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Some results about homoclinic orbits for a class of conservative second order Hamiltonian systems

Thesis submitted for the degree of
"Magister Philosophiæ"

CANDIDATE
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SUPERVISOR
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Introduction

In this thesis we study from a variational point of view the conservative second order Hamiltonian system

$$\text{(HS)} \qquad \ddot{q} + V'(q) = 0$$

where $q \in \mathbf{R}^N$ and V is a C^1 real function defined on some subset of $\mathbf{R}^N$. In particular we look for homoclinic orbits, i.e. solutions of (HS) defined on $\mathbf{R}$ and doubly asymptotic, with their derivatives, to some periodic solution of (HS). Actually, we consider the special case in which this periodic orbit is an equilibrium point p for V (i.e. a point where $V'(p) = 0$); in this case the conditions for a solution q of (HS) to be a homoclinic orbit to p reduce to $q(\pm\infty) = p$ and $\dot{q}(\pm\infty) = 0$ and obviously $q \not\equiv p$. Although these kind of orbits were first observed by Poincaré [11], it is only recently that they have been tackled with a variational approach, which seems quite natural for the structure of the problem. In fact, through this approach, several questions have been successfully investigated, also for more general hamiltonian systems as

$$\text{(H)} \qquad \dot{x} = JH'(x)$$

where $x \in \mathbf{R}^{2N}$, $J = \begin{pmatrix} 0_N & -I_N \\ I_N & 0_N \end{pmatrix}$ and a Hamiltonian of the following form:

$$H(x) = \frac{1}{2} x \cdot Ax + R(x)$$

being A a symmetric constant matrix such that JA is hyperbolic (i.e. $\text{sp}(JA) \cap i\mathbf{R} = \emptyset$) and $R(x) = o(|x|^2)$ as $x \to 0$, so that $x = 0$ is a hyperbolic point for H.

A first existence result of homoclinic solutions of (H) was established by Coti–Zelati, Ekeland and Séré [4] under the hypotheses R positive, convex and globally superquadratic (i.e. satisfying $R'(x) \cdot x \geq \alpha R(x)$ for all $x \in \mathbf{R}^{2N}$, with $\alpha > 2$); in this work the solution is obtained as critical point of the dual action functional, using the mountain–pass lemma. The lack of compactness due to the unboundness of the domain is overcome by the concentration–compactness principle of P. L. Lions. Then Hofer and Wysocki [9] could drop the convexity assumption and found a homoclinic orbit applying a linking theorem to the action functional

$$F(x) = \int_{\mathbf{R}} \left(\frac{1}{2} Jx \cdot \dot{x} - H(x) \right) dt$$

defined on $H^{1/2}(\mathbf{R}; \mathbf{R}^{2N})$. The same result was achieved by Tanaka [17] with the method of subharmonic orbits, where the homoclinic orbit $\bar{x}$ is obtained as limit in the C^1_{loc} topology of T–periodic solutions of (H) as $T \to \infty$. Each partial solution x_T is a critical point of the functional

$$F_T(x) = \int_0^T \left(\frac{1}{2} Jx \cdot \dot{x} - H(x) \right) dt$$

defined on $H^{1/2}([0,T];\mathbf{R}^{2N})$ and is obtained with a linking argument. Here the lack of compactness is reflected by the fact that $F(\bar{x}) \leq \lim_{T\to\infty} F_T(x_T)$ whereas the equality is not guaranteed. This method was introduced by Rabinowitz [12] to study the second order system (HS) with V of the form:

$$(\sharp) \qquad V(q) = -\frac{1}{2} q \cdot Lq + W(q)$$

being L a positive definite symmetric matrix and W globally superquadratic and such that $W'(q) = o(|q|)$ as $q \to 0$. Recently, Ambrosetti and Bertotti [1] and Rabinowitz and Tanaka [13], with different tecniques, were able to get the existence of a homoclinic solution for (HS) without the superquadraticity condition, but assuming that the component of $\{x : V(x) < 0\} \cup \{0\}$ containing 0 is bounded and $V'(x) \neq 0$ for any $x \in V^{-1}(0) \setminus \{0\}$. In some sense, the analogous thing was done by Séré [15] for the first order system as (H), supposing that $\Sigma \setminus \{0\}$ is compact and of restricted contact type, where $\Sigma = \{x : H(x) = 0\}$ is the zero–energy surface.

Some of the above–mentioned results are also valid if H —for (H)— or V —for (HS)— depend explicitly on time in a periodic way. This peculiarity gives rise to multeplicity results: in [15], supposing H 1–periodic in time, Séré shows that if x is a homoclinic orbit of (H), then for any finite sequence of integers $n_1 < n_2 < \ldots < n_k$ the function $\sum_{i=1}^{k} x(\cdot + n_i)$ is a quasi–solution provided that $n_{i+1} - n_i$ is large enough (clearly each addend solves (H) but coincides geometrically with x). This idea is used in [5] and [6] to give analogous results in different situations. Nevertheless, it is worth remarking that this multeplicity problem dates back to Poincaré [11], who proved the existence of infinitely many homoclinic orbits, geometrically distinct, provided that the stable and unstable manifolds intersect transversally. Moreover, we mention also the work of Melnikoff [10], who get the same result for the 1–dimensional system $\ddot{q} + f'(t,q) = 0$ with f periodic in time, using perturbation methods.

About the multeplicity of homoclinic solutions in the autonomous case, up to now, there is no literature, apart from a recent paper of Ambrosetti and Coti–Zelati [2] where the authors, with a variational approach and by means of the Ljusternik–Schnirelmann theory, prove the existence of two homoclinic orbits of (HS) for V of the form ($\sharp$), with W superquadratic, satisfying a "pinching" condition:

$$a|q|^{\alpha} \leq W(q) \leq b|q|^{\alpha}$$

provided that $b/a \leq 2^{\frac{\alpha-2}{2}}$.

This thesis presents a multeplicity result concerning a different situation; to be precise, we consider a conservative potential V possessing a strict local maximum at 0 and a singularity at a point $e \neq 0$. Hence, inspirated by [13] and [16], we find homoclinic solutions in two different ways; then we compare them and, under a geometric hypothesis, we can distinguish one from the other. The potential, near the singularity, is assumed satisfying the Gordon condition; however, since a homoclinic orbit is characterized as minimum of the lagrangian functional with respect to a suitable class, we investigate about the possibility to give a weakened condition and find one including the Keplerian case and generalizing an argument given in [7].

1. Existence of a generalized homoclinic orbit

In this section we will deal with a potential $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ with a singularity at a point $e \in \mathbf{R}^N$ and a strict local maximum at a point $p \in \mathbf{R}^N \setminus \{e\}$. Without loss of generality we can suppose $p = 0$ and $V(0) = 0$; so, the origin is an equilibrium for (HS). In addition, we will assume that the component of $\{x \in \mathbf{R}^N \ : \ V(x) < 0\} \cup \{0, e\}$ containing 0 is bounded and will do a suitable hypothesis for V' on $\partial\Omega$.

Definition 1.1. A function $q \in C(\mathrm{I}; \mathbf{R}^N)$, where I is an interval of $\mathbf{R}$, is called *generalized solution of (HS) in* I, *with energy h,* if

(i) $q^{-1}(e)$ is a nullset;

(ii) $q \in C^2(\mathrm{I} \setminus q^{-1}(e); \mathbf{R}^N)$ satisfies (HS) on $\mathrm{I} \setminus q^{-1}(e)$;

(iii) $\frac{1}{2}|\dot{q}(t)|^2 + V(q(t)) = h$ for any $t \in \mathrm{I} \setminus q^{-1}(e)$.

The cardinality of $q^{-1}(e)$ defines the number of collisions of q. If $\mathrm{I} = \mathbf{R}$, $q \not\equiv 0$ and $q(\pm\infty) = \dot{q}(\pm\infty) = 0$, q is said *generalized homoclinic orbit of (HS).* Clearly such a solution has energy zero.

We will find a generalized solution of (HS) on $\mathbf{R}_+ = [0, \infty)$ as critical point of the usual Lagrangian functional

$$I(u) = \int_0^\infty \left(\frac{1}{2}|\dot{u}|^2 - V(u)\right) dt$$

associated to (HS). To be precise, we consider the Hilbert space

$$E = \left\{u \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}_+; \mathbf{R}^N) \ : \ \int_0^\infty |\dot{u}|^2 dt < \infty\right\}$$

equipped with the norm

$$\|u\|^2 = |u(0)|^2 + \int_0^\infty |\dot{u}|^2 dt.$$

We define

$$\Gamma = \{u \in E \ : \ u(0) \in \partial\Omega, \ u(\infty) = 0, \ u(t) \in \overline{\Omega} \ \forall t \in \mathbf{R}_+\}$$

and look for a solution of (HS) on $\mathbf{R}_+$ as minimum of I on Γ.

Theorem 1.2. *If* $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ *verifies*

(V1) $V(0) = 0$ *and* $\lim_{x \to e} V(x) = -\infty$*;*

(V2) there is an open bounded set $\Omega \subset \mathbf{R}^N$ *containing* 0 *and* e *such that* $V(x) < 0$ *for any* $x \in \Omega \setminus \{0, e\}$ *and* $V'(x) \neq 0$ *for* $x \in \partial\Omega \setminus V^{-1}(0)$,

then (HS) *possesses a generalized solution* q *on* $\mathbf{R}_+$, *with energy 0, at most one collision and such that* $q(0) \in \partial\Omega$ *and* $q(\infty) = \dot{q}(\infty) = 0$. *Moreover* $q \in \Gamma$ *and* $I(q) = \inf I(\Gamma)$.

Using the invariance of (HS) under time reflection and considering the function $x \in C(\mathbf{R}; \mathbf{R}^N)$ defined by $x(t) = q(|t|)$, where q is the solution of (HS) given by theorem 1.2, the next result follows immediatly.

Corollary 1.3. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ verifies (V1) and*

(V3) there is an open bounded set $\Omega \subset \mathbf{R}^N$ containing 0 and e such that $V(x) < 0$ for any $x \in \Omega \setminus \{0, e\}$ and $V(x) = 0$ and $V'(x) \neq 0$ for $x \in \partial\Omega$,

then (HS) *possesses a generalized homoclinic orbit with at most one collision.*

To prove theorem 1.2, we follow Rabinowitz and Tanaka [13]; the following lemmas give the corresponding steps of the argument.

Lemma 1.4. *Let $K \subset \Omega \setminus \{0\}$ compact and $u \in E$ such that $u(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}_+$ and $u(t) \in K$ for $t \in \bigcup_{j=1}^{n} (r_j, s_j)$. Then*

$$I(u) \geq \sqrt{2\beta} \sum_{j=1}^{n} |u(r_j) - u(s_j)|$$

where $\beta = \min_{x \in K} -V(x)$.

Proof. By (V2) and for the compactness of K, $\beta > 0$. Moreover, using the Schwarz inequality, we obtain

$$\sum_{j=1}^{n} |u(r_j) - u(s_j)| \leq \tau^{\frac{1}{2}} \left\{ \int_0^\infty |\dot{u}|^2 \, dt \right\}^{\frac{1}{2}}$$

where $\tau = \sum_{j=1}^{n} (s_j - r_j)$. Then, bearing in mind the hypoteses about u, we find that

$$I(u) \geq \frac{1}{2\tau} \left(\sum_{j=1}^{n} |u(r_j) - u(s_j)| \right)^2 + \sum_{j=1}^{n} \int_{r_j}^{s_j} -V(u) \, dt \geq \frac{l^2}{2\tau} + \beta\tau$$

where $l = \sum_{j=1}^{n} |u(r_j) - u(s_j)|$. Therefore $I(u) \geq \inf_{\tau > 0} \left(\frac{l^2}{2\tau} + \beta\tau \right) = l\sqrt{2\beta}$.

q.e.d.

