



ISAS - INTERNATIONAL SCHOOL FOR ADVANCED STUDIES

On the Lavrentiev Phenomenon and the validity of Euler-Lagrange Equations for a class of integral functionals

Thesis submitted for the degree of
“Magister Philosophiæ”

CANDIDATE

Giulia Treu

SUPERVISOR

Prof. Arrigo Cellina

June 1992

TRIESTE

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1. INTRODUCTION

Euler-Lagrange Equations (*EL*) are well known necessary conditions for a function x to be a minimizer of the functional

$$\mathcal{I}(x) = \int_a^b f(t, x(t), x'(t)) dt$$

under given boundary conditions.

In order to prove the validity of such Equations one imposes that the Gateaux derivative in x of the functional along a certain class of directions is zero. More precisely, one imposes:

$$\left. \frac{d}{d\theta} \mathcal{I}(x + \theta\xi) \right|_{\theta=0} = 0,$$

for any ξ with essentially bounded derivative and zero boundary conditions. This procedure requires the differentiability of f with respect to the second and the third variable and the integrability of f and of its derivatives along trajectories close to the minimizer x . This last requirements can be satisfied by imposing some integrable bound on the growth of f , $\nabla_x f$, $\nabla_{x'} f$ in a neighbourhood of the graph of x . Actually, such assumptions are strong enough to ensure the continuity along a wider class of variations including, in particular, lipschitzian approximations of x . In other words, the hypotheses under which Euler-Lagrange Equations are usually derived, exclude Lavrentiev Phenomenon, which consists in the relevant fact that the infimum of \mathcal{I} on the class of admissible trajectories with essentially bounded derivative can be strictly larger than the minimum on the class of all admissible trajectories.

For this reason the work reported in this thesis moves from the search of conditions which prevent Lavrentiev Phenomenon to occur, in the aim of extending such a result to the problem of the derivation of Euler-Lagrange Equations. Indeed it is easy to exhibit simple examples (Manià's functional) in which the standard assumptions under which *EL* are derived are not satisfied; however one can check that formally equations *EL* are satisfied.

The study of conditions excluding Lavrentiev Phenomenon was considered since the first works on the subject ([L], [M] and [T]) and, more recently, by many authors ([A], [BuM], [CV] and [Lo]). In [T] Tonelli defined a kind of lipschitzian approximations of the minimizer and determined a class of functionals, characterized by some assumptions involving the differentiability properties of the integrand f , which are continuous along such approximations. In [A] the author, with a refinement of the idea of Tonelli, gives a very general condition on f which excludes Lavrentiev Phenomenon. In our first result, with a proof similar to that one of Angell, we provide a class of functionals for which the Lavrentiev Phenomenon does not occur, that is strictly larger than the class singled out by Tonelli.

The approximation procedure that we define, suggests to consider a class of variations around the minimizer, depending on a continuous parameter, along which the continuity of the functional

is ensured. Even though such variations are not taken along a fixed direction, we study the differentiability of \mathcal{I} along these variations, in the aim of deriving Euler-Lagrange Equations under assumptions weaker than the standard ones. Indeed, since such variations are obtained by truncating the derivative of x , some of the classical requirements on the behaviour of f near x can be removed. In such a way, we obtain a result which enlarge the range of validity of Equations EL .

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2. PRELIMINARIES AND NOTATIONS

We consider an open subset A of $\mathbb{R} \times \mathbb{R}^n$ and a compact interval of \mathbb{R} , $I = [a, b]$, and assume that, for any $t \in I$, the set $\{x \in \mathbb{R}^n : (t, x) \in \text{int}(A)\}$ is nonempty. Let $f : A \times \mathbb{R}^n \rightarrow \mathbb{R}$; we are interested in the study of the functional

$$\mathcal{I}(x) = \int_a^b f(t, x(t), x'(t)) dt$$

defined on the class of admissible trajectories with given boundary conditions, i.e. on the set

$$\Omega = \{x \in W^{1,1}(I, \mathbb{R}^n) : \text{for any } t \in I, (t, x(t)) \in A \text{ and } x(a) = x_a, x(b) = x_b\},$$

where $x_a, x_b \in \mathbb{R}^n$ are such that $(a, x_a), (b, x_b)$ belong to A and Ω is nonempty.

This thesis is mainly concerned with the problem

$$\mathcal{P} : \quad \text{Minimize } \{\mathcal{I}(x); x \in \Omega\}.$$

Given $x \in \Omega$ we call graph of x the set $\Gamma = \{(t, x(t)) : t \in I\}$ and given $\sigma > 0$ we call σ -neighbourhood of the graph of x the set $\Gamma_\sigma = \{(t, y) : t \in I, \text{ s. t. } |y - x(t)| \leq \sigma\}$. We say that the graph of x lies in the interior of A if there exists a σ -neighbourhood of the graph of x contained in A . We say that $x \in \Omega$ gives a strong local minimum for \mathcal{I} if there exists $\sigma > 0$ such that for any $y \in \Omega$ with graph contained in Γ_σ it is $\mathcal{I}(y) \geq \mathcal{I}(x)$. We say that $x \in \Omega$ gives a weak local minimum for \mathcal{I} if there exists $\sigma > 0$ and $\tau > 0$ such that for any $y \in \Omega$ with graph contained in Γ_σ and such that it is $|y'(t) - x'(t)| < \tau$ for a.e. $t \in I$, $\mathcal{I}(y) \geq \mathcal{I}(x)$.

We shall use the following standard notations. By $\langle \cdot, \cdot \rangle$ we denote the scalar product in \mathbb{R}^n and by $|\cdot|$ the associated norm; E^c is the complement of the set E and $\mu(\cdot)$ is the Lebesgue measure. We shall denote by $C(I)$, $L^p(I)$ and $W^{1,p}(I)$, the spaces $C(I, \mathbb{R}^n)$, $L^p(I, \mathbb{R}^n)$ and $W^{1,p}(I, \mathbb{R}^n)$, for $1 \leq p \leq \infty$, and by $\|\cdot\|_{C(I)}$, $\|\cdot\|_{L^p}$ and $\|\cdot\|_{W^{1,p}}$ the respective norms. By $\nabla_x f$ and $\nabla_{x'} f$ we denote the gradients of f with respect to the second and the third variable. We set also, for $p \geq 1$, $p' = p/(p-1)$.

