ISAS - INTERNATIONAL SCHOOL
FOR ADVANCED STUDIES

Gravitational Instantons and N=4 Superconformal Field Theories in the context of Heterotic String

Thesis submitted for the degree of
Magister Philosophiæ

Elementary Particle Sector

Candidate:

Marco Billò

Supervisor:

Professor
P. Fré

Academic Year 1991/92

TRIESTE

SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI
TRIESTE
Strada Costiera 11

Index

Introduction

While propagating in a given space (the "target space"), a string-like object describes a two-dimensional surface, known as its "world-sheet". This intuitive idea can be formalized as follows: to the propagation of a string in a given space it is associated a σ-model from the world-sheet to the space under consideration. The "background fields" describing the geometry of the target space are regarded as functions of the world-sheet coordinates.

For closed strings, the world-sheet is a Riemann surface. It is thus characterized topologically by a single number, its genus g. The string perturbative expansion (the expansion in string loops) is an expansion in g: the tree-level corresponds to g=0 (the Riemann sphere), the one-loop level corresponds to g=1 (the torus), and so on.

At the classsical level (classical from the string point of view) the theory is described by a σ-model defined on a two-dimensional sphere. Such a σ-model can represent (or to be a part of) a classical string vacuum only if it is exactly conformal invariant. Its β-functions must vanish at all orders in perturbation theory. We refer here to the perturbation expansion of the σ-model field theory; the parameter of this perturbative expansion is usually indicated as α' and represents the string tension. The β-functions are expressed in terms of the background fields. Their expressions involve derivatives of higher order as one considers higher orders in α'. Setting the β-functions to zero gives therefore a set of differential equations for the background fields. The solution of these equations determines in turn the "background", that is substantially determines the geometry of the target space.

Of course, it would be particularly appealing to have a non-perturbative control of the conformal nature of the σ-model. This happens, for example, when the σ-model represents since the beginning a "classical conformal theory".

String theory requires moreover that the total central charges (that furnished from the σ-model plus that of the internal theory) sum up to cancel the Virasoro anomaly of the reparametrization ghosts (and, if present, superghosts).

So far our considerations have been limited to the classical level from the string viewpoint. Suppose now that we have found a background such that the corresponding σ-model satisfies all the above requirements. The string loop contributions are described by

σ-models with the same target space, but defined on Riemann surfaces of higher genera. A fundamental consistency requirement is therefore met. This request is the modular invariance of the partition function. In particular the partition function must be invariant under the action of the genus one modular group $SL(2, \mathbf{Z})$.

Hence, when a space is a viable string vacuum, it is in correspondence with a conformal field theory (CFT), satisfying certain particular requirements. The Lagrangian formulation of this CFT is just the σ-model on the space itself.

It is therefore interesting to search for the "abstract" CFTs corresponding to non-trivial spaces which are thought to be of possible relevance (as classical string vacua) to physics. The idea is that of reformulating at the CFT level the geometrical properties of these spaces; note that at the CFT level it may be no longer necessary to refer to the Lagrangiam formulation.

That of CFT is an extremely powerful framework. In the case the CFT is solvable, one can in principle compute *exactly* the correlation functions for the conformal operators of the theory. Certain operators, called emission vertex operators, are put in correspondence with the incoming and outgoing particle states of the various fields appearing in the low-energy effective theory of the string. The correlators of the emission vertices give the amplitudes for the scattering of the particles on the corresponding background. Having a solvable CFT, one obtains therefore *exact* expressions for these amplitudes (where *exact* means exact up to string loop corrections).

Recently quite a lot of attention has been payed to the "search" of conformal field theories corresponding to non-trivial manifolds such as stringy Black Holes in unphysical dimensions (D=2) [1] and some progress has been made also for analogous solutions in D=4 [2]. Callan, Harvey and Strominger have discussed [3] the very interesting case of a four-dimensional instanton with torsion and dilaton, which solves the β-functions. The σ-model for this space exhibits an N=4 supersymmetry of world-sheet . This symmetry should protect the σ-model from renormalization, making it good as a classical string vacuum. Moreover, they have shown that (for a particular value of a constant appearing in the solution) the corresponding CFT is solvable. The last part of the thesis will focus on this solution.

Also the main issue treated in this thesis is related with the general programme just outlined of finding CFT corresponding to non-trivial spaces. It may be summarized as follows.
Consider what is hoped to be a "realistic" string model; an heterotic string [4] with gauge group $E_8' \times E_8$, compactified on a six-dimensional Calabi-Yau manifold [5]. It is well known that the effective low-energy theory for the reamining four dimensions is a particular matter-coupled N=1, D=4 supergravity.

Usually, one considers the vacuum of the effective theory given by the flat Minkowski space. That is, one considers a string vacuum given by a target space of the form $\mathcal{M}_6^{CY} \times \mathcal{M}_4^{Mink}$, plus heterotic fermions. However, it is possible to consider other solutions of the effective theory, given by topologically non-trivial spaces $\mathcal{M}_4$. It is then necerrary to investigate whether, and under which conditions, a consistent heterotic vacuum including the target space $\mathcal{M}_6^{CY} \times \mathcal{M}_4$ can be constructed. In particular, I will focus on four-dimensional spaces which constitute a generalizations of gravitational instantons. This generalization is allowed by the presence of other universal fields, beside the metric, in the effective theory, namely the dilaton and the torsion. First of all, it is necessary to characterize geometrically these spaces. This question is addressed in Chapter 1: they can be described as appropriate generalizations with torsion of HyperKähler manifolds, hereafter named for simplicity Generalized HyperKähler Manifolds (GHK).

It can be shown that such spaces are exactly those for which the supersymmetric (1,1)-σ-model admits actually (4,4) supersymmetry. This extremely strong symmetry protects it from renormalization; it constitutes thus a superconformal field theory. This matter is reviewed in Chapter 2. In this chapter a "geometrical" formalism is adopted which is shown to be well-suited (through the use of a first-order formulation) to the treatment of the dilatonic effects. The incorporation of the dilaton in the CFT is fundamental, as will be explicitely shown in the particular case of the Callan, Harvey and Strominger instanton, to ensure that the central charge of the theory be exactly 6 in both sectors (we are referring to 4-dimensional manifolds). This is what we expect. Indeed we do not want to loose the intepretation of the theory as representing the propagation of the string in the four-dimensional space. The value c=6 makes furthermore possible that the N=4 superconformal algebra involved be the "'standard" one (as opposed to the "extended" N=4 algebras appeared in the last years in the literature [6]). Recall indeed that the standard N=4 algebra requires c to be a multiple of six.

Summarizing: the stringy analogues of the gravitational instantons are the Generalized HyperKähler manifolds; the abstract CFTs corresponding to such manifolds are $(6,6)_{4,4}$ theories *.

In Chapter 3 some features of $(6,6)_{4,4}$ theories are recalled. It is particularly emphasized the identification of the conformal operators that abstractly represent, at the CFT level, the Dolbeault cohomology groups (or better, their generalization in presence of torsion). This identification is extremely similar to what happens in Gepner's identification of the compactification of Calabi-Yau manifolds with compactification by means of $(9,9)_{2,2}$ theories [7]. There certain *(chiral,chiral)* and *(chiral,antichiral)* operators are recognized as the "abstract (1,1) and (1,2) forms" of the $(9,9)_{2,2}$ theory.
In our case, also the N=4 moduli operators, i.e. those marginal operators that do not break the N=4 invariance of the theory, are naturally introduced in terms of the "abstract (1,1) forms" of the $(6,6)_{4,4}$ theory.

We want now to consider an heterotic string vacuum including an internal theory representing the six dimensions compactified on a Calabi-Yau manifold and a $(6,6)_{4,4}$ theory representing propagation on a gravitational instanton with torsion (GHK manifold). We must show that such a vacuum can be consistently constructed. This is the key point; the idea results to be very simple, and is described in Chapter 4. Our proposal stays at a generic level. As already said, we propose the identification of the $(6,6)_{4,4}$ theories with the description of string propagation on GHK manifolds in close analogy with the identification of the use of an internal $(9,9)_{2,2}$ theory with Calabi-Yau compactification.

* From now on we use the notation $(c_L,c_R)_{n_L,n_R}$ to mean a CFT of central charges $c_L(c_R)$ in the left (right) sector, possessing n_L,n_R left (right) supersimmetries.

In particular the analysis of how the original gauge group is broken by the embedding of the holonomies of the internal and space-time manifolds rensembles very much the usual compactification and in some sense corresponds to a sort of second "step" in the same process. Also the analysis of the modular invariance of the partition function rensembles the one carried in the usual compactification. The modular invariance is ensured since it is possible to obtain consistently the partition function from a type II one through a h-map mechanism.

If one knows the string vacuum corresponding to a certain background, then he has the possibility of computing the scattering amplitudes for the various particles on this background. In particular, consider asymptotically flat four-dimensional backgrounds, so that the definition of the scattering matrix elements matches that for flat space. The true amplitude for a certain process will include the contributions of all such backgrounds, on the top of the main contributions of the flat space. We can simbolically write

$$A(1,2,...,N) = \sum_{Bk}\sum_{RS} < V_{Bk}(1)V_{Bk}(2)...V_{Bk}(N) >$$

(one of the backgrounds Bk is the flat space, the others are asymptotically flat) where $V_{Bk}(i)$ is the emission vertex for the field (i) in the background Bk. The sum $\sum_{Bk}$ represents the string perturbation theory: a discrete sum over the genera of the Riemann surfaces spanned by the string , and, at fixed genus, the integration over the moduli space of the surface. In this continuous sum it is fundamental to utilize the proper measure. We have moreover indicated with $\sum_{Bk}$ the discrete sum over the topologies of the asymptotically flat backgrounds and, at fixed topology, the integration over their moduli spaces. Of course this expression remains too generic until we specify the measure to be used in $\sum_{Bk}$ (which will be essentially of four-dimensional nature); however, this task is left for further investigations. Here I just wanted to delineate "in priciple" the possible use in string theory of non-trivial spacetime backgrounds.
It is therefore fundamental to know the moduli space of the background one wants to treat. In the case of GHK manifolds there is a natural relation between the geometrical moduli and the deformations introduced by the N=4 moduli operators in the correspondent $(6,6)_{4,4}$ theories; this may provide the way to understand also the moduli space from the geometrical point of view.

Usual gravitational instantons are at most *locally* asymptotically euclidean (ALE) [8], but cannot be *globally* asymptotically euclidean. As a conseguence of the non-trivial torsion this possibility is instead present for the GHK manifolds. To this purpose we recall that ten years ago D'Auria and Regge [9] proposed a mechanism, based on the unsoldering of the $SO(4)$ principal bundle (the one in which the vielbeins and the spin connection are valued) from the tangent bundle through a torsion effect. This being the case, asymptotically flat topologically non-trivial configurations became possible. The price for this was that such configurations arose as solutions of an ad hoc constructed lagrangian, involving an extra scalar field. It is remarkable that the same configurations can instead arise naturally as instantonic solutions of the effective N=1, D=4 supergravity coming from Calabi-Yau compactification. An example of this is, as we will see, just the Callan, Harvey and Strominger instanton.

The contribution of asymptotically flat instantonic backgrounds may prove important is a renewed version of the Konishi-Magnoli-Panagopoulos mechanism [10]. In this mechanism, local supersymmetry breaks down dynamically because of the gravitino condensation in an instantonic background. However, purely gravitational instantons (such as the Eguchi-Hanson metric [11], for which the explicit computation was carried) were considered, and these cannot be globally asymptotically flat. It was then problematic to justify the consideration of the instantonic contributions in the same sector as flat space. This drawback is no longer present if asymptotically flat GHK manifolds are considered.

The fundamental point that one needs to treate in the abstract CFT corresponding to a given background is the construction of the vertex operators for the various fields. In Chapter 5 this subject is addressed for the GHK manifolds. In this chapter the emission vertices for the zero modes of the various fields appearing in the effective four-dimensional field theory are constructed. Their expressions are given in terms of the CFTs of which is composed the corresponding string vacuum.

First of all, an analysis "à la Kaluza-Klein " is performed to predict in terms of the topological numbers of the manifold the number of zero modes for the various kind of fields. These zero-modes turn out to be organized in representations of the "residual" symmetry group $SU(6)$. This group is the analogous of the E_6 symmetry group arising in the usual compactification. The construction of the emission vertices corresponding to the zero-modes is facilitated mainly by two considerations. On one side, flat space itself is a $(6,6)_{4,4}$ theory and thus, recasting the flat space expressions in a generic $(6,6)_{4,4}$ language, we have almost ready at hand the desired generic expressions. On the other side, one can rely on the identification of the operators which correspond abstractly to the cohomology classes (or,better, to their torsionful generalizations), identification already mentioned in Chapter 3. This identification is known from the attempts to use $(6,6)_{4,4}$ theories as internal theories; it is strongly reflected here, as the vertices counting and group arrangement is shown to coincide perfectly with the geometrical counting and group arrangement of the zero-modes. From Chapter 6 on the Callan, Harvey and Strominger instanton is examinated in detail, as an illustration of the general set-up previously developed. Thus, from our point of view, this configuration in retrieved in Chapter 6 as a particular solution of the equations of motion of the effective N=1, D=4 supergravity [12]; in doing so it is reviewed and emphasized the role of the New Minimal formulation of the theory, which is known to be the correct one for the string-derived case. The Callan, Harvey and Strominger solution presents many appealing features, that may be summarized as follows: it is in general an asymptotically flat GHK manifold (which implies all the consequences above mentioned, and developed in the thesis).

There is a particular value of a constant parameter appearing in the solution for which the associated conformal theory is solvable and quite simple indeed [3]. Unfortunately in this case asymptotic flatness is lost (the topology becomes that of $S^3 \times \mathrm{R} = SU(2) \times \mathrm{R}$, with torsion). In Chapter 7 the classical conformal field theory is discussed relying on the general formulædeveloped in Chapter 2, that allow an easy treatment of the dilatonic contributions, and on the formalism for σ-models on group-manifolds, in this case $SU(2) \times$ R. The quantum realization of the theory just requires some finite renormalizations. The goal is to recognize clearly the relevant geometrical structures characterizing the manifold as a GHK manifold, and the structures of its associated $(6,6)_{4,4}$ CFT. In particular we can explicitely single out the "abstract (1,1) and (0,1)" operators. On one side this permits to construct immediately the emission vertices on this background, just plugging the explicit expressions for these operators into the general form of the vertices, valid for any $(6,6)_{4,4}$ theory, of Chapter 5. On the other side, and this is the part on which Chapter 8 focuses, the knowledge of the form of the $N=4$ moduli operators permits to explicitely deform the Lagrangian still mantaining (4,4) supersymmetry, and thus still remaining within a $(6,6)_{4,4}$ theory. This in turns means that this deformed theory still corresponds geometrically to a GHK manifold. By examinating the σ-model lagrangian we can thus describe explicitely an entire class of GHK manifolds in dependence of certain parameters (moduli). These parameters appear as the coefficients governing the insertion of the possible $N=4$ moduli operators in the σ-model lagrangian; they turn out to be sixteen. Although explicitely carried through only for infinitesimal deformations, this represents a new result which extends the treatment given by Callan, Harvey and Strominger . It would be necessary to investigate analogously the finite deformations and to understand thus the geometry of the moduli space of this class of spaces. This is another point which deserves further future developement.

Finally in the Appendix it is shown in a particular case (the $SU(6)$ adjoint representation) that the expected $SU(6)$ symmetry of the zero-modes is actually implemented in a correct way in the expressions of the vertex operators.

Chapter 1

Generalized HyperKähler Spaces and Gravitational Instantons with Torsion

We introduce here, giving them the name of Generalized HyperKähler (GHK) manifolds, a class of spaces often considered in the literature in relation with the superconformal σ-models with extended supersymmetries.
After giving their definition, we show how these spaces, in four dimensions, can be naturally put in correspondence with the analogue of gravitational instantons in the case in which, beside the metric, other universal fields are present in the theory, such as the torsion and the dilaton.

1.1 Generalized HyperKähler Manifolds

With this name we mean a manifold $\mathcal{M}$, such that $dim\mathcal{M} = 4n$, on which two sets of three tensors, $\overset{-}{\mathcal{J}}{}^{x}_{ab}$ and $\overset{+}{\mathcal{J}}{}^{x}_{ab}$, ($x = 1, 2, 3$, a, b are tangent indices) can be defined, wich satisfy the following properties.
First of all,

$$(\overset{-}{\mathcal{J}}{}^{x})^2 = (\overset{+}{\mathcal{J}}{}^{x})^2 = -\mathbf{1} \tag{1.1}$$

i.e. all these tensors are almost complex structures on $\mathcal{M}$.
The Nijenhuis thensor relative to each almost complex structure vanishes:

$$N_{abc}(\mathcal{J}^x, \mathcal{J}^x) = \nabla_m \mathcal{J}^x_{a[b}\, \mathcal{J}^x_{mc]} + \mathcal{J}^x_{am} \nabla_{[b} \mathcal{J}^x_{cn]} = 0 \tag{1.2}$$

(here $\mathcal{J}^x$ can be $\overset{-}{\mathcal{J}}{}^{x}$ or $\overset{+}{\mathcal{J}}{}^{x}$) so that they are actually complex structures; moreover, also non-diagonal Nijenhuis conditions [13] are satisfied in each set.
The two sets of complex structures commute:

$$[\, \overset{-}{\mathcal{J}}{}^{x} \,,\, \overset{+}{\mathcal{J}}{}^{y} \,] = 0 \tag{1.3}$$

and separately each set gives a representation of the quaternionic algebra

$$\mathcal{J}^x \mathcal{J}^y = -\delta^{xy} + \epsilon^{xyz} \mathcal{J}^z \tag{1.4}$$

The metric must be hermitean with respect to all the complex structures, which in tangent indices means that

$$\mathcal{J}^x_{ab} = -\mathcal{J}^x_{ba} \qquad \text{i.e.} \qquad \mathcal{J}^{xT} = -\mathcal{J}^x \tag{1.5}$$

Define on $\mathcal{M}$ the two non-Riemannian connections $\omega^{\pm}_{ab} = \omega^{R}_{ab} \pm T_{abc}V^c$ where $T^a = T^a_{\ bc}V^bV^c$ is the torsion two-form * (whose components T_{abc} are assumed to be totally antisymmetric), ω^{R}_{ab} is the Riemannian spin-connection and V^c are the Vielbein on $\mathcal{M}$. Denote as $\stackrel{\pm}{\nabla}$ the covariant derivatives constructed with these two torsionful connections. The complex structures of one of the two sets are covariantly constant with respect to one of these connections:

$$\stackrel{-}{\nabla}_m \stackrel{-}{\mathcal{J}}{}^x_{ab} = 0 \tag{1.6a}$$

and those of the other set with respect to the other connection:

$$\stackrel{+}{\nabla}_m \stackrel{+}{\mathcal{J}}{}^x_{ab} = 0 \tag{1.6b}$$

In Chapter 2 we will see that spaces with the above preperties come naturally into the game when considering σ-models with (4,4) supersymmetry on the world-sheet. In particular, it is well known since a long time ago [13] that, in the case of zero torsion, the condition under which a σ-model admits (4,4)supersymmetry is that the target space must be HyperKähler . Thus, the name of Generalized HyperKähler manifolds given to the spaces satisfying properties (1.1)-(1.6) is justified; setting $T = 0$ we recover indeed the usual notion of a HyperKähler manifold.

For zero torsion, the two sets of complex structures coincide, and there is thus one set of complex structures, satisfying the quaternionic algebra, covariantly constant with respect to the Riemannian connection: this is precisely the definition of an HyperKähler space.
On such a manifold there exist three globally defined two-forms $\Omega^x = \mathcal{J}^x_{ab}V^aV^b$ which are closed: $d\Omega^x = 0$. The role of these forms is the generalization of that played for a Kähler space by the Kähler form $\Omega = \mathcal{J}_{ab}V^aV^b$.
Choosing a well-adapted basis of vielbein it is possible to show that the holonomy group $\mathcal{H}ol(HK_m)$ of a HyperKähler space HK_m with dimension $4m$ is contained in $Sp(2m)$ [14]. In particular a four-dimensional HyperKähler space has a holonomy group contained in $SU(2)$: the curvature 2-form is selfdual or antiselfdual. Note that this is the requirement a manifold must satisfy in order to be a gravitational instanton.
Let us see which is the analogue of this consideration when torsion is introduced into the game.

* Through all the thesis we shall not write explicitely the wedge symbol

1.2 GHK Spaces in Four Dimension: (Anti)Self-duality of the torsionful Curvatures

We refer now to a four-dimensional GHK space. Eq.(1.5) tells us that the $\overset{-}{\mathcal{J}}{}^x$ and $\overset{+}{\mathcal{J}}{}^x$ complex structures (with tangent indices) are expressed as antisymmetric 4×4 matrices. In 4-dimensions we can construct a basis for 4×4 antisymmetric matrices made by the following two sets of three constant matrices, respectively named $\hat{\mathcal{J}}^x$ and $\tilde{\mathcal{J}}^x$ $(x = 1, 2, 3)$*:

$$
\begin{aligned}
\hat{\mathcal{J}}^x_{ab} &= -(\delta_{a0}\delta_{bx} - \delta_{b0}\delta_{ax} + \epsilon_{xab}) \\
\tilde{\mathcal{J}}^x_{ab} &= (\delta_{a0}\delta_{bx} - \delta_{b0}\delta_{ax} - \epsilon_{xab})
\end{aligned}
\tag{1.9}
$$

that is

$$
\begin{aligned}
&\hat{\mathcal{J}}^1 = \begin{pmatrix} 0 & i\sigma^2 \\ i\sigma^2 & 0 \end{pmatrix} \quad ; \quad \hat{\mathcal{J}}^2 = \begin{pmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{pmatrix} \quad ; \quad \hat{\mathcal{J}}^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \\
&\tilde{\mathcal{J}}^1 = \begin{pmatrix} 0 & -\sigma^1 \\ \sigma^1 & 0 \end{pmatrix} \quad ; \quad \tilde{\mathcal{J}}^2 = \begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} \quad ; \quad \tilde{\mathcal{J}}^3 = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}
\end{aligned}
\tag{1.10}
$$

These matrices have the following properties:

- each set $\hat{\mathcal{J}}^x$, $\tilde{\mathcal{J}}^x$ gives a representation of quaternionic algebra (1.4)
- they are selfdual (resp anti-selfdual):

$$
\begin{aligned}
\hat{\mathcal{J}}^x_{ab} &= \frac{1}{2}\,\epsilon_{abcd}\,\hat{\mathcal{J}}^x_{cd} \qquad \leftrightarrow \qquad \epsilon_{ijk}\,\hat{\mathcal{J}}^k_{0} = -\,\hat{\mathcal{J}}^x_{ij} \\
\tilde{\mathcal{J}}^x_{ab} &= -\frac{1}{2}\,\epsilon_{abcd}\,\tilde{\mathcal{J}}^x_{cd} \qquad \leftrightarrow \qquad \epsilon_{ijk}\,\tilde{\mathcal{J}}^k_{0} = \tilde{\mathcal{J}}^x_{ij}
\end{aligned}
\tag{1.11}
$$

- all the $\hat{\mathcal{J}}^x$ commute with all the $\tilde{\mathcal{J}}^x$.

A priori we can expand both the $\overset{-}{\mathcal{J}}{}^x$ and $\overset{+}{\mathcal{J}}{}^x$ along the basis given by $\hat{\mathcal{J}}^x, \tilde{\mathcal{J}}^x$:

$$
\begin{aligned}
\overset{-}{\mathcal{J}}{}^x &= s^x{}_y\hat{\mathcal{J}}^y + a^x{}_y\tilde{\mathcal{J}}^y \\
\overset{+}{\mathcal{J}}{}^x &= \overset{+}{s}{}^x{}_y\hat{\mathcal{J}}^y + \overset{+}{a}{}^x{}_y\tilde{\mathcal{J}}^y
\end{aligned}
\tag{1.12}
$$

For all the coefficients in the expansion (1.12) there are two possibilities: they can be zero or, in order for the $\overset{-}{\mathcal{J}}{}^x$ and $\overset{+}{\mathcal{J}}{}^x$ to satisfy the quaternionic algebra, must be such that

$$
s^x{}_p s^y{}_p = \delta^{xy} \tag{1.13}
$$

$$
\epsilon^{xyz}\, s^z{}_t = \epsilon^{pqt}\, s^x{}_p s^y{}_q \tag{1.14}
$$

---

* We use the convention that ϵ_{xab} vanishes if one of the indices a,b takes the value 0. Otherwise in 3-space it is the usual Levi-Civita symbol

The same conditions hold for $a^x{}_y, s^x_{+\,y}, a^x_{+\,y}$. Relations (1.13-14) mean that each of these 3×3 matrices is orthogonal, namely they belong to the adjoint representation of $SO(3)$. We can use a vector notation $\mathbf{s}^x$ ($\mathbf{s}^x_+$) for the rows of the matrix $s^x{}_y$ ($s^x_{+\,y}$); if they are non zero, these vectors constitute an orthonormal basis in three-dimensional space.
Let us then consider the consequences of the fact that all the $\bar{\mathcal{J}}^x$ must commute with all the $\overset{+}{\mathcal{J}}{}^x$. Using the expansion (1.12) this means that

$$\mathbf{s}^x \wedge \mathbf{s}^y_+ = 0 \qquad ; \qquad \mathbf{a}^x \wedge \mathbf{a}^y_+ = 0 \qquad \forall x, y \tag{1.15}$$

(here the symbol $\wedge$ denotes the usual exterior product of three-dimensional vectors). We can expand the $\mathbf{s}^y_+$ in the basis $\{\mathbf{s}^x\}$:

$$\mathbf{s}^y_+ = c^y_p \mathbf{s}^p$$

Suppose now that the $\mathbf{s}^x$ are different from zero. Then the condition (1.15), upon use of eq.(1.14), states that

$$c^y_p \mathbf{s}^x \wedge \mathbf{s}^p = \epsilon^{xpq} c^y_p \mathbf{s}^q = 0$$

implying $c^y_p = 0$, that is $\mathbf{s}^y_+ = 0$.
If the $\mathbf{a}^x$ were non-zero, then an analogous argument would constrain the $\mathbf{a}^y_+$ to vanish as well; then the $\overset{+}{\mathcal{J}}{}^x$ would just be zero, which cannot be. The only allowed situation is the following

$$\begin{aligned} \bar{\mathcal{J}}^x &= s^x{}_y \hat{\mathcal{J}}^y \\ \overset{+}{\mathcal{J}}{}^x &= a^x_{+\,y} \tilde{\mathcal{J}}^y \end{aligned} \tag{1.16}$$

that is, the $\mathcal{J}^x$ are selfdual while the $\overset{+}{\mathcal{J}}{}^x$ antiselfdual (or viceversa).
Consider the curvature 2-form R^-_{ab}relative to the connection ω^-_{ab}. Let R^- be the matrix of components R^-_{ab}. It is an antisymmetric matrix and as such it can be expanded as follows:

$$R^- = A_x \hat{\mathcal{J}}^x + B_x \tilde{\mathcal{J}}^x$$

It must satisfy the integrability condition for the covariant constancy of $\bar{\mathcal{J}}^x$,

$$[R^- \,,\, \bar{\mathcal{J}}^x\,] = 0$$

Inserting the expansions of R^- and $\bar{\mathcal{J}}^x$ this means

$$A_p s^x{}_y [\hat{\mathcal{J}}^p \,,\, \hat{\mathcal{J}}^y] = 2\,\epsilon^{pyt} A_p s^x{}_y \hat{\mathcal{J}}^t = 0$$

The unique solution of this constraint is $A_p = 0$; this implies that R^- is antiselfdual.
Repeating an analogous argument for the curvature R^+_{ab} of the connection ω^+_{ab} we find that R^+ is selfdual.
Summarizing: on a four-dimensional GHK space the curvature two-form $R^- = R(\omega_R - T)$ is antiselfdual, while $R^+ = R(\omega_R + T)$ is selfdual.

This appears to be the natural generalization of the selfduality of the curvature two-form characterizing gravitational instantons in absence of torsion (recall that, via the cyclic Bianchi Identity, this selfduality implies that vacuum Einstein equations are satisfied). We assume therefore that four-dimensional GHK manifolds constitute the generalization of gravitational instantons to the case of non-vanishing torsion. In Chapter 6 we will check in a particular case that an instantonic solution of a theory including torsion (the instanton of Callan, Harvey and Strominger) actually turns out to be of the Generalized HyperKähler type.

Chapter 2

(4,4)-extended Sigma-models with Dilaton and Axion couplings and GHK Manifolds

Now we briefly review the subject of σ-models on a generic target space, taking into account the dilaton effects. To this purpose we utilize the rheonomy framework [15]. In the first part of this section we consider the bosonic σ-model, in the second part we extend the construction to locally supersymmetric σ-models of (1,1) type. Our results correspond to the generalization, with dilaton coupling, of the construction presented in [16]. Freezing the two-dimensional gravitinos one obtains the σ-model action with global (1,1) supersymmetry, that can be utilized to discuss the structure of the corresponding superconformal theory. The local construction, however, is essential to obtain the stress-energy tensor and the two supercurrents (left and right-moving). In the type II version of string theory, these supercurrents are coupled to the worldsheet gravitinos. After the h-map to the heterotic string, only the left-moving current corresponds to a local world-sheet symmetry. The right-moving supersymmetry ceases to be local and its role is the same for X-space as it is for the internal compactified space, namely it relates emission vertices of different particle modes. In the internal space this leads to remarkable consequences, in particular to the pairing between moduli fields and charged fields and to the special Kähler geometry of the moduli space. In subsequent sections we will discuss the analogue consequences for X-space. After having established the formalism for (1,1) σ-models we shall consider the conditions under which the global supersymmetry of the same model is accidentally extended to larger N. In particular we shall consider the conditions for (4,4) global supersymmetry. As we are going to see, in d=4 these conditions force the target space to be a Generalized HyperKähler Manifold, and this provides the link with instanton geometry.

