ma per seguir virtute e conoscenza

ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

Some existence results of Dunford densities for Banach-valued finitely additive measures

Thesis submitted for the degree of

"Magister Philosophiæ"

CANDIDATE

Pietro Celada

SUPERVISOR

Prof. D.Candeloro

Prof. G.Vidossich

June 1992

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

TRIESTE

Strada Costiera 11

Scuola Internazionale Superiore di Studi Avanzati

International School for Advanced Studies

Some existence results of Dunford densities for Banach-valued finitely additive measures

Thesis submitted for the degree of

"Magister Philosophiae"

CANDIDATE

Pietro Celada

SUPERVISORS

Prof. D.Candeloro
Prof. G.Vidossich

June 1992

0. Introduction.

This thesis is a report on an investigation on the existence of Radon-Nikodym derivatives for Banach-valued finitely additive measures (f.a.m. shortly) with respect to their total variations.
As is well known, in the classical case of scalar-valued, countably additive measures on σ-algebras, the Radon-Nikodym theorem asserts the equivalence between absolute continuity and the existence of a density function. This theorem is no longer true in its full generality as soon as we drop even one of the quoted hypotheses: see for instance examples 2.10 pg.50 and 3.2 pg.61 in [7] and example 6.3.5 pg.176 in [1].
The problem of the failure of Radon-Nikodym theorem in the previous settings has been given much attention by many authors and has raised several interesting questions.
On one hand, the characterization of Banach spaces having the so-called Radon-Nikodym property, that is those Banach spaces where all countably additive, absolutely continuous vector measures are indefinite Bochner integrals, has revealed deep interplay betweeen vector measures, Banach space structure theory and operator theory; detailed surveys of this topic with further references can be found in [6] and [7].
On the other hand, in the setting of finitely additive scalar measures, the failure of Radon-Nikodym theorem is tightly linked to the lack of completeness of L_p spaces with respect to f.a.m.'s so that only approximated Radon-Nikodym theorems may hold. A theorem of this kind was already proved by Bochner (see [2]) in the late 30's; roughly speaking, it asserts that, given a bounded f.a.m. ν, the indefinite integrals of simple functions with respect to ν are dense in the space of bounded, ν-continuous finitely additive measures. Since that time, several different proofs and improvements of this result have been given; see for instance [5], [8], [10], [13], [15].
As far as exact Radon-Nikodym theorems are concerned, we quote [14] and [11] where necessary and sufficient conditions are given in the case of bounded scalar f.a.m.'s and bounded monotone set functions respectively. Finally, [12] and [4] deal with Banach-valued f.a.m.'s.
In this thesis attention is focused on sufficient conditions on a pair of bounded, finitely additive measures (μ, ν) where μ is Banach-valued and ν is scalar-valued which do ensure the existence of a density function of μ with respect to ν. We shall confine ourselves to the case where ν is the total variation $|\mu|$ of μ and the representation of μ is accomplished by some weak form of integration.
We begin by setting some notations and hypotheses to be used throughout and by stating the problem in a precise form.
Let Ω be a (nonempty) abstract set, let $\mathcal{S}$ be a σ-algebra of subsets of Ω and let $\mu: \mathcal{S} \to X$ be a finitely additive measure where X is a real Banach space with separable dual space X^*.
We shall denote by $|\mu|$ and $\|\mu\|$ the total variation and the semivariation of μ respectively, by $\mathcal{N}(|\mu|)$ the ideal of $|\mu|$-null sets of $\mathcal{S}$ and by $ba(\Omega, \mathcal{S}, \mathbb{R})$ the Banach space of all real-valued, bounded f.a.m.'s on $\mathcal{S}$ endowed with the semivariation norm.
We shall assume throughout that

(P1) μ is of bounded variation;

(P2) $\mathcal{R}(\mu) = \{\mu(E): E \in \mathcal{S}\}$ is weakly closed in X;

(P3) $\mathcal{N}(|\mu|)$ is a σ-ideal, i.e. it is closed under countable union.

We aim at proving that under suitable conditions μ can be represented as a Dunford-type integral, that is, there exists a weakly $|\mu|$-integrable function $h: \Omega \to X^{**}$ such that the equation

$$< \mu(E), x^* >= \int_E < x^*, h(\omega) > d|\mu|(\omega)$$

holds for all $E \in \mathcal{S}$ and $x^* \in X^*$.
The idea of the proof goes as follows:

1. there exists $x_R^* \in X^*$, $\|x_R^*\| = 1$ such that $\nu = | < \mu, x_R^* > |$ (i.e. the total variation of $< \mu, x_R^* >$) is equivalent to μ in the sense that $|\mu| \ll \nu$ and $\nu \ll |\mu|$,

2. for each $x^* \in X^*$ there exists a ν-integrable function $f_{x^*}: \Omega \to I\!R$ such that

$$< \mu(E), x^* >= \int_E f_{x^*} d\nu \qquad E \in \mathcal{S},\ x^* \in X^*;$$

3. approximate $|\mu|$ in $ba(\Omega, \mathcal{S}, I\!R)$ by a collection $\{|\mu_n|\}\,(n \geq 1)$ of positive, bounded f.a.m.'s having Radon-Nikodym derivative with respect to ν by choosing a countable dense set $\{x_n^*\}\,(n \geq 1)$ in $\partial B_1^* = \{x^* \in X^*: \|x^*\| = 1\}$ and by setting

$$\begin{cases} \mu_n =< \mu, x_n^* > \\ f_n = f_{x_n^*} \end{cases} \qquad n \in I\!N_+;$$

4. set $f = \sup_{n \geq 1} |f_n|$; assume that f is ν-integrable and

$$|\mu|(E) = \int_E f d\nu \qquad E \in \mathcal{S}$$

and set

$$g(\omega) = \begin{cases} \infty & \text{if } f(\omega) = 0 \\ 1/f(\omega) & \text{if } 0 < f(\omega) < \infty \\ 0 & \text{if } f(\omega) = \infty \end{cases} \qquad ;$$

then g is $|\mu|$-integrable and

$$\nu(E) = \int_E g d|\mu| \qquad E \in \mathcal{S};$$

5. then, $(f_{x^*} g)$ is $|\mu|-$integrable for all $x^* \in X^*$ and

$$< \mu(E), x^* >= \int_E (f_{x^*} g) d|\mu| \qquad E \in \mathcal{S},\ x^* \in X^*;$$

6. reconstruct a weakly $|\mu|$-integrable function $h: \Omega \to X^{**}$ such that

$$< \mu(E), x^* >= \int_E < x^*, h(\omega) > d|\mu|(\omega) \qquad E \in \mathcal{S},\ x^* \in X^*.$$

The validity of the assumptions in (4) is discussed in section 3.