Definition 2.1. Let $E \subseteq \mathbb{R}^n$ be measurable, $h : E \rightarrow \mathbb{R}$ be measurable and $\alpha, \beta \in \bar{\mathbb{R}}$. We set

$$E_{\alpha, \beta}(h) := \{t \in E : h(t) \in]\alpha, \beta]\},$$

and $E_\alpha(h) := E_{\alpha, +\infty}(h)$. We set also $\omega(h, \alpha) := \mu(E_\alpha(h))$.

We will make use of the following theorem (see [WZ] pp. 81-83).

Theorem 2.1. Let $E \subseteq \mathbb{R}^n$ be measurable, $\mu(E) < \infty$, $h : E \rightarrow \mathbb{R}$ measurable, $\alpha, \beta \in \bar{\mathbb{R}}$, $\alpha \leq \beta$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and such that $\phi \circ h \in L^1(E)$. Then

$$\int_{E_{\alpha, \beta}(h)} \phi(h(x)) dx = - \int_{\alpha}^{\beta} \phi(\sigma) d\omega(h, \sigma) \quad (2.1)$$

(where the last is a Stieltjes integral). In particular

$$\int_{E_{\alpha, \beta}(h)} h^p = - \int_{\alpha}^{\beta} \sigma^p d\omega(h, \sigma) = -\beta^p \omega(h, \beta) + \alpha^p \omega(h, \alpha) + p \int_{\alpha}^{\beta} \sigma^{p-1} \omega(h, \sigma) d\sigma. \quad (2.2)$$

We recall Tchebyshev inequality (see for instance [WZ] p.82).

Theorem 2.2. Let $E \subseteq \mathbb{R}^n$ be measurable and h belong to $L^p(E, \mathbb{R}^n)$. Then

$$\omega(|h|, \sigma) \leq \frac{\|h\|_{L^p}^p}{\sigma^p} \text{ for any } \sigma > 0.$$

3. LAVRENTIEV PHENOMENON

We say that the functional \mathcal{I} exhibits the so called Lavrentiev phenomenon if

$$\inf_{x \in \Omega \cap W^{1, \infty}(I)} \mathcal{I}(x) > \min_{x \in \Omega} \mathcal{I}(x).$$

In the study of such phenomenon it is of particular interest the following example due to Manià (see [C] pp. 514-516, [D] pp. 92-95, [BuM] p.13), consisting in the minimum problem

$$\mathcal{P}_g : \text{Minimize } \left\{ \mathcal{I}_g(x) := \int_0^1 g(t, x(t), x'(t)) dt; x \in W^{1,1}([0,1], \mathbb{R}), x(0) = 0, x(1) = 1 \right\}$$

where $g(t, x, v) = (x^3 - t)^2 |v|^q$. It is easy to see that the solution of \mathcal{P}_g is $x_0(t) = t^{\frac{1}{3}}$ and $\mathcal{I}(x_0) = 0$. We have the following result.

Proposition 3.1.

- i) If $0 \leq q < \frac{9}{2}$ then $\inf \{ \mathcal{I}_g(x); x \in W^{1, \infty}([0,1], \mathbb{R}), x(0) = 0, x(1) = 1 \} = 0$.
- ii) If $q \geq \frac{9}{2}$ then $\inf \{ \mathcal{I}_g(x); x \in W^{1, \infty}([0,1], \mathbb{R}), x(0) = 0, x(1) = 1 \} > 0$.

Proof. We prove statement *i*), for *ii*) see [BuM].

Let us define the following sequence $\{x_k\}_{k \in \mathbb{N}}$ in $W^{1,\infty}(I, \mathbb{R})$ of lipshitz approximations of $x_0(t) = t^{\frac{1}{3}}$:

$$x_k(t) = \begin{cases} 3kt, & t \in [0, (3k)^{-\frac{3}{2}}[\\ t^{\frac{1}{3}}, & t \in [(3k)^{-\frac{3}{2}}, 1] \end{cases}$$

It is, by easy computations,

$$\mathcal{I}_g(x_k) = \int_0^1 g(t, x_k(t), x'_k(t)) dt = \frac{8}{105} (3k)^{q-\frac{9}{2}}.$$

Hence $\lim_{k \rightarrow \infty} \mathcal{I}_g(x_k) = 0$. □

The study of Manià example leads to the investigation of general properties of the integrand f which prevent Lavrentiev phenomenon to occur; Theorem 3.1 below provides a result in this direction extending the original work of Tonelli [T] (see Remark 3.2 below).

We shall need the following technical lemma.

Lemma 3.1 *Let $E \subseteq \mathbb{R}^n$ be measurable, $\mu(E) < \infty$, and let x belong to $L^p(E)$, $p > 0$. Let $q_1, q_2, \gamma_1, \gamma_2$ be positive numbers such that $q_2 \leq p$ and $\gamma_1(p - q_1) = (q_2 - p)\gamma_2$. Then, for any $\delta \geq 0$,*

$$\left(\int_{(E_\delta(|x|))^c} |x(t)|^{q_1} dt \right)^{\gamma_1} \left(\int_{E_\delta(|x|)} |x(t)|^{q_2} dt \right)^{\gamma_2} \leq \left(\int_{(E_\delta(|x|))^c} |x(t)|^p dt \right)^{\gamma_1} \left(\int_{E_\delta(|x|)} |x(t)|^p dt \right)^{\gamma_2}.$$

Proof. First of all the integrals in the l.h.s. do exist. For any $\sigma > 0$ we set $\psi(\sigma) = -\omega(|x|, \sigma)$; ψ is non decreasing, hence, if f and g are real valued continuous functions defined on $[0, +\infty[$ such that $f \leq g$, it is

$$\int_\alpha^\beta f(\sigma) d\psi(\sigma) \leq \int_\alpha^\beta g(\sigma) d\psi(\sigma)$$

for any $\alpha, \beta \in \bar{\mathbb{R}}^+$. Now using formula (2.1) we have

$$\begin{aligned} \left(\int_{(E_\delta(|x|))^c} |x(t)|^{q_1} dt \right)^{\gamma_1} \left(\int_{E_\delta(|x|)} |x(t)|^{q_2} dt \right)^{\gamma_2} = \\ \left(\int_0^\delta \sigma^{q_1} d\psi(\sigma) \right)^{\gamma_1} \left(\int_\delta^\infty \tau^p \tau^{q_2-p} d\psi(\tau) \right)^{\gamma_2}. \end{aligned} \tag{3.1}$$