2.1 The bosonic Sigma-model

In correspondence with a solution of the equations of motion derived from the effective bosonic lagrangian:

$$\mathcal{L}_{eff} = e^{-2\Phi}(\mathcal{R} - 4D_\mu\Phi D^\mu\Phi + ...)$$

that contains the metric $G_{\mu\nu}$ (i.e. equivalently the vielbeins V^a), the three form H and the dilaton Φ, we write the action for the bosonic σ-model utilizing a geometric first order formalism:

$$\begin{aligned}&\frac{1}{4\pi}\int_{\partial\mathcal{M}} V^a(\Pi^a_+ e^+ - \Pi^a_- e^-) + \Pi^a_+\Pi^a_- e^+e^- - 2\Phi R^{(2)} + p_+T^+ + p_-T^- + \\ &+\frac{1}{4\pi}\int_{\mathcal{M}} H_{abc}V^aV^bV^c\end{aligned} \tag{2.1}$$

Once rewritten in 2^{nd}order formalism, this action takes the more familiar form

$$S = \frac{-1}{4\pi}\int_{\partial\mathcal{M}} dzd\bar{z}\left[G_{\mu\nu}(X)\partial X^\mu\bar{\partial}X^\nu + B_{\mu\nu}(X)\partial X^\mu\bar{\partial}X^\nu\right] \tag{2.2}$$

where $ds^2 = G_{\mu\nu}dX^\mu \otimes dX^\nu = V^a \otimes V^a$ is the target space line element and the antisymmetric tensor $B_{\mu\nu}$ is such that $H = 2dB$.
Note that when the action is written as in eq.(2.2) it keeps no tracks of the dilaton which contributes only to the classical stress-energy tensor of the model. This contribution is obtained in a simple way in 1^{st}order formulation. In a similar way, when we consider the supersymmetric extensions of the above model, the contributions of the dilaton to the supercurrents are also easily retrieved from the 1^{st}order formulation.
Let us briefly explain the somewhat unusual notations and the meaning of the quantities appearing in eq.(2.1)[15,16]. In particular e^+ and e^- are the vielbein on the world-sheet $\partial\mathcal{M}$, whose geometry is described by the structure equations

$$\begin{aligned}de^+ - \omega^{(2)}e^+ &= T^+\\ de^- + \omega^{(2)}e^- &= T^-\\ d\omega^{(2)} &= R^{(2)}\end{aligned} \tag{2.3}$$

$\omega^{(2)}, T^\pm, R^{(2)}$ are the two-dimensional spin connection, torsion and curvature respectively. Classical conformal invariance of the model allows the choice of the "special conformal gauge":

$$e^+ = dz \quad ; \quad e^- = d\bar{z} \quad ; \quad \omega^{(2)} = R^{(2)} = 0 \tag{2.4}$$

where $z = x^0 + x^1$ and $\bar{z} = x^0 - x^1$. This is the choice we have used to obtain the 2^{nd}order form of the action (2.2). More specifically "after variation" we can use eq.(2.4). $\Pi^a_\pm, p_\pm$ are "1^{st}order fields": they can be reexpressed in terms of the usual dynamical fields upon use of the equations obtained by varying in $\Pi^a_\pm, p_\pm, \omega^{(2)}$.

$$\begin{aligned}&\text{Varying in } \Pi^a_\pm : \quad \Pi^a_\pm = V^a_\pm\\ &\text{Varying in } \omega^{(2)} : \quad p_\pm = \mp 2\partial_\pm\Phi = \mp 2\partial_a\Phi V^a_\pm\\ &\text{Varying in } p_\pm : \quad T_+ = T_- = 0\end{aligned} \tag{2.5}$$

In the present formalism, the general recipe to obtain the components of the stress-energy tensor is to vary the action with respect to the world-sheet vielbein, defining

$$\delta S = \frac{-1}{2\pi} \int \mathcal{T}_+ \, \delta e^+ + \mathcal{T}_- \, \delta e^- \tag{2.6}$$

and to consider the expansion $\mathcal{T}_+ = \mathcal{T}_{++} e^+ + \mathcal{T}_{+-} e^-$ (and the analogous one for $\mathcal{T}_-$). The conformal invariance of the model implies that $\mathcal{T}_{+-} = \mathcal{T}_{-+} = 0$ and one defines the usual holomorphic and antiholomorphic part of the stress-energy tensor to be

$$T(z) = \mathcal{T}_{++} \qquad ; \qquad \widetilde{T}(\bar{z}) = \mathcal{T}_{--} \tag{2.7}$$

For the model described by the action (4,1) varying, for example, in e^+, we obtain:

$$\delta S = \frac{-1}{2\pi} \left(-\frac{1}{2} \right) \int (V^a \Pi^a_+ - \Pi^a_+ \Pi^a_- e^-) \delta e^+ + p_+ d\delta e^+$$

Substituting eqs.(2.5), we obtain:

$$T(z) = \mathcal{T}_{++} = -\frac{1}{2} V^a_z V^a_z - \partial\partial\Phi \tag{2.8}$$

Actually, in order to discuss superstring theory, we rather need the supersymmetric version of the just described σ-model . Strictly speaking, heterotic string theory would require a (1,0) supersymmetrization; however, in view of the h-map, we can go one step beyond and consider the case of (1,1) local supersymmetry. We recall now some essential features of the geometrical formulation of (1,1) supersymmetric σ-model [16] and we include dilaton contributions.

2.2 The (1,1) locally supersymmetric Sigma-model

One realizes a classical superconformal invariant theory in terms of fields living on a super-world-sheet with two fermionic coordinates θ and $\bar{\theta}$ besides the two bosonic ones z and $\bar{z}$. The cotangent basis on the super world-sheet(the "supervielbein") is given by the already introduced 1-forms e^+, e^- and two bidimensional gravitinos ζ, χ.
The structure equations (2.3) are enlarged by the appearence of two fermionic torsion 2-forms:

$$\begin{aligned} T^\bullet &= d\zeta - \frac{1}{2}\omega^{(2)}\zeta \\ T^\circ &= d\chi + \frac{1}{2}\omega^{(2)}\chi \end{aligned} \tag{2.20}$$

The "curvatures" $T^+, T^-, T^\bullet, T^\circ, R^{(2)}$ must satisfy the Bianchi identities obtained by exterior differentiation of eqs.(2.3) and (2.20). This imposes a certain form for their parametrization, whose most relevant part is:

$$\begin{aligned} T^+ &= \frac{i}{2}\zeta\zeta \\ T^- &= -\frac{i}{2}\chi\chi \end{aligned} \tag{2.21}$$

The superconformal invariance of this construction allows for the choice of a "special superconformal gauge" where

$$
\begin{aligned}
e^{+} = dz + \frac{i}{2}\theta d\theta \qquad &; \qquad e^{-} = d\bar{z} + \frac{i}{2}\bar{\theta}d\bar{\theta} \\
\zeta = d\theta \qquad &; \qquad \chi = d\bar{\theta}
\end{aligned}
\tag{2.22}
$$

This is the choice we always use in 2^{nd}order formalism (see discussion after eq.(2.4)).
We describe superstring propagation on an arbitrary target manifold $\mathcal{M}_{\text{target}}$ by means of an embedding function $X^{\mu}(z, \bar{z}, \theta, \bar{\theta})$ mapping the super world-sheet into $\mathcal{M}_{\text{target}}$. Therefore we consider the quantities defining the geometry of $\mathcal{M}_{\text{target}}$, such as its vielbeins and spin-connection, as superfield on the super world-sheet, and thus they can be expanded on the cotangent basis of this latter. In particular we set

$$
V^{a} = V_{+}^{a} e^{+} + V_{-}^{a} e^{-} + \lambda^{a}\zeta + \mu^{a}\chi \tag{2.23}
$$

Also the torsion and curvature 2-forms of $\mathcal{M}_{\text{target}}$ can be expanded in the various "sectors" on the super world-sheet. For example, the torsion, defined by:

$$
dV^{a} + \omega^{ab}V^{b} = T^{a} = T^{abc}V^{b}V^{c} \tag{2.24a}
$$

yields

$$
\begin{aligned}
e^{+}e^{-}: \qquad & -\nabla_{-}V_{+}^{a} + \nabla_{+}V_{-}^{a} - 2T^{abc}V_{+}^{b}V_{-}^{c} = 0 \\
& \cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}
\tag{2.24b}
$$

(relations that we are always free to use because they are just the "pull-back" of the original definitions).
The key point are the Bianchi identities of $\mathcal{M}_{\text{target}}$ which become differential equations for V^{a} as a super-worldsheet function; that is, they determine the eqs. of motion for $V_{+}^{a}, V_{-}^{a}, \lambda^{a}, \mu^{a}$ [16]. The B.I. for the torsion of $\mathcal{M}_{\text{target}}$ is $\nabla T^{a} = \nabla^{2}V^{a} = R^{ab}V^{b}$ or, explicitly

$$
\nabla^{2}V_{+}^{a} = R^{ab}V_{+}^{b} \tag{2.25.a}
$$

$$
\nabla^{2}V_{-}^{a} = R^{ab}V_{-}^{b} \tag{2.25.b}
$$

$$
\nabla^{2}\lambda^{a} = R^{ab}\lambda^{b} \tag{2.25.c}
$$

$$
\nabla^{2}\mu^{a} = R^{ab}\mu^{b} \tag{2.25.d}
$$

Each of these equations can be analized in its various sectors. In particular the λ^{a} field equation, setting $\nabla_{o}\lambda^{a} = 0$, constraint compatible with the Bianchi identity:

$$
-\frac{i}{2}\nabla_{-}\lambda^{a} = -R^{ab}{}_{cd}\lambda^{b}\mu^{c}\mu^{d} \tag{2.26a}
$$

is retrieved in the $\chi\chi$ sector of eq.(2.25.c) and the μ^{a} field equation

$$
\frac{i}{2}(\nabla_{+}\mu^{a} + 2T^{abc}\mu^{b}V_{+}^{c}) = -R^{ab}{}_{cd}\mu^{b}\lambda^{c}\lambda^{d} \tag{2.26b}
$$

is retrieved in the $\zeta\zeta$ sector of eq.(2.25d). Bianchi identities for the curvature R^{ab} do not give any new information.

Next one tries to write down an action defined on super world-sheet from which both the definitions (2.24) and the field equations follow as variational equations. To this purpose one starts writing down the most general geometrical action defined on the super world-sheet which respects invariance under Weyl rescalings and two-dimensional Lorentz transformations, with undetermined coefficients; these latter are fixed by comparing the variational equations with parametrizations (2.24) and field equations.
It turns out that the projections of the variational equations in $\delta\lambda^a$ and $\delta\mu^a$ are sufficient to fix all the coefficients. The super world-sheet action takes then the form

$$\begin{aligned} S = \frac{1}{4\pi}\int_{\partial\mathcal{M}} & (V^a - \lambda^a\zeta - \mu^a\chi)(\Pi^a_+ e^+ - \Pi^a_- e^-) + \Pi^a_+\Pi^a_- e^+e^- + 2i\lambda^a \overset{-}{\nabla}\lambda^a e^+ + \\ & + 2i\mu^a \overset{+}{\nabla}\mu^a e^- + \lambda^a V^a\zeta - \mu^a V^a\chi - \lambda^a\mu^a\zeta\chi + \frac{4}{3}iT_{abc}\lambda^a\lambda^b\lambda^c\zeta e^+ - \\ & - \frac{4}{3}iT_{abc}\mu^a\mu^b\mu^c\chi e^- + 4R_{abcd}\lambda^a\lambda^b\mu^c\mu^d e^+e^- + \\ & - 2\Phi R^{(2)} + p_+T^+ + p_-T^- + p_\bullet T^\bullet + p_\circ T^\circ + \frac{1}{4\pi}\int_{\mathcal{M}} H \end{aligned} \tag{2.27}$$

The variation in δX^μ, restricted to the sectors $\zeta\zeta, \chi\chi$, where it really corresponds to a supersymmetry variation, fixes

$$T_{abc} = -3H_{abc} \tag{2.28}$$

justifying our assumption that T_{abc} is completely antisymmetric in its indices.
The action (2.26) is a geometrical one on the super world-sheet, and is therefore invariant against super-world-sheet diffeomorphisms. Its expression is however uniquely determined by its "bosonic" section $\zeta = \chi = 0$, due to the fact that the components of the curvatures along the "fermionic" directions are expressed by eqs.(2.24) in terms of those along the "bosonic" (or "inner") ones. This property is called "rheonomy". One can forget, if he wants to, about the super world-sheet and then the would-be diffeomorphisms in fermionic directions appear as supersymmetry transformations. For $\zeta = \chi = 0$ the action reduces to

$$\begin{aligned} S = \frac{1}{4\pi}\int_{\partial\mathcal{M}} & V^a(\Pi^a_+ e^+ - \Pi^a_- e^-) + \Pi^a_+\Pi^a_- e^+e^- + 2i\lambda^a \overset{-}{\nabla}\lambda^a e^+ + \\ & + 2i\mu^a \overset{+}{\nabla}\mu^a e^- + 4R^{ab}{}_{cd}\lambda^a\lambda^b\mu^c\mu^d e^+e^- + \\ & - 2\Phi R^{(2)} + p_+T^+ + p_-T^- + \frac{1}{4\pi}\int_{\mathcal{M}} H \end{aligned} \tag{2.29}$$

The above action possesses a global (1,1) supersymmetry that is the remainder of the local one present when the gravitino fields are switched on. In the next section we recall how, for suitable target manifolds, this global (1,1) SUSY extends to a global (4,4) supersymmetry.

From the complete form (2.26) of the action, one can derive the super-stress-energy tensor (i.e. the stress-energy tensor and the supercurrent) extending eq.(2.6) to

$$\delta S = \frac{-1}{2\pi}\int \mathcal{T}_+\,\delta e^+ + \mathcal{T}_-\,\delta e^- + \mathcal{T}_\bullet\,\delta\zeta + \mathcal{T}_\circ\,\delta\chi \tag{2.30}$$

Superconformal invariance requires

$$\begin{aligned}&\mathcal{T}_{+-}=\mathcal{T}_{-+}=\mathcal{T}_{\bullet-}=\mathcal{T}_{-\bullet}=\mathcal{T}_{\circ+}=\mathcal{T}_{+\circ}=0\\&\mathcal{T}_{+\bullet}=\frac{1}{2}\mathcal{T}_{\bullet+}\qquad;\qquad\mathcal{T}_{-\circ}=-\frac{1}{2}\mathcal{T}_{\circ-}\end{aligned}\tag{2.31}$$

The surviving four independent components define the classical holomorphic and antiholomorphic parts of stress-energy tensor and supercurrent:

$$\begin{aligned}T(z)&=\mathcal{T}_{++}\\G(z)&=2\sqrt{2}e^{-i\frac{\pi}{4}}\mathcal{T}_{+\bullet}\end{aligned}\;;\qquad\begin{aligned}\widetilde{T}(\bar{z})&=\mathcal{T}_{--}\\\widetilde{G}(\bar{z})&=2\sqrt{2}e^{-\frac{3i\pi}{4}}\mathcal{T}_{-\circ}\end{aligned}\tag{2.32}$$

In the action (2.26) or (2.28) two different covariant derivatives appear, $\overset{+}{\nabla}$ and $\overset{-}{\nabla}$, constructed with the two spin-connections $\omega^{\pm}$, defined as

$$\begin{aligned}\omega^{-}_{ab}&=\omega^{\mathrm{R}}_{ab}-T_{abc}V^{c}\\\omega^{+}_{ab}&=\omega^{\mathrm{R}}_{ab}+T_{abc}V^{c}=\omega^{-}_{ab}+2T_{abc}V^{c}\end{aligned}\tag{2.33}$$

where ω^{R}_{ab} is the Riemannian connection, i.e. is such that $dV^a+\omega^{\mathrm{R}}_{ab}V^b=0$. The connection appearing in eq.(2.23) (the one for which the torsion is T^a) is $\omega_{ab}=\omega^{-}_{ab}$. These connections play an important role in the sequel.

2.3 Complex Structures and extended SUSY

We review the conditions for the existence of additional global supersymmetries in the (1,1)-locally supersymmetric σ-model [13]. To discuss the additional supersymmetries, we formally introduce new fermionic directions of the super world-sheet, adding to the cotangent basis new "gravitinos" ζ^x and χ^x so that the parametrization (2.21) is extended to

$$\begin{aligned}T^{+}&=\frac{i}{2}(\zeta\zeta+\zeta^x\zeta^x)\\T^{-}&=-\frac{i}{2}(\chi\chi+\chi^x\chi^x)\end{aligned}\tag{2.34}$$

while the embedding of the extended super world-sheet in $\mathcal{M}_{\mathrm{target}}$ is described by expanding the target-space vielbeins as follows:

$$V^a=V^a_+e^{+}+V^a_-e^{-}+\lambda^a\zeta+\overset{-}{\mathcal{J}}{}^{x}_{ab}\lambda^b\zeta^x+\mu^a\chi+\overset{+}{\mathcal{J}}{}^{x}_{ab}\mu^b\chi^x\tag{2.35}$$

(Note that the new terms do not introduce any new dynamical quantities).
Consistency with the torsion definition and implementation of the Bianchi Identities leads to constraints on the tensors $\overset{\pm}{\mathcal{J}}{}^{x}_{ab}$, and therefore to a characterization of $\mathcal{M}_{\mathrm{target}}$.

The torsion definition (2.24a): $\bar{\nabla} V^a = T_{abc} V^b V^c$ can now be expanded in many sectors. Using the sectors *

$$
\begin{aligned}
\zeta\zeta: &\qquad \frac{i}{2} V^a_+ + \nabla_\bullet \lambda^a + T_{abc}\lambda^b\lambda^c = 0 \\
\zeta\zeta^x: &\qquad \nabla_\bullet(\mathcal{J}^x_{ab}\lambda^b) + \nabla^x_\bullet \lambda^a + 2T_{abc}\lambda^b \mathcal{J}^x_{cr}\lambda^r = 0 \\
\zeta^x\zeta^y: &\qquad iV^a_+\delta^{xy} + \nabla^x_\bullet(\mathcal{J}^y_{ab}\lambda^b) + \nabla^y_\bullet(\mathcal{J}^x_{ab}\lambda^b) + 2T_{abc}\mathcal{J}^x_{br}\mathcal{J}^y_{cs}\lambda^r\lambda^s = 0
\end{aligned}
\tag{2.36}
$$

by looking at terms containing V^a_+, one finds:
for $x = y$

$$\mathcal{J}^x_{ab}\mathcal{J}^x_{br} = -\delta_{ar} \qquad \text{i.e.} \qquad (\mathcal{J}^x)^2 = -\mathbf{1} \tag{2.37a}$$

for $x \neq y$

$$\{\mathcal{J}^x, \mathcal{J}^y\}_{ar} = 0 \tag{2.37b}$$

It follows that the $\mathcal{J}^x$ form a representation of the Clifford algebra. From the remaining terms in these equations, after some manipulations, one gets :
for $x = y$, the condition that the usual Nijenhuis tensor relative to each $\mathcal{J}^x$ should vanish:

$$N_{abn}(\mathcal{J}^x, \mathcal{J}^x) = \overset{\mathrm{R}}{\nabla}_m \mathcal{J}^x_{a[b}\,\mathcal{J}^x_{mn]} + \mathcal{J}^x_{am}\overset{\mathrm{R}}{\nabla}_{[b}\mathcal{J}^x_{mn]} = 0 \tag{2.38}$$

and for $x \neq y$ analogous non-diagonal Nijenhuis conditions [13].
From sectors $\chi\chi, \chi\chi^x, \chi^x\chi^y$ the same relations for $\overset{+}{\mathcal{J}}{}^x$ are retrieved:

$$
\begin{aligned}
&(\overset{+}{\mathcal{J}}{}^x)^2 = -\mathbf{1} \quad ; \quad \{\overset{+}{\mathcal{J}}{}^x, \overset{+}{\mathcal{J}}{}^y\} = 0 \\
&N_{abc}(\overset{+}{\mathcal{J}}{}^x, \overset{+}{\mathcal{J}}{}^y) = 0
\end{aligned}
\tag{2.39}
$$

Starting from the sector

$$\zeta^x\chi^y: \qquad \nabla^x_\bullet(\overset{+}{\mathcal{J}}{}^y_{ab}\mu^b) + \nabla^y_\circ(\mathcal{J}^x_{ab}\lambda^b) + 2T_{abc}\mathcal{J}^x_{br}\lambda^r\,\overset{+}{\mathcal{J}}{}^y_{cs}\mu^s = 0$$

and substituting the relations that follows from the other sectors $\zeta\chi^x, \chi\zeta, \zeta\chi$ by considering the terms that contain $\nabla_\bullet\mu^a$ we come to the conclusion that the two set of tensors should commute:

$$[\mathcal{J}^x, \overset{+}{\mathcal{J}}{}^y] = 0 \tag{2.40}$$

Now we can also consider the various sectors of the torsion Bianchi identities. In particular from eq.(2.25.c) in the sector $\zeta^x\zeta^x$:

$$\frac{i}{2}\nabla_+\mu^a + \nabla^x_\bullet\nabla^x_\bullet\mu^a = -R^{ab}{}_{cd}\mathcal{J}^x_{cr}\mathcal{J}^x_{ds}\lambda^r\lambda^s\mu^b$$

looking at the terms involving V^r_+ and using the field equation (2.26a) one ends with

$$\nabla_m \mathcal{J}^x_{ab} = 0 \tag{2.41}$$

---

* From now on we drop in all calculations the superscript $-$ for $\bar{\mathcal{J}}{}^x, \bar{\nabla}$ etc.

while the other terms impose the condition:

$$R = \mathcal{J}^{x\,T} R \mathcal{J}^x$$

$R^{ab} = R^{ab}{}_{cd} V^c V^d$ being the curvature two-form. This coincides with the integrability condition for eq.(2.41), namely

$$[R, \mathcal{J}^x] = 0 \tag{2.42}$$

if

$$\mathcal{J}^{x\,T} = -\mathcal{J}^x \tag{2.43}$$

which is just the hermiticity condition expressed in tangent indices.
Considering the sector $\chi^x \chi^x$ of eq.(2.26.d) and analizing terms proportional to V^a_- one sees that the torsion terms are such that the analogue of eq.(2.41) is given by:

$$\overset{+}{\nabla}_m \overset{+}{\mathcal{J}}{}^x_{ab} = 0 \tag{2.44}$$

At the same time in order for the other terms to reproduce the integrability condition

$$[\overset{+}{R}, \overset{+}{\mathcal{J}}{}^x] = 0 \tag{2.45}$$

(where $\overset{+}{R}_{ab}$ is the curvature 2-form of the connection ω^+_{ab}) the hermiticity condition

$$\overset{+}{\mathcal{J}}{}^{x\,T} = -\overset{+}{\mathcal{J}}{}^x \tag{2.46}$$

must be verified.

Summarizing: *The condition to have (N, N) supersymmetries is the existence of two sets of $N-1$ complex structures on the target space (whose Nijenhuis tensors vanish), each set realizing a representation of the Clifford algebra, and the two sets commuting with each other. One of the two sets, namely $\overset{-}{\mathcal{J}}{}^x$, must be covariantly constant with respect to the connection ω^-, while the other one, $\overset{+}{\mathcal{J}}{}^x$ is covariantly constant with respect to ω^+. The target space metric should be hermitean with respect to all complex structures.*

2.4 (4,4) SUSY and Generalized HyperKähler Manifolds

Consider the case of exactly 3+3 additional supersymmetries. It is easy to see that if $\mathcal{J}^1$ and $\mathcal{J}^2$ are two complex structures satisfying the above requirements then $\mathcal{J}^3 = \mathcal{J}^1 \mathcal{J}^2$ is another one. Due to the Clifford algebra requirement the set $\mathcal{J}^x$ closes a quaternionic algebra:

$$\mathcal{J}^x \mathcal{J}^y = -\delta^{xy} + \epsilon^{xyz} \mathcal{J}^z \tag{2.47}$$

The same holds true for $\overset{+}{\mathcal{J}}{}^x$.
This requirement, together with the ones in general necessary for (N, N) supersymmetries reproduces clearly just what in Chapter 1 we declared to be the characterizing features

of the Generalized HyperKähler Manifolds. These manifolds were identified, on the other side, with the instantonic geometries in presence of torsion and dilaton. Thus σ-models on gravitational instantons with torsion possess $(4,4)$ supersymmetry on the world-sheet.

2.5 The classical Supercurrents

Suppose that a $(1,1)$ σ-model on a manifold $\mathcal{M}_{\text{target}}$ admits an extended $(4,4)$ supersymmetry. We want now to obtain the classical form of the supercurrents generating these (4,4) supersymmetries, including explicitely the dilatonic contributions.
If the 3+3 additional supersymmetries are regarded just as global ones, the action on the bosonic world-sheet, namely eq.(2.29) is not modified at all: we just find that it is invariant against additional transformations. The novelty is that we can now search for the complete form of the action on the extended super world-sheet, i.e. the analogue of eq.(2.27) including terms proportional to ζ^x and χ^x. One should repeat the same steps needed to fix the form (2.27) taking into account all the possible new terms. Since from our point of view the only relevance of such an expression would be its use in the derivation of the classical supercurrents, we will confine ourselves to the terms involved in this derivation.
Let us note that the "dilatonic" terms will be enlarged to

$$\Phi R^{(2)} + p_+ T^+ + p_- T^- + p_\bullet T^\bullet + p_\bullet^x T_x^\bullet + p_\circ T_+^\circ p_\circ^x T_x^\circ \tag{2.48}$$

where (in perfect analogy with eq.(2.20)) $T_x^\bullet, T_x^\circ$ are the fermionic torsion two-forms relative to the new super world-sheet gravitinos:

$$T_x^\bullet = d\zeta^x - \frac{1}{2}\omega^{(2)}\zeta^x \quad ; \quad T_x^\circ = d\chi^x + \frac{1}{2}\omega^{(2)}\chi^x \tag{2.49}$$

Variations in the 1^{st} order fields p's sets all the torsions to zero. This allows the choice of an "enlarged" special superconformal gauge (the extension of eq.(2.22)).
Variation in the two-dimensional spin-connection $\omega^{(2)}$ yields

$$\begin{aligned} p_+ &= -2\,\partial_a \Phi V_+^a \quad &;\quad p_- &= 2\,\partial_a \Phi V_-^a \\ p_\bullet &= -4\,\partial_a \Phi \lambda^a \quad &;\quad p_\circ &= 4\,\partial_a \Phi \mu^a \\ p_\bullet^x &= -4\,\partial_a \Phi (\mathcal{J}^x \lambda)^a \quad &;\quad p_\circ^x &= 4\,\partial_a \Phi (\overset{+}{\mathcal{J}}{}^x \mu)^a \end{aligned} \tag{2.50}$$

The fermionic torsion terms in (2.51) will contribute to the variation of the action in the new gravitinos, as it is seen from expression (2.52). After variation we make use of eqs.(2.53).
The supercurrents are obtained by obvious extensions of eqs.(2.30) and following ones. Let

$$\delta S = \frac{-1}{2\pi} \int \mathcal{T}_+ \,\delta e^+ + \mathcal{T}_- \,\delta e^- + \mathcal{T}_\bullet \,\delta\zeta + \mathcal{T}_\circ \,\delta\chi + \mathcal{T}_\bullet^x \,\delta\zeta^x + \mathcal{T}_\circ^x \,\delta\chi^x \tag{2.51}$$

Then superconformal invariance imposes on the 1-forms $\mathcal{T}_\bullet^x$ and $\mathcal{T}_\circ^x$ the analogue of conditions (2.31), namely:

$$\mathcal{T}_{+\bullet}^x = \frac{1}{2}\mathcal{T}_{\bullet+}^x \quad ; \quad \mathcal{T}_{-\circ}^x = -\frac{1}{2}\mathcal{T}_{\circ-}^x$$

All the other components are zero.
Definition (2.32) is enlarged to include also the supercurrents G^x:

$$G^x(z) = 2\sqrt{2}e^{\frac{-i\pi}{4}}\mathcal{T}^x_{+\bullet} \quad ; \quad \widetilde{G}^x(\bar{z}) = -2\sqrt{2}e^{-\frac{3i\pi}{4}}\mathcal{T}^x_{-\circ} \tag{2.52}$$

From the action (2.27) we can extract $G^0(z) = G(z)$ and $\widetilde{G}^0(\bar{z}) = \widetilde{G}(\bar{z})$. For example, to get $G^0 \propto \mathcal{T}_{\bullet+}$ we vary in $\delta\zeta$ and we look for the terms proportional to e^+; the relevant terms are :

$$\begin{aligned}\delta S \rightarrow & \frac{1}{4\pi}\int_{\partial\mathcal{M}} \delta\zeta\left\{-\lambda^a\Pi^a_+e^+ - \lambda^aV^a_+e^+ + \frac{4}{3}iT_{abc}\lambda^a\lambda^b\lambda^c\right\} + \delta(p_\bullet T^\bullet) + \ldots = \\ & = \frac{-1}{2\pi}\int_{\partial\mathcal{M}} \delta\zeta\left\{\lambda^aV^a_+e^+ - \frac{2}{3}iT_{abc}\lambda^a\lambda^b\lambda^ce^+ - \frac{1}{2}\partial_+p_\bullet e^+ + \ldots\right\}\end{aligned}$$

where we have integrated by parts the last term after use of the definition (2.20). Using eq.(2.50) we get

$$\mathcal{T}_{+\bullet} = \frac{1}{2}\mathcal{T}_{\bullet+} = \frac{1}{2}\lambda^aV^a_+ - \frac{i}{3}T_{abc}\lambda^a\lambda^b\lambda^c + \partial_+(\partial_r\Phi\lambda^r) \tag{2.56}$$

so we finally obtain the expression for $G^0 = -2\sqrt{2}e^{-i\frac{\pi}{4}}\mathcal{T}_{+\bullet}$. In a similar way one obtains $\widetilde{G}^0(\bar{z})$.
To derive the other supercurrents we must analize the possible new terms that contribute to the relevant variations, and fix their coefficients by comparing the variational equations with the projections of the equation defining the target torsion (2.24a).
For example to get $G^x(z)$ through the computation of $\mathcal{T}^x_{\bullet+}$ the relevant terms in the extended super world-sheet action are (compare with eq.(2.27)):

$$\begin{aligned}S = \int_{\partial\mathcal{M}} & (V^a - \lambda^a\zeta - (\mathcal{J}^x\lambda)^a\zeta^x - \ldots)(\Pi^a_+e^+ - \ldots) + 2i\lambda^a\nabla\lambda^ae^+ + \ldots + \lambda^aV^a\zeta + \\ & + (\mathcal{J}^x\lambda)^aV^a\zeta^x + \ldots + \frac{4}{3}iT_{abc}\lambda^a\lambda^b\lambda^c\zeta e^+ + n_1T_{abc}(\mathcal{J}^x\lambda)^a\lambda^b\lambda^c\zeta^xe^+ + \ldots \\ & + p_\bullet T^\bullet + p^x_\bullet T^\bullet_x + \ldots\end{aligned} \tag{2.54}$$

A priori, besides the term of the form $T(\mathcal{J}\lambda)\lambda\lambda$, we could add to eq. (2.54) also two other kind of terms, namely $T(\mathcal{J}\lambda)(\mathcal{J}\lambda)\lambda$ and $T(\mathcal{J}\lambda)(\mathcal{J}\lambda)(\mathcal{J}\lambda)$. The reason why it suffices to add only the first term is the vanishing of the Nijenhuis tensor. Indeed the diagonal Nijenhuis tensor constructed from $\mathcal{J}^x$ or $\overset{+}{\mathcal{J}}{}^x$, (see eq.(2.38)), upon use of the covariant constancy condition $\overset{-}{\nabla}_m\mathcal{J}^x_{ab} = 0$, or $\overset{+}{\nabla}_m\overset{+}{\mathcal{J}}{}^x_{ab} = 0$ can be rewritten as follows:

$$N_{abc}(\mathcal{J},\mathcal{J}) = 3T_{rm[a}\mathcal{J}_{rb}\mathcal{J}_{mc]} - T_{abc} \tag{2.55}$$

(the antisymmetrization in abc is understood). By use of the Nijenhuis condition $N_{abc} = 0$ it is easy to show that

$$\begin{aligned}T(\mathcal{J}\lambda)(\mathcal{J}\lambda)\lambda &\propto T\lambda\lambda\lambda \\ T(\mathcal{J}\lambda)(\mathcal{J}\lambda)(\mathcal{J}\lambda) &\propto T(\mathcal{J}\lambda)\lambda\lambda\end{aligned}$$

Hence there is only one coefficient to fix in (2.54), namely n_1. To obtain its value, we consider the equation that follows from varying the action (2.54) in $\delta\lambda^a$. Focusing on its $\zeta^x e^+$ sector and comparing with the $\zeta^x\zeta^x$ sector of the torsion definition (see eq.(2.36)), we obtain

$$n_1 = 4i$$

Varying now (2.54) in $\delta\zeta^x$ and searching for $\mathcal{T}^x_{\bullet +}$, in analogy with the procedure utilized for G^0, we get

$$\mathcal{T}^x_{\bullet} = \left\{\frac{1}{2}(\mathcal{J}^x\lambda)^a\Pi^a_+ e^+ + \frac{1}{2}(\mathcal{J}^x\lambda)^a V^a_+ e^+ - 2iT_{abc}(\mathcal{J}^x\lambda)^a\lambda^b\lambda^c e^+ + \partial p^x_{\bullet} e^+ + ...\right\}$$

$$\mathcal{T}^x_{\bullet +} = \left\{(\mathcal{J}^x\lambda)^a V^a_+ - 2iT_{abc}(\mathcal{J}^x\lambda)^a\lambda^b\lambda^c + 2\partial(\partial_a\Phi(\mathcal{J}^x\lambda)^a)\right\}$$

Thus $G^x(z) = 2\sqrt{2}e^{-i\frac{\pi}{4}}\mathcal{T}^x_{+\bullet}$ is determined.
In a similar way one can calculate $\widetilde{G}^x(\bar{z})$.

