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Metric and Topological Properties of Non Decomposable Sets

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Sandro Zagatti

SUPERVISOR

Prof. Arrigo Cellina

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INTRODUCTION

Aim of this work is to study some topological and metric properties of a certain class of subsets of L^1. In order to motivate the subject we briefly review the main methods and techniques used in the literature in the context of existence of solutions of differential inclusions and of minima of certain functionals in the calculus of variations. We wish to point out that while for differential inclusions several methods have been developed, in the calculus of variations only recently some work has been devoted to overcome the original standard approach based on convexity, and going back to Tonelli. Indeed these two fields are in some sense related and have some points of contact, but the methods used until today to treat them are different. Our purpose is to present some results in the direction of bringing closer these two problems.

The following review is far from being complete and exaustive and we refer to the current literature for details, mainly to [AC] for differential inclusion and to [Da] for Calculus of Variations.

We consider the differential inclusion

$$x'(t) \in F(t, x(t)), \qquad x(t_0) = x_0 \tag{1}$$

where F is a multivalued function defined on some open subset Ω of R^{n+1} with values in some Banach space X (in many cases R^n).

With regard to the existence problem two kinds of assumptions on F are necessary: some regularity of the map (like continuity or semicontinuity) and topological or geometrical properties of its values (like compactness or convexity). Standard existence results need a compromise between these two kinds of hypothesis and an effort has been done in the recent years to weaken these assumptions.

Convex Valued Differential Inclusions [AC]. The simplest approach to the study of (1) is to use or to reproduce existence results for the differential equation

$$x'(t) = f(t, x(t)). \qquad x(t_0) = x_0. \tag{2}$$

This idea leads to the concept of selection. By a selection of the multifunction F we mean a function f whose values belong to the values of F on its domain. In general a measurable multifunction F admits measurable selections but this result is of no utility in the study of (2); on the other hand one is interested in finding conditions on F that ensure the existence of a continuous selection f. Indeed, in this case, the solutions of (2) turn out to be solutions of (1).

About continuous selections there exist two basic results: Michael's theorem and Cellina's theorem. The first states that a lower semicontinuous (l.s.c.) multifunction F from a metric space Y into the closed convex subsets of a Banach space X admits a continuous selection. As a straightforward consequence we can state that if F from $\Omega \in R^{n+1}$ into the closed convex nonempty subsets of R^n is l.s.c. then (1) has at least one solution.

Cellina's theorem states that, replacing in Michael's theorem lower semicontinuity with upper semicontinuity, there exists an approximate continuous selection, that is to say that given $\epsilon > 0$ there exists a continuous function f_ϵ such that

$$\text{graph}(f_\epsilon) \subset \text{graph}(F) \; + \; \epsilon B$$

where B is the unit ball in X.

Suppose now X to be a Hilbert space, F defined on $\Omega \subset R \times X$ into nonempty closed convex subsets of X, be upper semicontinuous (u.s.c.) and the map $(t,x) \to m(F(t,x))$ locally bounded (here $m(K)$ denotes the projection of zero onto the closed convex set K). Then (1) has at least one local solution. This result is in some sense analoguous to Peano's theorem for differential equations and can be proved with various methods. The simplest idea is to construct a sequence of approximate solutions x_n corresponding to the approximate selections of Cellina's theorem; then if the interval of existence is chosen sufficiently small the sequence of derivatives is pointwise bounded and the approximating solutions remain in a compact set. By a compactness argument based on Ascoli Arzela' and Alaoglu theorem it follows that there exists a subsequence x_{n_k} such that x'_{n_k} converges weakly in L^1. The primitive of the limit satisfying the initial condition is a solution of (1).

Non Convex Valued Differential Inclusions [AC]. As we have already remarked in the study of Differential Inclusions convergence of derivatives plays a relevant role, and this is a remarkable difference with respect to differential equations since in that case the convergence of the approximating solutions implies the convergence of derivatives themselves. If we remove convexity assumption on the values of the right hand side of (1), we lose standard results about continuous selections and we should use compactness arguments. Compactness is easier to be proved in weakened topologies and in fact in convex case we use weak convergence in L^1; unfortunately weak convergence does not imply pointwise convergence so the limit needs not be a solution and convexity is a device to pass from weak convergence to strong convergence. In non convex case compactness has to be used in strong sense and auxiliary compactness result are needed.

Consider a Hilbert space X and the Banach space $B(I,X)$ of bounded functions defined on some interval I with values in X and let H be a subset of $B(I, X)$. Suppose H to be equioscillating and pointwise precompact, then H is precompact.

Take now $\Omega \subset R^{n+1}$ and F a continuous map from Ω into the nonempty compact subsets of R^n, then (1) has at least one local solution. The proof is based on the following idea: construct piecewise linear approximations with derivatives equioscillating, boundedness of the values of F allows the use of the mentioned compactness result; going to the limit and taking the primitive satisfying the initial condition one obtains the result.

This procedure has a further development regarding regularity of solutions. Indeed suppose in addition F independent on time and absolutely continuous, and construct the approximate solutions in such a way that derivatives are equicontinuous; then the solution turns out to be continuously differentiable.