Since for $\tau \geq \delta$ it is $\tau^{q_2-p} \leq \delta^{q_2-p}$, and for $0 < \sigma \leq \delta$ it is $\delta^{(q_2-p)\frac{\gamma_2}{\gamma_1}} \leq \sigma^{(q_2-p)\frac{\gamma_2}{\gamma_1}}$, the right hand side of (3.1) can be estimated as

$$\begin{aligned} \left(\int_0^\delta \sigma^{q_1} d\psi(\sigma) \right)^{\gamma_1} \left(\int_\delta^\infty \tau^p \delta^{q_2-p} d\psi(\tau) \right)^{\gamma_2} &= \\ \left(\int_0^\delta \sigma^{q_1} \delta^{(q_2-p)\frac{\gamma_2}{\gamma_1}} d\psi(\sigma) \right)^{\gamma_1} \left(\int_\delta^\infty \tau^p d\psi(\tau) \right)^{\gamma_2} &\leq \\ \left(\int_0^\delta \sigma^{q_1+(q_2-p)\frac{\gamma_2}{\gamma_1}} d\psi(\sigma) \right)^{\gamma_1} \left(\int_\delta^\infty \tau^p d\psi(\tau) \right)^{\gamma_2}. \end{aligned}$$

Since $q_1 + (q_2 - p)\frac{\gamma_2}{\gamma_1} = p$, this ends the proof. \square

Following Angell and Cesari ([A], [C] and [CA]) we give the following

Definition 3.1. We say that $f : A \times \mathbb{R}^n \rightarrow \mathbb{R}$ satisfies Caratheodory condition (C) provided that given $\epsilon > 0$, there is a compact subset $K_\epsilon \subset I$ such that $\mu(I \setminus K_\epsilon) < \epsilon$, $A_{K_\epsilon} = A \cap (K_\epsilon \times \mathbb{R}^n)$ is closed, and the function f is continuous on $A_{K_\epsilon} \times \mathbb{R}^n$.

In the main result of this section we shall assume that f satisfies one of the following conditions.

H_1 : f satisfies condition (C) and maps bounded subsets of its domain into bounded subsets of \mathbb{R} .

H_2 : f is continuous on its domain.

Theorem 3.1. Let f satisfy either H_1 or H_2 and let x be an element of $\Omega \cap W^{1,p}(I)$ whose graph is contained in the interior of A (i.e. there exists a σ -neighbourhood Γ_σ of the graph of x contained in A) and such that $f(\cdot, x(\cdot), x'(\cdot)) \in L^1(I)$. Assume that

H_3 : there exist $m, M \geq 0$ and $\gamma > 0$ such that, for any $(t, y) \in \Gamma_\sigma$ it is

$$|f(t, y, x'(t)) - f(t, x(t), x'(t))| \leq (m + M|x'(t)|^q)|x(t) - y|^\gamma, \text{ where } q = p(\gamma + 1) - \gamma.$$

Then, given $\epsilon > 0$, there exists $y \in W^{1,\infty}(I) \cap \Omega$ such that

$$\|y - x\|_{W^{1,p}} \leq \epsilon$$

$$|\mathcal{I}(y) - \mathcal{I}(x)| \leq \epsilon.$$

Corollary 3.1. *Under the hypotheses of theorem 3.1, if x is a solution of \mathcal{P} , then*

$$\inf_{x \in \Omega \cap W^{1,\infty}(I)} \mathcal{I}(x) = \min_{x \in \Omega} \mathcal{I}(x);$$

that is to say, the hypotheses of theorem 3.1 exclude Lavrentiev phenomenon.

Remark 3.1. Hypothesis H_3 in theorem 3.1 includes as a special case, ($\gamma = 1$, $q = 2p - 1$), the following

H_4 : *f is continuously differentiable with respect to the second variable and there exist positive constants m, M such that $|\nabla_x f(t, y, x'(t))| \leq m + M|x'(t)|^{2p-1}$ for any $(t, y) \in \Gamma_\sigma$.*

Remark 3.2. Hypothesis H_4 provides an extension of condition (β) in [T] (see also [C] Remark, p. 512):

(β) : *f is continuously differentiable with respect to the second variable and there exist positive constants m, M such that $|\nabla_x f(t, y, v)| \leq m + M|v|$ for any $(t, y, v) \in \Gamma_\sigma \times \mathbb{R}^n$.*

We emphasize that, in the case in which x is in $W^{1,p}(I)$, the proof of Tonelli can be easily reproduced under the weaker assumption that there exist positive constants m, M such that $|\nabla_x f(t, y, v)| \leq m + M|v|^p$ for any $(t, y, v) \in \Gamma_\sigma \times \mathbb{R}^n$.

Remark 3.3. Application.

Let us consider the following Manià type functionals: $\mathcal{I}_{g_m}(x) = \int_0^1 g_m(t, x(t), x'(t))dt$; where $g_m(t, x, v) = (x^3 - t)^{2m}|v|^q$, $m \in \mathbb{N}$. The hypotheses of Theorem 3.1 are satisfied for $q < \frac{3}{2} + m$, while Tonelli's condition (β) holds only for $q < \frac{3}{2}$.

Proof of Theorem 3.1. We may assume $x' \in L^p(I) \setminus L^\infty(I)$ since when x' is essentially bounded there is nothing to prove. For any positive number ρ define the sets $I_\rho = \{t \in I : |x'(t)| > \rho\}$, take $R > 0$ such that the complement of I_R, I_R^c , has positive measure and for any $\delta \in \mathbb{R}$, $\delta \geq R$ we set

$$\beta_\delta = \frac{1}{\mu(I_R^c)} \int_{I_\delta} x'(\tau) d\tau.$$

Since x' belongs to $L^p(I)$ we have

$$\lim_{\delta \rightarrow \infty} \mu(I_\delta) = 0 \tag{3.2}$$

and, obviously,

$$\lim_{\delta \rightarrow \infty} \beta_\delta = 0. \tag{3.3}$$

Consider, for any $\delta \geq R$, the function y_δ defined by setting

$$y'_\delta(t) = \begin{cases} 0, & t \in I_\delta \\ x'(t), & t \in I_R \setminus I_\delta \\ x'(t) + \beta_\delta, & t \in I_R^c \end{cases}$$

and

$$y_\delta(t) = x_a + \int_0^t y'_\delta(\tau) d\tau.$$

Since y'_δ is bounded by δ , y_δ is in $W^{1,\infty}(I)$. We have $y_\delta(a) = x_a$ and

$$y_\delta(b) = x_a + \int_{I_\delta^c} x'_\delta(\tau) d\tau + \beta_\delta \mu(I_R^c) = x(a) + \int_I x'_\delta(\tau) d\tau = x(b) = x_b.$$