Summarizing: when a $(1,1)$ *supersymmetric* σ*-model described by the action (2.27) admits a global* $(4,4)$ *supersymmetry, its classical supercurrents have the following expression in terms of the 3+3 complex structures of* $\mathcal{M}_{\text{target}}$:

$$\begin{aligned} G^0(z) &= \sqrt{2}e^{-i\frac{\pi}{4}}\left\{\lambda^a V^a_z - \frac{2}{3}iT_{abc}\lambda^a\lambda^b\lambda^c + 2\partial\left[\partial_a\Phi\lambda^a\right]\right\} \\ G^x(z) &= \sqrt{2}e^{-i\frac{\pi}{4}}\left\{(\mathcal{J}^x\lambda)^a V^a_z - 2iT_{abc}(\mathcal{J}^x\lambda)^a\lambda^b\lambda^c + 2\partial\left[\partial_a\Phi(\mathcal{J}^x\lambda)^a\right]\right\} \end{aligned} \qquad (2.56a)$$

$$\begin{aligned} \widetilde{G}^0(\bar{z}) &= \sqrt{2}e^{-\frac{3i\pi}{4}}\left\{\mu^a V^a_{\bar{z}} - \frac{2}{3}iT_{abc}\mu^a\mu^b\mu^c + 2\bar{\partial}\left[\partial_a\Phi\mu^a\right]\right\} \\ \widetilde{G}^x(\bar{z}) &= \sqrt{2}e^{-\frac{3i\pi}{4}}\left\{(\overset{+}{\mathcal{J}}{}^x\mu)^a V^a_{\bar{z}} - 2iT_{abc}(\overset{+}{\mathcal{J}}{}^x\mu)^a\mu^b\mu^c + 2\bar{\partial}\left[\partial_a\Phi(\overset{+}{\mathcal{J}}{}^x\mu)^a\right]\right\} \end{aligned} \qquad (2.56b)$$

Chapter 3

Some remarks on c=6, N=4 Theories

We choose our conventions and notations to be that of [15,sec VI], so that the N=4 standard superconformal algebra looks as follows:

$$T(z)T(w) = \frac{c}{2}\frac{1}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + reg.$$

$$T(z)\mathcal{G}^a(w) = \frac{3}{2}\frac{\mathcal{G}^a(w)}{(z-w)^2} + \frac{\partial\mathcal{G}^a(w)}{(z-w)} + reg.$$

$$T(z)\overline{\mathcal{G}}^a(w) = \frac{3}{2}\frac{\overline{\mathcal{G}}^a(w)}{(z-w)^2} + \frac{\partial\overline{\mathcal{G}}^a(w)}{(z-w)} + reg.$$

$$T(z)A^i(w) = \frac{A^i(w)}{(z-w)^2} + \frac{\partial A^i(w)}{(z-w)} + reg.$$

$$A^i(z)\mathcal{G}^a(w) = \frac{1}{2}\frac{\mathcal{G}^b(w)(\sigma^i)^{ba}}{(z-w)} + reg.$$

$$A^i(z)\overline{\mathcal{G}}^a(w) = \frac{-1}{2}\frac{\overline{\mathcal{G}}^b(w)(\sigma^i)^{ab}}{(z-w)} + reg.$$

$$\mathcal{G}^a(z)\overline{\mathcal{G}}^b(w) = \frac{2c}{3}\frac{\delta^{ab}}{(z-w)^3} + \frac{4(\sigma_i^*)^{ab}A^i(w)}{(z-w)^2} + \frac{2\delta^{ab}T(w) + 2\partial A^i(w)(\sigma_i^*)^{ab}}{(z-w)} + reg.$$

$$A^i(z)A^j(w) = \frac{c}{12}\frac{\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}A^k(w)}{(z-w)} + reg. \tag{3.1}$$

where $T(z)$ is the stress-energy tensor , $\mathcal{G}^a(z), \overline{\mathcal{G}}^a(z)$, a=1,2 denote the supercurrents organized in two doublets of the $SU(2)$ generated by the currents $A^i(z)$.

In a (4,4)-theory there is a realization of these operators both in the left and in the right sector. The fields of the theory are organized in representations of $SU(2)_L \otimes SU(2)_R$. We denote by $\Phi\left[\begin{smallmatrix} h,\tilde{h} \\ J,\tilde{J}\end{smallmatrix}\right]^{m,\tilde{m}}$ a primary conformal field with left and right dimensions $h, \tilde{h}$ and isospins $J, \tilde{J}$, and with third components $m, \tilde{m}$.
Consider for example the left sector. The $SU(2)_1$ can be bosonized in terms of a single free boson $\tau(z)$:

$$A^3 = \frac{i}{\sqrt{2}} \partial \tau \qquad ; \qquad A^{\pm} = e^{\pm i \sqrt{2} \tau} \tag{3.2}$$

The spectral flow of the $N = 2$ theories is extended to a "multiplet of spectral flows":

$$\Phi \begin{bmatrix} h \\ J \end{bmatrix}^{m} = e^{im\sqrt{2}\tau} \hat{\Phi}^{(h-m^2)} \tag{3.3}$$

where $\hat{\Phi}^{(h-m^2)}$ is a singlet of $SU(2)$ of conformal weight $h - m^2$.
For example a doublet of $SU(2)$,$\Psi\left[\begin{smallmatrix} 1/2 \\ 1/2 \end{smallmatrix}\right]$, made of an $N = 2$ chiral and an antichiral field of weight $1/2$, (note that the charge respect to the $U(1)$ of the $N = 2$ contained in the N=4 is twice the third component of the isospin) in the NS sector is related by the spectral flow (3.3) to an $SU(2)$ singlet in the R sector:

$$\Psi \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}^{\pm\frac{1}{2}} = e^{\pm i \frac{\tau}{\sqrt{2}}} \Psi \begin{bmatrix} 1/4 \\ 0 \end{bmatrix} \tag{3.4}$$

We use the convention of giving the same name to fields related by spectral flow, distinghuishing them when necessary by their weight and isospin.
As explained in [15], the N=4 analogues of the (c, c) and (c, a) fields of weight $(\frac{1}{2}, \frac{1}{2})$, which play the role of "abstract" (1,1)- and (2,1)-forms in the $(9,9)_{2,2}$ theory, is given by those primary fields of the $(6,6)_{4,4}$ CFT that are of the form

$$\Psi_A \begin{bmatrix} \frac{1}{2}, \frac{1}{2} \\ \frac{1}{2}, \frac{1}{2} \end{bmatrix} \tag{3.5}$$

and correspond to the lowest components in a short representation of the N=4 algebra. In (3.5) the index A runs on $h^{1,1}$ values. Focusing on the left sector a short representation is made of the following set of fields

$$\Psi \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}^{a} (z) \ , \ \Phi \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z) \ , \ \Pi \begin{bmatrix} 1 \\ 0 \end{bmatrix} (z)$$

satisfying the OPEs

$$\begin{aligned}
\mathcal{G}^a(z)\Psi^b(w) &= \frac{\epsilon^{ab}\Phi(w)}{z-w} + reg. \\
\overline{\mathcal{G}}^a(z)\Psi^b(w) &= \frac{\delta^{ab}\Pi(w)}{z-w} + reg. \\
\mathcal{G}^a(z)\Phi(w) &= \overline{\mathcal{G}}^a(z)\Pi(w) = 0 \\
\overline{\mathcal{G}}^a(z)\Phi(w) &= 2\epsilon^{ab}\partial\left(\frac{\Psi^b(w)}{z-w}\right) + reg. \\
\mathcal{G}^a(z)\Pi(w) &= -2\delta^{ab}\partial\left(\frac{\Psi^b(w)}{z-w}\right) + reg.
\end{aligned} \tag{3.6}$$

The fields Φ and Π have dimension 1 and, being the last components of an N=4 representation (see the last two of the OPEs (3.6)), when added (in suitable combinations of the left and right sectors) to the Lagrangian they don't break its N=4 invariance. We call them the "N=4 moduli".
As already hinted, the fields $\Psi_A\left[\begin{smallmatrix}\frac{1}{2},\frac{1}{2}\\ \frac{1}{2},\frac{1}{2}\end{smallmatrix}\right]$ represent the abstract (1,1)-forms on the manifold described by the $(6,6)_{4,4}$ -theory.

As a first example of $(6,6)_{4,4}$ theory let's briefly consider that associated with flat space. The N=4 algebra (as an illustration we consider the left moving sector) is realized by the stress-energy tensor

$$T(z) = -\frac{1}{2}\partial X^\mu \partial X^\mu + \psi^\mu \partial \psi^\mu \tag{3.7a}$$

by the supercurrents

$$\begin{aligned} G^0(z) &= \sqrt{2}\psi^\mu \partial X^\mu \\ G^x(z) &= \sqrt{2}\left(\hat{\mathcal{J}}^x \psi\right)^\mu \partial X^\mu \end{aligned} \tag{3.7b}$$

and by the $SU(2)$ currents

$$A^i(z) = -\frac{i}{2}\psi^\mu \hat{\mathcal{J}}^i_{\mu\nu}\psi^\nu = i(\psi^0\psi^i + \frac{1}{2}\epsilon^{ijk}\psi^j\psi^k) \tag{3.7c}$$

Note that eq. (3.7b) is just the general formula (2.56a) applied to this particular case. Indeed, since the torsion is zero, the two sets of complex structures of Chapter 2 coincide, and the requirements of coveriant costancy reduces to that of constancy; we can thus chose them to be, for example, the $\hat{\mathcal{J}}^x_{\mu\nu}$ of eq. (1.9).
Explicitely, we have:

$$\begin{aligned} G^1 &= \sqrt{2}\left\{\psi^0\partial X^1 - \psi^3\partial X^2 + \psi^2\partial X^3 - \psi^1\partial X^0\right\} \\ G^2 &= \sqrt{2}\left\{\psi^3\partial X^1 + \psi^0\partial X^2 - \psi^1\partial X^3 - \psi^2\partial X^0\right\} \\ G^3 &= \sqrt{2}\left\{-\psi^2\partial X^1 + \psi^1\partial X^2 + \psi^0\partial X^3 - \psi^3\partial X^0\right\} \end{aligned}$$

In the left sector we can find two short representations , given by

$$\begin{aligned} &\Psi_1\begin{bmatrix}1/2\\1/2\end{bmatrix} = \begin{pmatrix}\psi^0 + i\psi^3\\ \psi^2 + i\psi^1\end{pmatrix} \;\; ; \; \Phi_1\begin{bmatrix}1\\0\end{bmatrix} = -\partial X^2 - i\partial X^1 \;\; ; \; \Pi_1\begin{bmatrix}1\\0\end{bmatrix} = -\partial X^0 - i\partial X^3 \\ &\Psi_2\begin{bmatrix}1/2\\1/2\end{bmatrix} = \begin{pmatrix}\psi^2 - i\psi^1\\ -(\psi^0 - i\psi^3)\end{pmatrix} \;\; ; \; \Phi_2\begin{bmatrix}1\\0\end{bmatrix} = \partial X^0 - i\partial X^3 \;\; ; \; \Pi_2\begin{bmatrix}1\\0\end{bmatrix} = -\partial X^2 + i\partial X^1 \end{aligned} \tag{3.8}$$

Two analogous ones exist in the right sector. Multiplying them in all possible ways we obtain four abstract (1,1)-forms $\Psi_{\mathcal{A}}\left[{1/2,1/2 \atop 1/2,1/2}\right]$. For instance we can set:

$$\begin{aligned}
\Psi_1\begin{bmatrix}\frac{1}{2},\frac{1}{2}\\ \frac{1}{2},\frac{1}{2}\end{bmatrix}(z,\bar{z}) &= \Psi_1\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)\,\Psi_1\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\bar{z})\\
\Psi_2\begin{bmatrix}\frac{1}{2},\frac{1}{2}\\ \frac{1}{2},\frac{1}{2}\end{bmatrix}(z,\bar{z}) &= \Psi_1\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)\,\Psi_2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\bar{z})\\
\Psi_3\begin{bmatrix}\frac{1}{2},\frac{1}{2}\\ \frac{1}{2},\frac{1}{2}\end{bmatrix}(z,\bar{z}) &= \Psi_2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)\,\Psi_1\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\bar{z})\\
\Psi_4\begin{bmatrix}\frac{1}{2},\frac{1}{2}\\ \frac{1}{2},\frac{1}{2}\end{bmatrix}(z,\bar{z}) &= \Psi_2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(z)\,\Psi_2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\bar{z})
\end{aligned} \tag{3.9}$$

This number of $N=4$-moduli agrees with the Hodge diamond of flat space *

$$\begin{array}{ccccc} & & 1 & & \\ & 2 & & 2 & \\ 1 & & 4 & & 1 \\ & 2 & & 2 & \\ & & 1 & & \end{array} \tag{3.10}$$

According to (3.10) we have also two holomorphic 1-forms and two antiholomorphic ones. At the level of CFT they are represented by operators of the form $\Psi_{\mathcal{A}}\left[{\frac{1}{2},0 \atop \frac{1}{2},0}\right]$ and $\Psi^\star_{\mathcal{A}^\star}\left[{0,\frac{1}{2} \atop 0,\frac{1}{2}}\right]$, respectively. The index $\mathcal{A}(\mathcal{A}^\star)$ runs on $2{=}h^{1,0}$ ($h^{0,1}$) values. The explicit expression of the two (0,1)-forms can be taken to be

$$\begin{aligned}
\Psi^\star_{1^\star}\begin{bmatrix}0,\frac{1}{2}\\ 0,\frac{1}{2}\end{bmatrix} &= \mathbf{1}\begin{bmatrix}0\\ 0\end{bmatrix}(z)\Psi_1\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\bar{z})\\
\Psi^\star_{2^\star}\begin{bmatrix}0,\frac{1}{2}\\ 0,\frac{1}{2}\end{bmatrix} &= \mathbf{1}\begin{bmatrix}0\\ 0\end{bmatrix}(z)\Psi_2\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix}(\bar{z})
\end{aligned} \tag{3.11}$$

The two (1,0)-forms have an analogous expression with the role of the left and right-moving sectors interchanged.

Another interesting point is the identification of the spin fields with the spectral flows of the identity operator and of the lowest component of the short representations (3.8). This is a very important point because the spin fields appear in the fermion emission vertices. If we are able to recast these vertices in an abstract $(6,6)_{4,4}$ language the extension from flat space to an instanton background is guaranteed. The gravitino emission vertex, for instance, that includes the proper gravitino and dilatino vertices, in flat space has the following expression [15, sec VI]:

$$V^\alpha_\mu\,(\,k,\,z,\,\bar{z}\,) \;=\; e^{\frac{1}{2}\,\phi^{sg}(z)}\,S_\alpha\,(z)\bar{\partial}\tilde{X}(\bar{z})e^{ik\cdot X(z,\bar{z})}\mathbf{1}\begin{pmatrix}3/8 & 0\\ -3/2 & 0\end{pmatrix} \tag{3.12a}$$

* We refer by this to the Hodge diamond of the flat space compactified to a torus.

$$V_{\dot\alpha\mu}\,(k,\,z,\,\bar z\,) \;=\; e^{\frac{1}{2}\,\phi^{sg}(z)}\,S^{\dot\alpha}\,(z)\bar\partial\tilde X(\bar z)e^{ik\cdot X(z,\bar z)}\mathbf{1}\begin{pmatrix}3/8 & 0\\ 3/2 & 0\end{pmatrix} \tag{3.12b}$$

the two formulae referring to the two possible chiralities. The last operator in the above formulæis a spectral flow of the identity in the internal theory. In order to convert these expressions to an abstract N=4 notation we need the interpretation of the operators $S_\alpha\,(z)$ $\bar\partial\tilde X(\bar z)$ $e^{ik\cdot X(z,\bar z)}$ and $S^{\dot\alpha}\,(z)$ $\bar\partial\tilde X(\bar z)$ $e^{ik\cdot X(z,\bar z)}$.
To this effect we note that $\bar\partial X^\mu\,(z,\bar z)$ is expressed by linear combinations of the operators $\tilde\Pi_1\left[{1\atop 0}\right]$, $\tilde\Pi_2\left[{1\atop 0}\right]$, $\tilde\Phi_1\left[{1\atop 0}\right]$ and $\tilde\Phi_2\left[{1\atop 0}\right]$, the right-sector counterparts of those appearing in (3.8). It remains to consider the spin fields. The four free fermions can be bosonized in terms of two free bosons as

$$\begin{aligned}\psi^0 \pm i\psi^3 &= \pm c^\pm\; e^{\pm i\varphi_2}\\ \psi^2 \pm i\psi^1 &= \pm c^\pm\; e^{\pm i\varphi_1}\end{aligned} \tag{3.13}$$

where the signs and the cocycle factor($c^\pm = e^{\mp\pi p^1}$) are arranged to reproduce the anticommutation properties of the fermions. The $SU(2)$ currents of eq.(3.7c) can be reexpressed via eq.(3.13). In particular

$$A^\pm = \pm c^\pm\; e^{\pm i\varphi_2}e^{\mp i\varphi_1}$$

However, we can rephrase all the algebra in terms of the vertex operators $e^{\pm i\varphi_1}, e^{\pm i\varphi_2}$, eliminating the need of preserving anticommutation relations (these operators anticommute with themselves and commute with each other). Then the $SU(2)$ currents are simply given by

$$\begin{aligned}A^3 &= \frac{i}{2}(\partial\varphi_2 - \partial\varphi_1)\\ A^\pm &= e^{\pm i\varphi_2}e^{\mp i\varphi_1}\end{aligned} \tag{3.14}$$

Comparison with the standard bosonized form (3.2) is immediate. We get:

$$\tau = \frac{1}{\sqrt2}(\varphi_2 - \varphi_1)$$

so that the spectral flow of eq.(3.3) is rewritten as

$$\Phi\begin{bmatrix}h\\ J\end{bmatrix}^m = e^{im(\varphi_2-\varphi_1)}\,\hat\Phi^{(h-m^2)} \tag{3.15}$$

The fields Ψ_1, Ψ_2 of eq.(3.9), as doublets with respect to the currents (3.14) are given by

$$\Psi_1 = \begin{pmatrix}e^{i\varphi_2}\\ e^{i\varphi_1}\end{pmatrix}\;;\;\Psi_2 = \begin{pmatrix}e^{-i\varphi_1}\\ e^{-i\varphi_2}\end{pmatrix} \tag{3.16}$$

We can single out the spectral flow and find their Ramond partners:

$$\Psi_1\begin{bmatrix}1/2\\ 1/2\end{bmatrix} = \begin{pmatrix}e^{i\varphi_2}\\ e^{i\varphi_1}\end{pmatrix} = \begin{pmatrix}e^{\frac{i}{2}(\varphi_2-\varphi_1)}\\ e^{-\frac{i}{2}(\varphi_2-\varphi_1)}\end{pmatrix}e^{\frac{i}{2}(\varphi_2+\varphi_1)} = spectral\ flow\cdot\Psi_1\begin{bmatrix}1/4\\ 0\end{bmatrix}$$

$$\Psi_2\begin{bmatrix}1/2\\1/2\end{bmatrix}=\begin{pmatrix}e^{-i\varphi_1}\\e^{-i\varphi_1}\end{pmatrix}=\begin{pmatrix}e^{\frac{i}{2}(\varphi_2-\varphi_1)}\\e^{-\frac{i}{2}(\varphi_2-\varphi_1)}\end{pmatrix}e^{-\frac{i}{2}(\varphi_2+\varphi_1)}=spectral\ flow\cdot\Psi_2\begin{bmatrix}1/4\\0\end{bmatrix}$$

Therefore we see that these R fields are just the two spin fields of positive chirality. Indeed, once the fermions have been bosonized as in eq.(3.13), the spin fields, corresponding to the weights of the $SO(4)$ spinor (s) and antispinor ($\bar{s}$) representations, are expressed as follows

$$\begin{aligned}+\text{chirality }(s\,\text{rep}):\quad & S^1=e^{\frac{i}{2}\varphi_2+\frac{i}{2}\varphi_1}\\ & S^2=e^{-\frac{i}{2}\varphi_2-\frac{i}{2}\varphi_1}\\ -\text{chirality }(\bar{s}\,\text{rep}):\quad & S^{\dot{1}}=e^{\frac{i}{2}\varphi_2-\frac{i}{2}\varphi_1}\\ & S^{\dot{2}}=e^{-\frac{i}{2}\varphi_2+\frac{i}{2}\varphi_1}\end{aligned}\tag{3.17}$$

Finally note that the spin fields of negative chirality form a doublet under the $SU(2)_L$ and are related through spectral flow to the identity operator:

$$\begin{pmatrix}S^{\dot{1}}\\S^{\dot{2}}\end{pmatrix}\begin{bmatrix}1/4\\1/2\end{bmatrix}=\begin{pmatrix}e^{\frac{i}{2}(\varphi_2-\varphi_1)}\\e^{-\frac{i}{2}(\varphi_2-\varphi_1)}\end{pmatrix}=spectral\ flow\cdot\mathbf{1}\begin{bmatrix}0\\0\end{bmatrix}\tag{3.18}$$

Comparing these results with equations (3.12) we see that, in flat space, the 8 gravitino zero-modes of positive chirality are given by the left spectral flow of the abstract (1,1)-forms (3.9), while the 8 zero-modes of negative chirality are given by the left spectral flow of the (0,1)-forms (3.11). In both cases, the right-moving part of the operator is SUSY-transformed to the last multiplet-components. This is in perfect agreement with formula giving the number of gravitino zero-modes in terms of the Hodge numbers of the manifold, reported in Chapter 5, eq.(5.17), and with the Hodge diamond of the flat space (3.10). In the case of the K_3-manifold only the positive chirality zero-modes are present, since the analogues of the (0,1)-forms (3.11) do not exist ($h^{1,0}=0$).

Chapter 4

Heterotic Vacua including Gravitational Instantons

As explained in the Introduction, we want to investigate the possibility of constructing consistent heterotic string vacua where the usual $c = (6,4)$ conformal field-theory (CFT) that represents four dimensional flat space is replaced by some new $c = (6,6)$ theory describing string propagation on a non-trivial four-dimensional geometry. In particular we focus on the $(6,6)_{4,4}$ theories corresponding to gravitational instantons with torsion (GHK manifolds).

The first part of our discussion is somehow heuristic: we use, as a guideline, the analogy of the scheme we propose with the procedure utilized to compactify string-theory on 6-dimensional manifolds of $SU(3)$ holonomy [5]. As it is well-known [7], from the abstract point of view, this operation is represented as the replacement of the $c = (9,6)$ theory, corresponding to six flat dimensions, by a $(9,9)_{2,2}$ conformal theory.

Let us briefly review the process of this compactification, in order to proceed in analogy with it also for the space-time part.
The "initial" situation is that required by critical heterotic string theory [4] in d=10, namely the vacuum is a CFT of central charges $(15,26)$ that can be realized as

$$(15,26) = (15,10) \oplus (0,16) \tag{4.1}$$

The $(15,10)$ theory is generated by 10 left-moving $\oplus$ 10 right-moving world-sheet bosons, together with 10 left-moving fermions: it represents the heterotic σ-model on flat 10-dimensional space. The $(0,16)$ theory is that generated by 32 right-moving fermions describing the gauge group G_{gauge} degrees of freedom, namely those of the Kač-Moody algebra $\hat{G}_{gauge}$. The choice of G_{gauge} is determined by the enforcement of modular invariance and we consider the version of the theory where $G_{gauge} = E_8' \times E_8$. We consider the 10-dimensional space to be split in a 6-dimensional internal submanifold and a 4-dimensional space-time manifold. At the level of conformal field-theories this means:

$$(15,26) = (6,4) \oplus (9,6) \oplus (0,16)$$

The main point of the h-map construction [7] is the possibility of considering heterotic string vacua where the above situation is modified as follows:

$$(15,26) = (6,4) \oplus (9,9) \oplus (0,13) \tag{4.2}$$

This is commonly expressed by saying that six of the heterotic fermions have been "eaten up" by the internal theory (which becomes left-right symmetric); the remaining thirteen generate the current algebra of $E_8' \times SO(10)$.
From the Kaluza-Klein viewpoint, one is considering a 10-dimensional manifold with the following structure:

$$\mathcal{M}_{10} = \mathcal{M}_6 \times \mathcal{M}_4^{\mathrm{flat}} \tag{4.3}$$

The "eating" of six heterotic fermions is due to the 10-dimensional axion Bianchi identity $dH = 0$ which (at 1^{st} order) requires

$$0 = \mathrm{Tr}\, F \wedge F - \mathrm{Tr}\, R_{(6)} \wedge R_{(6)} \tag{4.4}$$

and is solved by embedding the spin connection into the gauge connection. In this way the gauge group is broken to the normalizer of the internal manifold holonomy group $\mathcal{H}ol(\mathcal{M}_6)$. In the particular case of manifolds with $SU(3)$ holonomy (Calabi-Yau manifolds), the residual gauge group is $E_6 \otimes E_8'$, as it follows from the maximal subgroup embedding:

$$E_6 \times SU(3) \longrightarrow E_8 \tag{4.5}$$

Thus Kaluza-Klein analysis shows that the massless fields on $\mathcal{M}_4^{\mathrm{flat}}$ are organized in E_6 representations. From the abstract point of view, the case of $SU(3)$ holonomy corresponds to the particular case of the decomposition (4.2) in which the internal theory has (2,2)-supersymmetries:

$$(15,26) = (6,4) \oplus (9,9)_{2,2} \oplus (0,13)$$

One can show [7] that the $U(1)$ current appearing in the $N = 2$ algebra and the $SO(10)$ currents of the heterotic fermions combine, together with suitable spin fields, to yield the current algebra of E_6, in due agreement with the maximal subgroup embedding:

$$SO(10) \times U(1) \longrightarrow E_6 \tag{4.6}$$

Hence the emission vertices of the 4-dim fields are organized in E_6-representations as it is required by Kaluza-Klein analysis.
The question of consistency of these compactified theories and, in particular, the question of their (1-loop) modular invariance is better addressed by looking at their construction from a different viewpoint. Consider a modular-invariant type II superstring vacuum: for what concernes central charges we have:

$$(15,15) = (6,6) \oplus (9,9) \tag{4.7}$$

the $(6,6)$-theory corresponding to flat 4-dim space and the $(9,9)$-theory describing some non-trivial "internal" manifold. One shows that the "h-mapped" heterotic vacuum, obtained by replacing, in the partition function of (4.7), the subpartition function of the two

right-moving transverse fermions with that of 2+24 fermions (generating a $E'_8 \times SO(2+8) = E'_8 \times SO(10)$ current algebra) is also modular invariant.
When the internal theory has $N = 2$ supersymmetry, the fundamental implication of modular invariance is the projection onto odd-integer charge states with respect to the diagonal $U(1)$ group obtained by summing the $U(1)$ of the $N = 2$ algebra with the SO(2) generated by the transverse space-time fermions. This is just the rephrasing in the present context of the GSO projection [7,15].

