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BOSONIC STRINGS IN BACKGROUND FIELDS

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CHAPTER I

PHYSICS AND STRINGS

1 - INTRODUCTION.

For the last five years, a large part of the theoretical physics community has been focusing its work in a highly speculative approach to the fundamental structure of matter, the theory of relativistic strings. Proponents of this theory have claimed that it provides the basic laws unifying all known interactions, ***gravity included***, and that it promises the solution to some of the deepest remaining questions about Nature, including for example, the origin of the quark and lepton generations [1].

Such a fundamental description of Nature seems even more wonderful because it is built of elementary entities of a simple and concrete structure. These basic entities are, of course, the elementary strings, and all the particle spectrum can be visualized as different "excitation modes" of this fundamental object.

Before beggining the explication, it is worth reviewing the main properties of the string theories and, especially, that of supersymmetric versions, which shows the most promise of making contact with the fenomena of elementary particle physics. This version was originally formulated in 1970 by Neveu and Schwarz, Ramond and Thorn.

However is in the early of 1980's when Green and Schwarz clarified many of their properties and pressed its interpretation as a unifying theory for all interactions.

The main properties of this superstring theory which bolster its interpretation as a fundamental theory of Nature are the following :

i) The theory requires, as it was said before, that all particles (quarks, leptons, gauge bosons, gravitons, and their supersymmetric partners) are built of the same fundamental entities, the elementary strings. In this sense, string theories are the most elegant of all models of elementary particle substructure.

ii) The theory require that space-time (SP) be fundamentally supersymmetric. It also requires the existence of 10 SP dimensions. This would be an excesive number if all of these dimensions were extended to the size of the four dimensions that are part of our every day experience. However, the extra 6 dimensions may play a more subtle role, being probably "compactifyied" to a size of the characteristic length of an elementary string, of the order of the Planck length, 10^{-33} cm or (10^{19} Gev) .

iii) The theory naturally contains as a part of its structure the gauge invariance of Yang-Mills theory and gravity. In fact, this invariance is realized as a small part of an enormous group of generalized gauge symmetries.

iv) Although the theory contains within it a quantum theory of gravity, it is apparently free of ultraviolet divergences. The finiteness of the theory has been shown explicitily to one-loop order, and it's plausible this property holds at all order.

v) The theory restrics the possible choices for its Yang-Mills group to only two candidates : O(32) and $E_8 \times E_8$.

I would like to conclude this section by citing the major problems which must still be solved in order to bring this theory from the level of especulation to a point where it can make concrete predictions for experiment. The most pressing problems are those which concern the conversion of the 10-d SP of string theory into a form closer to experimental reality in which 6 of the 10 dimensions are curled up to a very small size, the so-called *compactification problem* . The geometry of this compactification of dimensions *determines all of the detailed properties of the system of elementary particles which would be visible at energies accesible to experiment* : the number of quarks and lepton generations, the gauge group which results from breaking the grand unification symmetry, the values of the strong and weak coupling constant, and the existence and number of the supersymmetric partners. The most basic aspects of how the geometry of the compact 6 dimensions determines these parameters have been clarified by Candelas, Horowitz, Strominger and Witten, among others. However, many issues, especially the mechanism of supersymmetry breaking and the relation of the weak interaction scale to the fundamental string length scale, remain obscure. In addition, we still have no idea *how* Nature chooses a particular geometry for the compact 6 dimensions from among a wealth of possibilities. The subject developed in this thesis is deeply related with the solution of this puzzle.

There are another problems of "technical" (mathematical) order in string theory . We still don't have a complete set of rules for computing the perturbation theory in string interactions ; also, we have almost *no idea* of how to discuss string dynamics beyond perturbation theory.

This last formal problem is a particularly important one, because it's known that many of the aspects of compactification that seem to us the most misterious (which compact space is chosen, for example), are simply not determinated at the level of the first perturbative loop corrections; quite plausible, these questions can only be settled by looking beyond perturbation theory.
In the following sections, we will restric ourselves to the bosonic string theory; the treatment of the supersymmetric case is similar (although more complicated, of course).
This will give the basic elements to introduce to Chapters II and III.

2 - THE CLASSICAL STRING.

We will work in this chapter in Minkowski space with metric signatures like (-1,1,...,1).
A string is, from a mathematical point of view, a bounded set of points continuously joined. In Fig.1 are showed the two possible configurations corresponding to "open" and "closed" strings, isomorphic to a segment of line and circle respectively.

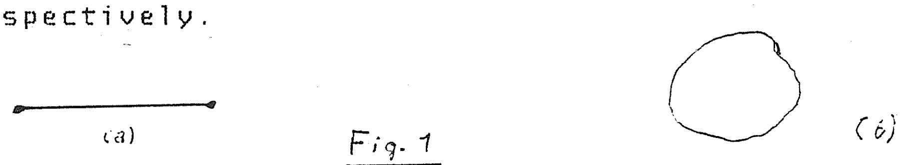
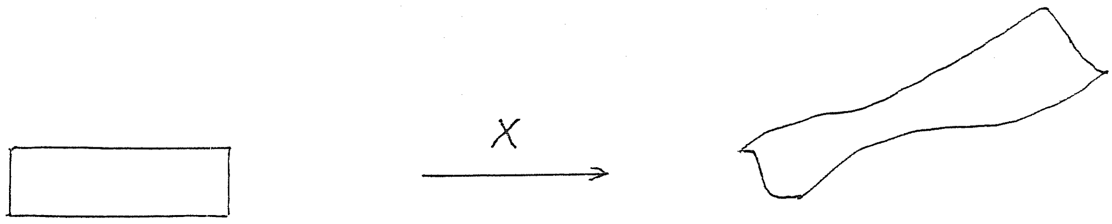


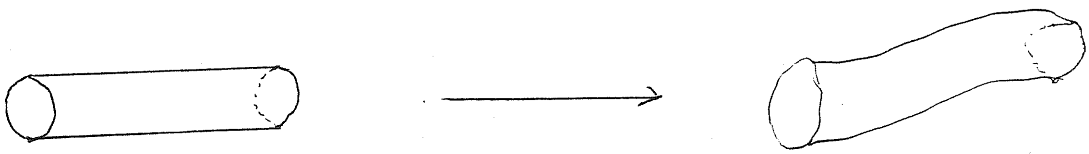
Fig. 1

We may consider an object of this characteristics moving in D-dimensional space-time (SP) ; in that case we describe it using some coordinates $X^{\mu}(\tau,\sigma)$, $\mu = 0,1,...,D-1$, with σ labelling the points of the string , and τ like a "proper time" (if $X^0(\sigma,\tau) = t$ is the time as measured by the observator, then $\tau = \tau(\sigma,t)$ defines the proper time corresponding to the point σ at time t).The study of the dynamics of this one-dimensional extended object define the bosonic string theory.

When moving, the string sweeps out a surface in SP, the "world-sheet", as illustrated in Fig.2.



(a) Open string



(b) Closed string

Fig. 2

In order to study the dynamics of the string, we must stress that ***the way in which we parametrize the world-seet cannot incide in the physics***, namely, the theory must be ***invariant under reparametrizations of the world-sheet***.

Also we ask that Poincare invariance holds.

There are two "natural" ways of guessing the action for the free string. The first way is to study the dynamics of the string considered as a set of relativistic point particles.

The second way rests in the knowledge of the action for a relativistic free point particle. This is proportional to the length of the trajectory of the particle in SP. The obvious generalization is proposing that the action for the string to be proportional to the area of the world sheet (WS) swept out by it in its trajectory. Remarkably, these two formulations are equivalent, as it was proved long time ago by Nambu and Goto. So, we propose the action :

$$S[X] = T \int_{WS} d^2\sigma \sqrt{\det M_{\alpha\beta}(X(\sigma))} \qquad (1)$$

$$M_{\alpha\beta}(X(\sigma)) = \partial_\alpha X^\mu(\sigma)\, \partial_\beta X_\mu(\sigma)$$

However, the awkward form of (1) led to Brink, Di Vecchia and Howe, and Polyakov to replace it by :

$$S[X,\gamma] = -\frac{T}{2} \int d^2\sigma \sqrt{\gamma(\sigma)}\, \gamma^{\alpha\beta}(\sigma)\, \partial_\alpha X^\mu(\sigma)\, \partial_\beta X_\mu(\sigma) \qquad (2)$$

where $T = (2\pi\alpha')^{-1}$ is the string tension, introduced in these equations by dimensional reasons, and $\gamma_{\alpha\beta}$ is a 2-d metric on the WS ($\gamma = \det(\gamma_{\alpha\beta})$).

Both (1) and (2) are manifestly invariant under reparametrization of the WS :

$$\sigma'^\alpha = \sigma'^\alpha(\sigma) \qquad (3-a)$$

$$\gamma'_{\alpha\beta}(\sigma') = \frac{\partial\sigma^\gamma}{\partial\sigma'^\alpha} \frac{\partial\sigma^\delta}{\partial\sigma'^\beta} \gamma_{\gamma\delta}(\sigma) \qquad (3-b)$$

$$X'^\mu(\sigma') = X^\mu(\sigma) \qquad (3-c)$$

$$S[X,\gamma] = S[X',\gamma'] \qquad (3-d)$$

In addition, (2) presents the ***Weyl symmetry*** :

$$\gamma_{\alpha\beta}(\sigma) \longrightarrow e^{2\varphi(\sigma)}\,\gamma_{\alpha\beta}(\sigma) \qquad (4)$$

for any $\varphi(\sigma)$. The possible breakdown of this symmetry at quantum level plays an essential role in string theory, as we shall see.

Also, of course, we have the "internal" (global) Poincare invariance :

$$X'^{\mu} = \Lambda^{\mu}{}_{\nu}\,X^{\nu} + b^{\mu} \qquad (5)$$

Classical covariant gauge fixing and the equations of motion.

The 2-d energy-momentum tensor is defined by :

$$T_{\alpha\beta}(\sigma) = -\frac{2}{T}\,\frac{1}{\sqrt{\gamma(\sigma)}}\,\frac{\delta S}{\delta\gamma^{\alpha\beta}(\sigma)} = \qquad (6)$$

$$= \partial_\alpha X^{\mu}(\sigma)\,\partial_\beta X_{\mu}(\sigma) - \frac{1}{2}\,\gamma_{\alpha\beta}(\sigma)\,\gamma^{\gamma\delta}(\sigma)\,\partial_\gamma X^{\mu}(\sigma)\,\partial_\delta X_{\mu}(\sigma)$$

which is traceless : $\gamma^{\alpha\beta}(\sigma)\,T_{\alpha\beta}(\sigma) = 0$ as a consecuence of (4). The classical field equations obtained by varying (2) are :

$$T_{\alpha\beta}(\sigma) = 0 \qquad (7-a)$$

$$\Delta_\gamma X^\mu_{(\sigma)} = 0 \qquad (7-b)$$

$$\Delta_\gamma = -\frac{1}{\sqrt{\gamma_{(\sigma)}}}\,\partial_\alpha \sqrt{\gamma_{(\sigma)}}\,\gamma^{\alpha\beta}_{(\sigma)}\,\partial_\beta$$ Laplace- Beltrami operator

It's easy to show that inserting (7-a) in (7-b) we obtain :

$$\partial_\alpha \sqrt{\det M_{\alpha\beta}(X_{(\sigma)})}\; M^{\alpha\beta}_{(X(\sigma))}\,\partial_\beta X^\mu_{(\sigma)} = 0 \qquad (8)$$

which is the same equation of motion derived from (1). From this result (1) and (2) are classically equivalents ; the metric $\gamma_{\alpha\beta}$ in (2) acts like a "Lagrange multiplier" that enforces the "constraint condition" (7-a).
The subsequent analysis of string dynamics is expedited by making a convenient choice gauge, ***neccesary*** due to the presence of three ***local*** invariances : two reparametrizations and one Weyl scaling.
By using the first one, we may choose a parametrization in which the so-called ***conformal gauge*** holds :

$$\gamma_{\alpha\beta\,(\sigma)} = e^{2\varphi(\sigma)}\,\eta_{\alpha\beta} \qquad (9)$$

In this gauge, the equations of motion (7) read :

$$T_{\alpha\beta\,(\sigma)} = \partial_\alpha X^\mu \partial_\beta X_\mu - \frac{1}{2}\,\eta_{\alpha\beta}\,\partial^\gamma X^\mu \partial_\gamma X_\mu \qquad (10-a)$$

$$\Box X^\mu_{(\sigma)} = 0 \quad , \quad \Box = \partial^\alpha \partial_\alpha \qquad (10-b)$$

At this point, we recall that, in order to get a stationary action (2), eqs. (10) must be supply by boundary conditions coming from the variation of (2), which in conformal gauge (4) reads :

$$-T \int d\tau \left[X'^{\mu}_{(\sigma)} \delta X_{\mu} \Big|_{\sigma=\pi} - X'^{\mu}_{(\sigma)} \delta X_{\mu} \Big|_{\sigma=0} \right] = 0 \qquad (11)$$

where we parametrized the string by :

$$\sigma \in [0,\pi] \quad , \quad \tau \in \mathbb{R}$$

and introduced the notation : $X'^{\mu} = \frac{\partial X^{\mu}}{\partial \sigma}$, $\dot{X}^{\mu} = \frac{\partial X^{\mu}}{\partial \tau}$.

The vanishing of this surface term implies the boundary conditions :

Open string : $$X'^{\mu}(\tau,0) = X'^{\mu}(\tau,\pi) = 0 \qquad (12\text{-a})$$

Closed string : $$X^{\mu}(\tau,0) = X^{\mu}(\tau,\pi) \qquad (12\text{-b})$$

Let's define the "ligth-cone" coordinates :

$$\sigma^{\pm} = \tau \pm \sigma \qquad (13)$$

In terms of them, eqs (10) read :

$$\partial_{+} \partial_{-} X^{\mu} = 0 \qquad (14\text{-a})$$

$$T_{\pm\pm} = \partial_{\pm} X^{\mu} \partial_{\pm} X_{\mu} = 0 \qquad (14\text{-b})$$

($T_{\pm\mp} = 0$ as consecuence of (4)).

The general solution of (14-a) with boundary conditions given by (12) is :

$$X^{\mu}(\tau,\sigma) = x^{\mu} + p^{\mu}\tau + i \sum_{n\neq 0} \frac{1}{n} \alpha_n^{\mu} e^{-in\tau} \cos n\sigma \qquad (15\text{-a})$$

$$X^{\mu}_{(\tau,\sigma)} = x^{\mu} + p^{\mu}\tau + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\left(\alpha^{\mu}_{n}\, e^{-2in\sigma^{-}} + \tilde{\alpha}^{\mu}_{n}\, e^{-2in\sigma^{+}}\right) \qquad (15-b)$$

respectivaly for open and closed strings (note the doubling of Fourier modes in the last case).

From the action (2) in conformal gauge (9) follow the ***Poisson brackets*** (taking $2\alpha' = 1$) :

$$[X^{\mu}(\tau,\sigma)\,,\, X^{\nu}(\tau,\sigma')]_{PB} = [\dot{X}^{\mu}_{(\tau,\sigma)}, \dot{X}^{\nu}_{(\tau,\sigma')}]_{PB} = 0 \qquad (16-a)$$

$$[\dot{X}^{\mu}_{(\tau,\sigma)}, X^{\nu}(\tau,\sigma')]_{PB} = \pi\,\eta^{\mu\nu}\,\delta(\sigma-\sigma') \qquad (16-b)$$

from where follows for the Fourier and zero modes :

$$[\alpha^{\mu}_{n}, \alpha^{\nu}_{m}]_{PB} = [\tilde{\alpha}^{\mu}_{n}, \tilde{\alpha}^{\nu}_{m}]_{PB} = i\,m\,\delta_{m+n}\,\eta^{\mu\nu} \qquad (17-a)$$

$$[\alpha^{\mu}_{n}, \tilde{\alpha}^{\nu}_{m}]_{PB} = 0 \qquad (17-b)$$

$$[p^{\mu}, x^{\nu}]_{PB} = \eta^{\mu\nu} \qquad (17-c)$$

However, the solutions of (14-a) must be constrained by (14-b). It's convenient to define Fourier components for $T_{\pm\pm}$. In closed strings, periodicity condition assures the well-definiteness for T_{++} and T_{--} separetely :

$$L_m = \frac{1}{2\pi}\int_0^{\pi} e^{-2im\sigma}\, T_{--}\, d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n \qquad (18-a)$$

$$\tilde{L}_m = \frac{1}{2\pi}\int_0^{\pi} e^{-2im\sigma}\, T_{--}\, d\sigma = \frac{1}{2}\sum_{n=-\infty}^{\infty}\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n \qquad (18-b)$$

where we introduced the convention $\alpha^{\mu}_{0} = p^{\mu} / 2$.

In open strings we define an extended periodic function from T_{++} and T_{--} ($\alpha_0^\mu = p^\mu$ in this case) :

$$L_m = \frac{1}{\pi}\int_0^\pi d\sigma \left(e^{im\sigma} T_{++} + e^{-im\sigma} T_{--} \right) = \frac{1}{2}\sum_{n=-\infty}^{\infty} \alpha_{m-n}\cdot\alpha_n \qquad (18-c)$$

These are the Virasoro generators, which satisfy the algebra :

$$[L_m , L_n]_{PB} = i\,(m-n)\, L_{m+n} \qquad (19)$$

At quantum level it will be modified by a "central charge".
Summarizing, solutions of (14) correspond to (15) with ;

$$L_m = \tilde{L}_m = 0 \quad , \qquad m \in \mathbb{Z}$$

Finally, we can use Poincare invariance (5) to get the canonical Noether currents and conserved Poincare generators :

$$P^\mu = \frac{1}{\pi}\int_0^\pi d\sigma\, \dot{X}^\mu(\sigma) \equiv p^\mu \qquad (20-a)$$

$$J^{\mu\nu} = \frac{1}{\pi}\int_0^\pi d\sigma \left(X^\mu \dot{X}^\nu - X^\nu \dot{X}^\mu \right) = l^{\mu\nu} + E^{\mu\nu} \qquad (20-b)$$

$$l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu \qquad (20-c)$$

$$E^{\mu\nu} = -i\sum_{n=1}^{\infty} \frac{1}{n}\left(\alpha_{-n}^\mu \alpha_n^\nu - \alpha_{-n}^\nu \alpha_n^\mu \right) \qquad (20-d)$$

with analogous contribution $\tilde{E}^{\mu\nu}$ from $\tilde{\alpha}_n^\mu$ -oscillators.

3 - THE QUANTIZED STRING.

A - Old covariant quantization.

As it's well-known, to quantize the theory in operatorial formalism we must consider the degrees of freedom $X^{\mu}(\sigma)$ as operators in a Hilbert space. A standard method of passing from classical to quantum physics is to replace the Poisson brackets by conmutators, *via* the substitution :

$$[\ldots,\ldots]_{PB} \longrightarrow i\,[\ldots,\ldots]$$

However, we must be careful in this step, because there may be problems in interpreting ***unambiguously*** operators which contain products of non-conmuting operators (clasically, numbers conmute among them !).

