Topological Sigma Model and Equivariant Cohomology

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1. Introduction

In the last four years a new class of theories, topological field theories developped extensively. This current started with the papers by Witten (1988). Topological field theories have though a longer history. At the end of the seventies, Schwarz expressed the Ray-Singer torsion (a topological invariant) as the partition function of a quantum field theory. This is a general feature of these models.
What defines such theories is the existence of a BRST-like operator, Q which is nilpotent ($Q^2 = 0$), such that the energy-momentum tensor is a Q-exact quantity, $T_{\mu\nu} = \{Q, \Lambda_{\mu\nu}\}$. It is well-known that in Chern-Simons theories the action does not depend on the gravitational background. As a consequence, the energy-momentum tensor of the theory vanishes, there are no physical excitations like particles and the theory is topological.
From quantum point of view, these models are of two types; **quantum** theories, in which the two-loop contribution (and possible higher loop) is nonzero (examples are Chern-Simons theories) and **cohomological** or **semiclassical** which are one-loop exact.
In this thesis I consider only cohomological (or Witten) theories, with emphasis on the topological sigma models. Observables in such models are Q-cohomology classes, i.e. operators $\mathcal{O}_i$ such that

$$\{Q, \mathcal{O}_i\} = 0 \quad , \quad \mathcal{O}_i \neq \{Q, B_i\}$$

for certain operators B_i. In simple situations, the topological structure of the target space (the classical cohomology ring) is reflected in the ring of observables which is a quantum deformation of the former. The correlators of the BRST invariant observables are expressed as integrals over a finite dimensional instanton moduli space of wedge products of forms. If the non-linear sigma model is not coupled with (topological) gravitation or topological YM , we are in the de Rham cohomology; we can use Poincaré duality to express the correlators as intersection numbers of dual cycles. When we couple the model with YM theories, the whole ring can be understood in terms of equivariant cohomology. The fact that the physics reflects the geometrical and topological structure of the target space is a more general feature of 2-dimensional models; Lerche, Vafa and Warner have shown that under general circumstances the chiral primary ring of N=2 SCFT is a quantum deformation of the classical cohomology ring of the target space. These rings were studied and classified in a (super) Landau-Ginsburg approach. It was shown that the superpotential allows a classification that is identical with that obtained in the catastrophe theory. The chiral ring encodes complete information of whether the theory is singular, with singular OPE's. The presence or the lack of singularities in QFT results from the presence or absence of singularities in the chiral ring. If the ring is finite, there are no singularities.

There are three approaches to the $N = 2$ supersymmetric (also for topological) sigma model; the first one (the usual) uses an explicit target space described by inhomogeneous coordinates; the second uses homogeneous coordinates and the momentum map. Finally, there is the Landau-Ginsburg approach.

There is increased interest in computing deformed rings corresponding to instanton effects in theories with potential added. Using the moment map description of the Lagrangian and taking symplectic quotients by means of the gauge freedom with spurion gauge fields introduced seems to be a very strong method.

The paper is organised as follows; in Chapter 2 the geometrical and topological notions are presented, with emphasis on the equivariant cohomology of compact Kähler manifolds. Chapter 3 contains information related to topological field theories, renormalization, coupling to YM; the effect of a potential added in topological sigma model is discussed; Chapter 4 contains an excursion in Landau-Ginsburg models, Gepner construction, the ring of the CP^n model , anomalies in sigma model with potential added. Finally, chapter 5 contains the conclusions. As modality of communication I addopted the 'naive' style (opposed to the 'expert' style).

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2. Equivariant Cohomology of Compact Kähler Manifolds

2.1. Cohomology

i) de Rham cohomology; homology; Poincaré duality

1. Simplicial homology

If M is a manifold which is smooth and connected, a p-chain is a (formal) sum

$$a_p = \sum_k c_k H_k \tag{2.1}$$

where H_k are smooth p-dimensional oriented submanifolds of M. An integral over a_P is defined as

$$\int_{\sum c_k H_k} = \sum_k c_k \int_{H_K} . \tag{2.2}$$

For $c_k \in R$ we have a *real* chain; otherwise we have to do with complex or integer chains and so on. Acting on M, ∂ is defined as the operation of taking the boundary. In this case $\partial^2 = 0$ because the boundary of a boundary is empty. Extending the ∂ operation to chains, we have

$$\partial a_p = \sum_k c_k \partial H_k. \tag{2.3}$$

cycle is a chain Z_p without boundary. If Z_p is the set of p-cycles and B_p is the set of boundaries,

$$B_p = \{a_p \mid a_p = \partial a_{p+1}\} \tag{2.4}$$

the *simplicial homology* of M is the quotient [1]

$$H_p = Z_p(M;R)/B_p(M;R). \tag{2.5}$$

If G is a field, R, C or Z_2, the homology group $H_p(M,G)$ is a vector space over G.

Example. For the torus $T_2 = S^1 \times S^1$ we have

$$H_0 = H_2 \cong R,$$

$$H_1 \cong R \oplus R. \tag{2.6}$$

2. Cohomology. Let Z^p be a set of closed p-forms

$$Z^p = \{\omega_p \mid d\omega_p = 0\} \tag{2.7}$$

and B^p the set of exact p-forms

$$B_p = \{\nu_p \mid \nu_p = d\lambda_{p-1}, \tag{2.8}$$

the *de Rham cohomology groups* are given by the quotient of each Z_p by the corresponding B^p :

$$H^p(M,R) = Z^p(M,R)/B^p(M,R). \tag{2.9}$$

$H^p(M,R)$ is the set of closed p-forms where two members are considered to be equivalent if they differ by an exact form

$$\omega_p \simeq \omega_p + d\lambda_{p-1}. \tag{2.10}$$

H^0 has a special meaning; it is the space of constant functions; hence $\dim H^0$ is the number of connected components of the manifold. de Rham has proved two classical theorems which show that H_p and H^p are dual to each other. On $H_p(M,R) \times H^p(M,R)$ we define

$$\pi : H_p \times H^p \longrightarrow R \tag{2.11}$$

as being

$$\pi(z_p, \omega_P) = \int_{z_p} \omega_p. \tag{2.12}$$

Using the Stokes lemma, it is trivial to show that this product does not depend on the choice of the representatives in each equivalence class. If d is the exterior derivative

$$d : \Omega_p \longrightarrow \Omega_{p+1} \tag{2.13}$$

and d^+ its adjoint (with respect to the inner product of p-forms , $(\alpha_p, \beta_p) = \int_m \alpha_p \wedge^* \beta_p, \beta_{Bp}$ being an (n-p) form related to β_p by Hodge * operation), the Hodge - de Rham " Laplacian " is

$$\Delta = (d + d^+)^2 = dd^+ + d^+ d. \tag{2.14}$$

Each cohomology class contains precisely *one harmonic form*; this form ω ($\Delta\omega = 0$) represents a cohomology class. The dimension of H^p is just the number of harmonic forms which are linearly independent (this is just the Betti number, b_p).

3. Poincaré duality

A p-form ω is harmonic if and only if $d\omega = d^+\omega = 0$.

Theorem. Let M be a compact, smooth manifold. Given a p-cycle a, there exists an (n-p)-form α, the Poincaré dual of $a_p \subset M$ such that

$$\int_{a_p} \omega_p = \int_M \alpha_{n-p} \wedge \omega_p \tag{2.15}$$

for any closed p-form ω. Given a basis $\{z^i\}$ for H_p and a dual basis $\{\omega_j\}$ for H^p such that

$$\int_{z^i} \omega_j = \delta^i_j, \tag{2.16}$$

there is also a basis $\{\lambda^i\}$ for H^{n-p} such that

$$\int_M \lambda^i \wedge \omega_j = \delta^i{}_j. \tag{2.17}$$

In this case $a_p = a_i z^i, \omega_p = \nu^i \omega_i, \alpha_{n-p} = a_i \lambda^i$ and eq. (2.15) follows, both members being equal to $a_i \nu^i$. Because ω is closed, α is defined up to an exact form. Sometimes it is useful to think that α is a current those support is concentrated on the cycle a_p.

ii) Intersection numbers

Two cycles a and b have transverse intersection if at each point p, the tangent spaces $T_p(a)$ and $T_p(b)$ have no vector in common. In this case, the intersection number of two cycles is [2]

$$N(a,b) = \Sigma_{p \in a \cap b}\, i_p(a,b) \tag{2.18}$$

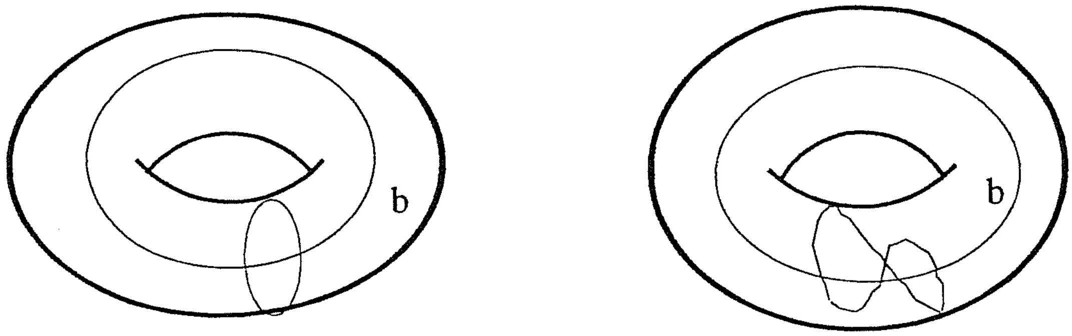
where $i_p = \pm 1$; if $\{u^1 ... u^i\}$ is an oriented basis in $T_p(a)$ and $\{v^1 ... v^j\}$ in $T_p(b)$, then $i_p = +1$ if the set $\{u^1 ... u^i, v^1 ... v^j\}$ is an oriented basis for $T_p(M)$ and -1 otherwise.

The intersection number $N(a,b)$ vanishes if either a or b is a boundary; this concept depends only on the homology classes of a and b. $N(a,b)$ can also be expressed in terms of the Poincaré duals α and β of a and b ;

$$N(a,b) = \int_M \alpha \wedge \beta \tag{2.19}$$

because α and β are basically distributions concentrated on a and b .

Example. Take on $T_2 = S^1 \times S^1$ two basic cycles a and b;



It follows that $N(a,b) = 1$ ($= 1 - 1 + 1$ in the second case).

The intersection number is an essential concept in cohomological theories where the correlators for the BRST-invariant observables (and which are nonzero due to instanton effects) are equal to integrals over the instanton(s) moduli space of wedge products of corresponding forms. Using the above arguments on Poincare duality, these correlators are equal to the intersection numbers of the dual cycles (on the moduli space).

iii) The Classical cohomology ring

The wedge product of two forms is also closed because

$$d(\alpha_p \wedge \beta_q) = d\alpha_p \wedge \beta_q + (-1)^p \alpha_p \wedge d\beta_q. \tag{2.20}$$

We see that the wedge product of a closed and an exact/closed form is exact/ closed.In this way the cohomological equivalence relation is preserved and a map

$$H^p(M,R) \otimes H^q(M,R) \longrightarrow H^{p+q}(M,R)$$

which defines a ring structure on

$$H^*(M,R) = \oplus_p H^p(M,R) \tag{2.21}$$

is introduced. In this case

$$\alpha_i \wedge \beta_j = c_{ij}{}^{k}\ \gamma_k \tag{2.22}$$

is the product operation in ring ($\alpha_i, \beta_j, \gamma_k$ are cohomology classes).

iv) Dolbeault Cohomology; Cohomology of Kähler manifolds

If M is a complex manifold of (complex) dimension n, in each patch V_α ($M = \bigcup_\alpha V_\alpha$) we have the coordinates $z_i = x_i + y_i$ and the conjugates $\bar{z}_i = x_i - y_i$. In the same time M is a real manifold of real dimension 2n. The difference lies in the fact that the transition functions are holomorphic. If a real manifold of dimension 2k contains a subatlas with holomorphic transition function, then the manifold is complex. The exterior derivative operator

$$d = dx^k \frac{\partial}{\partial x^k} + dy^k \frac{\partial}{\partial y^k}$$

can be decomposed as $d = \partial + \bar{\partial}$ where

$$\partial = dz^k \frac{\partial}{\partial z^k}, \bar{\partial} = d\bar{z}^k \frac{\partial}{\partial \bar{z}^k}.$$

We can define the space of complex exterior forms $\Omega^{p,q}$ which has a base containing p factors dz^k and q factors $d\bar{z}^k$. The action of the operators ∂ and $\bar{\partial}$ is the following

$$\partial : \Omega^{p,q} \longrightarrow \Omega^{p+1,q},$$

$$\tag{2.23}$$

$$\bar{\partial} : \Omega^{p,q} \longrightarrow \Omega^{p,q+1}.$$

$\bar{\partial}$ is called the Dolbeault operator. Obviously, $\partial^2 = \bar{\partial}^2 = 0$. We say that ∂ is an operator of (1,0) type while the Dolbeault operator is of (0,1) type.

For real manifolds, the Poincaré lemma appies; each closed form is locally exact. In the same way, on complex manifolds the Dolbeault-Grothendieck lemma applies; its content is the same (it suffices to replace d with the Dolbeault operator).