Lemma 1.5. *If $u \in E$, $u(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}_+$ and $I(u) < \infty$ then there exists $\lim_{t \to \infty} u(t) = \xi$ and $V(\xi) = 0$.*

Proof. The set of limit points of u in $\mathbf{R}^N$ as $t \to \infty$ is nonempty because $u(\mathbf{R}_+)$ is bounded. Let ξ be an element of this set. If there didn't exist $\lim_{t \to \infty} u(t)$, then u would cross infinitely often the corona $B_{2r}(\xi) \setminus B_r(\xi)$ with $r > 0$ small enough; hence, by lemma 1.4, it should be $I(u) \geq \sqrt{2\beta}\, r\, n$ for any positive integer n, in contradiction with the hypotesis $I(u) < \infty$. So there exists $\lim_{t \to \infty} u(t) = u(\infty) = \xi$ and since $\int_0^\infty -V(q) dt < \infty$, it must be $V(\xi) = 0$.

q.e.d.

For any $\varepsilon > 0$ sufficiently small call $\Omega_\varepsilon = \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega) > \varepsilon\}$ and

$$\Gamma_\varepsilon = \left\{ u \in E \ : \ u(0) \in \partial\Omega_\varepsilon, \ u(\infty) = 0, \ u(t) \in \overline{\Omega}_\varepsilon \ \forall t \in \mathbf{R}_+ \right\}.$$

Lemma 1.6. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ verifies (V1) and (V2) then there exists $q \in E$ such that:*

(i) $q \in \Gamma_\varepsilon$ and $I(q) = \inf I(\Gamma_\varepsilon)$;

(ii) $q(t) \in \Omega_\varepsilon$ for $t > 0$;

(iii) q is a generalized solution of (HS) on $\mathbf{R}_+$, with energy 0, at most one collision and such that $q(\infty) = \dot{q}(\infty) = 0$.

Proof. (i). Clearly $0 \leq \inf I(\Gamma_\varepsilon) < \infty$ and given a sequence $(u_n) \subset \Gamma_\varepsilon$ such that $I(u_n) \to \inf I(\Gamma_\varepsilon)$, it holds that (q_n) is bounded in E and so contains a subsequence converging to some $q \in E$ weakly in E and uniformly on the compact sets of $\mathbf{R}_+$. Then, by pointwise convergence, $q(0) \in \partial\Omega_\varepsilon$, $q(t) \in \overline{\Omega}_\varepsilon$ for all $t \in \mathbf{R}_+$ and, via Fatou lemma, $\int_0^\infty -V(q)\,dt \leq \liminf \int_0^\infty -V(u_n)\,dt$. On the other hand, using the weak convergence, $\liminf \int_0^\infty |\dot{u}_n|^2 dt \geq \int_0^\infty |\dot{q}|^2 dt$. Therefore $I(q) \leq \liminf I(u_n) = \inf I(\Gamma_\varepsilon)$; thus, by lemma 1.5, it follows that there exists $\lim_{t\to\infty} q(t) = q(\infty)$ and $q(\infty) \in V^{-1}(0) \cap \overline{\Omega}_\varepsilon = \{0\}$. Hence $q \in \Gamma_\varepsilon$ and $I(q) = \min I(\Gamma_\varepsilon)$.

(ii). By contradiction, if $q(t_o) \in \partial\Omega_\varepsilon$ for some $t_o > 0$, then, called $Q = q(\cdot + t_o)$, it would hold that $Q \in \Gamma_\varepsilon$ and $I(Q) < I(q)$, while $I(q) = \min I(\Gamma_\varepsilon)$.

(iii). To begin, it can be noticed that, by the part *(i)*, q is an injective function; otherwise, if it were $q(t_1) = q(t_2)$ for some $t_2 > t_1 \geq 0$, we could consider a new function Q defined by

$$Q(t) = \begin{cases} q(t) & 0 \leq t \leq t_1 \\ q(t + t_2 - t_1) & t > t_1 \end{cases}$$

and observe that $Q \in \Gamma_\varepsilon$ and $I(Q) < \min I(\Gamma_\varepsilon)$. Then, in particular, we deduce that $q^{-1}(e)$ is at most a singleton. Now, to show that q is a weak solution of (HS), we take an arbitrary $h \in C_c^\infty((0, +\infty); \mathbf{R}^N)$ and notice that $q + sh \in \Gamma_\varepsilon$ if $|s|$ is small enough. It's an ordinary exercise to see that

$$\lim_{s\to 0} \frac{1}{s}\{I(q + sh) - I(q)\} = \int_0^\infty (\dot{q} \cdot \dot{h} - V'(q) \cdot h)dt$$

and, being $I(q) = \min I(\Gamma_\varepsilon)$, it follows that $\int_0^\infty (\dot{q} \cdot \dot{h} - V'(q) \cdot h)\,dt = 0$. So, by the arbitrary of h, q is weak solution of (HS) on $\mathbf{R}_+$. To see that q is classical solution, take a generic interval $[r, s] \subset (0, \infty) \setminus q^{-1}(e)$ and point out that $\int_r^s (\dot{q} \cdot \dot{h} - V'(q) \cdot h)\,dt = 0$ for any $h \in W_0^{1,2}([r, s]; \mathbf{R}^N)$. If we consider the Cauchy problem (in x):

$$\begin{cases} \ddot{x} + V'(q) = 0 \\ x(r) = q(r) \\ x(s) = q(s) \end{cases}$$

this admits a classical solution $x \in C^2([r, s]; \mathbf{R}^N)$ satisfying $\int_r^s (\dot{x} \cdot \dot{h} - V'(q) \cdot h)\,dt = 0$ for any $h \in W_0^{1,2}([r, s]; \mathbf{R}^N)$. Then $\int_r^s (\dot{q} - \dot{x}) \cdot \dot{h}\ dt = 0$ for any $h \in W_0^{1,2}([r, s]; \mathbf{R}^N)$ and this, with the initial conditions of the previous Cauchy problem, implies $q = x$ on $[r, s]$. Hence, by the arbitrariness of $[r, s]$, q belongs to $C^2((0, +\infty) \setminus q^{-1}(e); \mathbf{R}^N)$ and satisfies (HS) on $(0, +\infty) \setminus q^{-1}(e)$ in a classical sense. Since V does not depend on t, the energy is constant, i.e.

$$\frac{1}{2}|\dot{q}|^2 + V(q(t)) = h_o \quad \text{for } t \in (0,\tau)$$
$$\frac{1}{2}|\dot{q}|^2 + V(q(t)) = h_\infty \quad \text{for } t > \tau$$

where $\tau = \max\{0, q^{-1}(e)\}$. Knowing that $q \in E$ and $I(q) < \infty$ and using the energy equation for $t > \tau$, we obtain $h_\infty = 0$ and $\dot{q}(\infty) = 0$. If $q^{-1}(e) = \tau > 0$ take the following function:

$$Q(t) = \begin{cases} q(\lambda t) & 0 \le t \le \frac{\tau}{\lambda} \\ q(t + \tau - \frac{\tau}{\lambda}) & t > \frac{\tau}{\lambda} \end{cases}$$

with a suitable $\lambda > 0$. It's clear that $Q \in \Gamma_\varepsilon$ and

$$\int_{\frac{\tau}{\lambda}}^{\infty} (\frac{1}{2}|\dot{Q}|^2 - V(Q))\,dt = \int_{\tau}^{\infty} (\frac{1}{2}|\dot{q}|^2 - V(q))\,dt$$
$$\int_0^{\frac{\tau}{\lambda}} (\frac{1}{2}|\dot{Q}|^2 - V(Q))\,dt = \lambda \int_0^{\tau} \frac{1}{2}|\dot{q}|^2 dt + \frac{1}{\lambda}\int_0^{\tau} -V(q)\,dt$$

To minimize the last expression we choose $\lambda = \sqrt{\frac{P}{K}}$ where $P = \int_0^\tau -V(q)\,dt$ and $K = \int_0^\tau \frac{1}{2}|\dot{q}|^2 dt$. In this way we find that $\int_0^{\frac{\tau}{\lambda}} (\frac{1}{2}|\dot{Q}|^2 - V(Q))\,dt = 2\sqrt{KP}$. Since $I(q) = \inf I(\Gamma_\varepsilon)$, it must be $K + P \le 2\sqrt{KP}$, that is $K = P$. But the conservation of the energy implies that $K - P = h_o\tau$; then $h_o = 0$ because $\tau > 0$.

q.e.d.

Proof of theorem 1.2. For any $\varepsilon > 0$ small enough let $q_\varepsilon \in E$ be the generalized solution of (HS) on $\mathbf{R}_+$ given by lemma 1.6. It can be readly seen that for any $u \in \Gamma$ there is $v \in \Gamma_\varepsilon$ with $I(v) < I(u)$. Then for any ε we have $I(q_\varepsilon) \le \inf I(\Gamma)$ and in particular $\|q_\varepsilon\| \le$ constant. Therefore there exist a sequence $\varepsilon_n \to 0$ and a function q to which (q_{ε_n}) converges, weakly in E and uniformly on the compact subsets of $\mathbf{R}_+$. Then, arguing as in the proof of lemma 1.6, part *(i)*, we obtain that $q(0) \in \partial\Omega$, $q(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}_+$, $I(q) \le \inf I(\Gamma)$ and, since $I(q) < \infty$, by lemma 1.5, there exists $\lim_{t\to\infty} q(t) = \xi$, $V(\xi) = 0$ and then $\xi \in \partial\Omega \cup \{0\}$. If $\xi \in \partial\Omega$, *(V2)* implies $V'(\xi) \ne 0$. Therefore there exists $r > 0$ such that $0, e \notin \overline{B_{2r}(\xi)}$ and $|V'(x) - V'(\xi)| \le \frac{1}{2}|V'(\xi)|$ for any $x \in B_{2r}(\xi)$. Moreover there is $t_o > 1$ such that $q(t) \in B_{\frac{r}{2}}(\xi)$ if $t \ge t_o$. Let T be an arbitrary positive number. By the uniform convergence on the compact sets, if n is sufficiently large $q_{\varepsilon_n}(t) \in B_r(\xi)$ for all $t \in [t_o, t_o + T]$ and so

$$|\dot{q}_{\varepsilon_n}(t) - \dot{q}_{\varepsilon_n}(t_o)| \ge \left|\int_{t_o}^t \ddot{q}_{\varepsilon_n} ds\right| \ge \left|\int_{t_o}^t V'(q_{\varepsilon_n})\,ds\right| \ge$$
$$\ge (t - t_o)|V'(\xi)| - \int_{t_o}^t |V'(\xi) - V'(q_{\varepsilon_n})|\,ds \ge \frac{1}{2}(t - t_o)|V'(\xi)|$$

and consequently

$$\int_{t_o}^{t_o+T} |\dot{q}_{\varepsilon_n}(t)|^2 dt \ge \frac{1}{T}\left\{\frac{1}{4}|V'(\xi)|T^2 - |\dot{q}_{\varepsilon_n}(t_o)|T\right\}^2 .$$

Since $|\dot{q}_{\varepsilon_n}(t_o)|^2 = -2V((q_{\varepsilon_n}(t_o)) \le \sup\{-2V(x) : x \in B_r(\xi) \cap \Omega\} = c^2$, we can say that for any $T > 0$ there is a positive integer n, in general depending on T, such that

$$\int_0^\infty |\dot{q}_{\varepsilon_n}|^2 dt \ge \frac{1}{T}\left\{\frac{1}{4}|V'(\xi)|T^2 - cT\right\}^2 .$$

From $V'(\xi) \ne 0$ we deduce that the sequence (q_{ε_n}) is not bounded in E, in contradiction with a relation found at the beginning. Hence $q(\infty) = 0$. It follows that $q \in \Gamma$, $I(q) = \min I(\Gamma)$, q is injective and $q^{-1}(e)$ is at most a singleton. Moreover, possibly for a subsequence, (q_{ε_n}) converges to q in $C^2_{\rm loc}(\mathbf{R}_+ \setminus q^{-1}(e); \mathbf{R}^N)$ so that q is a classical solution of (HS) on $\mathbf{R}_+ \setminus q^{-1}(e)$ with energy zero. In particular $\dot{q}(\infty) = 0$.

q.e.d.