With the coordinates this problem isn't present, so it follows from (16) and (17) :

$$[\hat{X}^{\mu}(\tau,\sigma),\ \dot{\hat{X}}^{\nu}(\tau,\sigma')] = i\pi\,\eta^{\mu\nu}\,\delta(\sigma-\sigma') \qquad (21\text{-}a)$$

$$[\hat{\alpha}_n^{\mu},\ \hat{\alpha}_m^{\nu}] = n\,\delta_{n+m}\,\eta^{\mu\nu} \qquad (21\text{-}b)$$

$$[\hat{x}^{\mu},\ \hat{p}^{\nu}] = i\,\eta^{\mu\nu}\ ,\ etc. \qquad (21\text{-}c)$$

The α_m^{μ}'s are therefore naturally interpreted as harmonic oscillators raising and lowering operators for negative and positive n . Indeed, if we define for n > 0 (hermiticity of $\hat{X}^{\mu}(\sigma)$ implies : $\hat{a}_m^{\mu\,+} = \hat{a}_{-n}^{\mu}$) :

$$\hat{a}_n^{\mu} = \hat{\alpha}_n^{\mu}/\sqrt{n}\ ,\quad n>0$$

$$\hat{a}_n^{\mu\,+} = \hat{\alpha}_{-n}^{\mu}/\sqrt{n} = \hat{\alpha}_n^{\mu\,+}/\sqrt{n} \qquad (22\text{-}a)$$

then the standard conmutation relations hold :

$$[\, \hat{a}_n^{\mu} , \hat{a}_m^{\nu} \,] = \eta^{\mu\nu} \delta_{n,m} \qquad (22\text{-}b)$$

The representation of the algebra (21) is well-known. We start with the ground state $|0;p\rangle$ which carries momentum p^{μ} and satisfies :

$$\hat{a}_n^{\mu} \,|0;p\rangle = \hat{\alpha}_n^{\mu} \,|0;p\rangle = 0 \;, \quad n > 0$$

$$\langle 0;p \,|\, 0;p'\rangle = C(p)\, \delta^D (p-p')$$

where C(p) is some (positive) normalization constant. Then we construct the orthonormal basis of the Hilbert space using the raising operators $\hat{a}_n^{\mu\,+}$:

$$|\,\{\epsilon_n^{\mu}\};p\rangle = \prod_{n=1}^{\infty} \frac{(\hat{a}_n^{\mu\,+})^{\epsilon_n^{\mu}}}{\sqrt{\epsilon_n^{\mu}!}} \;|0;p\rangle \qquad (23\text{-}a)$$

$$\langle \{\epsilon_n^{\mu}\};p \,|\, \{\epsilon_n^{\mu}\};p'\rangle = \delta^D (p-p')\, C(p)\, \delta_{\{\epsilon_n^{\mu}\},\{\epsilon_n'^{\mu}\}} \qquad (23\text{-}b)$$

In what follows, we will drop out the p-dependence.
At this point, we may observe that the Fock space is not positive definite due to the presence of the time-like operators $\hat{a}_m^{0}$. For example, the state $\hat{a}_n^{0\,+}|0\rangle$ has norm : $\langle 0|\, \hat{a}_n^{0} \; \hat{a}_n^{0\,+}|0\rangle = -1$; states of negative norm are called "ghosts", and are not desirable in a sensible causal theory [2]. However, we must not forget about the constraint conditions (14-b). At classical level, we saw they are equivalent to put their Fourier modes Lm ($\tilde{L}$m) equal to zero. At quantum level, we adopt the procedure "a

la Gupta-Bleuler" : we ask the positive frequency components of $T_{\pm\pm}$ annihilatte a "physical state". This requeriment is expressed by the Virasoro constraints :

$$(\hat{L}_0 - a)\ |phys> = 0 \qquad (24\text{-}a)$$

$$\hat{L}_m\ |phys> = 0\ ,\ m > 0 \qquad (24\text{-}b)$$

with analogous formulae for the $\tilde{L}m$'s.

So, the physical space will be the subspace of the complete Hilbert space (expanded by the states (23)) whose states obeys the conditions (24).

We have introduced an undetermined constant <u>a</u> by the following reason : all the Virasoro generators (18) can be interpreted unambiguously as quantum operators, ***except*** Lo, because $\hat{\alpha}^\mu_{-n}$ and $\hat{\alpha}^\mu_n$ don't conmute. Then, we ***define*** :

$$\hat{L}_0 = :\frac{1}{2}\sum_{n=-\infty}^{\infty} \hat{\alpha}_{-n}\cdot\hat{\alpha}_n : = \frac{1}{2}\hat{\alpha}_0^{\,2} + \hat{N} \qquad (25\text{-}a)$$

$$\hat{N} = \sum_{n=1}^{\infty} \hat{\alpha}_{-n}\cdot\hat{\alpha}_n = \sum_{n=1}^{\infty} n\ \hat{a}^{+}_n\cdot\hat{a}_n \qquad (25\text{-}b)$$

and introduce a "normal-ordering" constant <u>a</u> in the constraint equation (24-a). Due to this normal-ordering ambiguity, a careful evaluation of the Virasoro algebra yields to :

$$[\hat{L}_n, \hat{L}_m] = (n-m)\ \hat{L}_{n+m} + \frac{D}{12}(n^3 - n)\ \delta_{n+m} \qquad (26)$$

where a central charge proportional to D appears (in the modern covariant quantization including Fadeev-Popov ghost, the algebra of the complete Virasoro generators $L_m + L_m^{ghost}$ results the same, with D replaced by (D-26) ; only in the "critical dimension" : D = 26, we have no anomaly in the algebra).

Let's return to eq. (24-a). It has a very special status : it corresponds to the mass-shell condition because, being $M^2 = -p^2$, we get using (25) :

open string ($\alpha_0^\mu = p^\mu$) :

$$M^2 = 2 (N - a) \tag{27}$$

closed string ($\alpha_0^\mu = p^\mu/2$) :

$$M^2 = 4 (N + \tilde{N} - 2a) \tag{28}$$

$$N = \tilde{N}$$

Now, it's our hope that the constraints (24-b) define a physical Hilbert space free of ghost. We give without demostration the following result, known as "No-ghost theorem" : the constraints (24) define a physical space free of negative norm states if :

i) D = 26 , a = 1 , or

ii) D < 26 , a $\leq$ 1

So, at "tree level" we can consistently work in any of these cases. However, at one-loop level Lovelace proved that unitarity holds only in the first case. Furthermore, only for a = 1 we get massless particles in the spectrum (see below), which are the "salt and peper" of string theory. So, we will consider this case in the following,

Analysis of the spectra.

First, we note that conditions (24-b) hold if and only if they holds for $n = 1$ and $n = 2$, as consecuence of the algebra (26). We don't analyze higher massive levels here.

Open string : using (27) we have :

$N = 0 \qquad M^2 = -2 = -1/\alpha'$

Only the scalar ground state $|0;p\rangle$, a tachyon.

$N = 1 \qquad M^2 = 0$, massless level.

It corresponds to the states : $\xi(p)\cdot\alpha_{-1}^{\dagger}|0;p\rangle$ with $\xi^{\mu}(p)$ a polarization vector. The L_1 -constraint implies : $\xi(p)\cdot p = 0$, leaving (D-1) independent states. The norm of this states is $\xi(p)\cdot\xi(p)$; so, the longitudinal (proportional to p^{μ}) polarization corresponds to a physical state of zero norm, a so-called "spurious state". The abundant presence of these spurious states is a characteristic of the string theory in the critical parameters, and may indicate a huge underlying gauge symmetry, yet not well understood. Of course, the state analyzed before is the massless gauge boson of the open string theory.

Closed string : with (28) we have :

$N = \tilde{N} = 0 \qquad M^2 = -8 = -4\ 1/\alpha'$

A tachyonic state as before.

$N = \tilde{N} = 1$ $\qquad M^2 = 0$, massless level.

This is the more interesting. The states of this level are written, with $\xi_{\mu\nu}(p)$ a polarization tensor :

$$\xi_{\mu\nu}(p)\ \hat{\alpha}_{-1}^{\mu\dagger}\ \hat{\tilde{\alpha}}_{-1}^{\nu\dagger}\ |0;p\rangle$$

The L_1- constraint gives the transversality conditions :

$$\xi_{\mu\nu}(p)\ p^{\nu} = \xi_{\mu\nu}(p)\ p^{\mu} = 0$$

From the Lorentz point of view, these states descompose in the symmetric traceless, antisymmetric and trace parts, the famous spin-2 "graviton", "antisymmetric tensor" and scalar "dilaton" respectively.

B - The Feynmann-Polyakov quantization.

This quantization by path integral was introduced by Polyakov [3] few years ago, and it has became the most popular because of its geometrical appeal.

It has been clear that just as the amplitudes of free particles are defined by :

$$G(x,x') = \sum_{\{P(x,x')\}} e^{-m L[P(x,x')]} \tag{29}$$

where $P(x,x')$ is a path connecting points x and x', and $L[P(x,x')]$ is the length of the path (the action), one should define in string theory :

$$G(C,C') = \sum_{\{S(C,C')\}} e^{-T A[S(C,C')]} \tag{30}$$

where C and C' are the initial and final configurations of the string, S(C,C') is a trajectory (surface) connecting

C and C', and A[S(C,C')] is the area of S(C,C') (the Nambu-Goto action (1)). The urgent problem to face in this approach is to define in (30) the sum over "random surfaces" S(C,C').

We will restrict here our study to the closed bosonic string at "tree level", that is, considering the WS a sphere (which corresponds to a virtual string that appears in some place, propagates, and disappears).

First, we consider the action (as we made in Section 2) :

$$S[X,\gamma] = \frac{1}{2}\int_{S^2} d^2\sigma \sqrt{\gamma(\sigma)}\, X^\mu \Delta_\gamma X_\mu + \mu_0^2 \int_{S^2} d^2\sigma \sqrt{\gamma(\sigma)} \qquad (31)$$

where the "cosmological constant" μ_0^2 term is required by renormalization. The tree level vacuum amplitude will be given by :

$$Z = \int_{S^2} [D\gamma]\,[DX]\; e^{-S[X,\gamma]} \qquad (32)$$

The following step is to use the reparametrization invariance of (32) (integration measure included !) to fix the conformal gauge (9) (we make a stereographic projection of the sphere onto the complex plane) :

$$\gamma_{\alpha\beta} = e^{2\varphi}\,\delta_{\alpha\beta}$$

Under an infinitesimal diffeomorphism :

$$\sigma^\alpha \longrightarrow \sigma^\alpha + \varepsilon^\alpha(\sigma)$$

the metric change according its Lie derivative :

$$\gamma_{\alpha\beta} \longrightarrow \gamma_{\alpha\beta} - (\nabla_\alpha \varepsilon_\beta(\sigma) + \nabla_\beta \varepsilon_\alpha(\sigma))$$

So, the gauge-fixing condition is :

$$F(\gamma_{\alpha\beta}^{\epsilon}) = \gamma_{\alpha\beta} - \nabla_\alpha \epsilon_\beta - \nabla_\beta \epsilon_\alpha - e^{2\varphi}\delta_{\alpha\beta} \qquad (32)$$

Going to complex coordinates : $z = \sigma_1 + i\,\sigma_2$, the Fadeev-Popov determinant associated with (32) results [4] :

$$\Delta_{FP} = \det|\nabla_z| . \det|\nabla_{\bar z}| \qquad (33)$$

where ∇_z and $\nabla_{\bar z}$ are the covariant derivatives respect to (9). Therefore, we can write (32) in the form :

$$Z = \int [D\varphi]\, e^{-W[\varphi]}$$

$$W[\varphi] = \frac{1}{2} \ln \det \Delta_\gamma - \Delta_{FP} \qquad (34)$$

By regularizing according to the proper-time cutoff procedure (see Chapter II, Part B, Appendix) the evaluation of (34) gives :

$$W[\varphi] = \frac{26-D}{12\pi} \int d^2\sigma \left[\frac{1}{2}\partial_\alpha\varphi\,\partial_\alpha\varphi + \mu^2 (e^{2\varphi} - 1) \right] \qquad (35)$$

where μ^2 is a quadratically divergent renormalization of μ_0^2 . This action is known as "Liouville action", and shows very clearly the origin of the commonly known "critical dimension" of the bosonic string theory. It's only in D=26 that the conformal factor φ decouples of the theory , and then it is actually Weyl invariant. It can be shown the spectrum and scattering amplitudes (to be defined in the course of the following chapters) given by this quantization procedure coincide with those obtained in the operatorial formalism ***in the critical parameters*** D=26 and a=1.

For later convenience, we write down the expression derived from (35) :

$$\frac{\delta W[\varphi]}{\delta \varphi(\sigma)} = \frac{26-D}{12\pi}\left(-\Box \varphi(\sigma)\right) + \frac{26-D}{12\pi}\,\mu^{2}\, e^{2\varphi(\sigma)} \qquad (36)$$

CHAPTER II

STRINGS AND SIGMA MODELS

PART A : β-FUNCTIONS AND CONFORMAL ANOMALY.

1 - INTRODUCTION.

Let's consider the problem of coupling the bosonic closed string to background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and D, that we can interpret as "vacuum expectation values" of its massless modes graviton, antisymmetric tensor and dilaton respectively. The "natural" way of doing this is to generalize the free action to :

$$S[X,\gamma;\psi] = S_G[X,\gamma] + S_B[X,\gamma] + S_D[X,\gamma] \qquad (1-a)$$

$$S_G[X,\gamma] = \frac{1}{2a^2}\int_{M_2} d^2\sigma \sqrt{\gamma(\sigma)}\, G_{\mu\nu}(X(\sigma))\, \gamma^{\alpha\beta}(\sigma)\, \partial_\alpha X^\mu(\sigma)\, \partial_\beta X^\nu(\sigma) \qquad (1-b)$$

$$S_B[X,\gamma] = \frac{1}{2a^2}\int_{M_2} d^2\sigma \sqrt{\gamma(\sigma)}\, B_{\mu\nu}(X(\sigma))\, i\, \epsilon^{\alpha\beta}(\sigma)\, \partial_\alpha X^\mu(\sigma)\, \partial_\beta X^\nu(\sigma) \qquad (1-c)$$

$$S_D[X,\gamma] = -\frac{1}{8\pi}\int_{M_2} d^2\sigma \sqrt{\gamma(\sigma)}\, R_2(\gamma)\, D(X(\sigma)) \qquad (1-d)$$

where:

M_2 = compact oriented 2-manifold without boundary

$\sigma = (\sigma^1,\sigma^2)$, coordinates of M_2

$\gamma_{\alpha\beta}$ = metric on M_2, $R_2(\gamma)$ = scalar curvature of M_2

$\epsilon^{\alpha\beta}$ = antisymmetric tensor on M

$a^2 = 2\pi\alpha'$, α' is the string tension

All in euclidean notation (the reason of the "i").

The action (1) is manifestly covariant in WS, and also it is in SP if we change the fields according the tensor

rules (however the true symmetries of this type are those genereted by the *Killing vectors* of the fields [5]).
Also it presents the "gauge invariance":

$$B_{\mu\nu}(X) \longrightarrow B_{\mu\nu}(X) + \partial_{[\mu}\Lambda_{\nu]}(X)$$

for an arbitrary vector $\Lambda_\mu(X)$. This action defines the "so-called "generalized σ-model" on a 2-d curved space.
The quantization of the string in presence of the backgrounds is described by the Polyakov functional integral :

$$Z[\Psi] = \sum_{g=0}^{\infty} K^g \int_{M_2(g)} [\mathcal{D}\gamma][\mathcal{D}X]\, e^{-S[X,\gamma;\Psi(X)]} \qquad (2)$$

where g is the genus (number of handles) of M_2 , and k is the coupling constant of the theory. Due to the particular coupling of the dilaton , and the relation [6]

$$\varkappa = 2 - 2g = \frac{1}{4\pi}\int_{M_2(g)} d\sigma^2 \sqrt{\gamma}\, R_2[\gamma]$$

($\varkappa$ is the Euler characteristic of M_2) , e^{-D_0} acts like an effective coupling constant, where Do is the constant mode of the dilaton field ; [$\mathcal{D}\gamma$] and [$\mathcal{D}X$] are some measures to be (covariantly) defined.
The VEV of some operator $O(X,\gamma)$ is given by :

$$< O(X,\gamma) > = \sum_{g=0}^{\infty} K^g \int [\mathcal{D}\gamma][\mathcal{D}X]\, e^{-S[X,\gamma;\Psi(X)]}\, O(X,\gamma) \qquad (3)$$

Let's consider now some fixed genus g :

$$Z_g[\Psi] = \int_{M_2(g)} [\mathcal{D}\gamma][\mathcal{D}X]\, e^{-S[X,\gamma;\Psi(X)]} \qquad (4)$$

By using the well-known fact [4,6] that all metric on a

Riemann surface is conformally equivalent by diffeomorphisms to some reference metric $\hat{\gamma}(\tau)$ where τ are the moduli or Teichmuller parameters of M_2 which label the possible conformally non-equivalent metrics, we can fix the so-called conformal gauge:

$$\gamma_{\alpha\beta}(\sigma) = e^{2\varphi(\sigma)}\,\hat{\gamma}_{\alpha\beta}(\sigma) \qquad (5)$$

So, the integration over metrics of some function $f(\gamma)$ will be given (discarding the volume of the connected part of the diffeomorphisms group) by :

$$\int [D\gamma]\, f(\gamma) = \int_{Teich} [D\tau] \int [D\varphi]\ \Delta_{FP}(\gamma)\ f(\gamma)\Big|_{\gamma = e^{2\varphi}\hat{\gamma}} \qquad (6)$$

where $\Delta_{FP}(\gamma)$ is the Fadeev-Popov determinant corresponding to the gauge fixing (5) (see Chapter I, Section 3-B). Then, (4) can be written as :

$$Z_g[\Psi] = \int_{Teich} [D\tau] \int [D\varphi]\ e^{-W[\gamma]}\, Z[\Psi,\gamma]\Big|_{\gamma = e^{2\varphi}\hat{\gamma}} \qquad (7\text{-}a)$$

$$Z[\Psi,\gamma] = e^{-(W[\gamma,\Psi] - W_0[\gamma])} = e^{W_0[\gamma]} \int [D_{\gamma,G}X]\ e^{-S[X,\gamma,\Psi(X)]} \qquad (7\text{-}b)$$

$$e^{-W_0[\gamma]} = \int [DX]\ e^{-S_0[X,\gamma]} \qquad (7\text{-}c)$$

where So is the free action and W $[\gamma]$ is given in Chapter I, eq.(35). We shall consider in what follows the 2-d QFT defined by (7-b) discarding Wo, that is, the σ-model itself.

2 - A WARD IDENTITY FOR THE TRACE ANOMALY.

In this section we shall use dimensional regularization *assuming there is a consistent prescription* [5] ; the reasons will become obvious below. So, we extend the dimension of the parameter space to : $d = 2 + \varepsilon$, $\varepsilon \to 0^{-}$. Let's consider then in some d-dimensional manifold with metric $\gamma_{\alpha\beta}$ infinitesimal coordinates transformations with parameters ($k^{\alpha}(\sigma)$) and Weyl transformations with parameter $2\lambda(\sigma)$, under which the metric transforms like :

$$\delta \gamma_{\alpha\beta}(\sigma) = \nabla_{\alpha} K_{\beta}(\sigma) + \nabla_{\beta} K_{\alpha}(\sigma) + 2\lambda(\sigma)\, \gamma_{\alpha\beta}(\sigma) \qquad (8)$$

If we choose some fixed metric $\overline{\gamma}_{\alpha\beta}$, we can seek the transformations leaving invariant, i.e., the parameters (k^{α}, λ) such that $\delta\overline{\gamma}_{\alpha\beta} = 0$. The vectors k^{α} for which some λ can be found in the way this condition be fulfilled are the *conformal Killing vectors* and the set of combined diffeomorphisms and local scalings which preserve $\overline{\gamma}_{\alpha\beta}$ form the *conformal group* (of $\overline{\gamma}_{\alpha\beta}$). We shall be interested in the conditions under which the σ-model defined by (7) results in a local scale invariant field theory , that is, invariant under local rescalings of the metric.