Let's now consider an extension of the above described mechanism. We start from the conformal field-theory describing the heterotic string compactified on a Calabi-Yau manifold,

$$(15,26) = (6,4) \oplus (9,9)_{2,2} \oplus (0,13)$$

and we let the four-dimensional theory eat four of the heterotic fermions, so that

$$(15,26) = (6,6) \oplus (9,9)_{2,2} \oplus (0,11) \tag{4.8}$$

The remaining heterotic fermions generate a current algebra $E'_8 \times SO(6)$.
From the geometrical σ-model point of view, what we have done is to consider a target space of the form

$$\mathcal{M}_{10} = \mathcal{M}_6 \times \mathcal{M}_4$$

where $\mathcal{M}_6$ is still a manifold of $SU(3)$ holonomy but $\mathcal{M}_4$ is no longer flat space. Condition (4.4) extends to

$$0 = \mathrm{Tr}\, F \wedge F - \mathrm{Tr}\, R_{(6)} \wedge R_{(6)} - \mathrm{Tr}\, R_{(4)} \wedge R_{(4)}$$

which can be solved by embbeding also the holonomy group $\mathcal{H}ol(\mathcal{M}_4)$ into the gauge group. In particular consider the case where $\mathcal{H}ol(\mathcal{M}_4) \subset SU(2)$: this happens for gravitational instantons, whose curvature is either self-dual or antiself-dual. In this situation the gauge group is broken to $SU(6)$, as it follows from the maximal subgroup embedding:

$$SU(6) \times SU(3) \times SU(2) \longrightarrow E_8 \tag{4.9}$$

From the abstract viewpoint, this is reproduced if the $c = (6,6)$ theory possesses a (4,4) supersymmetry:

$$(15,26) = (6,6)_{4,4} \oplus (9,9)_{2,2} \oplus (0,11) \tag{4.10}$$

Indeed the $U(1)$ current of the $N = 2$ algebra associated with $\mathcal{M}_6$, the $SU(2)$ currents of the $N = 4$ algebra associated with $\mathcal{M}_4$ and the $SO(6)$ currents of the heterotic fermions combine together with suitable spin fields to yield the $SU(6)$-current algebra, according to the maximal embedding

$$U(1) \times SU(2) \times SO(6) \longrightarrow SU(6) \tag{4.11}$$

Thus, on this background, the emission vertices for particle-modes (both massive and massless) are organized in $SU(6)$-representations, as it is requested by Kaluza-Klein analysis.

4.1 The issue of Modular Invariance

As we already recalled, in the case of compactifications on Calabi-Yau manifolds, that is by means of a $(9,9)_{2,2}$ theory, one starts from a type II modular invariant partition function, corresponding to a CFT:

$$(15,15) = (6,6) \oplus (9,9)$$

The $(6,6)$ part, corresponding to 4-dimensional flat space, contains the world-sheet bosons $X^\mu, \tilde{X}^\mu$ and the fermions $\psi^\mu, \tilde{\psi}^\mu$ and the complete partition function has the structure:

$$Z_{tot} = \sum_{i,\bar{\imath}} Z^{(9,9)}_{i,\bar{\imath}} Z(X^\mu, \tilde{X}^\mu) B^{(4)}_i \left(B^{(4)}_{\bar{\imath}}\right)^* B^{(-2)}_i \left(B^{(-2)}_{\bar{\imath}}\right)^* \tag{4.12}$$

where

i) $Z(X^\mu, \tilde{X}^\mu)$ is the usual partition function for the four free bosons,

ii) $B^{(4)}_i$ are the $SO(4)$-characters in which we can organize the partition function for the four free fermions $\psi^\mu, \tilde{\psi}^\mu$ (the index i taking the values $0, v, s, \bar{s}$, for the scalar, vector and spinor conjugacy class, respectively),

iii) $B^{(-2)}_i$ are the partition functions for the superghosts,

iv) $Z^{(9,9)}_{i,\bar{\imath}}$ is the partition function for the internal theory which couples to reps $(i,\bar{\imath})$ of the space-time $SO(4)$ and of the superghosts.

The reason why we have denoted as $B^{(-2)}_i$ the superghost partition function becomes clear from the following considerations. If the superghosts have boundary conditions $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$, ($\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right] = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$) their partition function can be computed to be [17]:

$$Z^{sg}\begin{bmatrix}a\\b\end{bmatrix}(\tau|z) = \frac{\eta(\tau)}{\theta\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau|z)} = \frac{1}{Z\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right](\tau|z)} \tag{4.13}$$

which is exactly the reciprocal of the partition function for two free fermions with spin-structure $\left[\begin{smallmatrix}a\\b\end{smallmatrix}\right]$ *

Since the superghosts are forced by world-sheet supersymmetry to have the same spin-structure as the space-time fermions, dealing with the theory described by eq.(4.12) one can use the cancellation of the superghost partition function with the partition function for two fermions. Instead of (4.12) one can simply write

$$Z_{tot} = \sum_{i,\bar{\imath}} Z^{(9,9)}_{i,\bar{\imath}} Z(X^\mu, \tilde{X}^\mu) B^{(2)}_i \left(B^{(2)}_{\bar{\imath}}\right)^* \tag{4.14}$$

* This reciprocity holds only at genus g=1. For higher genera it is amended by a phase factor that amounts to a correct assignment of spin statistics [17]. In all known constructions if one fixes 1-loop modular invariance plus spin statistics, higher loop modular invariance is also ensured. We assume that this will go through also in our construction.

that is, one considers only the transverse fermions.
The h-map construction of the associated heterotic theory is based on an isomorphism between the $SO(2n)$-characters and those of $SO(2n+24)$ or $E_8' \times SO(2n+8)$. It works as follows. The action of the modular transformations S and T on the characters of $SO(2n)$ in the basis labeled by $0, v, s, \bar{s}$ is given by

$$
\begin{aligned}
B_i^{(2n)} &\xrightarrow{T} T_{ij}^{(2n)} B_j^{(2n)} \\
B_i^{(2n)} &\xrightarrow{S} S_{ij}^{(2n)} B_j^{(2n)}
\end{aligned}
\tag{4.15}
$$

where

$$
\begin{aligned}
T^{(2n)} &= \mathrm{diag}(1, 1, e^{in\frac{\pi}{4}}, e^{in\frac{\pi}{4}}) e^{-in\frac{\pi}{12}} \\
S^{(2n)} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & e^{i\frac{n\pi}{12}} & -e^{i\frac{n\pi}{12}} \\ 1 & -1 & -e^{i\frac{n\pi}{12}} & e^{i\frac{n\pi}{12}} \end{pmatrix}
\end{aligned}
\tag{4.16}
$$

The isomorphism is realized by

$$
\begin{aligned}
T^{(2n)} &= M\, T^{(2n+24)}\, M \\
S^{(2n)} &= M\, S^{(2n+24)}\, M
\end{aligned}
\tag{4.17}
$$

where the idempotent matrix M is given by:

$$
M = \begin{pmatrix} \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix} & \mathbf{0} \\ \mathbf{0} & \begin{matrix} -1 & 0 \\ 0 & -1 \end{matrix} \end{pmatrix}
$$

It interchanges the scalar and the vector characters, besides flipping the sign of the spinor and antispinor characters. The characters of $E_8' \times SO(2m)$ transform as those of $SO(2m+16)$, so that the isomorphism permits also to reach these groups.
Due to (4.17), if one replaces the two right-moving transverse $\tilde{\psi}^\mu$ fermions with 26 heterotic fermions that generate the gauge group $E_8' \times SO(10)$, by taking proper account of the matrix M, the resulting theory has a modular invariant partition function.

In eq.(4.12) we made no explicite use of the cancellation between superghosts and longitudinal fermions for the following reason. We wanted to enphasize the possibility of constructing, out of the $Z^{sg}\left[{a \atop b}\right]$ and by means of combinations analogous to those used for the free fermions, new characters labeled by an index $i = 0, v, s, \bar{s}$, whose modular transformations are very similar to those in eq.(4.16).
Indeed, in analogy with the characters of $2n$ fermions let the superghost characters be:

$$
\begin{aligned}
B_0^{(-2)} &= \frac{1}{Z\left[{0 \atop 0}\right]} + \frac{1}{Z\left[{0 \atop 1}\right]} \; ; \; B_v^{(-2)} = \frac{1}{Z\left[{0 \atop 0}\right]} - \frac{1}{Z\left[{0 \atop 1}\right]} \\
B_s^{(-2)} &= \frac{1}{Z\left[{1 \atop 0}\right]} + \frac{1}{Z\left[{1 \atop 1}\right]} \; ; \; B_{\bar{s}}^{(-2)} = \frac{1}{Z\left[{1 \atop 0}\right]} - \frac{1}{Z\left[{1 \atop 1}\right]}
\end{aligned}
\tag{4.18}
$$

Eq.(4.18) is obtained from the definition of the $B_i^{(2n)}$ characters [7,15] by the replacement $(Z\left[{a \atop b}\right])^n \longrightarrow 1/Z\left[{a \atop b}\right]$. Using the modular transformations of the $Z\left[{a \atop b}\right](\tau)$, already utilized to obtain eq.(4.16) , we find :

$$
\begin{aligned}
B_i^{(-2)} &\xrightarrow{T} T_{ij}^{(-2)} B_j^{(-2)} \\
B_i^{(-2)} &\xrightarrow{S} S_{ij}^{(-2)} B_j^{(-2)}
\end{aligned}
\tag{4.19}
$$

where

$$
\begin{aligned}
T^{(-2)} &= \mathrm{diag}(1,1,e^{-i\frac{\pi}{4}},e^{-i\frac{\pi}{4}})e^{i\frac{\pi}{12}} \\
S^{(-2)} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & e^{-i\frac{\pi}{12}} & -e^{-i\frac{\pi}{12}} \\ 1 & -1 & -e^{-i\frac{\pi}{12}} & e^{-i\frac{\pi}{12}} \end{pmatrix}
\end{aligned}
\tag{4.20}
$$

Formally, these matrices are obtained from those in eq.(4.16) by setting $2n = -2$, which explains the chosen notation. Moreover it is manifest that we can use the h-map isomorphism to substitute the characters of the superghosts with those of 22 heterotic fermions with gauge group $E_8' \times SO(-2+8) = E_8' \times SO(6)$.

Consider now a modular invariant type II vacuum in which the $c = (6,6)$ part represents a four-dimensional space with non trivial geometry. The partition function of such a theory is

$$
Z_{tot} = \sum_{i,\bar{\imath}} Z_{i,\bar{\imath}}^{(9,9)} Z_{i,\bar{\imath}}^{(6,6)} B_i^{(-2)} \left(B_{\bar{\imath}}^{(-2)}\right)^* \tag{4.21a}
$$

$Z_{i,\bar{\imath}}^{(6,6)}$ being the partition function for the $(6,6)$ theory which couples to the characters $(i,\bar{\imath})$ of the superghosts.

Although the $SO(4)$ characters have disappeared from the game, we can still perform the h-map construction of an associated modular invariant heterotic theory. The result is just of the form (4.8). If, in addition we choose a space-time with $SU(2)$ holonomy, the result is of the form (4.10). After h-map the partition function (4.21a) becomes

$$
Z_{tot} = \sum_{i,\bar{\imath}} Z_{i,\bar{\imath}}^{(9,9)} Z_{i,\bar{\imath}}^{(6,6)} B_i^{(-2)} \left(B_{\bar{\imath}}^{(E_8' \times SO(6))}\right)^* \tag{4.21b}
$$

Chapter 5

The construction of the Vertex Operators

In this Chapter we proceed in the analysis of the heterotic theory (4.21b) by examinating the subject of the corresponding vertex operators; in particular we construct explicitely those for zero-modes emission. The importance of such a step has already been stressed in the Introduction.

In the case where $\mathcal{M}_4$ is flat space, the emission vertex for any particle-field has the general form:

$$V_\epsilon^{\cdot\cdot,\bullet}\,(k,\,z,\,\bar{z}) \;=\; \Phi_\epsilon(z,\bar{z})\; e^{ik\cdot X(z,\bar{z})}\;\Psi^{\cdot\cdot}(z,\bar{z})\;\Lambda^\bullet(z,\bar{z})$$

where $\Phi_\epsilon(z,\bar{z})\; e^{ik\cdot X(z,\bar{z})}$, $\Psi^{\cdot\cdot}(z,\bar{z})$ and $\Lambda^\bullet(z,\bar{z})$ are conformal fields respectively belonging to the theories $(6,4)$, $(9,9)_{(2,2)}$ and $(0,13)$. The last two factors determine the internal quantum numbers $(\cdot\cdot,\bullet)$ of the particle one considers. The first factor, instead, determines its space-time character, namely its spin, its polarization ϵ, and its momentum k. The compound $\Phi_\epsilon(z,\bar{z})\; e^{ik\cdot X(z,\bar{z})}$ is the conformal field-theory corresponding of a pure-state wave-function $\psi_{k,\epsilon}(X)$ satisfying the wave equation:

$$\Delta\psi_{k,\epsilon}(X) \;=\; m^2\;\psi_{k,\epsilon}(X) \tag{5.1}$$

where Δ is the relevant wave-operator (Dirac, Rarita-Schwinger, Einstein, Yang-Mills,...) and $m^2 = k^2$ is the squared mass. The reason why polarization and momentum are utilized to label this part of the vertex operator is that they are good quantum numbers in flat space. Indeed on a flat background a complete set of solutions of (5.1) can always be expressed in terms of plane waves. Massless-particles have $k^2 = 0$ and are the zero-modes of the wave operator Δ . When we deal with some non trivial space-time geometry, the eigenfunctions of the operators Δ are no longer plane-waves and their spectrum is labeled by a new set of quantum numbers replacing the momentum k and the polarization

ϵ. Correspondingly the compound $\Phi_\epsilon(z,\bar z)\ e^{[i\,k\,\cdot\,X(z,\bar z)]}$ is replaced by suitable operators $\Theta_i(z,\bar z)$ of the $(6,6)_{4,4}$ theory. A finite number of these operators correspond to the zero modes of Δ and can be used to calculate the scattering of massless particles in the non-trivial background under consideration.

Another important remarque concerns the moduli: the non-trivial space-time one considers usually admits continuous deformations that preserve both its topology and its holonomy. The parameters of these deformations are named moduli and, from the CFT viewpoint, they correspond to suitable marginal operators one can add to the 2-dimensional lagrangian preserving its (4,4)-supersymmetry. Exactly as in Calabi-Yau compactifications, the spectrum of zero modes for the various operators Δ depends on the number of these deformations: furthermore the corresponding vertex must be thought as a function of the moduli.

In the construction of the relevant vertices we proceed in analogy with what one does for the internal dimensions. We relate the counting and the group-theoretical indexing of the possible conformal operators that possess the correct dimensions and charges to the counting of zero-modes for the fields appearing in the low-energy effective supergravity, when this latter is expanded around the particular background, abstractly described by the CFT under investigation. The procedure is like a Kaluza-Klein compactification to zero dimensions.

On the other hand, in order to gain a more intuitive comprehension of the role of the operators appearing in the $(6,6)_{4,4}$ theory, it is instructive to compare the vertices with those of flat space. To this purpose it is useful to recall that flat four-dimensional space possesses an $N{=}4$ world-sheet supersymmetry (see Chapter 3). Hence we can recast the operators appearing in the vertices in a form suitable of generalization to any $(6,6)_{4,4}$.
In what follows we will proceed in a general way. However, to give a concrete example of how the counting goes, we will also specify the formulæto the $K3$ manifold, the unique non-trivial Calabi-Yau manifold in four dimensions. Under some respects, $K3$ may appear as a good candidate to insert in our scheme for treating non-trivial four-dimensional geometries. Compactification on $K3$-surfaces has been extensively studied in the past [18] and it is known to be represented by a $(6,6)_{4,4}$ theory, which in some points of moduli space is solvable, being given by a tensor product of $N=2$ minimal models. The knowledge of $K3$ cohomology, described by the Hodge diamond

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \tag{5.2}$$

makes the counting of the zero-modes explicit yielding non-trivial results that can be compared with the CFT counting of vertices.

On the contrary, for physical reasons, $K3$ is not the most appealing possibility. It is a gravitational instanton, but it is compact. As already stressed, our goal is to extend the same techniques to four-dimensional instantons of the effective lagrangian that are asymptotically flat (this last feature seems to be realizable only with torsion [9]).

5.1 Geometrical analysis of the zero-modes

In order to analyse the vertex-operators for the zero-modes we need the field content of the effective four-dimensional theory, which is a matter-coupled $D=4,N=1$ supergravity arising from compactification on the internal Calabi-Yau manifold. This field content is described in Chapter 6.
We begin with the E_6 charged fields given by the gauge multiplet (gauge bosons and gauginos, transforming in the 78 representation), by $h^{2,1}$ WZ multiplets transforming in the 27 and $h^{1,1}$ transforming in the $\overline{27}$-representation (these Hodge numbers being those of the compactified CY space). We consider the zero-modes of these fields in the classical background provided by a non-trivial four-dimensional manifold (and we consider $K3$ as an illustration). We refer in the following to the Hodge diamond of the manifold in question, with the following caveats: the manifold can also be non-compact (and actually this is the case we are interested in), and in this case we refer to the diamond of its compactified version, which gives the correct local counting, i.e. the dimensionality of the space of zero-modes, these latters being no longer normalizable; if the manifold has torsion, we refer to the analogous diamond for the relevant fiber bundle whose connections are the $\omega_R \pm T$ and not simply the Riemannian one ω_R as in the zero-torsion case.
As already enphasized, we embed the space-time spin connection into the gauge connection, breaking the gauge group as follows:

$$E_6 \longrightarrow SU(6) \times SU(2) \tag{5.3}$$

To investigate the zero-modes we must take into account the branching of the representations of E_6 under (5.3).
The adjoint representation is decomposed as

$$78 = (35,1) + (1,3) + (20,2) \tag{5.4}$$

Consider the gaugino field. Its index in the adjoint of E_6 is split accordingly to eq.(5.4); it also has a spinorial index on the manifold. Thus the possible cases are :

- $\lambda_\alpha^{A,}$ A being an index in the adjoint (35) of $SU(6)$, α being the spinorial index. The zero-modes are in correspondence with the Dolbeaut cohomology $H^{0,q}$. Since the chirality is determined by $(-1)^q$, there are 2 zero modes with (-) chirality and $h^{0,1}$ with (+) chirality. For $K3$, looking at the Hodge diamond (5.2) we see that there are just two zero-modes both with the same chirality (-).

- $\lambda_\alpha^{,X}$ X in the adjoint of the $SU(2)$ holonomy group of the manifold. The zero-modes should be related to the cohomology groups $H^{0,q}(EndT)$ of Endomorphism-(of the tangent bundle)-valued antiholomorphic forms. For example, by the explicit realization of $K3$ as an algebraic surface one can evaluate the dimension of this cohomology group, case by case.

- $\lambda_\alpha^{a,x}$ a belongs to the 20 of E_6; x in the 2 of $SU(2)$ is the same as a contravariant holomorphic index which can be lowered by means of the holomorphic $(2,0)$ form. Because of the spinorial index, the zero-modes correspond to $(1,q)$ harmonic forms. We can therefore have just $h^{1,1}$ zero-modes (20 for $K3$) with chirality (+).

Consider then the gauge bosons. According to the decomposition (5.4) we have:

• $A_\mu^{A,}$ μ can be a holomorphic or antiholomorphic index. There can thus be $2h^{0,1}$ zero-modes. In the $K3$ case since, due to the vanishing of $h^{1,0}$ and $h^{0,1}$, the holomorphic $SU(6)$ bundle is trivial, there is no zero-mode of this kind.

• $A_\mu^{,X}$ Zero-modes are related to the Dolbeaut cohomology $H^1(EndT)$.

• $A_\mu^{a,x}$ Again, x behaves as a holomorphic index that can be lowered by the holomorphic $(2,0)$ form or by the metric according to the necessity to obtain again an antisymmetric form. Then the zero-modes can be set in corrispondence with $(1,1)$ forms, for both the type of μ. We have thus 2 $h^{1,1}$ zero-modes of this kind (40 for $K3$).

The 27 of E_6 is decomposed as

$$27 = (15,1) + (6,2) \tag{5.5a}$$

Consider the fermion field belonging to any of the WZ multiplets that transform in the 27 representation (these are the charged fields paired to the complex structure deformations of the Calabi-Yau manifold). The decomposition (5.5a) gives rise to the following cases:

• $\chi_\alpha^{A,}$ A belonging to the 15 of $SU(6)$, α the spinorial index. Zero-modes correspond to the Dolbeaut cohomology $H^{0,q}$ so there are 2 zero-modes with chirality (-) and $h^{0,1}$ with chirality (+); just the 2 with chirality (-) are present on $K3$.

• $\chi_\alpha^{a,x}$ a is in the 6 of $SU(6)$; x in the 2 of $SU(2)$ is like an (anti-)holomorphic index; once lowered by the (2,0) (or the (0,2))form the zero-modes are put into correspondence with $H^{1,q}$ (or $H^{q,1}$ so that there are $h^{1,1} = 20$ modes ($H^{1,1}$ has to be counted only once)with chirality (+) and $2h^{0,1}$ with chirality (-). This means 20 of chirality (+) on $K3$.

The possibilities for the scalars of these 27 families are:

• $\varphi^{A,}$ for which there is just $h^{0,0} = 1$ zero mode.

• $\varphi^{a,x}$ Lowering the index, the correspondence is with $H^{0,1}$ and so $h^{0,1}$ zero-modes exist (none for $K3$).

The $\overline{27}$ of E_6 decomposes as

$$\overline{27} = (\overline{15},1) + (\bar{6},\bar{2}) \tag{2.26b}$$

For the $\overline{27}$-spinors we have, analogously to the 27-ones, two zero-modes with (-) chirality and $h^{0,1}$ with (+) chirality in the $\overline{15}$ and $h^{1,1}$ with (+) chirality and $2h^{0,1}$ of (-) chirality in the $\bar{6}$; for the $\overline{27}$-scalars, one mode in the $\overline{15}$ and $h^{0,1}$ in the $\bar{6}$.

Consider now the fields of the gravitational multiplet

To look for the zero-modes of the graviton field, i.e. of the metric, means to look for the solutions of the Lichnerowitz equation; for example on $K3$ these are well known to be 58. This number is of course determined by the cohomology of $K3$, and to this purpose a discussion is necessary, about the separated counting of the metric and torsion zero-modes. It goes as follows (we refer to $K3$ but really the formulæare valid under the only assumption that $h^{0,2}=0$). From the K_3 Hodge diamond (5.2) we know that $h^{2,0}=1$ and $h^{1,1}=20$. Let Ω_{ij} be the (2,0)-holomorphic form and let $g_{ij^\star}$ be the fiducial Ricci flat Kähler metric $(i,j=1,2)$ that, for each Kähler class, is guaranteed to exist by the Calabi-Yau condition $c_1(K_3)=0$. Furthermore let $U^{(\alpha)}_{ij^\star}$ be a basis for the (1,1)-forms $(\alpha=0,1.....,19)$. A variation of the reference metric which keeps it Ricci-flat is given by:

$$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \delta g_{\mu\nu} \quad ; \quad \delta g_{\mu\nu} = \begin{cases} \delta g_{ij} \\ \delta g_{ij^\star} \\ \delta g_{i^\star j^\star} \end{cases} \tag{5.6}$$

where δg_{ij}, $\delta g_{ij^\star}$ and $\delta g_{i^\star j^\star}$ are harmonic tensors of the type specified by their indeces. Hence we can immediately write:

$$\delta g_{ij^\star} = c_\alpha\, U^{(\alpha)}_{ij^\star} \tag{5.7}$$

where c_α are 20 real coefficients. They parametrize the deformations of the Kähler class. On the other hand, using the holomorphic 2-form, any harmonic tensor with two antiholomorphic indeces $t_{i^\star j^\star}$ can be written as the following linear combination:

$$t_{i^\star j^\star} = -d^\star_\alpha \frac{1}{||\Omega||^2}\, \overline{\Omega}^k_{i^\star}\, U^\alpha_{kj^\star} \tag{5.8}$$

where raising and lowering of the indeces is performed by means of the fiducial metric and where $d^\star_\alpha$ are constant complex coefficients. Since $h^{2,0}=1$ it follows that, of the 20 independent linear combinations appearing in (5.8), only one leads to an antisymmetric $t_{i^\star j^\star}$; all the other combinations produce a symmetric tensor $t_{i^\star j^\star}$. Hence we can choose a basis of the (1,1)-harmonic forms such that:

$$\overline{\Omega}_{i^\star j^\star} = -\frac{1}{||\Omega||^2}\, \overline{\Omega}^k_{i^\star}\, U^0_{kj^\star} \tag{5.9a}$$

$$-\frac{1}{||\Omega||^2}\, \overline{\Omega}^k_{i^\star}\, U^a_{kj^\star} = S^a_{i^\star j^\star} = S^a_{j^\star i^\star} \qquad (a=1,.....,19) \tag{5.9b}$$

The 19 symmetric tensors $S^a_{i^\star j^\star}$ provide a basis for the expansion of the antiholomorphic part of the metric deformation

$$\delta g_{i^\star j^\star} = d_a\, S^a_{i^\star j^\star} \tag{5.10}$$

The holomorphic part just is the complex conjugate and it is expanded along the complex conjugate basis S^a_{ij} : $\delta g_{ij} = d_a\, S^a_{ij}$. The 19 complex coefficients d_a parametrize the complex structure deformations of the K_3 manifold. Summarizing the 58 zero-modes of the metric emerge from the following counting:

$$\# \text{ metric zero} - \text{modes} = h^{1,1} + 2 \left(h^{1,1} - 1 \right) \quad (5.11)$$

This formula is just a consequence of $h^{2,0} = 1$ and it has a meaning also for non-compact manifolds, like the instanton we consider later in this thesis, as a counting of local deformations. For global deformations one has still to check if they can be reabsorbed by diffeomorphisms.
In string-theory, the metric is not the only background field. We have also the antisymmetric axion $B_{\mu\nu}$, whose curl $H_{\lambda\mu\nu}$ is identified with the torsion $T_{\lambda\mu\nu}$, as we are going to see while discussing the σ-model formulation (see section 4). The zero-modes of the field $B_{\mu\nu}$ are counted in a similar way to the case of the metric. From the linearized field equation around the reference background, one concludes that δB_{ij}, $\delta B_{i^\star j^\star}$ and $\delta B_{ij^\star}$ must be harmonic tensors. Because of the different symmetry of the indices, this time we have:

$$\delta B_{ij} = A\, \Omega_{ij} \quad (5.12a)$$

$$\delta B_{ij^\star} = b_\alpha\, U^\alpha_{ij^\star} \quad (5.12b)$$

where A is a complex parameter and b_α are real parameters. Hence we have 22 axion zero-modes that emerge from the following counting:

$$\# \text{ axion zero} - \text{modes} = h^{1,1} + 2 \quad (5.13)$$

Altogether there are $3h^{1,1} - 2 \oplus h^{1,1} + 2 = 4h^{1,1}$ zero modes of the field $g_{\mu\nu} + iB_{\mu\nu}$. In the next subsection we see that this counting agrees with the counting of N=4 preserving marginal operators in a $(6,6)_{4,4}$-theory.

The gravitino zero-modes are the zero-modes of the Rarita-Schwinger operator. Utilizing the standard trick of writing spinors as differential forms we can relate the number of these modes to the dimensions of the cohomology groups. Let

$$\{\Gamma_i, \Gamma_j\} = 0 \quad ; \quad \{\Gamma_{i^\star}, \Gamma_{j^\star}\} = 0$$

$$\{\Gamma_i, \Gamma_{j^\star}\} = 2\, g_{ij^\star} \quad (5.14)$$

be the Clifford algebra written in a well-adapted basis. A spin $\frac{3}{2}$ field ψ_μ can be written as follows:

$$\psi_i = (\omega_i \mathbf{1} + \omega_{ij^\star}\Gamma^{j^\star} + \omega_{ij^\star k^\star}\, \Gamma^{j^\star k^\star}) | \zeta >$$

$$\psi_{i^\star} = (\omega_{i\star} \mathbf{1} + \omega_{i^\star j^\star}\Gamma^{j^\star} + \omega_{i^\star j^\star k^\star}\, \Gamma^{j^\star k^\star}) | \zeta > \quad (5.15)$$

where the spinor $| \zeta >$ satisfies the condition:

$$\Gamma_{i^\star} \,| \zeta > = \Gamma^i \,| \zeta > = 0 \quad (5.16)$$

The field ψ_μ is a zero mode if the coefficients $\omega_{....}$ in (5.15) are harmonic tensors. Hence from ω_i and $\omega_{i^\star}$ we get $h^{1,0}$ and $h^{0,1}$ zero-modes respectively. From $\omega_{ij^\star}$ and $\omega_{i^\star j^\star}$ we obtain $h^{1,1}$ + $h^{1,1}$ zero-modes. Finally 2 $h^{1,2}$ zero-modes arise from $\omega_{ij^\star k^\star}$ and $\omega_{i^\star j^\star k^\star}$. In view of the symmetries of the Hodge diamond the total number of zero-modes for the gravitino field is given by the formula

$$\#\,\mathrm{gravitino\ zero-modes} \;=\; 2\,h^{1,1} \;+\; 4\,h^{1,0} \tag{5.17}$$

In the case of K_3 the above number is 40.