Remark 0.1. We point out that hypothesis (P3) will be used only in the proof of proposition 3.7. Furthermore, D.Candeloro and A.Martellotti have recently proved in a still unpublished paper a generalization of this proposition where (P3) is dropped.

Remark 0.2. The approximation of $|\mu|$ in $ba(\Omega, \mathcal{S}, I\!\!R)$ by a sequence of positive, bounded f.a.m.'s having Radon-Nikodym derivative with respect to ν described in (3) can be accomplished even if X^* fails to be separable; see remark 1.3. Indeed, the requirement of separability of the dual space X^* is actually crucial only in the proof of proposition 3.7.

1. Some technical results.

The f.a.m. μ is s-bounded by (P1). Hence, there exists $x_R^* \in X^*$, $\|x_R^*\| = 1$ such that $| < \mu, x_R^* > |$ is a control measure for μ (see [3]). This means that ν is equivalent to the semivariation $\|\mu\|$ of μ in the following sense:

a) $\|\mu\|$ is ν-continuous;

b) ν is $\|\mu\|$-continuous.

We shall shortly denote (a) and (b) by $\|\mu\| \sim \nu$. Now, we claim that $|\mu| \sim \nu$.
First, it is clear that ν is $|\mu|$-continuous. Next, to see that the reversed relation holds as well, let $I\!\!P(\Omega)$ be the set of all finite, $\mathcal{S}$-measurable partitions of Ω and set

$$\mu_\pi(E) = \sum_{1 \le m \le n} \|\mu(E \cap E_m)\| \qquad E \in \mathcal{S}$$

for each $\pi = \{E_m\} (1 \le m \le n) \in I\!\!P(\Omega)$. Each μ_π is a bounded, ν-continuous submeasure satisfying

$$0 \le \mu_\pi(E) \le |\mu|(E) \qquad E \in \mathcal{S},\ \pi \in I\!\!P(\Omega).$$

Thus, $\{\mu_\pi\} (\pi \in I\!\!P(\Omega))$ is a uniformly s-bounded set of ν-continuous submeasures; by [16], $\{\mu_\pi\} (\pi \in I\!\!P(\Omega))$ is uniformly ν-continuous. Finally, recalling that the net $\{\mu_\pi\} (\pi \in I\!\!P(\Omega))$ converges pointwise on $\mathcal{S}$ to $|\mu|$, we see that $|\mu|$ is ν-continuous. Thus, $|\mu| \sim \nu$.
We now prove that for each $x^* \in X^*$ there exists Radon-Nikodym derivative of $< \mu, x^* >$ with respect to ν.
According to [4] or [11], it is enough to show that for each pair α, $\beta \in I\!\!R$, the f.a.m.

$$\alpha < \mu, x^* > + \beta < \mu, x_R^* > = < \mu, \alpha x^* + \beta x_R^* >$$

admits a Hahn decomposition.
Recalling that μ is s-bounded and that the range of μ is weakly closed, we derive that $im(\mu)$ is weakly compact and hence the range of $< \mu, \alpha x^* + \beta x_R^* >$ is a compact subset of $I\!\!R$. Thus, $< \mu, \alpha x^* + \beta x_R^* >$ attains its maximum on $\mathcal{S}$ and this yields a Hahn decomposition of it (see [1]).
For each $x^* \in X^*$, we shall denote by $f_{x^*} : \Omega \to I\!\!R$ an $\mathcal{S}$-measurable, ν-integrable function such that

$$< \mu(E), x^* > = \int_E f_{x^*}\, d\nu \qquad E \in \mathcal{S}.$$

Here and in the sequel, unless otherwise is stated, integration with respect to a f.a.m. is to be understood as in [9].
Furthermore, for any $x^* \in X^*$, the total variation $| < \mu, x^* > |$ also admits a Radon-Nikodym derivative with respect to ν given by $|f_{x^*}|$.
Next, we approximate $|\mu|$ in $ba(\Omega, \mathcal{S}, I\!R)$. Pick a countable, dense subset $\{x_n^*\}\,(n \geq 1)$ in the boundary ∂B_1^* of the closed unit ball of X^* and set for each $n \in I\!N_+$:

$$\begin{cases} \mu_n = < \mu, x_n^* > \\ f_n = f_{x_n^*}. \end{cases}$$

In addition, let

$$\begin{cases} \lambda_n = \max\limits_{1 \leq m \leq n} |\mu_m| \\ g_n = \max\limits_{1 \leq m \leq n} |f_m| \end{cases} \qquad n \in I\!N_+$$

where maxima are taken in the lattice structures of $ba(\Omega, \mathcal{S}, I\!R)$ and real-valued functions respectively. Hence, each λ_n is a positive, bounded f.a.m. such that

$$0 \leq \lambda_n(E) \leq \lambda_{n+1}(E) \leq |\mu|(E) \qquad E \in \mathcal{S},\ n \in I\!N_+$$

and $\{g_n\}\,(n \geq 1)$ is a nondecreasing sequence of nonnegative, $\mathcal{S}$-measurable and ν-integrable functions.
Then, we have the following:

Lemma 1.1. *$\lambda_n \to |\mu|$ as $n \to \infty$ in $ba(\Omega, \mathcal{S}, I\!R)$.*

Proof. It is enough to show that

$$\lambda_n(\Omega) \to |\mu|(\Omega)$$

as $n \to \infty$.
Let $\epsilon > 0$ be given and choose $\pi = \{E_k\}\,(1 \leq k \leq m) \in I\!P(\Omega)$ such that

$$|\mu|(\Omega) < \sum_{1 \leq k \leq m} ||\mu(E_k)|| + \epsilon/2.$$

For each $k = 1, ..., m$ let $x_{n_k}^*$ be such that

$$||\mu(E_k)|| < | < \mu(E_k), x_{n_k}^* > | + \epsilon/(2m) \qquad 1 \leq k \leq m$$

and then set $n_0(\epsilon) = \max_{1 \leq k \leq m} n_k$.
Then, for any $n \geq n_0(\epsilon)$ we have:

$$0 \leq |\mu|(\Omega) - \lambda_n(\Omega) < \sum_{1 \leq k \leq m} | < \mu(E_k), x_{n_k}^* > | + \epsilon - \lambda_n(\Omega) \leq$$

$$\leq \sum_{1\leq k\leq m} |\mu_{n_k}|(E_k) + \epsilon - \lambda_n(\Omega) \leq \sum_{1\leq k\leq m} \lambda_n(E_k) - \lambda_n(\Omega) + \epsilon = \epsilon.$$

Thus, $\lambda_n \to |\mu|$ as $n \to \infty$ in $ba(\Omega, \mathcal{S}, I\!R)$. q.e.d.