2. Non collision orbits

We consider again a potential $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ satisfying the hypotheses *(V1)* and *(V2)* of theorem 1.2; now we are interested in finding conditions for V near the singularity assuring the existence of non collision solutions. This will be possible in the case of a strong–force potential or when V is a radial function near the singularity, with a suitable behaviour.

Proposition 2.1. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ verifies (V1), (V2) and*

(SF) there is a neighborhood N_e of e in Ω and a function $U \in C^1(N_e \setminus \{e\}; \mathbf{R})$ such that $|U(x)| \to \infty$ as $x \to e$ and $-V(x) \ge |U'(x)|^2$ for any $x \in N_e \setminus \{e\}$,

then the solution q of (HS) *given by theorem 1.2 is a non collision orbit; therefore q is a classical solution on $\mathbf{R}_+$.*

Proof. It's enough to notice that, in general, if $u \in \Gamma$ and $I(u) < \infty$ then $u(t) \ne e$ for all $t \in \mathbf{R}_+$. In fact, by contradiction, if there were some $t > 0$ with $u(t) = e$ then there would exist $t_o \in (0, t)$ such that $u(t_o) \in \partial N_e$ and $u(s) \in N_e \setminus \{e\}$ for any $s \in (t_o, t)$. Therefore

$$\begin{aligned}
|U(u(s))| &\le |U(u(s)) - U(u(t_o))| + |U(u(t_o))| \\
&\le \left|\int_{t_o}^s U'(u(\tau)) \cdot \dot{u}(\tau)\, d\tau\right| + |U(u(t_o))| \\
&\le \left\{\int_{t_o}^s |U'(u(\tau))|^2 d\tau\right\}^{\frac{1}{2}} \left\{\int_{t_o}^s |\dot{u}(\tau)|^2 d\tau\right\}^{\frac{1}{2}} + |U(u(t_o))| \\
&\le \left\{\int_{t_o}^s -V(u(\tau))\, d\tau\right\}^{\frac{1}{2}} \left\{\int_0^\infty |\dot{u}(\tau)|^2 d\tau\right\}^{\frac{1}{2}} + |U(u(t_o))| \\
&\le \sqrt{2} I(u) + |U(u(t_o))| < \infty
\end{aligned}$$

while $|U(u(s))| \to \infty$ as $s \to t_-$.

q.e.d.

Proposition 2.2. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ verifies (V1), (V2) and*

(V4) there exist a constant $r_\circ > 0$ and a function $\phi \in C^1((0, r_\circ); \mathbf{R})$ such that $V(x) = \phi(|x-e|)$ for all $x \in B_{r_\circ}(e) \setminus \{e\}$ and $r\,\phi'(r) \to \infty$ as $r \to 0_+$

then the solution q of (HS) *given by theorem 1.2 is a non collision orbit.*

Remark 2.3. If (V4) holds, then $\phi(r)/\log r \to \infty$ as $r \to 0_+$ (but in general the viceversa is not true).
In the case $V(x) = -\big|\log|x-e|\big|^\beta$ for $x \in B_{r_\circ}(e) \setminus \{e\}$, the condition (V4) is satisfied if and only if $\beta > 1$. Moreover we note that (V4) is verified by potentials with the following behaviour:

$$V(x) = -\frac{1}{|x-e|^\alpha} + \varphi(|x-e|) \qquad \text{for} \quad x \in B_{r_\circ}(e) \setminus \{e\}$$

with $\alpha > 0$ and $\varphi \in C^1((0, r_\circ); \mathbf{R})$ such that $\lim_{r \to 0_+} r^{1+\alpha}\varphi'(r) \in (-\alpha, +\infty]$. It is also clear that this characterization does not exhaust all the cases described by *(V4)*.

Before proving the previous proposition, we recall the definition and the fundamental properties of the convexified of a real function.
Let I be an interval of $\mathbf{R}$ and let f be a function from I into $\mathbf{R}$, bounded from below. We call *convexified of f on* I the function $f_* : \mathrm{I} \to \mathbf{R}$ defined by:

$$f_*(x) = \sup_{(a,b) \in \mathcal{A}(f)} (ax + b) \qquad \text{for all } x \in \mathrm{I}$$

where $\mathcal{A}(f) = \{(a,b) \in \mathbf{R}^2 : ax + b \le f(x)\ \forall x \in \mathrm{I}\}$. It holds that $f_* \le f$ on I and $f_* = f$ if and only if f is convex and lower semicontinuous on I.
Moreover, given two applications $f, g : \mathrm{I} \to \mathbf{R}$ bounded from below, if $f \le g$ on I, then $f_* \le g_*$ on I.
In the sequel, we will make use of the following result.

Lemma 2.4. *Let $f \in C((0,a); \mathbf{R})$ such that $x\, f(x) \to \infty$ as $x \to 0$. Then $f_* \in C((0,a); \mathbf{R})$ and $x\, f_*(x) \to \infty$ as $x \to 0$.*

Proof. The continuity of f_* at an arbitrary point $x \in (0,a)$ follows by the convexity of f_* and by the fact that f_* is bounded from below; in fact $f_* \le f$ and f is continuous. In addition, it holds that for any $M > 0$ there is some $x_M \in (0,a)$ for which $f(x) \ge \frac{M}{x}$ if $x \in (0, x_M)$. Therefore, noticing that the function $x \mapsto \frac{M}{x}$ is convex, we obtain that $f_*(x) \ge \frac{M}{x}$ for any $x \in (0, x_M)$. Hence $x\, f_*(x) \to \infty$ as $x \to 0$.

q.e.d.

Proof of proposition 2.2. Let $q \in E$ be the solution of (HS) given by Theorem 1.2. The argument will be by contradiction, supposing $q^{-1}(e) = \{\tau\}$. For any $r \in (0, r_o)$, $t_1, t_2 > 0$ are uniquely determinated by these conditions: $t_1 < \tau < t_2$, $q(t_i) \in \partial B_r(e)$ for $i = 1, 2$ and $q(t) \in B_r(e)$ for $t \in (t_1, t_2)$. We point out that, since V is radial on $B_{r_o}(e)$, as long as $q(t) \in B_{r_o}(e)$, the angular momentum does not change throughout the motion, and its value is zero, because the orbit goes through the singularity. Thus q follows a straight line inside $B_{r_o}(e)$, even if along possibly different directions, before and after the collision. To be precise:

$$q(t) = e + \rho(t)\, e_1 \quad \text{if } t \in (t_1, \tau)$$
$$q(t) = e + \rho(t)\, e_2 \quad \text{if } t \in (\tau, t_2)$$

where $\rho(t) = |q(t) - e|$, $e_i = \frac{q(t_i)-e}{|q(t_i)-e|}$ $(i = 1, 2)$. The contradiction will be reached constructing a function $Q \in \Gamma$ such that $I(Q) < I(q)$.

To begin, we show that the singularity is crossed without change of direction. In fact, if not, consider the function Q different from q only for $t \in (t_1, t_2)$, where is defined as projection of the motion q along the segment joining $q(t_1)$ to $q(t_2)$. Explicitly:

$$Q(t) = \begin{cases} q(t) & t \in [0, t_1] \cup [t_2, +\infty) \\ \frac{1}{2}(q(t_1) + q(t_2)) + \hat{e} \cdot (q(t) - e)\, \hat{e} & t \in (t_1, t_2) \end{cases}$$

where $\hat{e} = \frac{e_2 - e_1}{|e_2 - e_1|}$ (notice that $e_1 \neq e_2$ since q is injective). It can easily checked that $Q \in \Gamma$, $\int_{t_1}^{t_2} |\dot{Q}|^2 dt = \frac{1 - e_1 \cdot e_2}{2} \int_{t_1}^{t_2} |\dot{q}|^2 dt$ and $\int_{t_1}^{t_2} -V(Q) dt \leq \int_{t_1}^{t_2} -V(q) dt$. Then, if $e_1 \neq -e_2$, $I(Q) < I(q)$. In the case $e_1 = -e_2$ take $Q = q + g e_o$ where e_o is a fixed vector of $\mathbf{R}^N$ with norm one, orthogonal to e_1 and g is a scalar function defined in the following way:

$$g(t) = \begin{cases} \frac{t - t_1}{\delta} \mu & t \in (t_1, t_1 + \delta) \\ \mu & t \in (t_1 + \delta, t_2 - \delta) \\ \frac{t_2 - t}{\delta} \mu & t \in (t_2 - \delta, t_2) \\ 0 & t \in [0, t_1] \cup [t_2, +\infty) \end{cases}$$

with appropriate $\mu, \delta > 0$. Observe that $Q \in \Gamma$ and $|Q(t) - e| \leq r$ for $t \in [t_1, t_2]$ if μ and δ are small enough. Moreover, called $h = g e_o$, it holds that

$$\int_{t_1}^{t_2} |\dot{Q}|^2 dt = \int_{t_1}^{t_2} |\dot{q}|^2 dt + 2\frac{\mu^2}{\delta}$$
$$\int_{t_1}^{t_2} \left(V(q) - V(Q)\right) dt = -\int_{t_1}^{t_2} \left(\int_0^1 V'(q + \lambda h) \cdot h \, d\lambda\right) dt.$$

Let f be the convexified of ϕ' on $(0, r_o)$. Since $r\phi'(r) \to +\infty$ as $r \to 0_+$, by lemma 2.4, $rf(r) \to +\infty$ and, changing r_o if necessary, we can say that $f(r) > 0$ for $r \in (0, r_o)$. Being $q(t) + \lambda h(t) \in B_r(e)$ for $t \in (t_1, t_2)$ and $\lambda \in (0, 1)$, it holds that

$$V'(q + \lambda h) \cdot h = \phi'(|q + \lambda h - e|) \frac{(q + \lambda h - e) \cdot h}{|q + \lambda h - e|} \geq$$
$$\geq \lambda |h|^2 \frac{f(|q + \lambda h - e|)}{|q + \lambda h - e|} \geq \lambda |h|^2 \frac{f(|q - e| + |h|)}{|q - e| + |h|}$$

and thus

$$\int_{t_1}^{t_2} (V(q) - V(Q))\, dt \le -\frac{1}{2} \int_{t_1}^{t_2} |h|^2 \frac{f(|q-e|+|h|)}{|q-e|+|h|}\, dt.$$

Since $|q(t) - e| \le \|\dot q\|_2 |t-\tau|^{\frac{1}{2}} = C|t-\tau|^{\frac{1}{2}}$, then $|q(t)-e| \le \mu$ if $|t-\tau| \le \sigma_\mu = \frac{\mu^2}{C^2}$. Therefore

$$\int_{t_1}^{t_2} (V(q) - V(Q))\, dt \le -\frac{1}{2} \int_{\tau-\sigma_\mu}^{\tau+\sigma_\mu} |h|^2 \frac{f(|q-e|+|h|)}{|q-e|+|h|}\, dt \le -\frac{1}{2}\frac{f(2\mu)}{2\mu} \int_{\tau-\sigma_\mu}^{\tau+\sigma_\mu} |h|^2 dt.$$

Taking μ so that $[\tau-\sigma_\mu, \tau+\sigma_\mu] \subseteq [t_1+\delta, t_2-\delta]$ we conclude that

$$\int_{t_1}^{t_2} (V(q) - V(Q)) dt \le -\frac{\mu^3}{2C^2} f(2\mu)$$

and finally:

$$I(Q) - I(q) \le -\mu^2 \left(-\frac{1}{\delta} + \frac{\mu}{2C^2} f(2\mu) \right)$$

But we know that $r\, f(r) \to +\infty$ as $r \to 0_+$; so, choosing μ sufficiently small, we find $I(Q) < I(q)$.

q.e.d.