Let $Z_{\bar{\partial}}^{p,q}(M)$ denote the space of $\bar{\partial}$ closed forms of type (p,q). Because $\bar{\partial}^2 = 0$ on $\Omega^{p,q}$ we have

$$\bar{\partial}(\Omega^{p,q}(m)) \subset Z_{\bar{\partial}}^{(p,q+1)}(M). \tag{2.24}$$

The Dolbeault cohomology groups are defined as being

$$H^{(p,q)}_{\bar\partial}(M) = Z^{(p,q)}_{\bar\partial}(M)/\bar\partial(\Omega^{(p,q-1)}(M)). \qquad (2.25)$$

On a Kähler manifold the de Rham cohomology and $\bar\partial$ cohomology are equivalent because

$$\partial\partial^{+} + \partial^{+}\partial = \bar\partial\bar\partial^{+} + \bar\partial^{+}\bar\partial = \frac{1}{2}(dd^{+} + d^{+}d). \qquad (2.26)$$

2.2. Equivariant Cohomology

i) The pullback of an exterior form

If α_p is an exterior p-form on M_2 and $f : M_1 \longrightarrow M_2$, the pullback of α_p $(f^*\alpha_p)$ is a p-form on M_1. For example, if $x^i \in M_1$, $Ay^j \in M_2$, $y^j = f^j(x^i)$ and $\alpha = h_j(y)dy^j$, then $f^*\alpha = h_j(f(x))\partial_i f^j dx^i$. Because $d(f^*\omega_p) = f^*d\omega_p$, f^* pulls back closed forms to closed forms and exact forms to exact forms, relating the cohomology groups $H^p(M_2;R)$ and $H^p(M_1;R)$;

$$f^* : H^p(M_2, R) \longrightarrow H^p(M_1, R). \qquad (2.27)$$

ii) Equivariant cohomology; the case of groups with free action on M

An isometry is a transformation $f \in Diff^0(M)$ such that $x' = f(x)$ leaves the metric invariant (g' = g). Consider a manifold on which a finite group of isometries G acts freely (there are no fixed points). If ω_p is an harmonic form on M and $g \in G$, the pullback $g^*\omega_p$ is also harmonic. ω_p is called G-invariant if $g^*\omega_p = \omega_p$ for all $g \in G$. In this case we can define harmonic forms on M/G as G-invariant harmonic forms. If we denote the equivariant cohomology groups as being $H^p_G(M)$, then

$$H^p_G(M) = H^p(M/G). \qquad (2.28)$$

Because the isometries have fixed points (the zeros of the Killing vectors) we have to introduce the general case.

iii) Equivariant cohomology; the general case

Let EG be an universal G-bundle and M_G the product

$$M_G = EG \times_G M. \qquad (2.29)$$

A point in M_G is obviously a pair (p,q) , where $p \in EG, q \in M$. The meaning of the above notation is that we can translate the right action of G on EG into the left action of G in M

$$(pg, q) = (p, gq) \tag{2.30}$$

for each $g \in G$. M_G is a bundle over the classifying space BG associated to the universal bundle, the fiber being M. If π defines the projection in the base space, this fiber bundle is given by

$$M_G \longrightarrow BG. \tag{2.31}$$

The equivariant cohomology of the G-space M is defined as the *ordinary* cohomology of the space M_G [3] ;

$$H^*_G(M) = H^*(M_G). \tag{2.32}$$

A more intuitive picture can be given using the Cartan-Borel diagramm

$$\begin{array}{ccccc} E & \longleftarrow & E \times M & \longrightarrow & M \\ \downarrow & & \downarrow & & \downarrow \\ B & \longleftarrow & E \times_G M & \longrightarrow & M/G \end{array} \tag{2.33}$$

where E=EG and B=BG. Take $(p, q) \in M_G$ and denote H_q the isotropy group of the point q (Hq = q). Because the orbit of q (Gq) is generated by G/H and not by G, the quotient space M/G is in fact the quotient with respect to G/H and by σ^{-1} we do not recover E but rather E/H_q where H_q is the stabilizer of q. If G is a compact group acting smoothly and freely on M, then $H_q = 1$ for all $q \in M$; as a consequence, M_G and M/G are equivalent , the equivalence relation being given by σ;

$$\sigma : EG \times q \longrightarrow Gq \in M, \tag{2.34}$$

$$\sigma^{-1}(Gq) = EG. \tag{2.35}$$

iv) Weil's de Rham model for Equivariant Cohomology

Consider g being the Lie algebra of G and define the tensor product

$$W(g) = \Lambda g^* \otimes S g^* \tag{2.36}$$

where Λg^* is the exterior algebra and Sg^* is the symmetric algebra of the dual g^* of g. This algebra is freely generated as a Z_2 graded commutative algebra

$$\omega_p \omega_q = (-1)^{pq} \omega_q \omega_p. \tag{2.37}$$

It is considered that generators in Λ have degree 1 and are denoted by $\{\theta^\alpha\}$ while generators in S have degree 2 and are denoted by $\{u_\alpha\}$. In this case

$$W(g) = R[\theta; u] \tag{2.38}$$

where $R[\theta; u]$ is by definition the polinomial ring (with real coefficients) in θ and u . If we consider θ^α as being left-invariant 1-forms on G obeying the Maurer-Cartan equation

$$d\theta^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} \theta^\beta \theta^\gamma = 0 \tag{2.39}$$

then in $W(g)$ we have a differential operator D such that

$$D\theta^\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} \theta^\beta \theta^\gamma + u_\alpha = 0, \tag{2.40}$$

$$Du_\alpha + \frac{1}{2} C^\alpha_{\beta\gamma} u_\beta \theta^\gamma = 0. \tag{2.41}$$

The Jacobi identity on $C^\alpha_{\beta\gamma}$ (equivalent to $d^2\theta^\alpha = 0$) is translated in $D^2 = 0$ in $W(g)$. **Example.** If G = T (a torus) the structure constants vanish and

$$D\theta^\alpha + u_\alpha = 0, \tag{2.42}$$

$$Du_\alpha = 0. \tag{2.43}$$

As a consequence, the D-cohomology of $W(g)$ (denoted $H_D(W(g))$) is R. If P is a principal bundle with structure group G and M is the base space, $X \in g$ appear naturally as vertical vector fields; our de Rham complex $\Omega^*(P)$ is natural to define the action $i(X)$ as an inner product

$$i(X)A = X^i A_{ii_2 \ldots i_n}. \tag{2.44}$$

if A is an n-form on P and the action of the Lie derivative $\mathcal{L}(X)$

$$\mathcal{L}(X) = i(X)d + di(X). \tag{2.45}$$

Under the projection $\pi : P \to M$, $\Omega^*(M)$ is identified with the basic elements of $\Omega^*(P)$;

$$i(X)\varphi = 0 \quad ; \quad \mathcal{L}\varphi = 0 \quad \forall X \in g. \tag{2.46}$$

In the Weil algebra , the same operations are given by

$$i(e_\alpha)\theta^\beta = \delta_\alpha{}^\beta \quad ; \qquad i(e_\alpha)u_\beta = 0 \quad , \tag{2.47}$$

$$\mathcal{L}(e_\alpha) = i(e_\alpha)D + Di(e_\alpha) \tag{2.48}$$

where $\{e_\alpha\}$ is a basis for g dual to $\{\theta^\alpha\}$. In analogy to eq.(2.46) we have the basic subcomplex $Bg \subset W(g)$; this is the ring of polinomials on g invariant under the coadjoint action of of g on g^* ;

$$Bg \cong Inv_g S(g^*). \tag{2.49}$$

For a compact connected Lie group G , the ring S is a polinomial ring in l=rang(G) generators (a de Rham model for $H^*(BG)$). In this way there is a natural isomorphism

$$Bg \cong H^*(BG). \tag{2.50}$$

When G reduces to a torus of rank l, BG reduces to $R[u_1u_2...u_l]$. We define $\Omega^*_g(M)$ as being the basic complex of $\Omega^*(M) \otimes W(g)$. I give without proof (see ref [5]) the following theorem: **Theorem** If G is a compact connected group acting smoothly on M, there is a natural isomorphism

$$H\{\Omega^*_g(M)\} \cong H^*_G(M). \tag{2.51}$$

In other words the equivariant cohomology ring $H^*_g(M)$ is (modulo this isomorphism) just the cohomology ring of $\Omega^*_g(M)$.

Example. When G is a circle S^1 , $W(g) = R[\theta, u]$;

$$\varphi \in \Omega^*(M) \otimes W(g)$$

$$\varphi = \sum a_k u^k + \sum b_l u^l \theta \tag{2.52}$$

$$a_k, b_l \in \Omega^*(M).$$

If X is the generator of $g \in R$ dual to θ ($i(X)\theta = 1$), a basic φ must obey

$$i(X)\varphi = \sum_k (i(X)a_k)u^k + \sum_l (b_l + i(X)b_l\theta)u^l = 0,$$

$$\tag{2.53}$$

$$\mathcal{L}(X)\varphi = \sum_k (\mathcal{L}(X)a_k)u^k + \sum_j \mathcal{L}(X)b_j u^j = 0.$$

which is equivalent to

$$\mathcal{L}(X)a_k = 0,$$

$$b_k = -i(X)a_k. \tag{2.54}$$

Denote by Ω^*_X the kernel of $\mathcal{L}(X)$ in $\Omega^*(M)$; if $\omega \in \Omega^*_X$, then ω is invariant under the action of $G = S$. Let $\Omega^*_X[u]$ a polinomial ring generated by an element (u) over Ω^*_X; we can define a ring homomorphism

$$\Omega^*_X[u] \longrightarrow \Omega^*(M) \otimes W(g) \tag{2.55}$$

by

$$\lambda(a) = a - i(X)a\theta$$

$$\lambda(u) = u \quad ; \tag{2.56}$$

more than (and maybe more important) λ induces a ring isomorphism

$$\Omega^*_X[u] \cong \Omega^*_g(M). \tag{2.57}$$

WAe can introduce a differential operator d_x on $\Omega^*_X[u]$ associated to D on $\Omega^*_g(M)$ such that

$$\lambda d_x = D\lambda \quad ,$$

$$D\lambda a = D(a - i(X)a\theta) = da - i(X)da\theta - i(X)au$$
$$= \lambda(da + i(X)au) \quad . \tag{2.58}$$

It follows that

$$d_x a = da + ui(X)a \quad . \tag{2.59}$$

Because u is closed in $\Omega^*_g(M)$ we have

$$d_x a = 0. \tag{2.60}$$

By setting u=0 in the action of λ (at the end) we obtain finally a natural map$\Omega^*_X[u] \rightarrow \Omega^*(M)$. This model extends naturally to a torus (G = T); when T has rank l one simply chooses a basis $u_1 ... u_l$ in its Lie algebra and define

$$d_x a = da + \sum_k i(X_k)au_k. \tag{2.61}$$

If G is non-abelian , one chooses firstly the maximal torus T in G (corresponding to the Cartan subalgebra) and then one describes $\Omega^*_g(M)$ in terms of the Weil group invariant terms in $\Omega^*_T(M)$.

v) Evaluation; reduction to the Witten complex

Let X be a Killing vector on M; we define the operator

$$d_s = d + si(X) \tag{2.62}$$

with s a real parameter acting on $\Omega^*(M)$. Witten [4] has studied the 'Hamiltonian'

$$H_s = d_s d_s^+ + d_s^+ d_s \tag{2.63}$$

ans proved that for $s \neq 0$ the dimension of the zero eigenspace W_s ($H_s\psi = 0, \psi \in W_s$) is the sum of Betti numbers for the **zeros** of the Killing fields. He has shown that $W_s \subset \Omega^*_X$,

$$\mathcal{L}(X)\psi = 0 \tag{2.64}$$

if $\psi \in W_s$. Because $d_s^2 = s\mathcal{L}(X)$, on Ω^*_X

$$d_s^2 = 0. \tag{2.65}$$

We can identify (canonically)

$$W_s \simeq H(\Omega^*_X, d_s) = Ker\ d_s/Im\ d_s. \tag{2.66}$$

The relation between the Witten groups W_s and equivariant cohomology groups is the following. If X generates the circle S , the equivariant cohomology ring $H^*_S(M)$ is a **module** over the polinomial ring C[u] and

$$H^*_S(M) = H(\Omega^*_X[u], d_X) \tag{2.67}$$

The only difference between the d_s cohomology and equivariant one is that in the first case s is a real parameter (it can be also complex) while in the second case u is an indeterminate of a polinomial.

vi) The 'Witten' complex and SQM

If d is the exterior derivative ($d : \Omega_p \longrightarrow \Omega_{p+1}$) then the Witten derivative $d_s = d + si(X)$ has an action

$$d_s : \Omega_p \longrightarrow \Omega_{p+1} \oplus \Omega_{p-1}. \tag{2.68}$$

On the Z_2 graduation of the de Rham complex $\Lambda^* = \Lambda_+ \oplus \Lambda_-$ where

$$\Lambda_+ = \sum_{p\ even} \oplus \Lambda_p,$$

$$\Lambda_- = \sum_{p\ odd} \oplus \Lambda_p \tag{2.69}$$

the action of d_s becomes

$$d_s : \Lambda_\pm \longrightarrow \Lambda_\mp. \tag{2.70}$$

We recognise the action of a fermionic charge (or a linear combination) on the Hilbert space of the SQM ($H = H_+ \oplus H_-$)

$$Q_i H_\pm = H_\mp. \tag{2.71}$$

With the identification

$$Q_s B = i^{1/2} d_s + i^{-1/2} d_s^+ \tag{2.72}$$

the Witten 'Laplacian' is equal (up to a numerical factor) to the Hamiltonian $H_s = Q_s^2 - 2is\mathcal{L}(X)$. Obviously, on Ω_X^* we have

$$H_s = Q_s^2. \tag{2.73}$$

If $F \subset M$ is the space of the zeros of the Killing vector X

$$F = \{x \in M | X(x) = 0\} \tag{2.74}$$

it was shown by Witten that for $H_s = d_s d_s^+ + d_s^+ d_s$ acting on invariant forms which belong to Ω_x^* the number of zero eigenvalues is **independent** on s if $s \neq 0$ and also independent on the choice of the X-invariant Riemann structure (metric) on M. This number is equal to the sum of Betti numbers for F. Because for $s = 0$, $d_s = d, H_{s=0}$ is just the usual Laplacian, the number of zero eigenvalues is equal to the sum of Betti numbers for M. The s-dependent term in H_s being a bounded operator the eigenvalues of H_s are smooth functions of s and the number of zero eigenvalues for small s is also equal to$\sum_{p=0}^n b_P(M)$. Using this model in the $s \to \infty$ limit ,Witten was able to prove the fixed-point theorem for the Euler number

$$\chi(M) = \sum (-1)^i b_i(M) = \sum (-1)^i b_i(F). \tag{2.75}$$

This result is closely connected to Hopf theorem which relates the Euler number to fixed points of arbitrary vector fields.

Consider the conjugation operation on d_s

$$d_s \longrightarrow e^{-\lambda f} d_s e^{\lambda f} \tag{2.76}$$

where $e^{\lambda f}$ is a multiplicative operator which acts on forms. We find

$$e^{-\lambda f} d_s e^{\lambda f} = e^{-\lambda} d_{s'} \tag{2.77}$$

where

$$s' = s e^{2\lambda}. \tag{2.78}$$

In this way s can be changed in an arbitrary way (but remaining nonzero). If we find a cocycle $\omega(d_{s'}\omega = 0)$ we have a cocycle $\omega' = e^{\lambda f}\omega$ with respect to $d_s(d_s\omega' = 0)$. Under conjugation,

$$d_s^+ \longrightarrow e^{-\lambda f} d_s^+ e^{\lambda f} \tag{2.79}$$

and the Hamiltonian (or the Laplacian) transforms as

$$H_s \longrightarrow e^{-\lambda f} H_s e^{\lambda f} = e^{-2\lambda} H_{s'}. \tag{2.80}$$

It follows that the number of zero eigenstates is independent of s (for $s \neq 0$).

Using the Riemann metric, it is useful to introduce the 1-form $\tilde{K}$ dual to the Killing vector K

$$\tilde{K}(K) = 1. \tag{2.81}$$

From the definition of the Killing vector

$$i(K)(d\tilde{K}) = -d(K^2). \tag{2.82}$$

The Hamiltonian becomes

$$H_s = dd^+ + d^+ d + s^2 K^2 + s(d\tilde{K} \wedge +i(d\tilde{K})). \tag{2.83}$$

The potential energy is $V(\phi) = s^2 K^2$. For $s \rightarrow \infty$ the zero eigenstates are concentrated near the zeros of K (the set F).

2.3. Isometries of Compact Kähler Manifolds

i) Poisson action and momentum map

Everytimes the action of a group G on a manifold M preserves a simplectic form (i.e. a 2-form which is closed and non-degenerate) ω on M we can define a momentum map. The idea is that the non-degeneracy of ω to introduce a duality relation between 1-forms and vector fields on M. For example, to each vector X we can associate a smooth function f_X on M such that

$$f_X = i(X)\omega. \tag{2.84}$$

Conversely, to a function f we can associate a vector X_f. In the first case, $(X \rightarrow f_X)$ we defined a Poisson action (or a comoment map on (M, ω). In general, a Poisson action is a function

$$\tilde{\mu} : g \longrightarrow \mathcal{F}_\omega M \tag{2.85}$$

where g is the Lie algebra of G and $\mathcal{F}_\omega M$ is the Lie algebra of the smooth functions on M whith respect to the Poisson bracket.

