and it is a crossing symmetric function. Therefore it is breather-like and a restriction to a subspace of scalar particles is possible. Now we want to analyze the bootstrap system which comes from (89).

The bootstrap closes with four particles and the full S-matrix is given by

$$\begin{array}{ll}
S_{11} = \; {}^2f_{\frac{7}{15}} \;\; {}^3f_{\frac{2}{15}} \;\; {}^1f_{\frac{2}{3}} f_{-\frac{1}{15}} f_{-\frac{2}{5}} & S_{12} = \; {}^1f_{\frac{23}{30}} \;\; {}^3f_{\frac{13}{30}} \\
S_{13} = \; {}^1f_{\frac{14}{15}} \;\; {}^2f_{\frac{11}{15}} \;\; {}^4f_{\frac{1}{5}} f_{-\frac{2}{15}} f_{-\frac{1}{3}} (f_{\frac{2}{5}})^2 & S_{14} = \; {}^3f_{\frac{13}{15}} f_{\frac{7}{15}} (f_{\frac{2}{3}})^2 \\
 & \\
S_{22} = \; {}^2f_{\frac{2}{3}} f_{\frac{7}{15}} \;\; {}^4f_{\frac{1}{5}} & S_{23} = \; {}^1f_{\frac{5}{6}} f_{\frac{1}{2}} f_{\frac{3}{10}} f_{\frac{11}{30}} \\
S_{24} = f_{\frac{7}{30}} f_{\frac{3}{10}} \;\; {}^2f_{\frac{9}{10}} f_{\frac{11}{30}} (f_{\frac{13}{30}})^2 & \\
 & \\
S_{33} = ({}^3f_{\frac{2}{3}})^3 \;\; (f_{\frac{2}{15}})^2 \;\; (f_{\frac{7}{15}})^2 \;\; f_{-\frac{1}{15}} f_{-\frac{2}{5}} & S_{34} = \; {}^1f_{\frac{1}{15}} f_{\frac{1}{5}} f_{\frac{7}{15}} (f_{\frac{4}{15}})^2 \;\; (f_{\frac{2}{5}})^3 \\
 & \\
S_{44} = ({}^4f_{\frac{2}{3}})^3 \;\; (f_{\frac{7}{15}})^3 \;\; f_{\frac{4}{15}} f_{\frac{2}{15}} (f_{\frac{1}{5}})^2 \;\; f_{\frac{2}{5}} \quad . &
\end{array} \tag{91}$$

In order to easily identify the appearance of the physical bound states, we have introduced a compact notation: a factor ${}^{a}f_x$ in a S matrix S_{bc}-matrix means that this pole gives rise to the physical particle a through $u_{bc}^{a} = x\pi$.

The particles A_1 and A_3 correspond respectively to the fundamental and the higher kink and their masses coincide with those given in (87). The other two particles A_2 and A_4 correspond, on the contrary, to the two breathers present in the $\mathcal{M}_{8,9} + \phi_{1,2}$ model and their masses coincide with those in (88).

As before, we realize that in the S-matrix of the fundamental particle poles appear in both sheets of the Riemann-surface. Hence, we expect spurious poles along the bootstrap procedure, which appear indeed. The same mechanism we already applied to the previous systems works successfully also here. We calculate for the zero's the corresponding fusion-angles and masses, and see that these "spurious" particles also appear on the physical sheet of the Riemann-surface. For example, consider the zero $u_{11}^{x} = -\frac{1}{15}$. This turns out to be exactly the "spurious" particle appearing in S_{22} at $u_{22}^{x} = \frac{8}{15}\pi$. Other singularities in (91) can be analyzed similarly.