3. Case Ω unbounded, $\Omega \ne \mathbf{R}^N$

In the previous sections we considered a system (HS) ruled by a potential V with a strict local maximum at 0, a singularity at $e \in \mathbf{R}^N \setminus \{0\}$ and such that the component of $\{x \in \mathbf{R}^N : V(x) < 0\} \cup \{0, e\}$ containing 0 was bounded.

Now we want to eliminate this last hypothesis and, to do this, we will adopt the following strategy: given an arbitrary $R > 0$ sufficiently large, the set $\Omega_R = \Omega \cap B_R$ is open, non-empty, bounded, contains 0 and $V(x) < 0$ for $x \in \Omega_R \setminus \{0, e\}$ (here $B_R = \{x \in \mathbf{R}^N : |x| < R\}$); thus, with respect to Ω_R, the hypotheses *(V1)* and *(V2)* of theorem 1.2 are satisfied. Then we find a (generalized) solution q_R of (HS) on $\mathbf{R}_+$ such that $q_R(0) \in \partial\Omega_R$, $q_R(\infty) = 0$ and $q_R(t) \in \Omega_R$ for $t > 0$. Hence, if $q_R(0) \in \partial\Omega$ for some R, we can reflect q_R, by defining $y(t) = q_R(|t|)$ for $t \in \mathbf{R}$, to obtain a homoclinic orbit for (HS).

So, the problem is to show that it cannot happen the situation $|q_R(0)| \to \infty$ as $r \to \infty$. Firstly we observe that $\int_0^\infty |\dot q_R|^2 dt \le$ constant. In fact, we know that $q_R \in \Gamma_R$ and $I(q_R) = \inf I(\Gamma_R)$ where $\Gamma_R = \{u \in E : u(0) \in \partial\Omega_R, u(\infty) = 0, u(t) \in \overline{\Omega}_R\ \forall t \in \mathbf{R}_+\}$. Given any $u \in \Gamma$ there is $v \in \Gamma_R$ such that $I(v) < I(u)$. Then $I(q_R) \le \inf I(\Gamma) < \infty$ for any $R > R_o$. Now, if we suppose $|q_R(0)| = R$ for all $R > R_o$, to obtain a contradiction, we must do some hypothesis about the behaviour of V at infinity (on Ω) and about V and V' in $\partial\Omega$.

It is useful to distinguish the case $\partial\Omega$ bounded, i.e. $\mathbf{R}^N \setminus \Omega$ bounded, and $\partial\Omega$ unbounded.

Theorem 3.1. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ satisfies (V1),*

(V3') there is an open connected set $\Omega \subset \mathbf{R}^N$ containing 0 and possibly e, with $\mathbf{R}^N \setminus \Omega$ bounded, such that $V(x) < 0$ for any $x \in \Omega \setminus \{0, e\}$, $V(x) = 0$ and $V'(x) \neq 0$ for $x \in \partial\Omega$,

(V5) there are a constant $R_o > 0$ and a function $U \in C^1(\mathbf{R}^N \setminus B_{R_o}; \mathbf{R})$ such that $|U(x)| \to \infty$ as $|x| \to \infty$ and $-V(x) \geq |U'(x)|^2$ for $|x| \geq R_o$,

then (HS) *admits a generalized solution q on $\mathbf{R}_+$, with energy 0, at most one collision and such that $q(0) \in \partial\Omega$ and $q(\infty) = \dot{q}(\infty) = 0$. Moreover the function $x \in C(\mathbf{R}; \mathbf{R}^N)$, defined by $x(t) = q(|t|)$, is a homoclinic generalized orbit for* (HS).

Proof. Since $q_R \in \Gamma_R$, for any $R > R_o$ there is $t_R \in \mathbf{R}_+$ such that $q_R(t_R) = |q_R|_\infty$; if $|q_R(t_R)| > R_o$ there is $s_R > t_R$ such that $|q_R(s_R)| = R_o$. Then, with the same passages used to prove proposition 2.1, we obtain

$$|U(q_R(t_R))| \leq \sqrt{2}I(q_R) + |U(q_R(s_R))| \leq \sqrt{2}\inf I(\Gamma) + \max_{|x|=R_o} |U(x)|.$$

By the previous argument, this uniform boundness on $|q_R|_\infty$ implies the thesis.

q.e.d.

Remark 3.2. Notice that *(V5)* is formally the strong–force condition at infinity; actually, in the case $V(x) = -\frac{1}{|x|^\alpha}$ for $|x|$ large, it is verified when $\alpha \leq 2$.

Theorem 3.3. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ satisfies (V1),*

(V3") there is an open connected set $\Omega \subset \mathbf{R}^N$, containing 0 and possibly e, with $\partial\Omega$ unbounded, such that $V(x) < 0$ for any $x \in \Omega \setminus \{0, e\}$, $V(x) = 0$ for $x \in \partial\Omega$, $|V'(x)| \geq c > 0$ for $x \in \partial\Omega$;

(V6) V and V' are Lipschitzian on $A_\delta = \{x \in \Omega : V(x) > -\delta,\ |x| > \delta\}$ for some $\delta > 0$;

(V7) $x_n \to \partial\Omega \cup \{0\}$ for any sequence $(x_n) \in \Omega$ such that $V(x_n) \to 0$;

then (HS) *admits a generalized solution q on $\mathbf{R}_+$, with energy 0, at most one collision and such that $q(0) \in \partial\Omega$ and $q(\infty) = \dot{q}(\infty) = 0$. Moreover the function $y \in C(\mathbf{R}; \mathbf{R}^N)$, defined by $y(t) = q(|t|)$, is a homoclinic generalized orbit for* (HS).

Remark 3.4. The hypothesis *(V7)* is equivalent to require that $\inf_{x \in \Omega_\varepsilon \setminus B_\varepsilon} |V(x)| > 0$ for any $\varepsilon > 0$ small enough.

Proof. Suppose $|q_R(0)| = R$ for all $R > R_o$; to reach a contradiction, we follow this scheme: for $\varepsilon \in (0, \delta)$ and $R > R_o$ let $A_\varepsilon = \{x \in \Omega : V(x) > -\varepsilon,\ |x| > \delta\}$ and $\Delta_R^\varepsilon = \{t \in \mathbf{R}_+ : q_R(t) \in A_\varepsilon\}$;
1) if $(t, t+T) \in \Delta_R^\varepsilon$ then $|V'(q_R(t))| \leq \left(M_1 T + \frac{2}{T}\right)\sqrt{2\varepsilon}$ where M_1 is the Lipschitz constant of V' on A_δ;
2) $\forall k > 0\ \forall \varepsilon \in (0, \delta)\ \exists r > R_o$ s.t. $\forall R \geq r\ |\Delta_R^\varepsilon| > k$;
let T_R^ε be the length of the largest interval of Δ_R^ε;

3) $\forall \varepsilon \in (0,\delta)\ \exists r > R_o$ s.t. $\forall R \geq r\ T_R^\varepsilon \geq 1$;
4) $\forall \varepsilon \in (0,\delta)\ \exists x_\varepsilon \in A_\varepsilon$ s.t. $|V'(x_\varepsilon)| \leq (M_1+2)\sqrt{2\varepsilon}$;
5) $\inf\{|V'(x)| : x \in \partial\Omega\} = 0$.
So the contradiction is obtained with respect to the hypothesis $|V'(x)| \geq c > 0$ for all $x \in \partial\Omega$. Now we prove in detail the statements 1) – 5).

1)

$$\begin{aligned}
|V'(q_R(t))| &= \frac{1}{T}\left|\int_t^{t+T}(V'(q_R(t)) - V'(q_R(s)))ds + \int_t^{t+T} V'(q_R(s))ds\right| \\
&\leq \frac{1}{T}\left\{\int_t^{t+T}|V'(q_R(t)) - V'(q_R(s))|ds + \left|\int_t^{t+T}\ddot{q}_R(s)ds\right|\right\} \\
&\leq \frac{1}{T}\left\{M_1\int_t^{t+T}|q_R(t)-q_R(s)|ds + |\dot{q}_R(t+T) - \dot{q}_R(t)|\right\} \\
&\leq \frac{1}{T}\left\{M_1\int_t^{t+T}ds\int_t^s d\sigma|\dot{q}_R(\sigma)| + |\dot{q}_R(t+T)| + |\dot{q}_R(t)|\right\} \\
&\leq \frac{1}{T}\left\{M_1T^2\sqrt{2\varepsilon} + 2\sqrt{2\varepsilon}\right\}
\end{aligned}$$

since $|\dot{q}_R(s)| = \sqrt{-2V(q_R(s))} \leq \sqrt{2\varepsilon}$ for $s \in \overline{\Delta_R^\varepsilon}$.

2) Let $s_R \in \mathbf{R}_+$ be the first instant at that q_R touches B_δ, i.e. $q_R(s_R) \in \partial B_\delta$ but $|q_R(t)| > \delta$ for $t \in [0, s_R)$. Fix an arbitrary $\varepsilon \in (0,\delta)$. On the one hand it holds:

$$\begin{aligned}
\int_0^\infty \frac{1}{2}|\dot{q}_R|^2dt \geq \int_{[0,s_R]\setminus\Delta_R^\varepsilon}\frac{1}{2}|\dot{q}_R|^2dt \geq \int_{[0,s_R]\setminus\Delta_R^\varepsilon} -V(q_R)dt \\
\geq \varepsilon\,|[0,s_R]\setminus\Delta_R^\varepsilon| \geq \varepsilon\,(s_R - |\Delta_R^\varepsilon|).
\end{aligned}$$

On the other hand:

$$(R-\delta)^2 \leq |q_R(0) - q_R(s_R)|^2 \leq \left(\int_0^{s_R}|\dot{q}_R|dt\right)^2 \leq s_R\int_0^\infty |\dot{q}_R|^2dt.$$

Hence, knowing that $\int_0^\infty |\dot{q}_R|^2dt \leq$ constant, we obtain

$$|\Delta_R^\varepsilon| \geq A(R-\delta)^2 - \frac{B}{\varepsilon}$$

with A and B positive constants independent on R and ε. Since $A(R-\delta)^2 - \frac{B}{\varepsilon} \to \infty$ as $R \to \infty$ uniformly with respect to ε, 2) follows.