Directionally Continuous Selections [B]. Standard selection approach require convexity of the right hand side of (1), because otherwise the existence of a continuous selection is not guaranteed, and until today non convex differential inclusion were treated with different methods. A recent work of Bressan [B] recovers the classical selection method, bringing to a remarkable advance in the theory of differential inclusions. The idea is to find a property general enough so that weak assumptions on F guarantee the existence of a selection f with such a property, yet strong enough to ensure that the differential equation $x'(t) = f(t, x(t))$ has solutions. Such a property is directional continuity. To be more precise let M be a positive constant, and define the cone:

$$\Gamma_M = \{(t, x) \in R^{n+1} : |x| \leq Mt\},$$

we say that a map $f : R^{n+1} \rightarrow R^n$ is directionally Γ_M-continuous at a point (t, x) if $f(t_n, x_n)$ converges to $f(t, x)$ for every sequence (t_n, x_n) converging to (t, x) with $(t_n -$

$t, x_n - x)$ belonging to Γ_M.

Consider a l.s.c. map F defined on R^{n+1} with closed convex values in some metric space, then, for any M, F admits a directionally Γ_M-continuous selection f and if f is bounded by some constant L less than M then (1) has at least one solution obtained as solution of (2).

Fixed point Approach [AC]. In the study of the differential inclusion (2) the existence problem can be formulated in terms of fixed point theory. Define, on the Banach space $C(I)$ of continuous functions defined on an interval I with values in a Banach space X, the operator

$$(Tx)(t) \;=\; x_0 \;+\; \int_{t_0}^{t} f(s, x(s))ds.$$

A solution of (2) can be found as a fixed point of T and a basic tool is Schauder's theorem.

In order to extend the fixed point approach to the differential inclusion (1), consider the set $K \subset C(I)$ of lipschitzian functions with fixed Lipschitz constant satisfying the initial condition, and define the multivalued integral operator $S : K \to K$ by the formula

$$S(x) = \{z \in K : \; z'(t) \in F(t, x(t)) \quad \text{a.e.} \; \in I\}.$$

Obviously a fixed point of S is a solution of (1).

Now a fixed point theorem for multivalued map is needed. Kakutani's theorem, which is a consequence of Schauder's one, asserts that an upper semicontinuous map S from a compact convex subset K of a Banach space X into its compact convex subsets has at least one fixed point, i.e. there exists a y in K such that $y \in S(y)$.

Suppose now F from $\Omega \subset R^{n+1}$ into nonempty convex compact subset of R^n be upper semicontinuous; it can be proved that the operator S defined above is convex valued and that the map $x \to S(x)$ is upper semicontinuous; then there exist a fixed point, i.e. a solution of (1).

It should be remarked that in this procedure the assumptions on F are quite strong. In particular convexity is needed to prove upper semicontinuity of the operator S and the convexity of its values. Nevertheless also in nonconvex case it is possible to prove that S has a fixed point and this can be done by a selection argument.

Suppose F from Ω into nonempty compact subsets of R^n be continuous; a selection argument due to Antosiewicz and Cellina shows that for any u in K, the set defined above, there exists a continuous function $g(u)(\cdot)$ such that $g(u)(t) \in F(t, u(t))$ a.e. in I. Then for any u in K defined the map $\phi(u)(\cdot)$ as the primitive ov $g(u)(\cdot)$ satisfying the initial condition, and it turns out that $\phi : K \to K$ is a continuous selection of S. Since K is compact in $C(I)$, ϕ has a fixed point which is a fixed point of S too.

To end this remark on the fixed point approach it should be mentioned that the continuity assumption on F can be weakened and replaced by lower semicontinuity; the selection argument described above can be reproduced with no essential modification giving as a result that ϕ is in this case a continuous map from K into L^1.

Relaxation Theorem and Baire's Category method [AC], [DP], [BC1]. Consider the differential inclusions:

$$x'(t) \in F(x(t)) \qquad x(t_0) = x_0 \tag{4}$$

$$x'(t) \in \bar{co}F(x(t)) \qquad x(t_0) = x_0 \tag{5}$$

where $\bar{co}$ denotes the closure of the convex hull. The solutions of (4) are obviously solutions of (5) too; we wish to study to what extent the operation of convexifying the right hand side of (4) (relaxation) actually introduces new solutions. In other words one ask oneself:

under what conditions is the solution set of (4) dense in the solutions set of (5)? Conversely, suppose to have a compact convex valued map F, is it possible to find a multifunction with values contained in those of F in order essentially to retain the solution of the original problem?

Filippov Wazewsky theorem states that given a lipschitzian map F defined on R^n into compact subsets of R^n any solution of (5) can be pointwise arbitrarily approximated with a solution of (4).

This result is of qualitative type, but as we will see its development has brought to some relevant results in existence problem by virtue of Baire's Category theorem.

The starting point is the following consideration due to Cellina [C1]: consider the differential inclusions

$$x'(t) \in \{-1,+1\} \qquad x(0) = 0 \tag{6}$$

$$x'(t) \in [-1,+1] \;=\; \bar{co}\{-1,+1\} \qquad x(0) = 0. \tag{7}$$

Call $K \subset C([0,1],R)$ the solution set of (7) which is compact, and define, for any ϵ the sets

$$K_\epsilon \;=\; \{x \in K : \;\; \int_0^1 (1 - |x'(t)|)dt \;<\; \epsilon\}$$

which are open and dense in K. By Baire's category theorem $\cap K_{1/n}$ is a G_δ dense subset of K whose elements are solutions of (6). So one obtains simultaneously an existence and a density result.

