SISSA - SCUOLA INTERNAZIONALE SUPERIORE DI STUDI AVANZATI

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Form Factors Approach to the Sinh-Gordon Model

Thesis presented by

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for the degree of Magister Philosophiae

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S.I.S.S.A. – I.S.A.S.

Elementary Particle Sector

Academic Year 1991 - 92

TRIESTE

to Daniele Cabrini

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Introduction

A Quantum Field Theory (QFT) is completely described by the set of the correlation functions (CF's) of the fields. The CF's must satisfy some fundamental properties [1], [2], in order to recover the defining features of a QFT, i.e. the probabilistic interpretation, the Poincaré symmetry and locality.

The properties of the CF's can in their turn be translated into the language of the algebra $\mathcal{A}$ of the local [1] fields. For example, the fundamental operator product expansion gives the exact short distance structure of the CF's.

Using this algebra we can build a structure of Hilbert space, and obtain the quantistic interpretation of the theory in terms of states and of a scalar product. The algebra $\mathcal{A}$ must obviously contain the generators of the fundamental Poincaré symmetries, i.e. the hamiltonian, the linear momentum, the angular momentum and the boost generator.

In the physical Hilbert space we can usefully choose the bases of *in* and *out* states. These are defined as the eigenstates of the Poincaré generators, and of internal symmetries resolving possible degeneracy, in the very past ($t \to -\infty$) and in the very future ($t \to +\infty$). Besides the mentioned symmetry generators, $\mathcal{A}$ must contain also an operator S defined by

$$| \, \{\alpha\} \, , \; in \, \rangle \; = \; S \; | \; \{\alpha\} \, , \; out \, \rangle \; .$$

Hence every scattering amplitude $\langle \, \{\beta\} \, , \; out \; | \; \{\alpha\} \, , \; in \, \rangle$ is encoded in the S-matrix

$$S_{\{\beta\} \leftarrow \{\alpha\}} \; \equiv \; \langle \, \{\beta\} \, , \; out \; | \; S \; | \; \{\alpha\} \, , \; out \, \rangle \; .$$

As it is well known, the pole structure of the functions $S_{\{\beta\} \leftarrow \{\alpha\}}$ gives us the discrete part of the spectrum.

A generic operator $\mathcal{O}(x)$ will be then characterized by its matrix elements in the basis chosen. Under the assumptions of Poincaré symmetry and locality, every

[1] Here for local fields we don't mean what is usually understood in QFT (see Chapter 2), but only the requirement that these quantum fields guarantee microcausality of observables. So the fields of $\mathcal{A}$ can be strictly local, semilocal etc.etc.

piece of information is encoded into the form factors

$$\langle 0 \mid \mathcal{O}(0) \mid \{\beta\} \, , in \rangle \ .$$

If a lagrangian formulation in terms of some fundamental massive fields is available, the *in* and *out* states are described in terms of asymptotics free fields. In this case the LSZ-formalism [3] allows us to express both the S-matrix and the form factors as proper on-shell limits of general CF's of the theory. It is common to refer to the properties of the S-matrix as to "on-shell physics" since it directly determines the scattering processes of physical particles. Analogously for the form factors one refers to "off-shell physics" because they permits the reconstruction of the CF's at any value of external momenta.

In the last years many results have shed light in the interplay between on-shell physics and off-shell physics for two dimensional massive integrable models (2dMIM's). The factorization of the S-matrix implied by the complete integrability, allowed the classification of many possible S-matrices through the development of the boostrap program [16, 20]. On the other hand, the complete solution [4] in terms of Virasoro algebra representations of the conformal field theories (CFT's), provided us with the tool for the accurate study of the UV-behaviour of the 2dMIM's. Hence a 2dMIM can effectively be thought as relevant perturbation of some CFT [18, 19, 23, 20]. The UV behaviour of a large class of 2dMIM's was so identified as an equivalent description of a corresponding integrable statistical lattice model near criticality [5].

The natural subsequent step towards the complete solution of a theory has been the determination of its local structure, i.e. of the off-shell physics. A study of the monodromy properties and of the pole strucure of the form factors for 2dMIM's, led [25, 26, 27] to a set of equations, depending on the two-particle S-matrix. The assumption of an exact S-matrix together with some prescription on the UV-behaviour of the form factors, is thus in principle sufficient to determine the local structure of a 2dMIM.

Another completely independent route towards the local structure of a 2dMIM is through the inverse scattering method (ISM) [6, 7]. According to this method, one has to describe the model through a set of action-angle variables, in terms of which the classical time evolution becomes linear. Obviously the difficulty consists in expressing the inverse canonical transformation from these new variables to the original fields degrees of freedom. The quantization of this scheme brings in general further complications due to the operatorial character of the fields. The general outcome is that, in order to preserve the integrability also at quantum level, one has to find a *ad hoc* quantization procedure, different from the naive corresponding principle, which must be only recovered in a semiclassical limit. Now we know a great deal of quantized version of classical two dimensional integrable models, i.e. of *ad hoc* quantization procedures.

What is still lacking in general is the precise relation between the quantum action-angle variables and the quantum local fields, i.e. the setting and the solution of the Quantum Gelfand-Levitan-Marchenko equation.

For the Sine-Gordon model [8, 9] this program has been carried out by Smirnov. As discussed by Sklyanin [33] the same construction can't be straightforwardly mapped to the Sinh-Gordon model, for the impossibility to describe for this theory the Bethe vacuum state in the infinite volume limit. By modifying cleverly the algebraic Bethe ansatz technique, Sklyanin managed to find a representation of the algebra of the quantum action-angle variables. But the solution to the locality problem for this model is still unknown.

The possibility to determine the local structure of the Sinh-Gordon model, besides being interesting by itself as an example of a non-trivial solved QFT, could thus also throw some light on the quantum ISM, and more generally on the problem of the quantization of integrable models. In this thesis the solution to the equations for the form factors for the Sinh-Gordon model is presented [10].

We close this Introduction by mentioning very recent results which suggest a new classical interpretation of integrable models through the discovery of some non-local symmetries [11]. What is hoped is that a two dimensional integrable QFT can be solved using a suitable quantization of these symmetries [12]. Hence this approach could give a very fundamental explanation to the characterization of two dimensional integrable QFT's in terms of classical integrable differential equations satisfied by the CF's [13] and of the corresponding τ-functions [14, 15].

Chapter 1

S-matrix for Quantum Integrable Models

1.1 Asymptotic States in the Lagrangian Approach

Our aim is to describe a two-dimensional massive Quantum Field Theory whose Lagrangian is the sum of a free part and an interacting one

$$\mathcal{L} = \mathcal{L}_{\mathcal{F}} + \mathcal{L}_{\mathcal{I}}. \tag{1.1}$$

For simplicity we will treat the case of a simple scalar boson defined by the action

$$\mathcal{S} = \int d^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 - \mathcal{L}_{\mathcal{I}}(\phi)\right] \quad . \tag{1.2}$$

The short distance character of the interaction of the theory, allows us to consider the asymptotic behaviour in time of our theory as a free one. *In* and *out* states are thus defined *a la* Fock

$$\begin{aligned} \mid \beta_1 > \beta_2 > \ldots > \beta_n,\, in\, \rangle \quad &= a^\dagger_{in}(\beta_1)a^\dagger_{in}(\beta_2)\ldots a^\dagger_{in}(\beta_n) \mid 0 \,\rangle \;, \\ \mid \beta_1 > \beta_2 > \ldots > \beta_n,\, out\, \rangle \quad &= a^\dagger_{out}(\beta_1)a^\dagger_{out}(\beta_2)\ldots a^\dagger_{out}(\beta_n) \mid 0 \,\rangle \end{aligned} \tag{1.3}$$

where we have adopted the on mass-shell parametrization of the momentum p_i^μ of the particle with mass m , $m^2 = p_i^2 = (p_i^0)^2 - (p_i^1)^2$

$$p_i^0 \;=\; m\cosh\beta_i \;\; , \quad p_i^1 \;=\; m\sinh\beta_i \;\; . \tag{1.4}$$

β_i is called the rapidity of the particle. The creation operator in (1.3) is the hermitean conjugate of the destruction operator

$$a_{\substack{\mathrm{in} \\ out}}(\beta) \;=\; \frac{i}{\sqrt{2}}\int dx^1\, e^{ipx}\, \overleftrightarrow{\partial_0}\, \phi_{\substack{\mathrm{in} \\ out}}(x) \quad . \tag{1.5}$$

The asymptotic fields $\phi_{\substack{\text{in}\\ out}}(x)$ satisfy the free massive field equation

$$(\Box + m^2)\phi_{\substack{\text{in}\\ out}}(x) = 0 \tag{1.6}$$

and they are the weak limit of the renormalized interaction field

$$\lim_{x^0 \to \mp\infty} \phi(x) = \lim_{x^0 \to \mp\infty} \phi_{\substack{\text{in}\\ out}}(x) \ . \tag{1.7}$$

Canonical Quantization for free fields implies that $a_{in}(\beta)$, $a_{in}^{\dagger}(\beta)$ satisfy the algebra

$$\begin{aligned} \left[a_{in}(\beta), a_{in}^{\dagger}(\beta')\right] &= 2\pi\delta(\beta - \beta') \\ \left[a_{in}(\beta), a_{in}(\beta')\right] &= 0 \ . \end{aligned} \tag{1.8}$$

The same obviously holds for *out*-operators. We assume the existence of a unique vacuum $|\, 0 \,\rangle$ such that

$$a_{\substack{\text{in}\\ out}}(\beta) \,|\, 0 \,\rangle = 0 \ . \tag{1.9}$$

Our basis vectors are normalized as

$$\begin{aligned} \langle\, \beta_1' > \ldots > \beta_m', in \,|\, \beta_1 > \ldots > \beta_n, \, in \,\rangle &= \\ = \delta_{m,n} \prod_{i=1}^{n} 2\pi\delta(\beta_i' - \beta_i) \ . \end{aligned} \tag{1.10}$$

In order to correctly implement the fundamental translation symmetry, a $2d$ vector operator P^{μ} must be defined such that the following algebra holds

$$[P^{\mu}, \phi_{in}(x)] = -i\partial_{\mu}\phi_{in}(x) \tag{1.11}$$

or in terms of creation operators

$$\left[P^{\mu}, a_{in}^{\dagger}(\beta)\right] = \frac{m}{2}\left(e^{\beta} + (-1)^{\mu}e^{-\beta}\right) a_{in}^{\dagger}(\beta) \ . \tag{1.12}$$

This implies that our basis vectors are eigenstates of P^{μ}

$$\begin{aligned} P^{\mu} \,|\, \beta_1 > \ldots > \beta_n, \, in \,\rangle &= \\ = \frac{m}{2}\sum_{i=1}^{n}\left(e^{\beta_i} + (-1)^{\mu}e^{-\beta_i}\right) \,|\, \beta_1 > \ldots > \beta_n, \, in \,\rangle \end{aligned} \tag{1.13}$$

Note that the ordering on the rapidities of (1.3) is necessary in order to avoid over-counting of the states.

We can now define the operator S

$$|\, \beta_1 > \ldots > \beta_n, \, in \,\rangle = S \,|\, \beta_1 > \ldots > \beta_n, \, out \,\rangle \tag{1.14}$$

such that its matrix elements in the basis $\{ \mid out \rangle \}$ are the amplitudes of probability of the transitions $\{\beta_i\} \to \{\beta'_i\}$

$$\begin{aligned}
&\langle\, \beta'_1 > \ldots > \beta'_m,\ out \mid \beta_1 > \ldots > \beta_n,\ in\, \rangle = \\
&\quad = \langle\, \beta'_1 > \ldots > \beta'_m,\ out \mid S \mid \beta_1 > \ldots > \beta_n,\ out\, \rangle \\
&\quad \equiv S_{\{\beta'\}_m \leftarrow \{\beta\}_n} \,.
\end{aligned} \tag{1.15}$$

It is possible to express the on-shell objects $S_{\{\beta'\}_m \leftarrow \{\beta\}_n}$ in terms of a limit of the off-shell correlation functions of the theory, i.e.

$$\begin{aligned}
S_{\{\beta'\}_m \leftarrow \{\beta\}_n} &= \left(\frac{1}{\sqrt{2}}\right)^{m+n} \lim_{\substack{p_i^2 \to m^2 \\ p_i'^2 \to m^2}} \prod_{i=1}^{n} \left(\frac{p_i^2 - m^2}{i}\right) \prod_{j=1}^{m} \left(\frac{p_j'^2 - m^2}{i}\right) \\
&\times \ (2\pi)^2\, \delta^2\left(\sum_{i=1}^{n} p_i + \sum_{j=1}^{m} p'_j\right) G^{n+m}(p_1, \ldots, p_n, p'_1, \ldots, p'_m)
\end{aligned} \tag{1.16}$$

where

$$\begin{aligned}
&(2\pi)^2\ \delta^2\left(\sum_{i=1}^{n} p_i\right) G^n(p_1, p_2, \ldots, p_n) = \\
&\quad \int \prod_{i=1}^{n} dx_i\ e^{-i \sum q_i x_i}\ \langle\, 0 \mid T(\phi(x_1)\phi(x_2) \ldots \phi(x_n)) \mid 0\, \rangle
\end{aligned} \tag{1.17}$$

are the correlation functions in the momentum representation. In order to obtain (1.16) we have used eqs. (1.3)-(1.10). S is an unitary operator, since $\{ \mid in \rangle \}$ and $\{ \mid out \rangle \}$ are considered as complete sets

$$\mid \beta_1 > \ldots > \beta_n,\ in\, \rangle = S \mid \beta_1 > \ldots > \beta_n,\ out\, \rangle \tag{1.18}$$

$$\langle\, \beta'_1 > \ldots > \beta'_n,\ in \mid = \langle\, \beta'_1 > \ldots > \beta'_n,\ out \mid S^\dagger \ . \tag{1.19}$$

Hence

$$\begin{aligned}
&\langle\, \beta'_1 > \ldots > \beta'_n,\ in \mid \beta_1 > \ldots > \beta_n,\ in\, \rangle = \\
&\quad = \prod_{i=1}^{n} 2\,\pi\ \delta(\beta'_i - \beta_i) \\
&\quad = \langle\, \beta'_1 > \ldots > \beta'_n,\ out \mid \beta_1 > \ldots > \beta_n,\ out\, \rangle \\
&\quad = \langle\, \beta'_1 > \ldots > \beta'_n,\ out \mid S^\dagger S \mid \beta_1 > \ldots > \beta_n,\ out\, \rangle \\
S^\dagger S &= 1.
\end{aligned} \tag{1.20}$$

Furthermore, S must in general satisfies crossing symmetry. This is the relation that exists between amputated correlation functions of particles and antiparticles and can be proved using locality properties. We will be more precise in the next section, when two-particles S-matrix elements properties for quantum integrable models will be discussed.

1.2 Quantum Integrable Scattering

In the case of integrable models scattering amplitudes acquire a very restricted form, due to the very simple kinematics and the existence of an infinite number of non trivial conserved currents. A quantum integrable model is a system which possesses an infinite number of local conserved currents [1], among them the stress energy tensor. The corresponding conserved charges have in general tensor properties. It is therefore useful to organize them as eigenstates, in the algebra of operators, of the Lorentz Spin, i.e. the generator of the Lorentz boosts. In the light-cone system of coordinates

$$(x^0, x^1) \to (x^+ = \frac{x^0 + x^1}{2}, x^- = \frac{x^0 - x^1}{2}) \tag{1.21}$$

the charges $\mathcal{Q}_s$ satisfy

$$[\Sigma, \mathcal{Q}_s] = s\mathcal{Q}_s \tag{1.22}$$

where Σ is the Lorentz spin and $s \in \mathcal{Z}$ since we restrict ourselves to mutually local operators [2]. The $\mathcal{Q}_s$'s also satisfy

$$[\mathcal{Q}_s, \mathcal{Q}_{s'}] = 0 \tag{1.23}$$

$$[\mathcal{Q}_s, S] = 0. \tag{1.24}$$

First representatives of them are

$$\mathcal{Q}_1 = P^+ = P^0 + P^1$$
$$\mathcal{Q}_{-1} = P^- = P^0 - P^1 \quad .$$

Lorentz invariance and locality also imply

$$\mathcal{Q}_{\pm s} \mid \beta \rangle = M_s e^{\pm s\beta} \mid \beta \rangle, \quad M_1 = m \tag{1.25}$$

and

$$\mathcal{Q}_{\pm s} \mid \beta_1 > \ldots > \beta_n, in \rangle = M_s \sum_{i=1}^{n} e^{\pm s\beta_i} \mid \beta_1 > \ldots > \beta_n, in \rangle \quad . \tag{1.26}$$

Since $[\mathcal{Q}_s, H] = 0$, the *in* and *out* states of a scattering process have the same eigenvalues of $\mathcal{Q}_{\pm s}$:

$$\sum_{i=1}^{n} e^{\pm s\beta_i} = \sum_{j=1}^{m} e^{\pm s\beta'_j}, \quad \forall s. \tag{1.27}$$

[1] Integrability is simply guaranteed by the existence ot these local conserved currents. The existence of also non-local conserved currents has recently put in evidence the promising possibility to solve the models according to representations of proper quantum algebras (see Introduction).

The only solution to this system of equations is

$$m = n, \quad \beta'_i = \beta_i \tag{1.28}$$

In other words, scattering processes take place in a purely elastic way and the matrix elements of S can be written as

$$S_{\{\beta'\}_m \leftarrow \{\beta\}_n} = \delta_{m,n} \prod_{i=1}^{n} 2\pi\, \delta(\beta'_i - \beta_i)\, S_n(\beta_{12}, \beta_{13}, \ldots) \tag{1.29}$$

where the dependence on the differences of the rapidities is a consequence of Lorentz invariance. From the definition of $S_{\{\beta'\}_m \leftarrow \{\beta\}_n}$, the eqs. (1.14) and (1.15) one can also write

$$|\, \beta_1 > \ldots > \beta_n,\, in \,\rangle = S_n(\beta_{12}, \beta_{13}, \ldots)\; |\, \beta_1 > \ldots > \beta_n,\, out \,\rangle \tag{1.30}$$

where $\beta ij = \beta_i - \beta_j$. The direct implication of the purely elasticity of the scattering is the factorization of the S-matrix [16, 17] into all the possible two-particles scattering processes:

$$S_n(\beta_{12}, \beta_{13}, \ldots) = \prod_{i<j} S_2(\beta_i - \beta_j) \quad . \tag{1.31}$$

Obviously, factorization must be independent of the order in which each two-particles scattering process is taken into account. This consistency principle is trivially satisfied in the case of one single self-conjugated boson. In fact the two-particles S-matrix of a self-conjugated boson is a pure phase, and therefore is commutative (diagonal S-matrix case). To illustrate the situation when this consistency principle becomes crucial, we must complicate the model, introducing n species of particles with masses m_i , not necessarily different one from each other. Let's denote in and out states as

$$|\, A_{i_1}(\beta_1) \ldots A_{i_n}(\beta_n), \beta_1 > \ldots > \beta_n,\, in \,\rangle = a^{\dagger}_{i_1\, in}(\beta_1) \ldots a^{\dagger}_{i_n\, in}(\beta_n) \mid 0 \,\rangle \tag{1.32}$$

$$|\, A_{i_1}(\beta_1) \ldots A_{i_n}(\beta_n), \beta_1 > \ldots > \beta_n,\, out \,\rangle = a^{\dagger}_{i_1\, out}(\beta_1) \ldots a^{\dagger}_{i_n\, out}(\beta_n) \mid 0 \,\rangle \tag{1.33}$$

where obviously

$$\begin{aligned} \left[a_{i_1\, in}(\beta_1),\, a^{\dagger}_{i_2\, in}(\beta_2)\right] &= \delta_{i_1 i_2}\, 2\pi\, \delta(\beta_1 - \beta_2) \\ \left[a_{i_1\, in}(\beta_1),\, a_{i_2\, in}(\beta_2)\right] &= 0 \quad . \end{aligned} \tag{1.34}$$

Shortening the notation, the normalization is now

$$\begin{aligned} \langle\, A_{i'_1}(\beta'_1) \ldots A_{i'_m}(\beta'_m),\, in \,\rangle \mid A_{i_1}(\beta_1) \ldots A_{i_n}(\beta_n),\, in \,\rangle = \\ \delta_{m,n} \prod_{j=1}^{n} \delta_{i'_j, i_j}\; 2\pi\; \delta(\beta'_j - \beta_j) \end{aligned} \tag{1.35}$$

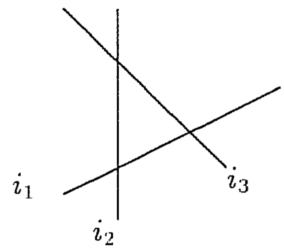

Figure 1.1: Elastic three-particle scattering process as a fixed sequence of two-particle ones.