A non trivial check of our conclusions has already been done using the Truncation Method and, actually, this was the way how the S matrix of the fundamental particle has been conjectured [49]. We start by checking the ultraviolet behaviour of the S matrices

91. The ultraviolet limit is taken solving Eqs. (51), with the S matrices given in (91), for $R \to 0$ [80]. In this limit the scaling function is easily written in terms of the Rogers dilogarithmic function

$$\tilde{c} = \frac{6}{\pi^2} \sum_{i=1}^{4} L(x_i); \qquad L(x) = -\frac{1}{2} \int_0^x dy \left[\frac{\ln(1-y)}{y} + \frac{\ln y}{1-y} \right] \tag{92}$$

where

$$\begin{aligned} x_1^{-1} &= 4\cos^2\left(\frac{\pi}{18}\right) \\ x_2^{-1} &= \left(2\cos\left(\frac{2\pi}{9}\right) + 1\right)^2 \\ x_3^{-1} &= 4\cos^2\left(\frac{\pi}{18}\right)\left(2\cos\left(\frac{\pi}{9}\right) + 1\right) \\ x_4^{-1} &= 64\cos^2\left(\frac{2\pi}{9}\right)\cos^4\left(\frac{\pi}{9}\right) \quad . \end{aligned} \tag{93}$$

Using the properties of the Rogers dilogarithmic function we can exactly perform the sum of Eq. (92) and our final result is $\tilde{c} = \frac{2}{3}$, which agrees with the corresponding conformal field theory (see Eq.(50)).

5.6 Restricted S-Matrices of $\Phi_{2,1}$ Perturbed Models

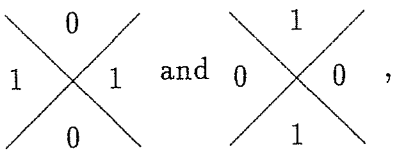
In [66] Smirnov proposed S matrices for the $\phi_{2,1}$ perturbations. Using the arguments of the quantum Liouville theory (see page 69) he simply exchanged the rôle of r and s in constructing the restricted models. That is, he proposed

$$S_{12} = \tilde{S}_0(\beta) R_{12}(e^{\frac{2\pi}{\xi}\beta}, e^{2i\gamma}) \quad .$$

Let's call the the new parameters $\tilde{\gamma}, \tilde{\xi}$. That is, we take $q = e^{\frac{2i\pi^2}{\gamma}}$ and $x = e^{\frac{2\pi}{\xi}\beta}$. Since $\gamma = \frac{r}{s}\pi$ we obtain that $\tilde{\gamma} = \frac{s}{r}\pi$ and $\tilde{q}^2 = e^{2i\gamma}$. Finally we obtain also $\tilde{\xi} = \frac{2}{3}\frac{\pi^2}{2\gamma - \pi}$. That is, all the formalism developed in section 5.1 can be carried over substituting the respective $\tilde{}$ values. Note that also in the restriction (70) now r enters instead of s. As an example we calculate the S matrix of the Ising model perturbed by the operator ϕ_{21}. We put the parameter $\tilde{\xi} = \frac{4}{3}$ into the formula for S_0 (see equation (65)) and find

$$S_0 = \frac{1}{\sinh(\frac{\beta}{\xi} - \frac{i\pi}{4}) \sinh(\frac{\beta}{\xi} - \frac{i\pi}{2})} \quad .$$

From the RSOS restriction (70) we find that there are only 2 allowed amplitudes



which take the same value $S = -1$. Since for the theory $\mathcal{M}_{3,4}$ the operators $\phi_{2,1}$ and $\phi_{1,3}$ coincide, we can compare the result with the one deriving from sine-Gordon theory and find that it corresponds.

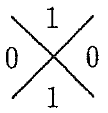
Encouraged by this, we turn to a more complicated case: the tricritical Ising Model (TIM). We leave aside now the twiddles over the parameters, but they are implicitly understood. In the TIM perturbed by the subleading magnetization operator, $r = 4$ and $\xi = \frac{10\pi}{9}$. From eq. (70), the only possible values of a_i are 0 and 1 and the one-particle states are the vectors: $| K_{01}\rangle$, $| K_{10}\rangle$ and $| K_{11}\rangle$. All of them have the same mass m. Notice that the state $| K_{00}\rangle$ is not allowed. A basis for the two-particle asymptotic states is

$$| K_{01}K_{10}\rangle, \quad | K_{01}K_{11}\rangle, \quad | K_{11}K_{11}\rangle, \quad | K_{11}K_{10}\rangle, \quad | K_{10}K_{01}\rangle \ . \tag{94}$$