Lemma 1.2. $\lambda_n(E) = \int_E g_n d\nu \qquad E \in \mathcal{S},\ n \in I\!N_+$.

Proof. Let $n \in I\!N_+$ be fixed. By the optimality of λ_n, we get

$$\lambda_n(E) \leq \int_E g_n d\nu \qquad E \in \mathcal{S}.$$

Conversely, let

$$\begin{aligned}
E_1 &= \{\omega \in \Omega: g_n(\omega) = |f_1(\omega)|\}\\
E_2 &= \{\omega \in \Omega: g_n(\omega) = |f_2(\omega)|\} \setminus E_1\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
E_n &= \{\omega \in \Omega: g_n(\omega) = |f_n(\omega)|\} \setminus \{E_1 \cup \ldots \cup E_{n-1}\}
\end{aligned}$$

so that $\{E_m\}\,(1 \leq m \leq n) \in I\!P(\Omega)$. Then,

$$0 \leq \int_E g_n d\nu = \sum_{1\leq m\leq n} \int_{E\cap E_m} |f_m| d\nu =$$

$$= \sum_{1\leq m\leq n} |\mu_m|(E \cap E_m) \leq \sum_{1\leq m\leq n} \lambda_n(E \cap E_m) = \lambda_n(E)$$

for each $E \in \mathcal{S}$ and this shows that λ_n is the indefinite integral of g_n for each $n \in I\!N_+$. q.e.d.

Remark 1.3. In remark 0.2 we pointed out that the construction of the approximating sequence $\{\lambda_n\}(n \geq 1)$ can be accomplished even if the dual space X^* is not assumed to be separable. Let us see how this can be done.
Let $\mathcal{F}(\partial B_1^*)$ be the filter of all finite subsets of $\mathcal{F}(\partial B_1^*)$ and for each $I \in \mathcal{F}(\partial B_1^*)$ set

$$\lambda_I = \max_{x^*\in I} | < \mu, x^* > |.$$

Hence, $\{\lambda_I\}(I \in \mathcal{F}(\partial B_1^*)$ is a net of positive, bounded f.a.m.'s converging to $|\mu|$ in $ba(\Omega, \mathcal{S}, I\!R)$. Now, choose an increasing sequence $I_n \in \mathcal{F}(\partial B_1^*)$, $n \geq 1$ such that

$$|\mu|(\Omega) - \frac{1}{n} \leq \lambda_{I_n}(\Omega) \leq |\mu|(\Omega)$$

and set $\lambda_n = \lambda_{I_n}$ for all n. It is plain that $\{\lambda_n\}$ converges in $ba(\Omega, \mathcal{S}, I\!R)$ to a f.a.m. which does not exceed $|\mu|$. To see that the limit of the λ_n's is actually $|\mu|$, suppose there exists $E \in \mathcal{S}$ such that $\lim_{n\to\infty} \lambda_n(E) < |\mu|(E)$; then, the strict reversed inequality holds for the complementary set $\Omega \setminus E$ and this yields a contradiction. Thus, $\lambda_n \to |\mu|$ as $n \to \infty$ in $ba(\Omega, \mathcal{S}, I\!R)$.
Finally, it is easy to construct the density functions of the λ_n's with respect to ν as in the case X^* is separable.

2. Representation of $|\mu|$ as an indefinite integral with respect to ν.

So far, we have approximated $|\mu|$ in $ba(\Omega, \mathcal{S}, I\!R)$ by an increasing sequence $\{\lambda_n\}$ $(n \geq 1)$ of bounded f.a.m.'s each of which is an indefinite integral with respect to ν. Hence, it is natural to inquire if

$$f = \sup_{n\geq 1} |f_n| = \sup_{n\geq 1} g_n$$

is a Radon-Nikodym derivative of $|\mu|$ with respect to ν.
That this is the case, provided a Radon-Nikodym of $|\mu|$ with respect to ν does exist, is proved by the following lemma which, roughly speaking, asserts that the density functions of the λ_n's with respect to ν can always be arranged in such a way that their supremum is actually a density function of $|\mu|$ with respect to ν.

Lemma 2.1. *Suppose that $|\mu|$ has a Radon-Nikodym derivative with respect to ν; then, there exist $\mathcal{S}$-measurable, ν-integrable functions g', $g'_n\colon \Omega \to [0,\infty]$ $\quad n \in I\!N_+$ such that:*
1) $\{g'_n\}$ $(n \geq 1)$ *is nondecreasing and* $g' = \sup_{n\geq 1} g'_n$;
2) $\lambda_n(E) = \int_E g'_n d\nu \qquad E \in \mathcal{S},\ n \in I\!N_+$;
3) $|\mu|(E) = \int_E g' d\nu \qquad E \in \mathcal{S}$.

Proof. By assumption, there exists a nonnegative, $\mathcal{S}$-measurable and ν-integrable function f' such that

$$|\mu|(E) = \int_E f' d\nu \qquad E \in \mathcal{S}.$$

Let $g'_n = g_n \wedge f'$, $n \in I\!N_+$. Then, $\{g'_n\}$ $(n \geq 1)$ is a nondecreasing sequence of nonnegative, $\mathcal{S}$-measurable and ν-integrable functions.
Fix $n \in I\!N_+$ and let $F_i^n \in \mathcal{S}$ $i = 1,2$ be a partition of Ω such that

$$\begin{cases} g'_n(\omega) = g_n(\omega) & \omega \in F_1^n \\ g'_n(\omega) = f'(\omega) & \omega \in F_2^n; \end{cases}$$

since $g'_n \leq g_n$, we get for each $E \in \mathcal{S}$

$$0 \geq \int_E (g'_n - g_n)\, d\nu = \int_{E\cap F_1^n} (g'_n - g_n)\, d\nu + \int_{E\cap F_2^n} (g'_n - g_n)\, d\nu =$$

$$= \int_{E\cap F_2^n} (g'_n - g_n)\, d\nu = |\mu|(E \cap F_2^n) - \lambda_n(E \cap F_2^n) \geq 0$$

that is, g'_n is also a Radon-Nikodym derivative of λ_n with respect to ν.
Next, set $g' = \sup_{n\geq 1} g'_n$ so that $g' \leq f'$, g' is nonnegative, $\mathcal{S}$-measurable and ν-integrable. Then, we have

$$|\mu|(E) = \int_E f' d\nu \geq \int_E g' d\nu \geq \int_E g'_n d\nu = \lambda_n(E) \qquad E \in \mathcal{S},\ n \in I\!N_+$$

and this completes the proof. q.e.d.