Usual definition of S is

$$| A_{i_1}(\beta_1) \dots A_{i_n}(\beta_n), \, in \rangle = S \, | A_{i_1}(\beta_1) \dots A_{i_n}(\beta_n), \, out \rangle \, . \tag{1.36}$$

Pure elasticity still holds, with the further property that the number of species with different masses is conserved in the scattering process. The matrix element of S is thus

$$S^{i'_1 \dots, i'_m}_{i_1 \dots, i_n}(\{\beta'\}, \{\beta\}) = \delta_{m,n} \prod_{k=1}^{n} 2\pi \, \delta(\beta'_k - \beta_k) \, S^{i'_1 \dots, i'_m}_{i_1 \dots, i_n}(\{\beta_i - \beta_j\}) \tag{1.37}$$

which can be written as

$$| A_{i_1}(\beta_1) \dots A_{i_n}(\beta_n), \, in \rangle = \sum_{i'_1, \dots, i'_n} S^{i'_1 \dots, i'_m}_{i_1 \dots, i_n}(\{\beta_i - \beta_j\}) \, | A_{i'_1}(\beta_1) \dots A_{i'_n}(\beta_n), \, out \rangle \, . \tag{1.38}$$

Consistency of factorization of 3-scattering processes is necessary and sufficient to guarantee consistency for $\forall n$; it reads, with $\beta ij = \beta_i - \beta_j$

$$S^{i'_1 i'_2 i'_3}_{i_1 i_2 i_3}(\beta_{12}, \beta_{13}, \beta_{23}) = S^{i''_1 i''_2}_{i_1 i_2}(\beta_{12}) S^{i'_1 i''_3}_{i''_1 i_3}(\beta_{13}) S^{i'_2 i'_3}_{i''_2 i''_3}(\beta_{23}) \tag{1.39}$$

(see the diagram representation of fig. 1.1 which fixes the sequence of 2- scattering processes). The same 3-scattering process can be obtained by a different sequence of 2-scattering ones shown in fig. 1.2 and therefore we arrive to the famous star-triangle or Yang-Baxter equation [5]

$$S^{i''_1 i''_2}_{i_1 i_2}(\beta_{12}) S^{i'_1 i''_3}_{i''_1 i_3}(\beta_{13}) S^{i'_2 i'_3}_{i''_2 i''_3}(\beta_{23}) = S^{i''_2 i''_3}_{i_2 i_3}(\beta_{23}) S^{i''_1 i'_3}_{i_1 i''_3}(\beta_{13}) S^{i'_1 i'_2}_{i''_1 i''_2}(\beta_{12}) \, . \tag{1.40}$$

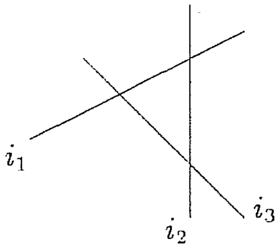

Figure 1.2: Elastic three-particle scattering process as a fixed sequence of two-particle ones, different from that shown in fig. 1.1.

All this is conveniently formulated by a non-commutative algebraic approach [16]. Let each particle be represented by the non commutative symbol $A_i(\beta)$ and each asymptotic in-state by the product of these symbols in decreasing order of the rapidities (the opposite conventions for the out-states):

$$\begin{aligned} \mid A_{i_1}(\beta_1)\ldots A_{i_n}(\beta_n),\ \beta_1 > \ldots > \beta_n,\ in\ \rangle &\sim A_{i_1}(\beta_1)\ldots A_{i_n}(\beta_n) \\ \mid A_{i_1}(\beta_1)\ldots A_{i_n}(\beta_n),\ \beta_1 > \ldots > \beta_n,\ out\ \rangle &\sim A_{i_n}(\beta_n)\ldots A_{i_1}(\beta_1)\ . \end{aligned} \tag{1.41}$$

The commutation property of these symbols represents the two-particles scattering

$$A_{i_1}(\beta_1)A_{i_2}(\beta_2) = \sum_{i'_1,i'_2} S_{i_1\,i_2}^{i'_1\,i'_2}(\beta_{12})A_{i'_2}(\beta_2)A_{i'_1}(\beta_1)\ . \tag{1.42}$$

So the three-particle state

$$A_{i_1}(\beta_1)A_{i_2}(\beta_2)A_{i_3}(\beta_3) = \sum_{i'_1,i'_2,i'_3} S_{i_1\,i_2\,i_3}^{i'_1\,i'_2\,i'_3}(\{\beta_{ij}\})A_{i'_3}(\beta_3)A_{i'_2}(\beta_2)A_{i'_1}(\beta_1) \tag{1.43}$$

can be obtained in terms of two-particles S-matrix by commuting iteratively pairs of A_i

$$\begin{aligned} A_{i_1}(\beta_1)A_{i_2}(\beta_2)A_{i_3}(\beta_3) &= S_{i_1\,i_2}^{i''_1\,i''_2}(\beta_{12})A_{i''_2}(\beta_2)A_{i''_1}(\beta_1)A_{i_3}(\beta_3) \\ &= S_{i_1\,i_2}^{i''_1\,i''_2}(\beta_{12})S_{i''_1\,i_3}^{i'_1\,i''_3}(\beta_{13})A_{i''_2}(\beta_2)A_{i''_3}(\beta_3)A_{i'_1}(\beta_1) \\ &= S_{i_1\,i_2}^{i''_1\,i''_2}(\beta_{12})S_{i''_1\,i_3}^{i'_1\,i''_3}(\beta_{13})S_{i''_2\,i''_3}^{i'_2\,i'_3}(\beta_{23})A_{i'_3}(\beta_3)A_{i'_2}(\beta_2)A_{i'_1}(\beta_1) \\ &= S_{i_2\,i_3}^{i''_2\,i''_3}(\beta_{23})A_{i_1}(\beta_1)A_{i''_3}(\beta_3)A_{i''_2}(\beta_2) \\ &= S_{i_2\,i_3}^{i''_2\,i''_3}(\beta_{23})S_{i_1\,i''_3}^{i''_1\,i'_3}(\beta_{13})A_{i'_3}(\beta_3)A_{i''_1}(\beta_1)A_{i''_2}(\beta_2) \\ &= S_{i_2\,i_3}^{i''_2\,i''_3}(\beta_{23})S_{i_1\,i''_3}^{i''_1\,i'_3}(\beta_{13})S_{i''_1\,i''_2}^{i'_1\,i'_2}(\beta_{12})A_{i'_3}(\beta_3)A_{i'_2}(\beta_2)A_{i'_1}(\beta_1)\ . \end{aligned} \tag{1.44}$$

Comparing the third line with the sixth line we obtain again the YB equation. This non commutative picture can be furtherly formalized by the assumption that for quantum integrable models there exists a set of creation and destruction operators denoted by $Z^i(\beta)$ such satisfying the following algebra

$$\begin{aligned} Z^\dagger_{i_1}(\beta_1)\, Z^\dagger_{i_2}(\beta_2) &= S^{i'_1\, i'_2}_{i_1\, i_2}(\beta_{12})\, Z^\dagger_{i'_2}(\beta_2)\, Z^\dagger_{i'_1}(\beta_1) \\ Z^{i_1}(\beta_1)\, Z^\dagger_{i_2}(\beta_2) &= Z^\dagger_{i'_2}(\beta_2)\, Z^{i'_1}(\beta_1)\, S^{i_1\, i'_2}_{i'_1\, i_2}(\beta_{12}) + \delta^{i_1}_{i_2}\, 2\pi\, \delta(\beta_1 - \beta_2)\ , \end{aligned} \tag{1.45}$$

with

$$Z^i(\beta) \mid 0 \rangle = 0. \tag{1.46}$$

Hence we can define

$$\begin{aligned} \mid \beta_1 > \beta_2 > \ldots > \beta_n,\, in \,\rangle &= Z^\dagger_{i_1}(\beta_1) Z^\dagger_{i_2}(\beta_2) \ldots Z^\dagger_{i_n}(\beta_n) \mid 0 \,\rangle \\ &\equiv |\overleftarrow{\{\beta\}}\,\rangle\ , \\ \mid \beta_1 > \beta_2 > \ldots > \beta_n,\, out \,\rangle &= Z^\dagger_{i_n}(\beta_n) Z^\dagger_{i_{n-1}}(\beta_{n-1}) \ldots Z^\dagger_{i_1}(\beta_1) \mid 0 \,\rangle \\ &\equiv |\overrightarrow{\{\beta\}}\,\rangle\ . \end{aligned} \tag{1.47}$$

These definitions automatically take into account the factorization property of quantum integrable models. The connection between the operators $Z(\beta)$ and $a_{in}(\beta)$, $a_{out}(\beta)$ is

$$\begin{aligned} a_{in}(\sigma)\, |\overleftarrow{\{\beta\}}\,\rangle &= \sum_{i=1}^{n} \delta(\sigma - \beta_i)\, |\overleftarrow{\{\beta - \beta_i\}}\,\rangle\ , \\ a_{out}(\sigma)\, |\overrightarrow{\{\beta\}}\,\rangle &= \sum_{i=1}^{n} \delta(\sigma - \beta_i)\, |\overrightarrow{\{\beta - \beta_i\}}\,\rangle\ , \end{aligned} \tag{1.48}$$

i.e.

$$\begin{aligned} |\overleftarrow{\{\beta\}}\,\rangle &= a^\dagger_{in}(\beta_1) a^\dagger_{in}(\beta_2) \ldots a^\dagger_{in}(\beta_n) \mid 0 \,\rangle \\ |\overrightarrow{\{\beta\}}\,\rangle &= a^\dagger_{out}(\beta_1) a^\dagger_{out}(\beta_2) \ldots a^\dagger_{out}(\beta_n) \mid 0 \,\rangle\ . \end{aligned} \tag{1.49}$$

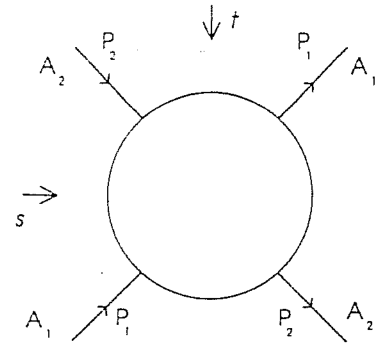

Figure 1.3: Two-particles scattering process $A_1 \times A_2 \longrightarrow A_1 + A_2$ and kinematical variables.

1.3 Analytical Properties of the Two-particles S-matrix

For two-dimensional two-particles elastic scattering

$$A_1 \times A_2 \longrightarrow A_1 + A_2 \tag{1.50}$$

depicted in fig. 1.3, with external momenta

$$p_i^0 = m \cosh \beta_i \ , \ \ p_i^1 = m \sinh \beta_i \quad , i = 1, 2 \ , \tag{1.51}$$

there is only one independent kinematical variable. The Mandelstam variable $s = (p_1 + p_2)^2$ becomes

$$s = m_1^2 + m_2^2 + 2\, m_1\, m_2 \cosh(\beta_1 - \beta_2) \tag{1.52}$$

Scattering obviously occurs for $\beta_{12} \in \mathcal{R}$, $s \geq (m_1 + m_2)^2$. But locality of Quantum Field Theory has some implications which can be exploited on the analytical continuation of $S_{i_1\, i_2}^{i_1'\, i_2'}(s)$ to the complex plane. For example, crossed scattering, i.e. the scattering process

$$\overline{A}_1 \times A_2 \longrightarrow \overline{A}_1 + A_2 \tag{1.53}$$

can be expressed, due to locality, in terms of the crossed $\bar{s}$ variable

$$\bar{s} = t = (-p_1 + p_2)^2 = m_1^2 + m_2^2 - 2\, m_1\, m_2 \ \cosh \beta_{12} \tag{1.54}$$

as argument of the analytically continued direct scattering S-matrix. Crossed scattering process thus occurs for $s \leq (m_1 - m_2)^2$ which corresponds to the line $\mathrm{Im}\beta_{12} = \pi$. We thus expect $S_{i_1\, i_2}^{i_1'\, i_2'}(s)$ to have two branching point at $s = (m_1 \pm m_2)^2$ with cuts going to infinity. Elasticity of the scattering, eq.(1.28), forbids n-particle thresholds with $n > 2$. In the intermediate region bound states, i.e. poles of $S_{i_1\, i_2}^{i_1'\, i_2'}(s)$, can occur. The intermediate region corresponds to the segment $\mathrm{Re}\beta_{12} = 0$, $0 < \mathrm{Im}\beta_{12} < \pi$

in the rapidity variables. Parametrization (1.52) tells us that the complex s-plane is mapped into the half-cylinder

$$0 \leq \mathrm{Im}\beta_{12} < 2\pi \,, \mathrm{Re}\beta_{12} \geq 0 \quad , \tag{1.55}$$

where the lines $\mathrm{Im}\beta_{12} = 0, \pi$ are the two cuts with branchings in $\mathrm{Re}\beta_{12} = 0$. Since the delta function in eq. (1.16) with $n, m = 2$ transforms on the mass-shell as

$$\begin{aligned} &\lim_{\substack{p_i^2 \to m_i^2 \\ p_j'^2 \to m_j^2}} (2\pi)^2 \, \delta^2(p_1 + p_2 + p_1' + p_2') = \\ &= \frac{1}{m_i m_j \sinh(\beta_{12})} \, \delta(\beta_1 - \beta_1') \, \delta(\beta_2 - \beta_2') \\ &= \frac{2}{\sqrt{(s - (m_1 - m_2)^2)\,(s - (m_1 + m_2)^2)}} \, \delta(\beta_1 - \beta_1') \, \delta(\beta_2 - \beta_2') \quad , \end{aligned} \tag{1.56}$$

the parametrization (1.52) uniformizes $S_{i_1\, i_2}^{i_1'\, i_2'}(\beta_{12})$ as defined in (1.29).

Another fundamental property is the unitarity (1.20), which becomes for the two-particles S-matrix

$$\sum_{i_1', i_2'} (S_{k_1\, k_2}^{i_1'\, i_2'}(s))^* \; S_{i_1\, i_2}^{i_1'\, i_2'}(s) = \delta_{i_1\, k_1} \; \delta_{i_2\, k_2} \quad . \tag{1.57}$$

Real analyticity is the property that $S_{i_1\, i_2}^{i_1'\, i_2'}(s)$ is real in the intermediate region, i.e. under the threshold; for complex s it becomes

$$S_{i_1\, i_2}^{i_1'\, i_2'}(s) = (S_{i_1\, i_2}^{i_1'\, i_2'}(s^*))^* \quad . \tag{1.58}$$

In the physical sheet $0 \leq \mathrm{Im}\beta \leq \pi$ it becomes

$$S_{i_1\, i_2}^{i_1'\, i_2'}(\beta) = (S_{i_1\, i_2}^{i_1'\, i_2'}(-\beta^*))^* \tag{1.59}$$

and in the scattering region $\mathrm{Im}\beta = 0$

$$S_{i_1\, i_2}^{i_1'\, i_2'}(\beta) = (S_{i_1\, i_2}^{i_1'\, i_2'}(-\beta))^* \quad . \tag{1.60}$$

Unitarity can thus be expressed as

$$\sum_{i_1', i_2'} S_{k_1\, k_2}^{i_1'\, i_2'}(-\beta) \; S_{i_1\, i_2}^{i_1'\, i_2'}(\beta) = \delta_{i_1\, k_1} \; \delta_{i_2\, k_2} \quad . \tag{1.61}$$

This is continued for any value of β .

For diagonal S-matrices, which occur in the case of n species of particles with different masses, we have

$$S_{k_1\, k_2}^{i_1\, i_2}(\beta) \equiv \delta_{k_1}^{i_1} \, \delta_{k_2}^{i_2} \; S_{k_1\, k_2}(\beta) \quad . \tag{1.62}$$

This implies that $S_{k_1 k_2}(\beta)$ is a pure phase.

Charge conjugation is implemented in our notation by requiring that among the fundamental particles which are assumed to exist $\{A_i\}$, for each particle A_i exists another particle A_j with the same mass such that $\overline{A}_i = A_j$. When $i=j$ A_i is a self-conjugated particle. In this notation crossing symmetry becomes

$$S_{k_1 k_2}^{i_1 i_2}(\beta) = S_{\overline{k_1} k_2}^{\overline{i_1} i_2}(i\pi - \beta) \tag{1.63}$$

We end this section summarizing the fundamental properties of two-particles S-matrix $S_{i_1 i_2}^{i'_1 i'_2}(\beta)$ of quantum integrable models so far discussed:

- Monodromy: the function $S_{i_1 i_2}^{i'_1 i'_2}(\beta)$ has no branching points.
- Unitarity:
$$\sum_{i'_1, i'_2} S_{k_1 k_2}^{i'_1 i'_2}(-\beta)\, S_{i_1 i_2}^{i'_1 i'_2}(\beta) = \delta_{i_1 k_1}\, \delta_{i_2 k_2} \ . \tag{1.64}$$
- Crossing symmetry:
$$S_{k_1 k_2}^{i_1 i_2}(\beta) = S_{\overline{k_1} k_2}^{\overline{i_1} i_2}(i\pi - \beta) \ . \tag{1.65}$$
- Factorization:
$$S_{i_1 i_2}^{i''_1 i''_2}(\beta_{12}) S_{i''_1 i_3}^{i'_1 i''_3}(\beta_{13}) S_{i''_2 i''_3}^{i'_2 i'_3}(\beta_{23}) = S_{i_2 i_3}^{i''_2 i''_3}(\beta_{23}) S_{i_1 i''_3}^{i''_1 i'_3}(\beta_{13}) S_{i''_1 i''_2}^{i'_1 i'_2}(\beta_{12}) \ . \tag{1.66}$$

If we restrict ourselves to the diagonal case last three properties become

- Unitarity:
$$S_{a\,b}(-\beta)\, S_{a\,b}(\beta) = 1. \tag{1.67}$$
- Crossing symmetry:
$$S_{a\,b}(\beta) = S_{\bar{a}\,b}(i\pi - \beta). \tag{1.68}$$
- Factorization is trivially satisfied.