For later use, we write down some useful formulae ; under a Weyl scaling in $d=2+\varepsilon$ dimensions we have :

$$\begin{aligned} \gamma_{\alpha\beta}(\sigma) &\longrightarrow e^{2\lambda(\sigma)}\, \gamma_{\alpha\beta}(\sigma) \\ \varepsilon_{\alpha\beta}(\sigma) &\longrightarrow e^{2\lambda(\sigma)}\, \varepsilon_{\alpha\beta}(\sigma) \end{aligned} \qquad (9-a)$$

so that :

$$\sqrt{\gamma(\sigma)} \longrightarrow e^{d\lambda(\sigma)} \sqrt{\gamma(\sigma)}$$

$$\sqrt{\gamma(\sigma)}\,(\gamma^{\alpha\beta}_{(\sigma)} + \varepsilon^{\alpha\beta}_{(\sigma)}) \longrightarrow e^{\varepsilon\lambda(\sigma)} \sqrt{\gamma(\sigma)}\,(\gamma^{\alpha\beta}_{(\sigma)} + \varepsilon^{\alpha\beta}_{(\sigma)}) \qquad (9-6)$$

$$\sqrt{\gamma(\sigma)}\, R_2[\gamma] \longrightarrow e^{\varepsilon\lambda(\sigma)} \sqrt{\gamma}\,[\, R_2(\gamma) +$$

$$+ 2(1+\varepsilon)\, \Delta_\gamma \Lambda - \varepsilon(1+\varepsilon)\, \partial^\alpha\lambda(\sigma)\, \partial_\alpha\lambda(\sigma)\,]$$

The σ-model will be local scale invariant if :

$$\sqrt{\gamma(\sigma)}\, < T(\sigma) > = - \frac{\delta\, W[\, e^{2\lambda}\gamma ; \varphi\,]}{\delta\, \lambda(\sigma)}\Bigg|_{\lambda=0} = 0$$

$$T_{\alpha\beta}(\sigma) = -\frac{2}{\sqrt{\gamma(\sigma)}}\, \frac{\delta S}{\delta \gamma^{\alpha\beta}(\sigma)} \qquad (10)$$

$$T(\sigma) = \gamma^{\alpha\beta}_{(\sigma)}\, T_{\alpha\beta}(\sigma)$$

and the expectation value is defined by :

$$< \dots > = \frac{\int [\mathcal{D}X]\, e^{-S} \dots}{\int [\mathcal{D}X]\, e^{-S}} \qquad (11)$$

It's widely believed that the Weyl anomaly is closely related to the β-functions of the theory. It's our purpose now to make precise this relation.

For convenience in the handling of expressions, we redefine the background fields as follows :

$$\frac{1}{a^2}\, G_{\mu\nu}(X) \longrightarrow G_{\mu\nu}(X)$$

$$\frac{1}{a^2}\, i\, B_{\mu\nu}(X) \longrightarrow B_{\mu\nu}(X) \qquad (12)$$

$$-\frac{1}{4\pi}\, D(X) \longrightarrow \frac{1}{d-1}\, D(X)$$

Then, we will consider the "bare" action :

$$S[X;\gamma,\psi(X)] = \int_{M_2} d^d\sigma\, \sqrt{\gamma(\sigma)}\, \Big[\, \frac{1}{2}\,(G_{\mu\nu}(X) + B_{\mu\nu}(X))\, \times$$
$$\times\, (\gamma^{\alpha\beta}(\sigma) + \varepsilon^{\alpha\beta}(\sigma))\, \partial_\alpha X^\mu(\sigma)\, \partial_\beta X^\nu(\sigma) \;+\; \frac{R_2[\gamma]}{2\pi(d-1)}\, D(X)\,\Big] \qquad (13)$$

As Friedan showed [7], the model is renormalizable in a generalized sense, changing the backgrounds and mixing among themselves under renormalization. In order to explicit this fact, we define a vector of backgrounds :

$$\psi = (\, G_{\mu\nu},\, B_{\mu\nu},\, D\,) \qquad (14)$$
$$\psi_R = (\, G^R_{\mu\nu},\, B^R_{\mu\nu},\, D^R\,)$$

calling ψ_R to the corresponding "renormalized" vector. Then, working in *minimal substraction* scheme :

$$\psi = \mu^{\varepsilon}\,(\,\psi_R + L^{\psi}(\psi_R)\,) \qquad (15)$$
$$L^{\psi}(\psi_R) = \sum_{K=1}^{\infty} \varepsilon^{-K}\, T^{(K)}(\psi_R)$$

the earlier statement is traduced in the fact that we can find the vector of counterterms $L^{\psi}(\psi_R)$ which rends the action (13) finite.

The β-functions associated with the couplings are defined by :

$$\hat{\beta}^{\psi}(\psi_R) = \mu \frac{\partial \psi_R}{\partial \mu} \qquad (16)$$

After some careful massage in (15) we get [8] :

$$\hat{\beta}^{\psi}(\psi_R) \simeq -\varepsilon \psi_R + \beta^{\psi}(\psi_R) \qquad (17)$$

$$\beta^{\psi}(\psi_R) = -\left[\left(1 + \Lambda \frac{\partial}{\partial \Lambda}\right) T^{(1)}(\Lambda^{-1}\psi_R)\right]\Big|_{\Lambda=1}$$

where, as usual, the β-functions are determined by the residues of the first order poles, and they are *functionals* of the renormalized fields.

Global scale invariance.

Being : $W[\gamma, \psi] = W_R[\gamma, \psi_R]$ independent of μ, we can get the renormalization group equation (RGE) for W_R :

$$\left[\mu \frac{\partial}{\partial \mu} + \int d^D x \; \hat{\beta}^{\psi}(\psi_R) \cdot \frac{\delta}{\delta \psi_R(x)}\right] W_R[\gamma; \psi_R] = 0 \qquad (18)$$

If, instead of considering $\Lambda(\sigma)$ in (10) we take Λ to be a constant (independent of σ), we get the integrated trace anomaly :

$$\int d^2\sigma \sqrt{\gamma(\sigma)} \, \langle T(\sigma) \rangle = -\frac{\partial W[e^{2\lambda}\gamma; \psi]}{\partial \lambda}\Big|_{\lambda=0} = -\frac{\partial W_R[e^{2\lambda}\gamma; \psi]}{\partial \lambda}\Big|_{\lambda=0} \qquad (19)$$

Under the rescaling : $\gamma \rightarrow \gamma e^{2\lambda}$ the (bare) action (13) changes (see (9)) by :

$$s \longrightarrow e^{\varepsilon\lambda} s$$

But, considering (15), we see that this is precisely the way in which the renormalization mass μ enters explicitely in W_R !. From this observation we obtain the key identity:

$$\frac{\partial W_R[\gamma;\psi_R]}{\partial\lambda} = \mu \frac{\partial W_R[\gamma;\psi_R]}{\partial\mu} \qquad (20)$$

By combining (18), (19) and (20), we get:

$$\int d^2\sigma \sqrt{\gamma(\sigma)} \langle T(\sigma)\rangle = -\frac{\delta W_R[e^{2\lambda}\gamma;\psi_R]}{\delta\lambda}\Big|_{\lambda=0} = -\frac{\delta W[e^{2\lambda}\gamma;\psi]}{\delta\lambda}\Big|_{\lambda=0} =$$

$$= \int d\hat{x}^D \hat{\beta}^{\psi}(\psi_R(x)) \frac{\delta W_R[\gamma;\psi_R]}{\delta\psi_R(x)} = \int d\hat{x}^D \beta^{\psi}(\psi_R) \frac{\delta W_R[\gamma;\psi_R]}{\delta\psi_R(x)} \qquad (21)$$

where in the last step we used (17) and the fact that

$W_R\ [\gamma;\psi_R]$ is finite.

From (21) we get the following ***sufficient*** conditions to assure that ***global scale invariance*** holds :

$$\beta^G_{\mu\nu} + 2\nabla_{\{\mu}V_{\nu\}} = 0$$

$$\beta^D_{\mu\nu} + 2H^R_{\mu\nu}{}^{\rho}V_\rho + 2\nabla_{[\mu}W_{\nu]} = 0 \qquad (22)$$

$$\beta^D + V^\mu\nabla_\mu D^R = 0$$

for ***any vectors*** V_μ (because the integrand results in a total SP derivative) and W_μ (due to the gauge invariance $B_{\mu\nu} \longrightarrow B_{\mu\nu} + \partial_{[\mu}W_{\nu]}$ explained at the beggining of the chapter).

Local scale invariance.

We are actually interested in the trace anomaly itself, more than in its integral. So, we would like to repeat the earlier steps (with some modifications, of course) for obtaining an expression for it.

The "key" equations before were the RGE (18) and the scaling relation (20). The trick for getting an apropppiated RGE, a *local* RGE, is to regard the renormalization mass as a *local* parameter $\mu(\sigma)$. Let's introduce local renormalized coupling functions as follows: the renormalization group guarantees that the explicit μ-dependence in the bare quantities is compensated by the implicit μ-dependence in the renormalized ones (background fields or local couplings in our case). Now, the same RG argument tells us that if we let the renormalized mass becomes σ-dependent, this can be compensated by a corresponding σ-dependence in the renormalized couplings ψ_R . We therefore define local renormalized coupling functions $\psi_R(\sigma)$ by :

$$\Psi = \mu^{\varepsilon}\,\hat{\Psi}(\Psi_R) = \mu(\sigma)^{\varepsilon}\,\hat{\Psi}(\Psi_R(\sigma))$$
$$\hat{\Psi}(\Psi_R) = \Psi_R + L^{\Psi}(\Psi_R) \qquad (23)$$
$$\Psi_R(x,\mu(\sigma)) \equiv \Psi_R(\sigma) = \epsilon(\Psi_R, \mu(\sigma)), \qquad \epsilon(\Psi_R, \mu) = \Psi_R$$

and *local* β-functions :

$$\hat{\beta}^{\Psi}(\epsilon(\Psi_R,\mu(\sigma))) = t\,\frac{\partial \epsilon(\Psi_R, t)}{\partial t}\bigg|_{t=\mu(\sigma)} \equiv \hat{\beta}(\Psi_R)\bigg|_{\Psi_R(\sigma)} \qquad (24)$$

By using these definitions, we can inmediately write down the *local* RGE similar to (18) :

$$\left[\mu(\sigma) \frac{\delta}{\delta \mu(\sigma)} + \int d\hat{x} \; \hat{\beta}^{\psi}(\psi_R(\sigma)) \cdot \frac{\delta}{\delta \psi_R(X, \mu(\sigma))} \right] W_R[\gamma ; \psi_R(\sigma)] = 0 \qquad (25)$$

Now, we look for some scaling relation like (20). From (9) and (23) it follows :

$$\sqrt{\gamma(\sigma)} < T(\sigma) > = - \frac{\delta W_R[e^{2\lambda}\gamma ; \psi_R(\sigma)]}{\delta \lambda(\sigma)} \Big|_{\substack{\mu(\sigma) = \mu \\ \lambda = 0}} = \qquad (26)$$

$$= - \mu(\sigma) \frac{\delta W_R[e^{2\lambda}\gamma ; \psi_R(\sigma)]}{\delta \mu(\sigma)} \Big|_{\substack{\mu(\sigma) = \mu \\ \lambda = 0}} + \sqrt{\gamma(\sigma)} < \Delta_\gamma D(X(\sigma)) >$$

The last term which involves the bare dilaton expectation value comes from the inhomogeneous scaling of $\sqrt{\gamma(\sigma)}\, R_2[\gamma]$ in (9). This is due to the fact that, at classical level, the dilaton coupling breaks Weyl invariance (unless D(X) be a constant). From the identity (which gives the classical equations of motion):

$$\frac{\delta S[X, \gamma, \psi(X)]}{\delta X_\mu(\sigma)} = \Delta_\gamma X^\mu(\sigma) + \frac{1}{2(d-1)} R_2(\gamma) \nabla^\mu D(X) - \qquad (27)$$

$$- (\gamma^{\alpha\rho}(\sigma) \Gamma^\mu_{\nu\varphi}(X) - \varepsilon^{\alpha\rho} H^{\mu}_{\nu\varphi}(X)) \, \partial_\alpha X^\nu(\sigma) \partial_\rho X^\varphi(\sigma)$$

$$H_{\mu\nu\varphi}(X) = \frac{1}{2} (\partial_\mu B_{\nu\varphi}(X) + \partial_\varphi B_{\mu\nu}(X) + \partial_\nu B_{\varphi\mu}(X))$$

it's obtained the expression :

$$\sqrt{\gamma(\sigma)} \Delta_\gamma D(X) = \frac{1}{2(d-1)} \sqrt{\gamma(\sigma)} R_2(\gamma) \nabla^\mu D(X) \nabla_\mu D(X) + \qquad (28)$$

$$+ (\nabla_\mu \nabla_\nu D(X) \gamma^{\alpha\rho} + H_{\mu\nu}{}^{\varphi}(X) \nabla_\varphi D(X) \varepsilon^{\alpha\rho}) \partial_\alpha X^\mu \partial_\rho X^\nu -$$

$$- \nabla^\mu D(X) \frac{\delta S[X, \gamma; \psi(X)]}{\delta X^\mu}$$

Taking its expectation value we get :

$$\sqrt{\gamma} < \Delta_\gamma D(X) > = \sqrt{\gamma} < \nabla_\mu \nabla_\nu D(X)\, \partial_\alpha X^\mu \partial^\alpha X^\nu > +$$
$$+ < H_{\mu\nu}^{\varphi}(X) \nabla_\varphi D(X)\, \varepsilon^{\alpha\rho} \partial_\alpha X^\mu \partial_\rho X^\nu > +$$
$$+ \frac{1}{2(d-1)} R_{\gamma}[\gamma] < \nabla_\mu D(X)\, \nabla^\mu D(X) > + \qquad (29\text{-}a)$$
$$+ \; 0$$

which can be expressed (with (16)) :

$$\sqrt{\gamma(\sigma)} < \Delta_\gamma D(X) > = \int d\hat{x}^\rho \Big(2 \nabla_\mu \nabla_\nu D(X) \frac{\delta}{\delta G_{\mu\nu}(X)} + \qquad (29\text{-}b)$$
$$+ 2 H_{\mu\nu}{}^{\varphi}(X) \nabla_\varphi D(X) \frac{\delta}{\delta B_{\mu\nu}(X)} + \nabla_\mu D(X) \nabla^\mu D(X) \Big) W[\gamma;\Psi] \; .$$

By combining (25), (26) and (29) we obtain :

$$\sqrt{\gamma(\sigma)} < T(\sigma) > = \int d\hat{x}^\rho \Big[\beta^{\Psi} \cdot \frac{\delta W_R[\gamma;\Psi_R(\sigma)]}{\delta \Psi_R(\sigma)} \Big|_{\mu(\sigma)=\mu} - \varepsilon \Psi_R(\sigma) \cdot \frac{\delta W_R[\gamma;\Psi_R(\sigma)]}{\delta \Psi_R(\sigma)} \Big|_{\mu(\sigma)}$$
$$+ \int d\hat{x}^\rho \Big(2 \nabla_\mu \nabla_\nu D(X) \frac{\delta}{\delta G_{\mu\nu}(X)} + 2 H_{\mu\nu}{}^{\varphi}(X) \nabla_\varphi D(X) \frac{\delta}{\delta B_{\mu\nu}} + \qquad (30)$$
$$+ \nabla_\mu D(X) \nabla^\mu D(X) \frac{\delta}{\delta D(X)} \Big) W[\gamma, \Psi] .$$

In order to arrive to our main result, two remarks are in order :

i) The renormalization theory states that (functionally) differentiating a *finite* Green's function respect a *finite* parameter gives another *finite* quantity. From (10) we see that $T(\sigma)$ and its insertions into any finite Green's function are finite. This is also true for $\frac{\delta W_R[\gamma,\Psi_R]}{\delta \Psi_R(X)}$; so :

$$\varepsilon \frac{\delta W_R[\gamma;\Psi_R]}{\delta \Psi_R(X)} \xrightarrow{\varepsilon \to 0} 0$$

as it was used to get (22).But this is generally *not* true of derivatives respect to the *local* couplings functions. Precisely, these divergences cancel those coming from the dilaton term in (30) (if there are) to render finite the trace anomaly ;

ii) The derivatives in (29) and (30) must be reexpressed in terms of those which involve the local renormalized couplings $\psi_R(\sigma)$. Then, since the trace anomaly is guaranteed to be finite, we know that all terms which involve explicit poles must cancel. We therefore can rewrite (30) as follows :

$$\sqrt{\gamma(\sigma)}\ \langle T(\sigma)\rangle = f.p\Big\{ -\varepsilon \int d^D x\ \psi_R \cdot \frac{\delta W_R[\gamma;\psi_R(\sigma)]}{\delta \psi_R(\sigma)}\Big|_{\mu(\sigma)=\mu} +$$

$$+ \int d^D x \Big[\big(\beta^G_{\mu\nu} + 2\nabla_\mu \nabla_\nu D^R(x)\big)\frac{\delta}{\delta G^R_{\mu\nu}(\sigma)} + \tag{31}$$

$$+ \big(\beta^B_{\mu\nu} + 2 H^{R\ \rho}_{\mu\nu}(x)\nabla_\rho D^R(x)\big)\frac{\delta}{\delta B^R_{\mu\nu}(\sigma)} +$$

$$+ \big(\beta^D + \nabla_\mu D^R(x)\nabla^\mu D^R(x)\big)\frac{\delta}{\delta D^R(\sigma)}\Big] W_R[\gamma;\psi_R(\sigma)]\Big|_{\mu(\sigma)=\mu}.$$

where f.p. stands for "finite part". From this equation we can assert that *sufficient* conditions for the vanishing of the trace anomaly are :

$$\beta^G_{\mu\nu} + 2\nabla_\mu \nabla_\nu D^R(x) = 0$$

$$\beta^B_{\mu\nu} + 2 H^{R\ \rho}_{\mu\nu}(x)\nabla_\rho D^R(x) = 0 \tag{32-a}$$

$$\beta^D + \nabla_\mu D^R(x)\nabla^\mu D^R(x) = 0$$

if the counterterms are choosing such that the Green's functions :

$$\int d\overset{p}{x} \ \psi_R \cdot \frac{\delta W_R[\gamma; \psi_R(\sigma)]}{\delta \psi_R(\sigma)}\Bigg|_{\mu(\sigma)=\mu} \quad \text{are finite.} \qquad (32\text{-}b)$$

But, if (32-b) doesn't hold, it is not difficult to see (using (21) and (31)) that such divergence must be a total derivative, of the general form [9] :

$$\frac{1}{\sqrt{\gamma(\sigma)}}\partial_\alpha \left(\sqrt{\gamma(\sigma)}\, \gamma^{\alpha\beta}_{(\sigma)} \partial_\beta X^\mu_{(\sigma)} \bar{V}_\mu(X) - \varepsilon^{\alpha\beta} \partial_\beta X^\mu_{(\sigma)} W_\mu(X) \right) \qquad (33)$$

for some *fixed vectors* $\bar{V}_\mu$ and W_μ . With (27) we can reexpress (33) as :

$$\left[2\,\nabla_{\{\mu} \bar{V}_{\nu\}}\, \gamma^{\alpha\beta}_{(\sigma)} + \left(\bar{V}_\varphi H^\varphi_{\mu\nu}(x) + 2\,\nabla_{[\mu} W_{\nu]} \right) \varepsilon^{\alpha\beta}_{(\sigma)} \right] \partial_\alpha X^\mu_{(\sigma)} \partial_\beta X^\nu_{(\sigma)} +$$

$$+ \frac{1}{2(d-1)} R_2(\gamma)\, \bar{V}^\mu \nabla_\mu D(x) - \frac{1}{\sqrt{\gamma}} \bar{V}^\mu \frac{\delta S}{\delta X^\mu} \qquad (34)$$

By taking the expectation value of (34), and reexpressing it as derivatives of W_R , we get a general set of *sufficient* conditions for absence of the trace anomaly :

$$\begin{aligned} &\beta^G_{\mu\nu} + 2\,\nabla_{[\mu} V_{\nu]} = 0 \\ &\beta^B_{\mu\nu} + 2H^R_{\mu\nu}{}^{\varphi}(x)\, V_\varphi + 2\,\nabla_{[\mu} W_{\nu]} = 0 \\ &\beta^D + V^\mu \nabla_\mu D^R = 0 \\ &V_\mu = \bar{V}_\mu + \nabla_\mu D^R \end{aligned} \qquad (35)$$

and $\bar{V}_\mu$, W_μ are defined in (33).