Let's now go back to f. Since f is the pointwise limit of $\{g_n\}\,(n \geq 1)$, we see that f is $\mathcal{S}$-measurable and nonnegative so that its monotone integral is well-defined:

$$(M)-\int_E f d\nu = \int_0^\infty \nu\left(E \cap \{\omega \in \Omega: f(\omega) > t\}\right) dt \qquad E \in \mathcal{S}.$$

In addition, as soon as the monotone integral is finite, f is also ν-integrable according to [9] and these two integrals coincide. Furthermore, since f dominates all g_n's and $\lambda_n \to |\mu|$ as $n \to \infty$ it is easy to see that

$$|\mu|(E) \leq (M)-\int_E f d\nu \qquad E \in \mathcal{S}.$$

We can now prove that the problem of representing $|\mu|$ as an indefinite integral with respect to ν reduces to a problem of integrability and convergence in measure.
Indeed, an application of lemma 1 and Vitali's convergence theorem shows that the following are equivalent:

1) there exists a Radon-Nikodym derivative of $|\mu|$ with respect to ν;

2) f is ν-integrable and $g_n \to f$ as $n \to \infty$ in ν-measure.

Furthermore, $\{g_n\}\,(n \geq 1)$ is a Cauchy sequence in ν-measure as the following lemma shows:

Lemma 2.2. *$\{g_n\}\,(n \geq 1)$ is a Cauchy sequence in ν-measure.*

Proof. Suppose not. Then, there exist $\eta > 0$, $\delta > 0$ and an increasing sequence of integers $\{n_m\}\,(m \geq 1)$ such that

$$\begin{cases} E_m = \{\omega \in \Omega: g_{n_{m+1}}(\omega) - g_{n_m}(\omega) \geq \eta\} \\ \nu(E_m) \geq \delta \end{cases} \qquad m \in \mathbb{N}_+.$$

Note that $\{E_m\}\,(m \geq 1) \subset \mathcal{S}$ and

$$\lambda_{n_{m+1}}(E_m) - \lambda_{n_m}(E_m) = \int_{E_m} \left(g_{n_{m+1}}(\omega) - g_{n_m}(\omega)\right) d\nu \geq \eta\delta \qquad m \in \mathbb{N}_+.$$

This yields a contradiction since $\{\lambda_n\}\,(n \geq 1)$ converges in $ba(\Omega, \mathcal{S}, \mathbb{R})$. q.e.d.

Unfortunately, no completeness result for convergence in measure is available in the context of finite additivity.
That's all we can say in the general case. However, there are cases where everything goes well. Here there is one of them.

Let μ satisfy (P1), (P2) and in addition assume there exist $\{x_n^*\}\,(n \geq 1) \subset \partial B_1^*$ such that

(P4) $\mu_n = <\mu, x_n^*> \geq 0$ and $\mu_{n+1} \geq \mu_n$, $n \in \mathbb{N}_+$;

(P5) $\mu_n \to |\mu|$ as $n \to \infty$ in $ba(\Omega, \mathcal{S}, I\!R)$.

Note that in this case the separability of X^* is not required. Then, $\lambda_n = \mu_n$, $g_n = |f_n| = f_n$ for each $n \in I\!N_+$ and we can assume that $\{f_n\}\,(n \geq 1)$ is increasing with respect to n. Moreover, we can choose

$$x_R^* = \left(\| \sum_{m\geq 1} 2^{-m} x_m^* \| \right)^{-1} \sum_{n\geq 1} 2^{-n} x_n^*,$$

$$\nu = < \mu, x_R^* > = \sum_{n\geq 1} \frac{\mu_n}{2^n \| \sum_{m\geq 1} 2^{-m} x_m^* \|}.$$

Then, we have the following:

Proposition 2.3. *There exists a Radon-Nikodym derivative of $|\mu|$ with respect to ν.*

Proof. According to [11], it is enough to show that $|\mu| - r\nu$ attains its maximum on $\mathcal{S}$ for each $r > 0$. Fix $r > 0$ and set

$$\sup_{E \in \mathcal{S}} (|\mu| - r\nu)(E) = \sup_{E \in \mathcal{S}} \sup_{n \geq 1} (|\mu| - r\nu)(E) = a < \infty,$$

$$A_n = \{\omega \in \Omega : f_n(\omega) > r\} \qquad n \in I\!N_+$$

so that $\{A_n\}\,(n \geq 1) \subset \mathcal{S}$ is an increasing sequence of sets such that

$$\sup_{E \in \mathcal{S}} (\mu_n - r\nu)(E) = (\mu_n - r\nu)(A_n) \qquad n \in I\!N_+.$$

Since $\{\mu_m - r\nu\}\,(n \geq 1)$ is a uniformly s-bounded sequence of f.a.m.'s, then $(\mu_m - r\nu)(A_n) \to a_m$ as $n \to \infty$ uniformly with respect to $m \in I\!N_+$.
Furthermore, the s-boundnedness of μ implies that $\{\mu(A_n)\}\,(n \geq 1)$ converges in the norm topology and hence weakly in X; by (P2), there exists $A \in \mathcal{S}$ such that $\mu(A_n) \to \mu(A)$ as $n \to \infty$.
Thus,

$$\begin{aligned} a_m &= \lim_{n\to\infty} (\mu_m - r\nu)(A_n) = \lim_{n\to\infty} < \mu(A_n), x_m^* - r x_R^* > = \\ &= < \mu(A), x_m^* - r x_R^* > = (\mu_m - r\nu)(A) \end{aligned}$$

uniformly with respect to $m \in I\!N_+$.
This, together with

$$(\mu_m - r\nu)(A_n) \to (|\mu| - r\nu)(A_n) \qquad m \to \infty, \quad n \in I\!N_+$$

shows that the net $\{(\mu_m - r\nu)(A_n)\}_{(m,n) \in I\!N_+^2}$ converges so that the limits can be interchanged:

$$\begin{aligned} a &= \sup_{E \in \mathcal{S}} \sup_{m \geq 1} (\mu_m - r\nu)(E) = \lim_{m\to\infty} (\mu_m - r\nu)(A_m) = \lim_{(m,n)} (\mu_m - r\nu)(A_n) = \\ &= \lim_{m\to\infty} \lim_{n\to\infty} (\mu_m - r\nu)(A_n) = \lim_{m\to\infty} (\mu_m - r\nu)(A) = (|\mu| - r\nu)(A) \end{aligned}$$

and so we are done. q.e.d.