These two equations imply periodicity of S_{ab} . It can be shown [21] that the general solution of (1.67) is

$$S_{a\,b}(\beta) = \prod_{x \in I_{a\,b}} u_x(\beta) \tag{1.69}$$

$$u_x(\beta) = \frac{\sinh \frac{1}{2}(\beta + i\pi x)}{\sinh \frac{1}{2}(\beta - i\pi x)} \tag{1.70}$$

where x can be restricted to the segment $-1 < x \leq 1$ due to the periodicity of $u_x(\beta)$. Simple poles of $u_x(\beta)$ in $\beta = i\pi x$ will determine the presence of bound states. In the case of self-conjugated particles

$$S_{ab}(\beta) = S_{ab}(i\pi - \beta). \tag{1.71}$$

we have the solution of (1.67) and (1.71) in the form

$$S_{ab}(\beta) = \prod_{x \in I_{ab}} f_x(\beta) \tag{1.72}$$

$$f_x(\beta) = \frac{\tanh \frac{1}{2}(\beta + i\pi x)}{\tanh \frac{1}{2}(\beta - i\pi x)} \tag{1.73}$$

Now the two poles of $f_x(\beta)$ in $\beta = i\pi x$ and $\beta = i\pi(1-x)$ corresponds to the occurence of bound states in the direct and crossed channels.

The set I_{ab} must be fixed by dynamics. Determining I_{ab} for $\forall\ (a,b)$ of the model means to determine completely the analytical structure of the S-matrix, and so to resolve on-shell physics of the model under consideration. The boostrap principle which we are going to describe in the next section establishes fundamental relations between different poles of the S-matrix which have to be fulfilled for sake of consistency of the theory.

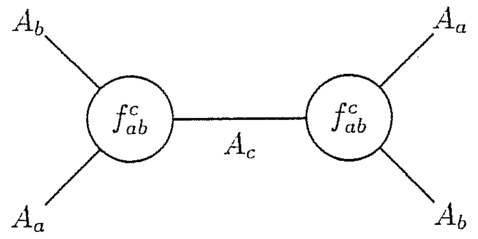

Figure 1.4: Bound state pole in scattering amplitude.

1.4 Bootstrap Principle for the S-matrix

Let's put ourselves for simplicity in the case of a system whose S-matrix is diagonal and possesses a single pole in $s = m_c^2$

$$S_{ab} \sim \Gamma_{abc} \frac{i}{s - m_c^2} \Gamma_{abc} \quad . \tag{1.74}$$

This pole can be interpreted as a propagator with mass m_c and this means that the system $\{A_a\}$ must contain a particle A_c with mass m_c. The location of this pole on the segment $0 < \mathrm{Im}\beta < \pi$ is at $\beta = iu^c_{ab}$ which is fixed in terms of the masses by

$$m_c^2 = m_a^2 + m_b^2 + 2m_a m_b \cos u^c_{ab} \quad . \tag{1.75}$$

We can describe diagrammatically this kinematical situation as shown in fig. 1.4. The residue R_{abc} of S_{ab} at the angle u^c_{ab}

$$S_{ab} \sim \frac{iR_{abc}}{(\beta - iu^c_{ab})} \tag{1.76}$$

can thus be interpreted as $R_{abc} = f^c_{ab} f^c_{ab}$, being f^c_{ab} a three vertex function $\Gamma^c_{ab}(p_1, p_2, p_3)$ at a fixed value of external momenta and so playing the role of a coupling constant. Since each particle of the set $\{A_a\}$ is at the same level of fundamentality of the others, f^c_{ab} must be symmetric in the particle indices, $f^c_{ab} = f_{abc}$, and we must find simple poles also in S_{ac} and S_{bc} at the values of the angles u^b_{ac}, u^a_{bc} given by

$$\begin{aligned} m_b^2 &= m_a^2 + m_c^2 + 2\, m_a\, m_c \; \cos u^b_{ac} \\ m_a^2 &= m_b^2 + m_c^2 + 2\, m_b\, m_c \; \cos u^a_{bc} \end{aligned} \tag{1.77}$$

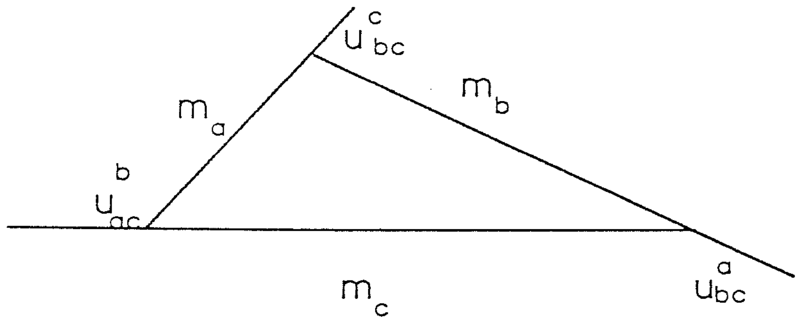

Figure 1.5: Geometric relation between masses and angles of the bound state $a\ b \rightarrow c$.

We can give a geometric picture of the mass-angle relations as in fig. 1.5. The angles thus are not independent

$$u_{ab}^{c} + u_{bc}^{a} + u_{ac}^{b} \;=\; 2\pi \quad . \tag{1.78}$$

These poles must show themselves also in the multiparticle scattering amplitudes; we are allowed to determine the two-particles scattering amplitude S_{cd} of the particle c with any other particle d by

$$S_{cd}(\beta_{cd}) \;=\; \left[\mathrm{Res}_{\beta_{ab}=iu_{ab}^{c}} S_{abd}\right] / \left[\mathrm{Res}_{\beta_{ab}=iu_{ab}^{c}} S_{ab}\right] \tag{1.79}$$

where $\beta_c = \beta_a - i\overline{u}_{ac}^{b}$ and $\overline{u}_{ac}^{b} = \pi - u_{ac}^{b}$. Taking into account factorization, we end up with the boostrap principle equation

$$S_{cd}(\beta) \;=\; S_{ad}(\beta + i\overline{u}_{ac}^{b}) S_{bd}(\beta - i\overline{u}_{bc}^{a}) \tag{1.80}$$

We can express this fusion structure $a\,b \rightarrow c$ in terms of formal analytical continuation of asymptotic states

$$\mid A_c(\beta)\ \rangle \;=\; \lim_{\epsilon\rightarrow 0}\ \epsilon \mid A_a(\beta + i\overline{u}_{ac}^{b} - \frac{\epsilon}{2})\ A_b(\beta - i\overline{u}_{bc}^{a} + \frac{\epsilon}{2}),\ in\ \rangle \tag{1.81}$$

If we apply the conserved charge operators to eqs $\mathcal{Q}_s$ and (1.81), using eq. (1.26) as well, we obtain the consistency boostrap equations

$$M_s^c \;=\; M_s^a e^{is\overline{u}_{ac}^{b}} \;+\; M_s^b e^{-is\overline{u}_{bc}^{a}} \tag{1.82}$$

This set of equations are satisfied trivially by $M_s^c = 0$, but this solution does not correspond to a quantum integrable model because all higher charges would be absent. For non-trivial values of M_s^c they impose severe limitations to the possible values of conserved spin s , once the structure of the model in terms of fundamental particles is guessed. Let's show this with a very simple example. Let's consider a theory of only one self-conjugated particle with the fusion structure

$$A_1\ A_1 \;\longrightarrow\; A_1 \tag{1.83}$$

The only bound state angle is $u_{11}^1 = \frac{2\pi}{3}$, from (1.78). Bootstrap consistency (1.82) becomes

$$2\cos\left(s\,\frac{\pi}{3}\right) = 1 \tag{1.84}$$

which is solved for only $s = 1\,,5\,(\mathrm{mod}6)$.

As another example we add to the previous model a particle A_2 such that

$$\begin{aligned} A_1 \times A_1 &\longrightarrow A_1 + A_2 \\ A_2 \times A_2 &\longrightarrow A_1 \end{aligned} \tag{1.85}$$

So in addition to the constraint (1.84) coming from the pole $A_1 \times A_1 \to A_1$, we have from (1.82)

$$\begin{aligned} M_s^2 &= 2M_s^1\cos(s\overline{u}_{12}^1) \\ M_s^1 &= 2M_s^2\cos(s\overline{u}_{21}^2) \end{aligned} \tag{1.86}$$

or

$$\cos(s\overline{u}_{12}^1)\cos(s\overline{u}_{21}^2) = \frac{1}{2} \tag{1.87}$$

This equation have two solutions:

$$\overline{u}_{12}^1 = \frac{\pi}{12}\;,\overline{u}_{21}^2 = \frac{5\pi}{12},\quad s = 1,4,5,7,8,11\,(\mathrm{mod}12) \tag{1.88}$$

and

$$\overline{u}_{12}^1 = \frac{\pi}{5}\;,\overline{u}_{21}^2 = \frac{2\pi}{5},\quad s = 1,3,7,9\,(\mathrm{mod}10) \tag{1.89}$$

which become, including $\overline{u}_{11}^1 = \frac{\pi}{3}$, $s = 1\,,5\,(\mathrm{mod}6)$

$$s = 1,5,7,11\,(\mathrm{mod}12) \tag{1.90}$$

and

$$s = 1,7,11,13,17,19,23,29\,(\mathrm{mod}30)\;. \tag{1.91}$$

We have thus seen that guessing the qualitative particle content of a quantum integrable model, thanks to the boostrap principle we are able to determine the scattering angles, mass ratios and the spin of the effectively conserved charges. But the power of the boostrap principle can give more. The very dynamical content of the boostrap principle is that, starting from the lightest particle A_1 and its S-matrix S_{11} , it gives us

$$S_{i1}(\beta) = S_{11}(\beta + i\overline{u}_{1i}^1)S_{11}(\beta - i\overline{u}_{1i}^1) \tag{1.92}$$

where A_i are the fundamental particles which appear as bound states of the scattering process

$$A_1 \times A_1 \longrightarrow \sum_i A_i \tag{1.93}$$

This argument can now be iterated to the new particles appearing as bound states of the S_{i1} 's. The Boostrap program is devoted to exploit this iteration consistently to end up with perfectly well defined on-shell physics of some quantum integrable model. We stop here and for more details on the Boostrap program we refer to the references [16, 20].

The Boostrap program provides us with a method of classifying possible S-matrices. But in order to make a theory complete we need its local structure, i.e. off-shell physics. In other words we need to connect a possible S-matrix with a Lagrangian, or more generally with a complete set [2] of correlation functions. Many results has already been reached in this direction for many models. In order to get some collection of them again we refer to the references [20].

Next chapter is devoted to show how the formalism of form factors can be a very useful approach towards off-shell physics.

Chapter 2

Form Factors Approach

A complete description of a given theory is equivalent to the knowledge of the whole set of correlation functions

$$\langle\, \mathcal{O}_{i_1}(x_1)\ldots\mathcal{O}_{i_n}(x_n)\,\rangle \equiv \langle\, 0 \mid T(\mathcal{O}_{i_1}(x_1)\ldots\mathcal{O}_{i_n}(x_n)) \mid 0\,\rangle \tag{2.1}$$

where $\mathcal{O}_i(x)$ are the local operators of the theory on which the group of translations acts with the unitary operator U_{T_a}

$$U_{T_a}\mathcal{O}_i(x)U_{T_a}^{-1} = \mathcal{O}_i(x+a) \ . \tag{2.2}$$

Making use of the base of the in-states (1.3), (1.47) in order to split the product of field operators for $x_1^0 > x_2^0 > \ldots > x_n^0$

$$\begin{aligned} &\langle\, 0 \mid T(\mathcal{O}_{i_1}(x_1)\ldots\mathcal{O}_{i_n}(x_n)) \mid 0\,\rangle = \\ &\sum_{j=1}^{n-1}\sum_{\{\alpha^{(j)}\}} \langle\, 0 \mid \mathcal{O}_{i_1}(x_1) \mid \{\alpha^{(1)}\},\, in\,\rangle\langle\, \{\alpha^{(1)}\},\, in \mid \mathcal{O}_{i_2}(x_2) \mid \{\alpha^{(2)}\},\, in\,\rangle \\ &\langle\, \{\alpha^{(2)}\},\, in \mid \mathcal{O}_{i_3}(x_3)\ldots\langle\, \{\alpha^{(n-1)}\},\, in \mid \mathcal{O}_{i_n}(x_n) \mid 0\,\rangle \end{aligned} \tag{2.3}$$

it becomes clear that the knowledge of the matrix elements

$$\langle\, \{\alpha\},\, in \mid \mathcal{O}_i(x) \mid \{\beta\},\, in\,\rangle \tag{2.4}$$

is equivalent to the knowledge of the correlation functions. By means of crossing symmetry and translation symmetry, every information is encoded in the set of the so called form factors

$$\langle\, 0 \mid \mathcal{O}_i(0) \mid \beta_1,\beta_2,\ldots,\beta_n, in\,\rangle \equiv F_n^{\mathcal{O}_i}(\beta_1,\beta_2,\ldots,\beta_n) \ . \tag{2.5}$$

The inverse formula, i.e. from correlation functions to form factors, is straightforwardly obtained thanks to LSZ formalism (1.3)-(1.13),[3]

$$F_n^{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_n) = \left(\frac{1}{\sqrt{2}}\right)^n \lim_{p_i^2\to m^2}\prod_{i=1}^{n}\left(\frac{p_i^2-m^2}{i}\right) G^{n,\mathcal{O}}(q=-\sum_{i=1}^{n}p_i, p_1,p_2,\ldots,p_n) \tag{2.6}$$

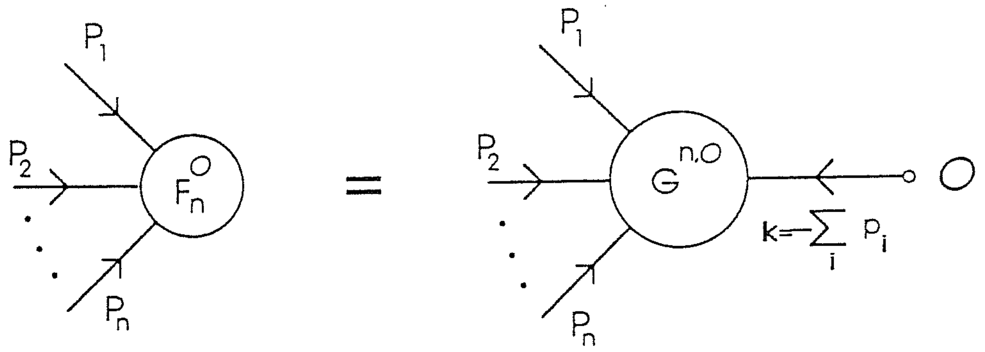

Figure 2.1: Diagrammatic representation of the form factors in terms of the correlation functions.

where

$$(2\pi)^2\delta^2(q+\sum p_i)G^{n,\mathcal{O}}(q,p_1,p_2,\ldots,p_n) = \int\prod_{i=1}^{n} dx_i\,dy e^{-i\sum p_i x_i}e^{-iqy}\langle\ 0\mid T(\mathcal{O}(y)\phi(x_1)\phi(x_2)\ldots\phi(x_n))\mid 0\ \rangle \tag{2.7}$$

for a theory of one scalar self-conjugated boson, (fig. 2.1).

The form factor $F_n^{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_n)$ is, strictly speaking, a function of the n variables $(\beta_1,\ldots,\beta_n)$ in the region $\beta_1 > \beta_2 > \ldots, > \beta_n$. In terms of the Zamolodchikov-Faddeev vertex operators $Z^i(\beta)$ (1.45), it is natural to extend this definition to the whole $\mathcal{R}^n$

$$F_n^{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_n) = \langle\ 0\mid \mathcal{O}(0)Z^\dagger(\beta_1)Z^\dagger(\beta_2)\ldots Z^\dagger(\beta_n)\mid 0\ \rangle \tag{2.8}$$

which reduces to (2.5) in the region $\beta_1 > \beta_2 > \ldots, > \beta_n$. With this definition we immediately realize of the property

$$F_n^{\mathcal{O}}(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) = S(\beta_i-\beta_{i+1})F_n^{\mathcal{O}}(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n) \tag{2.9}$$

which follows directly from the Zamolodchikov-Faddeev algebra (1.45).

Locality of the theory imposes well defined analytical properties of the form factors. We take them into account stating

$$F_n^{\mathcal{O}}(\beta_1+2\pi i,\beta_2,\ldots,\beta_n) = \prod_{j=2}^{n} S(\beta_j-\beta_1)F_n^{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_n) \tag{2.10}$$

which becomes, using (2.9)

$$F_n^{\mathcal{O}}(\beta_1+2\pi i,\beta_2,\ldots,\beta_n) = F_n^{\mathcal{O}}(\beta_2,\ldots,\beta_n,\beta_1)\ . \tag{2.11}$$

For the case $n=2$ (2.10) is

$$\begin{aligned} F_2^{\mathcal{O}}(\beta_1+2\pi i,\beta_2) &= S(\beta_2-\beta_1)F_2^{\mathcal{O}}(\beta_1,\beta_2) \\ &= F_2^{\mathcal{O}}(\beta_2,\beta_1) \ . \end{aligned} \tag{2.12}$$

For a scalar field $\mathcal{O}(x)$ Lorentz invariance implies $F_2^{\mathcal{O}}(\beta_1,\beta_2)=F_2^{\mathcal{O}}(\beta_1-\beta_2)$; so we can rewrite (2.12) as

$$F_2^{\mathcal{O}}(\beta_{12}) = S(\beta_{12})F_2^{\mathcal{O}}(-\beta_{12}) \tag{2.13}$$

$$F_2^{\mathcal{O}}(i\pi-\beta_{12}) = F_2^{\mathcal{O}}(i\pi+\beta_{12}) \ . \tag{2.14}$$

As an example we prove these equations following mainly [26]:

$$\begin{aligned} F_2^{\mathcal{O}}(\beta_{12}) &= \langle\, 0 \mid \mathcal{O}(0) \mid \beta_1,\beta_2,\ in\,\rangle \\ &= S(\beta_1-\beta_2)\langle\, 0 \mid \mathcal{O}(0) \mid \beta_1,\beta_2,\ out\,\rangle \end{aligned} \tag{2.15}$$

due to the complete integrability of the theory. Now if $\mathcal{O}^\dagger=\mathcal{O}$ we have

$$\langle\, 0 \mid \mathcal{O}(0) \mid \beta_1,\beta_2,\ out\,\rangle = (\langle\, \beta_1,\beta_2,\ out \mid \mathcal{O}(0) \mid 0\,\rangle)^*\,, \tag{2.16}$$

and CPT symmetry implies

$$(\langle\, \beta_1,\beta_2,\ out \mid \mathcal{O}(0) \mid 0\,\rangle)^* = \langle\, 0 \mid \mathcal{O}(0) \mid i\pi-\beta_1, i\pi-\beta_2,\ in\,\rangle \tag{2.17}$$

$$= F_2^{\mathcal{O}}(-\beta_{12}) \tag{2.18}$$

where $i\pi-\beta$ implements the CPT-transformation

$$P^\mu = (m\cosh(\beta), m\sinh(\beta)) \xrightarrow{\text{CPT}} (-m\cosh(\beta), m\sinh(\beta)). \tag{2.19}$$

Eq. (2.13) comes putting all together.