3) Suppose 3) false, so that there is $\varepsilon_o > 0$ with the following property: for any $r > R_o$ there exists $R \geq r$ such that $T_R^{\varepsilon_o} < 1$. Put $k = 1, 2, \ldots, n, \ldots$ in 2), finding a sequence (r_n) with $r_n > R_o$ and such that $|\Delta_R^\varepsilon| > n$ if $R \geq r_n$, for any $n \in \mathbf{N}$. Now, by the previous assumption with $r = r_1, r_2, \ldots$, we have a sequence (R_n), in general depending on ε, such that $\left|\Delta_{R_n}^\varepsilon\right| > n$ and $T_{R_n}^{\varepsilon_o} < 1$ for any

$n \in \mathbf{N}$. We choose $\varepsilon = \frac{1}{2}\varepsilon_o$ and study the passage of each orbit q_{R_n} through the strip $A_\varepsilon \setminus A_{\frac{\varepsilon}{2}}$. Since $\Delta_{R_n}^{\varepsilon_o/2}$ is not empty, there exists $\tau_n = \inf \Delta_{R_n}^{\varepsilon_o/2}$; let $s_n = \sup\{t > \tau_n : s \in \Delta_{R_n}^{\varepsilon_o/2} \ \forall s \in (\tau_n, t)\}$ and $t_n = \inf\{t > s_n : t \neq \Delta_{R_n}^{\varepsilon_o}\}$. We point out that $q_{R_n}(s_n) \in A_{\frac{\varepsilon_o}{2}}$, $q_{R_n}(t_n) \in A_{\varepsilon_o}$ and $q_{R_n}(t) \in A_{\varepsilon_o} \setminus A_{\varepsilon_o/2}$ for $t \in (s_n, t_n)$, i.e. $V(q_{R_n}(s_n)) = -\frac{\varepsilon_o}{2}$, $V(q_{R_n}(t_n)) = -\varepsilon_o$ and $\frac{\varepsilon_o}{2} \leq -V(q_{R_n}(t)) \leq \varepsilon_o$ for any $t \in (s_n, t_n)$. Therefore:

$$\text{(1)} \qquad \int_{s_n}^{t_n} |\dot{q}_{R_n}|^2 dt = \int_{s_n}^{t_n} -2V(q_{R_n}) dt \geq \varepsilon_o(t_n - s_n).$$

For V is Lipschitzian on A_{ε_o} (with constant M_o), we deduce that:

$$\text{(2)} \qquad \begin{aligned} \frac{\varepsilon_o}{2} &= V(q_{R_n}(s_n)) - V(q_{R_n}(t_n)) \leq \\ &\leq M_o|q_{R_n}(t_n) - q_{R_n}(s_n)| \leq M_o \int_{s_n}^{t_n} |\dot{q}_{R_n}| dt \leq M_o\sqrt{2\varepsilon}(t_n - s_n). \end{aligned}$$

(1) and (2) give

$$\int_{s_n}^{t_n} |\dot{q}_{R_n}|^2 dt \geq \frac{1}{M_o}\left(\frac{\varepsilon_o}{2}\right)^{\frac{3}{2}} = c_o > 0.$$

Since $T_{R_n}^{\varepsilon_o} < 1$, $t_n - s_n < 1$ and the previous argument must be repeated at least n times, because $|\Delta_{R_n}^{\varepsilon_o/2}| > n$. Then $\int_0^\infty |\dot{q}_{R_n}|^2 dt \geq c_o n$ for any $n \in \mathbf{N}$, in contradiction with the boundness $\int_0^\infty |\dot{q}_{R_n}|^2 dt \leq$ constant.

4) The statement 3) implies that for any $\varepsilon \in (0, \delta)$ there are $R_\varepsilon > R_o$ and $t_{R_\varepsilon} \in \mathbf{R}_+$ such that $(t_{R_\varepsilon}, t_{R_\varepsilon} + 1) \subseteq \Delta_{R_\varepsilon}^{\varepsilon}$. Then, apply 1), call $x_\varepsilon = q_{R_\varepsilon}(t_{R_\varepsilon})$ and, after having noticed that $x_\varepsilon \in A_\varepsilon$, 4) readly follows.

5) For any $\varepsilon \in (0, \delta)$ we determinate $x_\varepsilon \in \Omega$ according to the statement 4). Since $x_\varepsilon \in A_\varepsilon$, $V(x_\varepsilon) \to 0$ as $\varepsilon \to 0$ and so, by *(V7)* and being $A_\varepsilon \subset \Omega \setminus B_\delta$, $x_\varepsilon \to \partial\Omega$. Let $\delta_\varepsilon = \text{dist}(x_\varepsilon, \partial\Omega)$. Since $\delta_\varepsilon > 0$, there is $y_\varepsilon \in \partial\Omega$ such that $|x_\varepsilon - y_\varepsilon| < 2\delta_\varepsilon$. For V' is Lipschitzian on A_δ, it holds that $|V'(y_\varepsilon)| \leq M_1|y_\varepsilon - x_\varepsilon| + |V'(x_\varepsilon)|$. Hence, passing to the limit $\varepsilon \to 0$, we find $\inf_{y \in \partial\Omega} |V'(y)| = 0$.

q.e.d.

Remark 3.5. It can be possible to prove theorem 3.3 in a similar way, taking the sets $S_\varepsilon = \Omega \setminus \Omega_\varepsilon$ instead of A_ε. With this choice, we obtain the thesis, on the one hand relaxing the hypothesis (V6) in

(V6') V' is Lipschitzian on S_δ for some $\delta > 0$,

on the other hand, adding the condition that $\sup_{x \in \partial\Omega} |V'(x)| < \infty$.

4. Existence of a second homoclinic orbit

We come back to the problem of section 1 (finding a homoclinic orbit for (HS)) and give a new proof of corollary 1.3, using a minimax method; actually, besides the hypotheses *(V1)* and *(V3)*, we assume the strong–force condition for the potential V near the singularity (in fact we will obtain a classical solution) and a condition of quasi–concavity of V near the strict local maximum.

Theorem 4.1. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$, with $e \in \mathbf{R}^N \setminus \{0\}$, verifies (V1), (V3), (SF) and, for $N > 2$,*

(V8) there is some $\delta > 0$ such that $V(x) + \frac{1}{2}V'(x) \cdot x \leq 0$ for any $x \in B_\delta$ and $V \in C^{1,1}(B_\delta; \mathbf{R})$,

then (HS) *admits a homoclinic orbit.*

The proof consists of three main steps: firstly, we set up an approximating Dirichlet problem on $[0, T]$ using a potential V_T obtained cutting V out of Ω to a level T^{-1}; with a minimax method we get a solution q_T describing a sort of loop with initial and final point at 0, around the singularity. In the second part we give some uniform estimates for the approximating solutions that allow to pass to the limit $T \to \infty$ (last step) and, after some remarks, find the homoclinic orbit.

I – APPROXIMATING PROBLEM

Lemma 4.2. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ satisfies (V1) and (V3) then, fixed $\lambda > 0$, for any $T > 0$ there is a function $V_T \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ with the following properties:*

(i) $V_T(x) = V(x)$ for $x \in \Omega \setminus \{e\}$;

(ii) $V_T(x) = \frac{\lambda}{T}$ for $x \notin \Omega_T^+$;

(iii) $V_T(x) \leq \frac{\lambda}{T}$ for $x \in \Omega_T^+ \setminus \Omega$;

(iv) $|V_T'(x)| \leq c_1$ for any $x \notin \Omega$,

where Ω_T^+ is a bounded, open neighbourhood of Ω such that $\text{dist}(\Omega_T^+, \Omega) \to 0$ *as $T \to \infty$, and $c_1 > 0$ is a constant independent on x and T.*

Proof. To begin, we prove that given $\varepsilon > 0$ there is a function $f_\varepsilon \in C^\infty(\mathbf{R}^N; \mathbf{R})$ such that $0 \leq f_\varepsilon \leq 1$, $f_\varepsilon(x) = 0$ for $x \in \overline{\Omega}$, $f_\varepsilon(x) = 1$ if $\text{dist}(x, \Omega) \geq \varepsilon$ and $|f_\varepsilon'(x)| \leq \frac{A_N}{\varepsilon}$ for any $x \in \mathbf{R}^N$ with A_N constant with respect to x and T but depending on the dimension N. In fact, pick $\varphi \in C^\infty(\mathbf{R}^N; \mathbf{R})$ such that $0 \leq \varphi \leq 1$, $\varphi(x) = 0$ if $|x|_\infty \leq \frac{1}{2}$ and $\varphi(x) = 1$ if $|x|_\infty \geq 1$; for any $x \in \mathbf{R}^N$ let $\varphi_\varepsilon(x) = \varphi\left(\frac{x}{\varepsilon}\right)$ and

$$g_\varepsilon(x) = \prod_{y \in Y_\varepsilon} \varphi_\varepsilon(x - y)$$

where

$$Y_\varepsilon = \{y = (y^1, \ldots, y^N) \in \mathbf{R}^N : y^1, \ldots, y^N \in \varepsilon\mathbf{Z},\ \text{dist}_\infty(y, \Omega) \leq \tfrac{1}{2}\varepsilon\}.$$

One can readly see that $g_\varepsilon(x) = 0$ for $x \in \overline{\Omega}$, $g_\varepsilon(x) = 1$ if $\text{dist}_\infty(x, \Omega) \geq \frac{3}{2}\varepsilon$ and clearly $0 \leq g_\varepsilon \leq 1$ and $g_\varepsilon \in C^\infty(\mathbf{R}^N; \mathbf{R})$. In addition, taken $x \in \Omega^c$ with $\text{dist}_\infty(x, \Omega) < \frac{3}{2}\varepsilon$ the derivative of g_ε at x involves at most 2^N terms corresponding to some points $y_1, \ldots, y_{2^N} \in Y_\varepsilon$ so that

$$\left|\frac{\partial g_\varepsilon}{\partial x_i}(x)\right| = \left|g_\varepsilon(x) \sum_{j=1}^{2^N} \frac{1}{\varphi_\varepsilon(x - y_j)} \frac{1}{\varepsilon} \frac{\partial \varphi}{\partial x_i}\left(\frac{x - y_j}{\varepsilon}\right)\right| \leq \frac{A_N}{\varepsilon}.$$

Since $|x|_\infty \leq \sqrt{N}|x|$, taking $f_\varepsilon = g_\eta$ with $\eta = \frac{2}{3\sqrt{N}}\varepsilon$ we obtain the desired function. Now, by the continuity of V, for any $T > 0$ we can find $\varepsilon_T > 0$ such that

$$(3) \qquad |V(x)| \leq \frac{\lambda}{T} \quad \text{if } x \in \Omega_T^+ \setminus \Omega$$

being $\Omega_T^+ = \{x \in \mathbf{R}^N : \text{dist}(x, \Omega) < \varepsilon_T\}$. Finally, define for any $x \in \mathbf{R}^N \setminus \{e\}$:

$$V_T(x) = (1 - f_{\varepsilon_T}(x))V(x) + \frac{\lambda}{T} f_{\varepsilon_T}(x).$$

The properties of f_ε and (3) imply the thesis. Moreover $\text{dist}(\Omega_T^+, \Omega) = \varepsilon_T \to 0$ as $T \to \infty$.

q.e.d.