In general consider

$$x'(t) \in \;\partial F(t,x(t)) \qquad x(t_0) = x_0 \tag{7}$$

and

$$x'(t) \in \;F(t,x(t)) \qquad x(t_0) = x_0 \tag{8}$$

where ∂ here denotes the boundary. By an analogous argument it can be proved that if F is continuous and bounded with values in the closed convex subsets of a reflexive Banach space X then (7) and (8) have solutions and the solution set of (7) is a G_δ dense subset of the solution set of (8).

If the right hand side of (8) has nonconvex values this result is no longer true as it was shown with a well known counterexample by Pliss. In order to extend Baire's Category method to more general case it can be introduced a weaker notion of density and the definition of the so called Choquet functions. Roughly speaking, given a continuous map F defined on R^{n+1} with bounded values, the Choquet function $\Phi = \Phi(t, x, v)$ defined on the graph of F is a nonnegative bounded function vanishing if and only if v belongs to $ext(F(t,x))$, the set of extreme points of $F(t,x)$. So let us study the relation between

$$x'(t) \in ext(F(t,x)) \qquad x(t_0) = x_0 \tag{9}$$

and

$$x'(t) \in F(t,x) \qquad x(t_0) = x_0 \tag{10}$$

where F is continuous bounded has closed convex (or compact) values in a Banach space.

In general the map $(t,x) \to ext(f(t,x))$ is neither continuous nor closed valued, and one tries to reproduce the Baire's procedure sketched above. For this case one replaces the set K_ϵ with the set of solution of (10) such that the integral

$$\int_I \Phi(t, x(t), x'(t))dt$$

on the interval I of definition of solution is less than ϵ, and then it is possible to prove that (9) has solution and that for any function f belonging to a certain class of continuous selections of F including α-lipschitzian selections, denoting with P_f the solution set of $x'(t) = f(t, x(t))$ any neighbourhood of P_f contains a solution of (9).

Let us devote ourselves to the Calculus of variations and consider the so called scalar case, consisting in minimizing

$$I(x) = \int_a^b f(t, x(t), x'(t))dt, \quad x(a) = \alpha; \; x(b) = \beta$$

or

$$I(u) = \int_\Omega f(y, u(y), \nabla u(y))dy, \quad u \in u^* + W_0^{1,1}(\Omega)$$

where x is a vector valued function defined on the interval $[a, b]$ of R^1, and u is a scalar function defined on $\Omega \subset R^n$. The basic reasoning used in the proofs of existence of minima for the functionals $I(x)$ or $I(u)$ is the following: by imposing growth condition at infinity on the integrand with respect to the last argument, one obtains that the functions that make the integral finite are contained in a weakly compact set in the topology of AC([a,b]) or $W^{1,1}(\Omega)$ where the accent is on the L^1- convergence of derivatives. As a consequence of this compactness, out from a minimizing sequence one can extract a weakly converging subsequence x_n and the problem is then to show that the value of the functional on the limit point x^* is not larger than the inferior limit of the values of the functional along the sequence. It is so natural to require that I satisfies

$$I(x) \leq \lim \inf I(x_n) \tag{11}$$

as x_n converges weakly to x^*. If we impose this property for any weakly converging sequence to any point x^* the functional is called weakly lower semicontinuous and in terms of epigraphs this condition is equivalent to weak closure of the epigraph of I.

Weak lower semicontinuity plays a fundamental role in the literature and a basic result about this property is the following: a necessary and sufficient condition for the weak lower semicontinuity of the functional I, under suitably regularity and growth condition, is that the map $v \rightarrow f(\cdot, \cdot, v)$ is convex.

Apart from the procedure that we have sketched above, few attempts have been made to prove existence of minima for functionals like I, even though weak lower semicontinuity is far from be a necessary condition. Indeed for our purpose there is no need to impose inequality (11) on every weakly converging sequence; one can limit oneself to impose

1) that the inequality (11) is true for any minimizing sequence

2) that there exists a minimizing sequence such that (11) is true.

The idea suggested by 1) has been developed in [M], [AT], [R] with methods based on relaxation. One starts by considering a functional with non convex integrand and defines the so called relaxed problem obtained by replacing the integrand with its convexified. Under analogous regularity and growth conditions, the relaxed problem has solution and one then tries to prove that it is actually a solution of the original nonconvex problem itself. This can be done by adding regularity conditions as to ensure that the solution satisfies Euler-Lagrange equations and imposing that the solution stays in the region where the convexified and the original integrand coincide. As a consequence,along the solution the relaxed functional and the original functional have to be equal, and since the relaxed one is weakly lower semicontinuous, the original one is weakly lower semicontinuous on any minimizing sequence.

An application of 2) is due to Cellina and Colombo [CC] and is about the problem of minimizing the functional

$$I(x) \;=\; \int_a^b h(x(t))dt \;+\; \int_a^b g(x'(t))dt, \quad x(a)=\alpha,\; x(b)=\beta; \tag{12}$$

assuming standard regularity and growth conditions and imposing concavity on h instead of convexity on g. The idea of the existence proof is to consider the solution of the relaxed problem obtained replacing g with its convexified g^{**}, and to construct from it a solution of the original non convex problem. The main tool used in this method is Liapunov's theorem on the range of non atomic vector measures, and it should be remarked that concavity of h is a sufficient but not necessary condition.