These equations display the subtle relation between the σ-model β-functions and the conditions for vanishing of the Weyl anomaly. An analog result was obtained by Tseytlin, but following a method different to the functional one sketched here [10].

Let's remark that the β-functions are not uniquely determined, but they present ambiguities associated with the SP diffeomorphisms [8,9,10,11] ; however the conditions (35) aren't [5,10].

PART B : COMPUTATION OF THE TRACE ANOMALY.

As we saw at the beginning (eq.(7)), the Polyakov path integral on the sphere (without moduli) can be written in the form :

$$Z[\varphi]\Big|_{S^2} = \int_{S^2} [D\varphi]\; e^{-\left(W[\gamma] + W[\gamma;\varphi]\right)} \tag{36-a}$$

$$e^{-W[\gamma;\varphi]} = \left.\frac{\int [DX]\, e^{-S[X,\gamma;\varphi(x)]}}{\int [DX]\, e^{-S_0[X,\gamma]}}\right|_{\gamma_{\alpha\beta} = e^{2\varphi}\delta_{\alpha\beta}} \tag{36-b}$$

$$[DX] = \prod_{\sigma \in M_2} \left(\int d^D X^\mu_{(\sigma)} \sqrt{G(X(\sigma))} \right), \quad G \equiv \det(G_{\mu\nu}) \tag{36-c}$$

where So is the free action and W $[\varphi]$ is given in Chapter I, eq.(35). It's our purpose in this section to calculate W $[\varphi, \psi]$. We start writting the field X^μ as the sum of a "classical" background $\bar{X}^\mu$ and a quantum fluctuation π^μ :

$$X^\mu(\sigma) = \bar{X}^\mu(\sigma) + \pi^\mu(\sigma) \tag{37}$$

where $\bar{X}^\mu$ satisfies the equation of motion (27).

The problem with this splitting is that, not being π^μ a vector in a general reference system, but a difference of coordinates, the expansion of S will not be manifestaly covariant. However, there is an special reference frame known as *Riemann normal coordinates* (RNC) [8] in which it is. In order to define it, let's consider two points X and X' belonging to some D-dimensional manifold M_D with metric $G_{\mu\nu}(X)$, with coordinates $\bar{X}^\mu$ and ($\bar{X}^\mu + \pi^\mu$) respectively. Assuming they are quite near (reasonable hypothesis thinking about perturbation theory around $\bar{X}^\mu$), there exist and it's unique the geodesic which joins them.

Parametrizing it by $\lambda^{\mu}(t)$, we have :

$$\ddot{\lambda}^{\mu}(t) + \Gamma^{\mu}_{\nu\rho}(\lambda(t))\,\dot{\lambda}^{\nu}(t)\,\dot{\lambda}^{\rho}(t) = 0 \qquad (38-a)$$

$$\lambda^{\mu}(0) = \bar{X}^{\mu}$$
$$\lambda^{\mu}(1) = \bar{X}^{\mu} + \pi^{\mu} \qquad (38-b)$$

where $\Gamma^{\mu}_{\nu\rho}$ is the Christoffel conextion for $G_{\mu\nu}$.

If $\xi^{\mu} = \dot{\lambda}^{\mu}(0)$ is the tangent *vector* to the geodesic line in X (t=0), in virtue of (38) it will be related to π^{μ}.

Indeed, by sucessively differentiation of (38-a) we get the solution in power series :

$$\lambda^{\mu}(t) = \bar{X}^{\mu} + t\,\xi^{\mu} - \sum_{n=2}^{\infty} \frac{t^{n}}{n!}\,\Gamma^{\mu}_{\nu_1\ldots\nu_n}(\bar{X})\,\xi^{\nu_1}\ldots\xi^{\nu_n} \qquad (39)$$

that at t=1 gives :

$$\pi^{\mu} = \xi^{\mu} - \sum_{n=2}^{\infty} \frac{1}{n!}\,\Gamma^{\mu}_{\nu_1\ldots\nu_n}(\bar{X})\,\xi^{\nu_1}\ldots\xi^{\nu_n} \qquad (40-a)$$

$$\Gamma^{\mu}_{\nu_1\nu_2\nu_3} = \nabla^{\text{lower indices}}_{\{\nu_3}\,\Gamma^{\mu}_{\nu_1\nu_2\}}(\bar{X})$$

$$\Gamma^{\mu}_{\nu_1\ldots\nu_{n+1}} = \nabla^{\text{l.i.}}_{\{\nu_{n+1}}\,\Gamma^{\mu}_{\nu_1\ldots\nu_n\}}(\bar{X}) \qquad (40-b)$$

Now, let's consider the RNC in $\bar{X}^{\mu}$; in this coordinate system holds : $\pi^{\mu} \equiv \xi^{\mu}$

So, it must be verified : $\Gamma^{\mu}_{\nu_1\ldots\nu_n} = 0$

Due to ξ^{μ} is a vector, we hope the expansion in RNC of any tensor *may be expressed covariantly*. Indeed, it occurs. For example, we get [8] :

$$V_\mu(\bar{X}+\pi) = V_\mu(\bar{X}) + \xi^\nu \nabla_\nu V_\mu(\bar{X}) + \frac{1}{2}\xi^{\mu_1}\xi^{\mu_2}\nabla_{\mu_1}\nabla_{\mu_2}V_\mu(\bar{X}) + \ldots \qquad (41\text{-}a)$$

$$G_{\mu\nu}(\bar{X}+\pi) = G_{\mu\nu}(\bar{X}) + \frac{1}{3}R_{\mu\mu_1\mu_2\nu}(\bar{X})\xi^{\mu_1}\xi^{\mu_2} + \ldots \qquad (41\text{-}b)$$

We are going to compute W $[\gamma,\psi]$ at leading order *in derivatives of background fields*, considering terms with *no more* than two derivatives of the backgrounds (being they adimensional in string theory, we shall obtain an expansion in α'as we'll see). The expansion of (13) in RNC gives the following relevants terms [8,12] :

$$S[\bar{X}+\pi;\gamma,\psi(\bar{X}+\pi)] = S[\bar{X}] + S_0^{(2)} + S_1^{(2)} + S_2^{(2)} + S_3^{(2)} + S^{(3)} + S^{(4)}, \qquad (42\text{-}a)$$

$$S_0^{(2)} = \int d^2\sigma \sqrt{\gamma}\, \frac{1}{2}\nabla^\alpha\xi^a \nabla_\alpha \xi^a \quad , \qquad (42\text{-}b)$$

$$\nabla_\alpha \xi^a = \partial_\alpha \xi^a + \partial_\alpha \bar{X}^\mu \omega_\mu{}^{ab}(\bar{X})\xi^b , \quad \omega_\mu^{ab}(\bar{X}) \equiv \text{spin connection in } M_D$$

$$S_1^{(2)} = \frac{1}{2}\int d^2\sigma \sqrt{\gamma}\, \gamma^{\alpha\beta}\partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu \underbrace{\mathcal{R}_{a\{\mu\nu\}b}}(\bar{X})\xi^a\xi^b , \qquad (42\text{-}c)$$

$$S_2^{(2)} = \frac{1}{2}\int d^2\sigma \sqrt{\gamma}\, \varepsilon^{\alpha\beta}\partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu \underbrace{\mathcal{R}_{a[\mu\nu]b}}\xi^a\xi^b , \qquad (42\text{-}d)$$

$$\mathcal{R}_{\mu\nu\varrho\sigma} = R_{\mu\nu\varrho\sigma} + \nabla_\sigma H_{\mu\nu\varrho} - \nabla_\varrho H_{\mu\nu\sigma} + H_{\mu\sigma}{}^\varepsilon H_{\varepsilon\nu\varrho} - H_{\mu\varrho}{}^\varepsilon H_{\varepsilon\nu\sigma}$$

$$\underbrace{\mathcal{R}_{a\{\mu\nu\}b}} = R_{a\mu\nu b} + H^\varepsilon{}_{\nu a} H_{\varepsilon\mu b}$$

$$\underbrace{\mathcal{R}_{a[\mu\nu]b}} = R_{a\mu\nu b} + \nabla_a H_{\mu\nu b} + H^\varepsilon{}_{\nu a} H_{\varepsilon\mu b}$$

$$S^{(3)} = \frac{1}{3}\int d^2\sigma \sqrt{\gamma(\sigma)}\, \varepsilon^{\alpha\beta}_{(\sigma)} H_{abc}(\bar{X})\xi^a \partial_\alpha \xi^b \partial_\beta \xi^c \qquad (42\text{-}e)$$

$$S^{(4)} = \int d^2\sigma \sqrt{\gamma}\, M^{\alpha\beta}_{abcd}(\bar{X})\partial_\alpha \xi^a \xi^b \xi^c \partial_\beta \xi^d \qquad (42\text{-}f)$$

$$M^{\alpha\beta}_{abcd} = \gamma^{\alpha\beta} S_{abcd}(\bar{X}) + \varepsilon^{\alpha\beta}\mathcal{R}_{abcd}$$

$$S_{abcd}(\bar{X}) = \frac{1}{6}\left(R_{abcd} + H_{ab}{}^{e} H_{ecd}\right)$$

$$S_3^{(3)} = \frac{1}{2}\int d^2\sigma \sqrt{\gamma}\; R_2[\gamma]\; \xi^a \xi^b\; \nabla_a \nabla_b \Phi(\bar{X}) \qquad (42\text{-}g)$$

where we have introduced a "vielbein" :

$$e^a_\mu(\bar{X})\, e^a_\nu(\bar{X}) = G_{\mu\nu}(\bar{X})$$

$$G^{\mu\nu}(\bar{X})\, e^a_\mu(\bar{X})\, e^b_\nu(\bar{X}) = \delta^{ab} \qquad (43)$$

$$\xi^a = e^a_\mu\, \xi^\mu$$

$$etc\ldots$$

Finally, we must consider the expression of the X-measure in the new variables ξ^a :

$$[\mathcal{D}X] = \prod_\sigma \left(\int d^D X^\mu \sqrt{G(X)}\right) = \prod_\sigma \left(\int d^D \xi^\mu_{(\sigma)} \sqrt{G(\bar{X}+\xi)}\right) =$$

$$= \prod_\sigma \left(\int d^D \xi^a_{(\sigma)}\right) e^{-S_m} \qquad (44)$$

$$S_m = \frac{1}{6}\int d^2\sigma \sqrt{\gamma(\sigma)}\; K(\sigma,\sigma;\varepsilon)\; R_{ab}(\bar{X})\, \xi^a \xi^b$$

In the last step we used eq.(41-b), and introduced a "δ-function" in the form of the "heat-kernel" (see appendix) because we are going to regularize using the proper-time cutoff method, and according it [13] :

$$\prod_{\sigma\in M_2} e^{\frac{1}{6} R_{ab}(\bar{X})\xi^a\xi^b} = e^{\text{"}\sum_\sigma \frac{1}{6} R_{ab}(\bar{X})\xi^a\xi^b\text{"}} \simeq$$

$$= e^{\frac{1}{6}\int_{M_2} d^2\sigma \,\cdot\, \text{"Volume}(M_2)\text{"}\, R_{ab}(\bar{X})\xi^a\xi^b} \simeq e^{\frac{1}{6}\int_{M_2} d^2\sigma \sqrt{\gamma}\; K(\sigma,\sigma;\varepsilon)\, R_{ab}(\bar{X})\xi^a\xi^b}$$

From (36), (42) and (44) we can write W $[\gamma,\psi]$ as follows :

$$W[\gamma;\varphi] = S[\bar{X},\gamma,\varphi(\bar{X})] + \langle S_1^{(2)}\rangle_0 + \langle S_2^{(2)}\rangle_0 + \langle S_3^{(2)}\rangle_0 +$$
$$+ \langle S_m\rangle_0 + \langle S^{(4)}\rangle_0 - \frac{1}{2}\langle (S^{(3)})^2\rangle_0 \qquad (45)$$
$$\langle \dots \rangle_0 = \frac{\int [D\xi]\, e^{-S_0} \dots}{\int [D\xi]\, e^{-S_0}}$$

Carrying out the computation (see appendix for the necessary coincidence limits of heat-kernel, propagator and its derivatives) we finish with the result :

$$W[\gamma;\varphi] = S[\bar{X};\gamma,\varphi(\bar{X})] + \int d^2\sigma \Big[+\frac{1}{2}\partial_\alpha \bar{X}^\mu \partial^\alpha \bar{X}^\nu \left(R_{\mu\nu} - H_\mu{}^{\varrho\varepsilon} H_{\nu\varrho\varepsilon}\right) -$$
$$- \nabla^2 D\, \Box\varphi + \frac{1}{2}\varepsilon^{\alpha\beta}\partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu \nabla^\varrho D\, H_{\varrho\mu\nu} \Big] +$$
$$- \frac{1}{3}\int d^2\sigma\, \varphi\Box\varphi\, R[\bar{X}] + \frac{1}{12}\int d^2\sigma\, \Box\varphi(\sigma) \int d^2\sigma'\, \Box\varphi(\sigma')\, G(\sigma,\sigma')\, H^{\mu\nu\varrho}_{(\sigma)} H_{\mu\nu\varrho}(\sigma')$$
$$(46)$$

By using the equation of motion $\frac{\delta S[\bar{X};\gamma,\varphi(\bar{X})]}{\delta \bar{X}^\mu} = 0$ (eq. (27)), we can express $S[\bar{X},\gamma,\psi]$ in the form :

$$S[\bar{X};\gamma,\varphi(\bar{X})] = -2\int d^2\sigma\, \varphi(\sigma) \big(\nabla_\mu \nabla_\nu D(\bar{X})\, \delta^{\alpha\beta} -$$
$$- \nabla_\varrho D(\bar{X})\, H^\varrho{}_{\mu\nu}(\bar{X})\, \varepsilon^{\alpha\beta} \big) \partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu \big) + \qquad (47)$$
$$+ 2\int d^2\sigma\, \frac{1}{2}\varphi(\sigma)\Box\varphi(\sigma)\, \nabla_\mu D(\bar{X}) \nabla^\mu D(\bar{X})$$

Finally, by imposing the trace anomaly cancelation :

$$\sqrt{\gamma(\sigma)}\, \langle T(\sigma)\rangle = -\frac{\delta W}{\delta\varphi(\sigma)} = 0$$

we get from eqs. (12), (13), and (36) of Chapter I the final result :

$$\sqrt{\gamma}\,\langle T(\sigma)\rangle = B^{G}_{\mu\nu}\,\partial_\alpha \bar{X}^\mu \partial^\alpha \bar{X}^\nu + B^{B}_{\mu\nu}\, i\varepsilon^{\alpha\beta}\partial_\alpha \bar{X}^\mu \partial_\beta \bar{X}^\nu + B^{D}\frac{R_2(\gamma)}{2}$$

$$B^{G}_{\mu\nu} = \alpha'\,[\,R_{\mu\nu} - H_{\mu\rho\varepsilon} H_\nu{}^{\rho\varepsilon} - \nabla_\mu \nabla_\nu D\,] + O(\alpha'^2)$$

$$B^{B}_{\mu\nu} = \alpha'\,[\,\nabla^\rho H_{\rho\mu\nu} + \nabla^\rho D\, H_{\rho\mu\nu}\,] + O(\alpha'^2) \qquad (48)$$

$$B^{D} = \frac{D-26}{3\alpha'} + \alpha'\,[\,R - \tfrac{1}{3}H^2 + 2\nabla^2 D - (\nabla D)^2\,] + O(\alpha'^2)$$

where we have came back to "string notation" according to (12). These "famous" equations have been obtained first in Ref.[14]. A remarkable property of them is that they can be derived from a SP action (D=26) :

$$I[\psi] = \int d^{26}x\,\sqrt{G(x)}\; e^{D(x)}\,[\,R - \tfrac{1}{3}H^2 - (\nabla D)^2 - 2\nabla^2\Phi\,] \qquad (49)$$

It's guessed this effective action generates the tree level string S-matrix elements, that is, it represents the "classical effective action " for the massless modes of the string. An argument for this holds is given in Ref.[15], and suscintly is as follows (see also Refs.[16,17]) :
in conventional QFT, given the effective action $\Gamma[\Phi]$, we get a vacuum solution by imposing the quantum equations of motion : $\frac{\delta\Gamma[\Phi]}{\delta\Phi} = 0$
Some solution ϕ_0 of these equations corresponds to vacuum expectation values (VEV) of the fields, and the amplitudes must be evaluated in this vacuum solution, so that to assure the *quantum field* ξ , $\phi = \phi_0 + \xi$, has VEV equal to zero (the amplitudes must be defined as expectation values of ξ-products !) :

$$\langle \xi \rangle = 0 \qquad (50)$$

The analog object in string theory to the field ξ associated with some particle state Λ is the *vertex operator* for absorbing (or emitting) the state Λ (see Chapter III). So, given the σ-model action (1), the condition analog to (50) would be :

$$\langle V_\Lambda \rangle = 0 \qquad (51)$$

Now, let's consider the tree level case, which corresponds to world-sheets with the topology of the sphere S . It has a conformal group isomorphic to SL(2,C) [18], which includes the scalings and traslations ($z \in S^2$) :

$$z \longrightarrow a\, z$$
$$z \longrightarrow z + b \qquad (52)$$

for arbitrary $a,b \in \mathbb{C}$. Under a scaling, a physical vertex operator transforms with conformal weigth $J = 1$ [18], that is :

$$V_\Lambda(z) \longrightarrow a^{-1}\, V_\Lambda(a.z) \qquad (53)$$

Now, let's suposse the σ-model (ghost included !) is conformal invariant. Then, it inmediately follows from (52) and (53) that :

$$\langle V_\Lambda \rangle = 0 \qquad (54)$$

From here, we conclude that the conformal invariance of the σ-model implies the equations of motion for the backgrounds.

Let's stress this argument is valid only at tree level. At higher string-loop orders doesn't hold, because of the absence of a "nice" conformal group (for example, the torus presents only traslations as conformal group, isomorphic to U(1) * U(1)). An ansatz for including string-loop corrections to the effective action (49) will be explored in Chapter III.

APPENDIX :The proper-time cutoff regularization.