The moment map is the dual of the Poisson action

$$\Phi : M \longrightarrow g^* \tag{2.86}$$

where g^* is the algebra dual of g. **Example.** Consider the action of a circle S on M and denote by X the infinitesimal generator (the Killing vector) of this action. This can be lifted to a Poisson action f_X according to equation (2.84) . There is an obstruction in finding such an f_X; this is just the cohomology class of $i(X)\omega$ in $H^1(M)$. If the Betti number $b_1 = 0$ then f_X is unique up to an element in $H^0(M)$ (i.e. a constant). ω being supposed invariant under S and annihilated by the Lie derivative $\mathcal{L}(X)$ belongs to $\Omega^2_X[u]$. There is a unique way to extend ω to an equivariant closed form in $\Omega_X[u]$. If we introduce the form

$$\omega^* = \omega - fu \tag{2.87}$$

where f is a smooth function on M, then

$$d_X\omega^* = (i(X)\omega - df)u. \tag{2.88}$$

It is obvious that ω^* is an equivariant closed form if and only if f_X defines a Poisson lifting of the action. The above result is generalized in the following theorem; **Theorem.** For any group G , there is a natural one-to-one correspondence between Poisson liftings of a simplectic action and equivariantly closed extensions ω^* of the simplectic form ω to $\Omega^*_g(M)$.

The proof can be found in ref . I will sketch only the main steps. If $X \to f_X$ is a Poisson lifting of the G-action, in a base $\{X_a\}$ for g this becomes

$$X_a \to f_a.$$

Consider an element ω^* in $\Omega_*(M) \otimes W(g)$

$$\omega^* = \omega - D\sum_a f_a \theta^a \tag{2.89}$$

where $\{\theta^a\} \in g^*$ is a dual base to $\{X_a\}$ and define $i_a = i(X_a)$ then

$$i_a \omega^* = 0, \tag{2.90}$$

$$D\omega^* = d\omega + D^2(\sum f_a \theta^a). \tag{2.91}$$

So, ω is a basic closed form in $\Omega^*_g(M)$ because $d\omega = 0$ on M and $D^2 = 0$ in $\Omega^*(M) \otimes W(g)$.

ii) Symplectic quotients and the moment map

Take the basic differential form on the phase-space

$$\omega = \sum_j dp_j \wedge dq_j. \tag{2.91}$$

Each function F(q,p) has a differential which can be converted into the vector field

$$X_F = \sum_j \frac{\partial F}{\partial p_j}\frac{\partial}{\partial q_j} - \frac{\partial F}{\partial q_j}\frac{\partial}{\partial p_j}. \tag{2.92}$$

We have the inner product

$$dF = < X_F, \omega > . \tag{2.93}$$

This equation determines X_F uniquely because ω is non-degenerate. X_F preserves ω. The converse is also true, at least locally. **Example.** If F is the Hamiltonian H , X_F is the Hamiltonian flow whose integral curves give the dinamical evolution. Take [5]

$$H = \frac{1}{2}\sum_{j=1}^{n}(p_j^2 + q_j^2); \tag{2.94}$$

it corresponds to n uncoupled oscillators. For each pair (q_j, p_j) there is a flow

$$X_{H_j} = p_j \partial/\partial q_j - q_j \partial/\partial p_j \tag{2.95}$$

which generates rotations in the plane (q_j, p_j) . For $j \neq k$,we have

$$[X_{H_j}, X_{H_k}] = 0. \tag{2.96}$$

Because these flows commute, we have the action of the torus T_n on the system. Using complex variables, $z_j = p_j + iq_j$, the n functions H_j define a map

$$\mu : C^n \longrightarrow R^n$$

given by

$$(z_1, z_2, ..., z_n) \longrightarrow \frac{1}{2}(|z_1|^2, ..., |z_n|^2). \tag{2.97}$$

This is an example of the **moment map**. The image is the positive quadrant in R^n . Let us restrict to a fixed energy taking H=1. The energy surface is a sphere S^{2n-1} ;

$$\sum_{j=1}^{n} |z_j|^2 = 2. \tag{2.98}$$

On this surface the Hamiltonian flow acts,

$$z_j \rightarrow e^{i\theta} z_j. \tag{2.99}$$

The orbit space (or the quotient space) is just the complex projective space CP^{n-1}. When restricted to the energy surface S^{2n-1}, the original simplectic form ω becomes degenerate , but the degeneracy is just along the orbits of the Hamiltonian flow; a non-degenerate flow ω_1 on CP^{n-1} is induced. This is the Kähler form on CP^{n-1}. After some computation [5] we find

$$\int_{CP^{n-1}} {\omega_1}^{n-1} = (2\pi)^{n-1}. \tag{2.100}$$

It follows that

$$\Omega = \frac{\omega_1}{2\pi} \tag{2.101}$$

is the generator of $H^2(CP^{n-1}; Z)$. If we restrict the Hamiltonian to the symplectic quotient CP^{n-1} such that $\sum H_j = 1$,then the moment map projects CP^{n-1} into an (n-1)-simplex:

$$\mu : CP^{n-1} \rightarrow (n-1) simplex\{(1, 0..0), (0, 1, 0..0), ..., (0, 0, ..1)\}. \tag{2.102}$$

More general, if a torus acts symplectically on M, we have a moment map

$$\mu : M \rightarrow R^n$$

provided that the n Hamiltonians are single-valued. R^n in this case should be viewed as the dual t^* of the Lie algebra t of the torus T_n. At the end of this paragraph I should quote the main convexity result (Atiyah [6] ,Guillermin, Sternberg [7]): The image of a compact symplectic manifold M acted on by a torus T_n is a convex polyhedron.

iii) The Frankel theorem, relation with Morse theory

When a 1-parameter group acts by isometries on a Riemann manifold M, the fixed point set F is such that each component F_α of F is a totally geodesic submanifold of M whose domension has the same parity as the dimension of M. (This is a theorem due to Kobayashi.) If M is a compact Kähler manifold, the isometries are holomorphic transformations and F_α are compact Kähler submanifolds (points are also possible) [8] . The Frankel theorem [9] gives us the Betti numbers of the manifold M in terms of the Betti numbers of the fixed foint set F, using the Morse inequalities. Let M be a compact, connected, Kähler manifold of real dimension n=2k and I a connected one-parameter group of isometries of M. The following lemma holds; **Lemma 1.** If $b_1(M) = 0$, the fixed set F coincides with the non-degenerate critical set of a real smooth function ϕ on M. Proof. The group I acting complex anallitically on M, denote by X the Killing vector of the isommetry I. By applying eq. ( ) to a Kähler 2-form ω on M (since ω is Kähler $d\omega = 0$ and ω is an harmonic form). Harmonic forms being invariant under connected groups of isommetries (particularly I) we have $\mathcal{L}(X)\omega = 0$; so $di(X)\omega = 0$ or $i(X)\omega$ is a closed 1-form which is also *exact* because $b_1(M) = 0$. This implies the existence of a function ϕ such that

$$d\phi = i(X)\omega. \tag{2.103}$$

ϕ can be taken as real-valued (only the real structure was used in fact). In local coordinates we have

$$X = X_i\partial_i \quad ; \quad \omega = \omega_{ij}dx^i \wedge dx^j; \tag{2.104}$$

so

$$i(X)\omega = \omega_{ij}X^j dx^i. \tag{2.105}$$

On components,

$$\frac{\partial \phi}{\partial x^i} = \omega_{ij} X^j \tag{2.106}$$

The critical points of $\phi (d\phi = 0)$ are just the fixed-points of I (X=0) because ω is a non-degenerate form $(det(\omega_{ij}) = 0\ )$. The degeneracy of the critical point is determined by the Hessian matrix

$$H_{ij} = \frac{\partial^2 \phi}{\partial x^i \partial x^j} = \omega_{il} \frac{\partial X^l}{\partial x^j} \tag{2.107}$$

at the critical point (i.e. where X=0). Let p a critical point of ϕ; it belongs to a compact Kähler submanifold $F(p)$ of critical points . Take dim$F(p) = 2r, r \geq 0$ and denote T_p the tangent space to M at p and $H_p$the subspace of T_p tangent to F_p. Since I leaves p fixed and operates by isometries, it induces a one-parameter subgroup I_* of rotations in T_p. We can write it as

$$I_*(t) = e^{tS}. \tag{2.108}$$

Define a linear map$R : T_p \to T_p$ such that in the point p the action is

$$Y^i \partial_i \to \frac{\partial X^j}{\partial x^i} Y^i \frac{\partial}{\partial x^j}.$$

Extending this action (canonically) to a vector field in the neighborhood of p

$$R(Y) = [Y, X] = lim_{t \to 0} \frac{1}{t}(I_*(t) - 1)Y = SY \tag{2.109}$$

the Hessian matrix becomes

$$H_{ij} = \omega_{il} S^l_j. \tag{2.110}$$

As a linear transformation, the previous equation becomes [9]

$$H = JS \tag{2.111}$$

where J is the complex structure tensor ($J^2 = -1$). It follows that JS=SJ (X keeps invariant the complex structure). We can put J and S simultaneously in the canonical form, which is equivalent to a choice of (k-r) invariant 2-planes (at p) such that

$$T_p = e_1 \oplus e_2 \oplus ... \oplus e_{k-r} \oplus h_p$$

is invariant under the action of the 2kx2k matrices J and S

$$J = diag(V, V, ..., V) \ \ ; \ \ S = diag(\Theta_1 ... \Theta_{k-r} 0, ..., 0) \tag{2.112}$$

where

$$AV = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \Theta_i = \begin{pmatrix} 0 & -\theta_i \\ \theta_i & 0 \end{pmatrix}. \tag{2.113}$$

The Hessian operator (JS) is given by

$$H = diag(-\theta_1, -\theta_1, ..., -\theta_{k-r}, -\theta_{k-r}, 0...0). \tag{2.114}$$

The number of zeros in H is the dimension of the critical manifold F_P (=2r). This proves the lemma. We recognise ϕ as being the Poisson action of X. As a corollary, if $b_1(M) = 0$ and ϕ is different from a constant , then the fixed set is F not empty and not connected. The proof is based on the fact that since ϕ is not constant it has unequal extrema. The disjoint sets $\phi = Max$ and $\phi = min$ contain two components of F. **Lemma 2** In Lemma 1 the condition $b_1(M) = 0$ can be replaced by the statement that F is non-empty. Proof. Consider the Hodge decomposition

$$i(X)\omega = H(i(X)\omega) + d\phi \tag{2.115}$$

We need to show that $H[i(X)\omega] = 0$, H being the harmonic part. Take the 1-form associated to the Killing vector X

$$\chi = X_i dx^i \tag{2.116}$$

It was proved that $H[\chi] = 0$. For any harmonic form h we have

$$0 = \mathcal{L}(X)h = i(X)dh + di(X)h = di(X)h. \tag{2.117}$$

It follows that $i(X)h =$ constant (Bochner theorem). Because the Killing vector X has zeros somewhere, $i(X)h = 0$. In this way

$$(\chi, h) = \int_M *i(X)h = 0 \tag{2.118}$$

and $H[\chi] = 0$. Using the fact that $i(X)\omega = C\chi$ (S.Kobayashi), $H[i(X)\omega] = h[C\chi] = CH[\chi] = 0$. C is the complex structure operator applied to forms. The index of a critical manifold F_α denoted λ_α is the number of negative eigenvalues of the Hessian (it is clear from the form of the Hessian presented above that this number must be even). With these prerequisites , the Frankel theorem can be introduced: **theorem.** If F is non-empty, then

$$b_i(M; K) = \sum_\alpha b_{i-\lambda_\alpha}(F_\alpha; K) \tag{2.119}$$

for all i and for the field K being Q or Z_p ,where p is prime. The proof is based on the Morse-Bott inequalities

$$b_i(M;K) \leq \sum_\alpha b_{i-\lambda_\alpha}(F_\alpha;K) \tag{2.120}$$

and on a result of Floyd [10]

$$\sum b_i(M;K) \geq \sum b_i(F;K). \tag{2.121}$$

The above inequalities give directly the theorem. **Example 1.** Take $M = CP^2$ with the homogenous coordinates (z_0, z_1, z_2) and define the action of the $U(1)$ group as being

$$(z_0, z_1, z_2) \rightarrow (z_0, e^{it}z_1, e^{it}z_2) \tag{2.122}$$

The fixed set F is given by the point $F_1 = (1, 0, 0)$ and the complex projective line (2-sphere) $F_2 = (0, z_1, z_2)$. We can choose the sign of ϕ such that F_1 is a minimum and F_2 is the maximum set. In this case $\lambda_1 = 0$ and $\lambda_2 = 2$, the theorem being checked.

3. Topological field theories

3.1. Generalities about topological field theories

i) BRST

Consider a fermionic operator Q which is nilpotent ($Q^2 = 0$) such that the energy-momentum tensor is [11]

$$T_{\alpha\beta} = \{Q, \lambda_{\alpha\beta}\}. \tag{3.1}$$

In this case there are no phisical excitations which propagates because

$$< \psi'|H|\psi >= \int d^{d-1}x < \psi'|T_{00}|\psi >= 0 \tag{3.2}$$

because $Q|\psi >= 0$ for each physical state $|\psi >$.

ii) Twisting

Topological σ models in d=2 can be viewed as twisted versions of the N=2 supersymmetric nonlinear σ model if the target space is a Kähler manifold . The N=2 model has two chiral symmetries U_L and U_R acting on the left-handed modes (L=V-A) and on the right-handed modes (R=V+A). Because the Lorentz group in d=2 has only one generator J, there are no quadratic constraints. There are four supercharges which transform under $J \otimes U_L \otimes U_R$ as [12]

$$(-1/2, 1, 0) \oplus (-1/2, -1, 0) \oplus (1/2, 0, 1) \oplus (1/2, 0, -1). \tag{3.3}$$

We can define a new Lorentz generator as being

$$J' = J + \frac{1}{2}U_L - \frac{1}{2}U_R. \tag{3.4}$$

In the singlet sector ($J' = 0$) there are two supercharges Q_L and Q_R transforming under $U_L \times U_R$ as $(0,1) \oplus (1,0)$. Both are BRST-like and anticommute

$$Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0. \tag{3.5}$$

Because the representation of the Lorentz group is twisted ($J \to J'$) the fermions which have spins $\pm 1/2$ with respect to J will have spins 0 and± 1 with respect to J'.