Description of the present results

As we have seen convexity is a fundamental concept in the study of Differential Inclusions and of Calculus of Variations and in the last several years, in order to avoid convexity assumptions, an effort has been produced to find classes of sets that inherit some of the properties of convex sets from less geometric and more analytic considerations. The theory of decomposable sets is the product of such an effort. A decomposable set D is a subset of $L^1 = L^1(T, E)$, where T is a measure space and E a Banach space, having the property that $u\chi_A + v\chi_{T-A} \in D$ for every $u, v \in D$ and for every measurable subset A of T. One can find in [O1], [BC2], [CCF] and in [F1] results in which decomposability is in some sense a substitute of convexity. In particular the analysis of selection problems, which is fundamental in the theory of differential inclusions, makes a large use of the concept of decomposability since, as it turned out, a decomposable, closed and bounded set is the set of mesurable selections of an integrably bounded multifunction (that could describe the constraint of the problem). This means that dealing with a bounded decomposable set, actually we deal with a set of functions pointwise a.e. bounded by some function in L^1. On the contrary in variational problems this limitation is not met in general, and the techniques we have mentioned above lead to consider bounds coming from growth conditions or coercivity conditions. In variational problems one meets so called sublevel sets which in general can be defined as sets of functions $u \in L^1$ such that $\int \Phi \circ \|u\|$ is bounded by a constant M, possibly infinite, where the function $\Phi : [0, \infty[\to [0, \infty[$ has the only property that $\Phi \circ \|u\|$ is measurable for any $u \in L^1$. Since the assumptions on Φ are very weak, the class of sublevels contains a certain amount of subsets of L^1, such as Orlicz classes, L^p spaces $(0 < p < \infty)$, balls in such spaces and so on. However these sets, denoted by Φ_M, are *not decomposable*. Purpose of our work is to extend the theory developed for decomposable sets to sets of the type $\Phi_M \cap D$ i. e. to sets which are the intersection of a sublevel and of a set of selections of a measurable multifunction (but not necessarily bounded). In this way we cover the case of a weakly relatively compact family of functions in L^1, taking values in a closed subset M of E, assuming neither the boundedness of M nor the existence of an integral bound for the family itself, as it would be required within the framework of decomposable sets. It should be remarked that Φ_M or D can be chosen to be the whole space, including, as a special case, *the theory for decomposable sets and for sublevels*.

We present a metric and a topological result: first we prove that Kuratowski index α of a set of the type specified above coincides with its diameter, then we give a version of Dugundji extension theorem. In [BC2] the authors give an extension theorem in which decomposability in L^1 is used instead of convexity in a generic Banach space, obtaining, as consequences, that a closed decomposable set in a separable space L^1 is a retract, and that, in general, it has the compact fixed point property. We prove analogous statement for our case.

1. NOTATIONS AND PRELIMINARY RESULTS

We consider a measure space $(T, \mathcal{F}, \mu)$ where $\mathcal{F}$ is a σ-algebra of subsets of T and μ is a bounded positive nonatomic measure on T. If $f : T \to \mathbf{R}$ is a μ-measurable function we denote by $f \cdot \mu$ the measure having density f with respect to μ. E is a Banach space with norm $\|\cdot\|$ and $p \geq 1$ $L^p(T, E)$ is the Banach space of Bochner μ-integrable E-valued functions defined on T endowed with the norm $\|f\|_p = (\int_T \|f\|^p d\mu)^{1/p}$. If A and B are two sets, $A - B$ is their difference and $A \triangle B$ is their symmetric difference $(A - B) \cup (B - A)$. By χ_A we denote the characteristic function of the set A and by S^n the set $\{x = (x_0, ..., x_n) \in \mathbf{R}^{n+1}$ s. t. $\sum_{i=0}^{n} |x_i| = 1\}$. For a metric space X with distance d and for $A \subset X$, we set $d(x, A) \equiv \inf\{d(x, y),\ y \in A\}$, and $\operatorname{diam}(A) \equiv \sup\{d(x, y),\ x, y \in A\}$.

Definition 1.

A set $D \subseteq L^1(T, E)$ is called *decomposable* if for every $A \in \mathcal{F}$ and for every $u,\ v \in D$ it is

$$u\chi_A + v\chi_{T-A} \in D.$$

The decomposable hull of a subset S of $L^1(T, E)$ is defined as the smallest decomposable set containing S and is denoted by dec(S).

Definition 2.

Let $F : T \to 2^E$ be a multifunction. F is said measurable if for every closed $C \subseteq E$ it is $F^{-1}(C) \in \mathcal{F}$. The set of measurable selections of F is denoted by S_F.

The following statement is a characterization of decomposable sets (see [HU]).

Theorem 1.

Let $S \subset L^1(T,E)$ be nonempty and closed. Then S is decomposable if and only if there exists a measurable multifunction F such that $S = S_F$. Moreover, if S is bounded, F is integrably bounded i.e. there exists $h \in L^1(T,\mathbf{R})$ such that $\|u(t)\| \le h(t)$ $\mu - a.\ e.$ for all $u \in S_F$.

We now list some well known results which in some sense justify our work.

Definition 3.

A family S of $L^1(T,E)$ is called absolutely equiintegrable if for any $\epsilon > 0$ there exists $\delta > 0$ such that for any $\Delta \in \mathcal{F}$ with $\mu(\Delta) < \delta$ it is $\int_\Delta \|u\| d\mu < \epsilon$ for all $u \in S$.