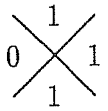
The scattering processes are

$$\begin{aligned}
| K_{01}(\beta_1)K_{10}(\beta_2)\rangle &= S_{00}^{11}(\beta_1-\beta_2) \mid K_{01}(\beta_2)K_{10}(\beta_1)\rangle \\
| K_{01}(\beta_1)K_{11}(\beta_2)\rangle &= S_{01}^{11}(\beta_1-\beta_2) \mid K_{01}(\beta_2)K_{11}(\beta_1)\rangle \\
| K_{11}(\beta_1)K_{10}(\beta_2)\rangle &= S_{10}^{11}(\beta_1-\beta_2) \mid K_{11}(\beta_2)K_{10}(\beta_1)\rangle \\
| K_{11}(\beta_1)K_{11}(\beta_2)\rangle &= S_{11}^{11}(\beta_1-\beta_2) \mid K_{11}(\beta_2)K_{11}(\beta_1)\rangle + S_{11}^{10}(\beta_1-\beta_2) \mid K_{10}(\beta_2)K_{01}(\beta_1)\rangle \\
| K_{10}(\beta_1)K_{01}(\beta_2)\rangle &= S_{11}^{00}(\beta_1-\beta_2) \mid K_{10}(\beta_2)K_{01}(\beta_1)\rangle + S_{11}^{10}(\beta_1-\beta_2) \mid K_{11}(\beta_2)K_{11}(\beta_1)\rangle
\end{aligned} \tag{95}$$

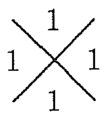
Explicitly, the above amplitudes are given by



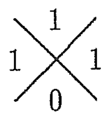
$$= S_{00}^{11}(\beta) = \frac{i}{2}\, S_0(\beta)\, \sinh\left(\frac{9}{5}\beta - i\frac{\pi}{5}\right)$$



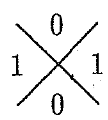
$$= S_{01}^{11}(\beta) = -\frac{i}{2}\, S_0(\beta)\, \sinh\left(\frac{9}{5}\beta + i\frac{\pi}{5}\right)$$



$$= S_{11}^{11}(\beta) = \frac{i}{2}\, S_0(\beta)\, \frac{\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)}\, \sinh\left(\frac{9}{5}\beta - i\frac{2\pi}{5}\right)$$



$$= S_{11}^{01}(\beta) = -\frac{i}{2}\, S_0(\beta)\, \left(\frac{\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)}\right)^{\frac{1}{2}}\, \sinh\left(\frac{9}{5}\beta\right)$$



$$= S_{11}^{00}(\beta) = -\frac{i}{2}\, S_0(\beta)\, \frac{\sin\left(\frac{\pi}{5}\right)}{\sin\left(\frac{2\pi}{5}\right)}\, \sinh\left(\frac{9}{5}\beta + i\frac{2\pi}{5}\right)$$

where we compute $S_0(\beta)$ using (65). It is given by

$$\begin{aligned} S_0(\beta) &= -\left(\sinh\frac{9}{10}(\beta - i\pi)\sinh\frac{9}{10}\left(\beta - \frac{2\pi i}{3}\right)\right)^{-1} \\ &\quad \times w\left(\beta, -\frac{1}{5}\right) w\left(\beta, +\frac{1}{10}\right) w\left(\beta, \frac{3}{10}\right) \\ &\quad \times t\left(\beta, \frac{2}{9}\right) t\left(\beta, -\frac{8}{9}\right) t\left(\beta, \frac{7}{9}\right) t\left(\beta, -\frac{1}{9}\right) \quad , \end{aligned} \tag{96}$$

where

$$w(\beta, x) = \frac{\sinh\left(\frac{9}{10}\beta + i\pi x\right)}{\sinh\left(\frac{9}{10}\beta - i\pi x\right)} \quad ;$$

$$t(\beta, x) = \frac{\sinh\ \frac{1}{2}(\beta + i\pi x)}{\sinh\ \frac{1}{2}(\beta - i\pi x)} \quad .$$

It is easy to check the unitarity equations:

$$\begin{aligned} &S_{11}^{00}(\beta)\ S_{11}^{00}(-\beta) + S_{11}^{01}(\beta)\ S_{11}^{10}(-\beta) = 1 \ ; \\ &S_{11}^{10}(\beta)\ S_{11}^{01}(-\beta) + S_{11}^{11}(\beta)\ S_{11}^{11}(-\beta) = 1 \ ; \\ &S_{11}^{10}(\beta)\ S_{11}^{00}(-\beta) + S_{11}^{11}(\beta)\ S_{11}^{10}(-\beta) = 0 \ ; \\ &S_{10}^{11}(\beta)\ S_{10}^{11}(-\beta) = 1 \ ; \\ &S_{00}^{11}(\beta)\ S_{00}^{11}(-\beta) = 1 \ . \end{aligned} \tag{97}$$