Analogously for eq. (2.14), let's start from

$$\begin{aligned} \langle\, \beta_1,\ out \mid \mathcal{O}(0) \mid \beta_2,\ in\,\rangle &= \langle\, 0 \mid \mathcal{O}(0) \mid \beta_2, \beta_1-i\pi,\ in\,\rangle \\ &= F_2^{\mathcal{O}}(i\pi-\beta_{12})\,, \end{aligned} \tag{2.20}$$

where we have implemented the crossing symmetry this way

$$(m\cosh(\beta), m\sinh(\beta))_{out} \xrightarrow{\text{Crossing}} (-m\cosh(\beta), -m\sinh(\beta))_{\overline{in}} \tag{2.21}$$

(so that $\beta_{12}\to i\pi-\beta_{12}$). For one-particle state the distinction between *in* and *out* is immaterial, $\mid \beta,\ out\,\rangle = \mid \beta,\ in\,\rangle$,

$$\begin{aligned} \langle\, \beta_1,\ out \mid \mathcal{O}(0) \mid \beta_2,\ in\,\rangle &= \langle\, \beta_1,\ in \mid \mathcal{O}(0) \mid \beta_2,\ out\,\rangle \\ &= (\langle\, \beta_2,\ out \mid \mathcal{O}(0) \mid \beta_1,\ in\,\rangle)^*\,. \end{aligned} \tag{2.22}$$

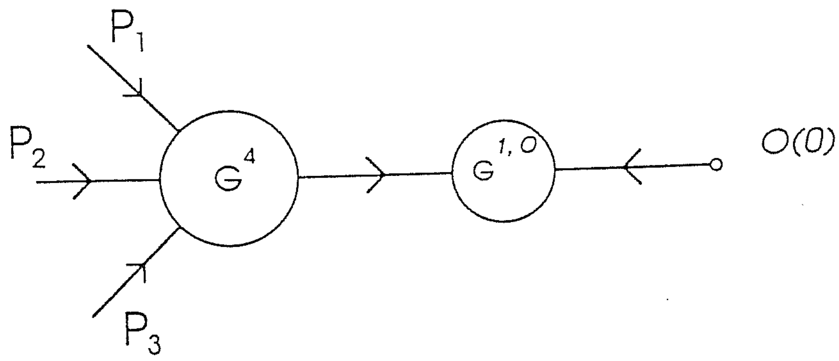

Figure 2.2: Diagrammatic representation of the contribution to the kinematical pole of F_3 .

Again CPT implies

$$(\langle\ \beta_2,\ out \mid \mathcal{O}(0) \mid \beta_1,\ in\ \rangle)^* = \langle\ i\pi - \beta_1,\ out \mid \mathcal{O}(0) \mid i\pi - \beta_2,\ in\ \rangle \tag{2.23}$$

and crossing

$$\begin{aligned}\langle\ i\pi - \beta_1,\ out \mid \mathcal{O}(0) \mid i\pi - \beta_2,\ in\ \rangle &= \langle\ 0 \mid \mathcal{O}(0) \mid i\pi - \beta_2, -\beta_1,\ in\ \rangle \\ &= F_2^{\mathcal{O}}(i\pi + \beta_{12})\ .\end{aligned} \tag{2.24}$$

Comparing eq. (2.20) with eq. (2.24) we recover eq. (2.14). In other words, we have shown that using locality properties, i.e. CPT and crossing symmetries, we can determine the analytical properties of the theory in the language of form factors. The result is that discontinuity (2.10) on the cuts $\beta_{1i} = 2\pi i$ is determined by scattering.

In order to have a complete description of the analytical structure of the theory, we must state precisely the location of the possible poles of the form factors. The origin of the poles is twofold. Some poles surely occur kinematically, as we will see. Other poles can occur if bound states are present. To show how kinematical poles appear, let's consider the LSZ-formula for form factors (2.6). In a perturbative approach, let's turn our attention to the contribution to the $F_3^{\mathcal{O}}$ form factor from the diagrams of the kind shown in fig. 2.2. Clearly the propagator

$$\frac{i}{(\sum_{i=1}^{3} p_i)^2 - m^2} \tag{2.25}$$

factorizes. So a pole can appear when $(\sum_{i=1}^{3} p_i)^2 = m^2$. This happens when $p_1 + p_2 = 0$ or $p_1 + p_3 = 0$ or $p_2 + p_3 = 0$. In terms of the rapidities, $p_i + p_j = 0$ implies $\beta_i = \beta_j + i\pi$. So near the pole $p_2 + p_3 = 0$, $F_3^{\mathcal{O}}(p_1, p_2, p_3)$ from LSZ can be written as

$$F_3^{\mathcal{O}}(p_1, p_2, p_3 \sim -p_2) \sim \Gamma^4_{\text{Trunc}}(-p_1, p_1, p_2, -p_2)\frac{i}{(\sum_{i=1}^{3} p_i)^2 - m^2} G^{1,\mathcal{O}}(p_1)\ . \tag{2.26}$$

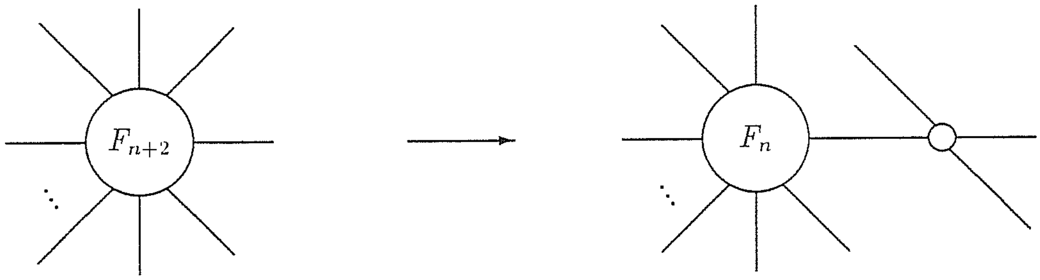

Figure 2.3: Kinematical recursive equation for the form factor F_n.

Recalling the LSZ formula for the S-matrix (1.16), we can take the residue in the rapidities variables

$$-i \lim_{\tilde{\beta} \to \beta} (\tilde{\beta} - \beta) F_3^{\mathcal{O}}(\tilde{\beta} + i\pi, \beta, \beta_1) = [1 - S(\beta - \beta_1)] F_1^{\mathcal{O}}(\beta_1) . \tag{2.27}$$

The elasticity property of the scattering processes implies that the only possibility to have kinematical poles for $F_n^{\mathcal{O}}$, $\forall n$, is through subprocesses of four particles as the one shown above. Therefore in order to fix the kinematical pole structure is sufficient to fix it for the four particle subprocess. Taking the residue of $F_{n+2}^{\mathcal{O}}$ at this kinematical pole, making use of LSZ formalism, one obtains a recursive relation between $F_{n+2}^{\mathcal{O}}$ and $F_n^{\mathcal{O}}$ (fig. 2.3),

$$-i \lim_{\tilde{\beta} \to \beta} (\tilde{\beta} - \beta) F_{n+2}^{\mathcal{O}}(\tilde{\beta} + i\pi, \beta, \beta_1, \ldots, \beta_n) = [1 - \prod_{i=1}^{n} S(\beta - \beta_i)] F_n^{\mathcal{O}}(\beta_1, \ldots, \beta_n) . \tag{2.28}$$

The second type of poles in the $F_n^{\mathcal{O}}$ only arises when bound states are present in the model. These poles are located at the values of β_{ij} in the physical strip which correspond to the resonance angles. Let $\beta_{ij} = iu_{ij}^k$ be one of such poles associated to the bound state A_k in the channel $A_i \times A_j$. For the S-matrix we have (fig. 2.4)

$$-i \lim_{\beta \to iu_{ij}^k} (\beta - iu_{ij}^k)\, S_{ij}(\beta) = \left(\Gamma_{ij}^k\right)^2 \tag{2.29}$$

where Γ_{ij}^k is the three-particle vertex on mass-shell. Near the pole $\beta_{ij} = iu_{ij}^k$ the contribution to $F_{n+1}^{\mathcal{O}}$ will come from the diagrams of the kind shown in fig. 2.5.

The corresponding residue for the $F_{n+1}^{\mathcal{O}}$ is given by

$$-i \lim_{\epsilon \to 0} \epsilon F_{n+1}^{\mathcal{O}}(\beta_1 + i\overline{u}_{ik}^j - \epsilon, \beta_1 - i\overline{u}_{jk}^i + \epsilon, \beta_2, \ldots, \beta_n) = \Gamma_{ij}^k F_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n) , \tag{2.30}$$

which can be depicted as in fig. 2.6.

We now summarize the properties of the form factors so far discussed

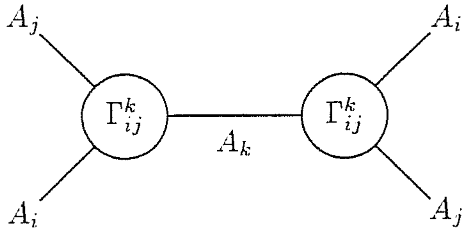

Figure 2.4: Bound state pole in scattering amplitude.

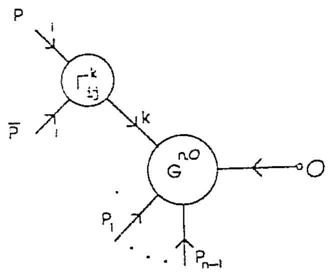

Figure 2.5: Diagrammatic representation of the contribution to the bound state pole of F_n .

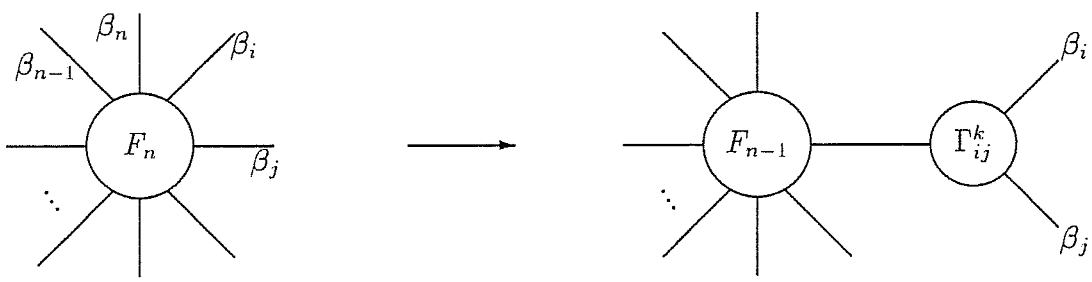

Figure 2.6: Bound state recursive equation for the form factor $\mathcal{F}_n$.

- $$F_n^{\mathcal{O}}(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) \;=\; S(\beta_i-\beta_{i+1})F_n^{\mathcal{O}}(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n) \quad (2.31)$$

- $$\begin{aligned} F_n^{\mathcal{O}}(\beta_1+2\pi i,\beta_2,\ldots,\beta_n) &= \prod_{j=2}^{n} S(\beta_j-\beta_1)F_n^{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_n) \\ &= F_n^{\mathcal{O}}(\beta_2,\ldots,\beta_n,\beta_1) \end{aligned}$$

- $$\begin{aligned} &-i \lim_{\tilde\beta\to\beta} (\tilde\beta-\beta)\; F_{n+2}^{\mathcal{O}}(\tilde\beta+i\pi,\beta,\beta_1,\ldots,\beta_n) \\ &\qquad = \left[1-\prod_{i=1}^{n} S(\beta-\beta_i)\right]\; F_n^{\mathcal{O}}(\beta_1,\ldots,\beta_n) \end{aligned} \quad (2.32)$$

- $$\begin{aligned} &-i \lim_{\epsilon\to 0} \epsilon\; F_{n+1}^{\mathcal{O}}(\beta_1+i\overline{u_{ik}^{j}}-\epsilon,\beta_1-i\overline{u_{jk}^{i}}+\epsilon,\beta_2,\ldots,\beta_n) \\ &\qquad = \Gamma_{ij}^{k}\; F_n^{\mathcal{O}}(\beta_1,\beta_2,\ldots,\beta_n)\ . \end{aligned} \quad (2.33)$$

Smirnov and Kirillov have shown that these equations can be regarded as a system of axioms for the whole local operator content of the theory. They have also shown that not only locality but also free asymptotics of the LSZ-approach can be recovered [27].

We close this chapter by mentioning that these equations are, strictly speaking, valid for spinless local operators. If $\mathcal{O}$ is an eigenstate, in the algebra of operators, of the Lorentz Spin operator with eigenvalues s

$$[\Sigma,\,\mathcal{O}] \;=\; s\mathcal{O} \quad (2.34)$$

then $F_n^{\mathcal{O}}$ will transform under Lorentz transformations as

$$F_n^{\mathcal{O}}(\beta_1+\Lambda,\ldots,\beta_n+\Lambda) \;=\; e^{s\Lambda}F_n^{\mathcal{O}}(\beta_1,\ldots,\beta_n)\ . \quad (2.35)$$

Spinless operators thus depend only on the difference of the rapidities $\beta_{ij}=\beta_i-\beta_j$.

Slight modifications of eq. (2.31) for general semilocal tensor operators can be found in [28].

Chapter 3

The Sinh-Gordon Model

3.1 Classical Integrability

As already anticipated in the introduction, in this thesis we present the solution to the problem of form factors for the Sinh-Gordon model. It is the simplest example of integrable model and it is described by the action

$$\mathcal{S} = \int d^2x \left[\frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{g^2}\cosh(g\phi)\right] \tag{3.1}$$

which gives rise to the equation of motion

$$\Box\phi(x) + \frac{m^2}{g}\sinh(g\phi(x)) = 0 \ . \tag{3.2}$$

Its Poisson structure is defined by the algebra

$$\left\{\phi(x^0,x^1),\, \partial_0\phi(x^0,x'^1)\right\} = \delta(x^1 - x'^1) \tag{3.3}$$

being $\phi(x)$ the classical variable and $\partial_0\phi(x)$ its canonically conjugate momentum. Boundary condition should also be precised, for example vanishing fields at infinity or periodicity on a cylinder. Its classical integrability can be established showing the equivalence of the equation of motion with a zero-curvature condition

$$\partial_0 U - \partial_1 V + [U, V] = 0 \tag{3.4}$$

where U and V are 2×2 matrices whose entries are functions of $(\phi(x), \partial_0\phi(x), \partial_1\phi(x))$ and of a spectral parameter λ

$$U = \frac{1}{2}\left[\begin{array}{ll} -\frac{g}{2}\partial_0\phi & -\sinh(\lambda - \frac{g}{2}\phi) \\ \sinh(\lambda + \frac{g}{2}\phi) & \frac{g}{2}\partial_0\phi \end{array}\right] \tag{3.5}$$

$$V = \frac{1}{2}\begin{bmatrix} -\frac{g}{2}\partial_1\phi & \cosh(\lambda - \frac{g}{2}\phi) \\ -\cosh(\lambda + \frac{g}{2}\phi) & \frac{g}{2}\partial_1\phi \end{bmatrix} \tag{3.6}$$

where we have put $m = 1$. U and V are determined modulo a gauge transformation

$$\begin{aligned} U &\rightarrow GUG^{-1} + (\partial_1 G)G^{-1} \\ V &\rightarrow GVG^{-1} + (\partial_0 G)G^{-1} \ . \end{aligned} \tag{3.7}$$

In terms of U we can define the transition matrices

$$\begin{aligned} T_L(\lambda) &= \mathcal{P}\left[e^{\int_0^L U(\lambda)dx}\right] \\ T(\lambda) &= \lim_{L\to+\infty} E_{-L}(\lambda)T_L(\lambda)E_L(\lambda) \end{aligned} \tag{3.8}$$

where $E_L(\lambda)$ is the asymptotic solution to the auxiliary linear problem

$$\partial_1 F = UF \ . \tag{3.9}$$

Using the Poisson algebra (3.3) one can show that

$$\frac{\partial}{\partial x^0}\operatorname{Tr} T(\lambda) = 0 \ . \tag{3.10}$$

So $\operatorname{Tr} T(\lambda)$ can be regarded as a generating functional of non trivial conserved charges.

A more direct way to obtain conserved currents is to use the so-called Backlund transformation. Let's introduce light-cone coordinates

$$(x^0, x^1) \rightarrow (x^+ = \frac{x^0 + x^1}{2}, x^- = \frac{x^0 - x^1}{2}) \ . \tag{3.11}$$

The metric tensor transforms as

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \ \mu = 0,1 \ \rightarrow \ g_{\mu'\nu'} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \ \mu' = +,- \ , \tag{3.12}$$

and

$$\Box = (\partial_0)^2 - (\partial_1)^2 = \partial_+\partial_- \ . \tag{3.13}$$

The equation of motion becomes

$$\partial_+\partial_-\phi(x) + \frac{m^2}{g}\sinh(g\phi(x)) = 0 \ . \tag{3.14}$$

Let's define a new field $\hat{\phi}(x^+, x^-, \epsilon)$ through the Backlund equations

$$\begin{aligned} \partial_+(\phi + \hat{\phi}) &= \frac{2m\epsilon}{g}\sinh(\frac{g}{2}(\phi - \hat{\phi})) \\ \partial_-(\phi - \hat{\phi}) &= -\frac{2m}{g\epsilon}\sinh(\frac{g}{2}(\phi + \hat{\phi})) \end{aligned} \tag{3.15}$$

It is immediate to see that ϕ is a solution of the equation of motion, the same is true for $\hat{\phi}$. One can check immediately that the following conservation law

$$\epsilon^{-1}\,\partial_+\left(\cosh\,\frac{g}{2}(\hat{\phi}+\phi)\right)+\epsilon\,\partial_-\left(\cosh\,\frac{g}{2}(\hat{\phi}-\phi)\right)=0\quad . \tag{3.16}$$

holds. Express now $\hat{\phi}$ as a formal power series in ϵ

$$\hat{\phi}(x^+,x^-,\epsilon)\,=\,\sum_{n=0}^{\infty}\phi^{(n)}(x^+,x^-)\,\epsilon^n \tag{3.17}$$

where the coefficient fields $\phi^{(n)}(x^+,x^-)$ are obtained in terms of ϕ and its derivatives through (3.15). Now from eq. (3.16) we can extract order by order the infinite set of conserved currents

$$\partial_+T_{s+1}\,+\,\partial_-\Theta_{s-1}\,=\,0 \tag{3.18}$$

whose conserved charges are

$$Q_s\,=\,\int dx^+\;T_{s+1}\,+\,\int dx^-\;\Theta_{s-1}\quad . \tag{3.19}$$

Indices means that Q_s , T_{s+1} and Θ_{s-1} are eigenvectors of the Lorentz Spin generator in the algebra of observables

$$\{\Sigma,\,Q_s\}\,=\,sQ_s \tag{3.20}$$

Being the Q_s 's of different spins, we are also sure that they are independent.