Let's consider the action :

$$S_0[\xi,\gamma] = \int_{M_2} d^2\sigma \sqrt{\gamma(\sigma)}\ \xi^a(\sigma)\ \Delta_\gamma\ \xi^a(\sigma) \qquad (A-1)$$

which define the propagator on a Riemann surface with metric $\gamma_{\alpha\beta}$, given as usual by :

$$< \xi^a(\sigma)\ \xi^b(\sigma') > = \frac{\int [D\xi]\ e^{-S_0[X,\gamma]}\ \xi^a(\sigma)\ \xi^b(\sigma')}{\int [D\xi]\ e^{-S_0[X,\gamma]}} \qquad (A-2)$$

In order to regularize :

i) The short-distance ("ultraviolet") behavior ;

ii) The "infrared" divergence due to the presence of the "zero mode" ,

we replace the action (A-1) by :

$$S_0^{r\beta g}[\xi,\gamma] = \frac{1}{2}\int d^2\sigma \sqrt{\gamma(\sigma)}\ \xi^a(\sigma)\ \Delta_\gamma^r\ \xi^a(\sigma) + \frac{\beta}{2}\sum_{a=1}^{D}\left[\int d^2\sigma \sqrt{\gamma(\sigma)}\ \xi^a(\sigma)\right]^2$$

$$\Delta_\gamma^r = \Delta_\gamma\ e^{\varepsilon \Delta_\gamma} \qquad (A-3)$$

$$\varepsilon, \beta \in \mathbb{R}$$

Now, let's take an orthonormal basis in the space of scalar functions on M_2 provided by the eigenfunctions of the Laplace-Beltrami operator :

$$\Delta_\gamma u_n = \lambda_n u_n \qquad (A-4-a)$$

$$(u_n | u_m) = \int_{M_2} d^2\sigma \sqrt{\gamma(\sigma)}\ u_n(\sigma)\ u_m(\sigma) = \delta_{n,m}$$

Therefore, the regularized laplacian satisfies :

$$\Delta_\gamma^r u_n = \lambda_n^r u_n \qquad (A-4-B)$$
$$\lambda_n^r = \lambda_n e^{\varepsilon \lambda_n}$$

By defining :

$$\xi^a(\sigma) = \sum_n \xi_n^a u_n(\sigma) \qquad (A-5)$$
$$[D\xi] = \prod_{n,a} \left(\int_{-\infty}^{\infty} d\xi_n^a \right)$$

it's straigthforward to get :

$$< \xi^a(\sigma)\, \xi^b(\sigma') > = \delta^{ab}\, G(\sigma,\sigma') \qquad (A-6)$$
$$G(\sigma,\sigma') = \sum_n \frac{u_n(\sigma)\, u_n(\sigma')}{\lambda_n^r}$$
$$\left(\lambda_0^r = \frac{1}{\beta V^2}, \quad V = \int_{M_2} d^2\sigma \sqrt{\gamma(\sigma)} \right)$$

In other hand, we define the "heat kernel" as :

$$K(\sigma,\sigma';\varepsilon) = \sum_n u_n(\sigma)\, u_n(\sigma')\, e^{-\varepsilon \lambda_n}, \qquad (A-7)$$

Coincidence limits.

i) Heat kernel :

By taking into account :

$$K(\sigma,\sigma';\varepsilon) \xrightarrow[\gamma_{\kappa\beta}=\delta_{\kappa\beta}]{\varepsilon \to 0} \frac{1}{4\pi\varepsilon} e^{-\frac{|\sigma-\sigma'|^2}{4\varepsilon}} \equiv \delta^2(\sigma-\sigma') \qquad (A-8)$$
$$\Delta_\gamma K(\sigma,\sigma';\varepsilon) = -\frac{\partial K(\sigma,\sigma';\varepsilon)}{\partial \varepsilon}$$

it's possible to prove [19] :

$$K(\sigma,\sigma;\varepsilon) = \frac{1}{4\pi\varepsilon} + \frac{1}{24\pi} R_2[\gamma] + O(\varepsilon) \qquad (A-9)$$

ii) Propagator :

By considering the metric : $\gamma_{\alpha\beta} = e^{2\varphi}\,\hat{\gamma}_{\alpha\beta}$ and the scaling property : $\Delta_{\gamma} = e^{-2\varphi}\,\Delta_{\hat{\gamma}}$, it's not difficult to prove using the definitions (A-3) and (A-4) :

$$\frac{\delta\, G(e^{2\varphi}\hat{\gamma}_{\alpha\beta};\sigma,\sigma)}{\delta\,\varphi(\sigma)} = \frac{1}{2\pi} \qquad (A-10)$$

$$G(\sigma,\sigma;\varepsilon)\Big|_{\gamma_{\alpha\beta}=e^{2\varphi}\hat{\gamma}} = \frac{1}{2\pi}\varphi + G(\sigma,\sigma;\varepsilon)\Big|_{\hat{\gamma}_{\alpha\beta}}$$

Finally, we take : $\hat{\gamma}_{\alpha\beta} = \delta_{\alpha\beta}$ (sphere) and write down the coincidence limits we are interested in, which follows from (A-9) and (A-10) :

$$K(\sigma,\sigma;\varepsilon) = \frac{1}{4\pi\varepsilon} - \frac{1}{12\pi}e^{-2\varphi}\,\Box\varphi(\sigma) + O(\varepsilon) \qquad (11-a)$$

$$G(\sigma,\sigma;\varepsilon) = -\frac{1}{4\pi}\ln\varepsilon + \frac{1}{2\pi}\varphi(\sigma) \qquad (11-b)$$

$$\left(G(\sigma,\sigma;\varepsilon)\Big|_{S^2} = -\frac{1}{4\pi}\ln\varepsilon \right)$$

$$\partial_\alpha G(\sigma,\sigma';\varepsilon)\Big|_{\sigma'=\sigma} = \frac{1}{4\pi}\partial_\alpha\varphi(\sigma) \qquad (11-c)$$

$$\gamma^{\alpha\beta}(\sigma)\,\partial_\alpha\partial'_\beta\, G(\sigma,\sigma';\varepsilon)\Big|_{\sigma'=\sigma} = e^{-2\varphi}\Big[\frac{1}{6\pi}\Box\varphi + \Big(\frac{1}{4\pi\varepsilon}-\frac{1}{V}\Big)e^{2\varphi} + O(\varepsilon)\Big] \qquad (11-d)$$

or

$$G_{\alpha\alpha} \equiv \partial_\alpha\partial'_\alpha\, G(\sigma,\sigma';\varepsilon)\Big|_{\sigma'=\sigma} = \frac{1}{6\pi}\Box\varphi + \Big(\frac{1}{4\pi\varepsilon}-\frac{1}{V}\Big)e^{2\varphi} + O(\varepsilon)$$

CHAPTER III

THE FRADKIN - TSEYTLIN APPROACH

1 - INTRODUCTION.

As we saw in Chapter I, a free string can be described in terms of an infinite number of its "oscilation modes" (tachyon, graviton, gauge vectorial boson, etc). It was observed time ago that the zero "string size" limit ($\alpha' \to 0$) of scattering amplitudes of different string modes coincides with on-shell scattering amplitudes in a theory of fields associated with the elementary string modes [20]. There is, however, a number of conceptual as well as technical problems in a string theory in this approach:

i) Indirect and not straightforward way (go to $\alpha' \to 0$ limit on-shell scattering amplitudes and then to guess a covariant action from which it could be derived).

ii) Expansions near a flat space-time ("trivial vacuum").

All this made it difficult to understand how a curved space-time could be built of the graviton string mode and thus how a spontaneous compactification from 26 (or 10 in the more "realistic" superstring models) to 4 space-time dimensions could be take place.

Recently, Fradkin and Tseytlin [21] have presented a covariant effective action for the infinite number of fields corresponding to string "excitations", with the hope that it helps to formulate the ground state problem for the string and therefore to look for solutions which solve the compactification problem and also the unitarity problem (tachyon).

2 - THE EFFECTIVE ACTION.

Using euclidean notation, we start with the Bose string action :

$$I_0[X,g] = \frac{1}{a^2} \int_{M_2} d\sigma^2 \sqrt{\gamma(\sigma)}\, \frac{1}{2}\, \gamma^{\alpha\beta}(\sigma)\, \partial_\alpha X^\mu(\sigma)\, \partial_\beta X^\nu\, \delta_{\mu\nu} \qquad (1)$$

The quantum theory is defined by the Polyakov integral:

$$\langle \ldots \rangle = \int_{\text{surfaces}} [D\gamma]\,[DX]\; e^{-I_0[X,\gamma]} \ldots \qquad (2)$$

Let's look at how scattering amplitudes (in D-dimensional space-time !) are usually defined.

Consider the "vertex operator" $V_\Lambda(X)$ associated with the (tree level) $|\lambda\rangle$ state of the string :

Closed string

$$V_\Lambda(x) = \int_{M_2} d\sigma^2 \sqrt{\gamma(\sigma)}\; \delta^D(x - X(\sigma))\; W_\Lambda[X(\sigma)] \qquad (3-a)$$

Open string

$$V_\Lambda(x) = \int_{\partial M_2 \equiv \{\sigma(t)\}} dt\; \sqrt{g(\sigma(t))}\; \delta^D(x - X(\sigma(t)))\; W_\Lambda[X(\sigma(t))] \qquad (3-b)$$

where $W_\Lambda[X]$ is some (covariant) polynnomial expression in derivatives of X^μ(to assure Poincare invariance from the starting). The difference between closed and open strings rests on the well-known fact that the emission (or absorption) of some state of the string can be made from all the world-sheet or from its boundary respectively [22].

For example (in closed strings) :

Tachyon $\qquad W_T[X] = 1$

Graviton $W_G^{\mu\nu}[X] = \gamma^{\alpha\beta}_{(\sigma)}\, \partial_\alpha X^\mu_{(\sigma)}\, \partial_\beta X^\nu_{(\sigma)}$

Antisymmetric tensor $W_B^{\mu\nu}[X] = \varepsilon^{\alpha\beta}_{(\sigma)}\, \partial_\alpha X^\mu_{(\sigma)}\, \partial_\beta X^\nu_{(\sigma)}$

Gauge boson (open str.)

$$W_A^\mu[X(t)] = \dot{X}^\mu_{(\sigma(t))} = \frac{d X^\mu_{(\sigma(t))}}{d t}$$

The Fourier transforms of (3) are :

$$V_\Lambda(p) = \int_{M_2} d^2_\sigma \sqrt{\gamma(\sigma)}\, e^{i p \cdot X(\sigma)}\, W_\Lambda[X(\sigma)] \qquad (3-c)$$

$$V_\Lambda(p) = \int_{\partial M_2} d t \sqrt{\gamma(\sigma(t))}\, e^{i p \cdot X(\sigma(t))}\, W_\Lambda[X(\sigma(t))] \qquad (3-d)$$

which coincides with the usual expression of a vertex operator. The definition of scattering amplitudes in string theory is [22] :

$$A_N^{(\Lambda)}(p_1, \ldots, p_N) = \langle V_{\Lambda_1}(p_1) \ldots V_{\Lambda_N}(p_N) \rangle \qquad (4-a)$$

which in space-time reads :

$$A_N^{(\Lambda)}(x_1, \ldots, x_N) = \langle V_{\Lambda_1}(x_1) \ldots V_{\Lambda_N}(x_N) \rangle \qquad (4-b)$$

(the expectation values are taken according to (2)).

Let's define a "source action" with the help of source fields Φ (X) (one for each mode) :

$$I_F[X, g; \Phi_\Lambda] = \sum_\Lambda \int d^D_x\, \Phi_\Lambda(x)\, V_\Lambda(x) \qquad (5-a)$$

or using (3) :

C.S. $$I_F[X,g;\Phi_\Lambda] = \sum_\Lambda \int_{M_2} d^2\sigma \sqrt{\gamma(\sigma)}\; \Phi_\Lambda(X(\sigma))\, W_\Lambda(X(\sigma)) \qquad (5\text{-}b)$$

O.S. $$I_F[X,g;\Phi_\Lambda] = \sum_\Lambda \int_{\partial M_2} dt \sqrt{\gamma(t)}\; \Phi_\Lambda(X(t))\, W_\Lambda(X(t)) \qquad (5\text{-}c)$$

So, it's straightforward to see that :

$$A_N^{(\Lambda)}(x_1,\dots,x_N) = (-g_c)^N \left.\frac{\delta^N \Gamma[\Phi_\Lambda]}{\delta\Phi_{\Lambda_1}(x_1)\dots\delta\Phi_{\Lambda_N}(x_N)}\right|_{\{\Phi_\Lambda=0\}} \qquad (6\text{-}a)$$

$$\Gamma[\Phi_\Lambda] = \int_{Surfaces} [D\gamma][DX]\, e^{-(I_0[X,\gamma]+I_F[X,\gamma;\Phi_\Lambda])} \qquad (6\text{-}b)$$

The last expression is the effective action proposed in Ref.[21] (g_c is a coupling constant). With it we can obtain the amplitudes in a analogous fashion as in QFT.

A crucial point in (6) is that it's not quite correct. The reason is that, like in QFT, ***the evaluation must be in the vacuum of the theory***, defined by some solution of :

$$\forall \Lambda : \quad \left.\frac{\delta\Gamma[\Phi_\Lambda]}{\delta\Phi_\Lambda(x)}\right|_{\{\Phi_\Lambda=\Phi_{0\Lambda}\}} = 0 \qquad (7\text{-}a)$$

which is the same thing to impose

$$\forall \Lambda : \quad \langle V_\Lambda(x)\rangle = \langle V_\Lambda(p)\rangle = 0 \qquad (7\text{-}b)$$

The expression (6) corresponds to take the trivial vacuum $\Phi_\Lambda(X) = 0$, $\forall\Lambda$; in principle it is not assured that this "vacuum" to be the true vacuum of the theory. Then, we are led to define:

$$A_N^{(\Lambda)}(x_1,\ldots,x_N) = (-g_c)^N \left. \frac{\delta^N \Gamma[\Phi_\Lambda]}{\delta\Phi_{\Lambda_1}(x_1)\ldots\delta\Phi_{\Lambda_N}(x_N)} \right|_{\{\Phi_\Lambda = \Phi_{0\Lambda}\}} \qquad (8)$$

where $\{\Phi_{0\Lambda}\}$ corresponds to some "good" solution of (7). We hope this vacuum to solve the compactification (associated with Λ = graviton) and tachyon problems. Let's rewrite the effective action in a more familiar way, like a string-loop expansion :

$$\Gamma[\Phi_\Lambda] = \sum_{\chi = 2,1,0,\ldots} e^{-\sigma_0 \chi} \; \Gamma_\chi[\Phi_\Lambda] \qquad (9-a)$$

$$\Gamma_\chi[\Phi_\Lambda] = \int_{M_2(\chi)} [D\gamma][DX] \; e^{-(I_0 + I_F)} \qquad (9-b)$$

where $g_c = e^{\sigma_0}$ is the dimensionless coupling constant of the theory and χ is the Euler characteristic of M_2 :

$$\chi(M_2) = \frac{1}{4\pi}\int_{M_2} d\sigma^2 \sqrt{\gamma(\sigma)}\, R_2[\gamma] + \frac{1}{2\pi}\int_{\partial M_2} dt \; K = \qquad (10)$$

$$= 2 - 2g - b$$

where g is the genus (or number of handles), b the number of boundaries and k the extrinsic curvature defined by [6] :

$$t^\alpha \nabla_\alpha t^\beta = K n^\beta \qquad (11)$$

being t^α and n^α the tangent and normal vector (respect to $\gamma_{\alpha\beta}$) to ∂M_2, respectively.

3 - THE OPEN STRING AND THE BORN-INFELD ACTION.

Let's consider now the bosonic (oriented) open string effective action for the massless mode, the gauge boson A_M

(ignoring in this way the other modes, *including* the closed string modes). For simplicity we also take U(1) as the gauge group. So, the effective action to calculate is :

$$\Gamma[A] = \sum_{\chi=1,0,-1} e^{-\sigma_0 \chi} \int_{M_2(\chi)} [D\gamma][DX]\, e^{-I[X,\gamma;A(X)]} \qquad (12)$$

$$I = \frac{1}{2a^2} \int_{M_2} d\sigma^2 \sqrt{\gamma(\sigma)}\, \gamma^{\alpha\beta}(\sigma)\, \partial_\alpha X^\mu(\sigma)\, \partial_\beta X^\mu(\sigma) + i \int_{\partial M_2} dt\, \dot{X}^\mu(t)\, A_\mu(X(t))$$

The metric on M_2 is : $e^2(t) = g_{\alpha\beta}(\sigma(t))\, \dot\sigma^\alpha(t)\, \dot\sigma^\beta(t)$.
As it's well-known, the Weyl mode decouples from the Polyakov integral when D=26 and in flat space. The proposal by Fradkin and Tseytlin [21, 22] is to forget the Weyl mode integration, " postulating that a "critical" string theory must be Weyl invariant in a true vacuum " [23], and define the effective action by :

$$\Gamma[A] = \sum_{\chi} e^{-\sigma_0 \chi} \int d\mu(\tau) \int [DX]\, e^{-I[X,\hat\gamma;A]} \qquad (13)$$

where $\hat\gamma_{\alpha\beta}(\tau)$ is a "standard metric on M_2 and $\{\tau\}$ stands for the Teichmuller parameters; $[D\mu(\tau)]$ is the measure of integration in the moduli space (see Chapter II, Part A, Section 1) to be properly defined .
In order to compute (13), we start splitting X in a constant and non-constant part :

$$X^\mu(\sigma) = y^\mu + \eta^\mu(\sigma) \qquad (14)$$

Then, (13) can be written as an integral over space-time coordinates in the following way :

$$\Gamma[A] = \sum_{\chi} e^{-\sigma_0 \chi} \, \Gamma_\chi[A] \qquad (15-a)$$

$$\Gamma_\chi[A] = \int d\hat{y}^{\rho} \int [\mathcal{D}\mu(\tau)] \, e^{-W[\hat{\gamma};A]} \qquad (15-b)$$

$$e^{-W[\hat{\gamma},A]} = \int [\mathcal{D}\xi^\mu] \, e^{-I[y+\xi;\,\hat{\gamma},\,A(y+\xi)]} \qquad (15-c)$$

where $[\mathcal{D}\xi^\mu]$ goes over non-constant functions on M_2 which satisfy Neumann boundary conditions. Now, we make a central assumption :

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \text{constant.} \qquad (16)$$

Thus our result will have to be considered as the leading order in derivatives of the strenght tensor F . With this assumption we have ($a^2 = 2\pi\alpha' = 1$):

$$I[\xi,\hat{\gamma};F] = I_2 + I_1 = \frac{1}{2}\int_{M_2} d^2\sigma \sqrt{\gamma}\,(\partial\xi)^2 + \frac{i}{2}F_{\mu\nu}(y)\int_{\partial M_2} dt\, \xi^\mu(t)\dot{\xi}^\nu(t)$$

$$e^{-W[\hat{\gamma};F]} = \int [\mathcal{D}\xi^\mu] \, e^{-I[\xi;\hat{\gamma};F]} \qquad (17)$$

Now, let's integrate over $\xi^\mu(\sigma)$ in all *internal* points of M_2 , that is, reducing (17) to a path integral over the boundary (note that (17) is quadratic in ξ^μ and then we "hope" to be possible calculating it). To do this, let's consider the boundary as an union of p simply connected components Ca (i.e., the two circunferences of an annulus) : $\partial M_2 = \bigcup_{a=1}^{p} C_a$ (p = 1 for a disk, 2 for an annulus, etc.); so : $\int_{\partial M} \ldots \equiv \prod_{a=1}^{p} \left(\int_{C_a}\right) \ldots$