An interesting property of this S matrices is that the crossing symmetry occurs in a non-trivial way, i.e.

$$\begin{aligned} S_{11}^{11}(i\pi - \beta) &= S_{11}^{11}(\beta) \ ; \\ S_{11}^{00}(i\pi - \beta) &= a^2\ S_{00}^{11}(\beta) \ ; \\ S_{11}^{01}(i\pi - \beta) &= a\ S_{01}^{11}(\beta) \ ; \end{aligned} \tag{98}$$

where

$$a = -\left(\frac{s\left(\frac{1}{5}\right)}{s\left(\frac{2}{5}\right)}\right)^{\frac{1}{2}} \quad , \tag{99}$$

and $s(x) \equiv \sin(\pi x)$.

The above crossing-symmetry relations may be seen as due to a non-trivial charge conjugation operator (see also [52]). In most cases, the charge conjugation is implemented

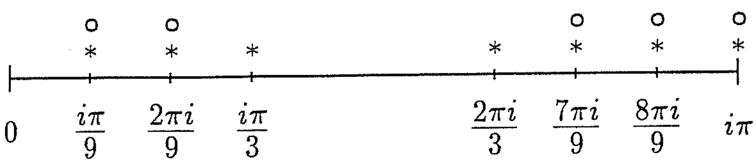


Figure 9: Pole structure of $S_0(\beta)$: $*$ are the location of the poles and $\mathbf{o}$ the position of the zeros.

trivially, i.e. with $a = \pm 1$ in eq. (98). Here, the asymmetric Landau-Ginzburg potential distinguishes between the two vacua and gives rise to the value (99).

The amplitudes are periodic along the imaginary axis of β with period 10 πi. The structure of poles and zeros is quite rich. On the physical sheet, $0 \leq \text{Im}\ \beta \leq i\pi$, the poles of the S-matrix are located at $\beta = \frac{2\pi i}{3}$ and $\beta = \frac{i\pi}{3}$ (fig.5.6). The first pole corresponds to a bound state in the direct channel while the second one is the singularity due to the particle exchanged in the crossed process. The residues at $\beta = \frac{2\pi i}{3}$ are given by

$$
\begin{aligned}
r_1 &= \text{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{00}^{11}(\beta) = 0 \ ; \\
r_2 &= \text{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{01}^{11}(\beta) = i \left(\frac{s\left(\frac{2}{5}\right)}{s\left(\frac{1}{5}\right)} \right)^2 \omega \ ; \\
r_3 &= \text{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{11}^{11}(\beta) = i \ \omega \ ; \\
r_4 &= \text{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{11}^{01}(\beta) = i \left(\frac{s\left(\frac{2}{5}\right)}{s\left(\frac{1}{5}\right)} \right)^{\frac{1}{2}} \omega \ ; \\
r_5 &= \text{Res}_{\beta=\frac{2\pi i}{3}} \quad S_{11}^{00}(\beta) = i \ \frac{s\left(\frac{2}{5}\right)}{s\left(\frac{1}{5}\right)} \ \omega \ ;
\end{aligned}
\tag{100}
$$

where

$$
\omega = \frac{5}{9} \ \frac{s\left(\frac{1}{5}\right) s\left(\frac{1}{10}\right) s\left(\frac{4}{9}\right) s\left(\frac{1}{9}\right) s^2\left(\frac{5}{18}\right)}{s\left(\frac{3}{10}\right) s\left(\frac{1}{18}\right) s\left(\frac{7}{18}\right) s^2\left(\frac{2}{9}\right)} \quad . \tag{101}
$$

In the amplitude S_{00}^{11} there is no bound state in the direct channel but only the singularity coming from to the state $\mid K_{11} \rangle$ exchanged in the t-channel. This is easily seen from figure 10 where we stretch the original amplitudes along the vertical direction