Moreover

$$\int_I |y'_\delta(t) - x'(t)|^p dt \leq \int_{I_\delta} |x'(t)|^p dt + |\beta_\delta|^p \mu(I_R^c).$$

Hence, by (3.2) and (3.3), y_δ is arbitrarily close to x in $W^{1,p}(I)$, and also in $C(I)$, when δ is sufficiently large; in particular we have the estimate

$$\|y_\delta - x\|_{C(I)} \leq 2 \int_{I_\delta} |x'(\tau)| d\tau. \quad (3.4)$$

Inequality (3.4) ensures that there exists δ_0 such that the set $\{(t, y_\delta(t)), t \in I\}$ is contained in $\Gamma_\sigma \subset A$ for every $\delta > \delta_0$. In particular, for any $\delta > \delta_0$, y_δ belongs to $W^{1,\infty}(I) \cap \Omega$.

To prove the theorem we shall show that $\mathcal{I}(y_\delta)$ is arbitrarily close to $\mathcal{I}(x)$ when δ is sufficiently large.

Let us write

$$\begin{aligned} |\mathcal{I}(y_\delta) - \mathcal{I}(x)| &= \left| \int_I (f(t, y_\delta(t), y'_\delta(t)) - f(t, x(t), x'(t))) dt \right| \leq \\ &\int_{I_\delta} |f(t, y_\delta(t), 0) - f(t, x(t), x'(t))| dt + \\ &\int_{I_R^c} |f(t, y_\delta(t), x'(t) + \beta_\delta) - f(t, x(t), x'(t))| dt + \\ &\int_{I_R \setminus I_\delta} |f(t, y_\delta(t), x'(t)) - f(t, x(t), x'(t))| dt = \\ &\Lambda_1(\delta) + \Lambda_2(\delta) + \Lambda_3(\delta). \end{aligned}$$

We claim that $\lim_{\delta \rightarrow \infty} \Lambda_i(\delta) = 0$, $i = 1, 2, 3$.

1.)

$$\Lambda_1(\delta) \leq \int_{I_\delta} |f(t, y_\delta(t), 0)| dt + \int_{I_\delta} |f(t, x(t), x'(t))| dt. \quad (3.5)$$

For any $\delta > \delta_0$, $|y_\delta(t)| \leq |x(t)| + \sigma$, $t \in I$, hence, since f maps bounded subsets of its domain into bounded subsets of \mathbb{R} , the first integrand in (3.5) is bounded by a constant. By hypothesis $f(\cdot, x(\cdot), x'(\cdot))$ belongs to $L^1(I)$; hence, by (3.2), absolute continuity of the integral implies that $\lim_{\delta \rightarrow \infty} \Lambda_1(\delta) = 0$.

2.) On the set I_R^c the family $\{x'(\cdot) + \beta_\delta, \delta \geq \delta_0\}$ is uniformly bounded by a constant. Hence the family

$$\{h_\delta(t) = |f(t, y_\delta(t), x'(t) + \beta_\delta) - f(t, x(t), x'(t))|, \quad \delta \geq \delta_0\}$$

is integrably bounded on I_R^c .

Assume first that f satisfies H_2 (i.e. f is continuous on its domain). By the pointwise convergence of y_δ to x on I_R^c , by (3.3) and by dominated convergence we have $\lim_{\delta \rightarrow \infty} \Lambda_2(\delta) = 0$.

Assume that f satisfies hypothesis H_1 . Given $\epsilon > 0$ we can take a compact set K_ϵ contained in I such that f is continuous on $A_{K_\epsilon} \times \mathbb{R}^n$ ($A_{K_\epsilon} = A \cap (K_\epsilon \times \mathbb{R}^n)$) and such that the measure of $I \setminus K_\epsilon$ is small enough so that, by the absolute equiintegrability of the family $\{h_\delta\}$,

$$\int_{(I \setminus K_\epsilon) \cap I_R^c} h_\delta(t) dt < \frac{\epsilon}{2} \quad \text{for any } \delta > \delta_0.$$

By the continuity of f on A_{K_ϵ} , the pointwise convergence of y_δ and by (3.3), there exists δ_ϵ such that, by dominated convergence,

$$\int_{K_\epsilon \cap I_R^c} h_\delta(t) dt < \frac{\epsilon}{2} \quad \text{for any } \delta > \delta_\epsilon.$$

Hence $\int_{I_R^c} h_\delta(t) dt < \epsilon$ for any $\delta > \delta_\epsilon$, and, also in this case, $\lim_{\delta \rightarrow \infty} \Lambda_2(\delta) = 0$.

3.) Hypothesis H_3 and (3.4) imply that

$$\begin{aligned} \Lambda_3(\delta) &\leq \int_{I_\delta^c} |f(t, y_\delta(t), x'(t)) - f(t, x(t), x'(t))| dt \leq \\ &\int_{I_\delta^c} (m + M|x'(t)|^q) |y_\delta(t) - x(t)|^\gamma dt \leq \\ &2^\gamma \int_{I_\delta^c} (m + M|x'(t)|^q) dt \left(\int_{I_\delta} |x'(\tau)| d\tau \right)^\gamma \leq \\ &2^\gamma m \left(\int_{I_\delta} |x'(\tau)| d\tau \right)^\gamma + 2^\gamma M \left(\int_{I_\delta^c} |x'(\tau)|^q d\tau \right) \left(\int_{I_\delta} |x'(\tau)| d\tau \right)^\gamma. \end{aligned} \tag{3.6}$$

Applying Lemma 3.1 with $q_1 = q$, $q_2 = 1$, $\gamma_1 = 1$, $\gamma_2 = \gamma$ to the second term in the r.h.s. of (3.6), we have

$$\Lambda_3(\delta) \leq 2^\gamma m \left(\int_{I_\delta} |x'(\tau)| d\tau \right)^\gamma + 2^\gamma M \left(\int_I |x'(\tau)|^p d\tau \right) \left(\int_{I_\delta} |x'(\tau)|^p d\tau \right)^\gamma.$$

Hence, by (3.2), $\lim_{\delta \rightarrow \infty} \Lambda_3(\delta) = 0$. □

4. EULER-LAGRANGE EQUATIONS

Euler-Lagrange equations are well known necessary conditions for a function x in Ω to be a local minimum for the functional \mathcal{I} , when f is assumed to be of class C^1 on its domain and satisfy some growth conditions in a neighbourhood of the graph of x .