Notice that an absolutely equiintegrable family of $L^1(T,E)$ is bounded. A bounded decomposable subset of L^1 is equiintegrable; however an equiintegrable set cannot be decomposable unless it is integrably bounded.

The following theorem is well known (see for example [Ce]).

Theorem 2.

Let S$\subset L^1(T,E)$. The following statements are equivalent:

i) S is sequentially weakly relatively compact in $L^1(T,E)$;

ii) S is absolutely equiintegrable;

iii) there exists $\Phi : [0,+\infty[\rightarrow \mathbf{R}$ nonnegative, increasing, satisfying $\Phi(t)/t \rightarrow +\infty$ as $t \rightarrow +\infty$, and a constant $L > 0$, such that $\int_T \Phi(\|u\|)d\mu \le L$ for all $u \in S$.

Since in the literature it is not easy to find the proof of the equivalence of ii) and of iii) we remind it.

For $u \in S$ set

$$T(u, \alpha, \beta) = \{t \in T : \alpha \leq \|u(t)\| < \beta\}$$

$$T(u, \alpha) = \{t \in T : \|u(t)\| \geq \alpha\}.$$

ii)$\Rightarrow$ iii). For $y \geq 0$ define

$$\gamma(y) = \sup_{u \in S} \int_{T(u,y)} \|u\| d\mu,$$

γ is a nonnegative, nonincreasing function and $\lim_{y \to +\infty} \gamma(y) = +\infty$. To prove the last assertion suppose that there exist $\epsilon > 0$, and $\{y_n\}_{n \in \mathbf{N}}$ increasing sequence in $\mathbf{R}$ with

$$\lim_{n \to \infty} y_n = +\infty$$

and $\{u_n\}_{n \in \mathbf{N}}$ in S such that $\int_{T(u_n, y_n)} \|u_n\| d\mu \geq \epsilon$; since $\lim_{n \to \infty} \mu(T(u_n, y_n)) = 0$ (otherwise it would be $sup_{u \in S} \int_T \|u\| d\mu = +\infty$) we get a contradiction.

Choose now two real increasing sequences $\{a_n\}_{n \geq 0}$, $\{y_n\}_{n \geq 0}$ satisfying:

$$y_0 = 0, \qquad \lim_{n \to \infty} y_n = +\infty,$$

$$a_0 \geq 0, \qquad \lim_{n \to \infty} a_n = +\infty, \qquad \sum_{n=0}^{\infty} a_n \gamma(y_n) = L < \infty$$

and define

$$\phi(y) = \sum_{n=0}^{\infty} a_n y \chi_{[y_n, y_{n+1}[}.$$

ϕ has the properties stated above and

$$\int_T \phi(\|u\|) d\mu = \sum_{n=0}^{\infty} \int_{T(u, y_n, y_{n+1})} \phi(\|u\|) d\mu = \sum_{n=0}^{\infty} a_n \int_{T(u, y_n, y_{n+1})} \|u\| d\mu \leq$$

$$\leq \sum_{n=0}^{\infty} a_n \int_{T(u, y_n)} \|u\| d\mu \leq \sum_{n=0}^{\infty} a_n \gamma(y_n) = L.$$

iii)$\Rightarrow$ ii) (De La Valleé Poussin).

Set $\psi(y) = \phi(y)\,/y$; fix $\epsilon > 0$, choose $A > 0$ such that $\psi(A) > 2L\,/\epsilon$ and δ such that $0 < \delta < \epsilon\,/2A$. For $\Delta \in \mathcal{F}$ with $\mu(\Delta) < \delta$ and $u \in S$, define

$$\Delta_1 = \Delta \cap T(u, A), \qquad \Delta_2 = \Delta \cap T(u, 0, A),$$

it is

$$\int_\Delta \|u\| d\mu \;=\; \int_{\Delta_1} \|u\| d\mu \;+\; \int_{\Delta_2} \|u\| d\mu \;\leq$$

$$\leq \; \frac{1}{\psi(A)} \int_{\Delta_1} \phi(\|u\|) d\mu \;+\; \int_{\Delta_2} A d\mu \;\leq\; \frac{L}{\psi(A)} + A\delta \;<\; \epsilon.$$

This ends the proof.

Functions satisfying the growth conditions specified in previous theorem play an important role in functional analysis, indeed the theory of Orlicz classes is based on such functions, the reader can found a wide exposition in [KJF].

These considerations justify the following

Definition 6.

Let $\Phi : [0, \infty[\to \mathbf{R}^+ \cup \{0\}$ having the property that $\Phi \circ \|u\|$ is measurable for every $u \in L^1(T, E)$. Let $M \in \mathbf{R}^+ \cup \{+\infty\}$; we set

$$\Phi_M \;\equiv\; \Phi_M(T, E) \equiv \{u \in L^1(T, E) s.\ t.\ \int_T \Phi \circ \|u\| d\mu < M\}$$

We call this set Φ-sublevel. If $M < \infty$ the strict inequality in the definition of Φ_M can be replaced by the weak one without affecting our results.

Among sets which can be described as sublevels in the sense specified above we mention, for example, balls in L^p-spaces and in Orlicz spaces (with radius possibly infinite).