In d=4 the rotation group is $SO(4) \simeq SU(2)_L \times SU(2)_R$. The Lagrangian of the N=2 SYM theory has a global $U(2) \simeq SU(2)_I \times U(1)_U$ symmetry. The interesting group in our problem is in this case $SU(2)_L \times SU(2)_R \times SU(2)_I \times U(1)_U$, under which the gauge fields are [13]

$$(1/2, 1/2, 0, 0)$$

the spinless bosons are

$$(0,0,0,2) \oplus (0,0,0,-2) \tag{3.6}$$

and the fermions are

$$(1/2, 0, 1/2, 1) \oplus (0, 1/2, 1/2, -1). \tag{3.7}$$

The exotic action of the rotation group is taken to be $SU(2)_L \times SU(2)'_R$ where the $SU(2)'_R$ is $SU(2)_R \oplus SU(2)_I$ (the diagonal sum). Under $SU(2)_L \times SU(2)'_R \times U(1)$ the bosons A_α, Φ, λ transform as

$$(1/2, 1/2, 0) \oplus (0,0,2) \oplus (0,0,-2) \tag{3.8}$$

and fermions $\psi_\alpha, \chi_{\alpha,\beta}, \eta$ as

$$(1/2, 1/2, 1) \oplus (0, 1, -1) \oplus (0, 0, -1). \tag{3.9}$$

The global symmetries of the N=2 SYM model transform as $(1/2, 0, 1/2, 1) \oplus (1/2, 0, 1/2, -1)$ under $SU(2)_L \times SU(2)_R \times SBU(2)_I \times U(1)_U$. Under $SU(2)_L \times SU(2)'_R$ they will transform as

$$(1/2, 1/2, -1) \oplus (0, 1, 1) \oplus (0, 0, 1). \tag{3.10}$$

The Lorentz singlet component (0,0,1) is just the BRST-like charge Q.

iii) Observables; the chiral ring

Because Q is a BRST-like symmetry, the observables $\mathcal{O}$ must be Q-closed ($\{Q, \mathcal{O}\} = 0$) [11] . The braces are notations for the graded commutators; we can use also square brackets. What is important is to keep in mind that the BRST variation of some operator is proportional to this bracket. In any N=2 theory there are two SUSY charges, Q^+ and Q^- which are nilpotent operators ($(Q^+)^2 = (Q^-)^2 = 0$) such that the Hamiltonian is $H = \{Q^+, Q^-\}$. The hermitean conjugate of Q^+ is Q^- and conversely. In a topological theory, the BRST cohomology of Q^+ is identified with the physical Hilbert space. This means that $Q^+|\psi>=0$, $|\psi>\sim|\psi>+Q^+|\rho>$. The ambiguity in defining the states can be fixed by choosing for example $Q^-|\psi>=0$; this is the equivalent of picking harmonic representatives in the standard cohomology. Using $\{Q^+,Q^-\}=H$ we can identify topological states with the ground states in the SUSY sector. Topological operators denoted Φ_i satisfy $[Q^+,\Phi_i]=0$. They are named also *chiral fields* . These chiral fields form a ring; to specify the ring we do *not* have to specify the points at which we put the fields, because a translation can be expressed as the graded commutator of two SUSY charges. Taking a basis $B=\{\Phi_i\}$ for the physical chiral fields, we get a ring

$$\Phi_i\Phi_j = C_{ij}{}^k\Phi_k + [Q^+,\rho] \tag{3.11}$$

which is usually a finite ring. If we can identify a unique vacuum $|0>$, states are **identified** with operators by $\Phi_i|0>=|i>$. It is a problem in identifying states with operators (chiral fields). How we identify the vacuum state? Usually there are several such ground states, their number being equal to the Witten index in the LG case. It seems impossible to pick a preferred one. Though, if dealing with SCFT there is a canonical choice. In this case we have two $U(1)$ charges, (q_L and q_R) which label the vacua; we look for the unique state with minimum value of $q_{L,R}$ and we identify it with the vacuum $|0>$. In the massive case there is only one $U(1)$, the fermion number and $q=q_L-q_R$. The LG ground states have $q=0$, or $q_L=q_R$. Using the **spectral flow** we can give an alternative definition of the vacuum. This is possible because the spectral flow connects the NS and R sectors, inBdependent whether the theory is conformal or massive (only $U(1)$ is involved). In this way, an isomorphism between operators in the NS sector and topological states in the R sector is established. If $|i>=\Phi_i|0>$ is a state in R, then

$$\Phi_i|j>=\Phi_i\Phi_j|0>=C_{ij}{}^k|k> \tag{3.12}$$

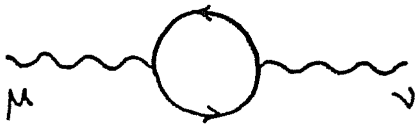
and $C_{ij}{}^k$ gives the action of the chiral fields on the 'ground' states $|j>$. We can repeat the above construction, with $Q^+\to Q^-$. In this case $\Phi_i\to\Phi_{\bar{i}}$, the adjoint, or anti-chiral field; the anti-chiral ring is obtained consequently (see refs. [34] [35]).

iv) The Renormalization problem

There are no physical excitations (as one-particle states, two-particle states,...) in topological field thories. The only physical state is the vacuum. A natural question emerges; **is the topological nature of this theories preserved by renormalisation ?**. In other words, is the topological nature valid at one-loop in theories of Witten-type ? One-loop computations in topological Yang-Mills (TYM) show in Feynman gauge the presence of divergences and a non-zero β-function [14] [15] which has the same value as in N=2 SYM

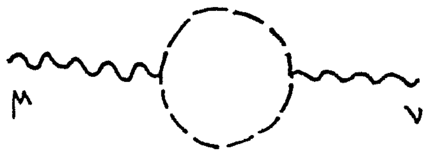
$$\beta(g) = -2g^3 C_2(G)/(4\pi)^2 \tag{3.13}$$

where $C_2(G)$ is the quadratic Casimir operator in adjoint representation and g is the coupling constant. The only new contributions are the $\chi\psi$ loop



$$-\frac{4}{3}\frac{C_2(G)g^2}{16\pi^2}\frac{1}{\epsilon}(k^\mu k^\nu - k^2 g^{\mu\nu})$$

and the $\phi\lambda$ loop



$$\frac{1}{3}\frac{C_2(G)g^2}{16\pi^2}\frac{1}{\epsilon}(k^\mu k^\nu - k^2 g^{\mu\nu}).$$

There is asymptotic freedom. Adding the pure YM contributions, there is no field strength renormalization in the Feynman gauge; only the coupling constant is supposed to renormalization. (For a different point of view see though [16] [17] [18] [19] .)

v) Non-renormalization theorems

Roughly speaking, topological field theories are not renormalized at all. The reason for this is the lack of UV divergences which results from the absence of local excitations, or local degrees of freedom. The correlators of BRST-invariant operators, or topological Green functions, cannot depend on the renormalization scale μ (topological invariants). This is not true in general for operators which are not BRST-invariant. Let s be the topological Slavnov (BRST) operator. The effective action of a TFT has in general the form

$$\Gamma_{top}(\mu) = \Gamma_0 + s\Delta\Gamma(\mu) \quad . \tag{3.14}$$

The above equation gives us information on the untwisted N=2 theories; the effective action of the dynamical N=2 theory obeys *the same* equation. As a consequence, the coupling constants which appear in Γ_0 are not renormalized.

The above argument has to be refined because 'twisting' requires also a field redefinition; usually this is anomalous and *explicitly* μ dependent. To understand this, one should note that in topological theories the observables are **adimensional** while in the 'untwisted' case their dimension is $\frac{q}{2}$, q being their $U(1)$ charge (chiral primary fields). So,

$$\Phi_{top} = \mu^{-q/2}\Phi_{dyn} \tag{3.15}$$

such that the two theories are related by a U(1) transformation with a parameter $\beta = -\frac{1}{2}log\mu$; the effective actions are related by

$$\Gamma_{top} = \Gamma - \frac{1}{2}log\mu \ \delta_{U(1)}\Gamma \ . \tag{3.16}$$

It is clear that the last term is the $U(1)$ anomaly, A. Finally, the effective action has the form

$$\Gamma = \Gamma_0 + \frac{1}{2}log\mu \ A + s\Delta\Gamma \ . \tag{3.17}$$

The above equation expresses the non-renormalization theorem in N=2 theories; both Γ_0 and A are not renormalized: Γ_0 by topological invariance and A by the Adler- Bardeen theorem. It is easy to show that, as a consequence, the superpotential W of the N=2 Landau-Ginsburg in d=2 is not renormalized. Indeed, if

$$\mathcal{L} = \int d^4\theta K(X,\bar{X}) + \int d^2\theta W(X) + \int d^2\bar{\theta}\bar{W}(\bar{(}X) \quad . \tag{3.18}$$

The BRST operator s ($s^2 = 0$) is

$$s(...) = \int d^2zd^2\bar{\theta}(...) \tag{3.19}$$

and (at classical level) we can identify

$$\Gamma_0 = \int d^2zd^2\theta W \quad , \tag{3.20}$$

$$\Delta\Gamma = \bar{W} + \int d^2\theta K \quad . \tag{3.21}$$

Using the general formula for the effective action, it follows that W is not renormalized by higher order contributions, because $A = 0$. (To understand more better the connection between the 'untwisted' and 'twisted' theories, one should *regularize* the dynamical theory, by introducing for example a cut-off, or by using the Pauli-Villars technique of massive auxiliary fields. In this case, the twist should give *the same* topological theory.)
The non-renormalization theorem can be used to show that in N=2 d=4 SYM the one-loop β function is **exact** . In this case the anomaly is

$$A = i2\frac{2N}{32\pi^2}\int F \wedge F \tag{3.22}$$

and the effective action becomes

$$\Gamma = (\frac{1}{4g^2} + i\theta - \frac{1}{2}\frac{4N}{32\pi^2}log\mu)\int F \wedge F + s(...) \quad . \tag{3.23}$$

This equation allows us to extract the exact β function which coincides to the one-loop value. Another important result is the fact that *the Káhler class is renormalized only at one-loop* . Let us consider a N=2 supersymmetric σ model on a Kähler target space M. The β function for the metric is

$$\mu\frac{dg_{i\bar{j}}}{d\mu} = R_{i\bar{j}} + higher\ loops \quad . \tag{3.24}$$

This β is a (1,1) form on M. This form is also *closed* because the metric remains Kähler. Which is its cohomology class? The answer is due to Alvarez-Gaumé and Ginsparg; The **exact** β lies in the cohomology class of the Ricci form and the Kähler class is not renormalized beyond one-loop.

3.2. The de Rham cohomology: Topological σ-models

i) Action; A and B models

The usual chiral (super)multiplet Φ is independent of $\bar{\theta}$; we express this as $\bar{D}_\alpha\Phi = 0$. The N=2 **twisted** chiral multiplet χ is independent of $P_+\theta$ and $P_-\bar{\theta}$, where $P_\pm$ are chiral projectors. This means

$$D_+\chi = \frac{1}{2}(1+\gamma_5)D\chi = 0 \quad , \tag{3.25}$$

$$\bar{D}_-\chi = \frac{1}{2}(1-\gamma_5)\bar{D}\chi = 0 \quad . \tag{3.26}$$

In the above notations,

$$D_{\pm} = (P_{\pm})_{\alpha}{}^{\beta} D_{\beta} \quad . \tag{3.27}$$

These constraints are consistent only for $d = 2$, because $\{D_+, \bar{D}_-\} = 0$ only for $d \leq 2$. In theories with **only** twisted multiplets, D and $\bar{D}$ cannot be distinguished. We encounter two kinds of theories: A. Theories of type A, in which the action is $S[\Phi, \bar{\Phi}]$; B. Theories of type B, in which the action is $S[\Phi, \bar{\Phi}; \chi, \bar{\chi}]$.

1. TNSM on an almost complex manifold The 2-dim nonlinear sigma model is a field theory for maps from a Riemann surface Σ to a target space M which is usually a compact manifold. To define the topological theory, the following fields have been introduced (D is the conformal dimension, U is the global charge)

Field	D	U	Statistics
u^i	0	0	+
χ^i	0	1	-
ρ^i_α	1	-1	-
H^i_β	1	0	+

In addition, both ρ^i_α and H^i_α are self-dual

$$\rho^i_\alpha = \epsilon_\alpha{}^\beta J^i{}_j \rho^j_\beta \quad , \tag{3.28}$$

$$H^i_\alpha = \epsilon_\alpha{}^\beta J^i{}_j H^j_\beta \quad . \tag{3.29}$$

The latin indices are indices on the target space while Greek indices are coordinate indices on the Riemann surface Σ. A connection on $\phi^*(T)$ is obtained by pulling back the connection of T from M to Σ giving the covariant derivative of χ^i

$$D_\alpha \chi^i = \partial_\alpha \chi^i + \partial_\alpha u^k \Gamma^i_{kl} \chi^l \tag{3.30}$$

The fermionic transformation laws can be chosen as being

$$\delta\chi^i = 0, \delta u^i = i\epsilon\chi^i \tag{3.31}$$

$$\delta\rho^i_\alpha = \epsilon(H^i_\alpha + \frac{i}{2}\epsilon_{\alpha\beta}(D_k J^i{}_j \chi^k \rho^{\beta j}) - i\epsilon\gamma^i_{jk}\chi^j\rho^k_\alpha. \tag{3.32}$$

Because $\delta_\eta\delta_\epsilon\Phi$ must be zero for any field, the transformation laws of H is

$$\delta H^{\alpha i} = -\frac{\epsilon}{4}\chi^k\chi^l(R_{kl}{}^i{}_t + R_{klpq}J^{pi}J^q{}_t)\rho^{\alpha t} +$$

$$\frac{i\epsilon}{2}\epsilon^{\alpha}{}_{\beta}(D_k J^i{}_j)\chi^k H^{\beta j} -$$

$$\frac{\epsilon}{4}(\chi^k D_k J^i{}_s)(\chi^l D_l J_t{}^s)\rho^{\alpha t} - i\epsilon\Gamma^i_{jk}\chi^j H^{\alpha k}. \tag{3.33}$$

A Q-invariant action could be

$$S = -i\{Q, V\} \tag{3.34}$$

for any V. As a consequence

$$\delta_Q S = i\epsilon[Q, S] = \epsilon[Q^2, V] = 0. \tag{3.35}$$

More than, we ask for a conformal invariant action which conserves U; V should be

$$V = \int d^2\sigma(\rho^i_\alpha \partial_\alpha u^i - \frac{1}{4}\rho^\alpha_i h^i_\alpha). \tag{3.36}$$

We find a Lagrangian

$$\mathcal{L} = \int d^2\sigma\{-\frac{1}{4}H^{\alpha i}H_{\alpha i} + H^{\alpha i}\partial_\alpha u^i - i\rho^\alpha_i(D_\alpha \chi^i +$$

$$\frac{i}{2}\epsilon_{\alpha\beta}\chi^k D_k J^i{}_j \partial^\beta u^j) - \frac{1}{8}\rho^\alpha_i \rho_{\alpha t}\chi^k \chi^l R_{kl}{}^{it} -$$

$$\frac{1}{6}\rho^\alpha_i \rho_{\alpha t}(\chi^k D_k J^i{}_s)(\chi^l D_l J^{ts})\}. \tag{3.37}$$

The conserved supercurrent obtained from $\delta S = i\int \partial_\alpha \epsilon J^\alpha$ is

$$J^\alpha = g_{ij}H^{\alpha i}\chi^j + \frac{1}{2}J^{is}\rho^\alpha_s D_k J_{ij}\chi^k \chi^j \quad . \tag{3.38}$$

2. A special case. If M is a Kähler manifold , ($D_k J^i{}_j = 0$), TNSM appear naturally as twisted versions of N=2 SNSM . If vector fields of type (1,0) or (0,1) are V^I and $W^{\bar{I}}$, the metric $g_{I\bar{J}}$, the action becomes

$$S = 2\int d^2\sigma\{g_{I\bar{J}}\partial_+ u^I \partial_- u^{\bar{J}} - \frac{i}{2}\rho^I_+ D_- \chi^{\bar{J}} g_{I\bar{J}}$$

$$-\frac{i}{2}\rho_-{}^{\bar{J}} D_+ \chi^I g_{I\bar{J}} - \frac{1}{4}\chi^I \chi^{\bar{I}} \rho_+{}^J \rho_-{}^{\bar{J}} R_{I\bar{I}J\bar{J}}\}. \tag{3.39}$$