Finally, for the E_6 neutral WZ multiplets, the fermion has two zero-modes of the same chirality (in correspondence with $H^{0,q}$), while the scalar has just the (trivial) zero-mode corresponding to $H^{0,0}$.

5.2 Construction of zero-modes Emission Vertices

For each field appearing in the effective four-dimensional theory it is possible to write the expressions of the vertices for the emission of its particle zero-modes on a certain four-dimensional background in terms of the conformal operators of the $(6,6)_{4,4}$ that abstractly corresponds to that background, and of the operators of the internal and heterotic fermions theories of course. We give a counting of the vertices for each E_6-charged or neutral field of a certain kind: for example we describe and count the various vertices arising from a set of fermions transforming in the 27 of E_6; it is to be recalled however that actually we have in the game $h^{2,1}$ (we mean here the number of (2,1) forms of the internal manifold) such 27's. Similarly in all other cases. The counting of the vertices reproduces completely the counting "à la Kaluza-Klein " of the zero-modes for the various fields on the corresponding background, topologically described by an Hodge diamond identical to the abstract Hodge diamond of the $(6,6)_{4,4}$, that has just been described.
In some cases the flat-4 dim-space expression is recalled to make easily understood its generalization to any $(6,6)_{4,4}$. We express the vertices in the so-called canonical picture.

• *Gravitational multiplet:*

Graviton

In the flat space, the graviton vertex is given by

$$e^{i\phi_{sg}(z)}\psi^\mu(z)\bar\partial\tilde X^\nu(\bar z)e^{ik\cdot X(z,\bar z)}\mathbf{1}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}(z,\bar z)$$

where, following the notations of ref [15] ϕ_{sg} is the field bosonizing the superghost system in the left sector, and $\mathcal{O}\begin{pmatrix}h & \tilde h\\ q & \tilde q\end{pmatrix}$ denotes an operator of the internal $(9,9)_{2,2}$ theory of dimensions $h,\tilde h$ and charges $q,\tilde q$ with respect to the $U(1)$ contained in the N=2 algebras of left and right sectors.
Note that left and right dimension of the operators are such that $e^{ik\cdot X}$ must have weight zero, i.e. $k^2 = 0$ and the vertex is really massless: indeed $\Delta(e^{i\phi_{sg}(z)}) + \Delta(\psi^\mu) = \frac{1}{2} + \frac{1}{2} = 1$ in the left sector and $\tilde\Delta(\bar\partial\tilde X^\mu) = 1$ in the right one. As recalled briefly in Chapter 3 the left

and right charges $Q, \tilde{Q}$ with respect to the total $U(1)$ must be respectively odd and even integers. This is the case for the above expression as $q(\psi^\mu) = 1$ (this is the value assigned to the vector representation of the space-time $SO(4)$), and no operator is charged in the right sector.

The graviton vertex in the generic case is given by

$$e^{i\phi_{sg}(z)}\Phi_{\mathcal{A}} \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$e^{i\phi_{sg}(z)}\Pi_{\mathcal{A}} \begin{bmatrix} 1/2 & 1 \\ 1/2 & 0 \end{bmatrix}^a \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$4h^{1,1}$ zero modes

The counting is explained as $\Phi_{\mathcal{A}}, \Pi_{\mathcal{A}}$ are the fields obtained by the action of the right moving supercurrents (see the OPEs ...) on the abstract (1,1)-forms $\Psi_{\mathcal{A}}\left[\begin{smallmatrix}1/2, 1/2 \\ 1/2, 1/2\end{smallmatrix}\right]$ $A = 1, ..., h^{1,1}$. A factor of 2 comes from the $SU(2)$ doublet in the left sector.

Gravitino

The flat space vertices for the gravitino massless mode emission of the two chiralities, already reported in Chapter 3, look like

$$V_\mu^\alpha\,(k,\,z,\,\bar{z}) \;=\; e^{\frac{1}{2}\,\phi^{sg}(z)}\,S_\alpha\,(z)\bar{\partial}\tilde{X}(\bar{z})e^{ik\cdot X(z,\bar{z})}\mathbf{1}\begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix}$$

$$V_{\dot{\alpha}\mu}\,(k,\,z,\,\bar{z}) \;=\; e^{\frac{1}{2}\,\phi^{sg}(z)}\,S^{\dot{\alpha}}\,(z)\bar{\partial}\tilde{X}(\bar{z})e^{ik\cdot X(z,\bar{z})}\mathbf{1}\begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix}$$

Note that again left and right dimensions are correct: $\Delta(e^{1/2\phi_{sg}}) + \Delta(S_{\alpha,\dot{\alpha}}) + 3/8 = 3/8 + 1/4 + 3/8 = 1$ and $\tilde{\Delta}(\bar{\partial}\tilde{X}^\mu) = 1$. For the left charge, note that to the spinor representation of $SO(4)$ is assigned th value 1/2, and -1/2 to the antispinor, so that the total charge is for the two chiralities $1/2 - 3/2 = -1$ or $-1/2 + 3/2 = 1$, both odd integers. The abstract $(6,6)_{\frac{1}{4},\frac{1}{4}}$ correspondent of the spin fields of the two chiralities having been investigated in Chapter 3, it is almost immediate to write the generic form of the gravitino vertices as follows:

$$e^{\frac{i}{2}\phi_{sg}(z)}\Phi_{\mathcal{A}} \begin{bmatrix} 1/4 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Pi_{\mathcal{A}} \begin{bmatrix} 1/4 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix}$$

$2h^{1,1}$ zero modes of $(+)$ chirality

$$e^{\frac{i}{2}\phi_{sg}(z)}\Phi^*_{\mathcal{A}^*} \begin{bmatrix} 1/4 & 1 \\ 1/2 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\mathbf{1} \begin{bmatrix} 1/4 \\ 1/2 \end{bmatrix}^a \Pi^*_{\mathcal{A}^*} \begin{bmatrix} 1/4 & 1 \\ 1/2 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix}$$

$4h^{0,1}$ zero modes of (-) chirality

Here Φ^*_{A*} and Π^*_{A*} are the operators obtained acting with the right-moving supercurrents on the abstract (0,1) operators Ψ^*_{A*} to obtain in the right sector the Φ or the Π upper component of the short representation, in the same way as Φ_A and Π_A are related to the abstract (1,1) forms Ψ_A.
Indeed for the positive chirality the 4-dim part of the flat space expression can be recast in the form $\Psi\left[\begin{smallmatrix}1/4\\0\end{smallmatrix}\right](z)\Pi(\text{or }\Phi)(\bar{z})$ i.e. there is the product of short reps. in both sectors, giving rise to the appropriate abstract-(1,1) operator; for the negative chirality, we have instead $\mathbf{1}\left[\begin{smallmatrix}1/4\\1/2\end{smallmatrix}\right](z)\Pi(\text{or }\Phi)(\bar{z})$, corresponding to abstract (0,1) operators.
let us also note that for having correct total $U(1)$ charges, the spacetime charge must be assigned consistently with what is done for the flat space case, and thus it is not (for these and for all the fermionic vertices that will follow) directly the charge respect to the U(1) of the N=2 contained in the N=4; this would be indeed 0 for the spin fields of negative chirality (singlets of $SU(2)$) and 1 mod2 for the positive chirality ones. The practical rule is to subtract 1/2 when dealing with the analogues of the flat space spin fields.

• *Neutral WZ multiplets*

SU(6)-singlet scalars

In the flat space the vertex espression is

$$e^{i\phi_{sg}(z)}e^{ik\cdot X(z,\bar{z})}\Omega_i\begin{pmatrix}1/2 & 1\\ 1 & 0\end{pmatrix}$$

The internal fields Ω_i are all the possible primary fields with the specified weights and charges.
The general analogue is simply

$$e^{i\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\Omega_i\begin{pmatrix}1/2 & 1\\ 1 & 0\end{pmatrix}$$

One zero mode.

where $\hat{\mathbf{1}}$ is the identity times maybe the dimension zero operator that plays the same role as $e^{ik\cdot X(z,\bar{z})}$ in the flat space case, and whose presence depends on the uncompactified geometry represented by the abstract $(6,6)_{4,4}$.

SU(6)-singlet fermions

From spectral flow in the left sector of the previous "scalar" flat space and generic vertices one gets the "fermionic" expression, both in flat space and in the general case. From now on we limit ourselves to write the generic expressions, as the practical "rules", following from the principle discussed in the previous chapters, for the generalization of flat vertices have been sufficiently illustrated.

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix}1/4 & 0\\ 1/2 & 0\end{bmatrix}^a\Omega_i\begin{pmatrix}3/8 & 1\\ -1/2 & 0\end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 0\\ 0 & 0\end{bmatrix}\Omega_i\begin{pmatrix}3/8 & 1\\ 1/2 & 0\end{pmatrix}$$

2 zero modes of (-) chirality and $h^{1,0}$ of (+) chirality.

• E_6 *Gauge bosons:*

We turn now our attention to the E_6 charged fields, starting with the E_6 gauge bosons. The vertices are grouped in $SU(6)$ representations as follows in the Kaluza-Klein analysis from the breaking $E_6 \longrightarrow SU(6)\times SU(2)$ due to the embedding of the spin connection of the 4-dim manifold, of $SU(2)$ holonomy, into the gauge connection. Their structure shows directly how these $SU(6)$ reps are reconstructed using the operators carrying $SO(6), SU(2)$ or $U(1)$ charges, as it follows from their branchings under $SO(6)\times SU(2)\times U(1)$; these branchings are indicated in the form

$$rep_{SU(6)} = (rep_{SO(6)}\,,\, rep_{SU(2)}\,,\, \tilde{q})$$

(we omit the $U(1)$ charge $\tilde{q}$ when it is equal to zero). For what concernes the adjoint (35) representation in appendix ... the reconstruction in terms of $SO(6), SU(2), U(1)$-charged operators is shown to be really correct from the group point of view.

Vertices in the 35 of SU(6), "SU(6)-gauge bosons"

$$35 = (15,1) + (1,1) + (1,3) + (4,2,\tilde{q}=3/2) + (\overline{4},2,\tilde{q}=-3/2)$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/2 & 0\\ 1/2 & 0\end{bmatrix}^a \mathbf{1}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\tilde{j}^A(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/2 & 0\\ 1/2 & 0\end{bmatrix}^a \mathbf{1}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\tilde{j}(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/2 & 0\\ 1/2 & 0\end{bmatrix}^a \mathbf{1}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}\tilde{A}^i(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/2 & 1/4\\ 1/2 & 1/2\end{bmatrix}^{a\tilde{a}} \mathbf{1}\begin{pmatrix}0 & 3/8\\ 0 & 3/2\end{pmatrix}\widetilde{\Sigma}_\alpha(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/2 & 1/4\\ 1/2 & 1/2\end{bmatrix}^{a\tilde{a}} \mathbf{1}\begin{pmatrix}0 & 3/8\\ 0 & -3/2\end{pmatrix}\widetilde{\Sigma}_{\dot{\alpha}}(\bar{z})$$

$2h^{1,0}$ zero modes.

$\tilde{j}$ are the currents in the adjoint (15) of $SO(6)$, $\tilde{A}$ the currents of the $SU(2)$ of the right N=4 algebra, $\tilde{j}$ is the internal $U(1)$ current. $\widetilde{\Sigma}_\alpha$ and $\widetilde{\Sigma}_{\dot{\alpha}}$ are the SO(6) spin fields of the two chiralities.

SU(6)-singlet scalars

$$e^{i\phi_{sg}(z)}\Omega_i\begin{bmatrix}1/2 & 1\\ 1 & 0\end{bmatrix}\mathbf{1}\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

The Ω_i are all the "space-time" fields with the indicated weights and isospins, so they are more than the Ψ_A (see sec.2) and their number should correspond to $\#End(T_{K3})$

"Scalar" vertices in the 20 of SU(6)

$$20 = (6,2) + (4,1,\tilde{q}=3/2) + (\overline{4},1,\tilde{q}=-3/2)$$

$$e^{i\phi_{sg}(z)}\Psi_A \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}^{a\tilde{a}} \mathbf{1} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \widetilde{\theta}_P(\bar{z})$$

$$e^{i\phi_{sg}(z)}\Psi_A \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 0 \end{bmatrix}^{a} \mathbf{1} \begin{pmatrix} 0 & 3/8 \\ 0 & 3/2 \end{pmatrix} \widetilde{\Sigma}_\alpha(\bar{z})$$

$$e^{i\phi_{sg}(z)}\Psi_A \begin{bmatrix} 1/2 & 1/4 \\ 1/2 & 0 \end{bmatrix}^{a} \mathbf{1} \begin{pmatrix} 0 & 3/8 \\ 0 & -3/2 \end{pmatrix} \widetilde{\Sigma}_{\dot{\alpha}}(\bar{z})$$

$2h^{(1,1)}$ zero modes.

The heterotic fermions $\widetilde{\theta}_P(\bar{z})$ transform in the fundamental of SO(6).

• E_6 *Gauginos:*

These vertices are related to the vertices for the gauge bosons much in the same way as the gravitino ones are related to the graviton; all the remarks done at that point apply here as well as to all fermionic vertices. From now on we will only more list the group decompositions and the vertices for the various fields.

Vertices in the 35, "SU(6) gauginos"

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}} \begin{bmatrix} 1/4 & 0 \\ 1/2 & 0 \end{bmatrix}^{a} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix} \tilde{j}^A(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}} \begin{bmatrix} 1/4 & 0 \\ 1/2 & 0 \end{bmatrix}^{a} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix} \tilde{j}(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}} \begin{bmatrix} 1/4 & 0 \\ 1/2 & 0 \end{bmatrix}^{a} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ -3/2 & 0 \end{pmatrix} \tilde{A}^i(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}} \begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}^{a\tilde{a}} \mathbf{1} \begin{pmatrix} 3/8 & 3/8 \\ -3/2 & 3/2 \end{pmatrix} \widetilde{\Sigma}_\alpha(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}} \begin{bmatrix} 1/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}^{a\tilde{a}} \mathbf{1} \begin{pmatrix} 3/8 & 3/8 \\ -3/2 & -3/2 \end{pmatrix} \widetilde{\Sigma}_{\dot{\alpha}}(\bar{z})$$

Two zero modes of (-) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_A \begin{bmatrix} 1/4 & 0 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix} J^A(\bar{z})$$

. . . .

. . . .

$h^{1,0}$ of (+) chirality.

SU(6) singlets fermions)

$$e^{i\phi_{sg}(z)}\Omega_i \begin{bmatrix} 1/4 & 1 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix}$$

The numbers of these vertices is again related to $\#End(T_{K3})$.

Fermions in the 20 of SU(6)

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_A \begin{bmatrix} 1/4 & 1/2 \\ 0 & 1/2 \end{bmatrix}^{\tilde{a}} \mathbf{1} \begin{pmatrix} 3/8 & 0 \\ 3/2 & 0 \end{pmatrix} \tilde{\theta}_P(\overline{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_A \begin{bmatrix} 1/4 & 1/4 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 3/8 \\ 3/2 & 3/2 \end{pmatrix} \tilde{\Sigma}_\alpha(\overline{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_A \begin{bmatrix} 1/4 & 1/4 \\ 0 & 0 \end{bmatrix} \mathbf{1} \begin{pmatrix} 3/8 & 3/8 \\ 3/2 & -3/2 \end{pmatrix} \tilde{\Sigma}_{\dot{\alpha}}(\overline{z})$$

$h^{1,1}$ zero modes, all of (+) chirality.

•*27-charged Scalars:*

Scalars in the 15 of SU(6)

$$15 = (6, 1, \tilde{q} = 1) + (4, 2, \tilde{q} = -\frac{1}{2}) + (1, 1, \tilde{q} = -2)$$

$$e^{i\phi_{sg}(z)}\hat{\mathbf{1}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Psi_k^+ \begin{pmatrix} 1/2 & 1/2 \\ 1 & 1 \end{pmatrix} \tilde{\theta}_P(\overline{z})$$

$$e^{i\phi_{sg}(z)}\hat{\mathbf{1}} \begin{bmatrix} 0 & 1/4 \\ 0 & 1/2 \end{bmatrix}^{\tilde{a}} \Psi_k^+ \begin{pmatrix} 1/2 & 3/8 \\ 1 & -1/2 \end{pmatrix} \tilde{\Sigma}_\alpha(\overline{z})$$

$$e^{i\phi_{sg}(z)}\hat{\mathbf{1}} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \Psi_k^+ \begin{pmatrix} 1/2 & 1 \\ 1 & -2 \end{pmatrix}$$

one zero mode.

The operators Ψ_k^+, which are of (chiral,chiral) type, are lowest components of a short representation and correspond to the abstract (1,2)-forms of the internal 6-dimansional Calabi-Yau manifold.

Scalars in the 6 of SU(6)

$$6 = (1, 2, \tilde{q} = 1) + (4, 1, \tilde{q} = -\frac{1}{2})$$

$$e^{i\phi_{sg}(z)}\Psi^*_{\mathcal{A}^*}\begin{bmatrix}0 & 1/2\\ 0 & 1/2\end{bmatrix}^{\tilde{a}}\Psi^+_k\begin{pmatrix}1/2 & 1/2\\ 1 & 1\end{pmatrix}$$

$$e^{i\phi_{sg}(z)}\Psi^*_{\mathcal{A}^*}\begin{bmatrix}0 & 1/4\\ 0 & 0\end{bmatrix}\Psi^+_k\begin{pmatrix}1/2 & 3/8\\ 1 & -1/2\end{pmatrix}\widetilde{\Sigma}_\alpha(\overline{z})$$

$h^{0,1}$ zero modes.

• *27-charged fermions*

Fermions in the 15

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix}1/4 & 0\\ 1/2 & 0\end{bmatrix}^{a}\Psi^+_k\begin{pmatrix}3/8 & 1/2\\ -1/2 & 1\end{pmatrix}\widetilde{\theta}_P(\overline{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix}1/4 & 1/4\\ 1/2 & 1/2\end{bmatrix}^{a\tilde{a}}\Psi^+_k\begin{pmatrix}3/8 & 3/8\\ -1/2 & -1/2\end{pmatrix}\widetilde{\Sigma}_\alpha(\overline{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{\mathbf{1}}\begin{bmatrix}1/4 & 0\\ 1/2 & 0\end{bmatrix}^{a}\Psi^+_k\begin{pmatrix}3/8 & 1\\ -1/2 & -2\end{pmatrix}$$

Two zero modes of (-) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 0\\ 0 & 0\end{bmatrix}\Psi^+_k\begin{pmatrix}3/8 & 1/2\\ 1/2 & 1\end{pmatrix}\widetilde{\theta}_P(\overline{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 1/4\\ 0 & 1/2\end{bmatrix}^{\tilde{a}}\Psi^+_k\begin{pmatrix}3/8 & 3/8\\ 1/2 & -1/2\end{pmatrix}\widetilde{\Sigma}_\alpha(\overline{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 0\\ 0 & 0\end{bmatrix}\Psi^+_k\begin{pmatrix}3/8 & 1\\ 1/2 & -2\end{pmatrix}$$

$h^{1,0}$ zero modes of (+) chirality.

Fermions in the 6

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 1/2\\ 0 & 1/2\end{bmatrix}^{\tilde{a}}\Psi^+_k\begin{pmatrix}3/8 & 1/2\\ 1/2 & 1\end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 1/4\\ 0 & 0\end{bmatrix}\Psi^+_k\begin{pmatrix}3/8 & 3/8\\ 1/2 & -1/2\end{pmatrix}\widetilde{\Sigma}_\alpha(\overline{z})$$

$h^{1,1}$ zero modes of (+) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi^*_{\mathcal{A}^*}\begin{bmatrix}1/4 & 1/2\\ 1/2 & 1/2\end{bmatrix}^{a\tilde{a}}\Psi^+_k\begin{pmatrix}3/8 & 1/2\\ -1/2 & 1\end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi^*_{\mathcal{A}^*}\begin{bmatrix}1/4 & 1/4\\ 1/2 & 0\end{bmatrix}^a \Psi^+_k \begin{pmatrix}3/8 & 3/8\\ -1/2 & -1/2\end{pmatrix}\widetilde{\Sigma}_\alpha(\bar{z})$$

$2h^{0,1}$ zero modes of (-) chirality.

•$\overline{27}$-*charged Scalars:*

Scalars in the $\overline{15}$ of SU(6)

$$\overline{15} = (6,1,\tilde{q}=-1) + (\bar{4},2,\tilde{q}=\frac{1}{2}) + (1,1,\tilde{q}=2)$$

$$e^{i\phi_{sg}(z)}\hat{1}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\Psi^-_k\begin{pmatrix}1/2 & 1/2\\ 1 & -1\end{pmatrix}\widetilde{\theta}_P(\bar{z})$$

$$e^{i\phi_{sg}(z)}\hat{1}\begin{bmatrix}0 & 1/4\\ 0 & 1/2\end{bmatrix}^{\tilde{a}}\Psi^-_k\begin{pmatrix}1/2 & 3/8\\ 1 & 1/2\end{pmatrix}\widetilde{\Sigma}_{\dot{\alpha}}(\bar{z})$$

$$e^{i\phi_{sg}(z)}\hat{1}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\Psi^-_k\begin{pmatrix}1/2 & 1\\ 1 & 2\end{pmatrix}$$

one zero mode.

The operators Ψ^-_k, which are of (chiral,antichiral) type, are lowest components of short representations, both is left and right sector, and correspond to the abstract (1,1)-forms of the internal 6-dimensional Calabi-Yau manifold.

Scalars in the $\bar{6}$ of SU(6)

$$6 = (1,2,\tilde{q}=-1) + (\bar{4},1,\tilde{q}=\frac{1}{2})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi^*_{\mathcal{A}^*}\begin{bmatrix}0 & 1/2\\ 0 & 1/2\end{bmatrix}^{\tilde{a}}\Psi^-_k\begin{pmatrix}3/8 & 1/2\\ 1/2 & -1\end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi^*_{\mathcal{A}^*}\begin{bmatrix}0 & 1/4\\ 0 & 0\end{bmatrix}\Psi^-_k\begin{pmatrix}3/8 & 3/8\\ 1/2 & 1/2\end{pmatrix}\widetilde{\Sigma}_\alpha(\bar{z})$$

$h^{0,1}$ zero modes

•$\overline{27}$-*charged fermions:*

Fermions in the $\overline{15}$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{1}\begin{bmatrix}1/4 & 0\\ 1/2 & 0\end{bmatrix}^a\Psi^-_k\begin{pmatrix}3/8 & 1/2\\ -1/2 & -1\end{pmatrix}\widetilde{\theta}_P(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{1}\begin{bmatrix}1/4 & 1/4\\ 1/2 & 1/2\end{bmatrix}^{a\tilde{a}}\Psi^-_k\begin{pmatrix}3/8 & 3/8\\ -1/2 & 1/2\end{pmatrix}\widetilde{\Sigma}_\alpha(\bar{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\hat{1}\begin{bmatrix}1/4 & 0\\ 1/2 & 0\end{bmatrix}^{a}\Psi_{k}^{-}\begin{pmatrix}3/8 & 1\\ -1/2 & 2\end{pmatrix}$$

Two zero modes of (-) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 0\\ 0 & 0\end{bmatrix}\Psi_{k}^{-}\begin{pmatrix}3/8 & 1/2\\ 1/2 & -1\end{pmatrix}\widetilde{\theta}_{P}(\overline{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 1/4\\ 0 & 1/2\end{bmatrix}^{\tilde{a}}\Psi_{k}^{-}\begin{pmatrix}3/8 & 3/8\\ 1/2 & 1/2\end{pmatrix}\widetilde{\Sigma}_{\alpha}(\overline{z})$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 0\\ 0 & 0\end{bmatrix}\Psi_{k}^{-}\begin{pmatrix}3/8 & 1\\ 1/2 & 2\end{pmatrix}$$

$h^{1,0}$ zero modes of (+) chirality.

Fermions in the $\overline{6}$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 1/2\\ 0 & 1/2\end{bmatrix}^{\tilde{a}}\Psi_{k}^{-}\begin{pmatrix}3/8 & 1/2\\ 1/2 & -1\end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}}\begin{bmatrix}1/4 & 1/4\\ 0 & 0\end{bmatrix}\Psi_{k}^{-}\begin{pmatrix}3/8 & 3/8\\ 1/2 & 1/2\end{pmatrix}\widetilde{\Sigma}_{\alpha}(\overline{z})$$

$h^{1,1}$ zero modes of (+) chirality.

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}^*}^{*}\begin{bmatrix}1/4 & 1/2\\ 1/2 & 1/2\end{bmatrix}^{a\tilde{a}}\Psi_{k}^{-}\begin{pmatrix}3/8 & 1/2\\ -1/2 & -1\end{pmatrix}$$

$$e^{\frac{i}{2}\phi_{sg}(z)}\Psi_{\mathcal{A}^*}^{*}\begin{bmatrix}1/4 & 1/4\\ 1/2 & 0\end{bmatrix}^{a}\Psi_{k}^{-}\begin{pmatrix}3/8 & 3/8\\ -1/2 & 1/2\end{pmatrix}\widetilde{\Sigma}_{\alpha}(\overline{z})$$

$2h^{0,1}$ zero modes of (-) chirality.

Chapter 6

An asymptotically flat instantonic solution of the stringy New Minimal N=1, D=4 Supergravity

We will now show how the instantonic solution proposed by Callan, Harvey and Strominger in [3] arises as a solution of the equations of motion of the effective four-dimensional theory (as it should to fit the general scheme outlined in Chapter 4).
The low-energy effective lagrangian of heterotic superstring theory is a supergravity lagrangian. If the superstring is compactified on a 6-dimensional Calabi-Yau manifold, then this effective lagrangian corresponds to that of an N=1, D=4 supergravity [12] which, when restricted to the bosonic fields, has the following well known general form:

$$\mathcal{L}^{(N=1)}_{Bosonic} = \sqrt{-g} \left[\mathcal{R} - g_{IJ^\star} \nabla_\mu z^I \nabla^\mu z^{J^\star} - \frac{1}{4} Re f_{\alpha\beta}(z) F^\alpha_{\mu\nu} F^{\beta\mu\nu} - V(z,\bar{z}) \right] - \\ - \frac{1}{8} Im f_{\alpha\beta}(z) F^\alpha_{\mu\nu} F^\beta_{\rho\sigma} \varepsilon^{\mu\nu\rho\sigma} \tag{6.1}$$

In (6.1), besides the gravitational field, described by the metric $g_{\mu\nu}$, one has the gauge fields A^α_μ belonging to the Lie algebra of a suitable gauge group G_{gauge} and a set of complex scalar fields z^I corresponding to the bosonic content of the Wess-Zumino scalar multiplets. The kinetic term of these scalars has a σ-model form in terms of a Kähler metric $g_{IJ^\star} = \partial_I \partial_{J^\star} G(z,\bar{z})$. The real Kähler function $G(z,\bar{z})$, besides determining the kinetic term, determines also the scalar potential term, via the celebrated formula [12,19]:

$$V(z,\bar{z}) = 4 \left(g^{IJ^\star} \partial_I G \, \partial_{J^\star} G - 3 \right) e^G - g^2 \left[Re f_{\alpha\beta} \right]^{-1} \mathcal{P}^\alpha \mathcal{P}^\beta \tag{6.2}$$

To be precise $G(z,\bar{z})$ is not exactly the Kähler potential of the metric $g_{IJ^\star}$, rather it is the norm squared

$$G(z,\bar{z}) = K(z,\bar{z}) + ln|W(z)|^2 = ln\,||W(z)||^2 \tag{6.3}$$

of a holomorphic section $W(z)$ in a line bundle $\mathcal{L}$, whose first Chern class is the Kähler class $\omega = i g_{IJ^\star} dz^I \wedge d\bar{z}^{J^\star}$ of that metric :

$$\omega = \partial\bar{\partial}||W||^2 \tag{6.4}$$

The holomorphic section $W(z)$ is named the superpotential and the hermitean metric $K(z,\bar{z})$ of this line bundle is the proper Kähler potential. In addition, if the gauge group has a linear action $\delta z^I = (T_\alpha)^I_J\, z^J$ on the scalar fields, then the contribution to the scalar potential (6.2) proportional to the gauge coupling constant g^2 is given in terms of Killing vectors prepotentials of the form

$$\mathcal{P}^\alpha = -\, i\, \partial_i G (T_\alpha)^I_J\, z^J \tag{6.5}$$

When the action of the gauge group is non linear, then the expression of $\mathcal{P}^\alpha$ is more complicated, but we shall not be interested in this case. Finally, the gauge coupling function $f_{\alpha\beta}(z)$ is some holomorphic function with *adjoint* $\otimes$ *adjoint* indices of the gauge group. In the case of Calabi-Yau compactifications [5] of the heterotic string the gauge group is $E_6 \otimes E_8^{'}$ and the scalar multiplets (all neutral under $E_8^{'}$) are of six different types [20,21]:

$$z^I = \begin{cases} S = \text{dilaton} - \text{axion field} \\ \mathcal{M}^a = (2,1) - \text{moduli}\ (a = 1, \ldots\ldots, h^{2,1}) \\ \mathcal{M}^i = (1,1) - \text{moduli}\ (i = 1, \ldots\ldots, h^{1,1}) \\ \mathcal{C}^a = 27 - \text{charged fields}\ (a = 1, \ldots\ldots, h^{2,1}) \\ \mathcal{C}^i = \overline{27} - \text{charged fields}\ (i = 1, \ldots\ldots, h^{1,1}) \\ \mathcal{Y}^u = \text{non} - \text{moduli singlets}\ (u = 1, \ldots\ldots, \#End(T)) \end{cases} \tag{6.6}$$

in correspondence with the cohomological properties of the internal space, dictated by its Hodge numbers $h^{1,1}, h^{2,1}$ and by the number of deformations of its tangent bundle $\#End(T)$. Of particular relevance are the moduli-fields, that describe the deformations of the compactified manifold, and their special Kähler geometry. Indeed, to lowest order in the charged fields and non-moduli singlets, the general forms of the complete Kähler potential and complete superpotential are respectively given by:

$$K = -\, log\,(S + \bar{S}) \;+ \hat{K}(\mathcal{M}, \overline{\mathcal{M}}) \;+\; \mathcal{G}_{ab^\star}(\mathcal{M}, \overline{\mathcal{M}})\, \mathcal{C}^a\, \overline{\mathcal{C}}^{b^\star}$$

$$+\, \mathcal{G}_{ij^\star}(\mathcal{M}, \bar{\mathcal{M}})\, \mathcal{C}^i\, \overline{\mathcal{C}}^{j^\star} \;+\; \ldots. \tag{6.7}$$

and

$$W = \frac{1}{3}\, W_{abc}(\mathcal{M})\, \mathcal{C}^a\, \mathcal{C}^b\, \mathcal{C}^c \;+\; \frac{1}{3}\, W_{ijk}(\mathcal{M})\, \mathcal{C}^i\, \mathcal{C}^j\, \mathcal{C}^k \;+\; \ldots\ldots\ldots\ldots\ldots \tag{6.8}$$

where $\hat{K}(\mathcal{M}, \overline{\mathcal{M}})$ is the Kähler potential of the moduli-space and $W_{abc}(\mathcal{M}), W_{ijk}(\mathcal{M})$ are the Yukawa couplings. These quantities are related by the peculiar identities of special geometry.