For $T > 0$ let $E_T = W_0^{1,2}([0, T]; \mathbf{R}^N)$ be the Hilbert space with the usual norm $\|u\|_T^2 = \int_0^T |\dot{u}|^2 dt$; consider its open subset $\Lambda_T = \{u \in E_T : u(t) \neq e \ \forall t \in [0, T]\}$ and the functional $I_T : \Lambda_T \to \mathbf{R}$ given by $I_T(u) = \int_0^T \left(\frac{1}{2}|\dot{u}|^2 - V_T(u)\right) dt$ for $u \in \Lambda_T$.
It can be shown in a standard way that $I_T \in C^1(\Lambda_T; \mathbf{R})$ and $q \in \Lambda_T$ is a critical point of I_T if and only if is classical solution of the following Dirichlet problem:

$$(P_T) \qquad \begin{cases} \ddot{q} + V_T'(q) = 0 & \text{in } (0, T) \\ q(0) = q(T) = 0 \end{cases}$$

We approach to the problem of the existence of critical points for I_T in a different way according to $N = 2$ or $N > 2$.

Case $N = 2$

Let $\Gamma_T^* = \{u \in \Lambda_T : \text{ind}(u) = 1\}$ where $\text{ind}(u)$ denotes the number of winding of u around the point e in some direction. To be precise $\text{ind}(u) = \int_\gamma \frac{dz}{z-e}$ with $\mathbf{R}^2$ identified to $\mathbf{C}$, $z \in \mathbf{C}$ and γ a closed curve in $\mathbf{C}$ parametrized by u. Clearly Γ_T^* is not empty, so that we can consider $\inf I(\Gamma_T^*)$.

Lemma 4.3. *For any $T > 0$ there exists $q_T \in \Gamma_T^*$ such that $I_T(q_T) = \inf I_T(\Gamma_T^*)$. Moreover q_T is a critical point of I_T.*

Proof. Let $(u_n) \subset \Gamma_T$ be a sequence such that $I_T(u_n) \to \inf I_T(\Gamma_T^*)$ as $n \to \infty$. Then for any $n \in \mathbf{N}$ it holds that $\frac{1}{2}\|u_n\|_T^2 = I_T(u_n) + \int_0^T V_T(u_n) dt \leq C + \lambda$ with C independent on n. Hence, possibly for a subsequence, (u_n) converges to some $q \in E_T$ weakly in E_T and uniformly on $[0,T]$. The strong–force condition implies that $q \in \Lambda_T$; otherwise, if $q(t) = e$ for some $t \in [0,T]$, then for any $n \in \mathbf{N}$ there is $t_n \in [0,t)$ such that $u_n(t_n) \in \partial N_e$ and $u_n(s) \in N_e$ for $s \in (t_n, t)$; hence, arguing as in proposition 2.1, we obtain for any $n \in \mathbf{N}$

$$|U(u_n(t))| \leq \sqrt{2} I_T(u_n) + |U(u_n(t_n))| \leq C' + \max_{x \in \partial N_e} |U(x)| < \infty$$

while $|U(u_n(t))| \to \infty$ as $n \to \infty$. In addition $\mathrm{ind}(q) = \lim \mathrm{ind}(u_n) = 1$, so that $q \in \Gamma_T^*$. As in the proof of lemma 1.6, it can be shown that $I_T(q) \leq \inf I_T(\Gamma_T^*)$ and so, for $q \in \Gamma_T^*$, $I_T(q) = \min I_T(\Gamma_T^*)$. To verify the second statement it suffices to notice that if $u \in \Gamma_T^*$, $\varphi \in C_c((0,T); \mathbf{R}^N)$ and $s \in \mathbf{R}$ then $u + s\varphi \in \Gamma_T^*$ for $|s|$ small enough. Therefore, since $q \in \Gamma_T^*$ and $I_T(q) = \min I_T(\Gamma_T^*)$, for any $\varphi \in C_c((0,T); \mathbf{R}^N)$ it holds that

$$\lim_{s \to 0} \frac{I_T(q + s\varphi) - I_T(q)}{s} = \frac{\partial I_T(q)}{\partial \varphi} = 0$$

that is $I'_T(q) = 0$.

q.e.d.

<u>Case $N > 2$</u>

Let $\Gamma_T = \{\gamma \in C(D^{N-2}; \Lambda_T) : \gamma(x) = 0 \ \forall x \in \partial D^{N-2}\}$ where $D^{N-2} = \{x \in \mathbf{R}^{N-2} : |x| \leq 1\}$. Given $\gamma \in \Gamma_T$, the function $(x,t) \mapsto \tilde{\gamma}(x,t) = \frac{\gamma(x)(t) - e}{|\gamma(x)(t) - e|}$ is well defined on $D^{N-2} \times [0,T]$. Being $\gamma(x)(t) = 0$ for $(x,t) \in \partial(D^{N-2} \times [0,T])$, we can consider as domain of $\tilde{\gamma}$ the quotient $D^{N-2} \times [0,T] / \partial(D^{N-2} \times [0,T]) \simeq S^{N-1}$ so that $\tilde{\gamma}$ is a continuous function from S^{N-1} into S^{N-1} (we use the notation $S^M = \{x \in \mathbf{R}^{M+1} : |x| = 1\}$). In this way the set $\Gamma_T^* = \{\gamma \in \Gamma_T : \deg(\tilde{\gamma}) \neq 0\}$ is well defined and non–empty (see lemma 1.2 of [3]) and the number

$$c(T) = \inf_{\gamma \in \Gamma_T^*} \max_{x \in D^{N-2}} I_T(\gamma(x))$$

is meaningful. Notice that, adopting the agreement $D^0 = \{0\}$ for $N = 2$, then Γ_T^* corresponds to that one defined in the case $N = 2$ and $c(T) = \inf I_T(\Gamma_T^*)$.

Lemma 4.4. *I_T satisfies the Palais–Smale condition.*

Proof. Let $(u_n) \in \Lambda_T$ be a Palais–Smale sequence. With the same argument of the previous proof, we obtain that, possibly passing to a subsequence, u_n converges to some $q \in \overline{\Lambda_T}$ weakly in E_T and uniformly on $[0,T]$ and, by the strong–force condition, $q \in \Lambda_T$. Now, using the fact that $I'_T(u_n) \to 0$, in a standard way one can verify that $u_n \to q$ strongly in E_T.

q.e.d.

Lemma 4.5. *For any $T > 0$ $c(T) > 0$ and $c(T)$ is a critical level for I_T.*

Proof. First, we show that $c(T) > 0$. In fact, taken a generic $\gamma \in \Gamma_T^*$, since $\deg(\gamma) \neq 0$ there exists $(x_o, t_o) \in D^{N-2} \times [0,T]$ such that $|\gamma(x_o)(t_o)| \geq 2\delta$ where $\delta > 0$ can be chosen so that $B_{2\delta} \subseteq \Omega$. Call $u = \gamma(x_o)$. For $u(0) = 0$, there are $t_1, t_2 \in (0, t_o]$ such that $t_1 < t_2$, $u(t_1) \in \partial B_\delta$, $u(t_2) \in \partial B_{2\delta}$ and $u(t) \in B_{2\delta} \setminus B_\delta$ for $t \in (t_1, t_2)$. Let $A = \{t \in [0,T] \, : \, u(t) \in \overline{\Omega}\}$ and $B = [0,T] \setminus A$. Then, called $m_\delta = \inf\{-V(x) \, : \, x \in B_{2\delta} \setminus B_\delta\}$ it holds that

$$\begin{aligned} I_T(u) &\geq \frac{1}{2}\int_{t_1}^{t_2} |\dot{u}|^2 dt + \int_A -V(u)dt + \int_B -V_T(u)dt \\ &\geq \frac{1}{2}\frac{\delta^2}{t_2 - t_1} + m_\delta(t_2 - t_1) - \lambda \geq \sqrt{2m_\delta}\delta - \lambda = \mu. \end{aligned}$$

Fixing $\lambda \in (0, \sqrt{2m_\delta}\delta)$, we deduce that $\max_{x \in D^{N-2}} I_T(\gamma(x)) \geq \mu > 0$ and so, by the arbitrariness of γ, $c(T) \geq \mu > 0$ follows for any $T > 0$.
To prove the second statement we use a standard deformation lemma according to that, since I_T satisfies the Palais–Smale condition, if c is not a critical value of I_T, then for any $\bar{\varepsilon} > 0$ there are an $\varepsilon \in (0, \bar{\varepsilon})$ and a deformation η of Λ_T in E_T that sends the sublevel $I_T^{c+\varepsilon}$ into the sublevel $I_T^{c-\varepsilon}$, lowers the values of the functional I_T and acts identically out of $I_T^{c+\bar{\varepsilon}} \setminus I_T^{c-\bar{\varepsilon}}$ (here $I_T^a = \{u \in \Lambda_T : I_T(u) \leq a\}$). By contradiction suppose that $c(T)$ is not a critical level of I_T. By the first part, we can apply the previous result choosing $c = c(T)$ and $\bar{\varepsilon} = \frac{1}{2}c(T)$ and find a deformation η of Λ_T and an $\varepsilon \in (0, \bar{\varepsilon})$ with the above-mentioned properties. Given an arbitrary $\gamma \in \Gamma_T^*$ we show that $\eta \circ \gamma \in \Gamma_T^*$. Clearly $\eta \circ \gamma \in C(D^{N-2}; E_T)$ and for any $x \in D^{N-2}$ $I_T((\eta \circ \gamma)(x)) \leq I_T(\gamma(x)) < \infty$ so that, by (SF), $\eta \circ \gamma(x) \in \Lambda_T$. Since $0 \notin I_T^{c+\bar{\varepsilon}} \setminus I_T^{c-\bar{\varepsilon}}$, if $x \in \partial D^{N-2}$ then $\eta \circ \gamma(x) = \eta(0) = 0$. Hence $\eta \circ \gamma \in \Gamma_T$. Moreover the invariance of the degree under homotopy implies that $\deg(\widetilde{\eta \circ \gamma})=\deg(\tilde{\gamma}) \neq 0$ and so $\eta \circ \gamma \in \Gamma_T^*$. Now take $\gamma_o \in \Gamma_T^*$ such that $\max_{x \in D^{N-2}} I_T(\gamma_o(x)) \leq c + \varepsilon$ and call $\gamma_1 = \eta \circ \gamma_o$. Then $\gamma_1 \in \Gamma_T^*$ and being $\eta(I_T^{c+\varepsilon}) \subseteq I_T^{c-\varepsilon}$, we get $\max_{x \in D^{N-2}} I_T(\gamma_1(x)) \leq c - \varepsilon$ in contrast with the definition of c.

q.e.d.

II – UNIFORM ESTIMATES

For any $T > 0$ let $q_T \in \Lambda_T$ be the solution of (P_T), founded as minimum point of I_T on Γ_T^* for $N = 2$, or as minimax point of I_T with respect to Γ_T^* in the case $N > 2$.