Now, lemma 2.1 shows that by a proper choice of the f_n's we have

$$|\mu|(E) = \int_E f d\nu \qquad E \in \mathcal{S}$$

where $f = \sup_{n\geq 1} f_n$.
Finally, let's provide an example of a measure satisfying (P4) and (P5). Let $\pi: \mathcal{S} \to [0,1]$ be a finitely additive probability and let $\{E_n\}(n \geq 1) \subset \mathcal{S}$ be a countable partition of Ω so that the series

$$\sum_{n\geq 1} \pi(E \cap E_n) \qquad E \in \mathcal{S}$$

unconditionally converges for all $E \in \mathcal{S}$. Then $\mu: \mathcal{S} \to l_1(\mathbb{N}_+)$ defined by $\mu(E) = (\pi(E \cap E_n))_{n\geq 1}$ for all $E \in \mathcal{S}$ turns out to be a f.a.m. whose total variation is easily seen to be

$$|\mu|(E) = \sum_{n\geq 1} \pi(E \cap E_n) \qquad E \in \mathcal{S}.$$

Now, for each $k \in \mathbb{N}_+$, let $x_k^* \in l_\infty(\mathbb{N}_+)$ be the sequence of unit $l_\infty(\mathbb{N}_+)$-norm whose first k entries are 1's followed by a tail of 0's and set $\mu_k(E) =< \mu(E), x_k^* > \; E \in \mathcal{S}$. It is easy to check that

$$\mu_k(E) = \pi(E \cap F_k) \qquad E \in \mathcal{S},\ k \in \mathbb{N}_+$$

where $F_k = \cup_{1\leq h\leq k} E_h,\ k \in \mathbb{N}_+$.
Then, it is clear that $\{\mu_k\}(k \geq 1)$ satisfies (P4) while (P5) follows from

$$0 \leq \sup_{E\in\mathcal{S}} \left| \mu_k(E) - |\mu|(E) \right| = (|\mu| - \mu_k)(\Omega) = \sum_{n\geq k+1} \pi(E_n) \to 0 \qquad k \to \infty$$

and the summability of $\{\pi(E_n)\}_{n\geq 1}$.

3. Reconstruction of μ as a Dunford-type integral.

Let f, f_n and g_n be fixed as in section 2. We now assume that f is ν-integrable and that $g_n \to f$ as $n \to \infty$ in ν-measure so that $|\mu|$ is the indefinite integral of f with respect to ν.
Set $g: \Omega \to [0, \infty]$, $g(\omega) = 1/f(\omega)$ with the usual conventions $1/0 = \infty$, $1/\infty = 0$. We claim that

a) g is $|\mu|$-integrable;

b) g is a Radon-Nikodym derivative of ν with respect to $|\mu|$.

First, we prove (b) assuming that (a) holds.

Lemma 3.1. *Let $\mathcal{A}$ be an algebra of subsets of Ω and let $\pi_i \in ba(\Omega, \mathcal{A}, I\!R)$, $i = 1, 2$ be positive measures such that:*

1) $\pi_1 \sim \pi_2$;

2) there exists a π_1-integrable function $p: \Omega \to [0, \infty]$ such that

$$\pi_2(E) = \int_E p d\pi_1 \qquad E \in \mathcal{S}.$$

Then, for each π_2-integrable function $q: \Omega \to I\!R$, (qp) is π_1-integrable and

$$\int_E q d\pi_2 = \int_E (qp) d\pi_1 \qquad E \in \mathcal{S}.$$

Proof. Choose a π_2-integrable function q and a determining sequence $\{q_n\}(n \geq 1)$ for q consisting of π_2-simple functions.

i) Let $\epsilon, \eta > 0$ be given. By (1), there exists $\delta(\epsilon) > 0$ such that $E \subset \Omega$, $\pi_2^*(E) < \delta(\epsilon)$ implies $\pi_1^*(E) < \epsilon$ where π_i^* denotes the submeasure associated to π_i by

$$\pi_i^*(E) = \inf_{E \in \mathcal{S},\, E \subset F} (F)$$

for $i = 1, 2$. Furthermore, there exists $n_0(\epsilon, \eta) \geq 1$ such that $n \geq n_0(\epsilon, \eta)$ implies

$$\pi_2^* \left(\{\omega \in \Omega: |q_n(\omega) - q(\omega)| > \eta\} \right) < \delta(\epsilon);$$

hence,

$$\pi_1^* \left(\{\omega \in \Omega: |q_n(\omega) - q(\omega)| > \eta\} \right) < \epsilon$$

for all $n \geq n_0(\epsilon, \eta)$ and this shows that $q_n \to q$ as $n \to \infty$ in π_1-measure.

ii) Let $s = \sum_{1 \leq m \leq n} s_m \chi_{E_m}$ be an arbitrary simple function such that $\{s_m\}_{1 \leq m \leq n} \subset I\!R \setminus \{0\}$ and the sets $\{E_m\}_{1 \leq m \leq n} \subset \mathcal{A}$ are pairwise disjoint; s is π_1- and π_2-integrable . Then, (sp) is π_1-integrable and

$$\int_\Omega s d\pi_2 = \sum_{1 \leq m \leq n} s_m \pi_2(E_m) = \sum_{1 \leq m \leq n} s_m \left(\int_\Omega \chi_{E_m} p d\pi_1 \right) =$$

$$= \int_\Omega \left(\sum_{1 \leq m \leq n} s_m \chi_{E_m} \right) p d\pi_1 = \int_\Omega (sp) d\pi_1.$$

Thus, the conclusions of the lemma hold for all π_2-simple functions.