3.2 Quantum Integrability

As pointed out by [34], direct canonical quantization of the Poisson algebra gives rise to a Quantum Field Theory which, in general, loses the fundamental property of integrability. In order to maintain this property, the quantized transition matrix must satisfy an equation which can be seen as a generalization of the quantization procedure. For the Sinh-Gordon this equation takes the form

$$R(\lambda-\mu)T(\lambda)T(\mu) \;=\; T(\mu)T(\lambda)R(\lambda-\mu) \tag{3.21}$$

where the 4×4 matrix $R(\lambda)$ is

$$R(\lambda) \;=\; \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & c & 0 \\ 0 & c & b & 0 \\ 0 & 0 & 0 & a \end{bmatrix} \tag{3.22}$$

$$a = \sinh(\lambda - i\gamma)$$
$$b = \sinh(\lambda)$$
$$c = -i\sin(\gamma)$$
$$\gamma = \tfrac{g^2}{8}$$

and satisfies the Yang-Baxter equation [33].

For the quantized gauge field $U(x,\lambda)$,(3.21) becomes

$$\begin{aligned} &R(\lambda-\mu) \\ &\left[U(x,\lambda)\otimes 1 + 1\otimes \mathrm{U}(\mathrm{x},\mu) + \lim_{\Delta\to 0}\frac{1}{\Delta}\int\int_{\mathrm{x}}^{\mathrm{x}+\Delta} \mathrm{dz}^1\mathrm{dy}^1\mathrm{U}(\mathrm{z},\lambda)\otimes\mathrm{U}(\mathrm{y},\mu)\right] = \\ &\left[U(x,\lambda)\otimes 1 + 1\otimes \mathrm{U}(\mathrm{x},\mu) + \lim_{\Delta\to 0}\frac{1}{\Delta}\int\int_{\mathrm{x}}^{\mathrm{x}+\Delta} \mathrm{dz}^1\mathrm{dy}^1\mathrm{U}(\mathrm{z},\mu)\otimes\mathrm{U}(\mathrm{y},\lambda)\right] \\ &R(\lambda-\mu) \;\; . \end{aligned} \tag{3.23}$$

It is a lenghty but straightforward exercise to see that if one quantizes $U(x,\lambda)$ naively with canonical commutation relation, (3.23) can't be satisfied. Then, one has to guess the form of $U(x,\lambda)$ which fulfills (3.23). This corresponds, as already said, to a generalized quantizaton procedure. We note that the old correspondence principle is recovered in the semiclassical limit $\hbar \to 0$, which is equivalent to a weak coupling limit $g \to 0$

$$\frac{1}{\hbar}S[\phi,g] \;=\; \frac{1}{\hbar g^2}S[g\phi=\phi',1] \; . \tag{3.24}$$

This is the reason why the new algebra which corresponds to this new quantization procedure is referred to as a deformation of the Poisson algebra.

Sklyanin has given a representation of the algebra(3.21) [33]. But in order to have a complete description of the model, we need the canonical transformation which connects the elements of the transition matrix $T(\lambda)$ with the fundamental field $\phi(x)$. This would imply the knowledge of the local structure of the quantized theory. This problem can be solved, in principle, through quantization of the Gelfand-Levitan-Marchenko equation [7], as already performed by Smirnov for the Sine-Gordon model [8, 9].

Another possibility could be to guess the quantized version of $U(x,\lambda)$ and construct $T(\lambda)$ .Then, comparing with Sklyanin's result we could extract the canonical transformation that we need.

However a completely different approach to the local structure of the Sinh-Gordon model can be followed through the form factors approach [10].

3.3 Form Factors Properties for the Sinh-Gordon Model

Since Sinh-Gordon is a quantum integrable model, its S-matrix is factorizable and so all on-shell information is encoded into two-particles S-matrix. The explicit expression was computed in [22]

$$S(\beta, g) = \frac{\tanh\left(\frac{1}{2}(\beta - i\frac{\pi}{2}B(g))\right)}{\tanh\left(\frac{1}{2}(\beta + i\frac{\pi}{2}B(g))\right)} \tag{3.25}$$

$$B(g) = \frac{g^2}{4\pi}\frac{1}{1+\frac{g^2}{8\pi}}$$

. This formula has been checked against perturbation theory in [22] and more recently by [24].

The spectrum of the theory consists in one self-conjugated boson which does not have bound states. This result is in agreement also with the $\mathcal{Z}_2$-symmetry of the Lagrangian. If spontaneously symmetry breaking is assumed not to occur, this symmetry forbids the existence of non-null vertices with an odd number of particles; therefore a bound state is forbidden, since it would imply

$$(\Gamma^{(3)})^2 \sim \mathrm{Res}\, S \neq 0 \ . \tag{3.26}$$

We will come again on this symmetry property later, when we will discuss the possible structure of the algebra of local fields.

The knowledge of the exact S-matrix is crucial for the approach to the form factors. We recall from the previous chapter the properties which we expect to hold for the form factors of local operators of spin s for a theory of a single scalar self-conjugated boson with no bound states

$$F_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n) = \langle\, 0 \mid \mathcal{O}(0) \mid \beta_1, \beta_2, \ldots, \beta_n, \mathrm{in}\, \rangle \tag{3.27}$$

whose analytical continuation to the strip $0 \leq \mathrm{Im}\beta < 2\pi$ satisfies

$$F_n^{\mathcal{O}}(\beta_1 + \Lambda, \ldots, \beta_n + \Lambda) = e^{s\Lambda} F_n^{\mathcal{O}}(\beta_1, \ldots, \beta_n) \tag{3.28}$$

and

-

$$F_n^{\mathcal{O}}(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n) = S(\beta_i - \beta_{i+1}) F_n^{\mathcal{O}}(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n) \ , \tag{3.29}$$

- $$F_n^{\mathcal{O}}(\beta_1 + 2\pi i, \beta_2, \ldots, \beta_n) \tag{3.30}$$
$$= \prod_{j=2}^{n} S(\beta_j - \beta_1)\, F_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n)$$
$$= F_n^{\mathcal{O}}(\beta_2, \ldots, \beta_n, \beta_1) \ ,$$

- $$-i \lim_{\tilde{\beta}\to\beta} (\tilde{\beta} - \beta)\, F_{n+2}^{\mathcal{O}}(\tilde{\beta} + i\pi, \beta, \beta_1, \ldots, \beta_n) = \tag{3.31}$$
$$\left[1 - \prod_{i=1}^{n} S(\beta - \beta_i)\right] F_n^{\mathcal{O}}(\beta_1, \ldots, \beta_n) \ . \tag{3.32}$$

Equations (3.29)-(3.31), reversing the logic, can be used *a la* Smirnov to define local operators. In order to prove locality, i.e.

$$[\mathcal{O}(x),\, \mathcal{O}(y)] = 0 \quad (x - y)^2 < 0 \tag{3.33}$$

or for Lorentz symmetry

$$\left[\mathcal{O}(0, x^1),\, \mathcal{O}(0,0)\right] = 0 \quad x^1 \neq 0 \ , \tag{3.34}$$

we need a further assumption: $F_n^{\mathcal{O}}(\beta_1, \beta_2, \ldots, \beta_n)$ are polynomially bounded in the momenta, i.e.

$$\exists\, K : \forall n \quad F_n^{\mathcal{O}}(\beta_1, \ldots, \beta_n) \ \to\ o(e^{K|\beta_i|}) \ \text{ as } \ |\,\beta_i\,| \to +\infty \ . \tag{3.35}$$

This hypothesis is equivalent to the requirement that the singularity in (3.34) is a distribution made of a finite number of derivatives of $\delta(x^1)$. One can show that this set of equations has unique solution up to the choice of the first non-null, if they behave like in (3.35) with $K < 0$. The classification of all the possible solution of the other case is still an open problem.

Under this hypothesis, we can bring solutions of (3.29)-(3.31) into the form [26]

$$F_n(\beta_1, \ldots, \beta_n) = K_n(\beta_1, \ldots, \beta_n) \prod_{i<j} F_{\min}(\beta_{ij}) \ , \tag{3.36}$$

where $F_{\min}(\beta)$ satisfies (3.29) and (3.30) for $n = 2$ and is analytical in $0 \leq \mathrm{Im}\beta < 2\pi$

$$\begin{aligned} F_{\min}(\beta) &= S(\beta) F_{\min}(-\beta) \\ F_{\min}(i\pi - \beta) &= F_{\min}(i\pi + \beta) \end{aligned} \tag{3.37}$$

and thus the K_n 's resume the pole structure (3.31) and satisfy (3.29) and (3.30) with $S = 1$, which means that they are completely symmetric, $2\pi i$-periodic functions of the β_i

$$\begin{aligned} K_n(\beta_1, \ldots, \beta_i, \beta_{i+1}, \ldots, \beta_n) &= K_n(\beta_1, \ldots, \beta_{i+1}, \beta_i, \ldots, \beta_n) \\ K_n(\beta_1 + 2\pi i, \ldots, \beta_n) &= K_n(\beta_1, \ldots, \beta_n) \; . \end{aligned} \tag{3.38}$$

Karowski and Weisz have shown that using an integral representation of $S(\beta)$ in the form

$$S(\beta) = \exp\left[\int_0^\infty \frac{dx}{x} f(x) \sinh\left(\frac{x\beta}{i\pi}\right)\right] \; . \tag{3.39}$$

then a solution of (3.37) is given by

$$F_{\text{min}}(\beta) = \mathcal{N} \exp\left[\int_0^\infty \frac{dx}{x} f(x) \frac{\sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)}{\sinh x}\right] \; . \tag{3.40}$$

where $\hat{\beta} \equiv i\pi - \beta$. The overall normalization constant $\mathcal{N}$ is chosen such that

$$\lim_{\beta \to \infty} F_{\text{min}}(\beta, B) = 1 \tag{3.41}$$

i.e.

$$\mathcal{N} = \exp\left[-\frac{1}{2}\int_0^\infty \frac{dx}{x} \frac{f(x)}{\sinh x}\right] \; . \tag{3.42}$$

For the Sinh-Gordon model

$$f(x) = \frac{\sinh\left(\frac{xB}{4}\right) \sinh\left(\frac{x}{4}(2-B)\right) \sinh\frac{x}{2}}{\sinh x} \; . \tag{3.43}$$

An useful identity satisfied by $F_{\text{min}}(\beta)$ for the Sinh-Gordon model is given by the functional equation

$$F_{\text{min}}(i\pi + \beta, B) F_{\text{min}}(\beta, B) = \frac{\sinh\beta}{\sinh\beta + \sinh\frac{i\pi B}{2}} \tag{3.44}$$

(see Appendix A).

Let's now discuss the pole structure of the theory. Since in this theory there are no bound states, the poles are all of kinematical origin. As already discussed in Chapter 2 in the LSZ formalism, they can be seen to occur in the limit $p_i \to -p_j$, or $\beta_i \to \beta_j$, from the singularity of the propagators

$$\frac{i}{(p_i + p_j + p_k)^2 - m^2} \; . \tag{3.45}$$

In terms of rapidity variables this means

$$\frac{i}{8m^2 \cosh\frac{1}{2}\beta_{12}\cosh\frac{1}{2}\beta_{13}\cosh\frac{1}{2}\beta_{23}} \tag{3.46}$$

All possible three particles poles are taken into account by the following parametrization of the function K_n

$$K_n(\beta_1,\ldots,\beta_n) = \frac{Q'_n(\beta_1,\ldots,\beta_n)}{\prod\limits_{i<j}\cosh\frac{1}{2}\beta_{ij}} , \tag{3.47}$$

where Q'_n is free of any singularity, and from (3.38) it satisfies

$$\begin{aligned} Q'_n(\beta_1,\ldots,\beta_i,\beta_{i+1},\ldots,\beta_n) &= Q'_n(\beta_1,\ldots,\beta_{i+1},\beta_i,\ldots,\beta_n) \\ Q'_n(\beta_1+2\pi i,\ldots,\beta_n) &= (-1)^{n-1}\,Q'_n(\beta_1,\ldots,\beta_n) . \end{aligned} \tag{3.48}$$

Since we require that F_n behaves like $o(e^{K|\beta_i|})$ as $|\beta_i| \to \infty$, and $F_{\text{min}} \to 1$ in that limit, Q'_n will be a symmetric function of the variables $x_i \equiv e^{\beta_i}$, with a finite number of terms. Using

$$\cosh\frac{\beta_{ij}}{2} = \frac{1}{2}\left(\sqrt{\frac{x_i}{x_j}} + \sqrt{\frac{x_j}{x_i}}\right) = \frac{1}{2}\frac{x_i + x_j}{\sqrt{x_i\,x_j}} \tag{3.49}$$

and redefining Q'_n into Q_n, from (3.36) and (3.47) we end up with the following parametrization of F_n

$$F_n(\beta_1,\ldots,\beta_n) = H_n\,Q_n(x_1,\ldots,x_n)\prod_{i<j}\frac{F_{\text{min}}(\beta_{ij})}{x_i + x_j} , \tag{3.50}$$

where H_n is a normalization constant which can be fixed to obtain a simplified kinematical recursive equation for the Q_n 's. This parametrization fulfills properties (3.29) and (3.30) of form factors, and takes into account the prescribed pole structure. All ignorance is now in the Q_n 's. Considering Lorentz properties of F_n

$$F_n^{\mathcal{O}}(\beta_1+\Lambda,\ldots,\beta_n+\Lambda) = e^{s\Lambda}F_n^{\mathcal{O}}(\beta_1,\ldots,\beta_n) \tag{3.51}$$

for a spin s operator, we see that for the functions Q 's they become

$$Q_n(e^{\Delta}x_1,\ldots,e^{\Delta}x_n) = e^{\left(s+\frac{n(n-1)}{2}\right)\Delta}Q_n(x_1,\ldots,x_n) . \tag{3.52}$$

So Q_n must be a symmetric function of the x_i 's, with a finite number of terms which transforms like (3.52), with $s+\frac{n(n-1)}{2} \in \mathcal{Z}$. It is therefore a symmetric polynomial in the x_i 's, homogeneous of total degree $s+\frac{n(n-1)}{2}$. It's convenient to introduce in this functional space the basis given by the elementary symmetric polynomials $\sigma_k^{(n)}(x_1,\ldots,x_n)$ which are generated by [37]

$$\prod_{i=1}^{n}(x+x_i) = \sum_{k=0}^{n} x^{n-k}\,\sigma_k^{(n)}(x_1,x_2,\ldots,x_n). \tag{3.53}$$

Conventionally the $\sigma_k^{(n)}$ with $k > n$ and with $n < 0$ are zero. The explicit expressions for the other cases are

$$\begin{aligned}
&\sigma_0 = 1 \ , \\
&\sigma_1 = x_1 + x_2 + \ldots + x_n \ , \\
&\sigma_2 = x_1 x_2 + x_1 x_3 + \ldots x_{n-1} x_n \ , \\
&\vdots \qquad\quad \vdots \\
&\sigma_n = x_1 x_2 \ldots x_n \ .
\end{aligned} \tag{3.54}$$

The $\sigma_k^{(n)}$ are homogeneous polynomials in x_i of total degree k and of degree one in each variable (except for $\sigma_0^{(n)}$ which is the constant 1).

So Q_n can be formally expressed as

$$Q_n(x_1, \ldots, x_n) = \sum_{\substack{k_1, \ldots, k_{M_n} = 0 \\ \sum_{i=1}^{M_n} k_i = s + \frac{n(n-1)}{2}}}^{n} \sigma_{k_1}^{(n)} \ldots \sigma_{k_{M_n}}^{(n)} q_{k_1 \ldots k_{M_n}} \tag{3.55}$$

where M_n is the degree in each variable. Since it should exist a constant K , such that

$$\forall n \ \ F_n \to x_i^K \quad \text{as } x_i \to \infty \ , \tag{3.56}$$

from the parametrization and the property $F_{\min} \to 1$ we determine M_n to be

$$M_n = K + n - 2 \ . \tag{3.57}$$

Note that this discussion is independent of the sign of K . From [27] we know that in the case $K < 0$ the solution of (3.29), (3.30) and (3.31) is unique, up to the choice of the first form factors different from zero (initial condition). Hence each $K < 0$, together with an initial condition, identifies a precise local operator $\mathcal{O}$ of the theory. The identification of $\mathcal{O}$ with some composite operator of the fundamental field ϕ of the Lagrangian approach is obtained through the LSZ formula (2.6), with which one also determine the asymptotic behaviour and F_1 or F_2 of each operator.

3.4 The Kinematical Recursive Equation

Recalling the properties discussed in the last section, the form factors $F_n^{\mathcal{O}}$ of a local operator $\mathcal{O}_s(x)$ of spin s can be put in the form

$$F_n(\beta_1,\ldots,\beta_n) = H_n\, Q_n(x_1,\ldots,x_n) \prod_{i<j} \frac{F_{\min}(\beta_{ij})}{x_i + x_j} , \tag{3.58}$$

with

$$Q_n(x_1,\ldots,x_n) = \sum_{\substack{k_1,\ldots,k_{M_n}=0 \\ \sum_{i=1}^{M_n} k_i = s + \frac{n(n-1)}{2}}}^{n} \sigma_{k_1}^{(n)} \ldots \sigma_{k_{M_n}}^{(n)}\, q_{k_1 \ldots k_{M_n}} \tag{3.59}$$

and with $M_n = K^{\mathcal{O}} + n - 2$, where $K^{\mathcal{O}}$ must be fixed by the asymptotic behaviour for $x_i \to \infty$. This parametrization takes into account the Lorentz properties of transformation (3.51), the locality axioms (3.29), (3.30), (3.35), together with the prescribed presence of the kinematical poles $x_i = x_j$, (3.31).