In each Ca we introduce a (non-constant) field $\eta^\mu_a(t_a)$, a = 1,...,p , and insert :

$$1 = \prod_{a=1}^{p} \left(\int [D\nu_a^\mu][D\eta_a^\mu] \, e^{i\int_{C_a} dt \, \nu_a^\mu (\xi^\mu|_{C_a} - \eta_a^\mu)} \right) \tag{18-a}$$

where we used the δ-representation :

$$\delta(\xi^\mu|_{C_a} - \eta_a^\mu) = \int_{C_a} [D\nu_a^\mu] \, e^{i\int_{C_a} dt \, \nu_a^\mu (\xi^\mu|_{C_a} - \eta_a^\mu)} \tag{18-b}$$

By inserting (18) in (17) we get :

$$e^{-W[\hat{g},F]} = \int [D\xi^\mu] \, e^{-I[\xi,\hat{g};F]} \int_{\partial M} [D\nu^\mu][D\eta^\mu] \, e^{i\int_{\partial M} dt \, \nu^\mu (\xi^\mu|_{\partial M} - \eta^\mu)} =$$

$$= \int [D\xi^\mu] \, e^{-(I_2 + I_1)} \prod_{a=1}^{p} \left(\int_{C_a} [D\nu_a^\mu][D\eta_a^\mu] \, e^{i\int dt \, \nu_a^\mu (\xi^\mu|_{C_a} - \eta_a^\mu)} \right) \tag{19}$$

$$I_2[\hat{g},\xi] = \frac{1}{2\alpha^2} \int_{M_2} d^2\sigma \sqrt{\hat{g}(\sigma)} \, g^{\alpha\beta}_{(\sigma)} \partial_\alpha \xi^\mu_{(\sigma)} \partial_\beta \xi^\mu_{(\sigma)}$$

$$I_1[\hat{g},\xi;A] = \int_{\partial M_2} dt \, \dot{\xi}^\mu \xi^\nu \, \frac{i}{2} F_{\mu\nu}(y)$$

By using the δ inserted with (18), we can replace $\xi^\mu \to \eta^\mu$ in $I_1[g,\xi,A]$. Then :

$$e^{-W[\hat{g},F]} = \int_{M_2} [D\xi^\mu] \int_{\partial M_2} [D\nu^\mu][D\eta^\mu] \, e^{-I_2[\hat{g};\xi] - I_1[\hat{g};\eta;A]} \, e^{i\int dt \, \nu^\mu (\xi^\mu|_{\partial M} - \eta^\mu)} =$$

$$= \int_{\partial M_2} [D\eta^\mu] \int_{\partial M_2} [D\nu^\mu] \, e^{-I_1[\hat{g},\eta;A] - i\int_{\partial M_2} dt \, \nu^\mu \eta^\mu} \cdot \int_{M_2} [D\xi^\mu] \, e^{-I_2[\hat{g};\xi] + i\int_{\partial M_2} \nu^\mu \xi^\mu dt} =$$

$$= \int_{\partial M_2} [D\eta^\mu] \, e^{-I_1[\hat{g},\eta;A]} \int_{\partial M_2} [D\nu^\mu] \, e^{-i\int_{\partial M_2} dt \, \nu^\mu \eta^\mu} \, e^{-F[\hat{g},\nu]} \, , \tag{20-a}$$

where

$$e^{-F[\hat{g},\nu]} = \int_{M_2} [D\xi^\mu] \, e^{-I_2[\hat{g};\xi] + i\int_{\partial M} dt \, \nu^\mu \xi^\mu|_{\partial M_2}} =$$

$$= \int_{M_2} [D\vec{\gamma}]\, e^{-\frac{1}{2}\int d^2\sigma \sqrt{\hat{\gamma}}\, \partial_\kappa \vec{\gamma}^\mu \partial^\kappa \vec{\gamma}^\mu + i\int_{\partial M} dt\, \nu^\mu(t)\, \vec{\gamma}^\mu(t)} =$$

$$= \int_{M_2} [D\vec{\gamma}^\mu]\, e^{-\frac{1}{2}\int d^2\sigma \sqrt{\hat{\gamma}}\, \vec{\gamma}^\mu \Delta_{\hat{\gamma}} \vec{\gamma}^\mu + i\sum_{a=1}^{p}\int_{C_a} dt_a\, \nu_a^\mu(t_a)\, \vec{\gamma}^\mu(\sigma(t_a))} \qquad (20-6)$$

For doing the integral, let's rewrite :

$$\sum_{a=1}^{p} i\int_{C_a} dt_a\, \nu_a^\mu(t_a)\, \vec{\gamma}^\mu(\sigma(t_a)) = \int_{M_2} d^2\sigma \sqrt{\hat{\gamma}(\sigma)}\, \vec{\gamma}^\mu(\sigma)\, J^\mu(\sigma)$$

$$J^\mu(\sigma) = \sum_a \int_{C_a} dt\; i\nu_a^\mu(t)\, \delta^2(\sigma - \sigma_a(t)) \qquad (21)$$

Then :

$$e^{-F[\hat{\gamma};\nu]} = \int_{M_2} [D\vec{\gamma}]\, e^{-\frac{1}{2}\int_{M_2} d^2\sigma \sqrt{\hat{\gamma}(\sigma)}\, \vec{\gamma}^\mu(\sigma) \Delta_{\hat{\gamma}} \vec{\gamma}^\mu(\sigma) + \int_{M_2} d^2\sigma \sqrt{\hat{\gamma}(\sigma)}\, \vec{\gamma}^\mu(\sigma) J^\mu(\sigma)} =$$

$$= e^{-\frac{D}{2}\ln \det' \hat{\Delta}_{\hat{\gamma}}}\; e^{\frac{1}{2}\langle J^\mu \Delta_{\hat{\gamma}}^{-1} J^\mu \rangle} \qquad (22-7)$$

$$\langle J^\mu \Delta_{\hat{\gamma}}^{-1} J^\mu \rangle = \int_{M_2} d^2\sigma \sqrt{\hat{\gamma}(\sigma)} \int_{M_2} d^2\sigma' \sqrt{\hat{\gamma}(\sigma')}\; J^\mu(\sigma)\, \hat{\Delta}^{-1}_{\hat{\gamma}\,\sigma,\sigma'}\, J^\mu(\sigma') =$$

$$= \int_{M_2} d^2\sigma \sqrt{\hat{\gamma}(\sigma)} \int_{M_2} d^2\sigma' \sqrt{\hat{\gamma}(\sigma')} \sum_{a,b} \int_{C_a} dt_a \int_{C_b} dt_b\; i\nu_a^\mu(t_a)\, i\nu_b^\mu(t_b)\, \delta^2(\sigma - \sigma_a(t)) \cdot \hat{\Delta}^{-1}_{\hat{\gamma}\,\sigma,\sigma'} \cdot \delta^2(\sigma' - \sigma_b(t)) =$$

$$= -\sum_{a,b} \int dt_a\, dt_b\; \nu_a^\mu(t_a)\, G_{ab}(t_a,t_b)\, \nu_b^\mu(t_b) \qquad (22-8)$$

where $G_{ab}(t_a, t_b)$ is the restriction on the components of ∂M_2 of $\Delta^{-1}_{\hat{\gamma}\,\sigma,\sigma'} = N(\sigma, \sigma')$ = Neumann function for the laplacian $\Delta_{\hat{\gamma}}$, that is :

$$\Delta_{\hat{\gamma}}\, N(\sigma,\sigma') = \delta^2(\sigma-\sigma')$$

$$n^\alpha\, \partial_\alpha N(\sigma,\sigma')\big]_{\partial M_2} = 0 \qquad (23)$$

$$G_{ab}(t_a, t_b) = N(\sigma=\sigma_a(t_a), \sigma'=\sigma_b(t_b))$$

Coming back to (20) we find another gaussian integral :

$$e^{-F[\hat{\gamma};\nu]} = e^{-D/2 \ln \det{}' \hat{\Delta}_{\hat{\gamma}} - \frac{1}{2}\sum_{a,b}\int dt\, dt'\, \nu_a^\mu(t)\, G_{ab}(t,t')\, \nu_b^\mu(t')} \qquad (24)$$

So :

$$e^{-W[\hat{\gamma},F]} = \int_{\partial M_2}[D\eta^\mu]\, e^{-\Gamma_1[\hat{\gamma},\eta;A]} (\det{}'\Delta_{\hat{\gamma}})^{-1/2} \int_{\partial M_2}[D\nu^\mu]\, e^{-\frac{1}{2}\langle \nu G \nu\rangle - i\langle \nu \eta\rangle} =$$

$$= (\det{}'\hat{\Delta}_{\hat{\gamma}})^{-D/2} \int_{\partial M_2}[D\eta^\mu]\, e^{-\Gamma_1[\hat{\gamma},\eta;A] - \frac{1}{2}\langle \eta^\mu G^{-1}\eta^\mu\rangle} \cdot (\det{}' G)^{-D/2} \qquad (25)$$

Now, with the assumption (16), and (17) :

$$\Gamma_1[\hat{\gamma},\eta;A] = \frac{i}{2}\tilde{F}_{\mu\nu}\int_{\partial M} dt\, \eta^\mu(t)\, \dot{\eta}^\nu(t)$$

$$\tilde{F}_{\mu\nu} = a^{2'} F_{\mu\nu} = 2\pi\alpha' F_{\mu\nu} \qquad (26)$$

In (25) we have :

$$e^{-W[\hat{\gamma};F]} = (\det{}'\Delta_{\hat{\gamma}} \cdot \det{}' G)^{-D/2} \,\cdot$$

$$\int_{\partial M_2}[D\eta^\mu]\, e^{-\frac{1}{2}\langle \eta^\mu G^{-1}\eta^\mu\rangle + \frac{i}{2}\tilde{F}_{\mu\nu}\int_{\partial M} dt\, \eta^\mu(t)\, \dot{\eta}^\nu(t)} \qquad (27)$$

Being quadratic, we hope it can be solved. As we will see, it is. Then let's consider :

$$Z[F] = \int_{\partial M_2} [D\eta^\mu]\, e^{-\frac{1}{2} < \eta^\mu G^{-1} \eta^\mu > + \frac{i}{2} \bar{F}_{\mu\nu} \int_{\partial M_2} dt\, \dot{\eta}^\mu(t)\, \eta^\nu(t)} \tag{28}$$

For computing it, let's make a O(D)- rotation (remember we are in euclidean space-time !) , putting $\bar{F}_{\mu\nu}$ in a standard block diagonal form :

$$(\bar{F}_{\mu\nu}) = \begin{pmatrix} 0 & \bar{\epsilon}_1 & & & & & \\ -\bar{\epsilon}_1 & 0 & & & & 0 & \\ & & 0 & \bar{\epsilon}_2 & & & \\ & & -\bar{\epsilon}_2 & 0 & & & \\ & & & & \ddots & & \\ & 0 & & & & 0 & \bar{\epsilon}_n \\ & & & & & -\bar{\epsilon}_n & 0 \end{pmatrix} = \tag{29}$$

$$= \begin{pmatrix} \bar{F}_1 & & & \\ & \bar{F}_2 & & \\ & & \ddots & \\ & & & \bar{F}_n \end{pmatrix} , \quad \underset{\sim}{\bar{F}}_K = \begin{pmatrix} 0 & \bar{\epsilon}_K \\ -\bar{\epsilon}_K & 0 \end{pmatrix} , \quad n = D/2$$

Then we can consider (28) in blocks because :

$$\frac{i}{2} \bar{F}_{\mu\nu}(y) \int_{\partial M_2} dt\, \dot{\eta}^\mu(t)\, \eta^\nu(t) = \sum_{K=1}^{n} \frac{i}{2} \int_{\partial M_2} dt\, \dot{\underset{\sim}{\eta}}_K^T\, \underset{\sim}{\bar{F}}_K\, \underset{\sim}{\eta}_K$$

$$\frac{1}{2} < \eta^\mu G^{-1} \eta^\mu > = \sum_{K=1}^{n} \frac{1}{2} < \underset{\sim}{\eta}_K^T\, G^{-1}\, \underset{\sim}{\eta}_K > \tag{30}$$

$$\underset{\sim}{\eta}_K = \begin{pmatrix} \eta_{2K-1} \\ \eta_{2K} \end{pmatrix}$$

Then (28) may be expressed as follows :

$$Z[F] = \prod_{K=1}^{n} Z_K(F) \tag{31-a}$$

$$Z_K[F] = \int [D\eta_1 D\eta_2] \, e^{-\frac{1}{2} \langle \eta^T G^{-1} \eta \rangle + \frac{i}{2} \int_{\partial M_2} dt \, \dot{\eta}^T \underset{\sim}{E}_K \eta} \tag{31-b}$$

$$\dot{\eta}^T \underset{\sim}{E}_K \eta = \frac{1}{2} \epsilon_K (\dot{\eta}_1 \eta_2 - \dot{\eta}_2 \eta_1) \tag{31-c}$$

By interchanging η_1 and η_2 in (31-b) we get :

$$Z_K[F] = \int [D\eta_1 D\eta_2] \, e^{-\frac{1}{2} \langle \eta^1 G^{-1} \eta^1 \rangle - \frac{1}{2} \langle \eta^2 G^{-1} \eta^2 \rangle + i \epsilon_K \langle \dot{\eta}_1 \eta_2 \rangle} =$$

$$= \int [D\eta^1] \, e^{-\frac{1}{2} \langle \eta^1 G^{-1} \eta^1 \rangle} \int [D\eta_2] \, e^{-\frac{1}{2} \langle \eta^2 G^{-1} \eta^2 \rangle + i \epsilon_K \langle \dot{\eta}^1 \eta^2 \rangle} =$$

$$= \int [D\eta] \, e^{-\frac{1}{2} \langle \eta G^{-1} \eta \rangle} \, e^{-\frac{1}{2} \epsilon_K^2 \langle \dot{\eta} G \dot{\eta} \rangle} (\det{}' G^{-1})^{-1/2} \tag{32}$$

The last integral is made to give :

$$Z_K[F] = (\det{}' G) \int [D\tilde{\eta}] \, e^{-\frac{1}{2} \langle \tilde{\eta} \Delta_K \tilde{\eta} \rangle} =$$

$$= (\det{}' G) \cdot (\det{}' \Delta_K)^{-1/2} \tag{33}$$

$$\Delta_K = 1 + \epsilon_K^2 \ddot{G} \cdot G$$

$$\ddot{G} = \frac{d^2 G(t,t')}{dt \, dt'}$$

So :

$$Z[F] = (\det{}' G)^n \prod_{K=1}^{n} (\det \Delta_K)^{-1/2} =$$

$$= (\det{}' G)^{D/2} \prod_{K=1}^{D/2} (\det \Delta_K)^{-1/2} \tag{34}$$

Returning to (27) with this result :

$$e^{-W[\bar{x}, F]} = (\det{}' \Delta_{\bar{x}})^{-D/2} \prod_{K=1}^{D/2} (\det \Delta_K)^{-1/2} \tag{35}$$

From (15) the final expression (exact under the assumption (16) !) is :

$$\Gamma[F] = \sum_{\chi} e^{-G\chi}\, \Gamma_{\chi}[F] \qquad (36\text{-}a)$$

$$\Gamma_{\chi}[F] = \int d\hat{y}^{D} \int_{Teich} [\mathcal{D}\mu(\tau)]\; Z[\hat{\gamma};0]\; \bar{Z}[\hat{\gamma};F] \qquad (36\text{-}b)$$

$$Z[\hat{\gamma};0] = (\det{}'\Delta_{\gamma})^{-D/2}\, \det(\text{Fadeev-Popov})\Big|_{\gamma} \qquad (36\text{-}c)$$

$$\bar{Z}[\hat{\gamma};F] = \prod_{K=1}^{D/2} (\det \Delta_K)^{-1/2} \qquad (36\text{-}d)$$

$$\Delta_K = \hat{1} + \bar{\epsilon}_K^{\,2}\, \ddot{G}\cdot G \;, \qquad \hat{1} \sim \delta_{ab}\,\delta(t_a - t_b) \qquad (36\text{-}e)$$

and G and Δ_K are defined over non-constant functions).

The next step is to carry out the computation over the Teichmuller space. For doing this we must obtain the Neumann function on M_2 , from which we can compute det(Δ_K) and also we must get det($\Delta_{\hat{\gamma}}$) . Finally we must integrate over the {τ} using the proper measure [$\mathcal{D}\mu(\tau)$].

The tree approximation.

In this case, we can take M_2 as the unit disk on the complex plane with the trivial metric : $\hat{g}_{\alpha\beta} = \delta_{\alpha\beta}$. Of course, there are no moduli. The only boundary is the circunference of unit radius , then we parametrize it by :

$$z(\theta) = e^{i\theta} , \quad \theta \in [0, 2\pi) \qquad (37)$$

The corresponding Neumann function is [24] :

$$N(z, z') = -\frac{1}{2\pi} \ln |z - z'| \, |z - \bar{z}'^{-1}| \qquad (38)$$

With the definition (23) we have :

$$G(\theta, \theta') = N(e^{i\theta}, e^{i\theta'}) = -\frac{1}{2\pi} \ln (2 - 2\cos\xi) , \quad \xi = \theta - \theta' \qquad (39)$$

By using the identity [25] :

$$\ln (1 + b^2 - 2b\cos\xi) = -2 \sum_{m=1}^{\infty} \frac{b^m}{m} \cos m\xi , \quad b \leq 1 \qquad (40)$$

we obtain :

$$G(\theta, \theta') = -\frac{2}{2\pi} \sum_{m=1}^{\infty} \frac{\cos m\xi}{m} = \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{\cos m\xi}{m} \qquad (41-a)$$

$$\ddot{G}(\theta, \theta') = \frac{1}{\pi} \sum_{m=1}^{\infty} m \cos m\xi \qquad (41-b)$$

$$\ddot{G} \cdot G(\theta, \theta') = \frac{1}{\pi} \sum_{m=1}^{\infty} \cos m\xi \equiv \bar{\delta}(\xi) \qquad (41-c)$$

where $\bar{\delta}$ is defined on non-constant functions (remember we are on the circunference, and so we can use Fourier expansions; in particular :

$$\delta(\theta-\theta') = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{n=0}^{\infty} \cos n(\theta-\theta')$$

is the Fourier representation of the δ-function; in $\bar{\delta}$ the constant part goes out).

From (36) :

$$\Xi[F] = \prod_{K=1}^{p/2} \left[\det\, (1+\bar{\epsilon}_K^{2})\, \ddot{\sigma}\cdot\sigma \right]^{-1/2} = \prod_{K=1}^{p/2} \left[\det (1+\bar{\epsilon}_K^{2})\, \bar{\delta}(\tau) \right]^{-1/2} =$$

$$= \prod_{K=1}^{p/2} \prod_{m=1}^{\infty} (1+\bar{\epsilon}_K^{2})^{-1} = \prod_{K=1}^{p/2} (1+\bar{\epsilon}_K^{2})^{1/2} \qquad (42)$$

where we have (carefully step !) used the Riemann ξ-function regularization [25] :

$$\prod_{n=1}^{\infty} c^{-1} = \lim_{s\to 0} \prod_{n=1}^{\infty} c^{n^{-s}} = e^{\sum_{n=1}^{\infty} \frac{\ln c}{n^s}} = e^{-\zeta(0)\ln c} =$$

$$= e^{\frac{1}{2}\ln c} \quad , \quad \zeta(0) = -1/2 \qquad (43)$$

With this result we finally get from (36) :

$$\Gamma[F] = g_0^{-2}\, Z_0\, \alpha'^{\,1-p/2} \int d\hat{y}^{p}\, \left[\det(\delta_{\mu\nu} + 2\pi\alpha' F_{\mu\nu}) \right]^{1/2}$$

$$Z_0 = (\det{}' \Delta_\gamma)^{-1/2} \det(F-P)\Big|_{disk} \qquad (44)$$

$$g_0 = e^{-\sigma_0/2}$$

This is the so-called "Born-Infeld action" .