The π_1-integrability of p together with (i) shows that $(q_n p) \to (qp)$ as $n \to \infty$ in π_1-measure. Furthermore, each $(q_n p)$ is π_1-integrable since each q_n is bounded on Ω.
Since $|q_n - q_m|$ is a π_2-simple function, we have

$$\int_\Omega |q_n p - q_m p| d\pi_1 = \int_\Omega |q_n - q_m| p d\pi_1 = \int_\Omega |q_n - q_m| d\pi_2$$

for each $m, n \in \mathbb{N}_+$ and this shows that the net $\{\int_\Omega |q_n p - q_m p| d\pi_1\}((m,n) \in \mathbb{N}_+^2)$ converges. Thus, (qp) is π_1-integrable and

$$\int_\Omega q d\pi_2 = \lim_{n\to\infty}\left(\int_\Omega q_n d\pi_2\right) = \lim_{n\to\infty}\left(\int_\Omega (q_n p) d\pi_1\right) = \int_\Omega (qp) d\pi_1.$$

This completes the proof. q.e.d.

Proposition 3.2. *Let g be $|\mu|$-integrable; then, $\nu(E) = \int_E g d|\mu| \quad E \in \mathcal{S}$.*

Proof. Set $\pi(E) = \nu(E) - \int_E g d|\mu| \quad E \in \mathcal{S}$; $\pi \in ba(\Omega, \mathcal{S}, \mathbb{R})$. Lemma 3.1 yields

$$\int_E g d|\mu| = \int_E (gf) d\nu \qquad E \in \mathcal{S}$$

and hence

$$\pi(E) = \nu(E) - \int_E (gf) d\nu = \nu(E) - \nu\left(E \setminus [\{\omega \in \Omega: f(\omega) = 0\} \cup \{\omega \in \Omega: f(\omega) = \infty\}]\right)$$

for all $E \in \mathcal{S}$ (here, the convention $0 \cdot \infty = 0$ has been used). Since

$$f \ \nu\text{-integrable} \Longrightarrow \nu\left(\{\omega \in \Omega: f(\omega) = \infty\}\right) = 0$$

$$|\mu|\left(\{\omega \in \Omega: f(\omega) = 0\}\right) = 0 \Longrightarrow \nu\left(\{\omega \in \Omega: f(\omega) = 0\}\right) = 0$$

we get $\pi = 0$. q.e.d.

So far, we are left to prove (a). We begin with some lemmas.

Lemma 3.3. *Let $\mathcal{A}$ be an algebra of subsets of Ω, let $\pi \in ba(\Omega, \mathcal{A}, \mathbb{R})$ be a positive measure and let $p, p_n: \Omega \to (0, \infty)$, $n \in \mathbb{N}_+$ be functions such that*

1) *$p_n \to p$ as $n \to \infty$ in π-measure;*
2) *for all $\epsilon > 0$ there exists $M(\epsilon) \in (1, \infty)$ such that*

$$\pi^*\left(\left\{\omega \in \Omega: 0 < p(\omega) \le \frac{1}{M}\right\} \cup \{\omega \in \Omega: M \le p(\omega) < \infty\}\right) < \epsilon.$$

Then, setting $q(\omega) = 1/p(\omega)$, $q_n(\omega) = 1/p_n(\omega)$, $\omega \in \Omega$, $n \in \mathbb{N}_+$ we get $q_n \to q$ as $n \to \infty$ in π-measure.

Proof. Let ϵ, $\eta > 0$ be given and pick $1 < M(\epsilon) < \infty$ such that

$$\pi^*\left(\left\{\omega \in \Omega: 0 < p(\omega) \le \frac{1}{M}\right\} \cup \{\omega \in \Omega: M \le p(\omega) < \infty\}\right) < \epsilon/2.$$

Since $t \in [\frac{1}{2M}, 2M] \to \frac{1}{t} \in [\frac{1}{2M}, 2M]$ is uniformly continuous, there exists a positive $\delta(\epsilon, \eta)$ such that $|s^{-1} - t^{-1}| < \eta$ for all $s, t \in [\frac{1}{2M}, 2M]$, $|s - t| < \delta(\epsilon, \eta)$. Assume that $0 < \delta(\epsilon, \eta) < \frac{1}{M}$. Next, choose $n_0(\epsilon, \eta) \ge 1$ such that $n \ge n_0$ implies

$$\pi^*\left(\{\omega \in \Omega\colon |p(\omega) - p_n(\omega)| \geq \delta(\epsilon,\eta)\}\right) < \epsilon/2.$$

Then, for all $n \geq n_0$ we get

$$\pi^*\left(\{\omega \in \Omega\colon |q(\omega) - q_n(\omega)| > \eta\}\right) \leq$$

$$\leq \pi^*\left(\left\{\omega \in \Omega\colon 0 < p(\omega) \leq \frac{1}{M} \text{ or } M \leq p(\omega) < \infty \text{ and } \left|\frac{1}{p(\omega)} - \frac{1}{p_n(\omega)}\right| > \eta\right\}\right) +$$

$$+\pi^*\left(\left\{\omega \in \Omega\colon \frac{1}{M} < p(\omega) < M,\ \left|\frac{1}{p(\omega)} - \frac{1}{p_n(\omega)}\right| > \eta,\ |p(\omega) - p_n(\omega)| < \delta\right\}\right) +$$

$$+\pi^*\left(\left\{\omega \in \Omega\colon \frac{1}{M} < p(\omega) < M,\ \left|\frac{1}{p(\omega)} - \frac{1}{p_n(\omega)}\right| > \eta,\ |p(\omega) - p_n(\omega)| \geq \delta\right\}\right) <$$

$$< \epsilon/2 + 0 + \epsilon/2$$

and this completes the proof. q.e.d.

Lemma 3.4. *For all $\epsilon > 0$ there exists $M(\epsilon) \in (0,\infty)$ such that:*
1) $|\mu|\left(\{\omega \in \Omega\colon 0 \leq f(\omega) \leq \frac{1}{M}\}\right) < \epsilon/2$;
2) $|\mu|\left(\{\omega \in \Omega\colon M \leq f(\omega)\}\right) < \epsilon/2$.