Our aim is to weaken these growth assumption. We begin by stating the classical theorem (see for example [C] Remark 2 pp.40-41, and Remark 1 p. 44) in order to compare it with our result (Theorem 4.2).

Theorem 4.1. *Let f belong to $C^1(A \times \mathbb{R}^n, \mathbb{R})$ and let x belong to $\Omega \cap W^{1,p}(I)$, $1 \leq p < \infty$. Assume that the graph of x lies in the interior of A , (i.e. there exist σ positive and a σ -neighbourhood Γ_σ of the graph of x contained in A), that x gives a weak local minimum for \mathcal{I} and that there exist positive constants m, M such that f satisfies the following conditions:*

$$C_1: |f(t, y, v)| \leq m + M|v|^p,$$

$$C_2: |\nabla_x f(t, y, v)| \leq m + M|v|^p,$$

$$C_3: |\nabla_{x'} f(t, y, v)| \leq m + M|v|^p$$

for any $(t, y, v) \in \Gamma_\sigma \times \mathbb{R}^n$.

Then

$$\frac{d}{dt} \nabla_{x'} f(t, x(t), x'(t)) = \nabla_x f(t, x(t), x'(t)) \quad \text{a.e. } t \in I. \quad (EL)$$

Or

$$\frac{d}{dt} \left(\frac{\partial}{\partial x_i} f(t, x(t), x'(t)) \right) = \frac{\partial}{\partial x_i} f(t, x(t), x'(t)) \quad \text{a.e. } t \in I \quad i = 1, \dots, n.$$

Let us consider Manià example introduced at the beginning of section 3. The assumptions C_1 - C_3 of theorem 4.1 are satisfied only for $q < \frac{3}{2}$, since the solution $x_0(t) = t^{\frac{1}{3}}$ belongs to $W^{1,p}(I)$ for $p < \frac{3}{2}$. On the other hand it is easy to check, by direct inspection, that x_0 satisfies equations (EL) for any q . This simple example shows that conditions C_1 - C_3 are far from being optimal, hence it is worth to make an effort in order to enlarge the range of validity of equations (EL). The following theorem goes in this direction.

Theorem 4.2. *Let f belong to $C^1(A \times \mathbb{R}^n, \mathbb{R})$ and let x belong to $\Omega \cap W^{1,p}(I)$, $1 < p < \infty$. Assume that the graph of x lies in the interior of A , (i.e. there exist σ positive and a σ -neighbourhood Γ_σ of the graph of x contained in A), that x gives a strong local minimum for \mathcal{I} and that f satisfies the following conditions:*

$$E_1 : f(\cdot, x(\cdot), x'(\cdot)) \in L^p(I);$$

$$E_2 : \nabla_x f(\cdot, x(\cdot), x'(\cdot)) \in L^1(I);$$

$$E_3 : \text{there exist } m_1, M_1 \geq 0 \text{ such that, for any } (t, y) \in \Gamma_\sigma, \text{ and for any } v \in \mathbb{R}^n$$

$$|\nabla_{x'} f(t, y, v)| \leq m_1 + M_1 |v|^p$$

$$E_4 : \text{there exist } m_2, M_2 \geq 0, \gamma \geq 1 \text{ such that for any } (t, z) \in \Gamma_\sigma$$

$$|\nabla_x f(t, z, x'(t)) - \nabla_x f(t, x(t), x'(t))| \leq (m_2 + M_2 |x'(t)|^q) |z - x(t)|^\gamma$$

where $p < q < p(\gamma + 1) - \gamma$.

Then equations (EL) hold true.

Remark 4.1. Comparison between hypotheses C_1 - C_3 and E_1 - E_4 .

The proof of Theorem 4.1 is performed by taking the Gateaux derivative of the functional along directions determined by elements of $W^{1,\infty}(I)$. To do this one needs integrability of f , $\nabla_x f$, $\nabla_{x'} f$ along trajectories whose graph is contained in a neighbourhood of the graph of the solution: conditions C_1 - C_3 ensure such property since guarantee that near the solutions f , $\nabla_x f$ and $\nabla_{x'} f$ are bounded by an integrable power of the derivative. If we consider Manià's functional, we notice that the integrand $g = g(t, x, v)$ and its derivatives with respect to x and v are zero along the solution but, if $q \geq \frac{3}{2}$, they are not integrable along trajectories contained in a neighbourhood of its graph. Hypotheses E_1 and E_2 are intended to take into account integrands which behaves "well" along the solution x , disregarding the behaviour in a neighbourhood of the graph of x .

While E_3 is analogous to C_3 , we replace C_2 by E_2 and E_4 , where E_4 involves some continuity of $\nabla_x f$ and E_2 guarantees its integrability along the solution.

As far as it concerns Manià example it is easy to see that

E_1, E_2 are satisfied for any q ;

E_3, E_4 are satisfied for $q < 2$.

Proof of Theorem 4.2. Our aim is to show that, given any $\xi \in W_0^{1,\infty}(I)$, it is

$$\int_I [\langle \nabla_x f(t, x(t), x'(t)), \xi(t) \rangle + \langle \nabla_{x'} f(t, x(t), x'(t)), \xi'(t) \rangle] dt = 0.$$

If this is so, by integration by parts, and by a standard argument (see for example [C] p. 42), it follows that

$$-\int_a^t \nabla_x f(s, x(s), x'(s)) ds + \nabla_{x'} f(t, x(t), x'(t)) = \text{const.} \quad t \in [a, b]$$

and then, by differentiation, one obtains equations (EL).

In the following we set, for the sake of brevity,

$$G(t) = \langle \nabla_x f(t, x(t), x'(t)), \xi(t) \rangle + \langle \nabla_{x'} f(t, x(t), x'(t)), \xi'(t) \rangle.$$

Hypotheses E_2 and E_3 imply that $G \in L^1(I)$.