It has also been obtained in Ref.[26] by calculating the β-function associated with the coupling A , showing that, at least in this case, the two methods coincide.

CHAPTER IV

STRING - LOOP CORRECTIONS

1 - INTRODUCTION

The close connection between conformally invariant two-dimensional σ-models and classical (tree level) string physics is by now well understood : the coupling constant functions of the σ-model may be identified with space-time expectation values of the massless string modes; the conformal invariance condition amounts to a set of space-time equations of motion for these modes, all of these equations may be derived from a single space-time effective action and , finally , this action is the one-particle irreducible generating functional for the massless particle string tree S-matrix [16,17].

The natural question at this stage is : how do string-loop effects modify the effective field theory action ? In other words, which procedure we must follow to take into account quantum corrections to the S-matrix generating functional ? As we saw in Chapter II, there is a well-defined string loop expansion parameter, exp(-D(X)), where D(X) is the background dilaton field, and it's plausible both that the background field equations of motion should have a power series expansion in exp(-D(X)) and that they be derivable from a space-time effective action, itself having a power series expansion in exp(-D(X)).

We will follow here the approach pionereed by Lovelace [27] and Fischler and Susskind [28], and best settled and extended by Callan et Al. [29]. We remark this is a subject under current research.

2 - BOSONIC CLOSED STRING THEORY.

That string loops must modify the effective action is obvious. The suggestion mentioned above is as follows : The idea is that the divergences associated with *string-loop perturbation theory* may be eliminated by new counterterm in the σ-model action, over and above those needed to renormalize the usual perturbative divergences, and that the new counterterms can be thought of as generating corrections to the usual renormalization group β-functions (and therefore to the conformal invariance conditions). In other words, the approach is based in letting the renormalization group flow be defined by the sum of the standard and non-standard counterterms; this strategy leads us to "loop-corrected β-functions". By imposing the conditions for vanishing of the conformal anomaly *including this corrections*, we shall get a set of loop-corrected equations (which maybe corresponds to the conformal invariance conditions of some generalized σ-model).
Hopefully, we will be able to find some space-time action we may derive them from. If this field theory action generates the appropiated loop-corrected S-matrix elements of the string theory is something *to be demonstrated.*
To start with, let's consider the closed bosonic string in metric and dilaton background fields $G_{\mu\nu}(X)$ and $D(X)$; the corresponding action is :

$$S = \frac{1}{4\pi\alpha'} \int_{M_2} d^2\sigma \sqrt{\gamma(\sigma)} \left[\gamma^{\alpha\beta}(\sigma)\, G_{\mu\nu}(X(\sigma))\, \partial_\alpha X^\mu(\sigma)\, \partial_\beta X^\nu(\sigma) - \frac{\alpha'}{2} R_2(\gamma)\, D(X(\sigma)) \right] \qquad (1)$$

As we saw in Chapter II, the conformal invariance conditions are (in D =26) :

$$R_{\mu\nu} - \nabla_\mu \nabla_\nu D = 0$$
$$- R + 2\nabla^2 D + (\nabla D)^2 = 0 \qquad (2)$$

Remarkably, this equations are equivalent to the equations of motions for the space-time action :

$$S_{eff}^{closed} = \int d^{26}x \sqrt{G(x)}\, e^{D(x)} \left[R - (\nabla D)^2 - 2\nabla^2 D \right] \qquad (3)$$

The results (2) correspond to the leading order in α' (or field derivatives). In general we'll have a counterterm of the form :

$$\delta S_{\sigma-m} = \frac{\log \Lambda}{2\pi} \int_{M_2} d^2\sigma \sqrt{\gamma(\sigma)} \left[\gamma^{\alpha\rho}\, \delta G_{\mu\nu}\, \partial_\alpha X^\mu \partial_\rho X^\nu - \frac{\alpha'}{2} R_2[\gamma]\, \delta D \right] \qquad (4)$$

where Λ is an ultraviolet cut-off. As it's sketched in the Appendix, considering the divergences coming from the Teichmuller space when we consider loops (torus, etc), we might construct a counterterm to cancel them. In flat space, this counterterm results proportional to the insertion of a simple local operator in a surface of lower genus. Not coincidently, this is the vertex operator for the emision of a zero-momentum dilaton; this divergence has to do with the existence of an amplitud for emitting such a dilaton into the vacuum. So, the divergences on the torus may be eliminated (and the string loop calculation renormalized) by adding a new counterterm of the form :

$$\delta S^{loop} = \frac{\log \Lambda}{2\pi} \int d\sigma^2 \; C \, \eta_{\mu\nu} \partial_\lambda X^\mu \partial^\lambda X^\nu \qquad (5)$$

to the free action , where C is the one-loop cosmological constant as calculated in Ref.[30] and [31] Let's quote that there exists another divergence associated with the tachyon, but being it a non physical feature of the theory will not be considered in what follows [29]

Since this is a dimension two operator, it's a perfectly legal counterterm from the point of view of two-dimensional field theory. When is evaluated on the *sphere*, it gives a divergent contribution which precisaly cancels the divergence of the *torus*. Explicitily, for any operator $\mathcal{O}$:

$$< O > = \int_{S^2} [D\gamma DX] \, e^{-(S + k\delta S^{loop})} O + \int_{T^2} [D\gamma DX] \, e^{-(S + k\delta S^{loop})} + O(k^2)$$

$$= \int_{S^2} [D\gamma DX] \, e^{-S} O + k [\int [D\gamma DX] \, e^{-S} O + \int_{S^2} [DX D\gamma] \, e^{-S} (-\delta S^{loop}) + O(k^2) =$$

$$= < O > |_{S^2} + k [< O > |_{T^2} + < (-\delta S^{loop} O > |_{S^2}] + O(k^2) \qquad (6)$$

The insertion of $(-)\delta S^{loop}$ cancels the divergence on the torus (k is the coupling constant).

Now, we would like to generalize (5) to the case of the string coupled to the backgrounds. This is made more easily taken into account two things :

i) In a general coordinatization of the space-time we must replace the flat metric by $G_{\mu\nu}$,

ii) As it's well-known, in the presence of a constant dilaton field, we must include a factor $e^{-D(x)}$ to take into account the topologically determinated dependence of the path integral on the dilaton zero-mode (the k-factor in (6)) ; locality requieres the inclusion of a factor for a general dilaton background D(X) [32].

So, the counterterm to be considered will be :

$$\delta S^{loop} = \frac{\log \Lambda}{2\pi} \, c \int d\sigma^2 \sqrt{\gamma}\, e^{-D(X)} G_{\mu\nu}(X)\, \gamma^{\lambda\rho} \partial_\lambda X^\mu \partial_\rho X^\nu \qquad (7)$$

Indeed, we must think it as the leading order in derivatives of the fields of the complete counterterm [29]. By combining (4) and (7), as it was explained before, we get the following "loop-corrected" equation for the background metric :

$$\beta^G_{\mu\nu}\Big|_{1-c} = R_{\mu\nu} - \nabla_\mu \nabla_\nu D - c\, e^{-D} G_{\mu\nu} = 0 \qquad (8)$$

This equation resembles the Einstein equation for non-zero cosmological constant C.

There is also a string loop-corrected dilaton equation, in principle, but it's not clear which is the counterterm that generates it. We will determine what it must be by an indirect procedure, that it may be used also at tree level for obtaining the B^D-equation [29], probably to all orders in σ-models perturbation theory [33].

The point is that eq.(8) cannot be compatible with any equation for D ; taking its divergence, and using the equation itself plus the Bianchi identities, it's possible to show that :

$$\nabla^{\nu} \beta^{S}_{\mu\nu}\Big|_{1c} = \nabla_{\mu} \left(\frac{1}{2} R - \nabla^{2} D - \frac{1}{2} (\nabla D)^{2} \right)$$

which means :

$$B^{D} = \frac{1}{2} R - \nabla^{2} D - \frac{1}{2} (\nabla D)^{2} \equiv constant \qquad (9)$$

Putting the constant in (9) equal to zero , the system of equations (8) and (9) results equivalent to that obtained by varying the space-time action :

$$S^{1-c}_{eff} = \int d^{26}x \sqrt{G(x)} \; [\, e^{D} (R - (\nabla D)^{2} - 2 \nabla^{2} D) + 2C \,] \qquad (10)$$

This action is the obvious one-loop, cosmological constant generalization of (3) ; the relative powers in exp(D) distinguish terms arising at different string-loop orders. The up-shot of all this is that the loop-corrected β-function for the metric implies a loop-corrected β-function for the dilaton (which indeed is not corrected according to this method, compare (2) and (9) !), and the mutual consistency of the two equations is guaranteed by the fact that both can be derived from a single space-time action (10).

In the following, we will extend the above construction for the much demanding case of interacting open and closed strings in a non-trivial gauge field background, and point out a remarkable relation.

3 - OPEN STRING-LOOP CORRECTIONS TO β-FUNCTIONS.

Following the strategy of the previous section, we will consider now the effect of open string-loop divergences on the same β-functions. The action corresponding to interacting open and closed strings coupled to all massless backgrounds is given by :

$$S[X; G, B, D, A] = \frac{1}{4\pi\alpha'} \int_{M_2} d^2\sigma \sqrt{\gamma} \left[\gamma^{\alpha\beta} G_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu + \right.$$

$$\left. + \, i \, \epsilon^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu B_{\mu\nu} - \frac{\alpha'}{2} R_2(\gamma) \, D(X) \right] +$$

$$+ \frac{1}{2\pi\alpha'} \oint_{\partial M_2} ds \left[i A_\mu(X(s)) \frac{\partial X^\mu}{\partial s} - \frac{\alpha'}{2} K(s) \, D(X) \right] \qquad (11)$$

where k(t) is the extrinsic curvature of the boundary, and A_μ has been rescaled to include a $(2\pi\alpha')$ factor. The coupling of the dilaton field D to the boundary curvature is needed because exp(-D) is the coupling constant of the theory, and then it must multiply the entire Euler density. Now, let's take M_2 = annulus, with inner radius a and outer radius one. We want to know the proper counterterm to cancel possible divergences coming from the integration on the a-parameter (which is the Teichmuller parameter for the annulus). In order to identify the (local) operator whose insertion *on the disk* reproduce this divergences, we must study the annulus partition function for the action (11) (it's worthwhile to say that this is the formal route to follow ; in the case of the closed string theory analyzed in the past section, we were able *to conjecture* the counterterm (7) generalizing the corresponding one to flat space, eq.(5)).

In Ref. [25] and [26] was calculated the partition function on the annulus for a constant strenght tensor. We quote the result :

$$Z_F^{ann}[F] = \int d^{26}x \; \det(1+F(x)) \; Z_0^{ann} \qquad (12)$$

where Z_0^{ann} is the zero-field partition function for the annulus (without considering the D-dimensional volume factor given by the zero mode, which is taken into account in the X-integration), given by :

$$Z_0^{ann} = \int_0^1 \frac{da}{a^3} \prod_{n=1}^{\infty} (1-a^{2n})^{-(D-2)} = \int_0^1 \frac{da}{a^3} \left[1 + (D-2)a^2 + O(a^4) \right] \qquad (13)$$

It's possible to show with a little bit of algebra that the divergent part of (13) is reproduced by [29] :

$$Z_F^{ann}\Big|_{div} = \int_0^1 \frac{da}{a^3} < (1-2a^2)\sqrt{\det(1+F)} >_{disk} + \int_0^1 \frac{da}{a^3} < 2\frac{a^2}{\alpha'} \sqrt{\det(1+F)} \left(\frac{1+F}{1-F}\right)_{\mu\nu} \partial_z X^\mu \partial_{\bar z} X^\nu >_{disk} \qquad (14)$$

where $z = \tau + i\sigma$, and $< O >_{disk} = \int_{disk} [DX] \, e^{-S} \, O$ for any operator O .

Now, if we cutoff the a-integration at a short distance Λ, we see that the counterterm action needed to compensate for all this divergences is :

$$\delta S^{loop} = \int d^2 z \left[\frac{2}{\alpha'} \log\Lambda \sqrt{\det(1+F)} \left(\frac{1+F}{1-F}\right)_{\mu\nu} \partial_z X^\mu \partial_{\bar z} X^\nu + \right.$$

$$\left. + \left(\Lambda^2 - 2\log\Lambda\right) \sqrt{\det(1+F)} \right] \qquad (15)$$

The second term is a counterterm for dimension zero operators, which generates loop-corrections to the tachyon β-function, that it would renormalize the tachyon background field. For reasons explained before, we will disregard this type of terms, and concentrate ourselves in the renormalization of "realistic" fields.

Our final task is to include the effect of closed string background fields $G_{\mu\nu}$, $B_{\mu\nu}$ and D. This is made as follows:

i) Being exp(-D/2) the coupling constant of the open string theory, a factor exp(-D/2) arises from the relative "weight" between the disk and the annulus (exp(D/2) and 1 respectivaly , because χ(disk) = 1, χ(annulus) = 0);

ii) The flat metric $\delta_{\mu\nu}$ must be replaced by $G_{\mu\nu}$;

iii) By noting that in the presence of a boundary, the "gauge" symmetry : $B_{\mu\nu} \to B_{\mu\nu} + 2\partial_{[\mu}\psi_{\nu]}$ is replaced by the generalized symmetry [23] :

$$\begin{aligned} \delta_\psi B_{\mu\nu} &= 2\,\partial_{[\mu}\psi_{\nu]} \\ \delta_\psi A_\mu &= -\psi_\mu \\ \delta_\psi F_{\mu\nu} &= -\delta_\psi B_{\mu\nu} \end{aligned} \qquad (16)$$

the F-dependence must be in the combination :

$$F_{\mu\nu} + B_{\mu\nu} \qquad (18)$$

(also it's possible to show this by expanding in Riemann normal coordinates the action (11) ; the F-dependence appears in the way given by (18) [29]).

Of course, there are also B-dependence in the form of the strenght tensor $H_{\mu\nu\rho} = 3\,\partial_{[\mu} B_{\nu\rho]}$, and G-dependence in curvatures and its covariant derivatives. So, the result we will write down must be thought as a leading order in derivatives of the background fields.

Taking into account the above remarks, the counterterm that generalize (16) to include closed string background fields will be :

$$\frac{2}{\alpha'} \log \Lambda \int d^2 z \sqrt{\det(G+B+F)}\; e^{-D/2} \left(\frac{G+B+F}{G-B-F}\right)_{\mu\nu} \partial_z X^\mu \partial_{\bar z} X^\nu \tag{19}$$

By following the approach sketched at the beggining, we treat it as a new counterterm for the σ-model. By separating the symmetric and antisymmetric parts we read from (19) the following open string loop corrections to the β-functions of $G_{\mu\nu}$ and $B_{\mu\nu}$:

$$\Delta \beta^G_{\mu\nu} = \frac{2\pi}{\alpha'} e^{-D/2} \sqrt{\det(G+B+F)} \left(\frac{G+(B+F)^2}{G-(B+F)^2}\right)_{\mu\nu} \tag{20}$$

$$\Delta \beta^B_{\mu\nu} = -\frac{4\pi}{\alpha'} e^{-D/2} \sqrt{\det(G+B+F)} \left(\frac{B+F}{G-(B+F)^2}\right)_{\mu\nu}$$

The effective action.

First, let's note that, since we have not kept track of the world-sheet curvature (we didn't consider the metric, or better, the conformal factor, as a dynamical variable) dependence of the counterterms, we cannot directly identify the loop correction to β^D. We can, however, infer what it must be by a consistency (compatibility) argument of the type which led us from (8) to (9). Instead, with some of magic, we will define an effective action that it will reproduce (20), and also will give us the missing dilaton part. The construction is as follows :

i) The effective action at tree level (sphere) for all the massless closed string background fields is (see Chapter II) :

$$S^{closed}_{eff} = \int dx^D \sqrt{G(x)}\, e^{D(x)} \left[R - \frac{1}{3} H^2 - (\nabla D)^2 - 2\nabla^2 D \right] \tag{21}$$

ii) Furthermore, the effective action for the gauge field A_μ at tree level (disk) coupled to the boundary of the world-sheet corresponding to the open string theory is the Born-Infeld action (see Chapter III, Section 3) :

$$S^{open}_{eff} = \int dx^D \sqrt{\det(1+F)} \tag{22}$$

If we consider now the open string coupled to open and closed string backgrounds (eq. (11)), using the arguments that led us from (15) to (19) (an explicit computation is also available in Ref. [23] and [29]) we obtain the generalized Born-Infeld action :

$$S_{eff}^{open} = \int d\hat{x}^{p}\ e^{D/2} \sqrt{\det(G+B+F)} \qquad (23)$$

which is exact for $G_{\mu\nu}$, $B_{\mu\nu}$ and $F_{\mu\nu}$ constants.

Now, although the closed and open string β-functions are logically independent each other (after all, they arise from different σ-models evaluated on different worl-sheets, sphere and disk respectively), it seems "natural" to guess that the "right" way of describing interacting closed and open strings is simply to add the associated effective actions (21) and (23) together, with an as-yet undetermined relative coefficient c. Thus the proposed full effective action is :

$$S_{eff}^{tot} = \int d\hat{x}^{p} \sqrt{G(x)}\ e^{D(x)} [\, R - \tfrac{1}{3} H^2 - (\nabla D)^2 - \nabla^2 D\,] +$$

$$+ \ c \int d\hat{x}^{p}\ e^{D(x)/2} \sqrt{\det(G+B+F)} \qquad (24)$$

This effective action don't exactly reproduce the equations for the closed bosonic sector, because the generalized Born-Infeld action (23) contributes to them. Specifically, we find :

$$\frac{\delta S_{eff}^{tot}}{\delta D(x)} = - e^{D} \beta^{D} + \frac{c}{2} e^{D/2} \sqrt{\det(G+B+F)} \qquad (25\text{-}a)$$

$$\frac{\delta S_{eff}^{tot}}{\delta B_{\mu\nu}(x)} = e^{D} \beta^{B}_{\mu\nu} + \frac{c}{2} e^{D/2} \sqrt{\det(G+B+F)} \left[\frac{B+F}{G-(B+F)^2} \right]_{\mu\nu} \qquad (25\text{-}b)$$

$$\frac{\delta S_{eff}^{tot}}{\delta G^{\mu\nu}} + \frac{1}{2} G_{\mu\nu} \frac{\delta S_{eff}^{tot}}{\delta D(x)} = - e^{D} \beta^{G}_{\mu\nu} - \frac{c}{4} e^{D/2} \sqrt{\det(G+B+F)} \left[\frac{G+(B+F)^2}{G-(B+F)^2} \right]_{\mu\nu}$$

(25-c)

where $B^{G}_{\mu\nu}$, $B^{B}_{\mu\nu}$ and B^{D} were calculated in Chapter II :

$$
\begin{aligned}
\beta^{G}_{\mu\nu} &= R_{\mu\nu} - H^{2}_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}D \\
\beta^{B}_{\mu\nu} &= \tfrac{1}{2}\nabla^{\lambda}H_{\lambda\mu\nu} + \tfrac{1}{2}\nabla^{\lambda}D\, H_{\lambda\mu\nu} \\
\beta^{D} &= -R + \tfrac{1}{3}H^{2} + 2\nabla^{2}D + (\nabla D)^{2}
\end{aligned}
\tag{26}
$$

Now, from (25) we can observe that the c-dependent factors may be considered as loop-corrections to the "old" B-functions. Indeed, if we take

$$c = -8\pi/\alpha' \tag{27}$$

then we reproduce the equations :

$$
\begin{aligned}
\beta^{G}_{\mu\nu}\Big|_{l-c} &= \beta^{G}_{\mu\nu} + \Delta\beta^{G}_{\mu\nu} = 0 \\
\beta^{B}_{\mu\nu}\Big|_{l-c} &= \beta^{B}_{\mu\nu} + \Delta\beta^{B}_{\mu\nu} = 0
\end{aligned}
\tag{28}
$$

with $\Delta\beta^{G}_{\mu\nu}$ and $\Delta\beta^{B}_{\mu\nu}$ given by (20) !