We end this section reminding the main tool used in establishing our results, that is to say the following version of Liapunov theorem about vector measure.

Theorem 3.

Let $\{g_1, ..., g_N\}$ be a finite family of nonnegative functions in $L^1(T, \mathbf{R})$. Then there exists a family $\{A_\alpha,\ \alpha \in [0,1]\}$ in $\mathcal{F}$ with the properties:

a) $A(0) = \emptyset,\ \ A(1) = T,\ \ A(\alpha) \subseteq A(\beta)\ \forall \alpha, \beta \in [0,1]\ \alpha \leq \beta$

b) $\int_{A(\alpha)} g_n d\mu = \alpha \int_T g_n d\mu\ \forall \alpha \in [0,1], \forall n = 1, ..., N.$

c) $\mu(A_\alpha \triangle A_\beta) \leq |\alpha - \beta|.$

Such family is called a refinement of T with respect to the measures $g_n \cdot \mu$, see [F1].

In what follows we shall consider sets that are intersections of decomposable sets and of sublevel sets. A typical application is to families of functions taking values in a closed subset M of E. To deal with such sets within the framework of bounded decomposable sets one had to impose that either M itself is bounded or that there exists an integral bound. In this paper we extend the theory to cover the case of a generic weakly relatively compact set of L^1 of maps taking values in M.

2. KURATOWSKI INDEX FOR BOUNDED SUBSETS OF L^1

We now recall the definition of Kuratowski's index α and a theorem which will be used later.

Definition 7.

Let X be a metric space, $A \subset X$ bounded.

$$\alpha(A) = \inf\{\epsilon > 0 : \ A = \bigcup_{i \in I} A_i \quad \text{where I is a finite set and } \operatorname{diam}(A_i) \leq \epsilon\}.$$

Theorem 4. (Lusternik, Schnirelman Borsuk).

Let $\{S_0, ..., S_n\}$ be a covering of closed sets of S^n. Then there is at least one set S_i that contains a pair of antipodal points.

See for example [DG].

In [CM] the following proposition is established.

Theorem 5.

Let $D \subset L^1(T, E)$ be decomposable and bounded. Then $\alpha(D) = \operatorname{diam}(D)$.

Corollary 1.

Let $S \subset L^1(T, E)$ bounded. Consider two sequences $\{u_n\}_{n \in \mathbf{N}}$ and $\{v_n\}_{n \in \mathbf{N}}$ such that $\lim_{n \to \infty} \|u_n - v_n\|_1 = \operatorname{diam}(S)$, and suppose $\operatorname{dec}(\{u_n, v_n\}) \subseteq S$ for every $n \in \mathbf{N}$. Then $\alpha(S) = \operatorname{diam}(S)$.

Proof.

Take $\epsilon > 0$, there exists $n \in \mathbf{N}$ such that $\|u_n - v_n\| \geq \operatorname{diam}(S) - \epsilon$; it follows:

$$\operatorname{diam}(S) \geq \alpha(S) \geq \alpha(\operatorname{dec}(\{u_n, v_n\}) = \operatorname{diam}(\operatorname{dec}\{u_n, v_n\}) =$$

$$= \|u_n - v_n\|_1 \geq \operatorname{diam}(S) - \epsilon,$$

Since ϵ is arbitrary we obtain the result.

The following is our first result.

Proposition 1.

Let $D \subseteq L^1(T,E)$ be decomposable and $\Phi_M \subseteq L^1(T,E)$ be a Φ-sublevel, suppose $D \cap \Phi_M$ nonempty and bounded.

Then $\alpha(D \cap \Phi_M) = \operatorname{diam}(D \cap \Phi_M)$.

Proof.

It is sufficient to prove that $\alpha(D \cap \Phi_M) \geq \operatorname{diam}(D \cap \Phi_M)$ (the converse inequality is obvious); for this purpose take $u,\ v \in D \cap \Phi_M$ and, using Liapunov theorem, consider a family $\{A(\alpha),\ \alpha \in [0,1]\}$, a refinement of T with respect to the measures $\|u\| \cdot \mu$, $\|v\| \cdot \mu$, $\Phi(\|u\|) \cdot \mu$, $\Phi(\|v\|) \cdot \mu$. Then, for $j \in \{0, 1, ..., n\}$ and $x \in S^n$, set

$$p_j(x) = \sum_{i=0}^{i=j} |x_i|,$$

$$N_0(x) = A(p_0(x)),$$

$$N_i(x) = A(p_i(x)) - A(p_{i-1}(x)), \quad i \geq 1,$$

$$I_x{}^+ = \{i :\ x_i > 0\}, \qquad I_x{}^- = \{i :\ x_i < 0\},$$

and define $\omega_n : S^n \longrightarrow L^1(T,E)$ by the formula:

$$\omega_n(x) = \sum_{i \in I_x{}^+} u\chi_{N_i(x)} + \sum_{i \in I_x{}^-} v\chi_{N_i(x)}.$$

Obviously $\omega(S^n) \subset D$ and

$$\int_T \Phi(\|\omega_n(x)\|)d\mu = \sum_{i\in I_x^+} \int_{N_i(x)} \Phi(\|u\|)d\mu + \sum_{i\in I_x^-} \int_{N_i(x)} \Phi(\|v\|)d\mu \leq \sum_{i=0}^{n} p_i(x) \cdot M \ = \ M.$$

hence $\omega(S^n) \subset D \cap \Phi_M$. The continuity of ω_n follows easily from property c) in Theorem 3.