In the proof of Theorem 4.1 one consider variations around the solution x of the form

$$x' \rightarrow x' + \theta \xi', \quad \text{and} \quad x \rightarrow x + \theta \xi,$$

for a real θ belonging to a neighbourhood of the origin; as we have already remarked, this requires some bounds on the growth of f , $\nabla_x f$, $\nabla_{x'} f$ in a neighbourhood of the graph of x (see hypotheses C_1 - C_3 in Theorem 4.1). Since hypotheses E_1 - E_4 do not guarantee such properties, we perform a different kind of variations which involve, as in the proof of Theorem 3.1, truncation of the derivative of x . This choice weakens the requirements on f and on $\nabla_x f$, and force us to assume that x is strong local minimum.

1.) Take $\xi \in W_0^{1,\infty}(I)$ and $0 < \alpha < \gamma$ such that $q = p + \frac{\gamma-\alpha}{1+\alpha}(p-1)$. We consider, as in previous section, the family of subsets of I , $I_\rho = \{t \in I : |x'(t)| > \rho, \rho \geq 0\}$, and define $\delta : \mathbb{R} \setminus \{0\} \rightarrow \mathbb{R}^+$ by setting $\delta(\theta) = |\theta|^{\frac{1+\alpha}{1-p}}$. Take $R > 0$ such that $\mu(I_R^c) > 0$ and θ_0 such that $\delta(\theta_0) \geq R$. For any $\theta \in [-\theta_0, \theta_0]$ we define the function η_θ by setting:

$$\beta_\theta = \frac{1}{\mu(I_R^c)} \left[\int_{I_{\delta(\theta)}} x'(\tau) d\tau + \theta \int_{I_{\delta(\theta)}} \xi'(\tau) d\tau \right]$$

$$\eta'_0(t) = 0 \quad \text{for } \theta = 0$$

$$\eta'_\theta(t) = \begin{cases} -x'(t), & t \in I_{\delta(\theta)} \\ \theta \xi'(t) + \beta_\theta, & t \in I_R^c \\ \theta \xi'(t), & t \in I_R \setminus I_{\delta(\theta)} \end{cases} \quad \text{for } \theta \in [-\theta_0, \theta_0] \setminus \{0\}$$

and

$$\eta_\theta(t) = \int_a^t \eta'_\theta(\tau) d\tau \quad t \in I = [a, b].$$

For any $\theta \in [-\theta_0, \theta_0]$, η_θ is in $W^{1,p}(I)$ and, remarking that $\int_I \xi'(\tau) d\tau = 0$, we have

$$\eta_\theta(b) = - \int_{I_{\delta(\theta)}} x'(\tau) d\tau + \theta \int_{I_{\delta(\theta)}^c} \xi'(\tau) d\tau + \mu(I_R^c) \beta_\theta = 0 = \eta_\theta(a).$$

Hence $\eta_\theta \in W_0^{1,p}(I)$.

We now list some properties useful in the following, denoting by c_1, c_2, c_3 suitable positive constants depending only on $\theta_0, \|x\|_{W^{1,p}}, \|\xi\|_{W^{1,\infty}}, R$ and $\mu(I)$.

i) Using Tchebishev inequality:

$$\mu(I_{\delta(\theta)}) \leq \|x'\|_{L^p}^p \delta(\theta)^{-p} \leq \|x'\|_{L^p}^p |\theta|^{(1+\alpha)p'}. \quad (4.1)$$

ii) By Hölder inequality, (4.1) implies that, for any $h \in L^p(I)$:

$$\int_{I_{\delta(\theta)}} |h(\tau)| d\tau \leq \mu(I_{\delta(\theta)})^{\frac{1}{p'}} \left(\int_{I_{\delta(\theta)}} |h(\tau)|^p d\tau \right)^{\frac{1}{p}} \leq \|x'\|_{L^p}^{p-1} \|h\|_{L^p} |\theta|^{1+\alpha}. \quad (4.2)$$

iii) By (4.2)

$$|\beta_\theta| \leq \frac{1}{\mu(I_R^c)} \left(\|x'\|_{L^p}^p |\theta|^{1+\alpha} + \|\xi'\|_{L^p} \|x'\|_{L^p}^{p-1} |\theta|^{2+\alpha} \right) \leq c_1 |\theta|^{1+\alpha}. \quad (4.3)$$

iv) It is

$$\frac{\eta'_\theta(t)}{\theta} - \xi'(t) = \begin{cases} -\frac{1}{\theta} x'(t) - \xi'(t), & t \in I_{\delta(\theta)} \\ \frac{\beta_\theta}{\theta}, & t \in I_R^c \\ 0, & t \in I_R \setminus I_{\delta(\theta)}. \end{cases}$$

By (4.1) and (4.3) we have that

$$\lim_{\theta \rightarrow 0} \left| \frac{\eta'_\theta(t)}{\theta} - \xi'(t) \right| = 0 \quad \text{a.e. } t \in I. \quad (4.4)$$

v) It is, by (4.2) and (4.3),

$$\begin{aligned} \left| \frac{\eta_\theta(t)}{\theta} - \xi(t) \right| &\leq \int_a^t \left| \frac{\eta'_\theta(\tau)}{\theta} - \xi'(\tau) \right| d\tau \leq \\ &\frac{1}{\theta} \int_{I_{\delta(\theta)}} |x'(\tau)| d\tau + \int_{I_{\delta(\theta)}} |\xi'(\tau)| d\tau + \frac{\beta_\theta}{\theta} \mu(I_R^c) \leq c_2 |\theta|^\alpha. \end{aligned}$$

Hence

$$\lim_{\theta \rightarrow 0} \left\| \frac{\eta_\theta}{\theta} - \xi \right\|_{L^\infty} = 0 \quad (4.5)$$

and, in particular,

$$\|\eta_\theta\|_{L^\infty} \leq c_3 |\theta| \quad (4.6)$$

2.) Consider now σ_1 such that $0 < \sigma_1 \leq \sigma$ and for any $y \in W^{1,p}(I) \cap \Omega$ with graph contained in Γ_{σ_1} ($\subseteq \Gamma_\sigma$) it is $\mathcal{I}(x) \leq \mathcal{I}(y)$. By (4.6) there exists θ_1 , $0 < \theta_1 \leq \theta_0$, such that for any $\theta \in [-\theta_1, \theta_1]$ the graph of $x + \eta_\theta$ is contained in Γ_{σ_1} . Hence $x + \eta_\theta$ belongs to Ω and $\mathcal{I}(x + \eta_\theta) \geq \mathcal{I}(x)$ for any $\theta \in [-\theta_1, \theta_1]$, and the function $\varphi : [-\theta_1, \theta_1] \rightarrow \mathbb{R}$, defined by $\varphi(\theta) = \mathcal{I}(x + \eta_\theta)$ has a minimum in $\theta = 0$. Our aim is to show that such a function is differentiable in zero and that $\varphi'(0)$ coincide with $\int_I G(t)dt$. This would prove the theorem.