Proof. Let $\epsilon > 0$ be given and choose $M_1(\epsilon) > \max\left\{1, \frac{2\nu(\Omega)}{\epsilon}\right\}$; then,

$$|\mu|\left(\left\{\omega \in \Omega\colon 0 \leq f(\omega) \leq \frac{1}{M_1}\right\}\right) = \int_{\left\{\omega\in\Omega:0\leq f(\omega)\leq \frac{1}{M_1}\right\}} f d\nu \leq$$

$$\leq \frac{1}{M_1}\nu\left(\left\{\omega \in \Omega\colon 0 \leq f(\omega) \leq \frac{1}{M_1}\right\}\right) \leq \frac{1}{M_1}\nu(\Omega) < \epsilon/2.$$

Furthermore, the ν-continuity of $|\mu|$ yields a $\delta(\epsilon) > 0$ such that $E \in \mathcal{S}$, $\nu(E) < \delta(\epsilon)$ implies $|\mu|(E) < \epsilon/2$ and the ν-integrability of f provides an $M_2(\epsilon) > 1$ such that

$$\nu\left(\{\omega \in \Omega\colon M_2 \leq f(\omega)\}\right) < \delta(\epsilon).$$

Thus,

$$|\mu|\left(\{\omega \in \Omega\colon M_2 \leq f(\omega)\}\right) < \epsilon/2.$$

Setting $M(\epsilon) = \max\{M_1, M_2\}$ we are done. q.e.d.

Proposition 3.5. *g is totally $|\mu|$-measurable.*

Proof. The sets $\{f = 0\}$, $\{f = \infty\}$ are both $|\mu|$- and ν-null sets. Thus, we may assume that $0 < f(\omega) < \infty$ for all $\omega \in \Omega$ and the same holds true for g as well.
Now, let $\{s_n\}(n \in I\!N_+)$ be a sequence of ν-simple functions such that $s_n \to f$ as $n \to \infty$ in ν-measure; it is plain that the s_n's are $|\mu|$-simple functions and that $s_n \to f$ as $n \to \infty$ in $|\mu|$-measure as well.

By lemma 3.4, the sets on wich f is small are small so we may assume that $s_n > 0$ for all $n \geq 1$. Finally, $t_n = 1/s_n$ is a $|\mu|$-simple function and an appeal to lemma 3.3 yields the total $|\mu|$-measurability of g. q.e.d.

Proposition 3.6. *g is $|\mu|$-integrable.*

Proof. The functions $f \vee \frac{1}{n}$, $g \wedge n$, $n \in I\!N_+$ are ν-integrable and $|\mu|$-integrable respectively. Moreover, $(f \vee \frac{1}{n})(g \wedge n) = 1$ on Ω for all n.
Then, set $E_n = \{\omega \in \Omega : 0 < f(\omega) \leq \frac{1}{n}\}$ and $g'_n = g(1 - \chi_{E_n})$ for $n \geq 1$ so that each g'_n is totally $|\mu|$-measurable by proposition 3.5 and $|\mu|$-integrable since it is bounded. It is plain that $g'_n \to g$ as $n \to \infty$ in $|\mu|$-measure since $|\mu|(E_n) \to 0$ as $n \to \infty$.
Now, we claim that the indefinite integrals of the g'_n's with respect to $|\mu|$ are μ-equicontinuous. To see this, let

$$\pi_n(E) = \int_E (f \vee \frac{1}{n}) d\nu = \int_{E \cap E_n} (f \vee \frac{1}{n}) d\nu + \int_{E \setminus E_n} (f \vee \frac{1}{n}) d\nu = $$
$$= \frac{1}{n}\nu(E \cap E_n) + |\mu|(E \setminus E_n)$$

for all $E \in \mathcal{S}$, $n \geq 1$. Since $\pi_n \sim \nu$ and $g \wedge n$ is ν-integrable for all n, an appeal to proposition 3.2 yields

$$\nu(E) = \int_E (g \wedge n) d\pi_n =$$
$$= \int_{E \setminus E_n} (g \wedge n) d|\mu| + \frac{1}{n} \int_{E \cap E_n} (g \wedge n) d\nu =$$
$$= \int_E g'_n d|\mu| + \frac{1}{n} n\nu(E \cap E_n) \geq \int_E g'_n d|\mu|$$

for all $E \in \mathcal{S}$, $n \geq 1$. Thus, the indefinite integrals of the g'_n's with respect to $|\mu|$ are ν-equicontinuous and hence μ-equicontinuous as well. The $|\mu|$-integrability of g follows by Vitali's convergence theorem. q.e.d.

We have thus proved (a) and (b). Now, recalling that

$$< \mu(E), x^* >= \int_E f_{x^*} d\nu \qquad E \in \mathcal{S}$$

for all $x^* \in X^*$ and applying lemma 3.1 we easily get:

$$< \mu(E), x^* >= \int_E (f_{x^*} g) d|\mu| \qquad E \in \mathcal{S},\ x^* \in X^*.$$

Finally, we can reconstruct a weakly $|\mu|$-integrable function which weakly represents μ as an integral. The proof given here is taken from [4].

Proposition 3.7. *Let X^* be separable, let μ satisfy (P1), (P3) and assume that for each $x^* \in X^*$ there exists an $\mathcal{S}$-measurable and $|\mu|$-integrable function $h_{x^*}: \Omega \to I\!R$ such that*

$$< \mu(E), x^* >= \int_E h_{x^*} d|\mu| \qquad E \in \mathcal{S}.$$

Then, there exists a weakly $|\mu|$-integrable function $h: \Omega \to X^{**}$ such that

$$< \mu(E), x^* >= \int_E < x^*, h(\omega) > d|\mu|(\omega) \qquad E \in \mathcal{S},\ x^* \in X^*.$$

Proof. Recall that X^* is separable by assumption and pick a countable total set $\{e_n^*\}(n \in \mathbb{N}_+)$ in X^* consisting of linearly independent vectors of unit norm. For each $n \geq 1$, set $h_n = h_{e_n^*}$. We may assume that $|h_n(\omega)| \leq 1$ for all $\omega \in \Omega$ and $n \geq 1$.
Indeed, let $h_n' = |h_n| \wedge 1$ for $n \geq 1$; for each n there exists an $\mathcal{S}$-measurable partition $\{\Omega_1^n, \Omega_2^n\}$ of Ω such that $h_n' = |h_n|$ on Ω_1^n and $h_n' < |h_n|$ on Ω_2^n. Since $h_n' \leq |h_n|$ on Ω, we get for any $E \in \mathcal{S}$:

$$0 \geq \int_E (h_n' - |h_n|) d|\mu| = \int_{E\cap\Omega_1^n} (h_n' - |h_n|) d|\mu| + \int_{E\cap\Omega_2^n} (h_n' - |h_n|) d|\mu| =$$

$$= \int_{E\cap\Omega_2^n} (h_n' - |h_n|) d|\mu| = |\mu|(E \cap \Omega_2^n) - |< \mu, e_n^* >|(E \cap \Omega_2^n) \geq |\mu|(E\cap\Omega_2^n) - \|\mu\|(E \cap \Omega_2^n) \geq 0.$$