There is an N=2 fermionic symmetry

$$\delta\chi^I = \delta\chi^{\bar{I}} = 0$$

$$\delta u^I = i\epsilon\chi^I, \delta u^{\bar{I}} = i\bar{\epsilon}\chi^{\bar{I}}$$

$$\tag{3.40}$$

$$\delta\rho_+^I = 2\tilde{\epsilon}\partial_+ u^I - i\epsilon g^{I\bar{S}}\partial_S g_{K\bar{S}}\chi^S\rho_+^K$$
$$\delta\rho_-^{\bar{I}} = 2\epsilon\partial_- u^{\bar{I}} - i\tilde{\epsilon} g^{S\bar{I}}\partial_{\bar{S}} g_{S\bar{K}}\chi^{\bar{s}}\rho_-^{\bar{K}}. \quad (3.41)$$

ϵ and $\tilde{\epsilon}$ are anticommuting constants; there are two fermionic charges Q_L and Q_R

$$Q_L^2 = Q_R^2 = \{Q_L, Q_R\} = 0. \quad (3.42)$$

We can also consistently set $\rho_-^J = \chi^{\bar{J}} = 0$; the action becomes

$$S = 2\int d_2\sigma\{g_{I\bar{J}}\partial_+ u^I\partial_- u^{\bar{I}} - \frac{i}{2}\rho_-^{\bar{J}} D_+\chi^I g_{I\bar{J}}\} \quad (3.43)$$

and has a single fermionic symmetry

$$\delta u^{\bar{I}} = \delta\chi^I = 0,$$
$$\delta u^I = i\epsilon\chi^I$$
$$(3.44)$$
$$\delta\rho_-^{\bar{I}} = 2\epsilon\partial_- u^{\bar{I}}$$
$$\delta\rho_+^I = -i\epsilon g^{I\bar{S}}\partial_S g_{K\bar{S}}\chi^S\rho_+^K.$$

What is obtained is a twisted version of the conventional (0,2) model.

ii) Observables

The non-trivial observables are cohomology classes of Q i.e. operators $\mathcal{O}$ which are Q-closed ($\{Q,\mathcal{O}\} = 0$) modulo operators which are Q-exact ($\mathcal{O} = \{Q, F\}$). Consider an n-form

$$A = A_{i_1 i_2 \ldots i_n} du^{i_1}\ldots du^{i_n} \quad (3.45)$$

on M. To A , we associate an operator

$$\mathcal{O}_A^0 = A_{i_1 i_2\ldots i_n}\chi^{i_1}\ldots\chi^{i_n} \quad (3.46)$$

Under symmetry ,its variation is equal to

$$\delta_Q\mathcal{O}_A^0 = i\epsilon\partial_j A_{i_1 i_2\ldots i_n}\chi^j\chi^{i_1}\chi^{i_2}\ldots\chi^{i_n} \quad (3.47)$$

Using the definition $\delta\mathcal{O} = -i\epsilon\{Q,\mathcal{O}\}$ it follows that

$$\{Q, \mathcal{O}_A^0\} = -\mathcal{O}_{dA}^0, \quad (3.48)$$

dA being the exterior derivative of A. One sees that $\mathcal{O}_A^0$ is BRST-invariant if and only if A is closed. If A is exact $A = dB$ then $\mathcal{O}_A^0$ is Q-exact. In this way, the Q-cohomology is equivalent to the de Rham cohomology on the target manifold M. Taking $A_1, A_2...A_j$ closed forms of degrees $m_1...m_j$ on M and choosing the points $P_1, P_2, ..., P_j \in \Sigma$, the correlator

$$Z(A_1...A_j) =< \mathcal{O}_{A_1}^{(0)}(P_1)...\mathcal{O}_{A_j}^{(0)}(P_j) > \tag{3.49}$$

is a topological invariant (it is unchanged under continuous changes in metric on Σ and M). Indeed, because

$$Z = \int D\Phi W[\phi]e^{-S}, \tag{3.50}$$

$$\delta Z = - < W[\Phi]\delta S >= 0 \tag{3.51}$$

because $\delta S = \{Q, \delta V\}$ for some δV . It is supposed that the measure $D\phi$ is invariant under such changes .

As a consequence, $Z(A_1...A_j)$ does not depend on the points P_i Considering $\mathcal{O}_A^{(0)}$ an operator-valued zero-form on Σ, we have

$$d\mathcal{O}_A^{(0)} = i\{Q, \mathcal{O}_A^{(1)}\} \tag{3.52}$$

where

$$\mathcal{O}_A^{(1)} = inA_{i_1...i_n}du^{i_1}\chi^{i_2}...\chi^{i_n} \tag{3.53}$$

is an operator-valued one-form on Σ. This happens because

$$\frac{\partial}{\partial\sigma^\alpha}\mathcal{O}_A^{(0)} = \partial_j A_{i_1 i_2...i_n}\partial u^j$$

$$+nA_{i_1 i_2...i_n}\frac{D\chi^{i_1}}{D\sigma^\alpha}\chi^{i_2}...\chi^{i_n}. \tag{3.54}$$

The difference $\mathcal{O}_A^{(0)}(P_1) - \mathcal{O}_A^{(0)}(P_2)$ is

$$\mathcal{O}_A^{(0)}(P_1) - \mathcal{O}_A^{(0)}(P_2) = \{Q, \int_\lambda \mathcal{O}_A^{(1)}\} \tag{3.55}$$

where λ is an arbitrary path from P_1 to P_2 . A new BRST-invariant observable can be introduced

$$W_A(\lambda) = \int_\lambda \mathcal{O}_A^{(1)}. \tag{3.56}$$

λ is a 1-dimensional homology cycle. We can repeat the process:

$$d\mathcal{O}_A^{(1)} = i\{Q, \mathcal{O}_A^{(2)}\} \tag{3.57}$$

An elementary computation gives for the last operator

$$\mathcal{O}^{(2)}_{A} = -\frac{n(n-1)}{2} A_{i_1 \ldots i_n} du^{i_1} \wedge du^{i_2} \chi^{i_3} \ldots \chi^{i_n}. \tag{3.58}$$

A new integral invariant is obtained:

$$W_A(\Sigma) = \int_{\Sigma} \mathcal{O}^{(2)}_{A}. \tag{3.59}$$

In this vay , to each homology class on Σ we can associate a BRST-invariant global observable .

iii) Non-Renormalization theorem

In the case of the Witten action the metric on the target space (Kähler) is renormalized [18]

$$(g_R)_{ij} = g_{ij} + \frac{1}{2\pi} R_{ij} \frac{1}{\epsilon} \tag{3.60}$$

It follows that

$$\beta^{g}_{ij} = \frac{1}{2\pi} R_{ij} \tag{3.61}$$

where R_{ij} is the Ricci tensor . The value of the β-function is the same as in ordinary sigma-model , the complex structure is not renormalized , all the renormalization of the Kähler form is due to the renormalization of the metric. At the end of this chapter it will be shown that this renormalization of the Kähler form ω plays an essential role in showing the non-renormalization of the Killing vectors in context of equivariant cohomology.

3.3. Topological Sigma Models coupled to Topological YM

This models were discussed in ref. [12] . The fermionic transformation laws for the u, χ, ρ multiplet must be modified such that the commutator of two fermionic transformations $\delta_\eta \delta_\epsilon - \delta_\epsilon \delta_\eta$ is not zero, but a gauge transformation. The new transformation laws are

$$\begin{aligned} \delta u^i &= i\epsilon\chi^i \quad , \\ \delta \chi^i &= \epsilon \Phi^a V^i_a \end{aligned} \tag{3.62}$$

where V_a are Killing vectors ($a = 1, ..., dimG$) and ϕ belongs to the N=2 vector supermultiplet. The transformation law of ρ is unchanged, but the transformation of H becomes

$$\delta H^i_\alpha =' old' + i\epsilon \Phi^a (D_j V^i_a) \rho^j_\alpha - i\frac{\epsilon}{2} \Phi^a V^k_a (D_k J^i{}_j) \epsilon_{\alpha\beta} \rho^{\beta j} \quad . \tag{3.63}$$

From these transformation rules, we see that the action of the BRST -charge Q is different ($Q \to Q^*$). The action in this sector becomes

$$S_\sigma = \{Q^*, V\} \tag{3.64}$$

where

$$V = \int d^2\sigma (\rho_i^\alpha D_\alpha u^i - \frac{1}{4}\rho_i^\alpha H_\alpha^i) \ . \tag{3.65}$$

D_α is the pullback covariant derivative. The problem is to couple this 'matter' lagrangian to topological YM in four dimensions [13] . The observables in the 'matter' sector are obtained starting from

$$\mathcal{O}_A^{(0)} = A_{i_1 \dots i_n} \cdot \chi^{i_1} \dots \chi^{i_n} \ . \tag{3.66}$$

Due to the modified transformation rule, $\mathcal{O}_A^{(0)}$ is not a BRST invariant operator. We have

$$\delta \mathcal{O}_A^{(0)} = \epsilon n \Phi^a v_a^{i_1} A_{i_1 i_2 \dots i_n} \chi^{i_1} \chi^{i_2} \dots \chi^{i_n} \ . \tag{3.67}$$

We have to add corrections in order to obtain a BRST-invariant operator. If i_a is the operator of contraction with V_a defined as

$$(i_a(B))_{i_2 i_3 \dots i_n} = n V_a^{i_1} B_{i_1 i_2 \dots i_n} \tag{3.68}$$

equation (3.60) can be written as

$$\delta \mathcal{O}_A^{(0)} = \epsilon \Phi^a \mathcal{O}_{i_a}(A) \ . \tag{3.69}$$

Let $\mathcal{L}_a$ be the Lie derivative with respect to the vector field V_a . The action on a form F is given by

$$\mathcal{L}_a(F) = (d i_a + i_a d) F \ . \tag{3.70}$$

If F is closed and V_a invariant, then $i_a(F)$ is also closed

$$d i_a(F) = 0 \ . \tag{3.71}$$

For F=A, $d i_a(A) = 0$; it is possible that

$$i_a(A) = d A_a \tag{3.72}$$

where A_a is an (n-2)-form. We can define a new operator

$$\tilde{\mathcal{O}}_A^{(0)} = A_{i_1 i_2 \dots i_n} \chi^{i_1} \dots \chi^{i_n} - \Phi^a (A_a)_{i_1 i_2 \dots i_{n-2}} \chi^{i_1} \dots \chi^{i_{n-2}} \ . \tag{3.73}$$

The BRST variation of $\tilde{\mathcal{O}}_A^{(0)}$ is

$$\delta\tilde{\mathcal{O}}_A^{(0)} = -(n-2)\Phi^a\Phi^b V_b^{i_1}(A_a)_{i_1 i_2 \ldots i_{2n-2}}\chi^{i_2}\ldots\chi^{i_{n-2}}. \tag{3.74}$$

The progress consists in the fact that the variation is of order (n-3) in χ. Continuing the process, after a finite number of steps a BRST-invariant operator is obtained. The whole process can be described using the Weil's model. Let $\Omega^*(M)$ be the de Rham complex of M and $S^*(\Phi^a)$ a polinomial algebra on Φ^a ; consider $\Omega_G^*(M) = \Omega^*(M)\otimes S^*(\Phi^a)$ and denote by W^* the G-invariant subcomoplex of $\Omega_G^*(M)$. W^* is the set of elements in $\Omega_G^*(M)$ which are annihilated by

$$L_a = \mathcal{L}_a + f_{ab}{}^c\Phi^b\frac{\partial}{\partial\Phi^c} \tag{3.75}$$

where f are the constant structures of G. Because $\Phi^a L_a = \Phi^a\mathcal{L}_a$, we have

$$\Phi^a\mathcal{L}_a \cdot W^* = 0. \tag{3.76}$$

Define $D = d + \Phi^a i_a$; then $D^2 = \Phi^a\mathcal{L}_a$ and obviously $D^2 = 0$ on W^*. Any element F of degree n on W^* has an expansion

$$F = F^n + \Phi^a F_a^{n-2} + \Phi^a\Phi^b F_{ab}^{n-4} + \ldots \tag{3.77}$$

where F^k is a k-form. The operator

$$\hat{\mathcal{O}}_F^{(0)} = \mathcal{O}_{F^n}^{(0)} + \Phi^a\mathcal{O}_{F_a^{n-2}}^{(0)} + \Phi^a\Phi^b\mathcal{O}_{F_{ab}^{n-4}}^{(0)} + \ldots \tag{3.78}$$

has the property that

$$\delta\hat{\mathcal{O}}_F^{(0)} = i\epsilon\hat{\mathcal{O}}_{DF}^{(0)}\,. \tag{3.79}$$

If $DF = 0$ then the operator is BRST-invariant; if $F = D\Lambda$ for $\Lambda \in W^*$ the operator is BRST-exact. In this way, it corresponds to the D-cohomology. This is just the G-equivariant cohomology of M [3] .

3.4. Sigma Models with Potential

i) Generalities

Starting with the bosonic model in which n scalar fields ϕ_i are inhomogenous coordinates on a target space M and the action is given by

$$S = \frac{1}{2}\int d^2x g_{ij}\partial_\mu\phi^i\partial^\mu\phi^j \tag{3.80}$$

where g_{ij} is the metric on the target space, it is possible to construct a supersymmetric N=1 extension by introducing the superfield

$$\Phi^i(x,\theta) = \phi^i(x) + \bar{\theta}\psi^i(x) + \frac{1}{2}\bar{\theta}\theta F^i(x) \tag{3.81}$$

where θ_α is a Grassman coordinate of superspace. Defining the supercovariant derivative and γ matrices

$$D_\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} - i(\gamma^\mu\theta)_\alpha\partial_\mu, \tag{3.82}$$

$$\gamma^0 = \sigma_y, \gamma^1 = i\sigma_x, \gamma_5 = \sigma_z, \bar{\psi} = \psi^T\gamma^0 \tag{3.83}$$

the superspace action is

$$S = \frac{1}{4i}\int d^2\theta d^2x g_{ij}(\Phi)\bar{D}\Phi^i D\Phi^j \tag{3.84}$$

and is invariant under supersymmetry transformation

$$\delta\Phi^i = \bar{\epsilon}_\alpha Q_\alpha \Phi^i \tag{3.85}$$

where

$$Q_\alpha = \frac{\partial}{\partial\bar{\theta}_\alpha} + i(\gamma^\mu\partial_\mu\theta)_\alpha. \tag{3.86}$$

After integrating over θ and eliminating the auxiliary fields F, the component action is

$$S = \frac{1}{2}\int d^2x(g_{ij}\partial_\mu\phi^i\partial_\mu\phi^j + ig_{ij}\bar{\psi}^i\gamma^\mu D_\mu\psi^j + \frac{1}{6}R_{ijkl}\bar{\psi}^i\psi^k\bar{\psi}^j\psi^l), \tag{3.87}$$

where

$$D_\mu\psi^j = \partial_\mu\psi^j + \Gamma^j_{kl}(\partial_\mu\phi)^k\psi^l. \tag{3.88}$$

On components, the supersymmetry transformation is

$$\delta\phi^i = \bar{\epsilon}\psi^i, \tag{3.89}$$

$$\delta\psi^i = -i\gamma^\mu\partial_\mu\phi^i\epsilon - \Gamma^i_{jk}\bar{\epsilon}\psi^j\psi^k. \tag{3.90}$$

It was shown [20] [21] that the above model admits a second supersymmetry if and only if the target space M is a Kähler manifold. If M is Kähler, the second supersymmetry is [20]

$$\delta\phi^i = \bar{\epsilon}J^i{}_j\psi^j, \tag{3.91}$$

$$\delta(J^i{}_j\psi^j) = -i\gamma^\mu\partial_\mu\phi^i\epsilon - \Gamma^i_{jk}J^j{}_lJ^k{}_m\bar{\epsilon}\psi^l\psi^m. \tag{3.92}$$

J is the complex structure. The metric $g_{a\bar{b}}$ is the second (mixed) derivative of the Kähler potential $K(z,\bar{z})$

$$g_{a\bar{b}} = \frac{\partial^2}{\partial z_a \partial z_{\bar{b}}} K(z,\bar{z}). \tag{3.93}$$

A potential in the N=1 sigma model can be introduced as follows. Add in the action a general term

$$\mathcal{V} = -(m^2V(\phi) + mW_{ij}(\phi)\bar{\psi}^i\psi^j + mW^5_A ij(\phi)\bar{\psi}^i\gamma_5\psi^j) \tag{3.94}$$

and modify the supersymmetry transformation of ψ^i as follows

$$\delta\psi^i = \delta(old) - mH^i(\phi)\epsilon - mG^i(\phi)\gamma_5\epsilon \tag{3.95}$$

where H^i and G^i are vector fields and $V(\phi)$ is a scalar on M. W_{ij} is a symmetric tensor and W^5_{ij} is antisymmetric . By imposing the invariance of the new action under the modified SUSY transformation we obtain the following constraints [20]

$$W_{ij} = D_iD_jW, \qquad H_i = D_iW, \tag{3.96}$$

$$W^5_{ij} = D_iG_j, \qquad D_iG_j + D_jG_i = 0. \tag{3.97}$$

It follows that G^i is a Killing vector of M. The scalar potential is obtained in the same way [20]

$$V(\phi) = g^{ij}(D_iWD_jW + G_iG_j). \tag{3.98}$$

The function $W(\phi)$ is the superpotential.