Let us write

$$\begin{aligned} \left| \frac{\varphi(\theta) - \varphi(0)}{\theta} - \int_I G(t)dt \right| &= \left| \frac{\mathcal{I}(x + \eta_\theta) - \mathcal{I}(x)}{\theta} - \int_I G(t)dt \right| \leq \\ &\int_{I_{\delta(\theta)}} \left| \frac{f(t, x(t) + \eta_\theta(t), 0) - f(t, x(t), x'(t))}{\theta} - G(t) \right| dt + \\ &\int_{I_R^c} \left| \frac{f(t, x(t) + \eta_\theta(t), x'(t) + \theta \xi'(t) + \beta_\theta) - f(t, x(t), x'(t))}{\theta} - G(t) \right| dt + \\ &\int_{I_R \setminus I_{\delta(\theta)}} \left| \frac{f(t, x(t) + \eta_\theta(t), x'(t) + \theta \xi'(t)) - f(t, x(t), x'(t))}{\theta} - G(t) \right| dt = \\ &\Lambda_1(\theta) + \Lambda_2(\theta) + \Lambda_3(\theta). \end{aligned}$$

We claim that $\lim_{\theta \rightarrow 0} \Lambda_i(\theta) = 0$ for $i = 1, 2, 3$.

3.) Estimate of $\Lambda_1(\theta)$.

$$\Lambda_1(\theta) \leq \frac{1}{\theta} \int_{I_{\delta(\theta)}} |f(t, x(t) + \eta_\theta(t), 0)| dt + \frac{1}{\theta} \int_{I_{\delta(\theta)}} |f(t, x(t), x'(t))| + \int_{I_{\delta(\theta)}} |G(t)| dt$$

Recalling (4.6), the set $\{(t, x(t) + \eta_\theta(t), 0), t \in I, \theta \in [-\theta_1, \theta_1]\}$ is contained in a fixed compact subset of $A \times \mathbb{R}^n$, and since f is continuous, there exists a positive constant M such that

$$|f(t, x(t) + \eta_\theta(t), 0)| \leq M \quad \text{for any } t \in I_{\delta(\theta)}.$$

Hence, recalling (4.1), (4.2) and E_1

$$\begin{aligned} \Lambda_1(\theta) &\leq \frac{1}{\theta} M \mu(I_{\delta(\theta)}) + \frac{1}{\theta} \|f(\cdot, x(\cdot), x'(\cdot))\|_{L^p} \|x'\|_{L^p}^{p-1} |\theta|^{1+\alpha} + \int_{I_{\delta(\theta)}} |G(t)| dt \leq \\ &M \|x'\|^p |\theta|^{(1+\alpha)p'-1} + \|f(\cdot, x(\cdot), x'(\cdot))\|_{L^p} \|x'\|^{p-1} |\theta|^\alpha + \int_{I_{\delta(\theta)}} |G(t)| dt. \end{aligned}$$

Since G is in $L^1(I)$ we have that $\lim_{\theta \rightarrow 0} \Lambda_1(\theta) = 0$.

4.) Estimate of $\Lambda_2(\theta)$.

$$\begin{aligned} \Lambda_2(\theta) \leq & \int_{I_R^c} \left| \frac{f(t, x(t) + \eta_\theta(t), x'(t) + \theta \xi'(t) + \beta_\theta) - f(t, x(t) + \eta_\theta(t), x'(t))}{\theta} - \right. \\ & \left. \langle \nabla_{x'} f(t, x(t) + \eta_\theta(t), x'(t) + \theta \xi'(t) + \beta_\theta), \xi'(t) \rangle \right| dt + \\ & \int_{I_R^c} \left| \frac{f(t, x(t) + \eta_\theta(t), x'(t)) - f(t, x(t), x'(t))}{\theta} - \right. \\ & \left. \langle \nabla_x f(t, x(t) + \eta_\theta(t), x'(t)), \xi(t) \rangle \right| dt. \end{aligned} \quad (4.7)$$

By mean value theorem there exist two functions, y_θ , z_θ , defined on I_R^c , such that $y_\theta(t)$ lies in the line segment joining $x'(t)$ and $x'(t) + \theta \xi'(t) + \beta_\theta$, for a.e. $t \in I_R^c$, $z_\theta(t)$ lies in the line segment joining $x(t)$ and $x(t) + \eta_\theta(t)$ for $t \in I_R^c$, and the right hand side of (4.7) is equal to

$$\begin{aligned} & \int_{I_R^c} \left| \langle \nabla_{x'} f(t, x(t) + \eta_\theta(t), y_\theta(t)), \xi'(t) + \frac{\beta_\theta}{\theta} \rangle - \langle \nabla_{x'} f(t, x(t), x'(t)), \xi'(t) \rangle \right| dt + \\ & \int_{I_R^c} \left| \langle \nabla_x f(t, z_\theta(t), x'(t)), \frac{\eta_\theta(t)}{\theta} \rangle - \langle \nabla_x f(t, x(t), x'(t)), \xi(t) \rangle \right| dt. \end{aligned} \quad (4.8)$$

We remark that both integrands in (4.8) equal a.e. measurable functions and then are measurable. On I_R^c $|x'|$ is bounded by R , hence, recalling (4.3) and (4.6) the sets $\{(t, x(t) + \eta_\theta(t), y_\theta(t)), t \in I, \theta \in [-\theta_1, \theta_1]\}$ and $\{(t, z_\theta(t), x'(t)), t \in I, \theta \in [-\theta_1, \theta_1]\}$ are contained in a fixed compact subset of $A \times \mathbb{R}^n$ and, since f is of class $C^1(A \times \mathbb{R}^n)$, there exists a positive constant L such that

$$|\nabla_x f(t, z_\theta(t), x'(t))| \leq L, \quad |\nabla_{x'} f(t, x(t) + \eta_\theta(t), y_\theta(t))| \leq L \quad \text{for } t \in I_R^c.$$

These inequalities and hypotheses E_2 , E_3 imply that both integrands in (4.8) are uniformly bounded by an integrable function. Moreover, (4.3) and (4.6) imply that they tend to zero a.e. on I_R^c and, by dominated convergence, we have $\lim_{\theta \rightarrow 0} \Lambda_2(\theta) = 0$.