(s-channel) and along the horizontal one (t-channel). Since the state $\mid K_{00}\rangle$ is not physical, the residue in the direct channel is zero. In the amplitude S_{01}^{11} we have the bound state $\mid K_{01}\rangle$ in the direct channel and the singularity due to $\mid K_{11}\rangle$ in the crossed channel. In S_{11}^{11}, the state $\mid K_{11}\rangle$ appears as a bound state in both channels. In S_{11}^{01} the situation is reversed with respect to that of S_{01}^{11}, as it should be from the crossing symmetry property (98): the state $\mid K_{11}\rangle$ appears in the t-channel and $\mid K_{01}\rangle$ in the direct channel. Finally, in S_{11}^{00} there is the bound state $\mid K_{11}\rangle$ in the direct channel but the residue on the t-channel pole is zero, again because $\mid K_{00}\rangle$ is unphysical. This situation is, of course, that obtained by applying crossing to S_{00}^{11}.

5.7 Energy Levels, Phase Shifts and Generalized Statistics

The one-particle line a of fig. (1.a) corresponds to the state $\mid K_{11}\rangle$. This energy level is not doubly degenerate because the state $\mid K_{00}\rangle$ is forbidden by the RSOS selection rules, eq. (70). With periodic boundary conditions, the kink states $\mid K_{01}\rangle$ and $\mid K_{10}\rangle$ are projected out and $\mid K_{11}\rangle$ is the only one-particle state that can appear in the spectrum. The results correspond with the ones found in the truncation space approach [16].

For real values of β, the amplitudes $S_{00}^{11}(\beta)$ and $S_{01}^{11}(\beta)$ are numbers of modulus 1. It is therefore convenient to define the following phase shifts

$$\begin{aligned} S_{00}^{11}(\beta) &\equiv e^{2i\delta_0(\beta)} \; ; \\ S_{01}^{11}(\beta) &\equiv e^{2i\delta_1(\beta)} \; . \end{aligned} \tag{102}$$

The non-diagonal sector of the scattering processes is characterized by the 2×2 symmetric S-matrix

$$\begin{pmatrix} S_{11}^{11}(\beta) & S_{11}^{01}(\beta) \\ S_{11}^{01}(\beta) & S_{11}^{00}(\beta) \end{pmatrix} \; . \tag{103}$$

We can define the corresponding phase shifts by diagonalizing the matrix (103). The eigenvalues turn out to be the same functions in (102),

$$\begin{pmatrix} e^{2i\delta_0(\beta)} & 0 \\ 0 & e^{2i\delta_1(\beta)} \end{pmatrix} \; . \tag{104}$$