It is then natural to obtain a recursive equation for the unknown polynomials Q_n simply plugging the parametrization (3.58) into the residue equation

$$-i \lim_{\tilde\beta\to\beta} (\tilde\beta - \beta)\, F_{n+2}^{\mathcal{O}}(\tilde\beta + i\pi, \beta, \beta_1, \ldots, \beta_n) = \tag{3.60}$$

$$\left[1 - \prod_{i=1}^{n} S(\beta - \beta_i)\right] F_n^{\mathcal{O}}(\beta_1,\ldots,\beta_n) \ . \tag{3.61}$$

We have to express the S-matrix (3.25) through the variables $x_i = e^{\beta_i}$

$$S(\beta - \beta_i) = \frac{\sinh(\beta - \beta_i) - \sinh\frac{i\pi B}{2}}{\sinh(\beta - \beta_i) + \sinh\frac{i\pi B}{2}} \tag{3.62}$$

$$S(\beta - \beta_i) = \frac{(x + \omega^{-1} x_i)\,(x - \omega x_i)}{(x - \omega^{-1} x_i)\,(x + \omega x_i)}$$

where $\omega \equiv e^{\frac{i\pi B}{2}}$, so that the S-matrix factor in (3.60) becomes

$$1 - \prod_{i=1}^{n} S(\beta - \beta_i) = \frac{\prod_{i=1}^{n} (x - \omega^{-1} x_i)\,(x + \omega x_i) - \prod_{i=1}^{n} (x + \omega^{-1} x_i)\,(x - \omega x_i)}{\prod_{i=1}^{n} (x - \omega^{-1} x_i)\,(x + \omega x_i)} \ . \tag{3.63}$$

Considering the left hand side of (3.60), we note that the pole as $\tilde\beta \to \beta$ of $F_{n+2}^{\mathcal{O}}(\tilde\beta + i\pi, \beta, \beta_1, \ldots, \beta_n)$ comes from the factor $\frac{1}{x(\tilde\beta + i\pi) + x(\beta)}$ in the parametrization (3.58), so that

$$-i \lim_{\tilde\beta\to\beta} \frac{\tilde\beta - \beta}{x(\tilde\beta + i\pi) + x(\beta)} = \frac{i}{x} , \tag{3.64}$$

and the left hand side reads

$$\frac{i}{x}\, F_{\min}(i\pi) \prod_{i=1}^{n} \frac{F_{\min}(\beta-\beta_i+i\pi)\, F_{\min}(\beta-\beta_i)}{(-x\,+\,x_i)(x\,+\,x_i)}$$
$$\prod_{i<j}^{n} \frac{F_{\min}(\beta_{ij})}{x_i+x_j}\, H_{n+2}\, Q_{n+2}(-x,x,x_1,\ldots,x_n)\ .$$

Using the (3.44) in the form

$$F_{\min}(i\pi+\beta,B)F_{\min}(\beta,B) \;=\; \frac{x^2\,-\,x_i^2}{(x-\omega^{-1}x_i)\,(x+\omega x_i)} \tag{3.65}$$

the left hand side transforms into

$$\frac{i}{x}\, F_{\min}(i\pi)\,(-1)^n \prod_{i=1}^{n} \frac{1}{(x-\omega^{-1}x_i)\,(x+\omega x_i)}$$
$$\prod_{i<j}^{n} \frac{F_{\min}(\beta_{ij})}{x_i+x_j}\, H_{n+2}\, Q_{n+2}(-x,x,x_1,\ldots,x_n)\ .$$

Comparing with the right hand side

$$\frac{\prod_{i=1}^{n}(x-\omega^{-1}x_i)\,(x+\omega x_i)\;-\;\prod_{i=1}^{n}(x+\omega^{-1}x_i)\,(x-\omega x_i)}{\prod_{i=1}^{n}(x-\omega^{-1}x_i)\,(x+\omega x_i)}$$
$$\prod_{i<j} \frac{F_{\min}(\beta_{ij})}{x_i+x_j}\, H_n\, Q_n(x_1,\ldots,x_n)$$

we finally obtain

$$H_{n+2}\, Q_{n+2}(-x,x,x_1,\ldots,x_n) \;=\; \frac{(-1)^n}{iF_{\min}(i\pi)}\mathcal{U}_n(x \mid x_1,\ldots,x_n)\, H_n\, Q_n(x_1,\ldots,x_n)\ , \tag{3.66}$$

where

$$\mathcal{U}_n(x \mid x_1,\ldots,x_n) \;=\; x\,\left\{\prod_{i=1}^{n}\left(x-\omega^{-1}x_i\right)(x+\omega x_i)\;-\;\prod_{i=1}^{n}\left(x+\omega^{-1}x_i\right)(x-\omega x_i)\right\}\ . \tag{3.67}$$

This function is usefully written as

$$\mathcal{U}_n(x \mid x_1,\ldots,x_n) \;= \tag{3.68}$$
$$2\,i\,x \sum_{k,l=0}^{n} x^{2n-k-l}\,(-1)^l\,\sin\frac{\pi B}{2}(k-l)\;\sigma_k^{(n)}(x_1,\ldots,x_n)\sigma_l^{(n)}(x_1,\ldots,x_n)$$

making use of the relations

$$\prod_{i=1}^{n}(x\,+\,ax_i) \;=\; \sum_{k=0}^{n} x^{n-k}\,\sigma_k^{(n)}(ax_1,\ldots,ax_n) \;=\; \sum_{k=0}^{n} x^{n-k}\,a^k\sigma_k^{(n)}(x_1,\ldots,x_n)\ . \tag{3.69}$$

The non-zero terms entering the sum (3.68) are only those with $(k-l)$ odd

$$
\begin{aligned}
&\mathcal{U}_n(x \mid x_1,\dots,x_n) \qquad (3.70)\\
&= 4\,i\,x \sum_{\substack{k,l=0\\ k-l=\text{odd}>0}}^{n} x^{2n-k-l}\,(-1)^l \sin\frac{\pi B}{2}(k-l)\;\sigma_k^{(n)}(x_1,\dots,x_n)\sigma_l^{(n)}(x_1,\dots,x_n)\\
&= 4\,i\,\sin\left(\frac{\pi B}{2}\right)\,x\,\mathcal{D}_n(x \mid x_1,\dots,x_n)
\end{aligned}
$$

where we have defined

$$
\begin{aligned}
&\mathcal{D}_n(x \mid x_1,\dots,x_n) = \qquad (3.71)\\
&\sum_{\substack{k,l=0\\ k-l=\text{odd}>0}}^{n} x^{2n-k-l}\,(-1)^l\,\frac{\sin\left(\frac{\pi B}{2}(k-l)\right)}{\sin\left(\pi B/2\right)}\;\sigma_k^{(n)}(x_1,\dots,x_n)\,\sigma_l^{(n)}(x_1,\dots,x_n)\ .
\end{aligned}
$$

Hence we can write the recursive equation as

$$
\begin{aligned}
&H_{n+2}\,Q_{n+2}(-x,x,x_1,\dots,x_n) = \qquad (3.72)\\
&\qquad \frac{4\sin\frac{\pi B}{2}}{F_{\min}(i\pi)}\,(-1)^n\,x\,\mathcal{D}_n(x \mid x_1,\dots,x_n)\,H_n\,Q_n(x_1,\dots,x_n)\ .
\end{aligned}
$$

This equation fixes the product $H_n\,Q_n$, up to the choice of the initial conditions. We can choose H_n such that

$$
H_{n+2} = \frac{4\sin\frac{\pi B}{2}}{F_{\min}(i\pi)}\,H_n\ , \qquad (3.73)
$$

which can be accomplished by

$$
\begin{aligned}
H_{2n+1} &= H_1\left(\frac{4\sin(\pi B/2)}{F_{\min}(i\pi,B)}\right)^{n-1} \qquad n\geq 1 \qquad (3.74)\\
H_{2n} &= H_2\left(\frac{4\sin(\pi B/2)}{F_{\min}(i\pi,B)}\right)^{n-1} \qquad n\geq 1
\end{aligned}
$$

The recursive equation takes the final form

$$
Q_{n+2}(-x,x,x_1,\dots,x_n) = (-1)^n\,x\,\mathcal{D}_n(x \mid x_1,\dots,x_n)Q_n(x_1,\dots,x_n)\ , \qquad (3.75)
$$

where $\mathcal{D}_n$, due to the relation

$$
\sin nx = n\,\sin x \prod_{j=1}^{\frac{n-1}{2}}\left(1-\frac{\sin^2 x}{\sin^2(j\pi/n)}\right) \qquad (3.76)
$$

is a polynomial also in the variable $\sin^2\left(\frac{\pi B}{2}\right)$.

Chapter 4

The Algebra of Local Operators of the Sinh-Gordon Model

4.1 $\mathcal{Z}_2$-grading and Descendent Structure

Having discussed so far in some details the properties of a general local operator of the theory, we come now to face the problem of the solution of the recursive equation (3.75), citenostro. With this aim, we characterize the local operator algebra $\mathcal{A}$ according to the symmetries of the theory. Lorentz invariance allows us to write $\mathcal{A}$ as a direct sum over subalgebras of fixed Lorentz spin

$$\mathcal{A} = \bigoplus_s \mathcal{A}_s \ . \tag{4.1}$$

Each subalgebra $\mathcal{A}_s$ with s odd contains the non trivial conserved charge

$$\mathcal{Q}_s = \int dx^+ \, T_{s+1} \, + \int dx^- \, \Theta_{s-1} \quad . \tag{4.2}$$

If we assume that the symmetry of the Lagrangian under $\phi \to -\phi$ holds also at quantum level, and with the further hypothesys that $\mathcal{A}$ is made only of composite operators of the elementary fields $\phi(x)$, we can split $\mathcal{A}$ according to this $\mathcal{Z}_2$-symmetry

$$\mathcal{A} = \mathcal{A}_{\rm odd} \oplus \mathcal{A}_{\rm even} \tag{4.3}$$

where obviously

$$\begin{aligned} \mathcal{A}_{\rm odd} &\xrightarrow{\mathcal{Z}_2} -\mathcal{A}_{\rm odd} \\ \mathcal{A}_{\rm even} &\xrightarrow{\mathcal{Z}_2} \mathcal{A}_{\rm even} \ . \end{aligned}$$

One can conclude, referring to (2.6), that

$$\mathcal{O} \in \mathcal{A}_{\rm odd} \quad \Rightarrow \quad F_{2n}^{\mathcal{O}} = 0 \tag{4.4}$$

$$\mathcal{O} \in \mathcal{A}_{\rm even} \quad \Rightarrow \quad F_{2n+1}^{\mathcal{O}} = 0 \quad . \tag{4.5}$$

Let's now consider an operator $\mathcal{O}_s$ of $\mathcal{A}$. Its form factors $F_n^{\mathcal{O}_s}$ then transforms under a Lorentz boost as

$$F_n^{\mathcal{O}_s}(\beta_1+\Lambda,\ldots,\beta_n+\Lambda) \;=\; e^{s\Lambda} F_n^{\mathcal{O}_s}(\beta_1,\ldots,\beta_n)\;. \tag{4.6}$$

Consider now polynomials $I_n^s(x_1,\ldots,x_n)$ with the following properties

$$\begin{aligned}
I_n^{s'}(x_1,\ldots,x_i,\ldots,x_j,\ldots,x_n) &= I_n^{s'}(x_1,\ldots,x_j,\ldots,x_i,\ldots,x_n) &\quad (4.7)\\
I_n^{s'}(e^{\Delta}x_1,\ldots,e^{\Delta}x_n) &= e^{s'\Delta}\, I_n^{s'}(x_1,\ldots,x_n) &\quad (4.8)\\
I_{n+2}^{s'}(-x,x,x_1,\ldots,x_n) &= I_n^{s'}(x_1,\ldots,x_n)\;. &\quad (4.9)
\end{aligned}$$

The functions F_n' defined by

$$F_n'(x_1,\ldots,x_n) \;=\; I_n^{s'}(x_1,\ldots,x_n)\, F_n^{\mathcal{O}_s}(x_1,\ldots,x_n) \tag{4.10}$$

can be thought as defining a local operator $\mathcal{O}' \in \mathcal{A}_{s+s'}$, since they satisfy the axioms (3.29), (3.30) and (3.31), together with (3.35). From the properties (4.7), (4.8) I_n^s is a polynomial which can be expressed in the basis $\sigma_k^{(n)}(x_1,\ldots,x_n)$ (3.53)

$$I_n^s(x_1,\ldots,x_n) \;=\; \sum_{\substack{k_1,\ldots,k_{M_{ns}}=0\\ \sum_{i=1}^{M_{ns}} k_i=s}}^{n} \sigma_{k_1}^{(n)}\ldots\sigma_{k_{M_{ns}}}^{(n)}\, i_{k_1\ldots k_{M_{ns}}}\;. \tag{4.11}$$

As shown by Cardy-Mussardo in [29], any polynomial $I_n^{s'}$ solution of (4.9) can be thought as a polynomial in the odd spin polynomials I_n^{2k+1} which satisfy the following recursive relation

$$\sigma_{2k+1}^{(n)} \;=\; I_n^{2k+1} + \sigma_2^{(n)}\, I_n^{2k-1} + \sigma_4^{(n)}\, I_n^{2k-3} + \ldots + \sigma_{2k}^{(n)}\, I_n^1\;. \tag{4.12}$$

Explicitly they are

$$\begin{aligned}
I_n^1 &= \sigma_1^{(n)}\;,\\
I_n^3 &= \sigma_3^{(n)} - \sigma_2^{(n)}\sigma_1^{(n)}\;,\\
I_n^5 &= \sigma_5^{(n)} - \sigma_2^{(n)}(\sigma_3^{(n)} - \sigma_2^{(n)}\sigma_1^{(n)}) - \sigma_4^{(n)}\sigma_1^{(n)}\;,\\
&\vdots
\end{aligned} \tag{4.13}$$

A closed expression of I_n^{2k-1} has been obtained in [32]

$$I_n^{2k-1} = (-1)^{k+1}\det\mathcal{I} \tag{4.14}$$

where the entries of the $(k\times k)$-matrix $\mathcal{I}$ for $j=1,\ldots,k$ and $i=2,\ldots,k$ are

$$\mathcal{I}_{1j} = \sigma_{2j-1} \qquad\qquad \mathcal{I}_{ij} = \sigma_{2j-2i+2} \tag{4.15}$$

i.e.

$$\mathcal{I} = \begin{pmatrix} \sigma_1 & \sigma_3 & \sigma_5 & \sigma_7 & \dots & \sigma_{2k-1} \\ 1 & \sigma_2 & \sigma_4 & \sigma_6 & \dots & \sigma_{2k-2} \\ 0 & 1 & \sigma_2 & \sigma_4 & \dots & \sigma_{2k-4} \\ 0 & 0 & 1 & \sigma_2 & \dots & \sigma_{2k-2} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \end{pmatrix} . \tag{4.16}$$

The determinant of $\mathcal{I}$ will always be of order $2k-1$ as required.

As it was first noticed in [29], eqs. (4.10) naturally provides a grading in $\mathcal{A}$. In fact, given an invariant polynomial I_n^s, eq. (4.10) defines form factors of an operator $\mathcal{O}'_s$ which, borrowing the terminology of Conformal Field Theories [4], is natural to call *descendent operator* of the spinless field $\mathcal{O}$. In particular, choosing $\mathcal{O}$ to be the trace of the stress-energy tensor, the form factors defined by eq. (4.10) are related to the matrix elements of the higher conserved currents, as can be easily seen by eq. (1.26) and by the fact that the symmetric polynomials which appear as eigenvalues of the conserved charges Q_s

$$s_k = x_1^k + x_2^k + \dots + x_n^k , \tag{4.17}$$

can be expressed in terms of the invariant polynomials I_n^s. Indeed they satisfy the recursive relation

$$s_k - s_{k-1}\sigma_1 + s_{k-2}\sigma_2 - \dots + (-1)^{k-1} s_1 \sigma_{k-1} + (-1)^k k \sigma_k = 0 , \tag{4.18}$$

that, together with eq. (4.12), permits to express s_k in terms of the invariant polynomials I_n^s.

We don't have the proof that any operator in $\mathcal{A}_s$ can be obtained by an operator in $\mathcal{A}_{s=0}$ by means of the invariant polynomials (4.10). Nevertheless, we will limit ourselves to the solution of the form factor equations for the two most significant fields, the elementary one $\phi(x)$ and the trace of the stress-energy tensor.

4.2 Asymptotic Behaviour and Initial Conditions for the Elementary Field and for the Trace of the Stress-Energy Tensor

The elementary field $\phi(x)$ is a $\mathcal{Z}_2$-odd spinless operator of A. The initial conditions for its recursive equation (3.60) are

$$F_0 = 0 \ , \tag{4.19}$$

$$F_1 = \frac{1}{\sqrt{2}} \ . \tag{4.20}$$

The first one is simply $\mathcal{Z}_2$-odd parity which implies

$$F_{2n} = 0 \ . \tag{4.21}$$

The other one is the normalization of one-particle wave function, which one can read from the LSZ-formula

$$\begin{aligned}
&(2\pi)^2\delta^2(p_1+p_2)G^2(p_1,p_2) = \\
&= \int \prod_{i=1}^{2} dx_i \, e^{-i\sum q_i x_i} \langle \, 0 \mid T(\phi(x_1)\phi(x_2)) \mid 0 \, \rangle \\
&= (2\pi)^2\delta^2(p_1+p_2) \left\{ \frac{i}{p_1^2 - m^2} + \text{Reg.} \right\} ,
\end{aligned} \tag{4.22}$$

where the pole in the physical mass comes from one-particle states, and the Reg. term collects the higher number of particles contribution. Now (4.20) is obtained plugging (4.22) into (2.6).

Using again (2.6) we can learn an useful factorization property of Q_n^ϕ

$$Q^\phi_{2n+1}(x_1,\ldots,x_{2n+1}) = \sigma^{(2n+1)}_{2n+1} \, P^\phi_{2n+1}(x_1,\ldots,x_{2n+1}) \quad n>0 \ . \tag{4.23}$$

The reason is that from any Feynman diagram which enters F^ϕ_{2n+1} we can factorize the propagator

$$\frac{i}{q^2-m^2}\Big|_{q=-\sum p_i,\ p_i^2=m^2} \tag{4.24}$$

that, written in terms of the variables x_i , becomes proportional to $\sigma^{(2n+1)}_{2n+1}$

$$\frac{i}{q^2-m^2}\Big|_{q=-\sum p_i,\ p_i^2=m^2} = \frac{i}{m^2}\frac{\sigma^{(2n+1)}_{2n+1}}{\sigma^{(2n+1)}_1\sigma^{(2n+1)}_{2n} - \sigma^{(2n+1)}_{2n+1}} \ . \tag{4.25}$$

The presence of the propagator (4.24) in front of any form factor of the elementary field $\phi(x)$ also implies that F^ϕ_{2n+1} behaves asymptotically as

$$F^\phi_{2n+1}(\beta_1,\beta_2,\ldots,\beta_{2n+1}) \to 0 \ \text{ as } \beta_i \to +\infty \ \ \beta_{j\neq i} \text{ fixed.} \tag{4.26}$$

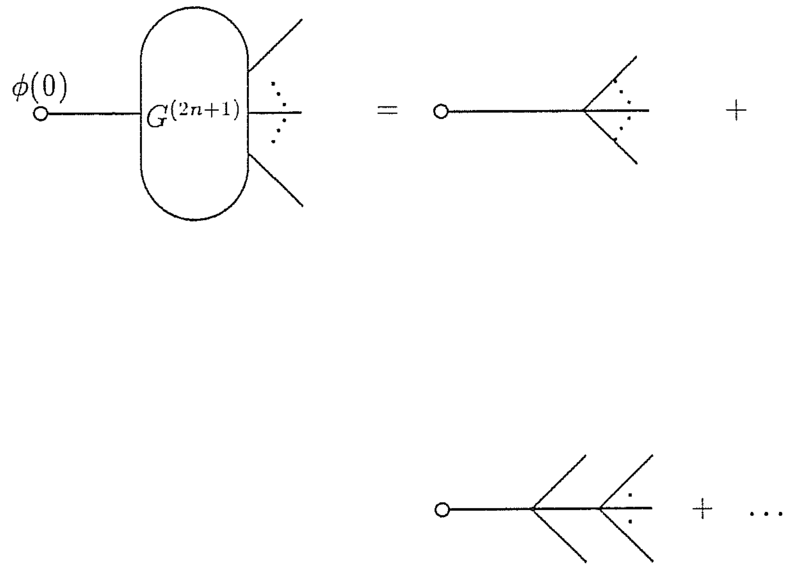

Figure 4.1: Lowest terms in the perturbative expression of the form factors of the elementary field $\phi(0)$. $G^{(2n+1)}$ is the Green function with $2n+1$ external legs.