Now suppose that there exists a finite covering $\{K_0, ..., K_n\}$ of $D \cap \Phi_M$ where each K_i is closed, it is

$$S^n = {\omega_n}^{-1}(K_0) \cup ... \cup {\omega_n}^{-1}(K_n),$$

each ${\omega_n}^{-1}(K_i)$ is closed and by Theorem 3 there exist x and i such that $\omega_n(x) \in K_i$ and $\omega_n(-x) \in K_i$, since $\|\omega_n(x) - \omega_n(-x)\|_1 = \|u - v\|_1$, $\text{diam}(K_i) \geq \|u - v\|_1$ and this implies $\alpha(D \cap \Phi_M) \geq \|u - v\|_1$. Since u and v are arbitrary we obtain the result.

3. AN EXTENSION THEOREM AND A CLASS OF ABSOLUTE RETRACTS

We recall Dugundji's extension theorem ([Du], p. 188).

Theorem 6.

Let A be a closed subset of a metric space X and K be a convex subset of a Banach space Z. Then every continuous map $f : A \to K$ has a continuous extension $\tilde{f} : X \to K$.

In [BC2] the authors prove an analogous result for $Z = L^1(T, E)$ assuming K decomposable instead of convex. Following their argument we state an extension theorem requiring neither convexity nor decomposability. To do this we need an extended version of Liapunov theorem due to Bressan and Colombo (see [BC2]).

Lemma 1.

Let $\{g_k,\ k \geq 0\} \subset L^1(T, \mathbf{R})$ be a sequence of nonnegative functions with $g_0 \equiv 1$. Then there exists a map $A : \mathbf{R}^+ \times [0,1] \to \mathcal{F}$ with the properties:

i) $A(\tau, \alpha_1) \subseteq A(\tau, \alpha_2)$, if $\alpha_1 \leq \alpha_2$,

ii) $\mu(A(\tau_1, \alpha_1) \triangle A(\tau_2, \alpha_2)) \leq |\alpha_1 - \alpha_2| + 2|\tau_1 - \tau_2|$,

iii) $\int_{A(\tau,\alpha)} g_k d\mu = \alpha \int_T g_k d\mu \quad \forall k \leq \tau$,

for all $\alpha,\ \alpha_1,\ \alpha_2 \in [0,1]$, $\tau,\ \tau_1,\ \tau_2 \geq 0$.

We are now ready to state our extension theorem whose proof is a slight modification of that one of theorem 1 in [BC2].

Proposition 2.

Let X be a metric space with distance d and $A \subseteq X$ be closed. Consider $D \subseteq L^1(T, E)$ decomposable, $\Phi_M \subseteq L^1(T, E)$ a Φ-sublevel and $D \cap \Phi_M \neq \emptyset$. If either X or $L^1(T, E)$ is separable and f is a continuous function from A to $L^1(T,E)$ such that $f(A) \subseteq D \cap \Phi_M$, then f has a continuos extension $\tilde{f} : X \longrightarrow L^1(T,E)$ with $\tilde{f}(X) \subseteq D \cap \Phi_M$.

Proof.

Assume $L^1(T,E)$ separable. For each $x \in X - A$ take an open ball $B(x, r_x)$ with radius $r_x < \frac{1}{2}d(x,A)$. The family $\{B(x,r_x),\ x \in X - A\}$ is an open covering of the paracompact space $X - A$ hence it admits an open neighbourhood-finite refinement $\{V_i,\ i \in I\}$ where I is possible uncountable index set. For any $i \in I$ select two points $x_i \in V_i$ and $y_i \in A$ such that $d(x_i, y_i) < 2d(x_i, A)$. Since $L^1(T,E)$ is separable it is possible to define a countable set $K = \{u_n,\ n \geq 1\} \subset f(A)$ dense in $f(A)$. For every $i \in I$ select $u_{\nu(i)} \in K$ such that $\|u_{\nu(i)} - f(y_i)\|_1 \leq d(x_i, y_i)$. Consider a continuous partition of unity $\{p_i, i \in I\}$ subordinate to $\{V_i\}$ and define, for every $n \geq 1$ the open set $W_n = \cup\{V_i,\ \nu(i) = n\}$ and the continuous partition of unity subordinate to the covering $\{W_n\}$ given by the formula $q_n(x) = \sum_{\nu(i)=n} p_i(x)$. Construct a sequence of continuous functions $\{h_n,\ n \geq 1\}$ such that $h_n \equiv 1$ on $supp(q_n)$ and $supp(h_n) \subseteq W_n$ and define on $X - A$ the continuous function

$$\lambda(x) = \sum_{m \leq n} q_m(x),\ n \geq 0$$

and

$$\tau(x) = \sum_{m,n \geq 1} h_m(x) h_n(x) 2^m 3^n.$$

Consider now the sequence $\{g_k,\ k \geq 0\}$ in $L^1(T,E)$ given by:

$$g_k(t) = \begin{array}{l} \{\|u_n(t) - u_m(t)\| \quad \text{if } k = 2^m 3^n, \\ \Phi(\|u_m\|) \quad \text{if } k = 2^m 3^n - 1, \\ \Phi(\|u_n\|) \quad \text{if } k = 2^m 3^n - 2, \\ 1 \quad \text{otherwise.}\} \end{array}$$

for $m,\ n \geq 1$.