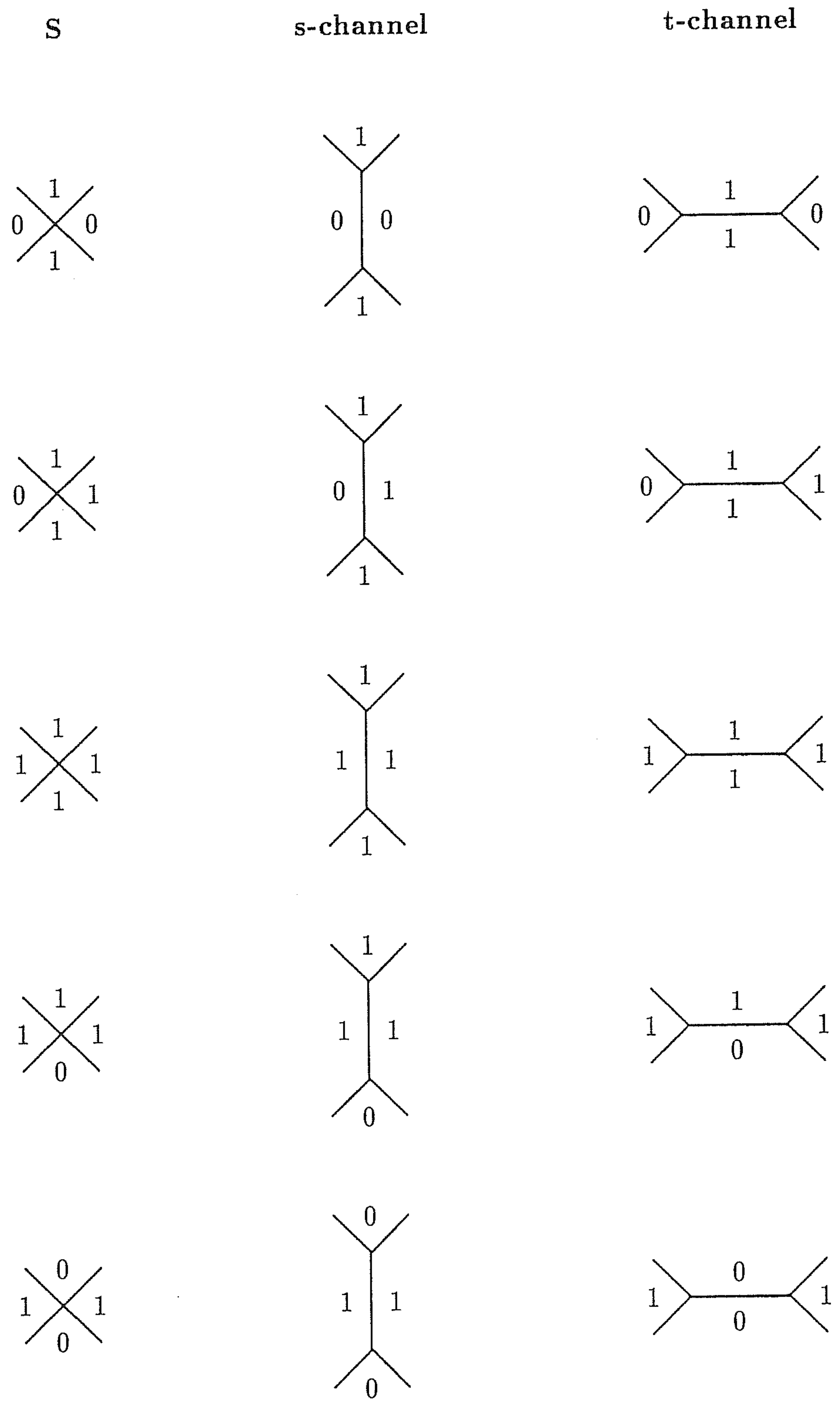


Figure 10: Intermediate states in the s-channel and t-channel of the RSOS S-matrix.

The phase shifts, for positive values of β, are shown in fig. 5. Asymptotically, they have the following limits

$$\begin{aligned} \lim_{\beta\to\pm\infty} \; e^{2i\delta_0(\beta)} &= e^{\pm\frac{6\pi i}{5}} \; ; \\ \lim_{\beta\to\pm\infty} \; e^{2i\delta_1(\beta)} &= e^{\pm\frac{3\pi i}{5}} \; . \end{aligned} \tag{105}$$

There is a striking difference between the two phase shifts: while $\delta_1(\beta)$ is a monotonic decreasing function, starting from its value at zero energy $\delta_1(0) = \frac{\pi}{2}$, $\delta_0(\beta)$ shows a maximum for $\beta \sim \frac{\pi}{3}$ and then decreases to its asymptotic value $\frac{3\pi}{5}$. Its values are always larger that $\delta_0(0) = \frac{\pi}{2}$. Such different behaviour of the phase shifts is related to the presence of a zero very close to the real axis in the amplitude $e^{2i\delta_0(\beta)}$, i.e. at $\beta = i\frac{\pi}{9}$. This zero competes with the pole at $\beta = i\frac{\pi}{3}$ in creating a maximum in the phase shift. Similar behaviour also occurs in non-relativistic cases [65] and in the case of breather-like S matrices which contains zeros [17]. The presence of such a zero is deeply related to the absence of the pole in the s-channel of the amplitude $e^{2i\delta_0(\beta)}$. For the amplitude $e^{2i\delta_1(\beta)}$, the zero is located at $\beta = \frac{4\pi i}{9}$ (between the two poles) and therefore its contribution to the phase shift is damped with respect to that one coming from the poles. The net result is a monotonic decreasing phase shift.