Lemma 4.6. *There is $\delta_o > 0$ such that $|q_T|_\infty \geq \delta_o$ for each $T > 0$.*

Proof. If $N = 2$ it suffice to notice that $|q_T|_\infty \geq |e|$ for any $T > 0$ because $\mathrm{ind}(q_T) = 1$. In the case $N > 2$ we use the hypothesis *(V8)* in the following way: let h_T be the total energy of q_T; it holds that $h_T = \frac{1}{2}|\dot{q}_T(0)|^2 \geq 0$. Now, if $h_T = 0$, then q_T would be solution of the Cauchy problem:

$$\begin{cases} \ddot{q} + V_T'(q) = 0 & \text{in } (0,T) \\ q(0) = \dot{q}(0) = 0 \end{cases}$$

and, by the uniqueness of the solution, for $V_T = V$ is $C^{1,1}$ in a neighborhood of 0, it follows that $q_T(t) = 0$ for any $t \in [0,T]$, in contrast with the fact that $c(T) > 0$. So, $h_T > 0$ for any $T > 0$ and consequently:

$$\frac{1}{2}\frac{d^2}{dt^2}|q_T(t)|^2 = -2V_T(q_T(t)) - V_T'(q_T(t)) \cdot q_T(t) + 2h_T > -2V_T(q_T(t)) - V_T'(q_T(t)) \cdot q_T(t).$$

By (V8) $\frac{d^2}{dt^2}|q_T(t)|^2 > 0$ if $q_T(t) \in B_{\delta_o}$ (we can always suppose $B_{\delta_o} \not\ni e$ and $B_{\delta_o} \subset \Omega$ so that $V_T = V$ on B_{δ_o}). Let $t_T^* \in [0,T]$ such that $|q_T(t_T^*)| = |q_T|_\infty$. Being $\frac{d^2}{dt^2}|q_T(t_T^*)|^2 \leq 0$, we deduce that $q_T(t_T^*) \notin B_{\delta_o}$, i.e. $|q_T|_\infty \geq \delta_o$.

q.e.d.

Lemma 4.7. *Chosen $T_o > 0$, for each $T \geq T_o$* $\|q_T\|_T \leq$ constant.

Proof. Firstly we notice that $\frac{1}{2}\|q_T\|^2 \leq I_T(q_T) + \lambda$ for any $T > 0$. Then we have to show that $c(T) <$ constant independent on T. For $e \in \Omega$ and Ω is open, fixed $T_o > 0$, there is $\gamma \in \Gamma_{T_o}^*$ such that $\gamma(D^{N-2} \times [0,T_o]) \subset \Omega$. Let $T \geq T_o$. For any $x \in D^{N-2}$ define $\gamma_T(x) = \gamma(x)\chi_{[0,T_o]}$, where $\chi_{[0,T_o]}(t) = 1$ if $t \in [0,T_o]$ and 0 otherwise. We point out that $\gamma_T \in \Gamma_T^*$ and $I_T(\gamma_T(x)) = I_{T_o}(\gamma(x))$ for all $x \in D^{N-2}$, so that $c(T) \leq \max_{x \in D^{N-2}} I_{T_o}(\gamma(x))$ for any $T \geq T_o$. Notice also that T_o is arbitrary.

q.e.d.

Lemma 4.8. *There is $\rho > 0$ such that $|q_T(t) - e| \geq \rho$ for each $t \in [0,T]$ and $T > 0$.*

Proof. Suppose the lemma is false. Then there are two sequences (T_n) and (t_n), with $0 < t_n < T_n$, such that $q_{T_n}(t_n) \to e$ as $n \to \infty$; moreover for any $n \in \mathbf{N}$ there is $s_n \in (0, t_n)$ such that $q_{T_n}(s_n) \in \partial N_e$ and $q_{T_n}(t) \in N_e$ for $t \in (s_n, t_n)$. The usual inequality, obtained with (SF),

$$|U(q_{T_n}(t_n))| \leq \sqrt{2} I_{T_n}(q_{T_n}) + |U(q_{T_n}(s_n))| \leq C + \max_{x \in \partial N_e} |U(x)| < \infty$$

gives the contradiction.

q.e.d.

III - LIMIT PROCESS

By lemma (4.6), for any $T > 0$ there is $\tau_T \in (0,T)$ such that $|q_T(\tau_T)| = \delta_o$ and $|q_T(t)| < \delta_o$ if $t \in (0, \tau_T)$. Now, define

$$y_T = \begin{cases} q_T(t + \tau_T) & \text{for } t \in [-\tau_T, T - \tau_T] \\ 0 & \text{for } t \in \mathbf{R} \setminus [-\tau_T, T - \tau_T] \end{cases}$$

All the functions y_T belong to the Hilbert space $E = \{u \in W^{1,2}_{\text{loc}}(\mathbf{R}; \mathbf{R}^N) : \int_{-\infty}^{\infty} |\dot{u}|^2 dt < \infty\}$ endowed with the norm $\|u\|^2 = |u(0)|^2 + \int_{-\infty}^{\infty} |\dot{u}|^2 dt$; moreover, by lemma (4.7), $\|y_T\| \leq$ constant for any $T > T_o$. Then there are a sequence $T_n \to \infty$ and a function $y \in E$ to that (y_{T_n}) converges, weakly in E and uniformly on the compact subsets of $\mathbf{R}$. Clearly $y \not\equiv 0$ because $|y(0)| = \delta_o$. Other properties of this function y are listed in the following lemma.

Lemma 4.9.
(i) $y \in C^1(\mathbf{R};\mathbf{R}^N)$ *and, possibly for a subsequence,* $y_{T_n} \to y$ *in* $C^1_{\text{loc}}(\mathbf{R};\mathbf{R}^N)$*;*
(ii) $y(t) \in \overline{\Omega} \setminus B_\rho(e)$ *for any* $t \in \mathbf{R}$*;*
(iii) $\frac{1}{2}|\dot{y}(t)|^2 + V(y(t)) = 0$ *for any* $t \in \mathbf{R}$*;*
(iv) $y \in C^2(\mathbf{R} \setminus \mathcal{T};\mathbf{R}^N)$*;*
(v) $\ddot{y}(t) + V'(y(t)) = 0$ *for any* $t \in \mathbf{R} \setminus \mathcal{T}$*,*
where $\mathcal{T} = \{t \in \mathbf{R} : y(t) \in \partial\Omega\}$.

Proof. (i). Fix $T_o > 0$ and let $C_1 = \sup\{|V_T(x)| : T > T_o,\ x \in \Omega_T^+ \setminus B_\rho(e)\}$ and $C_2 = \sup\{|V_T'(x)| : T > T_o,\ x \in \Omega_T^+ \setminus B_\rho(e)\}$. Clearly C_1, $C_2 < \infty$ and, by lemmas 4.5 and 4.8 (or 4.3 and 4.8 for the case $N = 2$), we deduce that for any $n \in \mathbf{N}$:

$$|\dot{y}_{T_n}|_\infty + |\ddot{y}_{T_n}|_\infty \le C_3$$

with C_3 positive constant independent on n. So, by Ascoli–Arzelà theorem, passing to a subsequence, if necessary, $\dot{y}_{T_n} \to z$ in $C_{\text{loc}}(\mathbf{R};\mathbf{R}^N)$; but $\dot{y}_{T_n} \to \dot{y}$ weakly in $L^2(\mathbf{R};\mathbf{R}^N)$, so that $y_{T_n} \to y$ in $C^1_{\text{loc}}(\mathbf{R};\mathbf{R}^N)$.

(ii) and *(iii)*. By lemma 4.8, $y(t) \notin B_\rho(e)$ for all $t \in \mathbf{R}$. Then, notice that $h_{T_n} \to \infty$ as $n \to \infty$, because $0 \le h_{T_n} T_n \le I_{T_n}(q_{T_n}) + 2\int_0^{T_n} V_{T_n}(q_{T_n})dt \le C_4 + 2\lambda < \infty$. Therefore, passing to the limit $n \to \infty$ in the energy equation for y_{T_n}, we get

$$\begin{array}{lll} \frac{1}{2}|\dot{y}(t)|^2 + V(y(t)) = 0 & \text{if} & y(t) \in \Omega \\ \dot{y}(t) = 0 & \text{if} & y(t) \notin \overline{\Omega}^c \end{array}$$

and this implies $y(t) \in \overline{\Omega}$ for any $t \in \mathbf{R}$, so that the energy equation is satisfied on $\mathbf{R}$.

(iv) and *(v)*. Pick a compact interval $[t_1, t_2]$ in an arbitrary component of $\mathbf{R} \setminus \mathcal{T}$. For any $t \in [t_1, t_2]$ $y(t) \in \Omega$ and then, for n sufficiently large $y_{T_n}(t) \in \Omega$, so that $V_{T_n}'(y_{T_n}(t)) = V'(y_{T_n}(t)) \to V'(y(t))$. By the dominated convergence theorem and by the result *(i)* we deduce that

$$\int_{t_1}^{t_2} -V'(y(t))dt = \lim_{n\to\infty} \int_{t_1}^{t_2} -V_{T_n}'(y_{T_n}(t))dt = \int_{t_1}^{t_2} \ddot{y}_{T_n}(t)dt = \dot{y}(t_2) - \dot{y}(t_1).$$

Hence, for the continuity of $-V' \circ y$ on $[t_1, t_2]$ we can say that $\dot{y}$ admits derivative and solves (HS) on $[t_1, t_2]$. By the arbitrariness of $[t_1, t_2]$, the proof is complete.

q.e.d.

Conclusion of the proof of theorem 4.1. By the result *(iii)* of previous lemma, since $y \in E$, $I(y) = \int_{-\infty}^{\infty}(\frac{1}{2}|\dot{y}|^2 - V(y))dt < \infty$. Then, by lemma 1.5, there exist $\lim_{t\to\pm\infty} y(t) = \xi_\pm$ and $\xi_\pm \in \partial U \{0\}$.

Consequently, using the energy equation, $\dot{y}(\pm\infty) = 0$. Notice that in lemma 4.6 we can always choose δ_o sufficiently small so that $\bar{B}_{\delta_o} \subset \Omega$; hence $|y(t)| \le \delta_o$ for any $t < 0$ and this implies $\xi_- = 0$. Now, if $\mathcal{T} \neq \emptyset$, called $t_o = \inf \mathcal{T}$, the function $t \mapsto Y(t) = y(t_o - |t - t_o|)$ is a homoclinic orbit for (HS). Instead, if $\mathcal{T} = \emptyset$, then necessarily $\xi_+ = 0$; in fact, if it were $\xi_+ \in \partial\Omega$ with $\mathcal{T} = \emptyset$, arguing as in the proof of theorem 1.2, we get for any $S > 0$ a $T = T(S) > 0$ such that

$$\int_{-\infty}^{\infty} |\dot{y}_T|^2 dt \ge \frac{1}{S}\left(\frac{1}{4}|V'(\xi_+)|S^2 - cS\right)^2$$

with $c > 0$. For S is arbitrary and $|V'(\xi_+) > 0$, by *(V3)*, we found a contradiction with the boundness $\|y_T\| \le$constant.

q.e.d.

Remark 4.10. The strategy adopted to prove theorem 4.1 can be applied every time that the approximating solutions are obtained making a minimax like $\inf_{\mathcal{A}_T} \sup_A I_T$, provided that $A_T \in \mathcal{A}_T$ for some $A \in \mathcal{A}_{T_o}$, being $A_T = \{u\chi_{[0,T_o]} \ : u \in A\}$. In particular this holds using the mountain–pass lemma to solve the approximating problem (P_T). In this sense, theorem 4.1 generalizes [1].

In the next result we eliminate the hypothesis (SF) and find a generalized homoclinic orbit.