In fact, the propagator (4.24) goes to zero in this limit whereas the remaining expression of the Feynman graphs entering F_{2n+1} is a perturbative series which starts from the tree level vertex diagram shown in fig. 4.1, which is a constant. Other tree level contributions at the lowest order and higher order corrections are either finite or they vanish in the limit (4.26). In fact, by dimensional analysis they must have external momenta in the denominator in order to compensate the increasing power of the mass in the coupling constants.

Using the transformation property

$$\sigma_k^{(n+2)}(-x, x, x_1, \ldots, x_n) = \sigma_k^{(n)}(x_1, x_2, \ldots, x_n) - x^2 \sigma_{k-2}^{(n)}(x_1, x_2, \ldots, x_n) \ , \qquad (4.27)$$

and (4.23), (3.75) becomes the recursive equation for the P_n^{ϕ} 's

$$(-)^{n+1} P_{n+2}^{\phi}(-x, x, x_1, \ldots, x_n) = \frac{1}{x} D_n(x \mid x_1, x_2, \ldots, x_n) \, P_n^{\phi}(x_1, x_2, \ldots, x_n) \ . \quad (4.28)$$

where in this case n is odd.

An analogous factorization property exists for the trace of the stress-energy tensor $\Theta(x)$. This field is classically defined through the stress-energy tensor

$$T_{\mu\nu} = \frac{4\pi}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}(x)} \tag{4.29}$$
$$= 2\pi \, (\partial_\mu \phi \partial_\nu \phi - g_{\mu\nu} \mathcal{L})$$

as

$$\Theta(x) = T^{\mu}_{\mu}(x) = 4\pi \frac{m^2}{g^2} \cosh g\phi \, . \tag{4.30}$$

It is spinless and $\mathcal{Z}_2$-even. For the definition of this operator at quantum level we need, in the form factors approach, to precise the initial conditions and the asymptotic behaviour. $\mathcal{Z}_2$-even parity implies

$$F^{\Theta}_{2n+1} = 0 \, . \tag{4.31}$$

The mentioned factorization property for F^{Θ}_{2n} can be proven if we consider the conservation laws satisfied by the stress-energy tensor

$$\partial_- T(x^+, x^-) + \partial_+ \Theta(x^+, x^-) = 0 \quad , \qquad \partial_+ \bar{T}(x^+, x^-) + \partial_- \Theta(x^+, x^-) = 0 \tag{4.32}$$

where T ($\bar{T}$) is the component of the stress-energy tensor which in the conformal euclidean limit becomes holomorphic (anti-holomorphic). Using eq. (2.2), (1.13) the identities

$$\sum_{i=1}^{n} e^{\beta_i} = \sigma_1^{(n)}(x_1, \ldots, x_n) \qquad , \qquad \sum_{i=1}^{n} e^{-\beta_i} = \frac{\sigma_{n-1}^{(n)}(x_1, \ldots, x_n)}{\sigma_n^{(n)}(x_1, \ldots, x_n)} \tag{4.33}$$

together with (4.32), we obtain

$$\sigma_1^{(2n)} \sigma_{2n}^{(2n)} F^T_{2n}(\beta_1, \ldots, \beta_{2n}) = \sigma_{2n-1}^{(2n)} F^{\Theta}_{2n}(\beta_1, \ldots, \beta_{2n}) \tag{4.34}$$
$$\sigma_{2n-1}^{(2n)} F^{\bar{T}}_{2n}(\beta_1, \ldots, \beta_{2n}) = \sigma_1^{(2n)} \sigma_{2n}^{(2n)} F^{\Theta}_{2n}(\beta_1, \ldots, \beta_{2n}). \tag{4.35}$$

Since F^T_{2n} , $F^{\bar{T}}_{2n}$ and F^{Θ}_{2n} are expected to have the same analytical structure, we conclude that $F^{\Theta}_{2n}(\beta_1, \ldots, \beta_{2n})$ is proportional to the product $\sigma_1^{(2n)} \sigma_{2n-1}^{(2n)}$ for $n > 2$. So we can state that

$$F^{\Theta}_{2n} = \sigma_1^{(2n)} \sigma_{2n-1}^{(2n)} \, P^{\Theta}_{2n} \tag{4.36}$$

One can convince himself of this property also through LSZ formalism, in a perturbative framework in which, for example, a total tadpole subtraction normal ordering can be adopted. Plugging (4.36) into (3.75), using again (4.27) we see that also in this case P^{Θ}_{2n} satisfies

$$(-)^{n+1} \, P^{\phi}_{n+2}(-x, x, x_1, \ldots, x_n) = \frac{1}{x} D_n(x \mid x_1, x_2, \ldots, x_n) \, P^{\phi}_n(x_1, x_2, \ldots, x_n) \, . \tag{4.37}$$

with n even.

It remains to discuss the non-null initial condition for $\Theta(x)$. It is taken to be

$$F_2^{\Theta}(\beta_{12}) \;=\; 2\pi m^2 \frac{F_{\rm min}(\beta_{12})}{F_{\rm min}(i\pi)} \tag{4.38}$$

where m is the physical mass, since it must fulfills the properties (3.29), (3.30) and the limit

$$F_2^{\Theta}(\beta_{12}=i\pi) \;=\; \lim_{\beta_2\to\beta_1} \langle\; out\,,\beta_2 \mid \Theta(0) \mid \beta_1,\; in \;\rangle \;=\; 2\pi\, m^2 \;, \tag{4.39}$$

in order to match the interpretation of $\Theta(x)$ as the generator of the dilatation transformation which controls the flow of the renormalization group in the space of the $2d$-theories [2, 40, 41]. Again, (4.39) can be easily checked perturbatively.

We conclude this section with the discussion of the form factors of a third operator, thanks to general properties. It is the composite field

$$\mathcal{O} \;=: \sinh g\phi : \tag{4.40}$$

whose form factors can be easily computed in terms of the form factors for ϕ . In fact, using eqs. (2.2), (1.13) we have

$$\begin{aligned} &\langle\; 0 \mid \partial_+\partial_-\phi(x^+,x^-) \mid \beta_1\ldots\beta_m,\; in \;\rangle \;= \\ &\qquad -\frac{m^2}{4}\sum_i e^{\beta_i}\sum_i e^{-\beta_i}\sum_i e^{-ixp_i}F_n^{\phi}(\beta_1,\ldots,\beta_n)\;\;. \end{aligned} \tag{4.41}$$

Employing the equation of motion and choosing $x^+ \;=\; x^- \;=\; 0$, together with the identities (4.33) we derive the relation

$$F_n^{\sinh g\phi} = g\frac{\sigma_1\sigma_{n-1}}{\sigma_n}F_n^{\phi}\;\;. \tag{4.42}$$

and thus

$$Q_n^{\sinh g\phi} \;=\; g\sigma_1\sigma_{n-1}P_n^{\phi}\;\;. \tag{4.43}$$

Hence we can describe the fundamental vertex operator $e^{g\phi(x)}$ using the solutions for the form factors of $\Theta \sim \cosh g\phi$ and ϕ .

4.3 Solutions of the Recursive Equations

Let us summarize the analysis carried out in the previous sections. The form factors F^{ϕ}_{2n+1} ($n > 0$) of the elementary field $\phi(x)$ are given by

$$F^{\phi}_{2n+1}(\beta_1,\ldots,\beta_{2n+1}) = \frac{1}{\sqrt{2}}\left(\frac{4\sin(\pi B/2)}{F_{\text{min}}(i\pi,B)}\right)^n \sigma^{(2n+1)}_{2n+1} P_{2n+1}(x_1,\ldots,x_{2n+1}) \prod_{i<j}\frac{F_{\text{min}}(\beta_{ij})}{x_i+x_j} \tag{4.44}$$

and the normalization of the field is fixed by

$$F_1^{\phi} = \frac{1}{\sqrt{2}}\ . \tag{4.45}$$

The form factors F^{Θ}_{2n} ($n > 1$) of the trace of the stress-energy tensor $\Theta(x)$ are given by

$$F^{\Theta}_{2n}(\beta_1,\ldots,\beta_{2n}) = \frac{2\pi m^2}{F_{\text{min}}(i\pi)}\left(\frac{4\sin(\pi B/2)}{F_{\text{min}}(i\pi)}\right)^{n-1} \sigma^{(2n)}_1\sigma^{(2n)}_{2n-1} P_{2n}(x_1,\ldots,x_{2n}) \prod_{i<j}\frac{F_{\text{min}}(\beta_{ij})}{x_i+x_j} \tag{4.46}$$

where the normalization is fixed by the matrix element of $\Theta(0)$ between the two-particle state and the vacuum

$$F_2^{\Theta}(\beta_{12}) = 2\pi m^2\frac{F_{\text{min}}(\beta_{12})}{F_{\text{min}}(i\pi)}\ . \tag{4.47}$$

Notice that (4.46) for $n=0$ leads to the expectation value of Θ on the vacuum

$$\langle\, 0 \mid \Theta(0) \mid 0\,\rangle = \frac{\pi m^2}{2\sin(\pi B/2)}\ . \tag{4.48}$$

Using the recursive equations (4.28) and the transformation property of the elementary symmetric polynomials (4.27), the explicit expressions of the first polynomials $P_n(x_1,\ldots,x_n)$ are given by[1]

$$\begin{aligned}
P_3(x_1,\ldots,x_3) &= 1\\
P_4(x_1,\ldots,x_4) &= \sigma_2\\
P_5(x_1,\ldots,x_5) &= \sigma_2\sigma_3 - c_1^2\sigma_5\\
P_6(x_1,\ldots,x_6) &= \sigma_2\sigma_3(\sigma_4-\sigma_6) - c_1^2(\sigma_4\sigma_5+\sigma_1\sigma_2\sigma_6)\\
P_7(x_1,\ldots,x_7) &= \sigma_2\sigma_3\sigma_4\sigma_5 - c_1^2(\sigma_4\sigma_5^2+\sigma_1\sigma_2\sigma_5\sigma_6+\sigma_2^2\sigma_3-c_1^2\sigma_2\sigma_5)+\\
&\quad -c_2(\sigma_1\sigma_6\sigma_7+\sigma_1\sigma_2\sigma_4\sigma_7+\sigma_3\sigma_5\sigma_6)+c_1c_2^2\sigma_7^2
\end{aligned} \tag{4.49}$$

[1]The upper index of the elementary symmetric polynomials entering P_n is equal to n and we suppress it, in order to simplify the notation.

where $c_1 = 2\cos(\pi B/2)$ and $c_2 = 1 - c_1^2$. Expression of the higher P_n are easily computed by an iterative use of eqs. (3.75). For practical application the first representatives of P_n are sufficient to compute with a high degree of accuracy the correlation functions of the fields. In fact, the n-particle term appearing in the euclidean correlation function of the fields

$$
\begin{aligned}
&\langle\, \mathcal{O}(x)\, \mathcal{O}(0)\, \rangle = \qquad\qquad (4.50)\\
&= \sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} \langle\, 0 \mid \mathcal{O}(x) \mid \beta_1, \ldots, \beta_n,\, in\, \rangle \langle\, in,\, \beta_1, \ldots, \beta_n \mid \mathcal{O}(0) \mid 0\, \rangle \\
&= \sum_{n=0}^{\infty} \int \frac{d\beta_1 \ldots d\beta_n}{n!(2\pi)^n} \mid F_n(\beta_1 \ldots \beta_n) \mid^2 \exp\left(-mr \sum_{i=1}^{n} \cosh\beta_i\right)
\end{aligned}
$$

where r denotes the radial distance, i.e. $r = \sqrt{x_0^2 + x_1^2}$, behaves as $e^{-n(mr)}$ and for quite large values of mr the correlator is dominated by the lowest number of particle terms. This behaviour is also confirmed by an application of the c-theorem which is discussed in sect. 6. Nevertheless, it is interesting to notice that closed expressions for P_n can be found for particular values of the coupling constant, as we demonstrate in the next subsections.

4.3.1 The Self-Dual Point

The self-dual point in the coupling constant manifold has the special value

$$B\left(\sqrt{8\pi}\right) = 1\,. \qquad (4.51)$$

The two zeros of the S-matrix merge together and the function $D_n(x, x_1, x_2, \ldots, x_n)$ acquires the particularly simple form

$$D_n(x|x_1, x_2 \ldots, x_n) = \left(\sum_{k=0}^{n}(-1)^{k+1} \sin\frac{k\pi}{2} x^{n-k} \sigma_k^{(n)}\right) \left(\sum_{l=0}^{n}(-1)^{l} \cos\frac{l\pi}{2} x^{n-l} \sigma_l^{(n)}\right)\,. \qquad (4.52)$$

In this case the general solution of the recursive equations (4.28) is given by

$$P_n(x_1, x_2, \ldots, x_n) = \det \mathcal{A}(x_1, x_2, \ldots, x_n) \qquad (4.53)$$

where $\mathcal{A}$ is an $(n-3) \times (n-3)$ matrix whose entries are

$$\mathcal{A}_{ij}(x_1, x_2, \ldots, x_n) = \sigma_{2j-i+1}^{(n)} \cos^2\left[(i-j)\frac{\pi}{2}\right]\,, \qquad (4.54)$$

i.e.

$$\mathcal{A} = \begin{pmatrix} \sigma_2 & 0 & \sigma_6 & 0 & \cdots \\ 0 & \sigma_3 & 0 & \sigma_7 & \cdots \\ 1 & 0 & \sigma_4 & 0 & \cdots \\ 0 & \sigma_1 & 0 & \sigma_5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \qquad (4.55)$$

This can be proved by exploiting the properties of determinants. i.e. their invariance under linear combinations of the rows and the columns. Let us consider the $(n-1)\times(n-1)$ matrix associated to $P_{n+2}(-x,x,x_1,\ldots,x_n)$

$$\mathcal{A}_{ij} = \left(\sigma^{(n)}_{2j-i+1} - x^2\,\sigma^{(n)}_{2j-i-1}\right)\cos^2\left[(i-j)\frac{\pi}{2}\right] \,, \tag{4.56}$$

where eq. (4.27) was used. Adding successively x^2 times the row $(i+2)$ to row i (starting with $i=1$), we obtain for the entries of the matrix $\mathcal{A}$

$$\mathcal{A}_{ij} = \left(\sigma^{(n)}_{2j-i+1} - x^4\,\sigma^{(n)}_{2j-i-3}\right)\cos^2\left[(i-j)\frac{\pi}{2}\right] . \tag{4.57}$$

Adding now x^4 times of the i^{th} column to column $(i+2)$ (starting with $i=1$), we obtain the following matrix:

$$\mathcal{A}^{(n-1)\times(n-1)} = \begin{pmatrix} & 0 & 0 \\ \mathcal{A}^{(n-3)\times(n-3)} & \vdots & \vdots \\ & 0 & \vdots \\ *\cdots* & \mathcal{A}_{(n-2)(n-2)} & 0 \\ *\cdots* & 0 & \mathcal{A}_{(n-1)(n-1)} \end{pmatrix} \tag{4.58}$$

where the entries in the lower right corner are given by

$$\begin{aligned} \mathcal{A}_{(n-2)(n-2)} &= \sum_{k=0}^{n}(-1)^k\cos\frac{k\pi}{2}x^{n-k-1}\sigma^{(n)}_k \\ \mathcal{A}_{(n-1)(n-1)} &= \sum_{l=0}^{n}(-1)^{l+1}\sin\frac{l\pi}{2}x^{n-l}\sigma^{(n)}_l \,. \end{aligned}$$

Developing the determinant of this matrix with respect to the last two columns and taking into account eqs. (4.52) and (4.53), we obtain the right hand side of equation (4.28), Q.E.D.

4.3.2 The "Inverse Yang-Lee" Point

A closed solution of the recursive equations (4.28) is also obtained for

$$B\left(2\sqrt{\pi}\right) = \frac{2}{3} \,. \tag{4.59}$$

The reason is that, for this particular value of the coupling constant the S-matrix of the Sinh-Gordon theory coincides with the inverse of the S-matrix $S_{\rm YL}(\beta)$ of the Yang-Lee model [30] or, equivalently

$$S(\beta,-\frac{2}{3}) = S_{\rm YL}(\beta) \,. \tag{4.60}$$

Since the recursive equations (4.28) are invariant under $B \to -B$ (see sect. 4.2), a solution is provided by the same combination of symmetric polynomials found for the Yang-Lee model [9, 31], i.e.

$$P_n(x_1, x_2, \ldots, x_n) = \det \mathcal{B}(x_1, x_2, \ldots, x_n) \tag{4.61}$$

with the following entries of the $(n-3) \times (n-3)$-matrix $\mathcal{B}$

$$\mathcal{B}_{ij} = \sigma_{3j-2i+1} \ . \tag{4.62}$$

The proof is similar to the one of the previous section and exploits the invariance of a determinant under linear combinations of the rows and the columns. In this case the function D_n is most conveniently expressed as determinant of a 2×2-matrix

$$D_n = \det \begin{pmatrix} \left(\sum\limits_{l=0}^{n}(-1)^l \cos\frac{l\pi}{3} x^{n-l}\sigma_l^{(n)}\right) & \left(\sum\limits_{l=0}^{n} \cos\frac{l\pi}{3} x^{n-l}\sigma_l^{(n)}\right) \\ \left(\sum\limits_{l=0}^{n}(-1)^l \sin\frac{l\pi}{3} x^{n-l}\sigma_l^{(n)}\right) & \left(\sum\limits_{l=0}^{n} \sin\frac{l\pi}{3} x^{n-l}\sigma_l^{(n)}\right) \end{pmatrix} \tag{4.63}$$

Let us consider the $(n-1)\times(n-1)$-matrix entering the expression $P_{n+2}(-x, x, x_1, \ldots, x_n)$, i.e.

$$\mathcal{B}_{ij} = \sigma_{3j-2i+1} - x^2\sigma_{3j-2i-1}, \tag{4.64}$$

By adding successively the i^{th} row to row $(i-1)$ (starting with $i = (n-1)$), we obtain $\mathcal{B}_{ij} = \sigma_{3j-2i+1} - x^{2(n-i)}\sigma_{3j-2n+1}$. Then by adding successively x^6 times the i^{th} column to column $(i+2)$, starting with $i = 1$, the entries for the matrix $\mathcal{B}$ read

$$\mathcal{B}_{ij} = \sum_{l=0} \sigma_{3j-2i-6l+1}x^{6l} - x^{2(n-i+3l)}\sigma_{3j-2n-6l+1}. \tag{4.65}$$

Subtracting x^6 times of the row $(i+3)$ from row i (starting with $i = 1$) we finally obtain the matrix

$$\mathcal{B}^{(n-1)\times(n-1)} = \begin{pmatrix} & 0 & 0 \\ \mathcal{B}^{(n-3)\times(n-3)} & \vdots & \vdots \\ & 0 & 0 \\ * \cdots * & \mathcal{B}_{(n-2)(n-2)} & \mathcal{B}_{(n-2)(n-1)} \\ * \cdots * & \mathcal{B}_{(n-1)(n-2)} & \mathcal{B}_{(n-1)(n-1)} \end{pmatrix} \tag{4.66}$$

where the entries of the (2×2) matrix in the lower right corner are still given by (4.65). It is easy to prove that the determinant of this (2×2) matrix in the lower right corner is equal to (4.63). Therefore, with the definition (4.61), the determinant of $\mathcal{B}^{(n-1)\times(n-1)}$ gives rise the right hand side of (4.28). Q.E.D.