Now, let F^* be the linear space spanned by $\{e_n^*\}(n \geq 1)$ over the field of rational numbers; F^* is a countable and dense subset of X^*. Then, define $h_0: \Omega \times F^* \to \mathbb{R}$ by

$$h_0 : (\omega, x^*) \in \Omega \times F^* \to h_0(\omega, x^*) = \sum_{1\leq m \leq n} \alpha_m h_m(\omega) \in \mathbb{R}$$

where $x^* = \sum_{1\leq m\leq n} \alpha_m e_m^*$ and note that

1) the mapping $\omega \in \Omega \to h_0(\omega, x^*) \in \mathbb{R}$ is $|\mu|$-integrable for each $x^* \in F^*$;

2) $< \mu(E), x^* >= \int_E h_0(\omega, x^*) d|\mu|(\omega) \quad E \in \mathcal{S},\ x^* \in F^*$;

3) the mapping $x^* \in F^* \to h_0(\omega, x^*) \in \mathbb{R}$ is linear over the field of rationals for all $\omega \in \Omega$.

These relations and (P3) show that

$$|\mu| - \text{supess}_{\omega\in\Omega} |h_0(\omega, x^*)| \leq \|x^*\|$$

holds for any fixed $x^* \in F^*$. Hence, as F^* is countable and (P3) holds, there exists $N \in \mathcal{N}(|\mu|)$ such that

$$\sup_{\omega\in\Omega} |h_0(\omega, x^*)|(1 - \chi_N(\omega)) \leq \|x^*\|$$

holds for all $x^* \in F^*$.
Then, redefine h_0 equal to 0 on $N \times F^*$ so that (1), (2) and (3) still hold and note that now $h_0(\omega, \cdot) : x^* \in F^* \to h_0(\omega, x^*)\mathbb{R}$ turns out to be a continuous linear functional on F^* of norm at most 1 for each $\omega \in \Omega$. Hence, by the denseness of F^*, for each $\omega \in \Omega$ there exists a unique norm-preserving linear extension of $h_0(\omega, \cdot)$, say $h(\omega) \in X^{**}$. The mapping $\omega \in \Omega \to h(\omega) \in X^{**}$ is the one we looked for.
q.e.d.

Remark 3.8. If X is $c_0(\mathbb{N}_+)$, hypothesis (P3) can be dropped.
Indeed, let $e_n^* \in l_1(\mathbb{N}_+)$ be the sequence whose entries are all 0's but the n-th which is 1 and let F^* and h_0 be defined as in proposition 3.7. For any fixed $\omega \in \Omega$, let $\varphi_\omega: F^* \to \mathbb{R}$ be defined by

$$\varphi_\omega : x^* = \sum_{1 \leq m \leq n} \alpha_m e_m^* \in F^* \to \varphi_\omega x^* = \sum_{1 \leq m \leq n} \alpha_m h_m(\omega) \in \mathbb{R};$$

as $|h_m(\omega)| \leq 1$ for all $m \geq 1$, it is plain that φ_ω is continuous and linear on F^* over the field of rationals with norm at most 1 for all $\omega \in \Omega$ so that

$$\sup_{\omega \in \Omega} |h_0(\omega, x^*)| = \sup_{\omega \in \Omega} |\varphi_\omega x^*| \leq \|x^*\|$$

holds for all $x^* \in F^*$. Then, the remaining part of the proof goes as in proposition 3.7.

Acknowledgements. I am very grateful to Prof. D.Candeloro for his constant attention and encouragement in the development of this research. I also wish to thank Prof.'s A.Cellina, G.Dal Maso and G.Vidossich of SISSA who, in some way or another, helped me with valuable discussions and advices.

References.

[1] K.P.S.Bhaskara Rao-M.Bhaskara Rao "Theory of charges." Academic Press, London (1983).

[2] S.Bochner "Additive set functions on groups." Ann. Math. 40 (1939) 769-799.

[3] J.K.Brooks "On the existence of a control measure for strongly bounded vector measures." Bull. Amer. Math. Soc. 77 (1971) 999-1001.

[4] D.Candeloro-A.Martellotti "A Radon-Nikodym theorem for vector-valued finitely additive measures with closed range." preprint Universitá di Perugia (1988).

[5] R.B.Darst-E.Green "On a Radon-Nikodym theorem for finitely additive set functions." Pacific J. Math. 27 (1968) 255-259.

[6] J.Diestel-J.J.Uhl jr. "The Radon-Nikodym theorem for Banach space valued measures." Rocky Mountain J. Math. 6 (1976) 1-46.

[7] J.Diestel-J.J.Uhl jr. "Vector measures." Amer. Math. Soc. Mathematical Surveys 15, Providence (Rhode Island) (1977).

[8] L.E.Dubins "An elementary proof of Bochner's finitely additive Radon-Nikodym theorem." Amer. Math. Monthly 76 (1969) 520-523.

[9] N.Dunford-J.T.Schwartz "Linear operators part I: general theory." Wiley-Interscience, New York (1958).

[10] C.Fefferman "A Radon-Nikodym theorem for finitely additive set functions." Pacific J. Math. 23 (1967) 35-45.

[11] G.H.Greco "Un teorema di Radon-Nikodym per funzioni d'insieme subadditive." Ann. Univ. Ferrara sez. VII (Scie. Mat.) 27 (1981) 13-19.

[12] J.W.Hagood "A Radon-Nikodym theorem and L_p-completeness for finitely additive measures." J. Math. Anal. Appl. 113 (1986) 266-279.

[13] S.S.Khurana "A note on Radon-Nikodym theorem for finitely additive measures." Pacific J. Math. 74 (1978) 103-104.

[14] H.B.Maynard "A Radon-Nikodym theorem for finitely additive bounded measures." Pacific J. Math. 82 (1979) 401-413.

[15] J.Pachl "An elementary proof of Radon-Nikodym theorem for finitely additive set functions." Proc. Amer. Math. Soc. 32 (1972) 225-228.

[16] T.Traynor "s-bounded additive set functions." in "Vector and operator valued measures and applications." edited by D.H.Tucker-H.B.Maynard, Academic Press, New York (1973) 353-365.

Trieste, June 20, 1992.