For N=2 models on Kähler manifolds we have to introduce isometries generated by holomorphic Killing vectors as follows;

$$\delta z^\alpha = V^\alpha(z), \delta\bar{z}^\alpha = \bar{V}^\alpha(\bar{z}) \tag{3.99}$$

where

$$D_\alpha V_\beta + D_\beta V_\alpha = 0. \tag{3.100}$$

Because the Killing vectors are holomorphic, we can introduce a real scalar function $U(z,\bar{z})$,such that

$$V_\alpha = i\partial_\alpha U, \tag{3.101}$$

$$V_{\bar{\alpha}} = -i\partial_{\bar{\alpha}} U \tag{3.102}$$

or , in real coordinates

$$V_i = J^j{}_i \partial_j U. \tag{3.103}$$

U is called the Killing potential. The Lie derivative of the Kähler potential is

$$\mathcal{L}_V K = V^\alpha \partial_\alpha K + \bar{V}^\alpha \partial_{\bar{\alpha}} K = f(z) + \bar{f}(\bar{z}). \tag{3.104}$$

In this case, the general expression of the Killing potential is

$$U(z,\bar{z}) = \frac{1}{2} i [V^\alpha \partial_\alpha K - f - \bar{V}^\alpha \partial_{\bar{\alpha}} K + \bar{f}]. \tag{3.105}$$

Example. In the case of CP^n in Fubini-Study coordinates , the Kähler potential is

$$K(z,\bar{z}) = ln(1 + \bar{z}^\alpha z^\alpha), \tag{3.106}$$

the holomorphic isometry group is $SU(n+1)$ and the Killing vector is

$$V^\alpha = C^{\alpha\beta} z^\beta + b^\alpha + (\bar{b}^\beta z^\beta) z^\alpha \tag{3.107}$$

where C is a hermitean matrix and b is a complex vector which contain together $2n^2 + n$ real parameters of $SU(n+1)$. The Killing potential is

$$U(z,\bar{z}) = -\frac{\bar{z}^\alpha C^{\alpha\beta} z^\beta + bz - b\bar{z}}{1 + \bar{z}z}. \tag{3.108}$$

In general, the superpotential can be written in the form

$$W(z,\bar{z}) = U(z,\bar{z}) + h(z) + \bar{h}(\bar{z}). \tag{3.109}$$

ii) Renormalization

If we consider the restriction of the N=2 model to the case of U=0 , G=0 we have a purely holomorphic superpotential, there is a superfield formulation of this model , the non-renormalization theorems of the d=4 N=1 model apply also in this case ; there are no radiative corrections to the N=2 superpotential. One should mention that if the holomorphic Killing vector contribute to the potential with the term U there is only an N=1 superfield formulation and a counterterm at one-loop level proportional to $g^{ij} D_i \partial_j U$. For general Kähler manifolds this counterterm is not the potential of an holomorphic Killing vector ; so the N=2 model is broken to N=1.(There is also an exception; if the target space is CP^n, this counterterm is proportional to U .) The above argument due to Alvarez-Gaumé and Freedman would spoil the topological invariance at quantum level; though it is not true. Firstly, breaking N=2 to N=1 would destroy the $U(1)$ symmetry associated to the Fermi number. This is a **vector** symmetry; hardly one could believe that radiative corrections could destroy it! This idea has a strong basis; even though using standard N=1 supergraph methods one has at one-loop

$$\mu\frac{dU}{d\mu} = \frac{1}{2}\Delta U \tag{1.110}$$

where $\Delta = d\delta + \delta d$ is the Laplacian, the above equation is consistent with N=2 supersymmetry because of the *magical* identities in Kähler geometry. In this way, the renormalized theory is N=2 invariant.

To prove this one has to use the relation between the Killing vector X, the Kähler form ω and the Killing potential U

$$i(X)\omega = dU \tag{3.111}$$

and to apply the Lichnerowicz theorem: *Let R be the Ricci form on a compact Kähler manifold M, ω the Kähler form and X a holomorphic Killing vector. Then,*

$$i(X)R = d[\frac{1}{2}\delta i(X)\omega] \quad . \tag{3.112}$$

If X has a Killing potential U , it results

$$i(X)R = d[\frac{1}{2}\delta U] \quad . \tag{3.113}$$

We have to use the fact that the potential U , being a soft perturbation, does not modify the renormalization of the metric. In this case, the renormalized Kähler form (at one-loop) is

$$\omega(\mu) = \omega + log(\mu/\Lambda)R \tag{3.114}$$

and we have

$$dU(\mu) \stackrel{\text{def}}{=} i(X)\omega(\mu) = dU + log(\mu/\Lambda)i(X)R \tag{3.115}$$

$$= d[U + \frac{1}{2}log(\mu/\Lambda)\Delta U] \quad . \tag{3.116}$$

It follows that

$$U(\mu) = U + \frac{1}{2}log(\mu/\Lambda)\Delta U \ , \tag{3.117}$$

as in the one-loop computation. This formula shows that the Killing vector X is not renormalized at all; the renormalization of the Kähler form is that compensates the change in the Killing potential.
We can understand this result in the following way; when Killing vectors X are introduced, the algebra has central charge, which is conserved. This central charge, being conserved is not renormalized; as a consequence, we should not expect a renormalization of the Killing vector X which expresses the action of the central charge.

The topological version obtained by twisting the N=2 SNSM was given in ref. [33] . The Witten action is modified by the introduction of the potential terms

$$V = \int_\Sigma \sqrt{g}(\lambda^2 G_{ij} V^i V^j - \lambda^2 \chi^i \chi^j D_i V_j - \frac{1}{4} g^{\alpha\beta} \rho^i_\alpha \rho^j_\beta D_i V_j). \tag{3.118}$$

We can write the first two terms in V as

$$V = \{Q, \int_\Sigma \sqrt{g}(i\lambda^2 G_{ij} V^i \chi^j)\}. \tag{3.119}$$

In the above expressions, V^i is a Killing vector, as G in [20] . The BRST algebra has to be also modified . I give only the modifications;

$$\{Q, \chi^i\} = iV^i,$$

$$[Q, H^i_\alpha] =' old' + D_k V^i \rho^k_\alpha - \frac{1}{2} \epsilon_\alpha{}^\beta V^k \rho^j_\beta D_k J^i{}_j. \tag{3.120}$$

Since the N=2 supersymmetric algebra has central charges, the BRST charge Q is not nilpotent ;

$$[Q^2, x^i] = V^i, \tag{3.121}$$

$$[Q^2, \chi^i] = \partial_j V^i \chi^j, \tag{3.122}$$

$$[Q^2, \rho^i_\alpha] = \partial_j V^i \rho^j_\alpha, \tag{3.123}$$

$$[Q^2, H^i_\alpha] = \partial_j V^i H^j_\alpha. \tag{3.124}$$

It follows that Q^2 acts as a local rotation. An example. If the target space is CP^1 , the only non-trivial observable is

$$X = \omega_{ij} \chi^i \chi^j - \Phi \tag{3.125}$$

where ω is the volume form on the target space

$$CP^1 = S^2$$

and Φ is the Killing potential (suitably normalized)

$$\Phi = \frac{1}{2\pi}(|z|^2/(1+|z|^2) - 1/2) \quad . \tag{3.126}$$

4. The Quantum Ring

4.1. Landau-Ginsburg Models

The N=2 SCFTs correspond to infrared fixed point of the renormalization group (RG) flows of the N=2 super Landau-Ginsburg [22] . The main property of the fixed point under the (RG) flow is the universality; if we perturb the action by adding operators with conformal dimension bigger than 2, the fixed point of the flow rests unchanged. (Such operators are called irrelevant.) There are only a finite number of operators which have the dimension smaller or equal two. These are named relevant operators: they can change the fixed point. **Example.** The N=0 minimal models were studied in ref. [23] . The LG action for the pth element of the A series with central charge

$$c = 1 - 6/p(p+1) \tag{4.1}$$

is

$$S = \int d^2 z (\partial\phi)^2 + gV(\phi) \tag{4.2}$$

where

$$V(\phi) = \phi^{2(p-1)}. \tag{4.3}$$

The relevant operators are $1, \phi, ..., \phi^{2(p-2)}$, whyle ϕ^{2p-3} is irrelevant because it corresponds to a shift in ϕ. Higher powers of ϕ are also irrelevant operators.

Example 2. The action of the N=2 models can be written as

$$\int d^2 z d^4\theta K(\phi_i, \bar{\phi}_i) + (\int d^2 z d^2\theta W(\phi_i) + c.c.). \tag{4.4}$$

Thanks to the non-renormalization theorem for the superpotential $W(\phi_i)$, which is an holomorphic function (the F term) , the analysis is simplified. It was shown that the D-term contains only irrelevants operators. It follows that the superpotential $W(\phi_i)$ dictates the fixed-point of the RG-group flows. The non-renormalization theorem tells us that W is an invariant of the RG-flow. This implies that for each W we have a conformal theory. (The D-term changes in a complicated way under renormalization.) When the metric is rescaled

$$g \longrightarrow \lambda^2 g \tag{4.5}$$

the F term changes as

$$\int d^2 z d^2\theta \longrightarrow \lambda \int d^2 z d^2\theta.$$

We have to rescale the fields to absorb this ;

$$\phi_i \longrightarrow \lambda^{\omega_i}\phi_i. \tag{4.6}$$

The numbers ω_i have to be chosen such that

$$\lambda W(\phi_i) = W(\lambda^{\omega_i}\phi_i). \tag{4.7}$$

The above scaling of fields makes the F term form-invariant. Functions which satisfy eq.(4.7) are named quasi-homogenous with weights ω_i. The left-right scaling dimension of ϕ_i is thus $(\omega_i/2, \omega_i/2)$. We can assume that

$$W(\phi_i = 0) = \partial_j W_{|0} = 0. \tag{4.8}$$

Also, we must impose $\partial_i\partial_j W_{|0} = 0$; otherwise massive modes are introduced. In the infrared limit these massive modes are frozen out [22] . A non-trivial theory must have a completely degenerate critical point at $\phi = 0$. Such a critical point is called singular. If we add to W a quadratic term in new fields, the IR point of the RG-flow is unchanged. Two superpotentials yield the same conformal theory if and only if they are related to one another by field redefinitions. To classify N=2 LG superconformal models, we have to classify quasi-homogenous holomorphic functions in n variables modulo an equivalence relation (two such potentials are equivalent if under a holomorphic change of coordinates they become identical). In plus, two holomorphic functions in m and n variables should be considered equivalent if adding a quadratic form in (k-m) variables to the first and a quadratic form in (k-n) variables to the second one they become equivalent. This type of relation is known as *stable equivalence.* The classification of the stable equivalence classes is the object of the singularity (or catastrophe) theory [27] . After we proceed to the scaling (4.5) , the change in the partition function At the fixed point of the RG flow is given by [24]

$$Z \longrightarrow exp[(c/48\pi) ln\lambda \int R] Z = \lambda^{c/6} Z. \tag{4.9}$$

The same computation must be given in the limit $\lambda \to \infty$ of the LG models. For each chiral superfield we get c=3 . After we redefine the fields

$$\phi_i \to \lambda^{-\omega_i}\phi_i \tag{4.10}$$

the partition function changes as

$$Z \longrightarrow \lambda^{\sum_i (1/2-\omega_i)} Z. \tag{4.11}$$

It follows that the central charge can be computed as

$$c = 6\sum_i (1/2 - \omega_i). \tag{4.12}$$

Denote by $C[\phi_i]$ the ring R of all the power series in ϕ_i and by J the ideal of R generated by the derivatives , i.e. $\partial_j W$. J is the space of power series in ϕ_i that have at least one of the partial derivative as a factor. The quotient

$$Q = \frac{r}{J} = \frac{C[\phi_i]}{\partial_j W} \tag{4.13}$$

is exactly the algebra of chiral operators of the theory (modulo descendents). The reasons are similar to that presented in ref. [26] . The dimension of Q is finite if the critical point is isolated (usually it is denoted by μ). If we perturb the superpotential, usually the critical point splits in μ independent critical points [22] . μ is named the multiplicity of the singularity and is equal to the Witten index . **Example.** If $W(x,y) = x^3 + y^3$, a basis of Q is $1, x, y, xy, y^2, xy^2$. The index is $\mu = 6$.

An important concept is that of modality. This is the mumber of complex parameters one can deform a singularity by (or the superpotential) without changing the dimension of the ring Q and which cannot be absorbed in a coordinate transformation. **Examples.** $W = x^k$ has modality zero because if we add x^m or we change the index $(m < k)$, or we remove it with an holomorphic transformation of coordinate$(m \geq k)$. On the other hand, $W = x^4 + y^4$ has modality one $(m = 1)$ because adding the monomial ax^2y^2 with $a^2 \neq 4$ the index is not changed while the modification cannot be removed.

The classification of the zero-modality $(m = 0)$ superpotentials was given in ref. [22] . They correspond to the A-D-E simply-laced algebras and are in one-to-one correspondence with the minimal N=2 models with $c = 3 - 6/N$, where N is the Coxeter number of the corresponding group. I give below this classification.

Algebra	Superpotential	Central charge c
A_k	x^{k+1}	$3 - 6/(k+1)$
D_k	$x^{k-1} + xy^2$	$3 - 6/2(k-1)$
E_6	$x^3 + y^4$	$3 - 6/12$
E_7	$x^3 + xy^3$	$3 - 6/18$
E_8	$x^3 + y^5$	$3 - 6/30$

For A_k and D_k the condition $k \geq 1$ has to be imposed.