Coming back to the 2×2 S-matrix of eq. (103), a basis of eigenvectors is given by

$$\begin{aligned} \mid \phi_1(\beta_1)\phi_1(\beta_2)\rangle &= A(\beta_{12})\left(\mid K_{11}(\beta_1)K_{11}(\beta_2)\rangle + \chi_1(\beta_{12}) \quad \mid K_{10}(\beta_1)K_{01}(\beta_2)\rangle\right) \\ \mid \phi_2(\beta_1)\phi_2(\beta_2)\rangle &= A(\beta_{12})\left(\mid K_{11}(\beta_1)K_{11}(\beta_2)\rangle + \chi_2(\beta_{12}) \quad \mid K_{10}(\beta_1)K_{01}(\beta_2)\rangle\right) \; . \end{aligned} \tag{106}$$

where $A(\beta_{12})$ is a normalization factor. In the asymptotic regime $\beta \to \infty$

$$\begin{aligned} \chi_1 &= -\frac{e^{-\frac{2\pi i}{5}}\left(a^2 + e^{\frac{6\pi i}{5}}\right)}{a} \\ \chi_2 &= -\frac{e^{-\frac{2\pi i}{5}}\left(a^2 + e^{\frac{3\pi i}{5}}\right)}{a} \; , \end{aligned} \tag{107}$$

and the probability P_{1001} to find a state $\mid K_{10}K_{01}\rangle$ in the vector $\mid \phi_2\phi_2\rangle$ w.r.t. the probability P_{1111} to find a state $\mid K_{11}K_{11}\rangle$ is given by the golden ratio

$$\frac{P_{1001}}{P_{1111}} = \frac{1}{a^2} = 2\cos\left(\frac{\pi}{5}\right) \; . \tag{108}$$

For the state $\mid \phi_1 \phi_1 \rangle$, we have

$$\frac{P_{1001}}{P_{1111}} = a^2 = \frac{1}{2\cos\left(\frac{\pi}{5}\right)} \quad . \tag{109}$$

The "kinks" ϕ_1 and ϕ_2 have the generalized bilinear commutation relation [70, 39, 68]

$$\phi_i(t,x)\phi_j(t,y) = \phi_j(t,y)\phi_i(t,x) \quad e^{2\pi i s_{ij}\epsilon(x-y)} \quad . \tag{110}$$

The generalized "spin" s_{ij} is a parameter related to the asymptotic behaviour of the S-matrix. A consistent assignment is given by

$$\begin{aligned} s_{11} &= \frac{3}{5} = \frac{\delta_0(\infty)}{\pi} \quad ; \\ s_{12} &= 0 \quad ; \\ s_{22} &= \frac{3}{10} = \frac{\delta_1(\infty)}{\pi} \quad . \end{aligned} \tag{111}$$

Notice that, interesting enough, the previous monodromy properties are those of the chiral field $\Psi = \Phi_{\frac{6}{10},0}$ of the original CFT of the TIM. The operator product expansion of Ψ with itself reads

$$\Psi(z)\Psi(0) = \frac{1}{z^{\frac{6}{5}}} \; 1 + \frac{C_{\Psi,\Psi,\Psi}}{z^{\frac{3}{5}}} \; \Psi(0) + \ldots \tag{112}$$

where $C_{\Psi,\Psi,\Psi}$ is the structure constant of the OPE algebra. Moving z around the origin, $z \to e^{2\pi i}z$, the phase acquired from the first term on the right hand side of (112) comes from the conformal dimension of the operator Ψ itself. In contrast, the phase obtained from the second term is due to the insertion of an additional operator Ψ. A similar structure appears in the scattering processes of the "kinks" ϕ_i: in the amplitude of the kink ϕ_1 there is no bound state in the s-channel (corresponding to the "identity term" in (112)) whereas in the amplitude of ϕ_2 a kink can be created as a bound state for $\beta = \frac{2\pi i}{3}$ (corresponding to the "Ψ term" in (112)). In the ultraviolet limit, the fields ϕ_i should give rise to the operator $\Psi(z)$, similarly to the case analyzed in [68].

The problem of finding a theoretical explanation for the energy levels of the ϕ_{21} perturbed TIM resulting from the truncation space approach, was first discussed by Zamolodchikov [79]. He explicitly constructed the S matrix having available these data and the requirement of factorized scattering. His result differs from the one above. Both solutions Are particular parameterization of the "hard square lattice gas" Boltzmann weights [3]. In [16] a comparison was made between the two different approaches, studying finite-size corrections of the energy levels obtained by the truncation space approach. The result suggests that the RSOS S matrix gives a more appropriate description of the scattering process of the massive excitations of the model.