5.) Estimate of $\Lambda_3(\theta)$.

$$\begin{aligned} \Lambda_3(\theta) &\leq \int_{I_{\delta(\theta)}^c} \left| \frac{f(t, x(t) + \eta_\theta(t), x'(t) + \theta \xi'(t)) - f(t, x(t) + \eta_\theta(t), x'(t))}{\theta} \right. \\ &\quad \left. \langle \nabla_{x'} f(t, x(t), x'(t)), \xi'(t) \rangle \right| dt + \\ &\int_{I_{\delta(\theta)}^c} \left| \frac{f(t, x(t) + \eta_\theta(t), x'(t)) - f(t, x(t), x'(t))}{\theta} \right. \\ &\quad \left. \langle \nabla_x f(t, x(t), x'(t)), \xi(t) \rangle \right| dt = \\ &\Lambda_{3.1}(\theta) + \Lambda_{3.2}(\theta). \end{aligned}$$

As in point 4., we can find $y_\theta(t)$ belonging, for a.e. $t \in I_{\delta(\theta)}^c$, to the line segment joining $x'(t)$ and $x'(t) + \theta \xi'(t)$ such that

$$\Lambda_{3.1}(\theta) = \int_{I_{\delta(\theta)}^c} |\langle \nabla_{x'} f(t, x(t) + \eta_\theta(t), y_\theta(t)), \xi'(t) \rangle - \langle \nabla_{x'} f(t, x(t), x'(t)), \xi'(t) \rangle| dt.$$

By E_3 , we have

$$\begin{aligned} &|\langle \nabla_{x'} f(t, x(t) + \eta_\theta(t), y_\theta(t)), \xi'(t) \rangle| \leq \\ &(m_1 + M_1 |x'(t) + \theta \xi'(t)|^p) |\xi'(t)| \leq m_1' + M_1' |x'(t)|^p \end{aligned} \quad (4.9)$$

where m_1' , M_1' are positive constants depending on $\|\xi'\|_{L^\infty}$. Since x' belong to $L^p(I)$, (4.5), (4.9) and dominated convergence imply that $\lim_{\theta \rightarrow 0} \Lambda_{3.1}(\theta) = 0$.

Let z_θ , defined on $I_{\delta(\theta)}^c$ be such that $z_\theta(t)$ lies in the line segment joining $x(t)$ and $x(t) + \eta_\theta(t)$ for any $t \in I_{\delta(\theta)}^c$. It is

$$\begin{aligned} \Lambda_{3.2}(\theta) &\leq \int_{I_{\delta(\theta)}^c} \left| \langle \nabla_x f(t, z_\theta(t), x'(t)), \frac{\eta_\theta(t)}{\theta} \rangle - \langle \nabla_x f(t, x(t), x'(t)), \frac{\eta_\theta(t)}{\theta} \rangle \right| dt + \\ &\int_{I_{\delta(\theta)}^c} \left| \langle \nabla_x f(t, x(t), x'(t)), \frac{\eta_\theta(t)}{\theta} - \xi(t) \rangle \right| dt = \\ &\Lambda'_{3.2}(\theta) + \Lambda''_{3.2}(\theta). \end{aligned}$$

Recalling E_2 and (4.5), $\lim_{\theta \rightarrow 0} \Lambda''_{3.2}(\theta) = 0$.

Using E_4 we have

$$\begin{aligned} \Lambda'_{3.2}(\theta) &= \int_{I_{\delta(\theta)}^c} \left| \langle \nabla_x f(t, z_\theta(t), x'(t)) - \nabla_x f(t, x(t), x'(t)), \frac{\eta_\theta(t)}{\theta} \rangle \right| dt \leq \\ &\int_{I_{\delta(\theta)}^c} \left| \frac{\eta_\theta(t)}{\theta} \right| (m_2 + M_2 |x'(t)|^q) |z_\theta(t) - x(t)|^\gamma dt. \end{aligned}$$

Now, by (4.6), $\left| \frac{\eta_\theta(t)}{\theta} \right| \leq c_3$ and $|z_\theta(t) - x(t)| \leq |\eta_\theta(t)| \leq c_3|\theta|$ for any $t \in I_{\delta(\theta)}^c$. Hence

$$\Lambda'_{3.2}(\theta) \leq c_3^{1+\gamma} m_2 \mu(I) |\theta|^\gamma + c_3^{1+\gamma} M_2 |\theta|^\gamma \int_{I_{\delta(\theta)}^c} |x'(t)|^q dt. \quad (4.10)$$

Recalling formula (2.2) and Tchebishev inequality, we have

$$\begin{aligned} \int_{I_{\delta(\theta)}^c} |x'(t)|^q dt &= -(\delta(\theta))^q \omega(|x'|, \delta(\theta)) + q \int_0^{\delta(\theta)} \sigma^{q-1} \omega(|x'|, \sigma) d\sigma \leq \\ & q \|x'\|_{L^p}^p \int_0^{\delta(\theta)} \sigma^{q-p-1} d\sigma = q \|x'\|_{L^p}^p |\theta|^{(q-p)\frac{1+\alpha}{1-p}}. \end{aligned} \quad (4.11)$$

Inserting (4.11) in (4.10) and denoting by c' and c'' suitable positive constants, we have:

$$\Lambda'_{3.2}(\theta) \leq c' |\theta|^\gamma + c'' |\theta|^{\gamma+(q-p)\frac{1+\alpha}{1-p}} = c' |\theta|^\gamma + c'' |\theta|^\alpha.$$

Hence $\lim_{\theta \rightarrow 0} \Lambda'_{3.2}(\theta) = 0$ and, finally, $\lim_{\theta \rightarrow 0} \Lambda_3(\theta) = 0$.

Collecting the results of points 3.), 4.) and 5.) we have the proof. \square

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