Notwithstanding the importance of these fields, in the present thesis, we are rather interested in the first term of eq.(6.7), namely in the universal S-field that includes both the dilaton and the axion. The structure of (6.7) implies that this field spans an $SU(1,1)/U(1)$ coset manifold and that the total scalar manifold is the direct product of this coset with some other Kähler manifold $\mathcal{K}'$. That this is the case follows from very general considerations we shall now review. Furthermore it is just the presence of S that allows for the existence of instantonic solutions that are asymptotically flat and not only locally asymptotically flat. To this effect we recall that according to a very interesting mechanism discovered by Konishi et al [10], gravitational instantons might induce a non-perturbative breakdown of supersymmetry via their contribution to the functional integral. An explicit calculation was in fact performed in [10], utilizing the Eguchi-Hanson metric [11]. The problem is that, for a correct implementation of this mechanism, the instanton should be asymptotically flat. This is not the case of the Eguchi-Hanson metric, for which asymptotic flatness is local and not global. The problem of finding asymptotically flat gravitational instantons was considered several years ago by D'Auria and Regge [9]. They realized that in order to reconcile the self-duality of the curvature with asymptotic flatness one needs an "unsoldering" of the principal Lorentz-bundle from the tangent bundle. This can be achieved by writing gravity in first order formalism and coupling it to a pseudoscalar field, whose derivative becomes the dual of the 3-index torsion. Indeed D'Auria and Regge proposed a certain configuration that realizes the desired instanton and that is a solution of an ad hoc constructed lagrangian. As we are going to see, their configuration is just equivalent to the dilaton-axion instanton discovered by Rey [22] to be an exact solution of the string derived Supergravity lagrangian (6.1) with Kähler potential (6.7). What D'Auria and Regge missed in their action and had to simulate with an ad hoc interaction term was just the dilaton. Indeed their pseudo-scalar was nothing else but the axion. In a certain limit the dilaton-axion instanton corresponds to an exactly solvable (4,4) superconformal theory that has been discovered by Callan [3]. In later sections of this thesis we use this examples and its associated (4,4)-theory to illustrate our ideas on the generalized h-map, studying also the corresponding moduli deformations. In the present section we discuss the derivation of this instanton in the context of the effective low-energy lagrangian, enphasizing the role of the New Minimal formulation of Supergravity.

The key point here is the observation that, indipendently from the compactification scheme the effective supergravity lagrangian should contain the coupling of a linear multiplet $(\phi, \chi, B_{\mu\nu})$ that arises directly via dimensional reduction from the dilaton and $B_{\mu\nu}$ field of the ten dimensional effective theory. In four dimension, this multiplet can be transformed into an ordinary WZ multiplet by a "duality transformation" relating the $B_{\mu\nu}$ field strength to an axion field:

$$\nabla_\mu A = \frac{1}{24}\frac{\epsilon_{\mu\nu\rho\sigma}}{\sqrt{|g|}} e^{-2\phi} H^{\nu\rho\sigma} \tag{6.9}$$

As we just recalled in matter-coupled 4-dim supergravity the complex bosonic matter fields are interpreted as the coordinates z^I of a Kähler manifold $\mathcal{K}$. For a generic theory, and if we derive the action from the Old Minimal off-shell formulation of supergravity [23], the manifold $\mathcal{K}$ is arbitrary: we recall that the Old minimal formulation is characterized by the presence of a scalar auxiliary field appearing in the SUSY-transformation rule of the gravitino. On the other hand if we adopt the New Minimal formulation [24], characterized by the absence of this scalar auxiliary field, then $\mathcal{K}$ cannot be arbitrary: it is constrained by conditions that imply the existence of a coordinate frame where the the Kähler function has the following form

$$G = \alpha \log(z+\bar{z}) + \hat{G}(z^i, \bar{z}^{i^*}) \tag{6.10}$$

the indices being split as follows $\{z^I\} = \{z, z^i\}$ and $\hat{G}(z^i, \bar{z}^{i^*})$ being an arbitrary Kähler function for the remaining scalar fields z^i, once the special field z has been subtracted. The parameter α is any real constant. In other words the existence of a New Minimal Formulation requires a factorization (at least a local one) of the scalar manifold into:

$$\mathcal{K}_{scalar} = \frac{SU(1,1)}{U(1)} \otimes \mathcal{K}' \tag{6.11}$$

These results were derived in [25]. In the same paper, it was also shown that the conditions for the existence of a New Minimal formulation are the same conditions that guarantee the possibility of duality-rotating one of the WZ-multiplets to a linear multiplet $(\phi, \chi, B_{\mu\nu})$, via equation (6.9).

In view of these very general results, it follows that a superstring derived supergravity, since it includes a linear multiplet, has necessarily a Kähler function of the form (6.10), and admits a New Minimal formulation. The second statement is further supported by the results of [26], showing that in heterotic string theory one cannot construct an emission vertex for the scalar auxiliary field.

Having clarified this crucial point we proceed to discuss the derivation of the dilaton-axion instanton in supergravities characterized by a scalar manifold of type (6.11). Using the New-Minimal Lagrangian we retrieve as an exact solution the Callan et al configuration [3], that is also of the same form as the one considered by D'Auria and Regge in [9]. Performing the generalized Weyl-transformation that maps the New into the Old Minimal theory, the Callan instanton flows into the Rey instanton, characterized by an exactly flat metric and a singular dilaton and axion.

Let us then go back to eq.(6.7) and concentrate on the Kähler function $G(S,\bar{S}) = -\log(S+\bar{S})$. When we consider the theory in Minkowski spacetime the fields of the dilaton multiplet

$$\begin{aligned} S &= f + ig \\ \bar{S} &= f - ig \end{aligned} \tag{6.12}$$

(f representing the original dilaton, g the axionic field) span the factorized $SU(1,1)/U(1)$ part of the scalar manifold, according to eq.(6.11). It turns out, however, that, while performing the Wick rotation to reach the Euclidean region, (due to the ϵ symbol appearing in the duality transformation eq.(6.9)), it is also necessary to perform a Wick rotation on the scalar manifold. Eq.(6.12) becomes

$$\begin{aligned} S &= f + g \\ \bar{S} &= f - g \end{aligned} \tag{6.13}$$

From now on we will consider the Euclidean case, since we search for an instantonic solution. However, for convenience, we continue to use the same "complex" notation as before the rotation.

Restricting our attention to the bosonic sector of the theory, in the New Minimal formulation, according to the results of [25], the curvature two-forms

$$\begin{aligned} R^{ab} &= d\omega^{ab} - \omega^a{}_c \omega^{cb} \\ R^a &= DV^a \\ R^{\otimes} &= d\mathcal{A} \end{aligned}$$

($\mathcal{A}$ being the Kähler connection on the scalar manifold) are parametrized as follows:

$$\begin{aligned} R^{ab} &= R^{ab}{}_{cd} V^c V^d \\ R^{\otimes} &= F_{ab} V^a V^b \\ R^a &= \kappa_2 \epsilon^{abcd} t_b V_c V_d \qquad\qquad (D_a t^a = 0) \\ dz^I &= Z^I_a V^a \end{aligned} \tag{6.14}$$

the parameter κ_2 being a free constant. The fields in this formulation are obtained from those in the Old Minimal one through a Weyl transformation,

$$V^a_{\text{new}} = e^{\phi/2} V^a_{\text{old}} \qquad \rightarrow \qquad Z_a^{I\text{new}} = e^{-\phi/2} Z_a^{I\text{old}} \tag{6.15a}$$

(for the bosonic fields).

In order for the transformation to be succesful, it is required that

$$\phi = \log \partial_{\bar{z}} G \tag{6.15b}$$

where $G(z, \bar{z})$ is given by equation (6.10). The auxiliary fields are then expressed as

$$t_a^{\text{new}} = Im(\partial_I \phi \; Z^I_a) \tag{6.16a}$$

$$\mathcal{A}^{\text{new}} = Im \; dG - (2\kappa_1 + 1) Im(\partial_I \phi \; Z^I_a) \tag{6.16b}$$

κ_1 is a constant appearing in the New Minimal parametrization of the fermionic curvatures for whose expression we refer to [25]. One sees that having the dilatonic WZ multiplet in the game, we are precisely in the situation of eq.(6.10) with $\alpha = -1, z = S$. Hence in the case of the superstring effective lagrangian we obtain the identification

$$\phi = \log \frac{1}{S + \bar{S}} = \log \frac{1}{2f} \tag{6.17}$$

The first order formulation of the bosonic New Minimal lagrangian is given by

$$\begin{aligned} \mathcal{L} =& e^{-\phi} \Big\{ R^{ab} V^c V^d \epsilon_{abcd} + 4\kappa_2 t_a R_b V^a V^b + (\partial_I \phi \; Z^I_a + \partial_{I^*} \phi \; Z^{I^*}_a) R_b V_c V_d \epsilon^{abcd} + \\ & + (\frac{2}{3} g_{IJ^*} - \partial_I \phi \; \partial_{J^*} \phi) \left[Z^I_a dz^{J^*} + Z^{J^*}_a dz^I \right] V_b V_c V_d \epsilon^{abcd} + \\ & - \left[\partial_I \phi \; \partial_J \phi \; Z^I_a dz^J + \partial_{I^*} \phi \; \partial_{J^*} \phi \; Z^{i^*}_a dz^{j^*} \right] V_b V_c V_d \epsilon^{abcd} + \\ & + \left[-\frac{1}{4} (\frac{2}{3} g_{IJ^*} - \partial_I \phi \; \partial_{J^*} \phi) Z^I_r Z^{J^* r} + \frac{1}{8} (\partial_I \phi \; \partial_J \phi \; Z^I_r Z^{Jr} \right. \\ & \left. + \partial_{I^*} \phi \; \partial_{J^*} \phi \; Z^{I^*}_r Z^{J^* r}) + \frac{1}{2} \kappa_2^2 \; t_r t^r - M \right] V^a V^b V^c V^d \epsilon_{abcd} \Big\} \end{aligned} \tag{6.18}$$

where the scalar potential takes the new form $M{=}{-}\frac{2}{3}\bar{e}^2(3{+}\alpha{-}g^{ij^*}\partial_i\hat{G}\;\;\partial_{j^*}\hat{G})\cdot e^G(z+\bar{z})^\alpha$, to be compared with eq.(6.2)

Recalling that in 2^{nd} order formalism

$$R^{ab}V^cV^d\epsilon_{abcd} = \frac{1}{2}\mathcal{R}\sqrt{|g|}d^4x$$

where $\mathcal{R}$ is the curvature scalar, and comparing the lagrangian in eq.(6.18) with the effective action used by Callan et al. [3],

$$S = \frac{1}{2}\int\sqrt{|g|}d^4xe^{2\Phi}\left(\mathcal{R}+...\right)$$

we have the correspondence

$$\phi = -2\Phi \tag{6.19}$$

We can consistently search for a particular solution in which only the dilaton and the axion field are relevant, setting the other fields z^i to constant values c^i such that

$$\partial_i M(c^i) = 0 \quad ; \quad M(c^i) = 0 \tag{6.20}$$

We furthermore impose the radial ansatz

$$\begin{array}{ll} V^a = e^{-\lambda(r)}e^a & (r^2 = x_a x^a) \\ S = S(r) \;\; \text{i.e.} & f = f(r), g = g(r) \end{array} \tag{6.21}$$

Recalling that

$$g_{S\bar{S}} = \frac{1}{(S+\bar{S})^2} = \frac{1}{4f^2}$$

and using as flat vierbeins the following ones

$$\begin{cases} e^0 = dr \\ e^i = -(r/\sqrt{2})\Omega^i \end{cases}$$

with $\Omega^i{=}SU(2)$-Maurer-Cartan forms such that $d\Omega^i = -(\epsilon^{ijk}/\sqrt{2})\Omega^j\,\Omega^k$, the variational equations obtained from the lagrangian (6.18) read:

- Matter equations ($g-, f-$*variations respectively*)

$$\frac{g''}{f} - 2\lambda'\frac{g'}{f} + \frac{3}{r}\frac{g'}{f} - \frac{g'}{f}\frac{f'}{f} = 0 \tag{6.22a}$$

$$-12\lambda'' + 12(\lambda')^2 - 36\frac{\lambda'}{r} + \left[\frac{f''}{f} - 2\lambda'\frac{f'}{f} + \frac{3}{r}\frac{f'}{f} - \frac{1}{2}\left(\frac{f'}{f}\right)^2\right] + \left(\frac{g'}{f}\right)^2 = 0 \tag{6.22b}$$

• Einstein equations (V^d-*variation*)

$$8\lambda" - 4(\lambda')^2 + 16\frac{\lambda'}{r} - 4\lambda'\frac{f'}{f} - \frac{8}{r}\frac{f'}{f} + 2\left(\frac{f'}{f}\right)^2 + \left(\frac{g'}{f}\right)^2 = 0 \qquad (6.22c)$$

$$-12(\lambda')^2 + 24\frac{\lambda'}{r} + 12\lambda'\frac{f'}{f} - \frac{12}{r}\frac{f'}{f} - 2\left(\frac{f'}{f}\right)^2 - \left(\frac{g'}{f}\right)^2 = 0 \qquad (6.22d)$$

primes meaning derivatives with respect to r. One sees that under the position $\lambda' = \frac{1}{2}(f'/f)$ the Einstein equations reduce to a single expression, $(f'/f)^2 - (g'/f)^2 = 0$, requiring

$$f' = \pm g' \qquad (6.23)$$

Inserting the above conditions into the matter equations, the following solution is obtained:

$$\lambda = \log\left(\frac{r/R_0}{\sqrt{1+(r/R_1)^2}}\right) \quad ; \quad f = \frac{1}{c}\frac{(r/R_0)^2}{1+(r/R_1)^2} \qquad (6.24)$$

where c, R_0, R_1 are arbitrary constant,which clearly reproduces the metric configuration of the one-instanton solution of Callan et al. [3]. For the choice $c = 2$ we have indeed

$$\begin{aligned} V^a &= e^{-\Phi}e^a \\ e^{-2\Phi} &= e^{-2\lambda} = \frac{1}{2f} = \left(\frac{R_0}{R_1}\right) + \left(\frac{R_0^2}{r^2}\right) \end{aligned} \qquad (6.25)$$

which gives the correspondence

$$\left(\frac{R_0}{R_1}\right)^2 = e^{-2\Phi_0} \quad ; \quad R_0^2 = n \qquad (6.26)$$

The configurations leading to the $SU(2)\times\mathbf{R}$ model, associated with a solvable (4,4)-theory is obtained in the limit $R_1 \longrightarrow \infty$.
For example in this last case it's easily checked that also the expression for the torsion agrees: making formula (6.16a) explicit we find

$$t_a = \frac{1}{2}\frac{1}{S+\bar{S}}\left(\partial_a S - \partial_a \bar{S}\right) = \frac{1}{S+\bar{S}}\partial_a g = \frac{\partial_a g}{2f}$$

and inserting this in the relation (6.23), we get $t_a = \frac{1}{2}\partial_a \log f$, that is

$$\begin{aligned} t_i &= 0 \\ t_0 &= \frac{1}{2}V_0^r(\log f)' = e^{\Phi}(\log f)' = \frac{r}{\sqrt{n}}\frac{1}{r} = \frac{1}{\sqrt{n}} \end{aligned}$$

From the parametrization (6.14) we obtain thus

$$
\begin{aligned}
T^0 &= 0 \\
T^i &= \kappa_2 \frac{\epsilon_{ijk}}{\sqrt{n}} V^j V^k
\end{aligned}
\tag{6.27}
$$

which agrees with the expression of the torsion for this particular solution, as obtained by Callan et al.,with the choice $\kappa_2 = 1$.

It is then clear that the configuration

$$
\begin{aligned}
ds^2 &= e^{-2\Phi}(dx)^2 & \qquad & V^a = e^{-\Phi} e^a \\
e^{-2\Phi} &= A + \frac{2k}{r^2} & \leftrightarrow \qquad & \Phi = \log\,(A + \frac{2k}{r^2})^{-\frac{1}{2}}
\end{aligned}
\tag{6.28}
$$

$$
H_{abc} = \frac{1}{3}\epsilon_{abcd}\,\partial_d \Phi
$$

is an exact euclidean solution of the effective superstring lagrangian in the New-Minimal formulation. When transformed back to the Old-Minimal formulation, by means of eq.s (6.15), this configuration becomes the dilaton-axion instanton found by Rey [22]. This is obvious from the fact that the metric in the Old-Minimal formulation becomes the flat one.

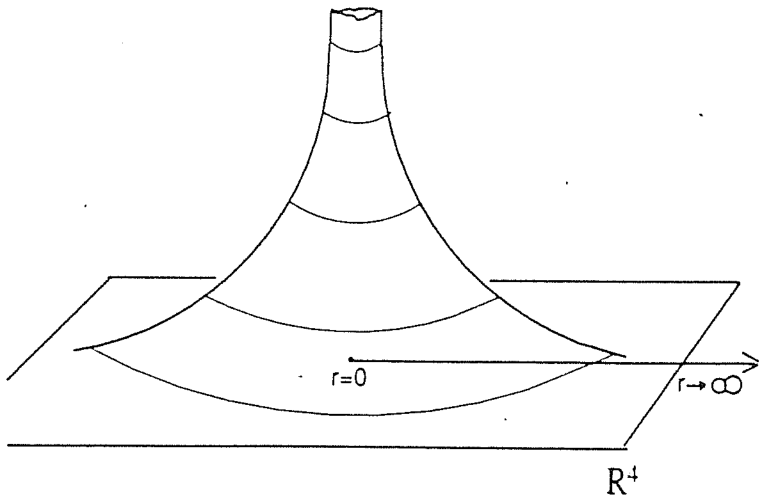


Fig 6.1 The metric of the configuration (6.28)

The configuration (6.28) is asymptotically flat, and it can be shown that its torsionful curvatures R^- and R^+ are respectively antiselfdual and selfdual; it corresponds thus to a GHK space. In Fig. 6.1 it is pictorially depicted the metric of (6.28): note the singularity of the vielbein in $r = 0$, giving rise to the detachment of the principal $SO(4)$ bundle from the tangent bundle.

Chapter 7

The SU(2) x R Instanton and its associated CFT

Let us now consider the limit $A \to 0$ of the configuration (6.28):

$$\begin{aligned} ds^2 &= e^{-2\Phi}(dx)^2 & \qquad & V^a = e^{-\Phi} e^a \\ e^{-2\Phi} &= \frac{2k}{r^2} & \leftrightarrow \qquad & \Phi = \log \frac{r}{\sqrt{2k}} \end{aligned} \tag{7.1}$$

$$H_{abc} = \frac{1}{3}\, \epsilon_{abcd}\, \partial_d \Phi$$

In this limit the manifold has the curious and somewhat unwanted topology of $S^3 \times \mathrm{R} \approx SU(2) \times \mathrm{R}$, which is not asymptotically flat. Asymptotic flatness is instead ensured when A is non vanishing.

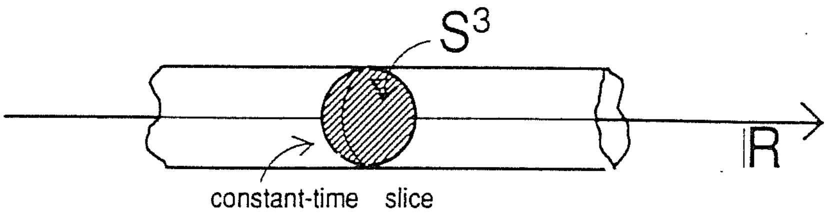


Fig. 7.1 The $S^3 \times \mathrm{R}$ topology of (7.1) rensembling an infinite "tube"

Yet as we are going to see at $A = 0$ the corresponding σ-model defines a solvable conformal and superconformal field-theory. Hence this limit is quite worth to be considered. In (7.1) $\{e^a\}$ is a set of vielbeins for the flat 4-dimensional space, r being a radial coordinate and the remaining 3 coordinates being the coordinates of a 3-sphere. Indeed we choose to write the flat metric as follows

$$dx^2 = dr^2 + \frac{r^2}{2}\Omega^i \otimes \Omega^i$$

where Ω^i are the Maurer-Cartan forms of $SU(2)$ which satisfy the equations

$$d\Omega^i = -\frac{\epsilon^{ijk}}{\sqrt{2}}\Omega^j \Omega^k$$

so that $\frac{1}{2}\Omega^i \otimes \Omega^i$ is the metric on the three-sphere of unit radius. The metric of the configuration (7.1) becomes

$$ds^2 = 2k\frac{dr^2}{r^2} + k\,\Omega^i \otimes \Omega^i \tag{7.2}$$

Redefining the radial coordinate as follows: $t = \sqrt{2k}\log(r/\sqrt{2k})$ we obtain:

$$ds^2 = dt^2 + k\,\Omega^i \otimes \Omega^i \tag{7.3}$$

(showing that the singularity in (7.2) is a coordinate artifact), while the dilaton is linear in the coordinate t:

$$\Phi = \frac{t}{\sqrt{2k}} \tag{7.4}$$

In correspondence with eq.(7.3) we choose the vielbeins as follows:

$$\begin{aligned} V^0 &= dt \\ dV^i &= -\sqrt{k}\Omega^i \end{aligned} \tag{7.5}$$

The only non-zero components of H in the Maurer-Cartan basis $\{\Omega^i\}$ turn out to be

$$H_{ijk} = \frac{1}{3}\epsilon_{ijk}(-\sqrt{k})^3\frac{\partial\Phi}{\partial t} = -\frac{k}{3}\frac{\epsilon_{ijk}}{\sqrt{2}} \tag{7.6}$$

(note that, with our choice of Maurer-Cartan forms , $\sqrt{2}\epsilon_{ijk}$ are just the structure constants of $SU(2)$).

7.1 The classical Sigma-model

Consider first of all the bosonic σ-model . The action corrisponding to the configuration we have described is $S = S_t + S_{\mathrm{WZW}}$, where

$$\begin{aligned} S_t &= \frac{1}{4\pi}\int_{\partial\mathcal{M}} dt\,(\Pi_+ e^+ - \Pi_- e^-) + \Pi_+\Pi_- e^+ e^- - \sqrt{\frac{2}{k}}tR^{(2)} + p_+T^+ + p_-T^- \\ S_{\mathrm{WZW}} &= \frac{1}{4\pi}\int_{\partial\mathcal{M}} -\sqrt{k}\Omega^i(\Pi^i_+ e^+ - \Pi^i_- e^-) + \Pi^i_+\Pi^i_- e^+ e^- - \frac{1}{3}\frac{k}{4\pi}\int_{\mathcal{M}}\frac{\epsilon_{ijk}}{\sqrt{2}}\Omega^i\,\Omega^j\,\Omega^k \end{aligned} \tag{7.7}$$

Once rewritten in 2^{nd} order formalism, these two actions take the simpler form

$$
\begin{aligned}
S_t &= \frac{-1}{4\pi}\int_{\partial\mathcal{M}} dz d\bar{z}\ \partial t \bar{\partial} t \\
S_{\mathrm{WZW}} &= \frac{-k}{4\pi}\int_{\partial\mathcal{M}} dz d\bar{z}\ \Omega^i{}_+ \Omega^i{}_- - \frac{1}{3}\frac{k}{4\pi}\int_{\mathcal{M}} \frac{\epsilon_{ijk}}{\sqrt{2}} \Omega^i \Omega^j \Omega^k
\end{aligned}
\tag{7.8}
$$

S_{WZW} is the correct expression for the action of the WZW model realized at level k, and corresponds to a CFT of central charge

$$
c_{\mathrm{WZW}} = \frac{3k}{k+2} \tag{7.9}
$$

The field $\varphi = -it$ is a free scalar boson with background charge $Q_{bk} = -i\sqrt{\frac{2}{k}}$. Indeed from the action (7.7), using the general recipe provided by eq. (2.8), we obtain the following stress-energy tensor:

$$
T_t(z) = -\frac{1}{2}(\partial t)^2 - \frac{1}{\sqrt{2k}}\partial\partial t \tag{7.10}
$$

which corresponds to a central charge

$$
c_t = 1 + \frac{6}{k} \tag{7.11}
$$

As one sees the σ-model on the configuration (7.1) is exactly conformal invariant at the quantum level and leads to a solvable conformal field theory: namely a tensor product of a Feigin-Fuchs model with a WZW-model (see ref.s [27,28,29]).

We can proceed a step further and analyze the (1,1) supersymmetric σ-model on this background, showing that indeed the conditions to admit (4,4) supersymmetries are matched.