Example. In the case of the E_8 algebra, eq. (4.12) gives for the central charge

$$c = 6(\frac{1}{2} - \frac{1}{3}) + 6(\frac{1}{2} - \frac{1}{5}) = \frac{14}{5}. \quad (4.14)$$

The ring Q is generated by $1, x, y, xy, y^2, xy^2, y^3, xy^3$ and is isomorphic with the chiral operator algebra. The superfieds x and y have dimensions $\frac{1}{6}$ and $\frac{1}{10}$ respectively. We see also that the E_8 model is the tensor product

$$E_8 = A_2 \otimes A_4. \quad (4.15)$$

In the same way,

$$E_6 = A_2 \otimes A_3. \quad (4.16)$$

One should mention that the 0-modal functions presented above do not exhaust the the class of N=2 SCLG without *physical* moduli . There are higher modal potentials without physical moduli because in the infrared fixed point the superpotential must be quasi-homogenous; it is possible that the deformation does not satisfy this, though being an irrelevant operator.

A complete classification is known for low modality ($m \leq 2$) [27] . For $m = 1$ there are three classes; parabolic, hyperbolic and exceptional. **Example.** The members of the parabolic class are denoted P_8, X_9 and J_{10}. These singularities have $c = 3$ and are given by

$$\begin{array}{lll} P_8: & x^3 + y^3 + z^3 + axyz , & a^3 + 27 \neq 0 , \\ X_9: & x^4 + y^4 + ax^2y^2 , & a^2 \neq 4 , \\ J_{10}: & x^3 + y^6 + ax^2y^2 , & 4a^3 + 27 \neq 0 . \end{array}$$

These are quasi-homogeneous functions and for a=0 correspond to compactifiations on $SU(3)/Z_3, SU(2) \times SU(2)/Z_4$ and $SU(3)/Z_6$ [22] .

What is interesting in this classification is the connection with Dynkin-like diagrams. They describe the topology of the vacuum near the conformal fixed point. The nodes of the Dynkin diagram correspond to the minima of the bosonic potential ($|\Delta W|^2$) and lines correspond to solitons tunelling between these vacua.

4.2. Topological Landau-Ginsburg models

These models were introduced in ref. [28] . The coincidence observed by

Witten between the chiral primary ring [25] of the N=2 SCFTs and BRST-invariant operator algebra of the topological sigma models is also true with respect to topological Landau-Ginsburg models [28] . The reason for this is that the observables of the topological LG theories have the same representation in bosonic fields as chiral primary fields in N=2 SCFTs. The classification in the simplest cases is the A-D-E, as in the previous paragraph. Consider a potential

$$W = [g/(n+1)]U^{n+1}. \tag{4.17}$$

An interesting problem discussed in ref. is the change in the topological LG model when interaction terms $U, U^2, ..., U^{n-1}$ are added to W. To each interaction term, there is a coupling constant $g_1, g_2....$ **Example.** For n=3 we can choose the perturbed superpotential as being

$$W = g(\frac{1}{4}U^4 + \frac{1}{2}\alpha_2 U^2 + \alpha_1 U). \tag{4.18}$$

If $\alpha_1 = \alpha_2 = 0$ the theory is of A_3 type. If α_i are non-zero, but satisfy the equation

$$(\frac{1}{2}\alpha_1)^2 + (\frac{1}{3}\alpha_2)^3 = 0 \tag{4.19}$$

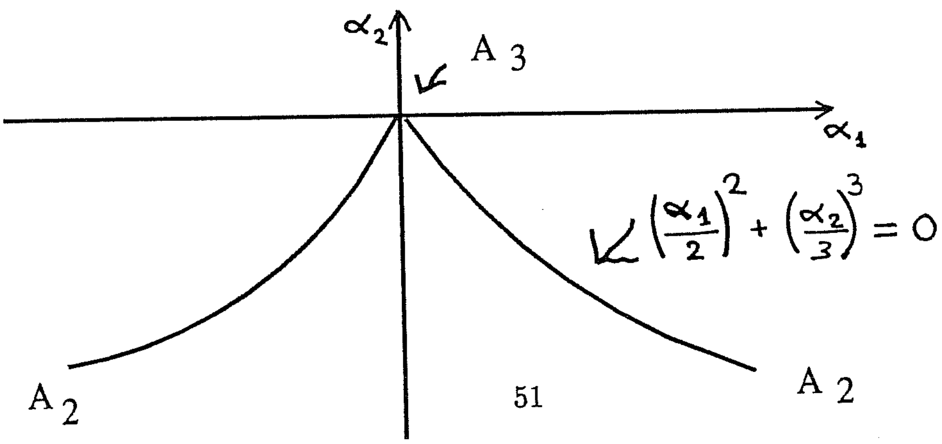
then the theory is of A_2 type. The reason for this is the following; $U = 0$ is not a critical point of the potential. The critical point of W is $U = a$ where

$$2a^3 = \alpha_1, \qquad -3a^2 = \alpha_2. \tag{4.20}$$

Making the transformation $U' = U - a$ the critical point will correspond to $U' = 0$. As function of U' the potential W becomes

$$W = g(aU'^3 + \frac{1}{4}U'^4 + \frac{3}{4}a^4). \tag{4.21}$$

One sees that W is diffeomorphic to U'^3 in a small neighborhood of the critical point. I give below the phase diagram of this model in the coupling-constant space.



It corresponds to the bifurcation sets of the catastrophe theory [27] .

4.3. The Gepner Construction

All kinds of compactified strings with N=1 space-time supersymmetry have two-dimensional N=2 superconformal invariance. Gepner [29] showed that tensor products of minimal (also discrete and unitary) N=2 superconformal models have the same massless spectrum and discrete symmetries as strings compactified on Calabi-Yau manifolds. A large class of Calabi-Yau manifolds were represented in terms of renormalization group fixed points of Landau-Ginsburg models [30] . **Example.** The LG superpotential $W(\Phi) = \Phi^{P+2}$ corresponds to the A-series of the modular invariant N=2 minimal model with central charge

$$c = \frac{3P}{P+2}. \tag{4.22}$$

For a tensor product of minimal models $(P_1, P_2, ..., P_r)$ we have a superpotential

$$W(\Phi_1 ... \Phi_r) = \Phi_1^{P_1+2} + ... \Phi_r^{P_r+2}. \tag{4.23}$$

The simplest Calabi-Yau manifold, $Y_{4;5}$ defined in CP^4 by

$$z_1^5 + z_2^5 + ... + z_5^5 = 0 \tag{4.24}$$

corresponds to the tensor product $(3,3,3,3,3)$ of five level-three minimal models. The relaton between CY manifolds and LG may be understood in terms of path integrals. Starting with

$$\int [d\Phi_1]...[d\Phi_5] e^{i\int d^2z d^2\theta (\Phi_1^5 + ... + \Phi_5^5)}, \tag{4.25}$$

where the D-term is neglected (in fact it is an irrelevant operator), we can change the variables

$$\xi_1 = \Phi_1^5, \qquad \xi_i = \Phi_i / \Phi_1; \tag{4.26}$$

the path integral becomes

$$\int [d\xi_1]...[d\xi_5] e^{i\int d^2z d^2\theta \xi_1 (1 + \xi_2^5 + ... + \xi_5^5)}. \tag{4.27}$$

After the ξ_1 integration is performed, we obtain a delta-function constraint

$$\delta(1 + \xi_2^5 + ... + \xi_5^5) \tag{4.28}$$

which is the equation of $Y_{4;5}$ in inhomogenous coordinates.

Example 2; Weighted Projective Spaces $WCP^N_{w_1...w_n}$ If we take the potential

$$W = \Phi_1^{l_1} + ... + \Phi_5^{l_5} \tag{4.29}$$

and change the variables

$$\xi 1 = \Phi_1^{l_1}, \qquad \xi_i^{l_i} = \Phi_i^{l_i}/\Phi_1^{l_1} \tag{4.30}$$

the path integral becomes

$$\int [d\xi_1]...[d\xi_5] J e^{i\int d^2z d^2\theta \xi_1 (1+\xi_2^{l_2}+...+\xi_5^{l_5})}. \tag{4.31}$$

The jacobian J is proportional to Φ_1^j where

$$j = 1 - l_1 + l_1(\sum_{i=2}^{5}(1/l_i)). \tag{4.32}$$

The jacobian does not depend on fields if $j = 0$, or

$$\sum_{i=1}^{5} \frac{1}{l_i} = 1. \tag{4.33}$$

When this condition is satisfied, the integral over ξ_1 becomes trivial, giving

$$\delta(1 + \xi_2^{l_2} + ... + \xi_5^{l_5}). \tag{4.34}$$

This correspond to an effective target space which is the CY manifold

$$\sum_{i=1}^{5} z_i^{l_i} = 0 \tag{4.35}$$

which coincides with $WCP^4_{d/l_1...d/l_5}$ where d is the least common multiple of $l_1...l_5$. The central charge of this model is

$$c = \sum_{i=1}^{5} 3(l_i - 2)/l_i. \tag{4.36}$$

Using eq.(4.33), we see that $c = 9$, which is the condition for the vanishing of the first Chern class.

4.4. Anomalies in the N=2 Supersymmetric CP^{n-1} Model

In N=2 superspace formulation, [31] the lagrangian has the form

$$L = \sum_{i=1}^{N} \bar{\Phi}_i e^{-V} \Phi_i + (n/2f)V. \tag{4.37}$$

Φ_i are chiral superfields and V is a vector superfield without kinetic term. We recognise in the above equation the moment map presented at pages 18-20. In this case the chiral fields Φ_i parametrize C^N in the same way as $z_i = p_i + iq_i$ in the Hamiltonian model presented by Atiyah and Bott. The super-covariant derivatives are

$$D_L = \frac{\partial}{\partial\theta_L} + \frac{1}{2}\bar{\theta}_L\partial_- \quad , \quad \bar{D}_L = \frac{\partial}{\partial\bar{\theta}_L} + \frac{1}{2}\theta_L\partial_- \quad ,$$

$$D_R = \frac{\partial}{\partial\theta_R} + \frac{1}{2}\bar{\theta}_R\partial_+ \quad , \quad \bar{D}_R = \frac{\partial}{\partial\bar{\theta}_R} + \frac{1}{2}\theta_R\partial_+ \tag{4.38}$$

where $\partial_\pm = \partial_1 \pm i\partial_2$. The chiral superfields have the following expressions

$$\Phi = e^{-u}(Z + \bar{\theta}_L\psi_R + \bar{\theta}_R\psi_L + \bar{\theta}_L\bar{\theta}_R G) \quad , \tag{4.39}$$

$$\bar{\Phi} = e^{u}(Z + \bar{\psi}_R\theta_L + \bar{\psi}_L\theta_R + \theta_R\theta_L\bar{G}) \quad . \tag{4.40}$$

The operator u is defined as

$$u = \frac{1}{2}(\bar{\theta}_L\theta_L\partial_- + \bar{\theta}_R\theta_R\partial_+) \quad . \tag{4.41}$$

G and $\bar{G}$ are auxiliary fields. The vector superfield in the Wess-Zumino gauge is given by

$$V = i\bar{\theta}_L\theta_L B_- + i\bar{\theta}_R\theta_R B_+ + \bar{\theta}_L\theta_R\bar{\phi} + \bar{\theta}_R\theta_L\phi + \bar{\theta}_L\theta_R\bar{\theta}_R\chi_L$$

$$+\bar{\theta}_R\theta_L\bar{\theta}_L\chi_R + \theta_L\bar{\theta}_R\theta_R\bar{\chi}_L + \theta_R\bar{\theta}_L\theta_L\bar{\chi}_R + \tag{4.42}$$

$$\bar{\theta}_R\theta_L\bar{\theta}_L\theta_R D \tag{4.43}$$

where D is an auxiliary field and $B_\pm = B_1 + iB_2$ is the vector component of the vector superfield. The algebraic equation of motion gives for the vector superfield

$$V = ln(\frac{2f}{n}\bar{\Phi}_i\Phi_i). \tag{4.44}$$

If we define the gauge-invariant chiral superfields as being

$$S = D_L\bar{D}_R V, \qquad \bar{S} = D_R\bar{D}_L V \tag{4.45}$$

the anomalous term in the effective action becomes [31]

$$\Gamma_{|an} = (n/4\pi)\int d^2x\{\int d\theta_R d\bar{\theta}_L S(log(S/\mu) - 1) + \int d\theta_L d\bar{\theta}_L \bar{S}(log(\bar{S}/\mu) - 1)\}. \tag{4.46}$$

We can check that the above anomaly written in terms of component fields coincides with the conformal, superconformal and chiral anomaly. μ is an adimensional parameter introduced in the renormalization of the functional determinants. Using the definition of the chiral superfield S we have

$$S_{|\theta=0} = \phi = log(1+\bar{\Phi}\Phi)_{|\bar{\theta}_R\theta_L} \simeq \psi_L\bar{\psi}_R \quad ; \tag{4.47}$$

After the Lorentz twist this field becomes proportional to $\psi^i\psi^j$; after a change in notation ($\psi \rightarrow \chi$) we recognise the topological observable X. The logaritm is a multi-valued function in complex variable ; we have to choose the physical branch . This is determined by the condition that the quantum phase $exp(i\Gamma)$ is single-valued. The minimum of the anomalous term is

$$S = \mu \quad , \quad or \quad S^n = \beta \tag{4.48}$$

where β is identified to μ^n. We see that the second form coincides with the quantum deformation of the classical cohomology ring of the CP^{n-1} manifold. It is interesting that, for the value given in eq. (4.49), the anomalous term in the effective action vanishes.

4.5. The Topological CP^n Model

It was shown in the first chapter that for Kähler manifolds the Dolbeault cohomology coincides with the de Rham cohomology. The simplest Kähler manifold which is compact (and suitable as target space) is the projective space CP^n. In this case the whole cohomology ring is generated by the 0-form 1 and the 2-form ω which is the Kähler form. The classical cohomology ring is

$$R = \{1, \omega, \omega^2, ..., \omega^n\} \tag{4.50}$$

and obviously

$$\omega^{n+1} = 0 \quad . \tag{4.51}$$

The wedge product is taken. In the topological sigma model, we have to introduce the operator

$$X = \omega_{ij}\chi^i\chi^j \quad . \tag{4.52}$$

It is BRST-exact, because ω is closed. The quantum ring of BRST- invariant **local** observables is

$$R_Q = \{1, X, X^2, X^3, ..., X^n\} \tag{4.53}$$

and is isomorphic with the geometrical ring. The only difference is that $X^{n+1} \neq 0$. To understand more better how things develop I will present the case n=1 [32] . For simplicity I consider fields defined on a genus-zero surface, i. e. a 2-sphere. The target space is $CP^1 = S^2$. The classical solutions of instanton number q are

$$w(z) = a\frac{\prod_{i=1}^{q}(z-b_i)}{\prod_{i=1}^{q}(z-c_i)} \tag{4.54}$$

where b_i and c_j are all different such that the fraction is irreducible. The only non-zero Betti numbers for the sphere are b_0 and b_2 , and are equal to one. Consequently there are only two homology classes; the point and the sphere. Consider the universal instanton Φ as a function

$$\Phi \quad : S^2 \times \mathcal{M} \longrightarrow CP^1 \tag{4.55}$$

where $\mathcal{M}$ is the q-instanton moduli space, parametrised by a, b_i and c_i. Denote by L_i homology cycles of $\mathcal{M}$; these are obtained by imposing

$$\Phi : (z_i) \times L_i \in H_i \tag{4.56}$$

where H_i are homology cycles of CP^1. For $H_i = CP^1$ we have $L_i = \mathcal{M}$; if H_i are points, denoted w_i, L_i can be extracted from the equation

$$w(z_i) = w_i \quad . \tag{4.57}$$

Now, we are ready to compute the correlator

$$< X(z_1)...X(z_N) > \quad . \tag{4.58}$$

The q-instanton moduli space has obviously the complex dimension $2q+1$. Using the ghost number (U) anomaly, we have the selection rule

$$\Delta U = dim\mathcal{M} \tag{4.59}$$

and the correlator (4.58) is non-zero only if $N = 2q+1$. (X has U=2.) On the other hand, the same correlator is the intersecton number of dual cycles L in $\mathcal{M}$. There are $2q+1$ cycles L_i those intersection we have to consider. This intersection number is just the number of distinct instanton configurations which contribute. Each cycle L_i is given by eq.(4.57). Their intersection is a system of $2q+1$ equations

$$w(z_i; a, b, c) = w_i \tag{4.60}$$

with $i = 1, ..., 2q+1$, in $2q+1$ unknowns a, b_i, c_i. There is only one solution [32] . Up to θ-vacuum factors, the correlator is one. The final result is

$$< X^{2k+1} > \; = \; \beta^k \quad ,$$

$$< X^{2k} > \qquad = 0. \tag{4.61}$$

β is the θ-vacuum contribution , given by

$$\beta = e^{-\theta_0} \tag{4.62}$$

where θ_0 is positive [13] . Only one type of instantons ($q \geq 0$) contribute.