Theorem 4.11. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$, with $e \in \mathbf{R}^N \setminus \{0\}$, verifies (V1), (V3) and (V8), for $N > 2$, then* (HS) *admits a generalized homoclinic orbit.*

Proof. For any $\varepsilon > 0$ small enough, let $V_\varepsilon \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$ satisfying *(V1)*, *(V3)*, *(SF)* and, for $N > 2$, *(V8)* and such that $V_\varepsilon(x) = V(x)$ if $|x - e| \ge \varepsilon$ and $V_\varepsilon \le V$ on $\mathbf{R}^N \setminus \{e\}$. Let $(q_{T,\varepsilon})_T$ be the family of the approximating solutions corresponding to the problem for V_ε, ruled by the equation

$$(\mathrm{HS})_\varepsilon \qquad \ddot{q} + V_\varepsilon'(q) = 0.$$

Coming back to lemmas 4.6 and 4.7, it is easy to recognize that for any $\varepsilon \in (0, \varepsilon_o)$ and $T > T_o$ it holds that

$$|q_{T,\varepsilon}|_\infty \ge \delta_o \ , \quad \|q_{T,\varepsilon}\|_T \le C$$

with δ_o and C positive constants independent on ε and T. We define $y_{T,\varepsilon}$ by shifting $q_{T,\varepsilon}$ so that $|y_{T,\varepsilon}(0)| = \delta_o$ and $|y_{T,\varepsilon}(t)| < \delta_o$ for $t < 0$. Finally let y_ε the homoclinic orbit of $(\mathrm{HS})_\varepsilon$ achieved as limit of the approximating solutions $y_{T,\varepsilon}$ as $T \to \infty$. For any $\varepsilon \in (0, \varepsilon_o)$ we get that

$$|y_\varepsilon(0)| = \delta_o \ , \quad \int_{-\infty}^{\infty} |\dot{y}_\varepsilon|^2 dt \le C'.$$

Hence there are a sequence $\varepsilon_n \downarrow 0$ and a function $y \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}; \mathbf{R}^N)$ such that $\int_{-\infty}^{\infty} |\dot{y}|^2 dt \le C'$ and $y_{\varepsilon_n} \to y$ uniformly on the compact subsets of $\mathbf{R}$. Then $|y(0)| = \delta_o$, $y(t) \in \overline{\Omega}$ for all $t \in \mathbf{R}$ and

$\int_{-\infty}^{\infty} -V(y)dt < \infty$; in fact $\int_{-\infty}^{\infty} -V(y)dt \leq \liminf \int_{-\infty}^{\infty} -V(y_{\varepsilon_n})dt = \liminf \int_{-\infty}^{\infty} \frac{1}{2}|\dot{y}_{\varepsilon_n}|^2 dt \leq C'$. Therefore $y^{-1}(e)$ is a nullset and $I(y) < \infty$. By lemma 1.5 there exist $\lim_{t\to\pm\infty} y(t) = \xi_\pm$ and $\xi_\pm \in \partial\Omega \cup \{0\}$. To see that $\xi_\pm = 0$ we argue indirectly, supposing, for instance, that $\xi_+ \in \partial\Omega$. Then we repeat the same argument used to prove theorem 1.2 and get a contadiction. So $\xi_+ = \xi_- = 0$. Now, keeping into account that for any $\varepsilon \in (0, \varepsilon_o)$ y_ε solves $(HS)_\varepsilon$ and its energy is zero, for any compact K contained in an arbitrary component of $\mathbf{R} \setminus y^{-1}(e)$ we get:

$$y_{\varepsilon_n} \to y \text{ in } C^2(K; \mathbf{R}^N)$$

$$\ddot{y}(t) + V'(y(t)) = 0 \text{ for any } t \in K$$

$$\frac{1}{2}|\dot{y}(t)|^2 + V(y(t)) = 0 \text{ for any } t \in K.$$

Since K is arbitrary, y is a generalized solution of (HS) on $\mathbf{R}$, with energy zero. Moreover $y \not\equiv 0$, $y(\pm\infty) = 0$ and, by the energy equation, $\dot{y}(\pm\infty) = 0$.

q.e.d.

Finally we give a multeplicity result; first, we introduce the following notations: for any $\delta > 0$ let $V_\delta = \{x \in \Omega \ : \ V(x) \leq -\delta\}$ and let V_δ^0 and V_δ^1 be the components of $\overline{\Omega} \setminus V_\delta$ containing 0 and $\partial\Omega$ respectively and call $r_\delta = \text{dist}(V_\delta^0, V_\delta^1)$. Then set $\Sigma = \{x \in \Omega \ : \ |x - e| = |e|\}$ and $v = \sup_{x\in\Sigma} |V(x)|$.

Theorem 4.12. *Let $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$, with $e \in \mathbf{R}^N \setminus \{0\}$, satisfying (V1), (V3), (SF) and, for $N > 2$, (V8). If there is some $\delta > 0$ such that*

$$(\star) \qquad 2\pi|e|\sqrt{2v} < r_\delta\sqrt{\delta}$$

then (HS) admits two geometrically distinct homoclinic orbits.

Proof. Fix $T_o > 0$ and let $\gamma_o \in \Gamma^*_{T_o}$ such that $\gamma_o(D^{N-2} \times [0, T_o]) = \Sigma$. Then, for any $x \in D^{N-2}$ define $\gamma_T(x) = \gamma_o(x)\chi_{[0,T_o]}$, where $\chi_{[0,T_o]}(t) = 1$ if $t \in [0, T_o]$ and 0 otherwise. Clearly $\gamma_T \in \Gamma^*_T$, $\gamma_T(D^{N-2} \times [0, T]) = \Sigma$ and $I_T(\gamma_T(x)) = K(x) + P_T(x)$ where $K(x) = \frac{1}{2}\int_0^{T_o} \left|\frac{d}{dt}\gamma_o(x)\right|^2 dt$ and $P_T(x) = \int_0^{T_o} -V_T(\gamma_o(x))dt$. Since $P_T(x) \leq T_o v$ for any $x \in D^{N-2}$, it follows that for $T \geq T_o$

$$c(T) \leq T_o v + \max_{x\in D^{N-2}} K(x).$$

We can always suppose that for any $x \in D^{N-2}$ $\gamma_o(x)$ describes a circular orbit passing for the origin, with radius $\rho_s \leq |e|$ and angular speed $\omega_o = \frac{2\pi}{T_o}$ so that $\max_{x\in D^{N-2}} K(x) = \dfrac{2\pi^2|e|^2}{T_o}$. Hence, with an appropriate choice of T_o we obtain:

$$c(T) \leq 2\pi|e|\sqrt{2v}$$

and, consequently

$$\|q_T\|_T^2 \leq 4\pi|e|\sqrt{2v} + 2\lambda$$

where q_T is the approximating solution solving (P_T). Therefore the homoclinic orbit y of (HS), found as weak limit of q_T as $t \to \infty$, satisfies the following inequality:

$$I(y) \leq 4\pi|e|\sqrt{2v} + 2\lambda.$$

Now, if $x \in W^{1,2}_{\mathrm{loc}}(\mathbf{R};\mathbf{R}^N)$ denotes the homoclinic orbit of (HS) given by corollary 1.3, for $x(0) \in \partial\Omega$ and x has energy zero, we get that

$$I(x) = \int_{-\infty}^{\infty} |\dot{x}|^2 dt \geq \int_{\mathcal{T}_\delta} |\dot{x}|^2 dt \geq \frac{(2r_\delta)^2}{|\mathcal{T}_\delta|}$$

where $\mathcal{T}_\delta = \{t \in \mathbf{R} : x(t) \in V_\delta\}$. On the other hand:

$$I(x) = \int_{-\infty}^{\infty} -V(x)dt \geq \int_{\mathcal{T}_\delta} -V(x)dt \geq \delta|\mathcal{T}_\delta|$$

and so

$$I(x) \geq 2r_\delta\sqrt{\delta}.$$

Then, by ($\star$), fixing $\lambda \in (0, r_\delta\sqrt{\delta} - 2\pi|e|\sqrt{2v})$, it follows that $I(y) < I(x)$ and $y(t) \notin \partial\Omega$ for any $t \in \mathbf{R}$. Otherwise, using the notation of section 1, called $x_+ = x|_{\mathbf{R}_+}$ and $y_+ = y|_{[t_o,\infty)}$ for a suitable $t_o \in \mathbf{R}$, it should be $y_+ \in \Gamma$ and $I(x_+) = \inf I(\Gamma)$, while $I(y_+) < I(x_+)$. Hence y forms a loop inside Ω.

q.e.d.

5. Case $\Omega = \mathbf{R}^N$

The case of a negative potential V with an absolute maximum at 0 and a singularity at $e \in \mathbf{R}^N \setminus \{0\}$, where V is assumed strong–forcelike, was studied by Tanaka in [16]; there, the author put the following hypothesis about the behaviour of V at infinity:

$$\limsup_{|x|\to\infty} V(x) < 0.$$

We can improve this assumption allowing V to go to 0 at infinity in a suitable manner.

Theorem 5.1. *If $V \in C^1(\mathbf{R}^N \setminus \{e\};\mathbf{R})$, with $e \in \mathbf{R}^N \setminus \{0\}$, is strictly negative apart from 0 and verifies (V1), (V3), (SF) and, for $N > 2$, (V8), then* (HS) *admits a homoclinic orbit.*

Proof. We follow the same argument used to prove theorem 4.1, with the obvious modification that, for V is non positive, the approximating problems are ruled by the same potential V. Then, all the passages hold again and the only new thing to prove is a boundness for the approximating solutions q_T with respect to the sup norm. Arguing indirectly, if it were $|q_{T_n}|_\infty \to \infty$ for some sequence (T_n), then, by *(V8)*, with the same passages of proposition 2.1, we would infer that for any $n \in \mathbf{N}$:

$$|U(q_{T_n}(t_n))| \leq \sqrt{2} I_{T_n}(q_{T_n}) + |U(q_{T_n}(s_n))|$$

where t_n is the time that achieves $|q_{T_n}(t_n)| = |q_{T_n}|_\infty$ and $s_n \in (0, t_n)$ is such that $|q_{T_n}(s_n)| = R_o$ while $|q_{T_n}(t)| > R_o$ if $t \in (s_n, t_n)$. But $I_T(q_T)$ is bounded independently on T and the same holds for $|U(q_{T_n}(s_n))|$; therefore $|U(q_{T_n}(t_n))| <$ constant in contrast with the fact that $|q_{T_n}(t_n)| \to \infty$ and so $|U(q_{T_n}(t_n))| \to \infty$. Now, since $|q_T|_\infty$ and $\|q_T\|_T$ are bounded by constants independent on T, the weak limit of the approximating solutions, possibly shifted, is a non-zero function $y \in C^2(\mathbf{R}; \mathbf{R}^N)$ satisfying (HS), with energy zero and such that $y(\pm\infty) = \dot{y}(\pm\infty) = 0$.

q.e.d.

As in section 4, we can state an analogous theorem without the strong-force condition, to find a generalized homoclinic orbit.

Theorem 5.2. *If $V \in C^1(\mathbf{R}^N \setminus \{e\}; \mathbf{R})$, with $e \in \mathbf{R}^N \setminus \{0\}$, is strictly negative apart from 0 and verifies (V1), (V3), and, for $N > 2$, (V8), then* (HS) *admits a generalized homoclinic orbit.*

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