Not only that, we can read from (25) the "missing" correction to β :

$$\Delta\beta^{D} = \frac{4\pi}{\alpha'}\, e^{-D/2}\sqrt{\det(G+B+F)} \tag{23}$$

Summarizing, the action (24) with c given by (27) provides a set of equations of motion for the massless closed string backgrounds which is equivalent to that one obtained by equating to zero the B-functions corrected by *open string-loop effects* given by eqs. (20) and (28).

4 - DISCUSSION AND CONCLUSIONS.

Let's now try to display some physical consecuences from the results obtained, and remark some troubles and uncertanties in the approach.

First, let's expand $\Delta\beta^{G}_{\mu\nu}$ and $\Delta\beta^{B}_{\mu\nu}$ in powers of $F_{\mu\nu}$ for vanishing D and $B_{\mu\nu}$; from (20) :

$$\Delta\beta^{G}_{\mu\nu} = \frac{2\pi}{\alpha'}\left(G_{\mu\nu} - G_{\mu\nu}\frac{F^2}{4} + 2F_{\mu\nu}^{2} + \ldots \right) \qquad (30\text{-}a)$$

$$\Delta\beta^{B}_{\mu\nu} = -\frac{4\pi}{\alpha'}F_{\mu\nu} + \ldots \qquad (30\text{-}b)$$

The first term in $\Delta\beta^{G}_{\mu\nu}$ corresponds to a *finite* contribution to the cosmological constant in the equation of motion for gravity, coming from the open string-loop divergences. This result contrasts with the contribution coming from the closed string-loops (torus), eq. (8), which is *divergent* (see Appendix).

The second term in $\Delta\beta^{G}_{\mu\nu}$ corresponds to the classical contribution of the gauge field to the energy-momentum tensor of matter.

Finally, the leading term in $\Delta\beta^{B}_{\mu\nu}$ expresses the well-known mixing of an abelian gauge field with the antisymmetric tensor field.

For ending, three remarks :

i) It's expected that the action (24) generates the appropiately loop-corrected string S-matrix, but it's worthwhile to stress that a prove of this is lacking ;

ii) A major deffect in this treatment of string-loop renormalization is the lack of a systematic regulation and renormalization procedure; in particular, the identification of both cut-off (that for regularizing σ-model divergences and that for handling modular divergences) is completaly ad-hoc ;

iii) If we think this procedure must work, how to explain the loop-corrections to the effective actions for Superstrings, which are presumibly free of modular divergences ?

APPENDIX : One loop amplitudes and modular divergences.

We shall give here a brief review about one-loop amplitudes in the bosonic closed string theory, mainly interested in the divergences coming from the modular integrations. For a complete treatment, see Ref.[31], where the operatorial formalism is used, as in this appendix. For a discussion following the Polyakov approach, see Ref.[30].

In QFT, a one-loop diagram may be built by "sewing" a propagator with the external legs attached, and summing over the states which circulate into it. By making the analogy with QFT we are led to define the one-loop amplitud in string theory according to :

$$A_M^{(\Lambda)}(k) = \kappa^M \, tr\left(\hat{\Delta} \, \hat{V}_{\Lambda_1}(k_1;1,1) \ldots \hat{\Delta} \, \hat{V}_{\Lambda_M}(k_M;1,1) \right) + \text{(non-cyclic permutations)} \tag{A-1}$$

where $\hat{\Delta}$ is the "propagator" of the string, κ is the coupling constant of the theory and V_Λ (k;1,1) is the vertex operator for absorbing a physical state Λ with momentum k (we use for convenience coordinates z, $\bar{z}$, where $z = \tau - i\sigma$ and τ is the "euclidean proper-time"). The trace is taken over the states which circulate around the loop, which must be ***the physical ones*** (if we work in the "old" covariant formalism described in Chapter I, then we must used ***projectors*** on the physical subespace; in the "new" covariant formalism this fact is taken into account by considering the (Fadeev-Popov !) ghost contribution), and *also includes* a momentum integration. Eq. (A-1) is depicted in Fig.A1 :

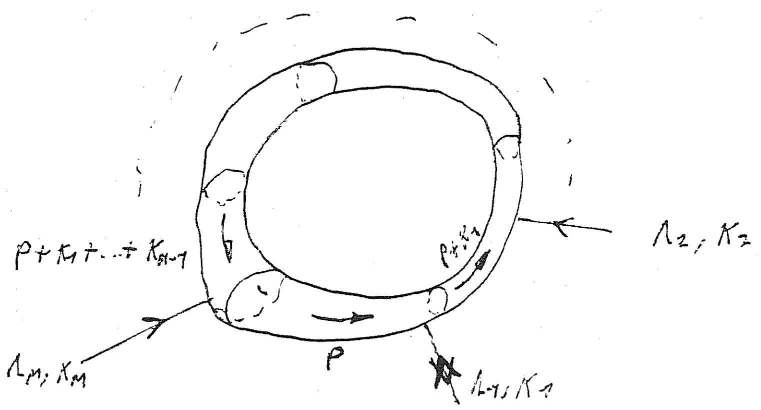


Fig. A1

Following the analogy with QFT, the inverse of the propagator should be the "mass-shell" condition. So, we postulate for the string :

$$\hat{\Delta} = \frac{1}{2}\left(\hat{L}_0 + \hat{\tilde{L}}_0 - 2\right)^{-1} = \frac{1}{2}\int_0^1 d\rho\ \rho^{(\hat{L}_0 + \hat{\tilde{L}}_0 - 3)} \qquad (A-2-a)$$

By taking into account the physical states must have : Lo = $\tilde{\mathrm{L}}$o , we introduce a "δ-function" in (A-2-a) :

$$\delta(\hat{L}_0 - \tilde{L}_0) = \int_0^{2\pi}\frac{d\phi}{2\pi}\ e^{i\phi(L_0 - \tilde{L}_0)} \qquad (A-2-b)$$

and redefine the propagator as ($z = \rho \exp(i\phi)$) :

$$\hat{\Delta} = \frac{1}{4\pi}\int_{|z|\leq 1} d^2z\ \frac{1}{|z|^4}\ z^{\hat{L}_0}\ \bar{z}^{\hat{\tilde{L}}_0} \qquad (A-3)$$

Using the property (which follows from the fact that :

$H = L_0 + \tilde{L}_0$, is the 2-d hamiltonian, and therefore gives the τ-evolution) :

$$z^{\hat{L}_0} \bar{z}^{\hat{\tilde{L}}_0} \hat{V}_\lambda (K;1,1) \bar{z}^{-\hat{\tilde{L}}_0} z^{-\hat{L}_0} = \hat{V}_\lambda (K;z,\bar{z}) \qquad (A-4)$$

eq.(A-1) can be writen as follows :

$$A_M^{(\lambda)}(K) = \left(\frac{K}{4\pi}\right)^M \int_{|w|\leq 1} \frac{d^2 w}{|w|^4} \prod_{r=1}^{M-1} \left[\int_{|w|\leq|\rho_r|\leq 1} \frac{d^2 \rho_r}{|\rho_r|^2} \theta(|\rho_r| - |\rho_{r+1}|) \right] I_M^{(\lambda)}(K;\rho,w)$$

$$+ \text{ (non-cyclic permutations)} \qquad (A-5)$$

$$\begin{aligned} \rho_K &= z_1 \ldots z_K \quad , K = 1, \ldots, M-1 \\ w &= \rho_M = z_1 \ldots z_M \end{aligned} \qquad (A-6)$$

$$I_M^{(\lambda)}(K;\rho,w) = tr\left(\hat{V}_{\lambda_1}(K_1;\rho_1,\bar{\rho}_1) \ldots \hat{V}_{\lambda_M}(K_M;\rho_M,\bar{\rho}_M)\, w^{\hat{L}_0} \bar{w}^{\hat{\tilde{L}}_0} \right)$$

Now, we'll consider the case of scattering of M tachyons, although the relevant conclusions can be applied to a general processes. The tachyonic vertex operator is :

$$\hat{V}_0 (K;z,\bar{z}) = :e^{iK\cdot\hat{X}(z,\bar{z})}: \quad , \quad K^2 = 4(1/\alpha') \qquad (A-8)$$

where $\hat{X}^\mu(z,\bar{z})$ is given in Chapter I, eq.(15).

With some oscillator algebra we can compute (A-7), being the result (in D=26) :

$$I_M(K;\rho,w) = |\epsilon(w)|^{-48} \left(\frac{-4\pi}{\ln|w|}\right)^{13} e^{\sum_{1\leq r<s\leq M} \frac{1}{2} K_r\cdot K_s \ln \chi_{rs}} \qquad (A-9)$$

where :

$$E(w) = \prod_{n=1}^{\infty} (1-w^n) \quad , \text{ Dedkind's function}$$

$$\chi(c,w) = e^{\ln^2|c|/2\ln|w|} \left| \frac{1-c}{c^{1/2}} \prod_{m=1}^{\infty} \frac{(1-w^m c)(1-w^m/c)}{(1-w^m)^2} \right|$$

$$\chi_{rs} = \chi(\rho_s/\rho_r , w) \qquad (A-10)$$

It's convenient to make another change of variables :

$$\nu_r = \ln \rho_r/2\pi i = \nu_{r1} + i\nu_{r2}$$

$$\tau = \nu_M = \ln w/2\pi i = \tau_1 + i\tau_2 \qquad (A-11)$$

In terms of them, we obtain from (A-5) and (A-9) :

$$A_M(K) = \int_F \frac{d^2\tau}{\tau_2^2} \, C(\tau) \, F(\tau) \qquad (A-12)$$

$$C(\tau) = \left(\tfrac{1}{2}\tau_2\right)^{-12} e^{4\pi\tau_2} \left| E(e^{i2\pi\tau}) \right|^{-48}$$

$$F(\tau) = 4(\pi\alpha')^M \tau_2 \prod_{r=1}^{M-1} \left(\int_D d^2\nu_r \right) \prod_{1\le r<s\le M} \chi_{rs}^{K_r \cdot K_s/2}$$

where :

$$D \equiv \{ \nu_1 \in [-1/2, 1/2] \, , \ \nu_2 \in [0, \tau_2] \}$$

$$\chi_{rs} = \chi(\nu_{sr}, \tau) \, , \quad \nu_{sr} = \nu_s - \nu_r \qquad (A-13)$$

$$\chi(\nu,\tau) = 2\pi \, e^{-\pi\nu_2^2/\tau_2} \left| \frac{\theta_1(\nu|\tau)}{\theta_1'(0|\tau)} \right|$$

and $\theta_n(\nu|\tau)$ are the Jacobi theta functions and ' indicates derivative respect to the ν-variable.

At this stage, two remarks are in order :

i) The sum over non-cyclic permutations in (A-5) is automatically taken into account considering the region of integration of the ρ-variables to be : $0 \leqslant |\omega| \leqslant |P_r| \leqslant 1$,

without any other restriction ;

ii) It's possible to show that (A-12-a) is invariant under the change of variables :

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \quad , \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \qquad \text{(A-14)}$$

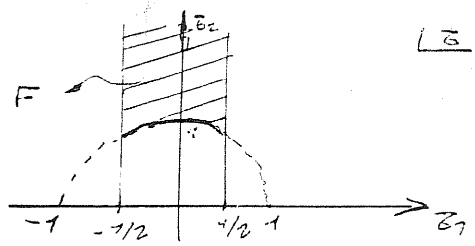
This is the so-called "modular invariance" ; indeed τ represents the usual Teichmuller parameter which labels the conformally non-equivalent metric structures of the torus. This invariance is fundamental for the geometric interpretation of the string theory, because the modular transformation group (SL(2,$\mathbb{Z}$) in this case) is the part of the diffeomorphisms group no connected with the identity, that is, no generated by exponenting infinitesimal reparametrizations.

If we write : $\tau = \tau_1 + i\tau_2$, then (A-11) tells us that the region of integration of τ would be :

$$0 \leqslant \tau_2 < \infty \quad , \quad \tau_1 \in [-1/2, 1/2]$$

But modular invariance prevent to us of considering the infinite strip, because we would get an infinite factor associated with the (infinite) volume of the modular group. Therefore, we are enforced to restrict "by hand" the integration region to a "fundamental one, denoted F in (A-12), so that it has only one copy of each "τ-class" defined by (A-14). In Fig.A2 is depicted the standard F :

Fig. A2



Analysis of divergences.

There exist several possible limits in the region of integration of (A-12) which may lead to divergences.

One possibility is when all ν_r-variables *minus one* are very near among them. This corresponds to isolate the loop on an external state leg (tachyon in our case). This kind of divergence may be interpreted as associated with a mass renormalization of the external state. Although there still are technical problems in computing the appropiated renormalized theory (maybe it will be possible in some framework of "loop-corrected" effective actions, as that we discussed before), there is no conceptual problem (see Ref.[17, 34] for some advances in the treatment of it).

The divergences we are really interested in coming from the region where all the particles approach each other on the torus, that is, $\nu_{rs} \longrightarrow 0$, $\forall$ r,s ($\nu_{rs} = \nu_r - \nu_s$).

To examine this region in a precise mathematical way it's convenient another change of variables :

$(\nu_1, \ldots, \nu_{M-1}; \tau) \quad > \quad (\eta_1, \ldots, \eta_{M-2}, \varepsilon, \phi; \tau)$, $\varepsilon, \phi \in \mathbb{R}$

given by ;

$$\varepsilon \eta_r = \nu_r - \nu_M \quad , \quad r = 1, \ldots, M-2$$

$$\eta_{M-1} = \varepsilon e^{i\phi} = \nu_{M-1} - \nu_M$$

(Note the $\varepsilon \longrightarrow 0$ limit coincides with the region we want to study). By rewritting (A-12) in terms of this new variables, and expanding the integrand in powers of ε, we get the "*dilaton*" *divergent part* [31] :

$$A_M^D(k)\Big|_{div.} = \left[\int_0^1 \frac{d\epsilon}{\epsilon} \int_{-\pi}^{\pi} d\phi \right] A_M^{(0)}(k)\ Z_{torus}$$

$$Z_{torus} = \int_F \frac{d^2\tau}{\tau_2^2}\ C(\tau) \qquad (A\text{-}17)$$

Here $A_M^0(k)$ is an M-point function *on the sphere* and Z_{torus} is the vacuum-to-vacuum amplitud on the torus [30,31]. This divergence comes from the WS configurations of the kind depicted in Fig.A3, where the factorized form of (A-17-a) is explicit. It can be interpreted as due to the absortion for the vacuum of a zero-momentum dilaton ; the integration along the "long-neck" in Fig.A3 corresponds to a propagator of dilaton evaluated a zero-momentum, which of course is divergent [28].

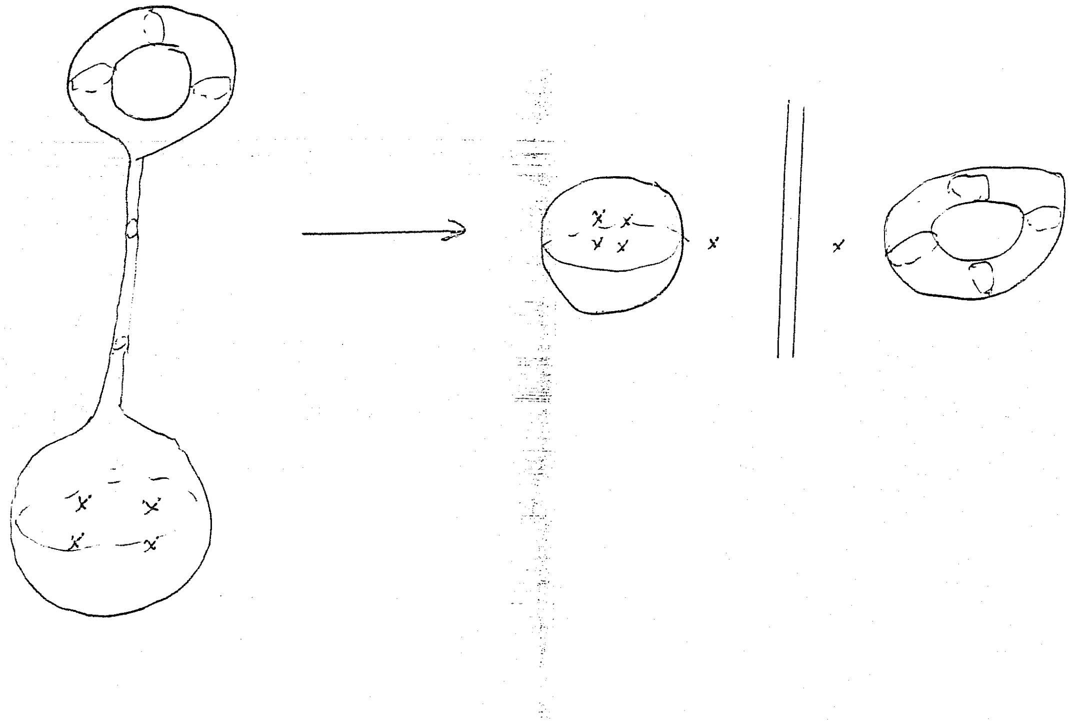


Fig. A3

Now, with the relation :

$$< A[X]\, V_0^P[X] >_{M_2} = \int_{M_2} [DX\,D\gamma]\, e^{-\frac{1}{4\pi\alpha'} V_0^P[X,\gamma]}\, A[X] = 8\pi\alpha'^2 \frac{\partial}{\partial\alpha'} < A[X] >_{M_2} \qquad (A-18)$$

where $V_0^P = \int d^2\sigma \sqrt{\gamma}\, \gamma^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X_\mu$ is the vertex operator for emitting a zero-momentum dilaton, we get :

$$< V_0^P[X] >_{T^2} = 8\pi\alpha'^2 \frac{\partial}{\partial\alpha'} Z_{torus} \propto Z_{torus} \propto C \qquad (A-19)$$

So, we can write (A-17) as :

$$A_M(K)\Big|_{div}^{torus} = const.\ \log\Lambda\ C\ A_M^{sphere}(K) \qquad (A-20)$$

Finally, let's consider a V_0^P -insertion *on the sphere* (thinking about (6)) ; using (A-18) again :

$$< \prod_{k_i} V_{k_i}(p_i)\, \frac{\log\Lambda}{2\pi}\, C\, V_0^P >_{S^2} = 8\pi\alpha'^2 \frac{\partial}{\partial\alpha'} < \prod_{k_i} V_{k_i}(p_i) >_{S^2} \frac{\log\Lambda}{2\pi} C =$$

$$= < \prod_{k_i} V_{k_i}(p_i)\, \delta S^{loop} >_{S^2} = const.\ \log\Lambda\ C\ A_M^{S^2}(p) \qquad (A-21)$$

from where we see, comparing with (A-**10**), that the "counterterm" δS given by (5) can cancel the dilaton divergence on the torus.

As a final remark, we can observe from (A-12) that the cosmological constant C is infrared ($\tau_2 \rightarrow \infty$) divergent, due to the presence of the exponential factor (associated with the tachyon circulating into the loop [31]).

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