Using the vierbein (7.5), we can write the Maurer-Cartan equations of the group-manifold $SU(2) \times \mathrm{R}$ as follows:

$$
dV^a = \frac{1}{2}\sqrt{\frac{2}{k}} f_{abc} V^b V^c \qquad a = 1,2,3,0 \tag{7.12}
$$

where the totally antisymmetric structure constants f_{abc} are given by

$$
\begin{aligned}
f_{0ab} &= 0 \\
f_{ijk} &= \epsilon_{ijk}
\end{aligned}
\tag{7.13}
$$

With these notations, an $SU(2) \times \mathrm{R}$ element in the adjoint representation is given by the 4×4 matrix

$$
\Gamma_{ab} = \begin{pmatrix} \Gamma_{ij} & 0 \\ 0 & 1 \end{pmatrix} \tag{7.14}
$$

the 3×3 submatrix Γ_{ij} being an $SU(2)$ element in its own adjoint representation. As such the matrix Γ has the properties that

$$
\begin{aligned}
\Gamma^T \Gamma &= \mathbf{1} \\
(\Gamma^T d\Gamma)_{ab} &= \sqrt{\frac{2}{k}} f_{abc} V^c
\end{aligned}
\tag{7.15}
$$

In 2^{nd} order formalism the action (2.29) for the (1,1)-supersymmetric σ-model on a generic manifold is written as follows:

$$
\begin{aligned}
S = \frac{-1}{4\pi} \int_{\partial \mathcal{M}} dz\, d\bar{z} \left\{ V_z^a V_{\bar{z}}^a + 2i\lambda^a \overset{-}{\nabla}_z \lambda^a - 2i\mu^a \overset{+}{\nabla}_{\bar{z}} \mu^a - 4R^{ab}{}_{cd} \lambda^a \lambda^b \mu^c \mu^d \right\} + \\
+ \frac{1}{4\pi} \int_{\mathcal{M}} H
\end{aligned}
\tag{7.16}
$$

For our particular background, as for any other group-manifold, this expression simplifies, due to the existence of two non-Riemannian spin-connections (the "zero" and the "one" connection in Cartan terminology [15]) that are proportional to the structure constants and that parallelize the manifold. These two connections coincide exactly with the ω^- and ω^+ discussed in the previous section. Indeed, utilizing the expression of the torsion that follows from eq.(7.12), we find:

$$
\begin{aligned}
\omega^-_{ab} &= 0 \\
\omega^+_{ab} &= \sqrt{\frac{2}{k}} f_{abc} V^c
\end{aligned}
\tag{7.17}
$$

so that

$$
R^-_{ab} = R^+_{ab} = 0 \tag{7.18}
$$

The "minus" covariant derivative is just an ordinary derivative, so the fermions λ^a are just free left-moving fermions. The μ^a, instead, are neither free nor right-moving. However we can rewrite the action in terms of right-moving quantities, using the 1-forms $\tilde{V}^a = \Gamma^{ab} V^b$, that provide an alternative set of vielbein for our manifold. They are given (compare with eq.(7.5)) by:

$$
\begin{aligned}
\tilde{V}^0 &= dt \\
\tilde{V}^i &= -\sqrt{k}\, \tilde{\Omega}^i
\end{aligned}
\tag{7.19}
$$

where the forms $\tilde{\Omega}^i$ are the components, along a Lie-algebra basis, of the right-invariant form on the group manifold: $\tilde{\Omega} = dg\, g^{-1}$. We expand these right-moving vielbeins on the superworld-sheet as follows:

$$
\tilde{V}^a = \tilde{V}^a_+ e^+ + \tilde{V}^a_- e^- + \tilde{\lambda}^a e^+ + \tilde{\mu}^a e^-
$$

Relying on the relation

$$
\tilde{\mu}^a = \Gamma^{ab} \mu^b, \tag{7.20}
$$

on the definition of $\stackrel{+}{\nabla}$ and on the properties (7.15) of the adjoint matrix we find that

$$-2i\mu^a \stackrel{+}{\nabla}_z \mu^a = -2i\tilde{\mu}^a\bar{\partial}\tilde{\mu}^a$$

Hence for group-manifolds the action (7.16) can be rewritten in such a way that involves only free fermions:

$$S = \frac{-1}{4\pi}\int_{\partial\mathcal{M}} dz\,d\bar{z}\left\{V^a_z V^a_{\bar{z}} + 2i\lambda^a\partial\lambda^a - 2i\tilde{\mu}^a\bar{\partial}\tilde{\mu}^a\right\} + \frac{1}{24\pi}\sqrt{\frac{2}{k}}\int_{\mathcal{M}} f_{abc}V^aV^bV^c \quad (7.21)$$

On the four-dimensional group-manifold $SU(2)\times \mathrm{R}$ it is now easy to show that the conditions for (4,4) supersymmetry are matched. Due to the vanishing of the ω^- connection, the set of complex structures $\mathcal{J}^x$ must be constant, and we can choose them to coincide with the $\hat{\mathcal{J}}^x$ of eqs.(1-9):

$$\mathcal{J}^x = \hat{\mathcal{J}}^x \quad (7.22)$$

The complex structures $\stackrel{+}{\mathcal{J}}{}^x$, that commute with the previous set and are covariantly constant with respect to ω^+ connection, are given by

$$\stackrel{+}{\mathcal{J}}{}^x = \Gamma^T\tilde{\mathcal{J}}^x\Gamma \quad (7.23)$$

This easily follows from the properties of the adjoint matrix. Substituting eqs.(7.22-7.23) into the general expressions (2.56), we can write down the explicit classical expression of the supercurrents in the case of the $SU(2)\times\mathrm{R}$ background.
Before doing this, we find it convenient to reformulate the theory in terms of free fermions $\psi^a(z), \tilde{\psi}^a(\bar{z})$ that satisfy the standard OPEs:

$$\psi^a(z)\psi^b(w) = -\frac{1}{2}\frac{\delta^{ab}}{z-w} \quad (7.24)$$

(and the same for the $\tilde{\psi}^a(\bar{z})$). This involves a simple renormalization of the original free fermions. Indeed from the classical Dirac brackets of the fields λ^a and the $\tilde{\mu}^a$, that translate into their quantum OPEs, we have:

$$\lambda^a(z)\lambda^b(z) = -\frac{i}{2}\frac{\delta^{ab}}{z-w}$$
$$\tilde{\mu}^a(\bar{z})\tilde{\mu}^b(\bar{z}) = \frac{i}{2}\frac{\delta^{ab}}{z-w}$$

Hence it suffices to set:

$$\lambda^a = e^{i\pi/4}\psi^a \quad ; \quad \tilde{\mu}^a = e^{i3\pi/4}\tilde{\psi}^a \quad (7.25)$$

For the left supercurrents, recalling the form of the dilaton, (eq.(7.4)), which implies that

$$\partial_a\Phi = \frac{\delta_{a0}}{\sqrt{2k}}$$

we immediately obtain the following expressions

$$
\begin{aligned}
G^0(z) &= \sqrt{2}\left\{\psi^a V_z^a + \frac{1}{3}\sqrt{\frac{2}{k}}\,\epsilon_{ijk}\,\psi^i\psi^j\psi^k + \sqrt{\frac{2}{k}}\partial\psi^0\right\} \\
G^x(z) &= \sqrt{2}\left\{(\hat{\mathcal{J}}^x\psi)^a V_z^a + \sqrt{\frac{2}{k}}\,\epsilon_{ijk}\,(\hat{\mathcal{J}}^x\psi)^i\psi^j\psi^k + \sqrt{\frac{2}{k}}\partial(\hat{\mathcal{J}}^x\psi)^0\right\}
\end{aligned}
\tag{7.26}
$$

For the right supercurrents, we must, first of all, give their expression in terms of right-moving quantities. To this purpose it suffices to make use of the properties (7.15) of the adjoint matrix and of the additional one

$$
f_{abc}\Gamma_{ar}\Gamma_{bs}\Gamma_{ct} = f_{rst} \tag{7.27}
$$

corresponding to the invariance of the group structure constants. These properties imply

$$
\begin{aligned}
\mu^a V_{\bar{z}}^a = \tilde{\mu}^a \tilde{V}_{\bar{z}}^a \qquad &; \qquad (\overset{+}{\mathcal{J}}{}^x\mu)^a V_{\bar{z}}^a = (\tilde{\mathcal{J}}^x\tilde{\mu})^a\tilde{V}_{\bar{z}}^a \\
\epsilon_{ijk}\,\mu^i\mu^j\mu^k = \epsilon_{ijk}\,\tilde{\mu}^i\tilde{\mu}^j\tilde{\mu}^k \qquad &; \qquad \epsilon_{ijk}\,(\overset{+}{\mathcal{J}}{}^x\mu)^i\mu^j\mu^k = \epsilon_{ijk}\,(\tilde{\mathcal{J}}^x\tilde{\mu})^i\tilde{\mu}^j\tilde{\mu}^k \\
(\overset{+}{\mathcal{J}}{}^x\mu)^0 &= (\tilde{\mathcal{J}}^x\tilde{\mu})^0
\end{aligned}
$$

so that, in our case, from eq.(5.31) we obtain, in terms of the fermions $\tilde{\psi}^a$:

$$
\begin{aligned}
\widetilde{G}^0(\bar{z}) &= \sqrt{2}\left\{\tilde{\psi}^a \tilde{V}_{\bar{z}}^a - \frac{1}{3}\sqrt{\frac{2}{k}}\,\epsilon_{ijk}\,\tilde{\psi}^i\tilde{\psi}^j\tilde{\psi}^k + \sqrt{\frac{2}{k}}\bar{\partial}\tilde{\psi}^0\right\} \\
\widetilde{G}^x(\bar{z}) &= \sqrt{2}\left\{(\tilde{\mathcal{J}}^x\tilde{\psi})^a \tilde{V}_{\bar{z}}^a - \sqrt{\frac{2}{k}}\,\epsilon_{ijk}\,(\tilde{\mathcal{J}}^x\tilde{\psi})^i\tilde{\psi}^j\tilde{\psi}^k + \sqrt{\frac{2}{k}}\bar{\partial}(\tilde{\mathcal{J}}^x\tilde{\psi})^0\right\}
\end{aligned}
\tag{7.28}
$$

7.2 Quantization and abstract Conformal Field-Theory

In the case of supersymmetric WZW models [30], the analysis of extended global SUSY can be also performed in purely algebraic terms; a complex structures is in one-to-one corrispondence with a Cartan decomposition of the Lie algebra. The group $SU(2)\times U(1)$ (this is our case) has actually three complex structures and so $N{=}4$ SUSY follows. We arrive at this algebraic description by quantizing our theory.

The quantization of the supersymmetric WZW on any group manifold and in particular on $SU(2)\times U(1)$ is straightforward [15,16]. Focusing on the left sector (we write the formulas for the right sector only when some difference is present) and using the currents J^a such that

$$
\partial t = -i\sqrt{2}J^0 \tag{7.29a}
$$

$$
\Omega^i = \frac{i\sqrt{2}}{k}J^i \tag{7.29b}
$$

we find, as result of a standard procedure,

$$J^i(z)J^j(w) = \frac{k}{2}\frac{\delta^{ij}}{(z-w)^2} + \frac{i\epsilon^{ijk}J^k}{z-w} \tag{7.30a}$$

$$J^0(z)J^0(w) = \frac{1}{2(z-w)^2} \tag{7.30b}$$

We will use also the notation $j^a = (J^0, \frac{J^i}{\sqrt{k+2}})$.
The correct quantum expression for the stress-energy tensor includes the Sugawara form for the level k $SU(2)$ WZW model, and is given by

$$T(z) = J^0J^0 + \frac{1}{k+2}J^iJ^i + \frac{i}{\sqrt{k+2}}\partial J^0 + \psi^a\partial\psi^a \tag{7.31}$$

Comparing this expression to eq.(7.10) we see that at quantum level a shift $k \to k+2$ is necessary in the background charge term. This shift of two unities in the value of k can be understood in the following way.
The term responsible for the background charge couples to the supersymmetrized version of the WZW-model at level k. From a purely algebraic point of view it is well known that a super Kac-Moody algebra of level k corresponds to an ordinary bosonic Kac-Moody algebra of level $k - C_V$ (where C_V is the value of the quadratic Casimir) plus a set of free fermions having regular OPEs with the Kac-Moody currents. The shift in k is due to this fact: the relevant value of k for the computation of the background charge is the central charge of the super Kac-Moody currents:

$$j^a_{super} \;=\; j^a \;+\; const.\; f^{abc}\,\psi^b\,\psi^c$$

and not the central charge of the Kac-Moody currents j^a.
The central charge attribuited to the Feigin-Fuchs boson t is shifted to $c_{FF} = 1+6/(k+2)$, the only value for which the total central charge sums up to 6, the correct one for a four dimensional supersymmetric solution:

$$c = c_{FF} + c_{WZW} + c_{ff} = 1 + \frac{6}{k+2} + \frac{3k}{k+2} + 2 = 6 \tag{7.32}$$

where c_{WZW} is the ordinary central charge of the bosonic $SU(2)$ WZW at level k and c_{ff}=2 is the contribution of the four free fermions.
In other words we have a $(6,6)_{4,4}$ in agreement with the general set up of section 2. Note that the dilaton, not necessary to obtain N=4 supersymmetry at the classical level, is essential at the quantum level to fix the central charge to its correct value.
The quantum expressions of the supercurrents (the classical ones were given in eq.(7.26-7.28)) are:

$$\begin{aligned} G^0(z) &= 2\left\{-ij^a\psi^a + \frac{1}{3}\frac{\epsilon_{ijk}}{\sqrt{k+2}}\psi^i\psi^j\psi^k + \frac{\partial\psi^0}{\sqrt{k+2}}\right\} \\ G^x(z) &= 2\left\{-ij^a\hat{\mathcal{J}}^x_{ab}\psi^b + \frac{\epsilon_{ijk}}{\sqrt{k+2}}(\hat{\mathcal{J}}^x\psi)^i\psi^j\psi^k + \frac{\partial(\hat{\mathcal{J}}^x\psi)^0}{\sqrt{k+2}}\right\} \end{aligned} \tag{7.33a}$$

$$\widetilde{G}^0(z) = 2\left\{ -i\tilde{j}^a\tilde{\psi}^a - \frac{1}{3}\frac{\epsilon_{ijk}}{\sqrt{k+2}}\tilde{\psi}^i\tilde{\psi}^j\tilde{\psi}^k + \frac{\partial\tilde{\psi}^0}{\sqrt{k+2}} \right\}$$

$$\widetilde{G}^x(z) = 2\left\{ -i\tilde{j}^a\tilde{\mathcal{J}}^x_{ab}\tilde{\psi}^b - \frac{\epsilon_{ijk}}{\sqrt{k+2}}(\tilde{\mathcal{J}}^x\tilde{\psi})^i\tilde{\psi}^j\tilde{\psi}^k + \frac{\partial(\tilde{\mathcal{J}}^x\tilde{\psi})^0}{\sqrt{k+2}} \right\} \tag{7.33b}$$

Without the dilatonic contributions (the last terms in the above eqs.), as already stressed, N=4 symmetry would be still present, but the supercurrents would not close the standard algebra; they would rather close the so called N=4 extended algebra [6], based on the Kac-Moody algebra of $SU(2)\times SU(2)\times U(1)$. The canonical way to reduce this extended algebra to the standard one is to add a background charge with a particular value. The solution we are considering automatically performs this reduction, assigning the needed background charge to the field t.

The $SU(2)_1$ currents of the two sectors are realized entirely in terms of free fermions:

$$A^i(z) = -\frac{i}{2}\psi^a\hat{\mathcal{J}}^i_{ab}\psi^b = i(\psi^0\psi^i + \tfrac{1}{2}\epsilon^{ijk}\psi^j\psi^k)$$

$$\tilde{A}^i(z) = -\frac{i}{2}\tilde{\psi}^a\tilde{\mathcal{J}}^i_{ab}\tilde{\psi}^b = -i(\tilde{\psi}^0\tilde{\psi}^i - \tfrac{1}{2}\epsilon^{ijk}\tilde{\psi}^j\tilde{\psi}^k) \tag{7.34}$$

i.e. they have the same expression as for the flat space (see eq.(2.44c)), except that, due to the non-vanishing torsion we are forced to use the two different sets of complex structures in the two sectors. The supercurrents $\mathcal{G}^a$, $\overline{\mathcal{G}}^a = (\mathcal{G}^a)^*$, organized in $SU(2)$ doublets as dictated by the above OPEs, are given by

$$\mathcal{G} = \frac{1}{\sqrt{2}}\begin{pmatrix} G^0 - iG^3 \\ -(G^2 + iG^1) \end{pmatrix} \quad ; \quad \overline{\mathcal{G}} = \frac{1}{\sqrt{2}}\begin{pmatrix} G^0 + iG^3 \\ -(G^2 - iG^1) \end{pmatrix} \tag{7.35}$$

for the left sector, and by the same tilded expressions in the right one.
Subsituting the explicit form of the tensors $\hat{\mathcal{J}}^x, \tilde{\mathcal{J}}^x$ into eqs.(7.33) we get

$$\begin{aligned}
G^0 &= 2\left[-iJ^0\psi^0 - \frac{i}{\sqrt{k+2}}J^i\psi^i + \frac{2}{\sqrt{k+2}}\psi^1\psi^2\psi^3 + \frac{\partial\psi^0}{\sqrt{k+2}}\right]\\
G^1 &= 2\left[iJ^0\psi^1 - \frac{i}{\sqrt{k+2}}(J^1\psi^0 - J^2\psi^3 + J^3\psi^2) + \frac{2}{\sqrt{k+2}}\psi^0\psi^2\psi^3 - \frac{\partial\psi^1}{\sqrt{k+2}}\right]\\
G^2 &= 2\left[iJ^0\psi^2 - \frac{i}{\sqrt{k+2}}(J^1\psi^3 + J^2\psi^0 - J^3\psi^1) + \frac{2}{\sqrt{k+2}}\psi^1\psi^0\psi^3 - \frac{\partial\psi^2}{\sqrt{k+2}}\right]\\
G^3 &= 2\left[iJ^0\psi^3 - \frac{i}{\sqrt{k+2}}(-J^1\psi^2 + J^2\psi^1 + J^3\psi^0) + \frac{2}{\sqrt{k+2}}\psi^1\psi^2\psi^0 - \frac{\partial\psi^3}{\sqrt{k+2}}\right]
\end{aligned} \tag{7.36}$$

while the $\tilde{G}$ have analogous but slightly different expressions.
The doublets of supercurrents can be written as

$$\begin{aligned}
\mathcal{G} =& \left[-i\sqrt{2}(j^0 - ij^3) + 2\sqrt{\frac{2}{k+2}}i(\psi^0\psi^3 - \psi^1\psi^2) + \sqrt{\frac{2}{k+2}}\partial\right]\begin{pmatrix}\psi^0 + i\psi^3\\ \psi^2 + i\psi^1\end{pmatrix} + \\
& - i\sqrt{2}(j^2 + ij^1)\begin{pmatrix}\psi^2 - i\psi^1\\ -(\psi^0 - i\psi^3)\end{pmatrix}
\end{aligned} \tag{7.37a}$$

$$\overline{\mathcal{G}} = (\mathcal{G})^*$$

and by

$$\begin{aligned}\widetilde{\mathcal{G}} =& \left[i\sqrt{2}(\tilde{j}^0 + i\tilde{j}^3) + 2\sqrt{\frac{2}{k+2}}i(\tilde{\psi}^0\tilde{\psi}^3 + \tilde{\psi}^1\tilde{\psi}^2) - \sqrt{\frac{2}{k+2}}\bar{\partial}\right]\begin{pmatrix}-(\tilde{\psi}^0 - i\tilde{\psi}^3)\\ \tilde{\psi}^2 + i\tilde{\psi}^1\end{pmatrix} + \\ & - i\sqrt{2}(\tilde{j}^2 + i\tilde{j}^1)\begin{pmatrix}\tilde{\psi}^2 - i\tilde{\psi}^1\\ \tilde{\psi}^0 + i\tilde{\psi}^3)\end{pmatrix} \\ \widetilde{\widetilde{\mathcal{G}}} =& (\widetilde{\mathcal{G}})^*\end{aligned} \tag{7.37b}$$

The relevant point is that we can easily obtain now the explicit form of the moduli operators for the conformal field theory we have just described. We need primary fields of dimension one which are the same time last components of an N=4 representation, namely we have to find solutions to the OPEs (2.43). Remarkably, in our case the solution of these OPEs is very similar in form to the solution (2.45) one obtains in the flat space case. Indeed consider the SU(2) doublets:

$$\Psi_1(z) = e^{-\sqrt{\frac{2}{k+2}}t}\begin{pmatrix}\psi^0 + i\psi^3\\ \psi^2 + i\psi^1\end{pmatrix}\ ;\ \Psi_2(z) = e^{-\sqrt{\frac{2}{k+2}}t}\begin{pmatrix}\psi^2 - i\psi^1\\ -(\psi^0 - i\psi^3)\end{pmatrix} \tag{7.38a}$$

$$\tilde{\Psi}_1(\bar{z}) = e^{-\sqrt{\frac{2}{k+2}}\tilde{t}}\begin{pmatrix}-(\tilde{\psi}^0 - i\tilde{\psi}^3)\\ \tilde{\psi}^2 + i\tilde{\psi}^1\end{pmatrix}\ ;\ \tilde{\Psi}_2(\bar{z}) = e^{-\sqrt{\frac{2}{k+2}}\tilde{t}}\begin{pmatrix}\tilde{\psi}^2 - i\tilde{\psi}^1\\ \tilde{\psi}^0 + i\tilde{\psi}^3)\end{pmatrix} \tag{7.38b}$$

These operators satisfy eq.s (2.43) with as last components the operators

$$\begin{aligned}\Phi_1(z) &= e^{-\sqrt{\frac{2}{k+2}}t}\left\{i\sqrt{2}(j^2 + ij^1) + 2\sqrt{\frac{2}{k+2}}(\psi^0 + i\psi^3)(\psi^2 + i\psi^1)\right\} \\ \Pi_1(z) &= e^{-\sqrt{\frac{2}{k+2}}t}\left\{i\sqrt{2}(j^0 + ij^3) + 2i\sqrt{\frac{2}{k+2}}(\psi^0\psi^3 - \psi^1\psi^2)\right\} \\ \Phi_2(z) &= e^{-\sqrt{\frac{2}{k+2}}t}\left\{-i\sqrt{2}(j^0 - ij^3) + 2i\sqrt{\frac{2}{k+2}}(\psi^0\psi^3 - \psi^1\psi^2)\right\} \\ \Pi_2(z) &= e^{-\sqrt{\frac{2}{k+2}}t}\left\{i\sqrt{2}(j^2 - ij^1) + 2\sqrt{\frac{2}{k+2}}(\psi^0 - i\psi^3)(\psi^2 - i\psi^1)\right\}\end{aligned} \tag{7.39a}$$

and

$$\begin{aligned}\tilde{\Phi}_1(z) &= e^{-\sqrt{\frac{2}{k+2}}\tilde{t}}\left\{i\sqrt{2}(\tilde{j}^2 + i\tilde{j}^1) + 2\sqrt{\frac{2}{k+2}}(\tilde{\psi}^0 - i\tilde{\psi}^3)(\tilde{\psi}^2 + i\tilde{\psi}^1)\right\} \\ \tilde{\Pi}_1(z) &= e^{-\sqrt{\frac{2}{k+2}}\tilde{t}}\left\{i\sqrt{2}(\tilde{j}^0 - i\tilde{j}^3) + 2i\sqrt{\frac{2}{k+2}}(\tilde{\psi}^0\tilde{\psi}^3 + \tilde{\psi}^1\tilde{\psi}^2)\right\} \\ \tilde{\Phi}_2(z) &= e^{-\sqrt{\frac{2}{k+2}}\tilde{t}}\left\{i\sqrt{2}(\tilde{j}^0 + i\tilde{j}^3) + 2i\sqrt{\frac{2}{k+2}}(\tilde{\psi}^0\tilde{\psi}^3 + \tilde{\psi}^1\tilde{\psi}^2)\right\}\end{aligned}$$

$$\tilde{\Pi}_2(z) = e^{-\sqrt{\frac{2}{k+2}}\tilde{t}} \left\{ i\sqrt{2}(\tilde{j}^2 - i\tilde{j}^1) + 2\sqrt{\frac{2}{k+2}}(\tilde{\psi}^0 + i\tilde{\psi}^3)(\tilde{\psi}^2 - i\tilde{\psi}^1) \right\} \tag{7.39b}$$

Note that, as expected from the purely fermionic form of the currents of $SU(2)$, the doublets are quite completely expressed in terms of the free fermions, the exponential term being only needed to cancel some unwanted poles. We stress that, due to the existence of the background charge, the operator of the F.F. theory

$$: e^{-\sqrt{\frac{2}{k+2}}t} : \tag{7.40}$$

has conformal dimension zero. Indeed in a F.F. theory with stress-energy tensor

$$T(z) = -\tfrac{1}{2}\partial t \partial t - \frac{i}{2} Q_{bk} \partial^2 t$$

the vertex operators : $\exp(i\alpha t)$: have a conformal weight $\Delta_\alpha = \frac{1}{2}\alpha(\alpha + Q_{bk})$ and in our case $Q_{bk} = -i\sqrt{\frac{2}{k+2}}$.

This factor is the counterpart of the plane-wave factor $e^{[i\, k \cdot X(z,\bar{z})]}$ appearing in the flat space case. Also there the exponential factor has conformal weight zero since $k^2 = 0$. Indeed we can say that $k_0 = \sqrt{\frac{2}{k+2}}$ is the energy component of the four-momentum. It is fixed to a constant value in terms of the space-like components $\mathbf{k}$. The difference resides in that $\mathbf{k}$ is a continuos variable for flat space, while its analogue is quantized to fixed values for the $SU(2) \times \mathrm{R}$ background, namely there is a finite number of zero-mode operators rather then a continuous infinity as in flat-space. This difference follows from the different topology of the constant-time slices in the two cases: noncompact $\mathbf{R}^3$ for flat-space, compact S^3 for the case under consideration.

The four fields Φ_a, Π_a, $a = 1, 2$ are the moduli of our conformal theory. Combining left and right fields, we find 16 infinitesimal deformations of our theory that preserve the N=4 superconformal algebra. These combinations are formally the same as the combinations (3.9). Moreover it is possible to construct two abstract (0,1)-forms $\Psi^\star_{\mathcal{A}^\star} \left[{}^{0,\frac{1}{2}}_{0,\frac{1}{2}}\right]$, analogously to eq.(3.11).
As an abstract $(6,6)_{4,4}$ -theory the $SU(2) \times \mathrm{R}$ background has the same Hodge-diamond as flat space (compare with eq.(2.46)). However since the torsion is different from zero, these abstract Hodge numbers are not the usual ones of compactified version of the underlying manifold $S^1 \times S^3$, whose Betti numbers

$$b^0 = 1\ ,\ b^1 = 1\ ,\ b^2 = 0\ ,\ b^3 = 1\ ,\ b^4 = 1$$

are obviously incompatible with such an Hodge decomposition.

Chapter 8

Deformations of the Target Space Geometry in the SU(2) x R case

The existence of non-trivial $N=4$ moduli implies that the geometrical data of the σ-model , namely its metric $g_{\mu\nu}$ and torsion (related to the axion $B_{\mu\nu}$) can be deformed in such a way as to mantain $N=4$ supersymmetry. In other words the existence of $h^{1,1}$ moduli implies that the generalized HyperKähler manifold we have considered is just an element in a continuous family of generalized HyperKähler manifolds, parametrized by 4 $h^{1,1}$ parameters. For instance in the case of the K_3 manifold the existence of 20 $N=4$ moduli follows from the fact that, as an algebraic surface, K_3 is described by a homogeneous equation with 19 nontrivial complex coefficients fixing the complex structure and that, for fixed complex structure, we still have a one parameter family of deformations for the Kähler class. These deformations of the metric and of the torsion fill an 80-dimensional moduli space whose global structure turns out to be $\mathcal{M}_{K_3} = SO(4,20)/SO(4) \times SO(20)/SO(4,20;\mathbf{Z})$. In a similar way flat space has four $N=4$ moduli because the constant metrics and constant torsions fills a space of dimension 16, namely the space of all 4×4 matrices (the symmetric parts is the metric, the antisymmetric part is the axion).

For the limit case of the $SU(2) \times \mathrm{R}$ instanton we have discovered from the algebraic approach that there are four $N=4$ moduli just as for flat space. Their geometrical interpretation, however, is less clear. In this section we explore the consequences of the $N=4$ moduli on the geometry of the target space. Namely we calculate the explicit form of the infinitesimal deformations of the metric and of the torsion due to these moduli. We show that the deformed space is still generalized HyperKähler as expected: the curvatures of ω^+ and ω^- are no longer zero but still self-dual (respectively antiselfdual) after the deformation and there exist deformed complex structures fulfilling all the requirements. A global characterization of this space of metrics and torsions is still an open and interesting problem. Let us discuss the infinitesimal deformations obtained by inserting the moduli operators in the σ-model Lagrangian.

We focus on the bosonic sector which suffices to give us informations about the new metric, new torsion and new complex structures. The bosonic parts of the moduli, reshifting the background charge to its classical value, are expressed, for the left sector, in terms of the components of the left-moving vielbeins:

$$\begin{aligned} \Phi_1(z) &= (V_z^2 + iV_z^1)e^{-\sqrt{\frac{2}{k}}t} & \Phi_2(z) &= (V_z^0 + iV_z^3)e^{-\sqrt{\frac{2}{k}}t} \\ \Pi_1(z) &= -(V_z^0 - iV_z^3)e^{-\sqrt{\frac{2}{k}}t} & \Pi_2(z) &= (V_z^2 - iV_z^1)e^{-\sqrt{\frac{2}{k}}t} \end{aligned} \tag{8.1a}$$

and for the right sector, in terms of the right-moving ones:

$$\begin{aligned} \tilde{\Phi}_1(\bar{z}) &= (\tilde{V}_{\bar{z}}^2 + i\tilde{V}_{\bar{z}}^1)e^{-\sqrt{\frac{2}{k}}t} & \tilde{\Phi}_2(\bar{z}) &= -(\tilde{V}_{\bar{z}}^0 - i\tilde{V}_{\bar{z}}^3)e^{-\sqrt{\frac{2}{k}}\tilde{t}} \\ \tilde{\Pi}_1(\bar{z}) &= (\tilde{V}_{\bar{z}}^0 + i\tilde{V}_{\bar{z}}^3)e^{-\sqrt{\frac{2}{k}}t} & \tilde{\Pi}_2(\bar{z}) &= (\tilde{V}_{\bar{z}}^2 - i\tilde{V}_{\bar{z}}^1)e^{-\sqrt{\frac{2}{k}}\tilde{t}} \end{aligned} \tag{8.1b}$$

Now we can construct conformal operators of weights $(1,1)$ to insert into the Lagrangian combining these $(1,0)$ and $(0,1)$ ones in all possible ways. Hence the most general expression we can add to the Lagrangian is simply

$$e^{-\sqrt{\frac{2}{k}}t(z,\bar{z})}V_z^a\, M_{ab}\, \tilde{V}_{\bar{z}}^b \tag{8.2}$$

M_{ab} being a constant matrix. The reality condition for this expression imposes $M \in GL(4,\mathrm{R})$. Thus our deformations depend on 16 real parameters as anticipated from the abstract counting.
In terms of the components of the undeformed vielbein, which we have chosen to be the left-moving ones, the term in (8.2) has the form:

$$e^{-\sqrt{\frac{2}{k}}t}V_z^a(M\Gamma)_{ab}V_{\bar{z}}^b \tag{8.3}$$

Γ being the variable $SU(2)\times\mathrm{R}$ element (point in the manifold) in the adjoint representation.
It is useful to separate the symmetric and antisymmetric part of the matrix $M\Gamma$ and to this purpose we introduce the notation

$$\begin{aligned} h_{ab} &= \frac{1}{2}e^{-\sqrt{\frac{2}{k}}t}(M\Gamma + \Gamma^T M^T)_{ab} \\ b_{ab} &= -\frac{1}{2}e^{-\sqrt{\frac{2}{k}}t}(M\Gamma - \Gamma^T M^T)_{ab} \end{aligned} \tag{8.4}$$

The overall normalization of the new term is of course irrelevant, since M is arbitrary, and we choose it in such a way that the bosonic part of the deformed σ-model action is:

$$S = \frac{-1}{4\pi}\int_{\partial\mathcal{M}} dz d\bar{z}\left\{V_z^a V_{\bar{z}}^a + 2V_z^a h_{ab} V_{\bar{z}}^b - 2V_z^a b_{ab} V_{\bar{z}}^b\right\} + \frac{1}{4\pi}\int_{\mathcal{M}} H \tag{8.5}$$

The torsion deformation (parametrized by the antisymmetric matrix b) can be recast in a shift of the 3-form H:

$$S = \frac{-1}{4\pi}\int_{\partial\mathcal{M}} dz d\bar{z}\left\{V_z^a V_{\bar{z}}^a + 2V_z^a h_{ab} V_{\bar{z}}^b\right\} + \frac{1}{4\pi}\int_{\mathcal{M}}(H + \delta H) \tag{8.6}$$

where $\delta H = dB$ with $B = b_{ab}V^aV^b$.
Following [31] it is also convenient to use the combinations:

$$\begin{aligned} G^+_{ab} &= h_{ab} + b_{ab} = e^{-\sqrt{\frac{2}{k}}t}(\Gamma^T M^T)_{ab} = (G^-)^T_{ab} \\ G^-_{ab} &= h_{ab} - b_{ab} = e^{-\sqrt{\frac{2}{k}}t}(M\Gamma)_{ab} \end{aligned} \tag{8.7}$$

Relying on the properties of the adjoint matrix (see eq.s(6.4)) simple expressions are obtained for the derivatives of the above matrices:

$$\begin{aligned} \partial_a G^-_{bc} &= \sqrt{\frac{2}{k}}(-\delta_{a0}G^-_{bc} + G^-_{br}f_{rca}) \\ \partial_a G^+_{bc} &= -\sqrt{\frac{2}{k}}(\delta_{a0}G^+_{bc} + f_{abr}G^+_{rc}) \end{aligned} \tag{8.8}$$

So far we have identified the deformation of the vielbein (i.e. of the metric):

$$V'^a = V^a + \delta V^a = V^a + h_{ab}V^b \tag{8.9.a}$$

and the components of the new torsion in the old basis which are given (this follows from the same supersimmetry variation argument as in the undeformed case) as

$$(T + \delta T)_{abc} = -3(H + \delta H)_{abc} \tag{8.9.b}$$

Now we must solve the relevant torsion equations for the two non-Riemannian connections we are interested in, these latter, in the undeformed situation, being given by eq.(6.6) The two torsion equations are, working at 1^{st}order in the moduli:

$$dV'^a + (\omega^\pm + \delta\omega^\pm)^{ab}\ V'^b = \mp(T')^a \tag{8.10}$$

The solutions of eqs.(8.10) are given by

$$\delta\omega^\pm_{ab|c} = -2\stackrel{\pm}{\nabla}_{[a}G^\pm_{b]c} \mp 4G^\pm_{[ar}T_{rb]c} - \stackrel{\pm}{\nabla}_c G^\pm_{[ab]} \pm 2G^\pm_{rc}T_{rab} \tag{8.11}$$

Making the covariant derivatives explicit, using eqs.(8.8) and the undeformed connections, we finally get

$$\delta\omega^\pm_{ab|c} = \sqrt{\frac{2}{k}}\left\{2\delta_{[a0}G^\pm_{b]c} + \delta_{c0}G^\pm_{[ab]} \pm G^\pm_{[ar}f_{rb]c} \pm G^\pm_{rc}f_{rab}\right\} \tag{8.12}$$

Next we look for the deformations of the associated curvatures. From the general formula $\delta R = \nabla\, \delta\omega$ we have

$$
\begin{aligned}
\delta R^-_{ab} &= d\,\delta\omega^-_{ab} = (\partial_p\, \delta\omega^-_{ab|q} + \frac{1}{2}\sqrt{\frac{2}{k}}\, \delta\omega^-_{ab|r} f_{rpq}) V^p V^q \\
\delta R^+_{ab} &= (\partial_p\, \delta\omega^+_{ab|q} + \frac{1}{2}\sqrt{\frac{2}{k}}\, \delta\omega^+_{ab|r} f_{rpq} + 2\sqrt{\frac{2}{k}} f_{[arp}\, \delta\omega^+_{rb]|q}) V^p V^q
\end{aligned}
\tag{8.13}
$$

Using eqs.(8.12) and (8.8), after some algebra one ends up with the following results:

$$
\begin{aligned}
\delta R^\pm_{0i} &= -\frac{2}{k}\left\{ G^\pm_{ij} V^0 V^j \mp G^\pm_{il}\, \epsilon_{ljk}\, V^j V^k \right\} \\
\delta R^\pm_{jk} &= \pm\, \epsilon_{ijk}\, \delta R^\pm_{0i}
\end{aligned}
\tag{8.14}
$$

We have that *the curvature of* $\omega^+ + \delta\omega^+$ (which is δR^+) *is selfdual*, while *that of* $\omega^- + \delta\omega^-$ (which is δR^-)*is antiselfdual*:

$$
\delta R^\pm_{ab} = \pm\frac{1}{2}\, \epsilon_{abcd}\, \delta R^\pm_{cd} \tag{8.15}
$$

Recall that this is a necessary condition for the deformed $\mathcal{M}_{\text{target}}$ to have N=4 supersymmetry, as we discussed in sec.5.