In the CP^1 model the quantum ring has only two generators, 1 and X. The multiplicative operation in ring gives

$$1 \cdot 1 = 1 \quad ,$$

$$1 \cdot X = X \cdot 1 = X \quad ,$$

$$X \cdot X = X^2 = a1 + bX \tag{4.63}$$

where a and b are constants which follow from

$$0 =< X^2 >= b < X >= b \quad ,$$

$$\beta =< X^3 >=< X^2 \cdot X >= a < X >= a \quad . \tag{4.64}$$

Finally,

$$X^2 = \beta \tag{4.65}$$

that looks like a quantum deformation of the classical cohomology relation

$$\omega^2 = \omega \wedge \omega = 0 \tag{4.66}$$

where ω is the volume form of CP^1.

4.6. The $CP^1 \times CP^1$ Model

If $R_1 = \{1, \omega_1\}$ and $R_2 = \{1, \omega_2\}$ are the cohomology rings of the first, respectively the second piece of the product target space, it follows that the cohomology ring of $CP^1 \times CP^1$ is

$$R_2 = \{1, \omega_1, \omega_2, \omega_1\omega_2\} \quad . \tag{4.67}$$

ω_i are volume forms on the component i (i=1,2) of the target space and the proof is based on the fact that the exterior product of two harmonic forms, each defined on the component i is an harmonic form on the product space. Using the representation of the metric tensor on the product space, as direct sum, it follows that the action is the sum $S = S_1 + S_2$. The model is the tensor product of two CP^1 models. Consider the quantum rings on each component as being

$$R_{1q} = \{1, X\} \quad ,$$
$$R_{2q} = \{1, Y\} \quad , \tag{4.68}$$

the quantum ring of the product theory is

$$R_{1\times 2q} = R_{1q} \otimes R_{2q} \quad , \tag{4.69}$$

or

$$R_{1\times 2q} = \{1, X, Y, XY\} \quad . \tag{4.70}$$

Using the decomposition of the action, there is clustering;

$$< X^N Y^M > = < X^N > < Y^M > \quad . \tag{4.71}$$

Because

$$X^2 = \beta_1 \quad , \quad Y^2 = \beta_2 \tag{4.72}$$

the only non-trivial correlator is

$$< X^{2p+1} Y^{2q+1} > = \beta_1^p \beta_2^q \tag{4.73}$$

where $p, q \geq 0$. The only quantum deformations are given by eq.(4.72) and as a consequence

$$X^2 Y^2 = \beta_1 \beta_2 \quad . \tag{4.74}$$

I considered that the θ-vacua can be in principle different in these sectors.

4.7. The Quantum Ring in the CP^{n-1} case with potential added

In section 4.4. the N=2 supersymmetric CP^{n-1} model was presented in the form of chiral (anti-chiral) superfields Φ_i ($\bar{\Phi}_i$) coupled to a vector superfield V which has no kinetic term. This is a formulation which uses homogeneous coordinates Φ supposed to the constraint $\bar{\Phi}\Phi = 1$. This constraint defines a sphere S^{2n-1} , but the gauge degree of freedom allows us to identify the target space with the symplectic quotient

$$S^{2n-1}/S^1 \sim CP^{n-1} \quad . \tag{4.75}$$

The above picture is extremely useful when a potential is added, as in the work by Labastida and Llatas. All we have to do is to introduce a new gauge superfield , U, coupled to Φ_i with the strength α_i such that the modified Lagrangian becomes

$$L' = \sum_{i=1}^{n} \bar{\Phi}_i e^{-(V'+\alpha_i U)} \Phi_i + (n/2f)V' \quad . \tag{4.76}$$

(We recognise the moment map, as in the case without potential.) U is called *spurion gauge field* . When we solve the equation of motion for $\alpha_i = \alpha$, we obtain $V' = V - \alpha U$ where V is the vector superfield in the case without potential. U is a vector superfield which contains the Killing vector V^i and the Killing potential Φ as components. What we have to do is to compute the effective action, following the same steps as in the paper by d' Adda, di Vecchia and Lüscher. At classical level, there are conformal, super-conformal and chiral symmetries, which at quantum level are broken by anomalies. The general expression for the contribution of *one* chiral superfield Φ_i to the effective Lagrangian in the case of CP^{n-1} is

$$L_{eff} \simeq S[log(\frac{S}{\beta^{1/n}}) - 1] + h.c. \tag{4.77}$$

where $S = D_L \bar{D}_R V$; after the Lorentz twist to the topological case, S becomes the topological observable. When the spurion gauge field U is introduced, we have to change

$$S \longrightarrow S - \alpha_i X \quad , \tag{4.78}$$

with

$$X = D_L \bar{D}_R U \quad . \tag{4.79}$$

The $\theta = 0$ component of X (or the $\bar{\theta}_R \theta_L$ component of U) is the Killing potential Φ. The minimum condition applied to the (new) effective action gives

$$\prod_{i=1}^{n} (S - \alpha_i X) = \beta \quad . \tag{4.80}$$

This gives the quantum ring in the case of CP^{n-1} with potential added. Taking the symplectic quotient is equivalent with identifying modulo a gauge transformation.

4.8. Sigma Models on Grassmannians

Complex Grassman manifolds are quotient spaces

$$G_C(p,q) = U(n)/U(p)\times U(q) = SU(n)/S(U(p)\times U(q)) \; , \; (p+q=n) \; , \; p \leq q \; . \tag{4.95}$$

while real Grassman manifolds are defined as being

$$G_R(p,q) = SO(n)/SO(p) \times SO(q) \; . \tag{4.96}$$

In the classical model [36] the bosonic field is a $n \times p$ matrix field $Z = Z(x)$ such that

$$Z^+Z = 1_p \; , \; or \;\; \bar{Z}_i^a Z_i^b = \delta^{ab} \; . \tag{4.97}$$

The Lagrangian is given by

$$L = g^{\mu\nu} Tr(D_\mu Z^+ D_\nu Z) = g^{\mu\nu} \overline{D_\mu Z_i^a} D_\nu Z_i^a \tag{4.98}$$

where the covariant derivative is defined as

$$D_\mu Z = \partial_\mu Z - Z A_\mu \; . \tag{4.99}$$

There is no kinetic term for the 'gauge' field A_μ ; in fact, using the equation of motion, we obtain

$$A_\mu = Z^+ \partial_\mu Z \; , \; or \;\; A_\mu^{ab} = \bar{Z}_i^a \partial_\mu Z_i^b \; . \tag{4.100}$$

The above model has the following properties:
i) Global $U(n)$ invariance;

$$Z \to g_0 Z \; , \; or \;\; Z_i^a = (g_0)_{ij} Z_j^a \; . \tag{4.101}$$

where g_0 is a unitary $n \times n$ matrix.
ii) Local $U(p)$ invariance;

$$Z \to Zh \; , \; or Z_i^a = h^{ba} Z_i^b \; . \tag{4.102}$$

In this case $h(x)$ is a unitary $p \times p$ matrix . iii) There is a conserved Noether current, $j_\mu = Z D_\mu Z^+ - D_\mu Z Z^+$

$$g^{\mu\nu} \partial_\mu j_\nu = 0 \; . \tag{4.103}$$

The current obeys the identity

$$\partial_\mu j_\nu - \partial_\nu j_\mu + 2[j_\mu, j_\nu] = 0 \; . \tag{4.104}$$

iv) There is an infinite set of non-local (classically conserved) charges; for example, the first of them is given by

$$Q^{(1)}(t) = \int dy_1 dy_2 \theta(y_1 - y_2)[j_0(t, y_1), j_0(t, y_2)] - \int dy j_1(t, y) \quad . \tag{4.105}$$

v) The model can be extended to include fermions in a minimal (M) or in a supersymmetric (S) way (see ref. [36] for details). The analogy with the CP^n case may be very useful. In this case the $Q^{(1)}$ symmetry survives at quantum level; this is useful in computing a factorised S-matrix (see ref. [37]).

Excepting the Lorentz indices, fermions ψ_i^a have an index 'i' in the fundamental representation of $U(n)$ plus an index 'a' in the fundamental representation of $U(p)$.

The classical cohomology ring of the complex Grassman manifold (which is interesting here) was studied in the mathematical literature [38] . The cohomology ring is generated by $X_1, ..., X_p$, where X_r is a (r,r)-form ($X_0 = 1$ by definition). One define

$$X^{(p)}(t) = \sum_{i=0}^{p} X_i t^i = \prod_{i=1}^{p}(1 + \rho_i t) \ , \tag{4.106}$$

$$Y^{(p)}(t) = (X^{(p)}(-t)^{-1} = \sum_n Y_n^p t^n \quad . \tag{4.107}$$

The ideal of Grassmannian cohomology is given by

$$Y_i^{(p)} = 0 \quad for \quad i = q+1, ..., q+p \quad . \tag{4.108}$$

For each vector $\mu = \sum_{i=1}^{p} n_i e_i$ such that

$$n(\mu) = \sum_{i=1}^{p} n_i \leq q \quad , \ n_i = 0, 1, 2, ...$$

there is a ($r(\mu), r(\mu)$) form Φ_μ in the Grassmannian cohomology such that

$$r(\mu) = \sum_{i=1}^{p} i n_i \quad . \tag{4.109}$$

In terms of X_i, this is given by

$$\Phi_\mu^{(p)} = det_{1 \leq i,j \leq n(\mu)} X_{a_i + i - j} \tag{4.110}$$

where a_i is such that $a_i \leq a_{i+1}$ with n_l of a_i equal to l for each $l = 1, ..., p$. There are

$$\frac{(p+q)!}{p!q!}$$

forms Φ_μ which are the cohomology elements of the Grassmanian $G(p,q)$. The Poincaré polinomial is given by

$$P_G(t\bar{t}) = \prod_{i=1}^{p} \frac{1-(t\bar{t})^{q+i}}{1-(t\bar{t})^i} \quad . \tag{4.111}$$

Following Gepner [39] , we define

$$W^{(p)}(t) = -log(X^{(p)}(-t)) = \sum_i W_i^{(p)} t^i \tag{4.112}$$

such that

$$\frac{\partial W^{(p)}(t)}{\partial X_i} = -Y^{(p)}(t)(-t)^i \tag{4.113}$$

where $i = 1, ..., p$. We see that the Grassmannian ideal is generated by the derivatives of $W^{(p)}_{p+q+1}(X_i)$, where

$$W^{(p)}_{p+q+1}(X_i) = \sum_{i=1}^{p} \frac{\rho_i^{p+q+1}}{p+q+1} \quad , \tag{4.114}$$

ρ_i being defined in eq.(4.106). We can use this expression as superpotential. The central charge of this model is

$$c = 3\frac{pq}{p+q+1} \tag{4.115}$$

When instanton effects are taken into account, we know (to be more precise, this is a conjecture introduced by C. Vafa [40]) the deformation which must be introduced in W (see K. Intriligator [41]);

$$W(X) = W^{(p)}_{p+q+1} + (-1)^p \beta X_1 \quad . \tag{4.116}$$

As in the CP^n model, X_1 being the unique (1,1) form is just the Kähler form (in the same time, the volume form is clearly X_p^q). (For p=1 (one variable) the form of the perturbed potential was obtained by Dijkgraaf, Verlinde and Verlinde [42] .) The chiral primary fields are unchanged under (4.116). The ideal $dW = 0$ gives the deformed cohomology relation $X_p Y_q = \beta$.

Example. In the case of $U(5)/U(2) \times U(3)$ the perturbed potential is

$$W(X) = X_1^6/6 - X_1^4 X_2 + (3/2)X_1^2 X_2^2 -$$

$$(1/3)X_2^3 + \beta X_1 \quad . \tag{4.117}$$

The deformed ring relation is given by

$$X_1^5 - 4X_1^3 X_2 + 3X_1 X_2^2 = -\beta \quad . \tag{4.118}$$

5. Conclusions

We have seen that the N=2 supersymmetric (also the topological) sigma models can be described physically in three (at least) equivalent ways. While in the pure (uncoupled to a potential) case it is useful to study the explicit multi-instanton solutions as done by Witten in ref.[32], to understand the geometry and the topology of the moduli space $\mathcal{M}$ and finally to compute topological correlators as intersection numbers, when we couple to potentials we change the cohomology (the de Rham cohomology is replaced to the equivariant one). We cannot describe symply these correlators as intersection numbers. Using the moment map and symplectic quotients is a strong method of computing the deformed ring in this case.

In topological sigma models on CP^n, we have classically conformal, superconformal and chiral invariances. At quantum level, these are spoiled by anomalies. The origin of these anomalies is the fact that products of field operators in the same point are not well-defined and we have to regularize. It happens that this regularization destroys the symetries. As example , what happens here is just the Konishi anomaly. By computing the effective potential, the minimum condition gives us the deformed ring.

In the Landau-Ginsburg approach, it was shown that the deformation is on the direction of the Kähler form. A classification of the LG potentials (not complete yet) was given. Using deformed LG potentials we can describe instanton effects in string theory, for example. We can (at least in principle) compute deformed chiral rings. There is a problem here. In the case of (closed) strings, the configuration space is not the target space M , but the loop space LM. What is important is the cohomology of LM. When the Hilbert space is constructed, it is possible (and it happens in fact) that two (or many) manifolds M_1 and M_2 which differ in geometry and topology may give the same vacuum; $S(M_1) = S(M_2)$. In this way (even a little bit against the classical intuition) we can associate two (or many) manifolds to a ground state. We have in fact two rings; the cohomology rings of M_1 and M_2. It was shown in ref.[25] that $b_{d-p,q}(M_1) = b_{p,q}(M_2)$. (Both M_1 and M_2 have the same complex dimension, d.)

In some cases (see the example given by K. Intriligator) there are some symmetries of the fusion rules induced by the spectral flow which give the integrability of the deformed model.

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