4.4 Form Factors and c-theorem

The Sinh-Gordon model can also be regarded as deformation of the free massless theory with central charge $c = 1$. This fixed point governs the ultraviolet behaviour of the model whereas the infrared behaviour corresponds to a massive field theory with central charge $c = 0$. Going from the short- to large-distances, the variation of the central charge is dictated by the c-theorem of Zamolodchikov [38]. An integral version of this theorem has been derived by Cardy [39] and related to the spectral representation of the two-point function of the trace of the stress-energy tensor in [40, 41], i.e.

$$\Delta c = \int_0^\infty d\mu \, c_1(\mu) \ , \tag{4.67}$$

where $c_1(\mu)$ is given by

$$\begin{aligned} c_1(\mu) &= \frac{6}{\pi^2}\frac{1}{\mu^3}\text{Im}\, G(p^2 = -\mu^2) \ , \\ G(p^2) &= \int d^2x \, e^{-ip\dot{x}} \langle \, 0 \mid \Theta(x)\Theta(0) \mid 0 \, \rangle_{\text{conn}} \ . \end{aligned} \tag{4.68}$$

Inserting a complete set of in-state into (4.68), we can express the function $c_1(\mu)$ in terms of the form factors F_{2n}^{Θ}

$$\begin{aligned} c_1(\mu) &= \frac{12}{\mu^3}\sum_{n=1}^{\infty}\frac{1}{(2n)!}\int \frac{d\beta_1 \ldots d\beta_{2n}}{(2\pi)^{2n}} \mid F_{2n}^{\Theta}(\beta_1,\ldots,\beta_{2n}) \mid^2 \\ &\quad \times \delta(\sum_i m \sinh\beta_i)\, \delta(\sum_i m\cosh\beta_i - \mu) \ . \end{aligned} \tag{4.69}$$

For the Sinh-Gordon theory $\Delta c = 1$ and it is interesting to study the convergence of this series increasing the number of intermediate particles. For the two-particle contribution, we have the following expression

$$\Delta c^{(2)} = \frac{3}{2F_{\text{min}}^2(i\pi)}\int_0^\infty \frac{d\beta}{\cosh^4\beta}\,|F_{\text{min}}(2\beta)|^2 \ . \tag{4.70}$$

The numerical results for different values of the coupling constant $g^2/4\pi$ are listed in 4.1.

It is evident that the sum rule is saturated by the two-particle form factor also for large values of the coupling constant. Hence, the expansion in the number of intermediate particles results in a fast convergent series, as it is confirmed by the computation of the next terms involving the form factor with four and six particles.

B	$\frac{g^2}{4\pi}$	$\Delta\, c^{(2)}$
$\frac{1}{500}$	$\frac{2}{999}$	0.9999995
$\frac{1}{100}$	$\frac{2}{199}$	0.9999878
$\frac{1}{10}$	$\frac{2}{19}$	0.9989538
$\frac{3}{10}$	$\frac{6}{17}$	0.9931954
$\frac{2}{5}$	$\frac{1}{2}$	0.9897087
$\frac{1}{2}$	$\frac{2}{3}$	0.9863354
$\frac{2}{3}$	1	0.9815944
$\frac{7}{10}$	$\frac{14}{13}$	0.9808312
$\frac{4}{5}$	$\frac{4}{3}$	0.9789824
1	2	0.9774634

Table 4.1: The first two-particle term entering the sum rule of the c-theorem.

Conclusion

In this thesis we have presented the solution to the locality problem for the Sinh-Gordon model solving the axiomatic equations for the form factors of the elementary field and of the trace of the stress-energy tensor. It has been proposed a characterization of the whole local operator algebra in terms of descendent of the above mentioned fundamental fields. This method can be in principle generalized to any two-dimensional massive integrable model whose exact S-matrix is known. The only obstacle to the solution of more complicated models, e.g. Affine Toda Field Theories [20, 23], would consist in the technical difficulty of the equations. The Bullogh-Dodd model [35] is perhaps the simplest of these theories, besides the Sinh-Gordon model [35, 36]. The method described in this thesis for this model seems to show very interesting features, like the existence of a decoupling point in which the model is equivalent to the Sinh-Gordon model at the inverse Yang-Lee point (we anticipate that in this approach the $\mathcal{Z}_2$-symmetry of the Sinh-Gordon model is seen to be dinamically recovered in a very natural way).

Since these models have an explicit dependence on some coupling constants, their solution allows to characterize the flow of the renormalization group along particular directions in the space of the two-dimensional theories, [2, 38, 40, 41]. In this thesis we have used this characterization by checking our result against the c-theorem finding a good agreement.

As a final remark we underl~~y~~ine the fact that the form factors can be expressed as determinants of some matrices of size proportional to the number of external particles. This seems to be very promising in the direction towards the interpretation of the correlation functions as τ-functions of some new classical integrable system [15, 13, 14].

Acknoledgments

The original part of this thesis is based on a collaboration with Andreas Fring and Giuseppe Mussardo. It is a pleasure for me to thank them for having put their experience at my disposal.

In particular I wish to express my deep gratitude to my supervisor Giuseppe Mussardo, not only for having patiently taught me many things on Integrable Models, but also for the moral support and friendship that he has given me. It has been a pleasure to work with him.

Finally I wish to thank Anni Koubek, Marta Nolasco and Adam Schwimmer for many fruitful discussions.

Appendix A

We prove in this appendix the eq. 3.44. We will refer to some useful formulas of [42]. In order to obtain 3.44 we need first to represent $F_{\text{min}}\beta, B$ as an infinite product of Euler's Γ-functions. Let's start from the following identity

$$8 \sinh \frac{Bx}{4} \sinh \frac{(2-B)x}{4} \sinh x/2 = 2 \left(\sinh x - \sinh \frac{Bx}{2} - \sinh \frac{(2-B)x}{2} \right) \tag{4.71}$$

which together with formulas 1.232.3 and 3.946.2 of [42] allows us to write

$$\begin{aligned}
8 \int_0^{+\infty} \frac{dx}{x} \sinh \frac{Bx}{4} \sinh \frac{(2-B)x}{4} \frac{\sinh x/2}{\sinh^2 x} \sin^2 \frac{\hat{\beta} x}{2\pi} &= \\
= 4 \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \int_0^{+\infty} \frac{dx}{x} e^{-(2k+2n+2)x} \left(e^{x} - e^{-x} + e^{-\frac{Bx}{2}} - e^{\frac{Bx}{2}} + e^{-\frac{(2-B)x}{2}} - e^{\frac{(2-B)x}{2}} \right) \sin^2 \frac{\hat{\beta} x}{2\pi} \\
= \sum_{n=0}^{+\infty} \sum_{k=0}^{+\infty} \log \left\{ \frac{1 + \left(\frac{\hat{\theta}/\pi}{2k+2n+1}\right)^2 1 + \left(\frac{\hat{\theta}/\pi}{2k+2n+2+B/2}\right)^2 1 + \left(\frac{\hat{\theta}/\pi}{2k+2n+3-B/2}\right)^2}{1 + \left(\frac{\hat{\theta}/\pi}{2k+2n+3}\right)^2 1 + \left(\frac{\hat{\theta}/\pi}{2k+2n+2-B/2}\right)^2 1 + \left(\frac{\hat{\theta}/\pi}{2k+2n+1+B/2}\right)^2} \right\} .
\end{aligned}$$

Using the functional relation 8.326.1 of [42] of the Euler Γ-function we arrive to

$$\begin{aligned}
F_{\text{min}}(\beta, B) = \mathcal{N} \prod_{k=0}^{+\infty} \left| \frac{\Gamma(k+1/2)\, \Gamma(k+1+B/4)\, \Gamma(k+3/2-B/4)}{\Gamma(k+3/2)\, \Gamma(k+1-B/4)\, \Gamma(k+1/2+B/4)} \right|^2 \\
\times \left| \frac{\Gamma\left(k+3/2+i\hat{\beta}/2\pi\right) \Gamma\left(k+1-B/4+i\hat{\beta}/2\pi\right) \Gamma\left(k+1/2+B/4+i\hat{\beta}/2\pi\right)}{\Gamma\left(k+1/2+i\hat{\beta}/2\pi\right) \Gamma\left(k+1+B/4+i\hat{\beta}/2\pi\right) \Gamma\left(k+3/2-B/4+i\hat{\beta}/2\pi\right)} \right|^2 .
\end{aligned}$$

Let's recall the normalization $\lim_{\beta\to\infty} F_{\text{min}}(\beta, B) = 1$ in order to obtain N

$$\lim_{\beta\to\infty} F_{\text{min}}(\beta, B) = \mathcal{N} \prod_{k=0}^{+\infty} \left| \frac{\Gamma(k+1/2)\, \Gamma(k+1+B/4)\, \Gamma(k+3/2-B/4)}{\Gamma(k+3/2)\, \Gamma(k+1-B/4)\, \Gamma(k+1/2+B/4)} \right|^2 = 1 .$$

With this result we simplify the above expression for $F_{\text{min}}(\beta, B)$

$$F_{\text{min}}(\beta, B) = \prod_{k=0}^{+\infty} \left| \frac{\Gamma\left(k+3/2+i\hat{\beta}/2\pi\right)}{\Gamma\left(k+1/2+i\hat{\beta}/2\pi\right)} \right|^2$$

$$\times \left| \frac{\Gamma\left(k+1-B/4+i\hat{\beta}/2\pi\right)\,\Gamma\left(k+1/2+B/4+i\hat{\beta}/2\pi\right)}{\Gamma\left(k+1+B/4+i\hat{\beta}/2\pi\right)\,\Gamma\left(k+3/2-B/4+i\hat{\beta}/2\pi\right)} \right|^2 .$$

The product of the left-hand side of 3.44 therefore becomes

$$\begin{aligned} F_{\text{min}}(i\pi+\beta,B)\,F_{\text{min}}(\beta,B) &= \\ &= \prod_{k=0}^{+\infty} \left| \left(\frac{1}{1+\frac{B/4}{k+1/2+i\beta/2\pi}} \right) \left(\frac{1}{1-\frac{B/4}{k+1+i\beta/2\pi}} \right) \right|^2 \\ &= \left| \frac{\Gamma\left(1/2+B/4+i\hat{\beta}/2\pi\right)\,\Gamma\left(1-B/4+i\hat{\beta}/2\pi\right)}{\Gamma\left(1/2+i\hat{\beta}/2\pi\right)\,\Gamma\left(1+i\hat{\beta}/2\pi\right)} \right|^2 , \end{aligned}$$

where we have used the property $\Gamma(1+x) = x\,\Gamma(x)$ and the formula 8.325.1 of [42]. Let's evaluate it on the line $\hat{\beta} = \frac{i\pi}{2} + \alpha \quad, \alpha \in \mathcal{R}$:

$$\begin{aligned} &F_{\text{min}}(i\pi+\beta,B)\,F_{\text{min}}(\beta,B) = \\ &= \frac{\Gamma\left(1/2+B/4+i\hat{\beta}/2\pi\right)\,\Gamma\left(1-B/4+i\hat{\beta}/2\pi\right)}{\Gamma\left(1/2+i\hat{\beta}/2\pi\right)\,\Gamma\left(1+i\hat{\beta}/2\pi\right)} \\ &\times \frac{\Gamma\left(B/4-i\hat{\beta}/2\pi\right)\,\Gamma\left(1/2-B/4-i\hat{\beta}/2\pi\right)}{\Gamma\left(-i\hat{\beta}/2\pi\right)\,\Gamma\left(1/2-i\hat{\beta}/2\pi\right)} \\ &= \frac{\cos\frac{i\beta}{2}\ \sin\frac{i\beta}{2}}{\cos\left(\frac{\pi B}{4}+\frac{i\beta}{2}\right)\ \sin\left(\frac{i\beta}{2}-\frac{\pi B}{4}\right)} , \end{aligned}$$

where we have used the formulas 8.334.2 and 8.334.3 of [42]. Simple trigonometric transformations then lead to the relation 3.44.

We close this appendix showing an useful expression for the numerical evaluation of $F_{\text{min}}(\beta,B)$ is given by

$$\begin{aligned} &F_{\text{min}}(\beta,B) = \\ &\mathcal{N} \prod_{k=0}^{N} \left[\frac{\left(1+\left(\frac{\hat{\beta}/2\pi}{k+\frac{1}{2}}\right)^2\right)}{\left(1+\left(\frac{\hat{\beta}/2\pi}{k+\frac{3}{2}}\right)^2\right)} \right]^{k+1} \\ &\times \left[\frac{\left(1+\left(\frac{\hat{\beta}/2\pi}{k+\frac{3}{2}-\frac{B}{4}}\right)^2\right)\left(1+\left(\frac{\hat{\beta}/2\pi}{k+1+\frac{B}{4}}\right)^2\right)}{\left(1+\left(\frac{\hat{\beta}/2\pi}{k+\frac{1}{2}+\frac{B}{4}}\right)^2\right)\left(1+\left(\frac{\hat{\beta}/2\pi}{k+1-\frac{B}{4}}\right)^2\right)} \right]^{k+1} \\ &\times e^{8\int_0^\infty \frac{dx}{x} \frac{\sinh\left(\frac{xB}{4}\right)\sinh\left(\frac{x}{4}(2-B)\right)\,\sinh\frac{x}{2}}{\sinh^2 x} (N+1-N\,e^{-2x})\,e^{-2Nx}\,\sin^2\left(\frac{x\hat{\beta}}{2\pi}\right)} . \end{aligned}$$

The rate of convergence of the integral may be improved substantially by increasing the value of N. Graphs of $F_{\text{min}}(\beta, B)$ are drawn in fig. 4.2.

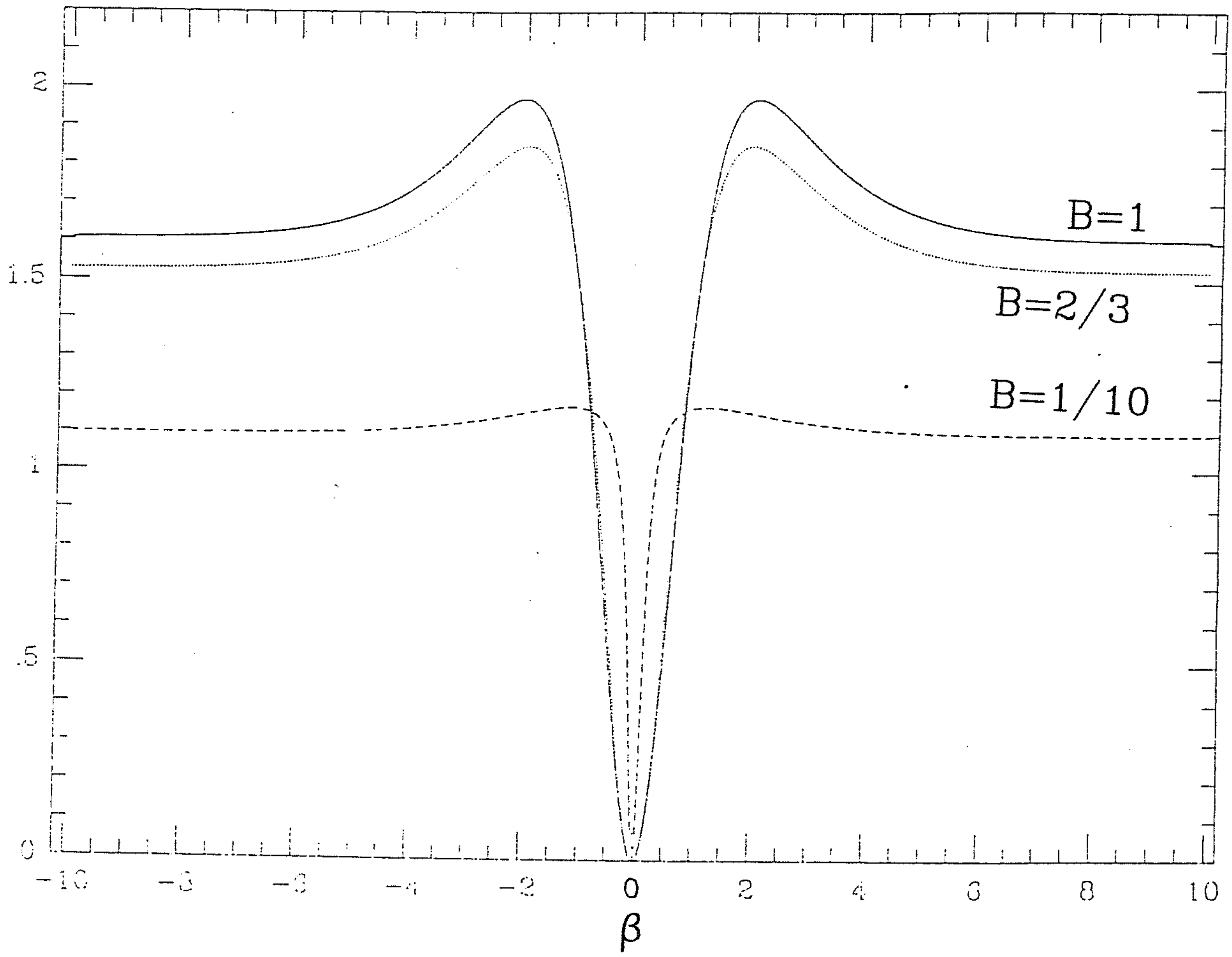

Figure 4.2: Graphs of $|F_{\min}(\beta, B)/\mathcal{N}|^2$ as function of β for different values of $B(g)$.

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