6 Conclusions and Outlook

We presented some results on the S matrix approach, in order to describe integrable deformations of conformal field theory. Our approach was based on the fact that the S matrix underlies constraints, which can be translated into properties of an R matrix of a quantum group. It turned out that the S matrix is proportional to this R-matrix. Perturbations of minimal models were related to RSOS restrictions of this S matrix, using the representation theory of the R matrix.

A conformal invariant system can be realized as a Toda field theory, corresponding to the group A_1. This is the only posibility to construct a Toda field theory with only one scalar field. The only affine extensions of this algebra (containing also only one scalar field) are $A_1^{(1)}$ and $A_2^{(2)}$. They are identified with the sine-Gordon model and the Izergin-Korepin model respectively. Using the correspondence to the Coulomb gas approach one realizes that they describe perturbations of conformal field theory: the sine-Gordon model is related to the $\phi_{1,3}$ perturbations and the Izergin-Korepin model to $\phi_{1,2}$ and $\phi_{2,1}$ perturbations. Therefore it is natural to assume, and is also confirmed, that the S matrices are proportional to the corresponding quantum group R matrices of $U_q(A_1^{(1)})$ and $U_q(A_2^{(2)})$.

Our results are twofold: First we considered the sine-Gordon model and compared the results with those coming from S matrix theory. That is, we considered the models $\mathcal{M}_{2,2n+3}$, which contain only one singularity in the S matrix amplitude of the basic particle, which is a scalar particle. We examined whether there can exist other consistent S matrices having only one singularity and found that the answer is negative.

This is an exiting result, but is understood to be only the beginning of a much vaster investigation. One would like to prove, that the class of consistent S matrices is equivalent with those found by the reduction mechanism, that is, intrinsically related to a Lie algebra. This hope arose since an analogous situation occurs in conformal field theory, where the models are classified by the ADE series. Considering basic amplitudes with more than one singularity, the analysis gets much more involved, and no analytic result is known up to now . In [44] a computer program was used to get information about theories with basic S matrices with 2 and 3 singularities, using only general constraints from the S matrix theory and the bootstrap principle. Also in this case the only consistent systems found are related to Lie algebras (E_7 and E_8) and can be obtained by the quantum group reduction. Unfortunately, a numerical analysis of this kind can only be taken as an indication in the right direction, but does not provide a conclusive result.

Secondly we examined RSOS restrictions of the Izergin-Korepin model, which is related to the $A_2^{(2)}$ quantum group. We explicitly calculated the full S matrix for $\phi_{1,2}$ perturbations of the $\mathcal{M}_{2,2n+3}$ series, which contains only breathers in this case. Using the TBA we found that the effective central charge in the ultraviolet limit, and indeed it corresponds to that of the critical theory.

Also the series $\mathcal{M}_{3,3n\pm1}$ and $\mathcal{M}_{4,4n\pm1}$ contain only breathers (or in the second case, breather-like particles). It is still an open problem what their full S matrix looks like. Also the S matrix of the unitary series $\mathcal{M}_{n,n+1}$ has still not been written down, though one knows in almost all cases the particle content.

For the $\phi_{2,1}$ perturbations only few examples are known up to now. This is due to the fact, that kink scattering amplitudes are more complicated to handle than breather ones. So also in this case there are still many things to do.

As one sees there are many open problems in the S matrix description of perturbations of conformal field theories. But the S matrix is a very basic ingredient for many further investigations. An open problem is to deal with the off-shell theory, that is to write down correlation fuctions of the theory, which in principle should be possible through the calculation of form-factors. Also for the TBA the S matrix is a basic ingredient. Also in this field there are many interesting open problems: the generalization to non-diagonal S matrices, how to reduce the enourmous numrical work, in order to solve the TBA equations, and more fundamentally, to understand the deeper reason how an on-shell description of the theory can give the right critical exponents and central charge in the ultraviolt limit. Though S matrix theory is only a first ingredient for these calculations, it plays such a fundamental rôle, that it is surely worth spending one's energy on this subject.

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