Applying Lemma 1 to such sequence we obtain a family $A(\tau, \alpha)$ and define, for any $x \in X - A$

$$\chi_n(x) = \chi_{A(\tau(x),\lambda_n(x)) - A(\tau(x),\lambda_{n-1}(x))}.$$

Then we extend f to the whole space X by the formula:

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if } x \in A, \\ \sum_{n \geq 0} u_n \cdot \chi_n(x) & \text{if } x \in X - A, \end{cases}$$

We remark that

$$\tau(x) \geq 2^m 3^n \quad \forall x \in supp(q_m) \cap supp(q_n) \tag{3.1}$$

and then it is easy to show that $\tilde{f}$ takes values in Φ_M, indeed, for $x \in X - A$, it is

$$\int_T \Phi(\|\tilde{f}(x)\|) d\mu \;=\; \sum_{n \geq 1} p_n(x) \int_T \Phi(\|u_n\|) d\mu \;<\; M$$

since the sum is taken over the finite set of indices n such that $p_n(x) \neq 0$, and for such indices,

$$\int_{A(\tau(x),\alpha)} \Phi(\|u_n\|) d\mu \;=\; \alpha \int_T \Phi(\|u_n\|) d\mu \quad \forall \alpha \in [0, 1],$$

by property iii) of Lemma 1. Hence, obviously, $\tilde{f}(X) \subseteq \Phi_M \cap D$. Continuity of $\tilde{f}$ in $X - A$ is a consequence of contunuity of $\lambda_n(\cdot)$ of $\tau(\cdot)$, of property ii) in Lemma 1 and of the fact that the summation defining $\tilde{f}$ is locally finite.

To prove continuity on A take $a \in A$ and $\epsilon > 0$. There exists $\delta > 0$ such that $\delta < \epsilon/12$ and $\|f(y) - f(a)\|_1 < \epsilon/2$ whenever $y \in A$ and $d(y, A) < 12\delta$. If $d(x, a) < \delta$ and $x \in V_i$ for some $i \in I$, then $\text{diam}(V_i) < 2\delta$, $d(x_i, A) < 3\delta$ and $d(x_i, y_i) < 6\delta$. Therefore $p_i(x) \neq 0$ implies $d(y_i, a) < 9\delta$, $\|f(y_i) - f(a)\|_1 < \epsilon/2$ and $\|u_{\nu(i)} - f(a)\|_1 < \epsilon$. Then

$$\|u_n - f(a)\| < \epsilon \;\; \forall n \text{ such that } q_n(x) \neq 0. \tag{3.2}$$

For any $x \in X - A$ with $d(x,a) < \delta$, fix j such that $q_j(x) \neq 0$. Using (4.1), (4.2) and property iii) of Lemma 1 we have:

$$\begin{aligned} \|f(a) - \tilde{f}(x)\|_1 &\leq \|f(a) - u_j\|_1 + \|u_j - \tilde{f}(x)\|_1 \leq \\ &\leq \epsilon + \sum_{n \geq 1} \int_T \|u_j - u_n\| \chi(x) d\mu \\ &= \epsilon + \sum_{n \geq 1} \int_T g_{2^j 3^n} \chi_n(x) d\mu \\ &= \epsilon + \sum_{n \geq 1} q_n(x) \int_T g_{2^j 3^n} d\mu \\ &= \epsilon + \sum_{n \geq 1} q_n(x) \|u_n - u_j\| \leq 3\epsilon. \end{aligned}$$

Since ϵ is arbitrary we obtain the result.

It is left to consider the case in which $L^1(T,E)$ is not separable. Since X-A is separable the covering $\{V_i, i \in I\}$ defined as in previous case, is countable. For each $i \in I$ choose $x_i \in V_i$ and $y_i \in A$ such that $d(x_i, y_i) < 2d(x_i, A)$. Then the analog of the set K of the previous case can be chosen to be $\{f(y_i),\ i \in I\}$ arranged in a sequence $\{u_n,\ n \geq 1\}$; then the proof proceeds in the same way.

Then, as in [BC2], we can state the following consequences.

Corollary 2.

Suppose that $L^1(T,E)$ is separable. Let $D \subseteq L^1(T,E)$ be decomposable and $\Phi_M \subseteq L^1(T,E)$ be a Φ-sublevel, then if $D \cap \Phi_M$ is nonempty and closed, it is a retract of the whole space.

The following is a generalization of results in [C1] and in [F2].

Corollary 3.

Let $D \subseteq L^1(T,E)$ be decomposable and $\Phi_M \subseteq L^1(T,E)$ be a Φ-sublevel. If $K \equiv$

$D \cap \Phi_M$ is nonempty and closed it has the compact fixed point property i. e. every continuous function $f : K \longrightarrow K$ with relatively compact image has a fixed point.

Proof.

Consider $f : K \to K$ with relatively compact image and continuous and call X the closure of the convex hull of $f(K)$; X is compact, hence it is separable. Using Proposition 2 extend the identity map $i : X \cap K \to X \cap K$ to a continuous function $\tilde{i} : X \to K$. The composed function $f \cdot \tilde{i}$ maps X into $X \cap K$. By Schauder Theorem it has a fixed point x_0 which is also a fixed point of f.

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