8.1 Deformations of the Complex Structures

Having singled out the deformations of the vielbein, of the torsion and, consequently, of the two non-Riemannian connections, our aim is now to find the deformations of the complex structures corresponding to the insertion of the N=4 moduli in the lagrangian. Indeed they must exist since N=4 symmetry is mantained.

Any infinitesimal deformation of one of the sets of complex structures must be such that the quaternionic algebra is preserved:

$$
(\mathcal{J}^x + \delta\mathcal{J}^x)(\mathcal{J}^y + \delta\mathcal{J}^y) = -\delta^{xy} + \epsilon^{xyz}(\mathcal{J}^z + \delta\mathcal{J}^z)
$$

that is

$$
\delta\mathcal{J}^x \mathcal{J}^y + \mathcal{J}^x\, \delta\mathcal{J}^y = \epsilon^{zxy}\, \delta\mathcal{J}^z \tag{8.16}
$$

The general ansatz solving this requirement is

$$
\delta\mathcal{J}^x = [\mathcal{J}^x, F] + \sum_z M_z\, \epsilon^{zxy}\, \mathcal{J}^y \tag{8.17}
$$

F being a generic (infinitesimal) matrix and M_z generic infinitesimal parameters.

We have to impose the "deformed" covariant-constancy conditions, different for the two sets of complex structures relevant in the left and in the right sector:

$$
\overset{\pm}{\nabla}\, \delta\, \overset{\pm}{\mathcal{J}}{}^x_{ab} + 2\,\delta\omega^\pm_{[ar}\, \overset{\pm}{\mathcal{J}}{}^x_{rb]} = 0 \tag{8.18}
$$

Inserting the ansatz (8.17) with M_z=0 into eq.(8.18) we get

$$\stackrel{\pm}{\mathcal{J}}{}^{x}_{[ar}(\stackrel{\pm}{\nabla} F^{\pm} - \delta\omega^{\pm})_{rb]} = 0 \tag{8.19}$$

Note that the deformations of the connections, (see eq.s(8.12)) can also be written as

$$\begin{aligned} \delta\omega^{+}_{ab} &= -\stackrel{+}{\nabla} G^{+}_{[ab]} - \sqrt{\frac{2}{k}}\hat{\mathcal{J}}^{x}_{ab}G^{+}_{xr}V^{r} \\ \delta\omega^{-}_{ab} &= -\stackrel{+}{\nabla} G^{-}_{[ab]} + \sqrt{\frac{2}{k}}\tilde{\mathcal{J}}^{x}_{ab}G^{-}_{xr}V^{r} \end{aligned} \tag{8.20}$$

where $\hat{\mathcal{J}}^{x}$ and $\tilde{\mathcal{J}}^{x}$ are the two sets of constant complex structures introduced in sec. 5. (see eq.(5.16)). Therefore if we start from the ansatz (17) with

$$F^{\pm}_{ab} = G^{\pm}_{[ab]} \tag{8.21}$$

and M_z=0, the requirement (8.19) reduces to

$$\begin{aligned} &\stackrel{+}{\mathcal{J}}{}^{x}_{[ar}\hat{\mathcal{J}}^{y}_{rb]}G^{+}_{yr}V^{r} = 0 \\ &\stackrel{-}{\mathcal{J}}{}^{x}_{[ar}\tilde{\mathcal{J}}^{y}_{rb]}G^{-}_{yr}V^{r} = 0 \end{aligned} \tag{8.22}$$

Recall that for our undeformed manifold, $\stackrel{-}{\mathcal{J}}{}^{x} \equiv \hat{\mathcal{J}}^{x}$. The above equations hold then true due to the commutations relations (see sec. 5)

$$\begin{aligned} \left[\stackrel{+}{\mathcal{J}}{}^{x}, \stackrel{-}{\mathcal{J}}{}^{y}\right] &= 0 \\ \left[\hat{\mathcal{J}}^{x}, \tilde{\mathcal{J}}^{y}\right] &= 0 \end{aligned} \qquad \forall x, y \tag{8.23}$$

Summarizing, we have obtained that the deformations of the left- and right-moving complex structures due to the insertion of the moduli in the original N=4 theory are given by

$$\delta\stackrel{\pm}{\mathcal{J}}{}^{x} = \left[\stackrel{\pm}{\mathcal{J}}{}^{x}, \pm b\right] \tag{8.24}$$

where

$$b_{ab} = \mp G^{\pm}_{[ab]} = e^{-\sqrt{\frac{2}{k}}t}(M\Gamma - \Gamma^{T}M^{T})_{ab}$$

8.2 Breaking of the old Isometries

As in general the deformed curvatures differ (as forms) from zero, the effect of the deformations cannot be trivially reabsorbed by a coordinate change. The deformed space is actually a new kind of manifold: it is no longer a group-manifold. Intuitively, the situation is depicted in Fig. 8.1. Due to the exponential factor, there is no longer a direct product between a "time" coordinate and three "spatial" ones. The "radius" of the constant-time slices increases as $t \to -\infty$; at the same time these slices get more and more deformed respect to a three-sphere along some appropriate harmonics of the group $SU(2)$ (recall the presence of the adjoint matrix in the deformed expressions). The deformations of the "radius" and of the "shape" of the constant-time slices interplay so as to mantain the properties characterizing the space as Generalized HyperKähler . The undeformed situation (the "tube") is recovered as $t \to +\infty$

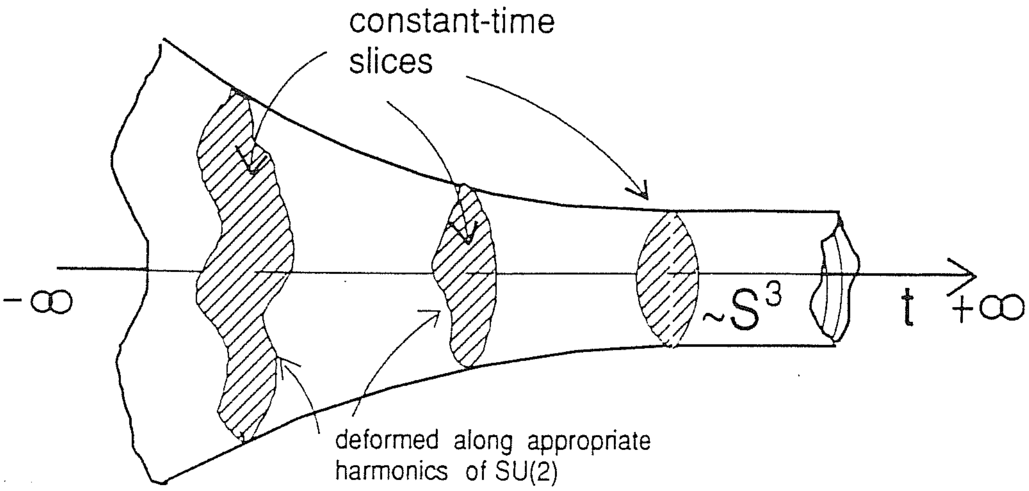


Fig. 8.1 The deformed geometry

In agreement with the above considerations, we show now that apparently none of the isometries is conserved by the above infinitesimal deformations. However, as stressed in the introduction, further study is needed to discuss the possibility, for some particular choice of the moduli, of modifying the old Killing vectors in such a way to become Killing vectors of the new metric, corresponding to the possibility of readsorbing the effect of thedeformation by coordinate changes.

The group of isometries of a group-manifold G is, for G a non-abelian Lie group, $G \times G$, corresponding to the existence of two basis of Killing vectors, the left-invariant ones k_A, generating right translations, and the right-invariant ones $\tilde{\mathrm{k}}_A$, generating left translations. The two sets are related by

$$\tilde{\mathrm{k}}_A = \Gamma_{AB} \mathrm{k}_B \tag{8.25}$$

Γ being as usual the adjoint matrix of the L.A. representing the group element g. The vector fields k_A are dual to the group-manifold vierbeins: *

$$i_{\mathrm{k}_A} \Omega^B = \delta^B_A \tag{8.26}$$

* We use the geometric formalism extensively developed, for instance, in [15]; we indicate in particular with i_{k} the "contraction" between vectors and forms, and with ℓ_{k} the Lie derivative along the vector k

For G abelian, the two translations coincide, and the isometry group is simply R or $U(1)$. The isometry group of the manifold $SU(2) \times \mathrm{R}$ is therefore

$$SU(2) \times SU(2) \times \mathrm{R}$$

and it is generated by the Killing vectors $\mathbf{k}_i, \tilde{\mathbf{k}}_i,\ i = 1,2,3$ and $\mathbf{k}_0$, which can be normalized so that their non-zero contraction with the vielbeins of the manifold are

$$\begin{aligned} i_{\mathbf{k}_i} V^j &= \delta_{ij} \\ i_{\tilde{\mathbf{k}}_i} V^j &= \Gamma_{ij} \\ i_{\mathbf{k}_0} V^0 &= 1 \end{aligned} \tag{8.27}$$

To check explicitely that these vectors correspond to isometries of the manifold it is sufficient to compute the Lie derivative of the line element $ds^2 = V^a \otimes V^a$ along each of them, finding in all cases that it vanishes.
To perform the computation one uses the fact that, from the formula for the Lie-derivative

$$\ell_{\mathbf{k}} V = d(i_{\mathbf{k}} V) + i_{\mathbf{k}} dV$$

one gets

$$\begin{aligned} \ell_{\mathbf{k}_0} V^a &= \ell_{\tilde{\mathbf{k}}_i} V^a = \ell_{\mathbf{k}_i} V^0 = 0 \\ \ell_{\mathbf{k}_i} V^j &= -\sqrt{\frac{2}{k}}\, \epsilon_{ijk}\, V^k \end{aligned} \tag{8.28}$$

Now we raise the question whether any of these isometries remains an isometry of the deformed manifold. To see if this is the case, we have simply to compute the Lie derivative along the above Killing vectors of the deformed line element:

$$ds'^2 = ds^2 + \delta ds^2 = V^a \otimes V^a + e^{-\sqrt{\frac{2}{k}}t} V^a \otimes (M\Gamma + \Gamma^T M^T)_{ab} V^b \tag{8.29}$$

By explicit computation we find:

$$\ell_{\mathbf{k}_0} \delta ds^2 = -\sqrt{\frac{2}{k}} e^{-\sqrt{\frac{2}{k}}t} (M\Gamma + \Gamma^T M^T)_{ab} V^a \otimes V^b \tag{8.30a}$$

$$\ell_{\mathbf{k}_i} \delta ds^2 = -2\sqrt{\frac{2}{k}} e^{-\sqrt{\frac{2}{k}}t}\, \epsilon_{ilk}\, (M\Gamma)_{la} V^a \otimes V^k \tag{8.30b}$$

$$\ell_{\tilde{\mathbf{k}}_i} \delta ds^2 = 2\sqrt{\frac{2}{k}} e^{-\sqrt{\frac{2}{k}}t}\, \epsilon_{pni}\, \Gamma_{pj} M_{an} V^a \otimes V^j \tag{8.30c}$$

Since in general none of these expression vanishes, none of the isometries is mantained after deformation. This does not exclude that some modified Killing vectors exist, as discussed before.

Appendix

We show now explicitely that the vertices declared in Chapter 5 to transform in the adjoint representation of SU(6)really do so. We focus on the right moving parts of the of the "$SU(6)$ Gauge bosons" or "Gauginos" vertices, as reported in Section 5.2. They involve the currents of the $SO(6)$ group of the heterotic fermions and the currents of the $U(1)$ of the N=2 supersymmetry (that of the internal space) and of the $SU(2)$ of the N=4 supersymmetry (that of the spacetime). Between these currents there is a subset of 5 mutually commuting (in the OPE sense) dimension-one operators, namely

- the currents in the Cartan subalgebra (CSA) of the $SO(6)$
- the $U(1)$ current $\tilde{j}(\bar{z})$, bosonized as $\frac{i}{\sqrt{3}}\partial\tilde{\phi}(\bar{z})$
- the CSA current of the $SU(2)$, $A^3 = \frac{i}{\sqrt{2}}\partial\tilde{\tau}(\bar{z})$

Let us consider the eigenvalues of these operators (which will constitute the CSA of some Lie Algebra) on the right-moving part of the vertices under discussion:

- $SO(6)$ currents $\tilde{J}^A(\bar{z})$ (between them there are 3 CSA generators), 15 fields
- the 3 $SU(2)$ currents (with one CSA generator)
- the $U(1)$ current, which is in the CSA
- $\hat{\mathbf{1}}\left[\begin{smallmatrix}1/4\\1/2\end{smallmatrix}\right](\bar{z})\Sigma_\alpha(\bar{z})e^{i\frac{\sqrt{3}}{2}\tilde{\phi}(\bar{z})}$, $4\times 2 = 8$ fields
- $\hat{\mathbf{1}}\left[\begin{smallmatrix}1/4\\1/2\end{smallmatrix}\right](\bar{z})\Sigma_{\dot{\alpha}}(\bar{z})e^{-i\frac{\sqrt{3}}{2}\tilde{\phi}(\bar{z})}$, $4\times 2 = 8$ fields

for a total of 35 fields, exactly the dimension of $SU(6)$, so that it is possible that these fields indeed form its adjoint representation. Note that in the last two cases $e^{\pm i\frac{\sqrt{3}}{2}\tilde{\phi}(\bar{z})}$ is the explicit bosonized expression of the internal operator $\mathbf{1}\begin{pmatrix}3/8\\ \pm 3/2\end{pmatrix}$.

If the ipotesis is correct, the weights (the CSA eigenvalues) correspondent to these fields should actually form the $SU(6)$ root system; we should then be able to find explicitely the simple roots and to work out the Dynkin diagram, recognizing it to be precisely the $SU(6)$ one.

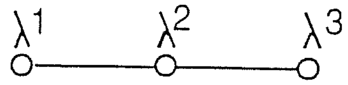
Let's introduce briefly some notations. The twelve $SO(6)$ roots can be described by the vectors

$$\pm\epsilon_i \pm \epsilon_j \qquad i,j = 1,2,3$$

$\{\epsilon_i\}$ being the versors of a three-dimensional Euclidean space. The $SO(6)$ simple root-system is made of

$$\lambda^1 = \epsilon_2 - \epsilon_3$$
$$\lambda^2 = \epsilon_1 - \epsilon_2$$
$$\lambda^3 = \epsilon_2 + \epsilon_3$$

so that its Dynkin diagram is simply



The foundamental weights π^i, such that $2\frac{(\pi^i,\lambda^j)}{(\lambda^i,\lambda^j)} = \delta^{ij}$ are expressed as

$$\pi^1 = \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3)$$
$$\pi^2 = \epsilon_1$$
$$\pi^3 = \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3)$$

The weights of the spinor (s) and antispinor ($\bar{s}$) representations of $SO(6)$ are given by:

$$s : \frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3) = \pi^3 \qquad \bar{s} : \frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3) = \pi^1$$
$$\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3) = \pi^3 - \lambda^3 \qquad \frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3) = \pi^1 - \lambda^1$$
$$\frac{1}{2}(-\epsilon_1 + \epsilon_2 - \epsilon_3) = \pi^3 - \lambda^3 - \lambda^2 \qquad \frac{1}{2}(-\epsilon_1 + \epsilon_2 + \epsilon_3) = \pi^1 - \lambda^1 - \lambda^2$$
$$\frac{1}{2}(-\epsilon_1 - \epsilon_2 + \epsilon_3) = \pi^3 - \lambda^3 - \lambda^2 - \lambda^1 \qquad \frac{1}{2}(-\epsilon_1 - \epsilon_2 - \epsilon_3) = \pi^1 - \lambda^1 - \lambda^2 - \lambda^3$$

Let's now collect the weights that we obtain acting with the CSA generators on those of the above 35 operators which are not in the CSA; in some cases we already know the result, in some other we perform the relevant OPEs. Beside the versors for the weight space of SO(6) we introduce the versors μ, dual to the CSA generator $\tilde{A}^3$ and κ, dual to $\bar{j} = \frac{i}{\sqrt{3}}\partial\bar{\phi}$. We have:

- from the 12 non-CSA currents of $SO(6)$ the weights

$$\pm\epsilon_i \pm \epsilon_j \qquad i,j = 1,2,3$$

- from the 2 non-CSA currents of $SU(2)$ the weights

$$\pm\sqrt{2}\mu$$

- from the operators $\hat{1}\left[{}^{1/4}_{1/2}\right](\bar{z})\Sigma_\alpha(\bar{z})e^{i\frac{\sqrt{3}}{2}\bar{\phi}(\bar{z})}$ the 4×2 weights

$$\frac{1}{2}(\delta_1\epsilon_1 + \delta_2\epsilon_2 + \delta_3\epsilon_3) + \frac{\eta}{\sqrt{2}}\mu + \frac{\sqrt{3}}{2}\kappa \qquad (\delta_1\delta_2\delta_3 = 1\ ;\ \eta = \pm 1)$$

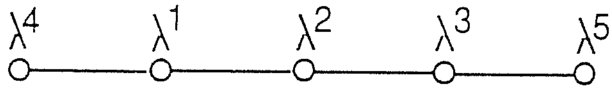
- from the operators $\hat{\mathbf{1}}\left[{}^{1/4}_{1/2}\right](\bar{z})\Sigma_{\dot{\alpha}}(\bar{z})e^{-i\frac{\sqrt{3}}{2}\tilde{\phi}(\bar{z})}$ the 4×2 weights

$$\frac{1}{2}(\delta_1\epsilon_1 + \delta_2\epsilon_2 + \delta_3\epsilon_3) + \frac{\eta}{\sqrt{2}}\mu - \frac{\sqrt{3}}{2}\kappa \qquad (\delta_1\delta_2\delta_3 = -1 \ ; \ \eta = \pm 1)$$

These vectors are in number of 30, that indeed equals $dimSU(6) - rankSU(6) = 35 - 5$ and we call them from now on roots (although we still have to show that they really are the roots of $SU(6)$). A simple root system for these roots is given by

$$\begin{aligned}
&\lambda^1, \lambda^2, \lambda^3 \qquad SO(6) \text{ simple roots}\\
&\lambda^4 = -\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3) + \frac{\sqrt{2}}{2}\mu + \frac{\sqrt{3}}{2}\kappa = \pi^1 - \lambda^1 - \lambda^2 - \lambda^3 + \frac{1}{\sqrt{2}}\mu + \frac{\sqrt{3}}{2}\kappa\\
&\lambda^5 = \frac{1}{2}(-\epsilon_1 - \epsilon_2 + \epsilon_3) + \frac{\sqrt{2}}{2}\mu - \frac{\sqrt{3}}{2}\kappa = \pi^3 - \lambda^3 - \lambda^2 - \lambda^1 - \frac{1}{\sqrt{2}}\mu - \frac{\sqrt{3}}{2}\kappa
\end{aligned}$$

The splitting in positive and negative roots follows of course the $SO(6)$ one for the $SO(6)$ roots and the $SU(2)$ one for the $SU(2)$ roots and depends moreover in the natural way from the sign of the coefficients in front of κ for the other vectors. One can show easily that all positive roots are obtained from positive integer coefficients-superpositions of the λ^i; for example, $\sqrt{2}\mu = \lambda^1 + \lambda^2 + \lambda^3 + \lambda^4 + \lambda^5$ and so the weights of "spinorial" origine are obtained from λ^4 or λ^5 adding in various steps $\lambda^1, \lambda^2, \lambda^3$.
Moreover, $(\lambda^i)^2 = 2 \ \forall i = 1, ..., 5$. For example $(\lambda^{4,5})^2 = \frac{1}{16}(9 + 1 + 1 + 1) + \frac{1}{2} + \frac{3}{4} = 2$.
Finally, the Dynkin diagram results to be



that is exactly the one of $SU(6)$

References

[1] E.Witten, Phys. Rev **D44** (1991) 314; R.Dijkgraf, E.Verlinde and H.Verlinde *"String propagation in a Black Hole geometry"* PUPT-1252; S.Elitzur, A.Forge, E.Rabinovici *"Some Global aspects of String Compactifications"* RI-143/90; G.Mandal, A.Sengupta and S.Wadia *"Classical solutions of 2D string theory"* IASSNS-HEP-91/10; M.Rocek, K.Schoutens, A.Sevrin *"Off-Shell WZW Models in Extended Superspace"* IASSNS-HEP-91/14; A.Giveon Mod. Phys. Lett. **A6** No 31 (1991) 2843.

[2] M.Rocek, C.Ahn, K.Schoutens, A.Sevrin *"Superspace WZW Models and Black Holes"* IASSNS-HEP-91/69; P.Ginsparg, F.Quevedo, *"Strings on Curved Space-Time: Black Holes, Torsion, and Duality"* LA-UR-92-640, hepth@xxx/9202092;
C.Kounnas, D.Lust *"Cosmological String Backgrounds from Gauged WZW Models"* CERN-TH.6494/92

[3] C.G.Callan, J.A.Harvey, A.Strominger Nucl. Phys. **B359** (1991) 611;
C.G.Callan, Lectures at Sixth J.A.Swieca summer school, PUPT-1278.

[4] D.J.Gross, J.A.Harvey, E.Martinec and R.Rohm Nucl. Phys. **B256** (1985) 253.

[5] P.Candelas, G.Horowitz, A.Strominger, E.Witten Nucl. Phys. **B258** (1985) 46; For a review see M.Green, J.Schwarz, E.Witten *"Superstring Theory"* (Cambridge U.P.) Cambridge , 1987.

[6] A.Sevrin,W.Troost and A.Van Proeyen, Phys.Lett. **B208** (1988) 447

[7] D.Gepner, Trieste Lectures 1989; Nucl. Phys. **B296** (1988) 757; Phys. Lett. **B199** (1987) 380.

[8] T.Eguchi, P.B.Gilkey, and A.J.Hanson, Phys. Rep. **66** (1980) 213.

[9] R.d'Auria, T.Regge Nucl. Phys. **B195** (1982) 308

[10] K.Konishi, N.Magnoli and H.Panagopoulis, Nucl. Phys. **B309** (1988) 201, Nucl. Phys **B323** (1989) 441.

[11] T.Eguchi, A.J.Hanson Phys. Lett. **74B**(1978) 249; Ann. Phys. **120** (1979) 82

[12] E.Cremmer, S.Ferrara, L. Girardello, A.van Proeyen Nucl. Phys. **B212** 413;
Phys. Lett. **116B** (1982) 231

[13] L.Alvarez-Gaumé and D.Z.Freedman, Comm. Math. Phys. **80** (1981) 443; S.J.Gates, C.M.Hull, M.Rocek Nucl. Phys. **B248** (1984) 157; P.S.Howe and G.Papadopoulos, Nucl. Phys. **B289** (1987) 264, Class. Quant. Grav. **5** (1988) 1647

[14] K.Galicki Comm. Math. Phys. **108** (1987) 117
[15] L.Castellani, R.d'Auria and P.Fré, *"Supergravity and Superstrings: A geometric Perspective"* (World Scientific ed.), 1990
[16] L.Castellani,R.d'Auria and D.Franco, Int.Jour.Mod.Phys. **A6** 4009
[17] E.Verlinde, H.Verlinde Nucl. Phys. **B288** (1987) 357.
[18] T.Eguchi, H.Ooguri, A.Taormina and S.K.Yang, Nucl. Phys. **B315** (1989) 193.
[19] R. Barbieri, S.Ferrara, D.V.Nanopoulos, K.S. Stelle Phys. Lett. **113B** (1982) 219
[20] S.Cecotti, S.Ferrara, L.Girardello Int. Jour. Mod. Phys. **A10** (1989) 2475; A.Strominger, Phys. Rev. Lett. **55** (1985) 2547; A.Strominger, E.Witten Comm. Math. Phys. **101** (1985) 341; P.Candelas Nucl. Phys. **B298** (1988) 458.
[21] L.Dixon, V.S.Kaplunovski, J.Louis Nucl. Phys. **B329** (1990) 27.
[22] S.J.Rey, Phys. Rev. **D43** (1991) 526; *"On String Theory and Axionic Strings and Instantons"* Slac-PUB-5659 (Sept. 1991).
[23] S. Ferrara, P.van Nieuwenhuizen Phys. Lett. **74B** (1978) 333; **76B** (1978) 404; **78b** (1978) 573; K.S.Stelle, P.C.West Phys. Lett. **74B** (1978) 330; **77B** (1978) 376; S. Ferrara, L.Girardello, T.Kugo, A.van Proyen Nucl. Phys. **B233** (1983) 191.
[24] V.P. Akulov, D.V.Volkov and V.A.Suroka Teor. Math. Phys. **31** (1977) 285; M.F.Sohnius, P.C.West Phys. Lett. **105B** (1981) 353; see also R.D'Auria, P.Fré, P.van Nieuwenhuizen , P.K.Townsend Ann. Phys **155** (1984) 423
[25] R.D'Auria, P.Fré, G.De Matteis, I.Pesando, Int. Jour. Mod. Phys. **A4** No14 (1989) 3577
[26] B.A.Ovrut, S.Kalyana Rama, Phys. Lett. **B245**(1990) 429; also Nucl. Phys **B343** (1990) 86.
[27] E.Witten, Comm. Math. Phys. **92** (1984) 455
[28] V.Dotsenko and V. Fateev, Nucl.Phys. **B240** (1984) 312
[29] I.Antoniadis, C. Bachas, J.Ellis and D.Nanopoulos, Phys. Lett. B211 (1988) 393; Nucl.Phys. **B328** (1989) 117
[30] Ph.Spindel, A.Sevrin, W.Troost, A.Van Proeyen Phys.Lett. **B206** (1988) 71, Nucl.Phys. **B308** (1988) 662
[31] R.d'Auria, P.Fré , F.Gliozzi and A.Pasquinucci, Nucl.Phys. **B334** (1990) 24.

Aknowledgements

I wish to thank sincerely Pietro Fré, Luciano Girardello and Alberto Zaffaroni with whom the work on which this thesis is entirely based has been written. Im am expecially grateful to Pietro Fré, who gave me his patient and invaluable help during all the last three years.